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The Invisible Renormalon

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Abstract

We study the structure of renormalons in the Heavy Quark Effective Theory, by expanding the heavy quark propagator in powers of $1/m_Q$. We demonstrate that the way in which renormalons appear depends on the regularisation scheme used to define the effective theory. In order to investigate the relation between ultraviolet renormalons and power divergences of matrix elements of higher-dimensional operators in the heavy quark expansion, we perform calculations in dimensional regularisation and in three different cut-off regularisation schemes. In the case of the kinetic energy operator, we find that the leading ultraviolet renormalon which corresponds to a quadratic divergence, is absent in all but one (the lattice) regularisation scheme. The nature of this "invisible renormalon" remains unclear.

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1 Introduction

Several recent papers have been devoted to the study of renormalons in the Heavy Quark Effective Theory (HQET) [1]–[5]. These studies exploit and extend the understanding of the relevance of renormalons in field theory, and for Operator Product Expansions (OPE) in particular, developed in refs. [6]–[14]. The present interest was initiated by the observation that the pole mass of a heavy quark, which is an important parameter in the heavy quark expansion, has an intrinsic ambiguity of order $\Lambda_{\rm QCD}$ due to the presence of infrared (IR) renormalons [1, 2]. In addition to the infrared renormalons and other non-perturbative effects present in QCD, renormalon ambiguities also arise in the HQET as a result of the expansion in powers of $1/m_Q$. A consequence of these additional renormalons is the appearance of non-perturbative effects in the Wilson coefficients which relate operators in the effective theory to operators in the full theory (QCD). This problem is common to all effective theories obtained from an expansion in inverse powers of a large scale. Since these effects are an artefact of the construction of the effective theory and are absent in the original theory, they have to cancel in predictions for physical quantities. In the context of the HQET, it has been established that the IR renormalon ambiguities in the Wilson coefficients are cancelled by ultraviolet (UV) renormalon ambiguities in the matrix elements of the operators in the effective theory. How this cancellation occurs was traced explicitly, to order $1/m_Q$ in the heavy quark expansion, for both inclusive [3, 4, 5] and exclusive [4] weak decays of hadrons containing a heavy quark. A non-perturbative approach for the elimination of the ambiguities in the definition of the pole mass, or more generally in the matrix elements of the operators appearing in the HQET, has been proposed in ref. [15].

In the present paper we study in more detail the structure of renormalons in the HQET. To be specific, we consider QCD Green functions that depend on a large scale m_Q (the heavy quark mass) and a small scale k (the so-called residual momentum). The purpose of the heavy quark expansion is to disentangle the physics on different length scales by introducing a factorisation scale λ such that $k \ll \lambda \ll m_Q$. Contributions from virtual momenta above λ are calculable in perturbation theory and attributed to Wilson coefficients, whereas contributions from virtual momenta below λ are contained in the matrix elements of the operators in the effective theory. If this program is performed with a "hard" factorisation scale, these matrix elements will, for dimensional reasons, diverge as powers of the UV cut-off λ. The appearance of power divergences leads to non-perturbative ambiguities, since factors such as

$$\lambda \exp\left(-\frac{2\pi}{\beta_0\,\alpha_s(\lambda)}\right) = \Lambda_{\rm QCD} \tag{1}$$

give non-vanishing contributions, which do not appear in perturbation theory [16]. For practical reasons, however, one usually calculates the Wilson coefficients using dimensional regularisation. In this case, by definition, power divergences do not appear, and the "hard" factorisation scale λ is replaced by a "soft" renormalisation scale μ. In such a scheme it is unavoidable that the Wilson coefficients receive contributions from momenta below μ (so-called IR renormalons), and the matrix elements receive contributions from momenta above μ (so-called UV renormalons). These contributions lead to a factorial growth of the coefficients in the perturbative expansion of the Wilson coefficients and matrix elements. The corresponding perturbative series are divergent and not Borel summable. Since the choice of the renormalisation scale is arbitrary, however, the effects of IR and UV renormalons must cancel each other if one combines the perturbative series for the coefficient functions and matrix elements.

Our main point is to demonstrate that the way in which renormalons appear in the HQET (and, by the same argument, in other effective field theories) is not universal, but depends on the regularisation scheme used to define the effective theory. To this end, we investigate in detail the relation between UV renormalons and power divergences of hadronic matrix elements of higher-dimensional operators. We study explicitly the $1/m_Q$ expansion of the heavy quark propagator in different regularisation schemes: dimensional regularisation and three different schemes with a hard UV cut-off (Pauli–Villars, momentum flow [17] and lattice regularisation).

In general, from the degree of divergence of the matrix elements of an operator in the HQET one can deduce the position of the UV renormalon singularities in the Borel transform of these matrix elements (see section 2 for a detailed explanation). By dimensional arguments we expect that the kinetic energy operator for a heavy quark is quadratically divergent, since it can mix with the identity operator under renormalisation. Indeed, this mixing has been observed in lattice perturbation theory[16]. The corresponding renormalon singularity is absent, however.[1] This "missing" singularity is what we have called the "invisible renormalon". Even more surprising is the absence, at one-loop order, of the quadratic divergence in the two other schemes considered in our study, i.e. the Pauli-Villars and momentum flow regularisations. We have been unable to understand whether there is a symmetry, broken by the lattice regularisation, which prevents the appearance of the quadratic divergence in the other cases, or whether this is an artefact of one-loop perturbation theory. In the latter case, the quadratic divergence would appear at higher orders. Given the relevance of the kinetic energy operator for heavy hadron spectroscopy and inclusive decays [18]–[26], an understanding of this puzzle is very important.

The remainder of this paper is organised as follows: In sect. 2 we summarise and discuss the main results of our study. Technical details and explicit calculations in different regularisation schemes are presented in sects. 3–5. In sect. 6 we

[1] The absence of the corresponding IR renormalon in the pole mass was previously noted in ref. [2].

give our conclusions.

2 Renormalons and power divergences

Perturbative expansions in QCD are asymptotic and, in general, not Borel summable. When one tries to resum a perturbative series to all orders, one encounters ambiguities, which indicate that perturbation theory is by itself incomplete and must be supplemented by non-perturbative corrections. A convenient way to analyse these ambiguities is to consider the Borel transform $\widetilde{S}(u)$ of a series $S(\alpha_s(\mu))$ with respect to the coupling constant [6]. Formally, the Borel sum of the series can be defined by the integral[2]

$$S(\alpha_s(\mu)) = \int_0^\infty \mathrm{d}u \, \exp\left(- \frac{4\pi u}{\beta_0 \alpha_s(\mu)} \right) \widetilde{S}(u) \,, \tag{2}$$

where $\beta_0 = 11 - \frac{2}{3} n_f$ is the first coefficient of the β-function. However, if the Borel transform contains singularities on the integration contour, the result of the integration depends on the regularisation prescription, and $S(\alpha_s(\mu))$ is not uniquely defined in terms of $\widetilde{S}(u)$. In QCD, one source of such singularities are the higher-order diagrams in which a virtual gluon line with momentum k is dressed by a number of fermion, gluon and ghost loops. More precisely, one has to consider a gauge-invariant generalisation of such diagrams, which is called a renormalon chain. Effectively, this introduces the running coupling constant $\alpha_s(k)$ at the vertices. When $\alpha_s(k)$ is expressed in terms of the coupling constant renormalised at a fixed scale μ, the appearance of powers of large logarithms leads to a factorial divergence in the expansion coefficients of the perturbative series. Associated with this are renormalon singularities in the Borel transform $\widetilde{S}(u)$. Depending on whether these singularities are related to the region of large or small virtual momenta, they are referred to as UV or IR renormalons [6]–[14].

Although the resummation of renormalon chains corresponds to only a partial resummation of the perturbative series, it elucidates many non-perturbative effects and thus provides an interesting, non-trivial approximation. If one accepts this so-called "bubble approximation", the renormalon singularities occur as poles on the real axis in the Borel plane.[3] Let us denote the position of the nearest pole on the positive u-axis by u_0, i.e. $\widetilde{S}(u) = r_0/(u - u_0) +$ terms that are regular at $u = u_0$. Then a measure of the ambiguity in the Borel integral (2) is given by

$$\Delta S = r_0 \, \exp\left(- \frac{4\pi u_0}{\beta_0 \, \alpha_s(\mu)} \right) = r_0 \left(\frac{\Lambda_{\rm QCD}}{\mu} \right)^{2u_0} . \tag{3}$$

[2]We follow the notations and definitions of ref. [4].

[3]When the calculations are extended beyond the "bubble approximation", the poles become replaced by branch points of cut singularities.

In the last step we have used the one-loop expression for the running coupling constant.

As mentioned above, in dimensional regularisation power divergences are hidden because of the absence of an intrinsic mass scale in the computation. However, perturbation theory "knows" about these divergences in the form of renormalon singularities in the Borel plane. In fact, there is a one-to-one correspondence between the structure of the UV renormalon poles in dimensional regularisation and the power divergences in regularisation schemes with a hard cut-off. For instance, if a quantity is linearly divergent in one-loop perturbation theory, then its Borel transform in the bubble approximation is logarithmically divergent at $u = 1/2$, corresponding to a pole singularity at this point. A similar argument is expected to hold for quadratic or higher-order power divergences. To illustrate this point we present results for the renormalised inverse heavy quark propagator $S_{\rm eff}^{-1}(k)$ in the HQET. Here $k = p_Q - m_Q v$ is the residual momentum, which stays finite in the limit $m_Q \to \infty$, and v is a four-velocity vector ($v^2 = 1$), which is usually identified with the velocity of the hadron containing the heavy quark. In dimensional regularisation, renormalon effects appear as singularities in the Borel transform $\widetilde{S}_{\rm eff}^{-1}(u)$. In the bubble approximation one finds[4] [2]

$$\widetilde{S}_{\rm eff}^{-1}(k,u) = v\cdot k \left\{ \delta(u) + \frac{6C_F}{\beta_0} \left[\left(\frac{-2v\cdot k}{\mu} \right)^{-2u} \frac{\Gamma(1-u)\,\Gamma(-1+2u)}{\Gamma(2+u)} + \frac{1}{2u} - R_Z(u) \right] \right\} \tag{4}$$

in the Landau gauge. The scheme-dependent function $R_Z(u)$, which for renormalisation schemes with analytic counterterms (such as MS-like schemes) is entire in the complex u-plane, is irrelevant for our purposes. Expanding the above expression around $u = 0$ and inserting the result into the Borel integral (2), one finds

$$S_{\rm eff}^{-1}(k) = v\cdot k \left\{ 1 + \frac{C_F \alpha_s}{\pi} \left(\frac{3}{2} \ln \frac{-2v\cdot k}{\mu} + \text{const.} \right) + O(\alpha_s^2) \right\}, \tag{5}$$

where $C_F = 4/3$ is a colour factor. The Γ-functions in the numerator of (4) define the positions of the renormalon singularities. The UV renormalons are given by the poles of $\Gamma(-1+2u)$ and are thus located at $u = \frac{1}{2}, 0, -\frac{1}{2}, \ldots$[5] They are due to the contributions of virtual momenta above μ in loop integrals. Note that there appear also IR renormalons (at $u = 1, 2, \ldots$), which are due to the contributions of virtual momenta below the soft scale k. Since, by construction, the effective theory and the full theory describe the same dynamics in the IR region, these renormalons reflect truly non-perturbative effects of QCD and are thus independent of the UV regularisation scheme. They are not related to the construction of the HQET and are therefore not the topic of our discussion. The

[4] In this paper we define the running coupling constant in the so-called V scheme [27], in which the one-loop counterterms differ from those of the MS scheme by an additive constant. Otherwise, the Borel transform has to be multiplied by a scheme-dependent factor e^{-Cu}.

[5] The pole at $u = 0$ is removed by renormalisation, however.

UV renormalon at $u = 1/2$ is the nearest singularity on the integration contour of the Borel integral in (2). An expansion of the Borel transform around this point leads to

$$\widetilde{S}_{\rm eff}^{-1}(k,u) = -\frac{2C_F}{\beta_0}\,\frac{\mu}{u-\frac{1}{2}} + \dots\,. \tag{6}$$

The linear dependence on μ reflects the linear UV divergence of the inverse propagator in the effective theory, which in dimensional regularisation is not seen in perturbation theory. Using the above result together with (3), we find that the renormalon ambiguity in the definition of the inverse propagator is of order $\Lambda_{\rm QCD}$.

Let us now compare these results to the one-loop calculation of the inverse propagator performed with a hard Pauli–Villars cut-off λ. In this case one finds (for details see sect. 5)

$$S_{\rm eff}^{-1}(k) = -\frac{C_F\alpha_s}{2}\,\lambda + v\cdot k\left\{1 + \frac{C_F\alpha_s}{\pi}\left(\frac{3-a}{2}\,\ln\frac{-2v\cdot k}{\lambda} + \text{const.}\right)\right\} + O(\alpha_s^2)\,. \tag{7}$$

We have given the result for an arbitrary covariant gauge. The Landau gauge corresponds to $a = 0$, whereas the Feynman gauge corresponds to $a = 1$. Note that the coefficient of the logarithmic term is the same in (5) and (7). The linearly UV divergent term in (7) is in correspondence with the term proportional to μ in (6), which leads to a renormalon ambiguity of order $\Lambda_{\rm QCD}$. Note that in the case of the cut-off regularisation the power divergence leads to an ambiguity of the same order, as explained above [see (1)].

This correspondence between UV renormalons and power divergences works in several cases. However, an apparent exception is provided by the quark matrix element of the kinetic energy operator $\bar h_v\,(iD_\perp)^2 h_v$, which appears at order $1/m_Q$ in the expansion of the inverse propagator.[6] This dimension-five operator has the same quantum numbers as the lower-dimensional operators $\bar h_v\,h_v$ (dimension-three) and $\bar h_v\,iv{\cdot}D\,h_v$ (dimension-four). Hence, in a generic regularisation scheme with a dimensionful regulator, one expects a mixing with these operators, leading to quadratic and linear UV divergences. Indeed, these divergences appear in the lattice formulation of the HQET, where, at one-loop order, one finds[7] [16]

$$S_{\rm kin}^{-1}(k) = \langle\, k\,|\,\bar h_v\,(iD_\perp)^2 h_v\,|\,k\,\rangle = \frac{W\,\alpha_s}{a^2} + \frac{2X\,\alpha_s}{a}\,v\cdot k + \dots\,. \tag{8}$$

Here W and X are dimensionless constants, and the UV cut-off is provided by the inverse lattice spacing, $\lambda = 1/a$. The puzzle that we wish to point out is that

[6] Here h_v denotes the velocity-dependent heavy quark field in the HQET, and $D_{\perp,\mu} = D_\mu - v\cdot D\,v_\mu$ contains the "spatial" components of the covariant derivative. For details see ref. [28] and references therein.

[7] The calculation was originally performed for a heavy quark at rest. The generalisation to an arbitrary four-velocity is possible, although there are considerable subtleties in formulating the effective theory at non-zero velocity in Euclidean space [29, 30].

the quadratic divergence is *not* seen in some other cut-off regularisation schemes (such as Pauli–Villars). Moreover, in the bubble approximation we do not find an UV renormalon pole corresponding to the quadratic divergence. The nearest UV singularity in the Borel transform of $S_{\rm kin}^{-1}(k)$ is located at $u = \frac{1}{2}$, and an expansion around this point gives

$$\widetilde{S}_{\rm kin}^{-1}(k,u) = \frac{6C_F}{\beta_0}\,\frac{\mu}{u-\frac{1}{2}}\,v\cdot k + \dots\,. \tag{9}$$

This behaviour corresponds to the linearly divergent term in (8). However, we do not find a term of the form $\mu^2/(u-1)$, which would correspond to the quadratic divergence. We shall refer to this "missing" renormalon pole at $u = 1$ as the *invisible renormalon*.

There are several possible explanations for this puzzle. The general argument given above is that there is a relation between the degree of divergence of the matrix elements of higher-dimensional operators and the position of the singularities in their Borel transform in dimensional regularisation. The coefficients of the divergent terms and of the renormalon singularities, however, are not universal and may be zero in some regularisation schemes. Indeed, there is no reason why renormalon singularities or power divergences should be the same for all versions of the theory, since the ambiguities related to these renormalons (or powers divergences) are spurious and cancel in the predictions for physical quantities. It is therefore conceivable to find some regularisation schemes in which at one-loop order the coefficient of the quadratic divergence of the kinetic operator happens to be zero.[8] However, it is surprising to us that we could find only a single regularisation scheme in which this coefficient *does not vanish*. Moreover, even if one accepts this point of view one still has to explain the absence of a renormalon pole at $u = 1$ in the Borel transform (9). A possibility would be that such a singularity appears only when one goes beyond the bubble approximation.

On the other hand, one may wonder whether the appearance of a quadratic divergence in the lattice regularisation scheme is due to the breaking of some symmetry preserved in other regularisation schemes. In this case the invisible renormalon would be nothing more than a lattice artefact. Examples of such broken symmetries are Lorentz symmetry and, as a consequence, the so-called reparametrisation invariance of the HQET [31, 32]. We note that the breaking of reparametrisation invariance is responsible for the fact that the renormalisation constants of the operators $\bar h_v\, iv\cdot D\, h_v$ and $\bar h_v\,(iD_\perp)^2 h_v$ differ by a finite term of order α_s in the lattice version of the HQET [16], whereas they are the same in the continuum version of the theory [31]. However, reparametrisation invariance does not forbid the mixing between the operators $\bar h_v\,(iD_\perp)^2 h_v$ and $\bar h_v\, h_v$, which is responsible for the quadratic divergence.

[8] If this is the case, a quadratic divergence is likely to appear at two-loop order, since at one-loop order the operator $\bar h_v\,(iD_\perp)^2 h_v$ mixes with $\bar h_v\, iv\cdot D\, h_v$, which itself mixes with $\bar h_v\, h_v$.

This concludes our main discussion. In the following sections we present the details of the calculations, whose results were referred to above.

3 Expansion of the inverse propagator

Let us investigate the appearance of renormalons in the $1/m_Q$ expansion of the inverse heavy quark propagator, which is the simplest Green function for a heavy quark. We shall generalise the analysis of Beneke and Braun [2] by including terms of order $1/m_Q$. We work in dimensional regularisation and adopt the bubble approximation to investigate the singularities in the Borel plane. For technical details of the calculation the reader is referred to refs. [2, 4, 12].

We start from the quark propagator in the full theory,

$$S(p,m) = \frac{1}{\not p - m - \Sigma(p,m)}\,, \tag{10}$$

where m denotes the bare mass, and

$$\Sigma(p,m) = m\,\Sigma_1(p^2,m^2) + (\not p - m)\,\Sigma_2(p^2,m^2) \tag{11}$$

is the self-energy. We write the heavy quark momentum as $p = m_Q v + k$, where m_Q is the expansion parameter of the HQET (in general $m_Q \neq m$), and consider $\Sigma_{1,2} \equiv \Sigma_{1,2}(k,m)$ as functions of the residual momentum k. Next we define a projected propagator $S_P(k,m_Q)$ from the relation[9]

$$\frac{1+\not v}{2}\,S_P(k,m_Q) \equiv \frac{1+\not v}{2}\,S(p,m)\,\frac{1+\not v}{2}\,. \tag{12}$$

Our goal is to construct the heavy quark expansion of $S_P^{-1}(k,m_Q)$. Including terms of order $1/m_Q$, we find

$$S_P^{-1}(k,m_Q) = (1-\Sigma_2)\left\{(m_Q-m)+v\cdot k+\frac{k_\perp^2}{2m_Q}\Big(1-\delta/2\Big)^{-1}\right\}-\Sigma_1\,m+O(1/m_Q^2)\,, \tag{13}$$

where $k_\perp^2 = k^2 - (v\cdot k)^2$, and

$$\delta = 1 - \frac{(1+\Sigma_1-\Sigma_2)\,m}{(1-\Sigma_2)\,m_Q}\,. \tag{14}$$

The result simplifies if we restrict ourselves to the bubble approximation, which corresponds to the first term in an expansion in powers of $1/\beta_0$. We can then use the fact that Σ_1, Σ_2, δ and $(m_Q - m)$ are of order $1/\beta_0$. At the same time we

[9]The authors of ref. [2] have projected the inverse propagator instead of the propagator itself. This procedure is not applicable if one wants to include $1/m_Q$ corrections in the HQET.

can substitute the bare mass m by the HQET expansion parameter m_Q in the expressions for Σ_i. This leads to

$$\begin{aligned} S_P^{-1}(k,m_Q) &= -\delta m - \Big[\Sigma_1(k,m_Q) - \Sigma_1(0,m_Q)\Big]\, m_Q \\ &\qquad + \Big[1 - \Sigma_2(k,m_Q)\Big] \left(v\cdot k + \frac{k_\perp^2}{2m_Q} \right) + O(1/\beta_0^2, 1/m_Q^2)\,, \end{aligned} \tag{15}$$

where $\delta m = m_{\rm pole} - m_Q$, and

$$m_{\rm pole} = m \Big\{ 1 + \Sigma_1(0,m_Q) \Big\} + O(1/\beta_0^2) \tag{16}$$

is the pole mass. The heavy quark expansion is consistent as long as the so-called residual mass δm is a parameter of order $\Lambda_{\rm QCD}$ [33].

In order to see the appearance of renormalons, one has to consider the Borel transform of the inverse propagator. In the bubble approximation, explicit expressions for Borel transforms of the pole mass and the functions $\Sigma_i(k,m_Q)$ have been derived by Beneke and Braun [2]. By expanding their results in powers of $1/m_Q$, we obtain

$$\begin{aligned} \widetilde S_P^{-1}(k,m_Q,u) &= -\delta\widetilde m(u) + \delta(u) \left(v\cdot k + \frac{k_\perp^2}{2m_Q} \right) \\ &\quad + \frac{6C_F}{\beta_0} \bigg\{ v\cdot k \Big[(1-u^2)\, R_1 + R_2 \Big] \\ &\qquad + \frac{k_\perp^2}{2m_Q} \Big[(1-u^2)\, R_1 + (1-2u)\, R_2 \Big] \\ &\qquad + \frac{(v\cdot k)^2}{2m_Q} \Big[X(u)\, R_1 - 3\, R_2 \Big] \bigg\} + O(1/\beta_0^2, 1/m_Q^2)\,, \end{aligned} \tag{17}$$

where

$$\begin{aligned} \delta\widetilde m(u) &= \widetilde m_{\rm pole}(u) - m_Q\, \delta(u)\,, \\ \widetilde m_{\rm pole}(u) &= m \left\{ \delta(u) + \frac{6C_F}{\beta_0} \left[(1-u)\, R_1 - \frac{1}{2u} + R_m(u) \right] \right\}, \\ R_1 &= \left(\frac{m_Q}{\mu} \right)^{-2u} \frac{\Gamma(u)\,\Gamma(1-2u)}{\Gamma(3-u)}\,, \\ R_2 &= \left(\frac{-2v\cdot k}{\mu} \right)^{-2u} \frac{\Gamma(1-u)\,\Gamma(-1+2u)}{\Gamma(2+u)}\,, \\ X(u) &= -\frac{6+5u-2u^2-8u^3-4u^4}{2(1+2u)}\,. \end{aligned} \tag{18}$$

The scheme-dependent function $R_m(u)$ is irrelevant to our discussion. It is convenient to rewrite (17) in the form of a convolution of Borel transforms, which is defined as

$$\widetilde{f}(u) * \widetilde{g}(u) \equiv \widetilde{f \cdot g}(u) = \int_0^u \mathrm{d}u' \, \widetilde{f}(u') \, \widetilde{g}(u-u') \,. \tag{19}$$

Then the result takes the form

$$\begin{aligned}
\widetilde{S}_P^{-1}(k, m_Q, u) = &-\delta\widetilde{m}(u) + \widetilde{Z}_Q^{-1}(m_Q, u) * \Big\{ \widetilde{S}_{\rm eff}^{-1}(k, u) \\
&+ \frac{1}{2m_Q} \Big[\widetilde{C}_{\rm kin}(m_Q, u) * \widetilde{S}_{\rm kin}^{-1}(k, u) + \widetilde{C}_{(v\cdot D)^2}(m_Q, u) * \widetilde{S}_{(v\cdot D)^2}^{-1}(k, u) \Big] \\
&+ O(1/m_Q^2) \Big\} \,,
\end{aligned} \tag{20}$$

in which the heavy quark expansion is explicitly realised, i.e. the dependence on the two scales m_Q and k is disentangled. The dependence on the heavy quark mass m_Q is contained in the functions

$$\begin{aligned}
\widetilde{Z}_Q^{-1}(m_Q, u) &= \delta(u) + \frac{6C_F}{\beta_0} \left[(1-u^2)\, R_1 - \frac{1}{2u} + R_Z(u) \right] , \\
\widetilde{C}_{\rm kin}(m_Q, u) &= \delta(u) \,, \\
\widetilde{C}_{(v\cdot D)^2}(m_Q, u) &= \frac{6C_F}{\beta_0} \left[X(u)\, R_1 + \frac{3}{2u} + R_C(u) \right] ,
\end{aligned} \tag{21}$$

which are the Borel transforms (in the bubble approximation) of the wave-function renormalisation factor Z_Q^{-1} and the Wilson coefficients of the operators $\bar{h}_v\,(iD_\perp)^2 h_v$ (kinetic energy operator) and $\bar{h}_v\,(iv{\cdot}D)^2 h_v$ in the effective Lagrangian of the HQET [34, 35]. Note that $\widetilde{C}_{\rm kin} = \delta(u)$ implies $C_{\rm kin} = 1$, as required by reparametrisation invariance [31]. The dependence on the small scale k in (20) resides in the quantities

$$\begin{aligned}
\widetilde{S}_{\rm eff}^{-1}(k, u) &= v\cdot k \left\{ \delta(u) + \frac{6C_F}{\beta_0} \left[R_2 + \frac{1}{2u} - R_Z(u) \right] \right\} , \\
\widetilde{S}_{\rm kin}^{-1}(k, u) &= k_\perp^2 \left\{ \delta(u) + \frac{6C_F}{\beta_0} \left[(1-2u)\, R_2 + \frac{1}{2u} - R_Z(u) \right] \right\} \\
&\quad + (v\cdot k)^2\, \frac{6C_F}{\beta_0} \left[-3R_2 - \frac{3}{2u} - R_C(u) \right] , \\
\widetilde{S}_{(v\cdot D)^2}^{-1}(k, u) &= (v\cdot k)^2\, \delta(u) + O(1/\beta_0) \,,
\end{aligned} \tag{22}$$

which are the Borel transforms (in the bubble approximation) of the quark matrix elements of the operators appearing in the effective Lagrangian of the HQET. Their calculation will be outlined in sect. 4. Note that it is sufficient to keep terms of order unity in $\widetilde{S}_{(v\cdot D)^2}$, since the corresponding Wilson coefficient $\widetilde{C}_{(v\cdot D)^2}$ in (21) starts at order $1/\beta_0$. The functions $R_Z(u)$ and $R_C(u)$ in (21) and (22) depend on the renormalisation scheme. They are irrelevant to our discussion.

It is instructive to trace how renormalons are introduced in the construction of the heavy quark expansion, and in which way they cancel between the coefficient functions and matrix elements. To start with, we note that even in the full theory the self-energy contains IR renormalons from the contributions of virtual momenta below the scale k. Their positions are determined by the factor $\Gamma(1-u)$ contained in R_2 in (18). Note that the leading IR renormalon at $u=1$ is not forbidden by gauge invariance, since the self-energy is not gauge invariant and there exists a gauge-variant operator of dimension two [2]. What is introduced in the process of constructing the heavy quark expansion are new renormalon singularities at half-integer values of u. In the coefficient functions there appear IR renormalons at positions $u=\frac{1}{2},1,\frac{3}{2},\dots$ determined by the factor $\Gamma(1-2u)$ contained in R_1. Likewise, in the matrix elements there appear UV renormalons at positions $u=\frac{1}{2},0,-\frac{1}{2},\dots$ determined by the factor $\Gamma(-1+2u)$ contained in R_2. Since these singularities are absent in the original theory, they must cancel, order by order in $1/m_Q$, in the effective theory. Consider now our results (17) and (20) to see how the IR and UV renormalon poles conspire. At order $1/m_Q$ only the renormalons at $u=1/2$ need to be considered. We observe that the IR renormalon in $\delta\widetilde{m}$ matches with the UV renormalon in $\widetilde{S}_{\rm eff}^{-1}$, and the IR renormalon in the coefficient $\widetilde{Z}_Q^{-1}$ which multiplies $\widetilde{S}_{\rm eff}^{-1}$ matches with the UV renormalon in $\widetilde{S}_{\rm kin}^{-1}$. This becomes explicit if we expand (17) around $u=1/2$:

$$\begin{aligned}\frac{\beta_0}{8C_F}\,\widetilde{S}_P^{-1}(k,m_Q,u) = &-m_Q\,\frac{1}{2(1-2u)}\,\frac{\mu}{m_Q}\\ &+v\cdot k\left[\frac{3}{4(1-2u)}\,\frac{\mu}{m_Q}-\frac{1}{2(2u-1)}\,\frac{\mu}{v\cdot k}\right]\\ &+\frac{(v\cdot k)^2}{2m_Q}\,\frac{3}{2(2u-1)}\,\frac{\mu}{v\cdot k}+\dots\,,\end{aligned}\tag{23}$$

where the ellipses represent terms that are regular at $u=\frac{1}{2}$. Notice the absence of an UV renormalon at $u=\frac{1}{2}$ in the term proportional to $k_\perp^2$ in $\widetilde{S}_{\rm kin}^{-1}$ given in (22). Such a renormalon is forbidden, since it would give rise to a non-local behaviour of the form $\mu\,k_\perp^2/v\cdot k$. On the other hand, nothing forbids an IR renormalon at $u=1$ in $\delta\widetilde{m}$, which could conspire with an UV renormalon at $u=1$ in the term proportional to $(v\cdot k)^2$ in $\widetilde{S}_{\rm kin}^{-1}$.[10] The fact that this does not appear is the puzzle mentioned in the introduction.

Before we proceed, let us note that from an expansion of the expressions in (21) and (22) around $u=0$ one can extract the one-loop expressions for the

[10]There is in fact a pole at $u=1$ in $\widetilde{S}_{\rm kin}^{-1}$, but it has nothing to do with the UV region. Computing the matrix element of $\bar h_v\,(iD_\perp)^2 h_v$ with a hard IR cut-off we find that this pole disappears. Hence, it is related to the region of momenta below k. This IR renormalon pole appears also in the full theory.

corresponding quantities. Keeping only logarithmic terms, we find

$$\begin{aligned}
Z_Q^{-1}(m_Q) &= 1 - \frac{3-a}{2}\,\frac{C_F\alpha_s}{\pi}\,\ln\frac{m_Q}{\mu}\,,\\
C_{(v\cdot D)^2}(m_Q) &= \frac{3(3-a)}{2}\,\frac{C_F\alpha_s}{\pi}\,\ln\frac{m_Q}{\mu}\,,\\
S_{\rm eff}^{-1}(k) &= v\cdot k\left\{1+\frac{3-a}{2}\,\frac{C_F\alpha_s}{\pi}\,\ln\frac{-2v\cdot k}{\mu}\right\},\\
S_{\rm kin}^{-1}(k) &= k_\perp^2\left\{1+\frac{3-a}{2}\,\frac{C_F\alpha_s}{\pi}\,\ln\frac{-2v\cdot k}{\mu}\right\}\\
&\quad - \frac{3(3-a)}{2}\,\frac{C_F\alpha_s}{\pi}\,(v\cdot k)^2\,\ln\frac{-2v\cdot k}{\mu}\,,
\end{aligned} \tag{24}$$

where we have given the results for an arbitrary covariant gauge. The expression for the Wilson coefficient $C_{(v\cdot D)^2}$ has been derived (in the Feynman gauge) in ref. [35].

4 Renormalons in HQET matrix elements

In this section we show in more detail that the functions in (22) can be identified with the Borel transforms of quark matrix elements of operators in the effective Lagrangian of the HQET. At order $1/m_Q$, this Lagrangian reads [34, 35]

$$\begin{aligned}
\mathcal{L}_{\rm HQET} = \bar h_v\, iv\!\cdot\! D\, h_v + \frac{1}{2m_Q}\Big[C_{\rm kin}\,\bar h_v\,(iD_\perp)^2 h_v + C_{(v\cdot D)^2}\,\bar h_v\,(iv\!\cdot\! D)^2 h_v\\
+ C_{\rm mag}\,\frac{g_s}{2}\,\bar h_v\,\sigma_{\mu\nu}G^{\mu\nu}h_v\Big] + O(1/m_Q^2)\,,
\end{aligned} \tag{25}$$

with $C_{\rm kin} = 1$ by reparametrisation invariance [31]. The so-called chromo-magnetic operator $\bar h_v\,\sigma_{\mu\nu}G^{\mu\nu}h_v$ plays no role for our discussion here, since its matrix element between quark states vanishes. We will now outline the calculation of the matrix elements of the other operators between heavy quark states with velocity v and residual momentum k, using the Landau gauge.

The Borel transform of the inverse heavy quark propagator in the effective theory has been calculated by Beneke and Braun [2]. The result is

$$S_{\rm eff}^{-1}(k) = \langle\, k\,|\,\bar h_v\, iv\!\cdot\! D\, h_v\,|\,k\,\rangle \;\stackrel{\rm B.T.}{\to}\; v\cdot k\left\{\delta(u) + \frac{6C_F}{\beta_0}\,R_2\right\}. \tag{26}$$

After UV renormalisation, which amounts to removing the pole at $u = 0$ contained in R_2, this leads to the expression for $\widetilde S_{\rm eff}(k,u)$ given in (22).

Let us now turn to the calculation of the Borel transform of the matrix element of the kinetic energy operator $\bar h_v\,(iD_\perp)^2 h_v$. The relevant diagrams are depicted in fig. 1. The shaded bubble represents the Borel transform of the resummed gluon propagator in the Landau gauge. In the bubble approximation, the Borel transform of any one-loop diagram is simply obtained by using this propagator instead of the usual one [2, 12]. We find that the contribution of the "vertex diagram" shown in fig. 1(a) is

$$\frac{6C_F}{\beta_0}\, R_2 \left[(1-2u)\, k_\perp^2 - 5\,(v\cdot k)^2\right]. \tag{27}$$

The two "sail diagrams" depicted in fig. 1(b) and (c), which have a gluon attached to the operator insertion, each give a contribution

$$\frac{6C_F}{\beta_0}\, R_2\, (v\cdot k)^2 . \tag{28}$$

In dimensional regularisation the tadpole diagram of fig. 1(d) vanishes. Adding the tree-level contribution to the above results, we obtain

$$S_{\rm kin}^{-1}(k) = \langle\, k\,|\, \bar h_v\, (iD_\perp)^2 h_v \,|\, k\,\rangle \stackrel{\rm B.T.}{\to} k_\perp^2\, \delta(u) + \frac{6C_F}{\beta_0}\, R_2 \left[(1-2u)\, k_\perp^2 - 3\,(v\cdot k)^2\right], \tag{29}$$

in agreement with the expression for $\widetilde S_{\rm kin}^{-1}(k,u)$ given in (22).

Repeating the same calculation for the operator $\bar h_v\,(iv\cdot D)^2 h_v$, we find that the vertex diagram vanishes, and only the sail diagrams give a non-vanishing contribution. The result is

$$S_{(v\cdot D)^2}^{-1}(k) = \langle\, k\,|\, \bar h_v\, (iv\cdot D)^2 h_v \,|\, k\,\rangle \stackrel{\rm B.T.}{\to} (v\cdot k)^2 \left\{\delta(u) + \frac{12C_F}{\beta_0}\, R_2\right\}, \tag{30}$$

which generalises the expression given in (22) to order $1/\beta_0$.

5 Power divergences in HQET matrix elements

In this section we present the results of several one-loop calculations of the HQET matrix elements $S_{\rm eff}^{-1}(k)$ and $S_{\rm kin}^{-1}(k)$ in regularisation schemes with a hard UV cut-off λ. The aim is to study the correspondence between the UV renormalon poles encountered in the previous section and the power divergences associated with the use of a dimensionful regulator. For simplicity, we shall perform the calculations in gauges suitable for the regularisation method of choice. We note that the structure of the power divergences is gauge invariant.

5.1 Pauli–Villars regularisation

In the simplest version of the Pauli–Villars regularisation scheme, one substitutes for the gluon propagator in the Feynman gauge the expression

$$G_{\mu\nu}(q) = -ig_{\mu\nu} \left(\frac{1}{q^2} - \frac{1}{q^2-\lambda^2}\right) = \frac{ig_{\mu\nu}\lambda^2}{q^2(q^2-\lambda^2)} . \tag{31}$$

Computing the one-loop self-energy in the effective theory and adding the tree-level expression for the inverse propagator, we find (assuming $v\cdot k<0$)

$$\begin{aligned} S_{\rm eff}^{-1}(k) &= v\cdot k\left\{1+\frac{C_F\alpha_s}{\pi}\left(\ln\frac{-2v\cdot k}{\lambda}+\sqrt{x^2-1}\,\arctan\sqrt{x^2-1}\right)\right\} \\ &= -\frac{C_F\alpha_s}{2}\,\lambda+v\cdot k\left\{1+\frac{C_F\alpha_s}{\pi}\left(\ln\frac{-2v\cdot k}{\lambda}-1\right)\right\}+O(1/\lambda)\,, \end{aligned} \tag{32}$$

where $x=\lambda/(-v\cdot k)$. The generalisation of this result to an arbitrary covariant gauge has been given in (7). The linear divergence corresponds to the UV renormalon pole at $u=\frac{1}{2}$ in (26). It is instructive to rewrite the result in a form that makes explicit the mixing of operators under renormalisation. We define

$$\begin{aligned} S_{\rm eff}^{-1}(k) &= \langle\,k\,|\,\bar h_v\,iv\cdot D\,h_v\,|\,k\,\rangle \\ &= Z_{\rm eff}^{-1}\,\langle\,k\,|\,\bar h_v\,iv\cdot D\,h_v\,|\,k\,\rangle_0+Z_{v\cdot D\to\hat 1}^{-1}\,\lambda\,\langle\,k\,|\,\bar h_v\,h_v\,|\,k\,\rangle_0\,, \end{aligned} \tag{33}$$

where $\langle\,k\,|\,\bar h_v\,iv\cdot D\,h_v\,|\,k\,\rangle_0=v\cdot k$ and $\langle\,k\,|\,\bar h_v\,h_v\,|\,k\,\rangle_0=1$ are the tree-level quark matrix elements. From (32) we then read off the renormalisation constants

$$\begin{aligned} Z_{\rm eff}^{-1} &= 1+\frac{C_F\alpha_s}{\pi}\left(\ln\frac{-2v\cdot k}{\lambda}-1\right), \\ Z_{v\cdot D\to\hat 1}^{-1} &= -\frac{C_F\alpha_s}{2}\,. \end{aligned} \tag{34}$$

In order to regulate higher power divergences, it is necessary to introduce a more general form of the Pauli–Villars regularisation. Instead of (31), we shall use two subtractions and write

$$\begin{aligned} G_{\mu\nu}(q) &= -ig_{\mu\nu}\left(\frac{1}{q^2}-\frac{\gamma}{\gamma-1}\,\frac{1}{q^2-\lambda^2}+\frac{1}{\gamma-1}\,\frac{1}{q^2-\gamma\lambda^2}\right) \\ &= \frac{-ig_{\mu\nu}\gamma\lambda^4}{q^2(q^2-\lambda^2)(q^2-\gamma\lambda^2)}\,. \end{aligned} \tag{35}$$

Repeating the above calculation, we obtain

$$\begin{aligned} S_{\rm eff}^{-1}(k) = v\cdot k\Bigg\{1+\frac{C_F\alpha_s}{\pi}\Bigg[\ln\frac{-2v\cdot k}{\lambda}+\frac{\ln\gamma}{2(\gamma-1)} \\ +\frac{1}{\gamma-1}\left(\gamma\sqrt{x^2-1}\,\arctan\sqrt{x^2-1}-\sqrt{x_\gamma^2-1}\,\arctan\sqrt{x_\gamma^2-1}\right)\Bigg]\Bigg\}\,, \end{aligned} \tag{36}$$

where $x_\gamma=\sqrt{\gamma}\,x=\sqrt{\gamma}\,\lambda/(-v\cdot k)$. This leads to

$$\begin{aligned} Z_{\rm eff}^{-1} &= 1+\frac{C_F\alpha_s}{\pi}\left(\ln\frac{-2v\cdot k}{\lambda}+\frac{\ln\gamma}{2(\gamma-1)}-1\right), \\ Z_{v\cdot D\to\hat 1}^{-1} &= -\frac{C_F\alpha_s}{2}\,\frac{\sqrt{\gamma}}{\sqrt{\gamma}+1}\,. \end{aligned} \tag{37}$$

Eqs. (34) are recovered in the limit $\gamma \to \infty$.

The calculation of the matrix element of the kinetic energy operator is more complicated. Consider again the diagrams in fig. 1, where now the shaded bubble represents the regulated propagator given in (35). We find that in the Feynman gauge only the vertex and the tadpole diagrams give non-vanishing contributions. For dimensional reasons the tadpole diagram is quadratically divergent. Its contribution to $S_{\rm kin}^{-1}(k)$ is

$$-\,\frac{3C_F\alpha_s}{4\pi}\,\lambda^2\,\frac{\gamma\ln\gamma}{\gamma-1}\,. \tag{38}$$

However, this quadratic divergence is exactly cancelled by the quadratic divergence of the vertex diagram. For the sum of all one-loop contributions we find

$$\begin{aligned} S_{\rm kin}^{-1}(k) = \frac{C_F\alpha_s}{\pi}\Bigg\{ & \frac{3\gamma\lambda^2}{\gamma-1}\left(\sqrt{x_\gamma^2-1}^{\,-1}\arctan\sqrt{x_\gamma^2-1}-\sqrt{x^2-1}^{\,-1}\arctan\sqrt{x^2-1}\right) \\ & +\left[k_\perp^2-3(v\cdot k)^2\right]\Bigg[\ln\frac{-2v\cdot k}{\lambda}+\frac{\ln\gamma}{2(\gamma-1)} \\ & +\frac{1}{\gamma-1}\left(\sqrt{x_\gamma^2-1}^{\,-1}\arctan\sqrt{x_\gamma^2-1}-\gamma\sqrt{x^2-1}^{\,-1}\arctan\sqrt{x^2-1}\right)\Bigg]\Bigg\}\,. \end{aligned} \tag{39}$$

Note that, in spite of the explicit factor of λ^2 in the first term on the r.h.s., this term diverges only linearly with λ.

We are now ready to compute the mixing of the kinetic energy operator with lower-dimensional operators. We define

$$\begin{aligned} S_{\rm kin}^{-1}(k) &= \langle\, k\,|\,\bar h_v\,(iD_\perp)^2 h_v\,|\,k\,\rangle \\ &= Z_{\rm kin}^{-1}\,\langle\, k\,|\,\bar h_v\,(iD_\perp)^2 h_v\,|\,k\,\rangle_0 + Z^{-1}_{D_\perp^2\to(v\cdot D)^2}\,\langle\, k\,|\,\bar h_v\,(iv\!\cdot\! D)^2 h_v\,|\,k\,\rangle_0 \\ &\quad + Z^{-1}_{D_\perp^2\to v\cdot D}\,\lambda\,\langle\, k\,|\,\bar h_v\, iv\!\cdot\! D\, h_v\,|\,k\,\rangle_0 + Z^{-1}_{D_\perp^2\to\hat 1}\,\lambda^2\,\langle\, k\,|\,\bar h_v\, h_v\,|\,k\,\rangle_0\,. \end{aligned} \tag{40}$$

From an expansion of (39) in the limit of large λ, we obtain

$$\begin{aligned} Z_{\rm kin}^{-1} &= 1+\frac{C_F\alpha_s}{\pi}\left(\ln\frac{-2v\cdot k}{\lambda}+\frac{\ln\gamma}{2(\gamma-1)}\right), \\ Z^{-1}_{D_\perp^2\to(v\cdot D)^2} &= -\frac{3C_F\alpha_s}{\pi}\left(\ln\frac{-2v\cdot k}{\lambda}+\frac{\ln\gamma}{2(\gamma-1)}-1\right), \\ Z^{-1}_{D_\perp^2\to v\cdot D} &= \frac{3C_F\alpha_s}{2}\,\frac{\sqrt{\gamma}}{\sqrt{\gamma}+1}\,, \\ Z^{-1}_{D_\perp^2\to\hat 1} &= 0\,. \end{aligned} \tag{41}$$

As mentioned above, there is no quadratic divergence in the sum of all diagrams, i.e. in Pauli–Villars regularisation there is no mixing of the kinetic energy operator with the operator $\bar h_v\, h_v$. We have checked that this result holds true in an arbitrary number of space-time dimensions. Note that the coefficient $Z_{\rm kin}^{-1}$ is related to the self-energy of the heavy quark by

$$Z_{\rm kin}^{-1} = \frac{\rm d}{{\rm d}(v\cdot k)}\,S_{\rm eff}^{-1}(k)\,. \tag{42}$$

This relation is a consequence of reparametrisation invariance [31]. It is satisfied also for the expressions given in (22), which were obtained in dimensional regularisation.

5.2 Momentum flow regularisation

We shall now discuss another regularisation scheme in which a hard UV cut-off is used to compute the quark matrix elements in the HQET. It is based on the observation that the resummation of renormalon chains is equivalent to performing one-loop calculations with a running coupling constant [17]. Consider again the matrix element of the operator $\bar h_v\,iv\!\cdot\!D\,h_v$. Imagine performing the one-loop calculation of this matrix element with a running coupling constant. In the Landau gauge, the result can be written in the form

$$S_{\rm eff}^{-1}(k) = v\cdot k\left\{ 1 + \int\limits_0^\infty {\rm d}\tau\,\widehat w_{v\cdot D}(\tau)\,\frac{\alpha_s(\sqrt{\tau}\omega)}{\pi} \right\}, \tag{43}$$

where we assume that $\omega=-2v\!\cdot\!k\gg\Lambda_{\rm QCD}$. It has been shown in ref. [17] that the distribution function $\widehat w_{v\cdot D}(\tau)$ is related to the (unrenormalised) Borel transform of the inverse propagator. One finds

$$\int\limits_0^\infty {\rm d}\tau\,\widehat w_{v\cdot D}(\tau)\,\tau^{-u} = \frac{3C_F}{2}\,\frac{\Gamma(1-u)\,\Gamma(-1+2u)}{\Gamma(2+u)}\,, \tag{44}$$

which can be inverted to give

$$\widehat w_{v\cdot D}(\tau) = \frac{C_F}{8\tau^2}\left\{ (1+4\tau)^{3/2} - 1 - 6\tau \right\}. \tag{45}$$

The representation (43) offers a natural way to introduce a hard UV cut-off, since the product $\sqrt{\tau}\omega$ has the interpretation of a physical scale. The distribution function $\widehat w_{v\cdot D}(\tau)$ controls the momentum flow in the one-loop self-energy diagram. For large values of τ, this function falls off proportional to $1/\sqrt{\tau}$, so that the integral is linearly UV divergent. Introducing a hard UV cut-off[11] $\tau_{\rm UV}=\lambda^2/\omega^2$

[11]In principle one should also introduce an IR cut-off, since the running coupling constant is not well-defined in the low-momentum region. This fact is irrelevant at one-loop order, however, where one can neglect the running of the coupling constant.

and expanding the result for $\lambda\to\infty$, we find for the renormalisation factors defined in (33)

$$\begin{aligned} Z_{\rm eff}^{-1} &= 1+\frac{3C_F}{2}\,\frac{\alpha_s}{\pi}\left(\ln\frac{-2v\cdot k}{\lambda}-\frac{1}{2}\right), \\ Z_{v\cdot D\to\hat{1}}^{-1} &= -\frac{C_F\alpha_s}{\pi}\,. \end{aligned} \tag{46}$$

Note the similarity to the results (34) obtained with a Pauli–Villars regulator.

Let us now consider the quark matrix element of the kinetic energy operator. We define

$$S_{\rm kin}^{-1}(k) = k_\perp^2\left\{1+\int\limits_0^\infty\!{\rm d}\tau\,\widehat{w}_{D_\perp^2}(\tau)\,\frac{\alpha_s(\sqrt{\tau}\omega)}{\pi}\right\}+(v\cdot k)^2\int\limits_0^\infty\!{\rm d}\tau\,\widehat{w}_{(v\cdot D)^2}(\tau)\,\frac{\alpha_s(\sqrt{\tau}\omega)}{\pi}\,. \tag{47}$$

Using again the relation between the distribution functions and the Borel transforms given in (22), we obtain

$$\begin{aligned} \widehat{w}_{D_\perp^2}(\tau) &= -\frac{3C_F}{16\tau^2}\left(\sqrt{1+4\tau}-1\right)^2, \\ \widehat{w}_{(v\cdot D)^2}(\tau) &= -3\,\widehat{w}_{v\cdot D}(\tau)\,. \end{aligned} \tag{48}$$

For large values of τ, the first function falls off proportional to $1/\tau$, so that the corresponding integral is logarithmically UV divergent. After a straightforward calculation we find for the renormalisation factors defined in (40)

$$\begin{aligned} Z_{\rm kin}^{-1} &= 1+\frac{3C_F}{2}\,\frac{\alpha_s}{\pi}\left(\ln\frac{-2v\cdot k}{\lambda}+\frac{1}{2}\right), \\ Z_{D_\perp^2\to(v\cdot D)^2}^{-1} &= -\frac{9C_F}{2}\,\frac{\alpha_s}{\pi}\left(\ln\frac{-2v\cdot k}{\lambda}-\frac{1}{2}\right), \\ Z_{D_\perp^2\to v\cdot D}^{-1} &= \frac{3C_F\alpha_s}{\pi}\,, \\ Z_{D_\perp^2\to\hat{1}}^{-1} &= 0\,. \end{aligned} \tag{49}$$

Once again we find that the quadratic divergence, which is expected on dimensional grounds, is absent. In the momentum flow regularisation scheme, the absence of the quadratic divergence is directly linked to the absence of an UV renormalon pole at $u=1$ in the Borel transform $\widetilde{S}_{\rm kin}^{-1}(k,u)$ in (22), since the Borel transform determines the distribution function through an integral relation of the type (44).

It is instructive to note the close similarity of the above results with (41). In both regularisation schemes and in an arbitrary covariant gauge, the results for the renormalisation factors can be written in the form

$$Z_{\rm eff}^{-1} = 1+\frac{C_F\alpha_s}{\pi}\left(\frac{3-a}{2}\,\ln\frac{-2v\cdot k}{\lambda}+c\right),$$

$$Z^{-1}_{v\cdot D\to\hat{1}} = -\frac{C_F\alpha_s}{\pi}\, d\,, \tag{50}$$

and

$$\begin{aligned}
Z^{-1}_{\rm kin} &= \left(1 + v\cdot k\,\frac{\mathrm{d}}{\mathrm{d}v\cdot k}\right) Z^{-1}_{\rm eff}\,,\\
Z^{-1}_{D^2_\perp\to(v\cdot D)^2} &= -3\Big(Z^{-1}_{\rm eff} - 1\Big)\,,\\
Z^{-1}_{D^2_\perp\to v\cdot D} &= -3Z^{-1}_{v\cdot D\to\hat{1}}\,,\\
Z^{-1}_{D^2_\perp\to\hat{1}} &= 0\,,
\end{aligned} \tag{51}$$

where

$$c_{\rm PV} = \frac{\ln\gamma}{2(\gamma-1)} - 1\,, \qquad d_{\rm PV} = \frac{\pi}{2}\,\frac{\sqrt{\gamma}}{\sqrt{\gamma}+1} \tag{52}$$

in Pauli–Villars regularisation (in the Feynman gauge), and

$$c_{\rm MF} = -\frac{1}{2}\,, \qquad d_{\rm MF} = 1 \tag{53}$$

in the momentum flow regularisation (in the Landau gauge). The coefficient c is gauge dependent, but the constant d is not.

5.3 Lattice regularisation

In this section we discuss the one-loop corrections to the propagator and to the kinetic energy operator in the lattice regularisation. All results reported in this section have been obtained in the static theory ($\vec{v} = 0$) in Euclidean space.

Among the possible lattice formulations of the HQET we choose the simplest one, in which the heavy quark propagator in a background gauge field U has the form [36]

$$S^0_h(x,y) = \theta(x^4 - y^4)\,\delta(\vec{x} - \vec{y})\,\mathcal{P}_{\vec{x}}(x^4, y^4)\,, \tag{54}$$

where $\mathcal{P}_{\vec{x}}(x^4, y^4)$ is the path-ordered exponential from $(\vec{x}, x^4)$ to $(\vec{x}, y^4)$ along a path whose position in space is constant:

$$\mathcal{P}_{\vec{x}}(x^4, y^4) \equiv U_4^\dagger(\vec{x}, x^4 - a)\,U_4^\dagger(\vec{x}, x^4 - 2a)\ldots U_4^\dagger(\vec{x}, y^4 + a)\,U_4^\dagger(\vec{x}, y^4)\,. \tag{55}$$

At one-loop order, the calculation of the heavy quark self-energy in the Feynman gauge gives [16, 37, 38]

$$\langle S^0_h(x,y)\rangle = \theta(x^4 - y^4)\,\delta(\vec{x} - \vec{y})\,\Big\{1 + \alpha_s(Xta^{-1} + Y)\Big\} + O(\alpha_s^2)\,, \tag{56}$$

where $t = x^4 - y^4$, $\alpha_s = g_0^2/4\pi$ is the bare lattice coupling constant, and $\langle\ldots\rangle$ denotes the average over the field configurations. The quantities X and Y are given by

$$X = -\frac{C_F}{4}\int_{-\pi}^{\pi}\frac{\mathrm{d}^3k}{2\pi^2}\,\frac{1}{A}\,,$$
$$Y = \frac{C_F}{4}\int_{-\pi}^{\pi}\frac{\mathrm{d}^3k}{2\pi^2}\,\frac{1}{A\sqrt{(1+A)^2-1}}\,, \tag{57}$$

with

$$A = \sum_{i=1}^{3}(1-\cos k_i)\,. \tag{58}$$

The constant Y is logarithmically divergent. It contributes to the wave-function renormalisation and hence to the renormalisation of any local operator containing the heavy quark field. The quantity X is gauge invariant and has to be identified with the coefficient of the operator $\bar{h}_v\,h_v$. Adopting the notation introduced in (33), where now $\lambda = 1/a$ plays the role of the UV cut-off, one obtains

$$Z^{-1}_{v\cdot D\to\hat{1}} = -\alpha_s\,X \tag{59}$$

for the coefficient of the linearly divergent term.

The corrections to the kinetic energy operator can be computed from

$$S_h^1(x,y) = \theta(x^4-y^4)\,\delta(\vec{x}-\vec{y})\,\sum_{w^4=y^4}^{x^4}\mathcal{P}_{\vec{x}}(x^4,w^4)\,\vec{D}^2(\vec{x},w^4)\,\mathcal{P}_{\vec{y}}(w^4,y^4)\,. \tag{60}$$

For the operator $\vec{D}^2$ we take

$$\vec{D}^2(\vec{z},z^4)\,f(z) = \frac{1}{a^2}\sum_{j=1}^{3}\Big(U_j(z)\,f(z+a\hat{\jmath}) + U_j^\dagger(z-a\hat{\jmath})\,f(z-a\hat{\jmath}) - 2f(z)\Big)\,, \tag{61}$$

where $\hat{\jmath}$ denotes the unit vector in the j-direction. At the tree level one finds

$$\langle S_h^1(x,y)\rangle_0 = \theta(x^4-y^4)\,\delta(\vec{x}-\vec{y})\,\frac{1}{a^2}\sum_{w^4=y^4}^{x^4}\sum_{j=1}^{3}\Big\{\delta(\vec{x}+a\hat{\jmath}-\vec{y})+\delta(\vec{x}-a\hat{\jmath}-\vec{y})-2\delta(\vec{x}-\vec{y})\Big\}\,. \tag{62}$$

At one-loop order one obtains [16]

$$\begin{aligned}\langle S_h^1(x,y)\rangle &= \theta(x^4-y^4)\,\delta(\vec{x}-\vec{y})\,\alpha_s\,(Wta^{-2}+2Xa^{-1})\\ &\quad + \Big[\Delta + \alpha_s\,(Xta^{-1}+Y)\Big]\,\langle S_h^1(x,y)\rangle_0\,,\end{aligned} \tag{63}$$

where X and Y are defined in (57), $\Delta = 1+\frac{1}{6}\,\alpha_s W$, and

$$W = -C_F\int_{-\pi}^{\pi}\frac{\mathrm{d}^3k}{2\pi^2}\,\frac{1}{\sqrt{(1+A)^2-1}}\,. \tag{64}$$

Using the notation introduced in (40), we find

$$
\begin{aligned}
Z^{-1}_{D_\perp^2\to v\cdot D} &= 2\alpha_s\, X\,, \\
Z^{-1}_{D_\perp^2\to \hat{1}} &= -\alpha_s\, W\,.
\end{aligned} \tag{65}
$$

Thus, in this case the mixing coefficient of $\bar{h}_v\,(iD_\perp)^2 h_v$ with $\bar{h}_v\,h_v$ does not vanish. Moreover, since reparametrisation invariance is broken on the lattice, the renormalisation of the kinetic energy operator differs from the wave function renormalisation by a factor $\Delta \neq 1$.

6 Conclusions

In the HQET the coefficients of higher-order terms in the expansion in inverse powers of the heavy quark mass are proportional to matrix elements of higher-dimensional operators. In general these matrix elements diverge in perturbation theory as powers of the UV cut-off, and using dimensional regularisation their perturbation series are not Borel summable due to the presence of UV renormalon singularities in the Borel plane. The ambiguities due to the presence of these renormalons are cancelled by those due to IR renormalons in the Wilson coefficient functions of lower-dimensional operators.

In this paper we have investigated in some detail the relation between power divergences and renormalons. This correspondence, which is based on dimensional arguments, appears to fail in one important case, that of the kinetic energy operator. In a lattice regularisation $\bar{h}_v\,(iD_\perp)^2 h_v$ mixes with $\bar{h}_v\,h_v$ with a quadratically divergent coefficient, as expected from the naive degree of divergence of the corresponding Feynman diagrams. However, at least to one-loop order in perturbation theory, this quadratic divergence is absent when using the Pauli-Villars and momentum flow regularisations. Moreover, the corresponding UV renormalon is absent using dimensional regularisation. The absence of the corresponding IR renormalon in the pole mass had already been noted in ref. [2]. These results suggest that there may be some symmetry, which prevents $\bar{h}_v\,(iD_\perp)^2 h_v$ from mixing with $\bar{h}_v\,h_v$, and that this symmetry is broken with the lattice regularisation. An obvious candidate for such a symmetry is Lorentz (or Euclidean) invariance, which takes the form of reparametrisation invariance in the effective theory [31, 32]. This symmetry does not seem to forbid such a mixing, however. Given the relevance of the kinetic energy operator to studies of the spectroscopy and inclusive decays of heavy hadrons [18]–[26], it is important to understand whether the absence of the power divergences and the corresponding renormalon is an accident of one-loop perturbation theory and the bubble approximation, or whether it is a consequence of a more general principle. We hope that this paper will stimulate further investigation of this puzzle.

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Figure

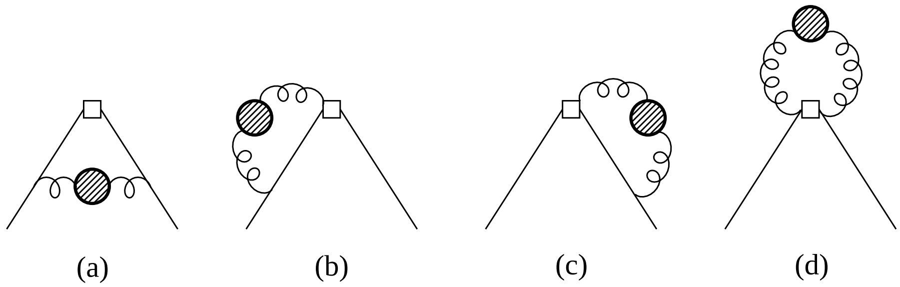

Figure 1: Diagrams contributing to the quark matrix element of the kinetic energy operator (indicated by a square).