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Perturbative Prepotential and Monodromies in N=2 Heterotic Superstring*

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Abstract

We discuss the prepotential describing the effective field theory of N=2 heterotic superstring models. At the one loop-level the prepotential develops logarithmic singularities due to the appearance of charged massless states at particular surfaces in the moduli space of vector multiplets. These singularities modify the classical duality symmetry group which now becomes a representation of the fundamental group of the moduli space minus the singular surfaces. For the simplest two-moduli case, this fundamental group turns out to be a certain braid group and we determine the resulting full duality transformations of the prepotential, which are exact in perturbation theory.

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1. Introduction

A N=2 supersymmetric gauge theory [1] is completely defined by its prepotential – an analytic function of vector superfields. This analytic structure is very restrictive and can be used to obtain interesting information about perturbative as well as non-perturbative behaviour of the theory [2]. Recently, Seiberg and Witten [3] constructed a complete solution of the $SU(2)$ model, and their analysis has been extended to larger gauge groups in refs.[4]. The central point of these studies is the prepotential describing the massless moduli fields whose vacuum expectation values break the gauge group down to an abelian subgroup. It is a very interesting question whether some similar methods could be employed to analyse the moduli space of superstring theories.

N=2 supersymmetric, (4,4) [5] orbifold compactifications of heterotic superstring theory provide some simplest examples of string moduli spaces analogous to the globally supersymmetric spaces considered in refs.[3, 4]. A special feature of these models is the existence of $U(1) \otimes U(1)$ gauge group associated with an untwisted orbifold plane. Such a plane is parametrized by two complex moduli, T and U, of (1,1) and (1,2) type, respectively. The tree-level duality group which leaves the mass spectrum and interactions invariant is $O(2,2;Z)$ [6], which is isomorphic to the product of $SL(2,Z)_T$ and $SL(2,Z)_U$ together with the Z_2 exchange of T and U. The $U(1) \otimes U(1)$ gauge group becomes enhanced to $SU(2) \otimes U(1)$ along the $T = U$ line, and further enhanced to $SO(4)$ or to $SU(3)$ at $T = U = i$ and $T = U = \rho(=e^{2\pi i/3})$, respectively [7]. In this work, we first analyse the perturbative dependence of the prepotential on this type of moduli, and determine its monodromy properties. Because of the N=2 non-renormalization theorems this amounts to computing the one-loop contributions to the prepotential, as all higher loop corrections vanish. At the one-loop level the prepotential develops logarithmic singularity due to the appearance of the additional massless states at the enhanced symmetry subspaces. As a result, we show that the duality group is modified to a representation of the fundamental group of the 4-dimensional space obtained by taking the product of the fundamental domains of the T and U moduli and removing the diagonal locus. One of the consequences of this modification is that at the quantum level the $SL(2,Z)_T$ and $SL(2,Z)_U$ duality transformations do not commute and also that the T, U exchange becomes an element of infinite order. The monodromies associated with moving a point around the singular locus generate a normal abelian subgroup of the full monodromy group depending on 9 integer parameters. In addition, there is the usual dilaton shift which commutes with the above duality group.

In N=2 heterotic superstrings in four dimensions, the T, U moduli together with the dilaton-axion S modulus belong to vector multiplets, so their effective field theory is described by a N=2 supergravity theory [8] coupled to these three vector multiplets. At a generic point of the moduli space and in the absence of charged massless matter (hypermultiplet) states, the effective field theory which is obtained by integrating out all massive string states is local. Its underlying geometric structure is "special geometry" [9], the same structure that appears in the discussion of the moduli sector of superstrings compactified on Calabi-Yau threefolds. The symplectic structure based on $Sp(2r)$ for rigid Yang-Mills theories with gauge group G broken to $U(1)^r$ (r being the rank of G) is here extended to $Sp(2r+4)$, due to the presence of the additional S-vector multiplet and the graviphoton. For a generic (4,4) compactification of the heterotic superstring on $T_2 \times K_3$, we expect 17 moduli ($r=17$) and a symplectic structure $Sp(38;Z)$. For a general (4,0) compactification one can also obtain other values of r up to a maximum of 22. The classical moduli space of vector multiplets in these theories is

$$\left.\frac{SU(1,1)}{U(1)}\right|_{\text{dilaton}} \times \left. \frac{O(2,r)}{O(2)\times O(r)} \middle/ \Gamma \right.$$

where $\Gamma = O(2,r;Z)$. At a generic point of this moduli space the gauge group is $U(1)^{r+2}$

and there are no massless charged hypermultiplets. As in the $O(2,2)$ case there are again complex co-dimension 1 surfaces where either one of the $U(1)$'s is enhanced to $SU(2)$ and/or some charged matter hypermultiplets appear. The one-loop prepotential develops logarithmic singularities near these surfaces. We study the modifications of the duality group due to these singularities.

This paper is organized as follows. In section 2, we derive the perturbative prepotential in N=2 orbifold compactifications of the heterotic superstring and study its dependence on the T, U moduli associated with the untwisted plane. In section 3, we determine the quantum monodromies of the one-loop prepotential. These monodromies are further exploited in section 4, by introducing the usual N=2 supergravity basis for the fields where all transformations act linearly. We thus find that the duality group $O(2,2;Z)$ is extended to a bigger group which is contained in $Sp(8,Z)$ symplectic transformations and depends on 15 integer parameters. In section 5, we generalize these results to the full vector moduli space ($r = 17$) for arbitrary N=2 (4,4) compactifications. In section 6, we discuss generalizations to (4,0) compactifications. We also give an explicit orbifold example of two moduli T, U of the untwisted 2-torus T^2, where the orbifold group acts as shifts on the T^2. In this case one encounters singularities associated with the appearance of charged massless hypermultiplets, as well. Finally, section 7 contains concluding remarks.

2. String computation of the one-loop prepotential

The simplest way to determine the one-loop correction to the prepotential is to reconstruct it from the Kähler metric of moduli fields. Indeed, the Kähler potential of a N=2 locally supersymmetric theory can be written as

$$K = -\ln(iY)\,, \qquad\qquad Y = 2F - 2\bar{F} - \textstyle\sum_Z (Z-\bar{Z})(F_Z + \bar{F}_Z)\,, \tag{2.1}$$

where F is the analytic prepotential, $F_Z \equiv \partial_Z F$, and the summation extends over all chiral (N=2 vector) superfields Z [8]. The part of the prepotential that depends on the moduli of the untwisted plane can be written as

$$F \;=\; STU + f(T,U)\,, \tag{2.2}$$

where the first term proportional to the dilaton, is the tree-level contribution, and the one-loop correction is contained in a dilaton-independent function $f(T,U)$. In our conventions S is defined such that $\langle S\rangle = \frac{\theta}{\pi} + i\frac{8\pi}{g^2}$ where g is the string coupling constant and θ the usual θ-angle. Thus the one loop moduli metric is

$$K^{(1)}_{Z\bar{Z}} \;=\; \frac{2i}{S-\bar{S}} G^{(1)}_{Z\bar{Z}} \tag{2.3}$$

with

$$G^{(1)}_{T\bar{T}} \;=\; \frac{i}{2(T-\bar{T})^2}(\partial_T - \frac{2}{T-\bar{T}})(\partial_U - \frac{2}{U-\bar{U}})f \;+\; \text{c.c.} \tag{2.4}$$

and similar expressions for other components. Our first goal is to extract the function $f(T,U)$ from the moduli metric obtained in ref.[10] by means of a direct superstring computation.

In ref.[10], the $G^{(1)}_{T\bar{T}}$ component of the metric has been written as

$$G^{(1)}_{T\bar{T}} = \mathcal{I}\, G^{(0)}_{T\bar{T}}\,, \tag{2.5}$$

where $G^{(0)}_{T\bar{T}} = -(T-\bar{T})^{-2}$ is the tree-level metric,[1] and the world-sheet integral

$$\mathcal{I} = \int \frac{d^2\tau}{{\tau_2}^2} \bar{F}(\bar{\tau})\, \partial_{\bar{\tau}}(\tau_2 \sum_{p_L,p_R} e^{\pi i\tau\,|p_L|^2}\, e^{-\pi i\bar{\tau}\,|p_R|^2}) \tag{2.6}$$

extends over the fundamental domain of the modular parameter $\tau \equiv \tau_1 + i\tau_2$. In eq.(2.6), $\bar{F}(\bar{\tau}) = \overline{F(\tau)}$, where $F(\tau)$ is a moduli-independent meromorphic form of weight -2 with

[1] A $Z \to iZ$ rescaling on the chiral fields of ref.[10] is necessary to recover the chiral fields as defined here.

a simple pole at infinity due to the tachyon of the bosonic sector. This in fact fixes F completely up to a multiplicative constant:

$$F(\tau) = -\frac{1}{\pi}\frac{j(\tau)[j(\tau)-j(i)]}{j_\tau(\tau)} \tag{2.7}$$

where j is the meromorphic function with a simple pole with residue 1 at infinity and a third order zero at $\tau = \rho$. The summation inside the integral extends over the left- and right-moving momenta in the untwisted orbifold plane. These momenta are parametrized as

$$p_L = \frac{1}{\sqrt{2\,\mathrm{Im}T\,\mathrm{Im}U}}(m_1 + m_2\bar{U} + n_1\bar{T} + n_2\bar{T}\bar{U}) \tag{2.8}$$

$$p_R = \frac{1}{\sqrt{2\,\mathrm{Im}T\,\mathrm{Im}U}}(m_1 + m_2\bar{U} + n_1 T + n_2 T\bar{U}) \tag{2.9}$$

with integer m_1, m_2, n_1 and n_2.

In ref.[10] it has been shown that the integral $\mathcal{I}$ satisfies the differential equation

$$[\partial_T\partial_{\bar{T}} + \frac{2}{(T-\bar{T})^2}]\mathcal{I} = -\frac{4}{(T-\bar{T})^2}\int d^2\tau\,\bar{F}(\bar{\tau})\,\partial_\tau(\partial_{\bar{\tau}}^2 + \frac{i}{\tau_2}\partial_{\bar{\tau}})(\tau_2\sum_{p_L,p_R} e^{\pi i\tau\,|p_L|^2}\,e^{-\pi i\bar{\tau}\,|p_R|^2})\,. \tag{2.10}$$

The r.h.s. being a total derivative with respect to τ vanishes away from the enhanced symmetric points $T = U$ (modulo $SL(2,Z)$). However, as it has been pointed out by Kaplunovsky [11], the surface term gives rise to a δ-function due to singularities associated with the additional massless particles at $T = U$. They correspond to lattice momenta (2.8), (2.9) with $m_1 = n_2 = 0$ and $m_2 = -n_1 = \pm 1$, so that $p_L = 0$ and $p_R = \pm i\sqrt{2}$. These are the two additional gauge multiplets which enhace the gauge symmetry to $SU(2)\times U(1)$. Expanding p_L, p_R around $T = U$ for these states, it is easy to show that the surface term becomes proportional to:

$$\lim_{\tau_2\to\infty}\tau_2 e^{-\frac{\pi\tau_2|T-U|^2}{2\mathrm{Im}T\mathrm{Im}U}} \sim \delta^{(2)}(T-U)\,.$$

Note that there are two special points on the $T=U$ plane (modulo $SL(2,Z)$) where the gauge symmetry is further enhanced: $T=U=i$ giving rise to $SO(4)$ and $T=U=\rho$ to $SU(3)$, ρ being the cubic root of unity. We will comment on these special points later. To solve eq.(2.10) we will stay away from the singular region and we will take into account the singularity structure by suitable boundary conditions. We therefore have the following equations:

$$[\partial_T\partial_{\bar T}+\frac{2}{(T-\bar T)^2}]\mathcal{I}=[\partial_U\partial_{\bar U}+\frac{2}{(U-\bar U)^2}]\mathcal{I}=0\,. \tag{2.11}$$

The general solution of eqs.(2.11) is

$$\mathcal{I}=\frac{1}{2i}(\partial_T-\frac{2}{T-\bar T})[(\partial_U-\frac{2}{U-\bar U})f(T,U)+(\partial_{\bar U}+\frac{2}{U-\bar U})\tilde f(T,\bar U)]+c.c.\,, \tag{2.12}$$

where f and $\tilde f$ depend only on the indicated variables. The above equation is not in the form (2.4) dictated by N=2 supersymmetry due to the presence of $\tilde f$ but we will now show that the latter vanishes. Taking appropriate derivatives of eq.(2.12) one finds the following identity:

$$D_{\bar U}\partial_{\bar U}D_T\partial_T\mathcal{I}=\frac{1}{2i}\partial_T^3\partial_{\bar U}^3\tilde f\ , \tag{2.13}$$

where the covariant derivative $D_T=\partial_T+\frac{2}{T-\bar T}$. Now we can evaluate the l.h.s. of the above equation by using the explicit string expression (2.6) for $\mathcal{I}$ with the forms (2.8) and (2.9) for the lattice momenta. The result is:

$$\partial_T^3\partial_{\bar U}^3\tilde f=-\frac{16\pi^2}{(T-\bar T)^2(U-\bar U)^2}\int\frac{d^2\tau}{{\tau_2}^2}\bar F(\bar\tau)\,\partial_{\bar\tau}(\tau_2^2\partial_\tau(\tau_2^2\partial_\tau(\tau_2\sum_{p_L,p_R}\bar p_R^4e^{\pi i\tau\,|p_L|^2}\,e^{-\pi i\bar\tau\,|p_R|^2}))) \tag{2.14}$$

One can show that the r.h.s. is a total derivative in τ and vanishes away from the enhanced symmetric points. As a result, the general solution for $\tilde f$ is a quadratic polynomial in T and $\bar U$. However such a polynomial can be reabsorbed in the function $f(T,U)$, as can be seen from the expression (2.12) for $\mathcal{I}$. Therefore without loss of generality we can set $\tilde f=0$. This result is compatible with N=2 supersymmetry, as seen from eqs.(2.5) and (2.4) and the function f appearing in (2.12) can be identified with the one loop correction to the prepotential (2.2).

Our next task is to determine f. Equation (2.12) has no simple holomorphic structure, therefore it is not suitable for exploiting the holomorphy property of the prepotential. However, a simpler equation can be obtained by taking appropriate derivatives as in the case of $\tilde{f}$ above. It can be shown that

$$-i(U-\bar{U})^2 D_T \partial_T \partial_{\bar{U}} \mathcal{I} = \partial_T^3 f \ . \tag{2.15}$$

A straightforward calculation utilizing eqs.(2.8) and (2.9) yields

$$f_{TTT} = 8\pi^2 \frac{U-\bar{U}}{(T-\bar{T})^2} \int \frac{d^2\tau}{{\tau_2}^2} \bar{F}(\bar{\tau})\, \partial_{\bar{\tau}} [{\tau_2}^2 \partial_\tau ({\tau_2}^2 \sum_{p_L,p_R} p_L \bar{p}_R^3\, e^{\pi i \tau\, |p_L|^2}\, e^{-\pi i \bar{\tau}\, |p_R|^2})] \, . \tag{2.16}$$

The r.h.s. can be further simplified by integrating by parts. The boundary term is vanishing away from the enhanced symmetry points and the result is:

$$f_{TTT} = 4\pi^2 \frac{U-\bar{U}}{(T-\bar{T})^2} \int d^2\tau\ \bar{F}(\bar{\tau}) \sum_{p_L,p_R} p_L \bar{p}_R^3\, e^{\pi i \tau\, |p_L|^2}\, e^{-\pi i \bar{\tau}\, |p_R|^2} \, . \tag{2.17}$$

The r.h.s. of the above equation is indeed an analytic function of T and U, as can be verified by taking derivatives with respect to $\bar{T}$ or $\bar{U}$. The resulting expressions are total derivatives in τ and vanish upon integration.

We now employ the $SL(2,Z)_T \otimes SL(2,Z)_U$ spacetime duality symmetry in order to further determine the r.h.s. of (2.17). Under $SL(2,Z)_T$ transformations,

$$T \to \frac{aT+b}{cT+d} \, , \tag{2.18}$$

the lattice momenta (2.8), (2.9) transform as $(p_L, \bar{p}_R) \to ((cT+d)/(c\bar{T}+d))^{1/2}(p_L, \bar{p}_R)$ modulo relabeling of the integers m_i, n_i. Similarly under $SL(2,Z)_U$ transformations, they transform as $(p_L, p_R) \to ((cU+d)/(c\bar{U}+d))^{1/2}(p_L, p_R)$. Using these properties one can verify that the r.h.s. of eq.(2.17) behaves like a meromorphic modular function of weight 4 in T and

-2 in U. Furthermore, the only sigularity in the T, U plane (including infinities) is a simple pole at $T = U$ (modulo $SL(2,Z)_U$). Indeed, by expanding p_L and p_R around $T = U$ for the additional massless states, one finds that the r.h.s. behaves like $\int d\tau_2 (\bar{T} - \bar{U}) e^{-\frac{\pi\tau_2 |T-U|^2}{2\mathrm{Im}T\mathrm{Im}U}} \sim 1/(T-U)$. Following the standard theorems of modular forms, we find

$$f_{TTT} = \frac{j(U)\,[j(U) - j(i)]}{j_U(U)\,[j(U) - j(T)]}\, h(T)\,, \tag{2.19}$$

where j is defined below eq.(2.7) and $h(T)$ is a meromorphic modular function of weight 4, with at most a first order pole at infinity. Inspection of the integral (2.17) shows that $f_{TTT} \to 0$ as $T \to i\infty$ which implies that $h(T)$ must be holomorphic everywhere. This therefore fixes f_{TTT} uniquely to:[2]

$$f_{TTT} = -\frac{2i}{\pi}\frac{j_T(T)}{j(T) - j(U)} \left\{\frac{j(U)}{j(T)}\right\} \left\{\frac{j_T(T)}{j_U(U)}\right\} \left\{\frac{j(U) - j(i)}{j(T) - j(i)}\right\} \equiv 2W(T,U). \tag{2.20}$$

The function f_{UUU} is obtained from eq.(2.20) by replacing $T \leftrightarrow U$. A tedious calculation shows that the result is consistent with the integrability condition

$$\partial_U^3 f_{TTT} = \partial_T^3 f_{UUU}\,, \tag{2.21}$$

which is necessary for the existence of the prepotential $f(T,U)$.

In order to find a solution f for the above differential equations, it is convenient to introduce the following closed meromorphic one-form ω:

$$\omega(T,U;T',U') = dT' Q(U,U')(T-T')^2 W(T',U') + dU' Q(T,T')(U-U')^2 W(U',T'), \tag{2.22}$$

where $Q(x,x')$ is the second order differential operator defined as:

$$Q(x,x') = \frac{1}{2}(x-x')^2 \partial_{x'}^2 + (x-x')\partial_{x'} + 1. \tag{2.23}$$

[2] This result was also known to V. Kaplunovsky [11], as recently reported by B. de Wit, V. Kaplunovsky, J. Louis and D. Lüst in hep-th/9504006.

Using the property $\partial_{x'}Q(x,x') = \frac{1}{2}(x-x')^2\partial^3_{x'}$ and the integrability condition (2.21), one can indeed prove that ω is closed, namely: $d'\omega = 0$, where $d' \equiv dU'\partial_{U'} + dT'\partial_{T'}$. For non-singular (T,U), one can show that the following line integral of ω satisfies the differential equations for $f(T,U)$, therefore defining the latter up to a quadratic polynomial in T and U:

$$f(T,U) = \int_{(T^0,U^0)}^{(T,U)} \omega(T,U;T',U'), \tag{2.24}$$

where (T^0,U^0) is an arbitrary base point (outside the singular locus of ω), different choices of the base point modifying $f(T,U)$ by a quadratic polynomial, as is evident from the fact that ω is quadratic in T,U. The path of integration in (2.24) is chosen such that it does not cross any singularity. Note that the complement of the singular locus is connected and therefore such a path always exists, however this complement is not simply connected, and as a result the above line integral depends on the homology class of the integration path. Different choices of homology classes of paths will alter f by quadratic polynomials in T,U. This ambiguity is related to the non-trivial quantum monodromies which will be discussed in the next section.

The other important point concerns the transformation properties of $f(T,U)$ under the action of $PSL(2,Z)$ on T and U. From the equation defining ω, it follows that under $T \to T_g \equiv \frac{aT+b}{cT+d}$ we have:

$$\omega(T_g,U;T'_g,U') = (cT+d)^{-2}\omega(T,U;T',U'). \tag{2.25}$$

Using this property in (2.24) one can derive the following equation:

$$f(T_g,U) = (cT+d)^{-2}[f(T,U) + \int_{(T^0_{g^{-1}},U^0)}^{(T^0,U^0)} \omega(T,U;T',U')]. \tag{2.26}$$

The homology class of path defining the second term of the r.h.s. of this equation is determined by those defining $f(T,U)$ and $f(T_g,U)$. We will be more precise on this point in the next section, however we note here that equation (2.26) implies that f transforms with weight -2 in T up to a quadratic polynomial in T, U coming from the second term in the r.h.s. of (2.26). The same transformation properties hold for the U variable. Similarly under T, U exchange one can show that:

$$f(U,T) = f(T,U) + \int_{(U^0,T^0)}^{(T^0,U^0)} \omega(T,U;T',U'), \tag{2.27}$$

implying again that f picks an additive quadratic polynomial.

When U is one of the fixed points of the modular group $SL(2,Z)_U$ (e.g. the order 2 fixed point $U = i$ or the order 3 fixed point $U = \rho$), f_{TTT} vanishes. Let us consider the behaviour of f_{TTT} at generic U away from these fixed points. As mentioned above, eq.(2.20) is singular as T approaches $U_g = \frac{aU+b}{cU+d}$ where g is an $SL(2,Z)$ element:

$$f_{TTT} \rightarrow -\frac{2i}{\pi}\frac{1}{T-U_g}(cU+d)^2 \ . \tag{2.28}$$

Note that if U_g is one of the fixed points then one must sum over the residues around the poles $1/(T-U_{gg'})$ where g' is an element of the little group of U_g. It is easy to verify that the resulting sum vanishes consistent with the fact that f_{TTT} is zero at these points. Upon integration, the limit (2.28) becomes

$$f(T,U) \rightarrow -\frac{i}{\pi}[(cU+d)T-(aU+b)]^2 \ln(T-U_g)\,, \tag{2.29}$$

giving rise to a branch cut starting at $T = U_g$. When U_g is not one of the fixed points, it follows from eq.(2.4) that

$$G^{(1)}_{T\bar{T}} \rightarrow \frac{1}{\pi} \ln|T-U_g|^2\, G^{(0)}_{T\bar{T}} \ . \tag{2.30}$$

When U_g is one of the fixed points then the summation over the little group of U_g introduces a multiplicative factor 2 or 3 for the fixed points of order 2 or 3, corresponding to the enhanced symmetries $SO(4)$ or $SU(3)$ respectively.

The singular behaviour (2.30) of the modulus (and its N=2 superpartners) wave function renormalization factor can be understood within the framework of effective field theory.

It is due to infrared divergences which arise in the presence of massless particles carrying non-zero charges with respect to the $U(1)$ gauge group associated with the N=2 vector multiplet of T. The field-theoretical result is

$$G^{(1)}_{T\bar{T}} \to \frac{1}{2\pi}\sum_a e_a^2 \ln m_a^2\, G^{(0)}_{T\bar{T}} \ , \tag{2.31}$$

where e_a and $m_a \propto |T-U_g|$ are the charges and masses, respectively, of N=2 vector multiplets that become massless in the $T\to U_g$ limit. These multiplets do indeed carry non-zero charges, and it is not difficult to show that eq.(2.31) agrees with eq.(2.30). The multiplicative factors of 2 and 3 at the fixed points of order 2 and 3 respectively arise due to the presence of additional charged massless states corresponding to the gauge groups $SO(4)$ and $SU(3)$. Indeed, the ratio 1:2:3 corresponds the the ratio of 1/2 of the $SU(2)$ β-function to the β-functions of $SO(4)$ and $SU(3)$. The factor 1/2 is due to the fact that the field which has well-defined quantum numbers under $SU(2)$ is not T itself but the combination $(T-U)$.

3. Monodromies of the one-loop prepotential

Now we turn to the question of the monodromy group that acts on f. At the classical level there is the usual action of the modular group acting on T and U upper half planes, namely $PSL(2,Z)_T\otimes PSL(2,Z)_U$. The $PSL(2,Z)_T$ subgroup of the $PSL(2,Z)_T\otimes PSL(2,Z)_U$ modular symmetry group is generated by the transformations

$$g_1:\ T\to -1/T \qquad\qquad g_2:\ T\to -1/(T+1)\ . \tag{3.1}$$

The $PSL(2,Z)_U$ subgroup is generated by

$$g_1':\ U\to -1/U \qquad\qquad g_2':\ U\to -1/(U+1)\ . \tag{3.2}$$

These generators obey the $SL(2,Z)$ relations

$$(g_1)^2 = (g_1')^2 = (g_2)^3 = (g_2')^3 = 1 \ , \tag{3.3}$$

and the relations implied by the fact that the two $PSL(2,Z)$'s commute. There is also an exchange symmetry generator, namely:

$$\sigma : \ T \leftrightarrow U, \tag{3.4}$$

which satisfies $\sigma^2 = 1$. Moreover σ relates the two $PSL(2,Z)$'s via $g_1' = \sigma g_1 \sigma$ and $g_2' = \sigma g_2 \sigma$. We expect that these relations do not hold in the quantum case, due to the singularities of the prepotential. For instance, since σ^2 corresponds to moving a point around $T = U$ singularity, it will not be equal to the identity. In order to understand the monodromy properties in the quantum case we have to find the new relations among the generators. To do that it is convenient to think of the above relations as relations among the generators of the fundamental group of the underlying moduli space. The classical monodromy group is then obtained by imposing the relation $\sigma^2 = 1$, while in the quantum case this relation is modified by the presence of a logarithmic branch cut.

At the classical level the underlying space is the product of two $PSL(2,Z)$ fundamental domains with an identification given by σ. Topologically each of these two fundamental domains can be thought of as a two-sphere S (S') with 3 distinguished points x_1 (x_1'), x_2 (x_2') and x_3 (x_3'), which can be taken to be the images of i, ρ and ∞ by the j-function. Associated with these three points we have generators g_i (g_i') of the fundamental group of orders 2, 3 and ∞ respectively, subject to the conditions $g_3 g_2 g_1 = 1$ and $g_3' g_2' g_1' = 1$. The total space is then the product of the two spheres S and S' minus $\{x_i\} \times S'$ and $S \times \{x_i'\}$, $i = 1, 2, 3$, and the fundamental group of the resulting 4-dimensional space is the product of the fundamental groups of the two punctured spheres. Including σ, we have the additional relations $g_i' = \sigma g_i \sigma$ and $\sigma^2 = 1$.

In the quantum case however, since we have singularities at $T = U$, we must remove the diagonal in the product of the two punctured spheres and this modifies the structure of the fundamental group. In general, when one takes a product of two (or more) identical Riemann surfaces and removes the diagonal, the fundamental group of the resulting space is called braid group and has been studied extensively [12]. One can adapt the results of ref.[12] to the present case, and obtain the following relations:

$$\begin{aligned}
&g_3 g_2 g_1 = \sigma^2, \qquad (g_1)^2 = (g_2)^3 = 1 \\
&g_i' = \sigma^{-1} g_i \sigma \\
&g_1 \sigma^{-1} g_2 \sigma = \sigma^{-1} g_2 \sigma g_1 \\
&\sigma g_i \sigma^{-1} g_i = g_i \sigma^{-1} g_i \sigma.
\end{aligned} \tag{3.5}$$

The full fundamental group is indeed generated by three elements σ, g_1, g_2 subject to the above relations. Notice that if one sets $\sigma^2 = 1$ one gets back the classical relations for the two commuting $PSL(2,Z)$'s. However, as mentioned earlier, in the quantum case $\sigma^2 \neq 1$ and the two $PSL(2,Z)$'s do not commute anymore. In fact, σ^2 corresponds to moving a point around the singularity at $T = U$ and therefore transforms the prepotential f non-trivially:

$$Z_1 \equiv \sigma^2 : f(T,U) \to f(T,U) + 2(T-U)^2 \tag{3.6}$$

Note that the additive piece above is uniquely fixed by the fact that it must be at most quadratic in T as well as U and by the behaviour of f near $T = U$ governed by the logarithmic term in eq.(2.29).

Actually one can explicitly check the non commutativity of T and U duality transformations using the integral representation for f given in (2.24). For instance one finds for the commutator $g_1 g_1'(g_1)^{-1}(g_1')^{-1}$:

$$g_1 g_1'(g_1)^{-1}(g_1')^{-1} : f(T,U) \to f(T,U) + 2(T-U)^2 - 2(1+TU)^2 \tag{3.7}$$

Notice also that we could redefine g_3 in the first equation of (3.5) by $\tilde{g}_3 = \sigma^{-2} g_3$, and then $\tilde{g}_3 g_2 g_1 = 1$, which is the usual $SL(2,Z)$ relation. We can do the same for g_i', showing that the quantum monodromy group contains the two $SL(2,Z)$'s as subgroups. However, as seen from (3.5) the two $SL(2,Z)$'s now do not commute.

Having the generators and relations of the fundamental group, we will now determine the monodromy transformations of the prepotential f. We can assume the following transformation properties of f under the generators g_1, g_2 and σ:

$$\begin{aligned}
g_1 : &\qquad T \to -1/T \quad ; \qquad f \to T^{-2}\left(f + P(T,U)\right) \,, \\
g_2 : &\qquad T \to -1/(T+1) \quad ; \qquad f \to (T+1)^{-2}\left(f + R(T,U)\right) \,, \\
\sigma : &\qquad T \leftrightarrow U \quad ; \qquad f \to f + K(T,U) \,,
\end{aligned} \tag{3.8}$$

As explained in the previous section the functions P, R and K are polynomials quadratic in T and U. Note that this property is consistent with the requirement that the quantity $\mathcal{I}$ which gives the physical metric (2.5) remains invariant under all three transformations. In fact, using eq.(2.12), one finds that these functions must satisfy

$$Im\{(\partial_T - \frac{2}{T-\bar{T}})(\partial_U - \frac{2}{U-\bar{U}})Q\} = 0 \quad ; \quad Q \equiv P, R, K$$

It is then straightforward to show that the most general solution to this equation is a general quadratic polynomial in both T, U with real coefficients.

The functions P, R, K must be compatible with the relations (3.5) and also with (3.6). The latter implies that:

$$K(T,U) + K(U,T) = 2(T-U)^2. \tag{3.9}$$

The general solution for $K(T,U)$ then is:

$$K(T,U) = (T-U)^2 + (T-U)(xUT + y(T+U) + z), \tag{3.10}$$

where x, y and z are complex numbers. The relation $(g_1)^2 = 1$ implies that P must be of the form $\alpha(T^2-1)+\beta T$ where α and β are quadratic polynomials in U. Similarly from the relation $(g_2)^3 = 1$ one finds that $R = AT^2 + 2(A+C)T + C$, with A and C quadratic in U. Using the freedom to add to f a quadratic polynomial in T and U (involving 9 parameters) we can set for example 9 parameters entering in α, β and $A+C$ to zero. Using the last two relations of (3.5), we can then show that all the remaining parameters get fixed, resulting into the following expressions for the 3 polynomials:

$$\begin{aligned} P &= 0 \\ R &= 2(T^2-1) \\ K &= (T-U)^2 + (T-U)(-2UT + T + U + 2). \end{aligned} \tag{3.11}$$

Notice that the coefficients of the polynomials are real, and as a result one can check, using (2.4), that the Kähler metric transforms covariantly.

The full monodromy group G contains a normal abelian subgroup H, which is generated by elements Z_g obtained by conjugating Z_1 by an element g which can be any word in the g_i's, g_i''s and their inverses. More explicitly, if g acts on the T,U space as $T \to T$ and $U \to \frac{aU+b}{cU+d}$, then Z_g acts as:

$$Z_g : (T,U) \to (T,U) \;\; ; \;\; f(T,U) \to f(T,U) + 2((cU+d)T - (aU+b))^2 \tag{3.12}$$

In other words Z_g corresponds to moving a point around the singularity $T = U_g$, where the prepotential behaves as shown in (2.29). Notice that the fact that H is abelian does not follow from the general group structure of (3.5), but from the specific logarithmic singularity (2.29), which implies that H acts on f by shifts as in (3.12). A general element of H is obtained by a sequence of such transformations and shifts f by:

$$f \to f + 2\sum_i N_i((c_iU + d_i)T - (a_iU + b_i))^2 \equiv f + \sum_{n,m=0}^{2} c_{nm}T^nU^m \quad N_i \in Z \qquad (3.13)$$

with a_i, b_i, c_i, d_i corresponding to some $SL(2,Z)$ elements for each i. Since the polynomial entering in (3.13) has 9 independent parameters c_{nm}, it follows that H is isomorphic to Z^9. The set of all conjugations of H by elements generated by g_i's and g_i''s defines a group of (outer) automorphisms of H which is isomorphic to $PSL(2,Z) \times PSL(2,Z)$, under which c_{nm} transform as $(3,3)$ representation (in this notation the two $PSL(2,Z)$'s act on the index n, m respectively). Moreover, the conjugation by σ defines an automorphism which interchanges the indices n and m in c_{nm}. Thus the set of all conjugations of H is isomorphic to $O(2,2;Z)$, under which the c_{nm}'s transform as a second rank traceless symmetric tensor. Finally, the quotient group G/H is isomorphic to $O(2,2;Z)$, therefore G is a group involving 15 integer parameters. On the other hand, G is not a semidirect product of $O(2,2;Z)$ and H, since $O(2,2;Z)$ is not a subgroup of G, as it follows from the quantum relations (3.5). Of course for physical on-shell quantities the group H acts trivially and therefore one recovers the usual action of $O(2,2;Z)$.

4. Linear basis for the monodromies and quantization

So far we have discussed the monodromies of f, which turned out to be consistent with the covariance of the Kähler metric. However, in order for the Kähler potential to transform by a Kähler transformation, the transformations of f must be supplemented by suitable transformations of the dilaton field S. From the form of the Kähler potential (2.1) and (2.2) one deduces that S must transform as:

$$g_1 : S \quad \to \quad S + \frac{f_U}{T} \qquad (4.1)$$

$$g_2 : S \quad \rightarrow \quad S + \frac{f_U}{T+1} \tag{4.2}$$

$$\sigma : S \quad \rightarrow \quad S - \frac{1}{2} K_{TU} \tag{4.3}$$

One can verify that the above transformations satisfy all the group constraints discussed earlier. The above equations therefore define the action of the monodromy group G on S. In addition to this, there is also the usual axionic shift which leaves T, U, and f invariant,

$$D : S \rightarrow S + \lambda \; , \tag{4.4}$$

where λ is a real number. The full perturbative group of monodromies is the direct product of G with the abelian translation group (4.4).

In order to better understand the group stucture and discuss quantization of the parameters due to non-perturbative effects, it is convenient to introduce a field basis where all monodromies act linearly. To this end we use the formalism of the standard N=2 supergravity [8] where the physical scalar fields Z^I of vector multiplets are expressed as $Z^I = X^I/X^0$, in terms of the constrained fields X^I and X^0. This is a way to include the extra $U(1)$ gauge boson associated with the graviphoton which has no physical scalar counterpart. In our case we have

$$S = \frac{X^s}{X^0} \quad T = \frac{X^2}{X^0} \quad U = \frac{X^3}{X^0} \tag{4.5}$$

and the prepotential (2.2) is the following homogeneous polynomial of degree 2:

$$F = \frac{X^s X^2 X^3}{X^0} + (X^0)^2 f(\frac{X^2}{X^0}, \frac{X^3}{X^0}) \tag{4.6}$$

The Kähler potential K is

$$K = -\log i(\bar{X}^I F_I - X^I \bar{F}_I) \; , \tag{4.7}$$

where F_I is the derivative of F with respect to X^I and $I=0,s,2,3$. This has a generalization in basis where F_I is not the derivative of a function F [13]. Then, the kinetic matrix for vector fields N_{IJ} is a 4×4 symmetric matrix completely determined by X^I and F_I through the formulae (4.7) and

$$F_I = N_{IJ}X^J \quad , \quad \mathcal{D}_I\bar{F}_J = N_{JL}\mathcal{D}_I\bar{X}^L$$

where $\mathcal{D}_I=\partial_I+K_I$. For the case in which $F_I=\partial_I F$, it reduces to the known expression of ref. [8].

It is clear that symplectic transformations acting on (X^I,F_I) leave the Kähler potential invariant. Since the monodromy group leaves K invariant, we expect it to be a subgroup of the symplectic group $Sp(8)$. In the following we will identify this subgroup. A general symplectic transformation is

$$\begin{pmatrix} X^I \\ F_I \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} X^I \\ F_I \end{pmatrix} \tag{4.8}$$

where a, b, c, d are 4×4 matrices and satisfy the defining relations of the symplectic group, namely

$$a^tc - c^ta = 0 \quad , \quad b^td - d^tb = 0 \quad , \quad a^td - c^tb = 1. \tag{4.9}$$

Under this transformation, however, the vector kinetic term $\mathrm{Im}\mathcal{F}^I_{\mu\nu}\overline{N}_{IJ}\mathcal{F}^{J\mu\nu}$ transforms as:

$$N \rightarrow (c+dN)(a+bN)^{-1}. \tag{4.10}$$

If $b\neq 0$ then from the above equation it follows that the gauge coupling gets inverted and therefore in a suitable basis the perturbative transformations must have $b=0$. When $b=0$ the symplectic contraints (4.9) imply that $d^t=a^{-1}$ and $c={a^t}^{-1}\tilde{c}$ with $\tilde{c}$ an arbitrary symmetric matrix. Furthermore, from eq.(4.10) we see that the vector kinetic term changes by $\tilde{c}_{IJ}\mathrm{Im}\mathcal{F}^I\mathcal{F}^J$ which, being a total derivative, is irrelevant at the perturbative level. However at the non-perturbative level, due to the presence of monopoles, the matrix $\tilde{c}$ must have integer entries.

In the absence of the one-loop correction f, one can verify that the $PSL(2,Z)_T$ transformation $T \to \frac{aT+b}{cT+d}$ transform X^I and F_I as:

$$\begin{array}{rclcrcl} X^0 & \to & cX^2 + dX^0 & \qquad & F_0 & \to & aF_0 - bF_2 \\ X^s & \to & cF_3 + dX^s & & F_s & \to & aF_s + bX^3 \\ X^2 & \to & aX^2 + bX^0 & & F_2 & \to & -cF_0 + dF_2 \\ X^3 & \to & cF_s + dX^3 & & F_3 & \to & aF_3 + bX^s \end{array} \tag{4.11}$$

and similarly $PSL(2,Z)_U$ transformation is given by interchanging X^2 with X^3 and F_2 with F_3 in the above equation. Note that these transformations act linearly and are in fact symplectic. However, in this basis the matrix $b \neq 0$ as X^I's get transformed to F^I's. It is therefore convenient to make a symplectic change of the basis into (X^I, F_I) where $I = 0,1,2,3$ with $X^1 = F_s$ and $F_1 = -X^s$. In the new basis the tree-level $O(2,2;Z)$ transformations are block diagonal, i.e. $b = c = 0$ and $d = a^{t^{-1}}$. For $PSL(2,Z)_T$ transformations a is given by

$$a = \begin{pmatrix} d & 0 & c & 0 \\ 0 & a & 0 & b \\ b & 0 & a & 0 \\ 0 & c & 0 & d \end{pmatrix} \tag{4.12}$$

while for $PSL(2,Z)_U$, a is obtained by interchanging the last two columns and rows. Finally T,U interchange corresponds to

$$a = \begin{pmatrix} 1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.13}$$

These matrices a are $O(2,2;Z)$ matrices which preserve the metric M

$$M = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix} \tag{4.14}$$

As explained in the last section, when one includes the one loop correction to the prepotential f, the $O(2,2;Z)$ group is replaced by the monodromy group G generated by the three elements g_1, g_2 and σ. The action of these elements on f and S is given by equations (3.8), (3.11) and (4.3). In the new symplectic basis introduced above, these transformations act linearly with the upper off-diagonal block $b=0$, that is they are of the form:

$$\begin{pmatrix} a & 0 \\ a^{t-1}\tilde{c} & a^{t-1} \end{pmatrix} \tag{4.15}$$

The matrices a, $\tilde{c}$ for the three generators are as follows:

$$\begin{aligned}
g_1: \quad a \quad &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \tilde{c}=0 \\
g_2: \quad a \quad &= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \qquad \tilde{c}=\begin{pmatrix} -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\sigma: \quad a \quad &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \tilde{c}=\begin{pmatrix} 0 & -1 & 2 & -2 \\ -1 & 0 & -2 & 2 \\ 2 & -2 & 4 & -1 \\ -2 & 2 & -1 & 0 \end{pmatrix}
\end{aligned} \tag{4.16}$$

Note that the matrices $\tilde{c}$ are symmetric and satisfy $\mathrm{Tr}M\tilde{c}=0$, where M is the metric (4.14).

The abelian group H introduced in (3.13) is generated by symplectic matrices (4.15) with $a=1$, and $\tilde{c}$:

$$\tilde{c}=\sum_i 2N_i g_i^t \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1-2 \end{pmatrix} g_i \tag{4.17}$$

where g_i can be chosen for instance as $PSL(2,Z)_T$ matrices of the form (4.12). Since g_i preserve the metric M it is clear that the symmetric matrices (4.17) are traceless with respect to M. Therefore, by suitable choices of g_i's and N_i's one can generate all symmetric 4×4 matrices which are traceless with respect to M, and therefore depending on 9 integer parameters. They form the 9-dimensional representation of $O(2,2;Z)$ corresponding to the second rank symmetric traceless tensors, as explained in the last section.

The full perturbative monodromy group contains also the axionic shift D (4.4) which in the above symplectic basis corresponds to

$$D: \quad \begin{pmatrix} 1 & 0 \\ -\lambda M & 1 \end{pmatrix}, \tag{4.18}$$

which commutes with the above matrices of G, as expected. The parameter λ should also be quantized at the non-perturbative level. In this way one generates all possible symmetric 4×4 lower off-diagonal matrices depending on 10 integer parameters, the trace part being generated by M in (4.18). The full monodromy group is generated by the 4 generators g_1, g_2, σ and D.

5. Generalization to arbitrary $(4,4)$ compactifications

The heterotic string compactified on $T^2\times K_3$ with spin connection identified with the gauge connection gives rise to N=2 supersymmetry having, besides the $U(1)^2$ associated with the dilaton and the graviphoton, a rank 17 gauge group $E_7\times E_8\times U(1)^2$.[3] There are also 20 massless hypermultiplets in the **56** representation of E_7. In the previous sections we discussed the dependence of the prepotential on the $U(1)^2$ vector multiplets corresponding to the moduli of the 2-torus T^2. However, the complete moduli space also includes the

[3] For special points in the hypermultiplet moduli space, as for example orbifold point of K_3, there could be extra massless vector multiplets increasing the rank of the gauge group. We will discuss such situations in the next section.

2×15 Wilson lines which enlarge the lattice deformations to $O(2,17)$. At a generic point of this moduli space the gauge group is broken to $U(1)^{17}$ and all charged hypermultiplets become massive. Complex co-dimension 1 singularities in the moduli space correspond either to the appearance of two extra massless vector multiplets which enlarge one of the $U(1)$ factors to $SU(2)$, or to massless hypermultiplets. These are the analogues of the $T = U$ singularities discussed in the previous sections. There are of course higher co-dimensional surfaces analogous to $T = U = i$ or ρ, which correspond to larger gauge groups and/or more massless hypermultiplets; they are not relevant in the following discussions.

At the classical level, the duality group is $O(2,17;Z)$ which leaves the mass spectrum and the interactions invariant. This is a subgroup of the symplectic transformations $Sp(38;Z)$ mentioned in the introduction. As in the last section, one can choose a field basis in which these transformations are linear and block diagonal at the tree level. For convenience we will choose here a basis [13] such that $O(2,17;Z)$ leaves invariant the diagonal metric $\eta = diag(-1,-1;1,1,\ldots,1)$:

$$
\begin{aligned}
X^I &= (X^0, X^1, X^\alpha)\,, \qquad X^I X^J \eta_{IJ} = 0 \\
F_I &= S\eta_{IJ}X^J
\end{aligned}
\tag{5.1}
$$

where $\alpha = 2,\ldots,18$ and S is the dilaton. The 17 physical coordinates y^α of the $O(2,17)/(O(2)\times O(17))$ manifold are given in terms of X's by $X^\alpha/X^0 = 2y^\alpha/(1+y_\alpha^2)$. X^I and F_I satisfy the constraints: $F_I\eta^{IJ}F_J = F_I X^I = 0$. Note that in this basis the prepotential does not exist, i.e. F_I is not I-th derivative of a function. This is exactly as in the case of $O(2,2)$ in the new basis introduced in section 4, where the role of X^s and F_s was interchanged to diagonalize the $O(2,2;Z)$ transformations. If one wishes, one could go back to a basis where a prepotential exists. The tree-level Kähler potential is given by

$$
K^{(0)} = -\log i(X^I\bar{F}_I - \bar{X}^I F_I) = -\log i(\bar{S}-S) - \log X^I\eta_{IJ}\bar{X}^J
\tag{5.2}
$$

and the $O(2,17)$ transformations in the symplectic basis (5.1) take the form:

$$\begin{pmatrix} X^I \\ F_I \end{pmatrix} \rightarrow \begin{pmatrix} a & 0 \\ 0 & a^{t-1} \end{pmatrix} \begin{pmatrix} X^I \\ F_I \end{pmatrix} , \tag{5.3}$$

where a is a $O(2,17)$ matrix which preserves the metric η.

The BPS mass formula [13] is

$$m = e^{K/2} |n_I^{(e)} X^I - n^I_{(m)} F_I| , \tag{5.4}$$

which is invariant under Kähler transformations. Here $n^{(e)}$ and $n_{(m)}$ are the electric and magnetic charge vectors. The elementary string states have $n_{(m)} = 0$ and $n^{(e)}$ lie in a lattice $\Gamma^{(e)}$ which for instance can be choosen to be the product of an even self-dual lattice $\Gamma^{(2,2)}$ corresponding to the two-torus with the weight lattices of $E_7 \times E_8$. For convenience we will choose for $\Gamma^{(2,2)}$ the $SO(4) \times SO(4)$ weight lattice with the conjugacy classes of the two factors being identified.[4] The conjugacy class of the scalar in E_7 corresponds to the vector multiplets while the one of **56** corresponds to hypermultiplets. In fact, for $n_{(m)} = 0$, the mass (5.4) is just the left moving momentum of the two-torus $|p_L|$, i.e. they correspond to the ground state of left-moving sector with momentum p_L. Massless states are the ones with $m = 0$ and $n_I^{(e)} \eta^{IJ} n_J^{(e)} = 2$ for vector multiplets and $=3/2$ for hypermultiplets. Thus the point $y^\alpha = 0$ corresponds to the gauge group $E_7 \times E_8 \times SO(4)$ with massless hypermultiplets in **56** representation of E_7 whose multiplicity is governed by the cohomology of K_3 and is 20. On the other hand it is clear from the constraints for massless states that at generic values of y^α's, there are no charged massless states and therefore the gauge group is $U(1)^{17}$. The symmetry group $O(2,17;Z)$ is the automorphism group of $\Gamma^{(e)}$.

The complex co-dimension 1 surface of singularity corresponding to the enhancement of one of the $U(1)$'s to $SU(2)$ (i.e. when two charged vector multiplets become massless)

[4] Here we normalise roots to have length $\sqrt{2}$.

is defined by the equation

$$n_I^{(e)} X^I = 0 \tag{5.5}$$

for a particular choice of $n^{(e)}$ vector obeying $n_I^{(e)} \eta^{IJ} n_J^{(e)} = 2$. Different choices of such charge vectors define different surfaces of singularity and they are related to different $U(1)$'s being enhanced to $SU(2)$. For different vectors $n^{(e)}$'s that are related by $O(2,17;Z)$ transformation, the corresponding surfaces are also $O(2,17;Z)$ transforms of each other. Similarly, the singular surfaces associated with the appearance of massless hypermultiplets are given by eq.(5.5) with $n_I^{(e)} \eta^{IJ} n_J^{(e)} = 3/2$. The appearance of these massless states gives rise to logarithmic singularities in the prepotential as in the $O(2,2)$ case discussed previously. In the following we will identify the coefficient of these logarithmic singularities as they enter in the monodromy matrices.

Let us denote by f_I the one-loop corrections to F_I of eq.(5.1). The one-loop correction to the Kähler potential is

$$K^{(1)} = -\frac{1}{S-\bar{S}} \frac{(X^I \bar{f}_I - \bar{X}^I f_I)}{X^K \eta_{KL} \bar{X}^L} \ . \tag{5.6}$$

Consider now the behaviour of $K^{(1)}$ near a singular surface $n_I^{(e)} X^I = 0$. The direction orthogonal to the surface, and subject to the constraint $X^I X^J \eta_{IJ} = 0$, is $\delta X^I = \eta^{IJ} n_J^{(e)} \epsilon$, where ϵ is an infisitesimal parameter. We are interested in the component of the metric along this direction, since it is this component which has a logarithmic singularity near the surface. Expanding the Kähler potential (5.2) and (5.6) in powers of ϵ and $\bar{\epsilon}$ and extracting the coefficient of $\epsilon\bar{\epsilon}$, one finds:

$$\begin{aligned} G^{(1)}_{\epsilon\bar{\epsilon}} &= \frac{i}{2} G^{(0)}_{\epsilon\bar{\epsilon}} [\frac{1}{{n^{(e)}}^2} n_I^{(e)} \eta^{IJ} (\delta_\epsilon f_J - \delta_{\bar{\epsilon}} \bar{f}_J) + \frac{X^I \bar{f}_I - \bar{X}^I f_I}{X^K \eta_{KL} \bar{X}^L}] \\ G^{(0)}_{\epsilon\bar{\epsilon}} &= -\frac{{n^{(e)}}^2}{X^I \eta_{IJ} \bar{X}^J} \end{aligned} \tag{5.7}$$

where ${n^{(e)}}^2 \equiv n_I^{(e)} \eta^{IJ} n_J^{(e)}$. Note that the tree-level metric $G^{(0)}$ does not mix the ϵ direction with the directions tangential to the singular surface since the linear terms in the expansion of $K^{(0)}$ vanish on the surface. The linear terms in the expansion of $K^{(1)}$ are proportional to

$$\frac{\epsilon}{X^K\eta_{KL}\bar{X}^L}[n_I^{(e)}\eta^{IJ}\bar{f}_J - \bar{X}^I\delta_\epsilon f_I - c.c.] \tag{5.8}$$

We know that the one-loop metric near the singular surface has a logarithmic singularity of the form $G^{(1)}_{\epsilon\bar{\epsilon}}/G^{(0)}_{\epsilon\bar{\epsilon}} = \frac{c}{\pi}\log|\frac{n_I^{(e)}X^I}{X^0}|^2$ with $c = {n^{(e)}}^2 = 2$ for vector multiplets, and $c = -10{n^{(e)}}^2 = -15$ for hypermultiplets. The appearance of ${n^{(e)}}^2$ can be understood from the fact that these are the square of the charges of the states that become massless with respect to the $U(1)$ defined by the ϵ-direction. The particular values 2 and 15 are associated with charges ± 1 for the $SU(2)$ adjoint representation, and $\pm\sqrt{3}/2$ for the 20 hypermultiplets. As mentioned before, the multiplicity 10 is related to the cohomology of K_3, and $O(2,17)$ deformations do not alter this value. As for the mixed components of the one-loop metric involving ϵ and a direction tangential to the surface, there is no logarithmic singularity since the sum over the charges vanishes. These requirements together with eqs.(5.7) and (5.8) imply that the singular part of f_I near the surface is:

$$f_I = -\frac{2iN}{\pi}n_I^{(e)}n_J^{(e)}X^J\log\frac{n_L^{(e)}X^L}{X^0} \tag{5.9}$$

where $N = 1$ or -10 for the case of vector multiplets or hypermultiplets, respectively.

The presence of logarithms in f_I modifies the classical monodromies just as in the $O(2,2;Z)$ case. The analogue of the $T \leftrightarrow U$ exchange corresponds now to the Weyl reflections $W_{n^{(e)}}$ defined by the vectors $n^{(e)}$'s satisfying ${n^{(e)}}^2 = 2$ (i.e. for the vector multiplets). $W_{n^{(e)}}$ is an automorphism of the charge lattice and, at the classical level, it satisfies $(W_{n^{(e)}})^2 = 1$. However at the quantum level this relation is no longer true due to the logarithmic singularities in f, as in the $O(2,2)$ case. Indeed, $(W_{n^{(e)}})^2 \equiv Z_{n^{(e)}}$ corresponds to moving a point around the singular surface $n_I^{(e)}X^I = 0$. Consider a vector $n^{(e)}$ lying in the α-directions. From equation (5.9) it is easy to see that $(W_{n^{(e)}})^2$ shifts F_I as

$F_I \to F_I + 4n_I^{(e)} n_J^{(e)} X^J$. This results in the following symplectic transformation:

$$Z_{n^{(e)}} = \begin{pmatrix} 1 & 0 \\ \tilde{c}^v & 1 \end{pmatrix} \qquad \tilde{c}^v = 4n^{(e)} n^{(e)^t} \tag{5.10}$$

It follows that $W_{n^{(e)}}$ must be of the form:

$$W_{n^{(e)}} = \begin{pmatrix} a & 0 \\ a^{t^{-1}} \tilde{c} & a^{t^{-1}} \end{pmatrix} \tag{5.11}$$

where a is the element of $O(2,17;Z)$ corresponding to the above Weyl reflection and $\tilde{c}$ is a symmetric matrix satisfying the condition $a^{t^{-1}} \tilde{c} a + \tilde{c} = -4n^{(e)} n^{(e)^t}$.

In the case of $n^{(e)^2} = 3/2$ corresponding to 56 of E_7 (i.e. for hypermultiplets) the reflection is not a symmetry of the lattice. However there is still a non-trivial monodromy $Z_{n^{(e)}}$ associated with moving a point around such singular surfaces:

$$Z_{n^{(e)}} = \begin{pmatrix} 1 & 0 \\ \tilde{c}^h & 1 \end{pmatrix}, \qquad \tilde{c}^h = 40n^{(e)} n^{(e)^t} \tag{5.12}$$

where the coefficient 40 appears due to the multiplicity 20 of the hypermultiplets that become massless.

Similarly to the $O(2,2)$ case discussed in sections 3 and 4, the fact that $(W_{n^{(e)}})^2$ is not equal to the identity implies that the classical group $O(2,17;Z)$ is replaced by a quantum monodromy group G. The latter is defined by the fundamental group of the space obtained after removing the singular surfaces from the fundamental domain of $O(2,17;Z)$ in $O(2,17)/O(2) \times O(17)$. Note that the number of singular surfaces in the fundamental domain is given by the number of distinct $O(2,n;Z)$ orbits among the lattice vectors satisfying $(n^{(e)})^2 = 2$ or $3/2$ and is finite. The fundamental group is finitely presented, and when $Z_{n^{(e)}}$ are set equal to identity, this group reduces to $O(2,17;Z)$. The subgroup generated by $Z_{n^{(e)}}$'s defines a normal abelian subgroup H of G. In the symplectic basis an arbitrary element of H is given by

$$\begin{pmatrix} 1 & 0 \\ \tilde{c} & 1 \end{pmatrix} \qquad \tilde{c} = \sum_i N_i g_i^t \tilde{c}^v g_i + \sum_j M_j g_j^t \tilde{c}^h g_j \tag{5.13}$$

where g_i are $O(2,17;Z)$ elements. In this way, we generate a general symmetric matrix $\tilde{c}$ depending on $19 \times 20/2$ integer parameters. It is decomposed into a sum of two irreducible representations of $O(2,17)$: a traceless symmetric tensor and a singlet corresponding to the trace. Note that the latter can be identified with the quantized dilaton shift having the form:

$$\begin{pmatrix} 1 & 0 \\ \eta & 1 \end{pmatrix} \tag{5.14}$$

Of course at the perturbative level, on top of this transformation one can add an arbitrary dilaton shift with η replaced by $\lambda\eta$. The quotient group G/H is isomorphic to $O(2,17;Z)$. A representative element in a class of G/H is given in the symplectic basis as:

$$\begin{pmatrix} a & 0 \\ a^{t-1}\tilde{c} & a^{t-1} \end{pmatrix} \tag{5.15}$$

where a is the corresponding $O(2,17;Z)$ matrix and $\tilde{c}$ is some symmetric matrix, whose precise form is determined by the relations satisfied by the generators of G as was done in the case of $O(2,2)$ in sections 3 and 4. For example, as stated above for the Weyl reflections $W_{n^{(e)}}$, $\tilde{c}$ is constrained by the group relation $(W_{n^{(e)}})^2 = Z_{n^{(e)}}$. Unfortunately at present we do not know the complete set of group relations defining the fundamental group and therefore we are unable to construct the $\tilde{c}$'s for various generators explicitly. For consistency at the non-perturbative level the entries of $\tilde{c}$ must be quantized such that $\tilde{c}\Gamma_{(m)} \subset \Gamma^{(e)}$, where $\Gamma_{(m)}$ is the magnetic charge lattice which, as we shall discuss in the next section, is the lattice dual to $\Gamma^{(e)}$ with respect to the metric η. One can see that the $\tilde{c}$'s appearing in H subgroup (5.13) satisfy this condition. Although we are unable to determine G completely, we can however say that it is some finite index subgroup of the group of matrices of the form (5.15) with $a \in O(2,17;Z)$ and $\tilde{c}$ an arbitrary symmetric matrix satisfying the quantization condition.

6. $(4,0)$ models

So far we have discussed generic $(4,4)$ models leading to rank $r=17$ gauge group. However in the moduli space of hypermultiplets, there are special points where additional vector multiplets become massless leading to an increase in the rank. For example at the Z_2 orbifold point one gets an extra $SU(2)$ factor increasing the rank to 18, while for special radii one can even get rank 22 gauge groups. At these special points the moduli space of vectors is usually increased to $O(2,r)/(O(2)\times O(r))$ and the classical symmetry group is $O(2,r;Z)$. The above analysis can again be repeated. We first introduce the symplectic basis (X^I,F_I) with $I=0,1,\dots r+1$ and $X^I\eta_{IJ}X^J=0$ on which the $O(2,r;Z)$ transformations act linearly by block diagonal symplectic matrices. The mass spectum is again given as in eq.(5.4) with the charge vectors $n^{(e)}$ living in a lattice $\Gamma^{(2,r)}$. We assume for simplicity that the sublattice Γ_v associated with the charges of vector multiplets is even and integral, which is the case for orbifolds. For orbifolds, it is also true that the full lattice $\Gamma^{(2,r)}$ is the dual of Γ_v, the non-trivial conjugacy classes C of $\Gamma^{(2,r)}$ with respect to Γ_v being associated with hypermultiplets. In the full string theory, each of these classes is coupled to a block of the internal conformal field theory which describes the remaining $(22-r)$ right movers. The data from the latter which is relevant here, is the multiplicity m_C of the number of operators in the Neveu-Schwarz sector carrying conformal dimension $(1/2,\Delta_C)$ with $\Delta_C\leq 1$ in the block coupled to the conjugacy class C. Of course, world-sheet modular invariance implies that $\Delta_C+\frac{1}{2}{n^{(e)}}^2$ is an integer for $n^{(e)}$ belonging to the class C. Obviously m_C and Δ_C do not change under $O(2,r)$ deformations. This is similar to the multiplicity 20 of the 56's of E_7 in the $(4,4)$ models. The classical symmetry group which should preserve the spectrum is $O(2,r,Z)$ which preserves the lattice Γ_v. At a generic point in the moduli space $O(2,r)/(O(2)\times O(r))$, the gauge group is $U(1)^r$ and there are no massless hypermultiplets.

At the one loop level the prepotential again develops logarithmic singularities near complex co-dimension 1 surfaces where extra massless particles appear. The ones associated with the enhancement of gauge symmetry to $U(1)^{r-1}\times SU(2)$ are given by the surfaces $n^{(e)}\cdot X = 0$ for $n^{(e)}\in\Gamma_v$ and ${n^{(e)}}^2 = 2$; the ones associated with the appearance of extra massless hypermultiplets correspond to $n^{(e)}\cdot X = 0$ with $n^{(e)}$ belonging to a non-trivial class C in $\Gamma^{(2,r)}$ with ${n^{(e)}}^2 + 2\Delta_C = 2$. As in the (4,4) case, one can show that the singular part of f_I's near such a surface is given by eq.(5.9), with N being 1 for vector multiplets and $-m_C$ for hypermultiplets associated with the conjugacy class C. As before the presence of logarithmic singularity gives rise to non-trivial monodromies. The Weyl reflections associated with $n^{(e)}\in\Gamma_v$ satisfying ${n^{(e)}}^2 = 2$ are again represented by the matrices $W_{n^{(e)}}$ of eq.(5.11). Similarly $W^2_{n^{(e)}}\equiv Z_{n^{(e)}}$ is given by (5.10). For hypermultiplets the reflections are not automorphisms of the lattice. However moving a point around such surfaces one gets monodromies that are given by the matrices $Z_{n^{(e)}}$ of eq.(5.12) with $\tilde{c}^h = 4m_C n^{(e)}{n^{(e)}}^t$ for $n^{(e)}$ in the conjugacy class C. The normal abelian subgroup H consists of elements of the type (5.13) with $\tilde{c} = \sum_i N_i g_i^t \tilde{c}^v g_i + \sum_j M_j g_j^t \tilde{c}^h g_j$, where g_i are $O(2,r;Z)$ elements. In this way, we generate a general symmetric matrix $\tilde{c}$ depending on $(r+2)(r+3)/2$ integer parameters. It is decomposed into a sum of two irreducible representations of $O(2,r)$: a traceless symmetric tensor and a singlet corresponding to the trace. The latter is identified with quantized axionic shift as before. The quotient G/H is isomorphic to $O(2,r,Z)$ and a general element of G is again of the form given in eq.(5.15) where $\tilde{c}$ is to be determined from the precise form of the relations defining the fundamental group.

Now let us discuss the consistency of the monodromy group when non-perturbative effects are taken into account. This means that the monodromy preserves the complete mass spectrum of BPS states involving electric as well as magnetic charges. The monodromy group G acts as symplectic transformation of electric and magnetic charge vectors $(n^{(e)}, n_{(m)})$. Dirac quantization condition for magnetic charges implies that magnetic charge vectors must be in the dual lattice of electric charge vectors $\Gamma^{(2,r)}$. This means that magnetic charges in fact lie in Γ_v. A general element of the perturbative monodromy group we have discussed so far consists of matrices whose upper off-diagonal block is zero. Morever the diagonal blocks are made up of $O(2,r,Z)$ matrices which by definition preserve Γ_v and therefore the electric and magnetic charge lattices separately. The non-trivial question is whether the lower off-diagonal block $\tilde{c}$ which mixes the magnetic charge lattice with the electric charge one, is consistent. In other words we must have $\tilde{c}n_{(m)} \in \Gamma^{(2,r)}$. Since $\tilde{c}$ appearing in H is made up of matrices of the form $2n^{(e)}n^{(e)^t}$ this condition is obviously satisfied. $\tilde{c}$ appearing in a general element of G must also satisfy this condition. Thus we see that again G is a finite index subgroup of the group of matrices of the form (5.15) with $a \in O(2,r;Z)$ and $\tilde{c}$ an arbitrary symmetric matrix satisfying the quantization condition. The non-perturbative consistency also implies the quantization of the dilaton shift: $S \to S +$ integer.

To illustrate the above let us consider Z_2 orbifold and restrict to a subspace of two moduli which generalize the $O(2,2)$ case discussed in sections 2, 3 and 4. More precisely, we start with a model defined from the usual toroidal compactification $T^2 \times T^4$ of the $E_8 \times E_8$ heterotic theory by a Z_2 twist on the T^4 together with a Z_2 shift δ acting on the $\Gamma^{(2,2)}$ momentum lattice corresponding to T^2. In order to satisfy the level matching condition δ^2 must be $1/2$. Note that this is in contrast with the usual orbifold constructions where the shift is embedded in one of the E_8 factors breaking it to $E_7 \times SU(2)$. Now the gauge group is $E_8 \times E_8 \times U(1)^2$ at a generic point in the moduli space of T^2. In terms of the integers n_i, m_i that define the momenta (2.8), (2.9), the effect of this shift is the following. In the untwisted sector, vector multiplets are associated with $m_2 + n_1$ even integers, while hypermultiplets correspond to $m_2 + n_1$ odd. In the twisted sector m_2 and n_1 are shifted by 1/2 and these states are hypermultiplets. With respect to the lattice Γ_v corresponding to $m_2 + n_1$ even, the charge lattice has now four classes. Besides the trivial class C_0, the non-trivial ones are C_1 associated with $m_2, n_1 \in Z$ and $m_2 + n_1$ odd, and C_2 and C_3 associated with $m_2, n_1 \in Z + 1/2$ and $m_2 + n_1$ even and odd, respectively. The data from the remaining conformal field theory (m_C, Δ_C) discussed above is (1,0) for C_1, (32,3/4) for C_2 and (8,1/4) for C_3. Furthermore the tree-level symmetry group $\tilde{O}(2,2;Z)$ is a subgoup of $O(2,2;Z)$ defined in section 4, which leaves these classes invariant. More precisely, its even part is the subgroup of $SL(2,Z)_T \times SL(2,Z)_U$ obtained by identifying the cosets of the two factors with respect to the $\Gamma(2)$ subgroup of $SL(2,Z)$; its odd part is obtained by including the $T \leftrightarrow U$ exchange.

Repeating the analysis of section 2, one can show that the third derivative of the one-loop prepotential f_{TTT} is given as a sum of contributions from the four classes, each of them being expressed by the r.h.s. of eq.(2.17):

$$f_{TTT} = 4\pi^2 \frac{U - \bar{U}}{(T - \bar{T})^2} \sum_{C_\ell} \int d^2\tau \, \bar{F}_\ell(\bar{\tau}) \sum_{p_L, p_R \in C_\ell} p_L \bar{p}_R^3 \, e^{\pi i \tau |p_L|^2} \, e^{-\pi i \bar{\tau} |p_R|^2} . \tag{6.1}$$

At $\tau_2 \to \infty$, $2i\pi^2 \bar{F}_\ell$ behaves as $\bar{q}^{-1}$ for $\ell = 0$, $-1\bar{q}^{-1}$ for $\ell = 1$, $-32\bar{q}^{-1/4}$ for $\ell = 2$ and $-8\bar{q}^{-3/4}$ for $\ell = 3$, where $q = e^{2i\pi\tau}$. One can verify from eq.(6.1) that in each class there is a simple pole singularity associated with the appearance of massless sates. The condition $p_L = 0$ gives the lines $m_1 + m_2 U + n_1 T + n_2 TU = 0$ while the massless condition for the right movers gives $m_1 n_2 - m_2 n_1 = 1, 1, 1/4, 3/4$ for the four classes C_0, C_1, C_2, C_3, respectively. For C_0 there are four distinct singular lines (modulo the automorphism group) $T = U$, $T = U + 1$, $T = -1/U$ and $T = U/(U + 1)$, where the gauge group becomes $SU(2) \times U(1)$. For the other classes there is one representative singular line each which we can choose to be $T = -1/(U + 1)$ for C_1, $T = U$ for C_2 and $T = 3U$ for C_3, where we have two massless hypermultiplets. Note that the singular line of class C_2 coincides with one of the lines for

C_0 implying that the two massless hypermultiplets come in one $SU(2)$ doublet.

To each of the above singular lines there is an associated non-trivial monodromy. For the $T=U$ singularity, where besides the $SU(2)$ gauge symmetry also 32 massless $SU(2)$ doublet hypermultiplets appear, we have the following monodromy for f:

$$T \quad around \quad U: \qquad\qquad f \to f - 62(T-U)^2, \tag{6.2}$$

where the coefficient -62 is due to the contribution $+2$ of the vectors and -64 of the hypermultiplets. For the other 3 $SU(2)$ lines the monodromies are:

$$\begin{aligned}
&T \quad around \quad (U+1): && f \to f + 2(T-U-1)^2\\
&T \quad around \quad -\frac{1}{U}: && f \to f + 2(1+TU)^2\\
&T \quad around \quad \frac{U}{U+1}: && f \to f + 2(TU+T-U)^2.
\end{aligned} \tag{6.3}$$

Finally, for the remaining two hypermultiplet lines we have:

$$\begin{aligned}
&T \quad around \quad 3U: && f \to f - 16(T-3U)^2\\
&T \quad around \quad -\frac{1}{U+1}: && f \to f - 2(TU+T+1)^2.
\end{aligned} \tag{6.4}$$

Here, we have used the particular values for the multiplicities of the various classes to get the multiplicative coefficients.

7. Concluding remarks

In this paper we studied the perturbative monodromies of the prepotential in N=2 heterotic string models in four dimensions. At the tree-level the duality group is a direct product of Z corresponding to the dilaton shift with $O(2,r;Z)$ given by the automorphisms of the charge lattice, where r is the rank of the gauge group. In some symplectic basis, the duality group acts in a block diagonal form. At the one-loop level, due to the presence of singularities associated with the appearance of massless states at complex co-dimension 1 surfaces in the moduli space of vector multiplets, its fundamental group gets modified. The resulting quantum monodromies associated with closed curves around the singular surfaces which acted as identity at the tree-level, now get modified by a lower off diagonal symmetric matrix which depends on $(r+2)(r+3)/2$ integer parameters. They define a normal abelian subgroup H of the monodromy group G. The quotient group G/H is isomorphic to the duality group $O(2,r;Z)$.

In order to find the quantum duality group G, it is necessary to find the fundamental group of the quantum moduli space. We have solved completely this problem in the $r=2$ case, where the fundamental group is known to be related to the braid group, but for $r \geq 3$ (and for the two-moduli case of section 6) we do not have a complete solution.

In view of the recent work of Seiberg and Witten in the rigid theory, one can ask the question whether at the non-perturbative level the monodromy group is further modified. On general grounds we know that a non-perturbative generator will be an element of $Sp(2r+4,Z)$, with a non-vanishing b entry (see eq.(4.8)).[5] The relation of monodromies to braid groups may be helpful in identifying the non-perturbative monodromy group and in studying the dynamics of N=2 superstrings.

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