ESTIMATES OF THE HIGHER-ORDER QCD CORRECTIONS TO $R(s)$, R_τ AND DEEP INELASTIC SCATTERING SUM RULES [1]

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ABSTRACT

We present the attempt to study the problem of the estimates of higher-order perturbative corrections to physical quantities in the Euclidean region. Our considerations are based on the application of the scheme-invariant methods, namely the principle of minimal sensitivity and the effective charges approach. We emphasize, that in order to obtain the concrete results for the physical quantities in the Minkowskian region the results of application of this formalism should be supplemented by the explicite calculations of the effects of the analytical continuation. We present the estimates of the order $O(\alpha_s^4)$ QCD corrections to the Euclidean quantities: the e^+e^--annihilation D-function and the deep inelastic scattering sum rules, namely the non-polarized and polarized Bjorken sum rules and to the Gross–Llewellyn Smith sum rule. The results for the D-function are further applied to estimate the $O(\alpha_s^4)$ QCD corrections to the Minkowskian quantities $R(s) = \sigma_{tot}(e^+e^- \to hadrons)/\sigma(e^+e^- \to \mu^+\mu^-)$ and $R_\tau = \Gamma(\tau \to \nu_\tau + hadrons)/\Gamma(\tau \to \nu_\tau\overline{\nu}_e e)$. The problem of the fixation of the uncertainties due to the $O(\alpha_s^5)$ corrections to the considered quantities is also discussed.

I. During the last few years, essential progress has been achieved in the area of the calculation of the next-next-to-leading order ($NNLO$) QCD corrections to the number of physical quantities. Indeed, the complete $NNLO$ $O(\alpha_s^3)$ QCD corrections are known at present for the characteristics of $e^+e^- \to$ hadrons process [1], [2], $\tau \to \nu_\tau+$hadrons decay [3] $Z^0 \to$ hadrons process [4] and for the deep inelastic scattering sum rules, namely the non-polarized Bjorken sum rule (BjnSR) [5], the Gross-Llewellyn Smith sum rule (GLSSR) and the polarized Bjorken sum rule (BjpSR) [6]. Amongst the physical information provided by these results is the estimate of the theoretical uncertainties of the corresponding next-to-leading-order (NLO) perturbative QCD predictions for $R(s) = \sigma_{tot}(e^+e^- \to hadrons)/\sigma(e^+e^- \to \mu^+\mu^-)$ [7], $\Gamma(Z^0 \to hadrons)$ (see. e.g., [8]), $R_\tau = \Gamma(\tau \to \nu_\tau + hadrons)/\Gamma(\tau \to \nu_\tau\overline{\nu}_e e)$ [3], BjnSR [9], GLSSR and BjpSR [10].

After gaining from the understanding of the phenomenological meaning of the effects of the $NNLO$ corrections to observable quantities one can of course stop at this level. However, in view of the fact that the precision of the experimental data for $R(s)$, R_τ, $\Gamma(Z^0 \to hadrons)$ and the DIS sum rules is continuously increasing, the questions about the possible values of the effects of non-calculated higher order terms is frequently raised both by theoreticians and experimentalists. This information can be useful in the studies of several important problems. It allows to present the concrete numerical estimate of the theoretical uncertainties of the $NNLO$ QCD approximations. Moreover, any new information, even empirical, about the possible values of the higher-order terms might be useful in understanding of the general structure of the perturbative expansions for different quantities in the concrete renormalization schemes and different energy regions.

A certain step in the direction of the estimates of the order $O(\alpha_s^4)$ QCD corrections was made in the case of R_τ in Ref. [3]. These estimates are based on the tendency, observed in Ref. [11], of the scheme-dependent uncertainties of the perturbative QCD predictions for $R(s)$ and R_τ to decrease as a result of taking into account the order $O(\alpha_s^3)$-terms. This effect already occurs at the $O(\alpha_s^2)$ level. The inclusion of the $O(\alpha_s^3)$ corrections [11] confirms this effect and makes it more vivid. The foundations of Ref. [11] were further improved in the process of phenomenological studies of the QCD predictions for R_τ, using the explicite separations of the expressions for the n-th order coefficients r_n^τ to the n-th order coefficients d_n of the Euclidean D-function and the certain contributions, which can be explicitly taken into account after completing the calculations in the $k \leq n-1$ order of the perturbation theory [12].

Notice, that starting from the $NNLO$ level these contributions contain π^2-factors which appear as the result of analytic continuation to the physical region. As was demonstarted in Refs.[13, 14],[12] in order to achieve better convergence of the corresponding approximants it is essential to treat them seperately from the coefficients of the D-function. Even the seperate renormalization-group [15] inspired summation of all additional terms in the expression for $R(s)$ [14] and R_τ [13, 12] was discussed.

This work is the attempt to address the important questions of the study of the higher-order perturbative approximations to physical quantities. We will use the approach based on the imrovement formula [16] of re-expansion of the expressions for the quantities obtained in the N-th order of perturbation theory within the principle of minimal sensitivity (PMS) [16] and the

effective charges (ECH) approach [17], which is equivalent *a posteriori* to the scheme-invariant perturbation theory [18]. The main aim of this work, which is based on the considerations of Refs. [19, 20], is to present the observations obtained in the process of the study of the problem of the possibilities of estimating higher-order QCD corrections to $R(s)$, R_τ and deep-inelastic scattering sum rules using the scheme-inariant procedures.

Of course, this approach to the estimation of the values of the uncalculated higher- order terms cannot be considered as the alternative to the direct analytical or numerical calculations. However, we hope that this method can give the concrete prescription for impression about the possible theoretical errors due to variation of the higher order terms. The first message came from the fact that the rather bold-guess application of this method for the estimate of the four-loop corrections to the expression for $(g-2)_e$ [21] gave results, which turned out to be in surprisingly good agreement with the latest results of the direct numerical calculations of Ref. [22] [4]. The second argument in favour of this procedure came from its successful application for the analysis of the Drell-Yan cross section at the $O(\alpha_s^2)$-level [24]. Moreover, as will be demonstrated in our work, the re-expansion formalism of Ref. [16] is also working quite well in QCD at the $NNLO$ for at least three independent quantities, namely for the e^+e^--annihilation D-function and the non-polarized and polarized Bjorken sum rules. Therefore, we prefer to consider all these facts as an arguments in favour of more detailed studies of the intrinsic features of this approach.

In this work we further develop the re-expansion formalism of Ref. [16], deriving new terms in the corresponding improvement formula. The previously-known terms are used to obtain the estimates of the next-to-next-to-next-to-leading order (N^3LO) QCD corrections to the D-function, the non-polarized Bjorken sum rule and polarized Bjorken sum rule, which is closely related by the structure of the corresponding perturbative series to the Gross–Llewellyn Smith sum rule. We will use the results obtained for the D-function to estimate the effects of the N^3LO corrections to the perturbative series for the Minkowskian quantities $R(s)$ and R_τ by adding the explicitly calculable terms, previously discussed in Refs.[25, 12], supplementing thus the related considerations of Refs. [16], [26] by the additional input information. The derived new term in the improvement formula is applied to touch on the problem of fixing the values of the $O(\alpha_s^5)$-corrections to the analysed quantities. We also specify the scheme which is related to our considerations and emphasize that the structure of the perturbative expansions in the Euclidean and Minkowskian regions differ essentially.

II.   Consider first the order $O(a^N)$ approximation of a Euclidean physical quantity

$$D_N = d_0 a (1 + \sum_{i=1}^{N-1} d_i a^i) \tag{1}$$

with $a = \alpha_s/\pi$ being the solution of the corresponding renormalization group equation for the β-function which is defined as

$$\mu^2 \frac{\partial a}{\partial \mu^2} = \beta(a) = -\beta_0 a^2 (1 + \sum_{i=1}^{N-1} c_i a^i) \ . \tag{2}$$

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[4]A detailed re-consideration of the $(g-2)_\mu$ analysis of Ref. [21] and its generalization to the five-loop order will be presented elsewhere [23].

In QCD, in the process of the concrete calculations of the coefficients $d_i, i \geq 1$ and $c_i, i \geq 2$, the $\overline{MS}$ scheme is commonly used. However, this scheme is not the unique prescription for fixing the RS ambiguities (for the recent discussions see e.g., Ref. [26]).

The PMS [16] and ECH [17] prescriptions stand out from various methods of treating scheme-dependence ambiguities. Indeed, they are based on the conceptions of the scheme-invariant quantities, which are defined as the combinations of the scheme-dependent coefficients in Eqs. (1) and (2). Both these methods pretend to be "optimal" prescriptions, in the sense that they might provide better convergence of the corresponding approximations in the non-asymptotic regime, and thus allow an estimation of the uncertainties of the perturbative series in the definite order of perturbation theory. Therefore, applying these "optimal" methods, one can try to estimate the effects of the order $O(a^{N+1})$-corrections starting from the approxima-tions $D_N^{opt}(a_{opt})$ calculated in a certain "optimal" approach [16], [21], [27]. This idea is closely related to the QED technique of Ref. [28], which was used to predict the renormalization-group controllable $ln(m_\mu/m_e)$-terms in the series for $(g-2)_\mu$ from the expression of $(g-2)_\mu$ through the effective coupling constant $\bar{\alpha}(m_\mu/m_e)$. In our work work we are using this technique to estimate also the constant terms of the higher-order corrections in QCD.

Let us following the considerations of Ref. [16] re-expand $D_N^{opt}(a_{opt})$ in terms of the coupling constant a of the particular scheme

$$D_N^{opt}(a_{opt}) = D_N(a) + \delta D_N^{opt} a^{N+1} \tag{3}$$

where

$$\delta D_N^{opt} = \Omega_N(d_i, c_i) - \Omega_N(d_i^{opt}, c_i^{opt}) \tag{4}$$

are the numbers which simulate the coefficients of the order $O(a^{N+1})$-corrections to the physical quantity, calculated in the particular initial scheme, say the $\overline{MS}$-scheme. The coefficients Ω_N can be obtained from the following system of equations:

$$\frac{\partial}{\partial \tau}(D_N + \Omega_N a^{N+1}) = O(a^{N+2}),$$

$$\frac{\partial}{\partial c_i}(D_N + \Omega_N a^{N+1}) = O(a^{N+2}), \ i \geq 2 \tag{5}$$

where the parameter $\tau = \beta_0 \ell n(\mu^2/\Lambda^2)$ represents the freedom in the choice of the renormal-ization point μ. The conventional scale parameter Λ will not explicitly appear in all our final formulas.

The explicit form of the coefficients Ω_2 and Ω_3 in which we will be interested can be obtained by the solution of the system of equations (5), following the lines of Ref. [16]. We present here the final already known expressions [16]:

$$\Omega_2 = d_0 d_1(c_1 + d_1), \tag{6}$$

$$\Omega_3 = d_0 d_1\left(c_2 - \frac{1}{2}c_1 d_1 - 2d_1^2 + 3d_2\right). \tag{7}$$

and the new term Ω_4 evaluated by us:

$$\Omega_4 = \frac{d_0}{3}(3c_3d_1 + c_2d_2 - 4c_2d_1^2 + 2c_1d_1d_2 - c_1d_3 + 14d_1^4 - 28d_1^2d_2 + 5d_2^2 + 12d_1d_3) \tag{8}$$

which reproduces the renormalization-group controllable logarithmic terms at the five-loop level [29].

It should be stressed that in the ECH approach $d_i^{ECH} \equiv 0$ for all $i \geq 2$. Therefore one gets the following expressions for the $NNLO$ and N^3LO corrections in Eq. (3):

$$\delta D_2^{ECH} = \Omega_2(d_1, c_1) \tag{9}$$

$$\delta D_3^{ECH} = \Omega_3(d_1, d_2, c_1, c_2) \tag{10}$$

$$\delta D_4^{ECH} = \Omega_4(d_1, d_2, d_3, c_1, c_2, c_3) \tag{11}$$

where Ω_2, Ω_3 and Ω_4 are defined in Eqs. (6),(7) and (8) respectively.

It is worth emphasizing that the general expressions for the correction coefficients $\delta D_N = d_0 d_N$ in Eqs. (9), (10), (11) can be obtained from the following exact relation for the process dependent but scheme-independent quantities

$$\frac{c_N^{ECH}}{N-1} = \frac{c_N}{N-1} + d_N - \frac{\Omega_N}{d_0} \tag{12}$$

which are related to the coefficients c_N^{ECH} of the ECH β-functions defined as $\beta_{eff}(a_{ECH}) = -\beta_0 a_{ECH}^2 (1 + c_1 a_{ECH} + \sum_{i \geq 2} c_i^{ECH} a_{ECH}^i)$. Therefore, the expressions for the corrections δD_N^{ECH} are the <u>exact</u> numbers which are related to the special scheme. This scheme is identical to the $\overline{MS}$ scheme at the lower order levels and is defined by the condition $c_N = c_N^{ECH}$ at the N-th order, where c_N^{ECH} are considered as unknown numbers.

In order to find similar corrections to Eq.(3) in the N-th order of perturbation theory starting from the PMS approach [16], it is necessary to use the relations obtained in Ref. [30] between the coefficients d_i^{PMS} and c_i^{PMS} ($i \geq 1$) in the expression for the order $O(a_{PMS}^N)$ approximation $D_N^{PMS}(a_{PMS})$ of the physical quantity under consideration. The corresponding corrections have the following form:

$$\delta D_2^{PMS} = \delta D_2^{ECH} + \frac{d_0 c_1^2}{4} \tag{13}$$

$$\delta D_3^{PMS} = \delta D_3^{ECH} . \tag{14}$$

Notice the identical coincidence of the N^3LO corrections obtained starting from both the PMS and ECH approaches. A similar observation was made in Ref. [21] using different (but related) considerations.

In the fourth order of perturbation theory the additional contribution to δD_4^{PMS} has more complicated structure. Indeed, the expression for $\Omega_4(d_i^{PMS}, c_i^{PMS})$ in Eq. (4) reads:

$$\Omega_4(d_i^{PMS}, c_i^{PMS}) = \frac{d_0}{3}[\frac{1}{4}c_1 c_3^{PMS} - \frac{4}{81}(c_2^{PMS})^2 - \frac{5}{81}c_1^2 c_2^{PMS} + \frac{7}{648}c_1^4] \tag{15}$$

where

$$c_2^{PMS} = \frac{9}{8}(d_2 + c_2 - d_1^2 - c_1 d_1 + \frac{7}{36}c_1^2) + O(a) \qquad (16)$$

and

$$c_3^{PMS} = 4(d_3 + \frac{1}{2}c_3 - c_2 d_1 - 3d_1 d_2 + 2d_1^3) + \frac{1}{2}c_1(d_2 + c_2 + 3d_1^2 - c_1 d_1 + \frac{1}{108}c_1^2) + O(a). \qquad (17)$$

The expressions for Eqs. (15)- (17) are pure numbers, which do not depend on the choice of the initial scheme. Note that we have checked that in the case of the consideration of the perturbative series for $(g-2)_\mu$ the numerical values of $\Omega_4(d_i^{PMS}, c_i^{PMS})$ are small and thus the *a posteriori* approximate equivalence of the ECH and PMS approaches is preserved for the quantities under consideration at this level also [23]. We think that this feature is also true in QCD.

III. Consider now the familiar characteristic of the $e^+ e^- \to \gamma \to hadrons$ process, namely the D-function defined in the Euclidean region:

$$D(Q^2) = Q^2 \int_0^\infty \frac{R(s)}{(s + Q^2)^2} \, ds \qquad (18)$$

Its perturbative expansion has the following form:

$$D(Q^2) = 3\Sigma Q_f^2 [1 + a + \sum_{i \geq 1} d_i a^{i+1}] + (\Sigma Q_f)^2 [\tilde{d}_2 a^3 + O(a^4)] \qquad (19)$$

where Q_f are the quark charges, and the structure proportional to $(\Sigma Q_f)^2$ comes from the light-by-light-type diagrams. The coefficients d_1 and $d_2, \tilde{d}_2$ were calculated in the $\overline{MS}$-scheme in Refs. [7] and [1, 2] respectively. They have the following numerical form:

$$
\begin{aligned}
d_1^{\overline{MS}} &\approx 1.986 - 0.115f \\
d_2^{\overline{MS}} &\approx 18.244 - 4.216f + 0.086f^2, \quad \tilde{d}_2 \approx -1.240 \ .
\end{aligned} \qquad (20)
$$

Following the proposals of Ref. [31], we will treat the light-by-light-type term in Eq. (19) separately from the "main" structure of the D-function, which is proportional to the quark-parton expression $D^{QP}(Q^2) = 3\Sigma Q_f^2$. In fact, one can hardly expect that it is possible to "predict" higher-order coefficients $\tilde{d}_i, i \geq 3$ of the second structure in Eq. (19) using the only explicitly-known term $\tilde{d}_2$. Therefore we will neglect the light-by-light-type structure as a whole in all our further considerations. This approximation is supported by the relatively tiny contribution of the second structure of Eq. (19) to the final $NNLO$ correction to the D-function.

The next important ingredient of our analysis is the QCD β-function (2), which is known in the MS-like schemes at the $NNLO$ level [32]. Its corresponding coefficients read

$$
\begin{aligned}
\beta_0 &= (11 - \frac{2}{3}f)\frac{1}{4} \approx 2.75 - 0.167f \\
c_1 &= \frac{153 - 19f}{66 - 4f} \\
c_2^{\overline{MS}} &= \frac{77139 - 15099f + 325f^2}{9504 - 576f} \ .
\end{aligned} \qquad (21)
$$

Using now the perturbative expression for the D-function, one can obtain the perturbtive expression for $R(s)$, namely

$$R(s) = 3\Sigma Q_f^2 [1 + a_s + \sum_{i \geq 1} r_i a_s^{i+1}] + (\Sigma Q_f)^2 [\tilde{r}_2 a_s^3 + ...] \tag{22}$$

where $a_s = \bar{\alpha}_s/\pi$, and

$$
\begin{aligned}
r_1 &= d_1 \\
r_2 &= d_2 - \frac{\pi^2 \beta_0^2}{3}, \quad \tilde{r}_2 = \tilde{d}_2 \\
r_3 &= d_3 - \pi^2 \beta_0^2 (d_1 + \frac{5}{6} c_1) \ .
\end{aligned} \tag{23}
$$

The corresponding π^2-terms come from the analytic continuation of the Euclidean result for the D-function to the physical region. The effects of the higher-order π^2-terms were discussed in detail in Ref. [25]. For example, the corresponding espression for the r_4-term reads :

$$r_4 = d_4 - \pi^2 \beta_0^2 (2d_2 + \frac{7}{3} c_1 d_1 + \frac{1}{2} c_1^2 + c_2) + \frac{\pi^4}{5} \beta_0^4 \tag{24}$$

The perturbative expression for R_τ is defined as

$$R_\tau = 2 \int_0^{M_\tau^2} \frac{ds}{M_\tau^2} (1 - s/M_\tau^2)^2 (1 + 2s/M_\tau^2) \tilde{R}(s) \simeq 3[1 + a_\tau + \sum_{i \geq 1} r_i^\tau a_\tau^{i+1}] \tag{25}$$

where $a_\tau = \alpha_s(M_\tau^2)/\pi$ and $\tilde{R}(s)$ is $R(s)$ with with $f = 3, (\Sigma Q_f)^2 = 0, 3\Sigma Q_f^2$ substituted for $3\Sigma \mid V_{ff'} \mid^2$ and $\mid V_{ud} \mid^2 + \mid V_{us} \mid^2 \approx 1$.

It was shown in Ref. [12] that it is convenient to express the coefficients of the series (25) through those ones of the series (19) for the D-function in the following form :

$$
\begin{aligned}
r_1^\tau &= d_1^{\overline{MS}}(f = 3) + g_1(f = 3) \\
r_2^\tau &= d_2^{\overline{MS}}(f = 3) + g_2(f = 3) \\
r_3^\tau &= d_3^{\overline{MS}}(f = 3) + g_3(f = 3)
\end{aligned} \tag{26}
$$

where in our notations

$$
\begin{aligned}
g_1 &= -\beta_0 I_1 = (f = 3) = 3.563 \\
g_2 &= -[2d_1 + c_1]\beta_0 I_1 + \beta_0^2 I_2 = (f = 3) = 19.99 \\
g_3 &= -[3d_2 + 2d_1 c_1 + c_2]\beta_0 I_1 + [3d_1 + \frac{5}{2} c_1]\beta_0^2 I_2 - \beta_0^3 I_3 = (f = 3) = 78.00
\end{aligned} \tag{27}
$$

and I_k are defined and calculated in Ref. [12]. Their analytical expressions read: $I_1 = -19/12$, $I_2 = 265/72 - \pi^2/3$ and $I_3 = -3355/288 + 19\pi^2/12$. One of the pleasant features of Eqs.(27) is that they are absorbing all effects of the analytical continuation.

Following the lines of Ref.[12] we derive the corresponding expression for the coefficient r_4^τ:

$$r_4^\tau = d_4^{\overline{MS}}(f=3) + g_4(f=3) \tag{28}$$

where

$$
\begin{aligned}
g_4 &= -[4d_3 + 3d_2c_1 + 2d_1c_2 + c_3]\beta_0 I_1 + [6d_2 + 7c_1d_1 + \frac{3}{2}c_1^2 + 3c_2]\beta_0^2 I_2 \\
&\quad -[4d_1 + \frac{13}{3}c_1]\beta_0^3 I_3 + \beta_0^4 I_4 = (f=3) \\
&= 3.562c_3(f=3) + 14.247d_3(f=3) - 466.73
\end{aligned}
\tag{29}
$$

and $I_4 = 41041/864 - 265\pi^2/36 + \pi^4/5 \approx -5.668$.

In order to estimate the values of the order $O(a^3)$, $O(a^4)$ and $O(a^5)$ corrections to $R(s)$ and R_τ, we will apply Eqs. (6) - (11) in the Euclidean region to the perturbative series for the D-function and then obtain the estimates we are interested in using Eqs. (23), (24), (26)-(29). This is the new ingredient of this analysis, which distinguishes it from the related ones of Refs. [16], [26].

We now recall the perturbative expression for the non-polarized Bjorken deep-inelastic scattering sum rule

$$BjnSR = \int_0^1 F_1^{\bar{v}p - vp}(x, Q^2)dx = 1 - \frac{2}{3}a(1 + \sum_{i \geq 1} d_i a^i) \tag{30}$$

where the coefficients d_1 and d_2 are known in the $\overline{MS}$ scheme from the results of calculations of Ref. [9] and Ref. [5] respectively:

$$d_1^{\overline{MS}} \approx 5.75 - 0.444f \tag{31}$$

$$d_2^{\overline{MS}} \approx 54.232 - 9.497f + 0.239f^2 \ . \tag{32}$$

The expression for the polarized Bjorken sum rule BjpSR has the following form:

$$BjpSR = \int_0^1 g_1^{ep-en}(x, Q^2)dx = \frac{1}{3} \mid \frac{g_A}{g_V} \mid [1 - a(1 + \sum_{i \geq 1} d_i a^i)] \tag{33}$$

where the coefficients d_1 and d_2 were explicitly calculated in the $\overline{MS}$ scheme in Refs. [10] and [6] respectively. The results of these calculations read

$$d_1^{\overline{MS}} \approx 4.583 - 0.333f \tag{34}$$

$$d_2^{\overline{MS}} \approx 41.440 - 7.607f + 0.177f^2 \ . \tag{35}$$

It should be stressed that since deep inelastic scattering sum rules are defined in the Euclidean region, we can directly apply to them the methods discussed in Section 2 without any additional modifications.

It is also worth emphasizing that, in spite of the identical coincidence of the NLO correction to the Gross-Llewellyn Smith sum rule $GLSSR = (1/2) \int_0^1 F_3^{\bar{\nu}p+\nu p}(x, Q^2) dx$ with the result of Eq. (34) [10], the corresponding $NNLO$ correction differs from the result of Eq. (35) by the contributions of the light-by-light-type terms typical of the GLSSR [6]. Since these light-by-light-type terms appear for the first time at the $NNLO$, it is impossible to predict the value of the light-by-light-type contribution at the N^3LO level using the corresponding $NNLO$ terms as the input information. However, noticing that at the $NNLO$ level the corresponding light-by-light-type contributions are small [6], we will assume that the similar contributions are small at the N^3LO level also. Only after this assumption can our estimates of the $NNLO$ and N^3LO corrections to the BjpSR be considered also as the estimates of the corresponding corrections in the perturbative series for the GLSSR . Note, that the Padé predictions of the order $O(a^4)$ contributions to the GLSSR [33], which do not take into account the necessity of the careful considerations of the light-by-light-type terms, deserve more detailed considerations. To our point of view, it is better to neglect the small light-by-light contribution as the whole and to consider the problem of estimates of the higher order corrections to BjpSR and GLSSR simultaniously, but not seperately, as was done in Ref. [33].

IV. The estimates of the coefficients of the order $O(a^3)$ and $O(a^4)$ QCD corrections to the D-function, $R(s)$, BjnSR and BjpSR/GLSSR, obtained following the discussions of Section 2 with the help of the results summarized in Section 3, are presented in Tables 1 - 4 respectively. Due to the complicated f-dependence of the coefficients Ω_2, Ω_3 in Eqs. (6) and (7), we are unable to predict the explicit f-dependence of the corresponding coefficients in the form respected by perturbation theory. The results are presented for the fixed number of quark flavours $1 \leq f \leq 6$. The estimates of the $NNLO$ corrections, obtained starting from both the ECH and PMS approaches, are in qualitative agreement with the results of the explicit calculations. The best agreement is achieved for $f = 3$ numbers of flavours.

Using the results of Table 1 and Eqs. (25),(26) we get the following estimates of the $NNLO$ coefficients R_τ: $(r_2^\tau)_{ECH}^{est} \approx 25.6$ and $(r_2^\tau)_{PMS}^{est} \approx 24.8$. One can see the agreement with the explicitly calculated result $(r_2^\tau)_{\overline{MS}} = 26.366$. Considering this agreement as the additional a $posteriori$ support of the methods used, we use the estimate of the N^3LO coefficient for the D-function with $f = 3$ numbers of flavours as presented in Table 1, namely:

$$d_3^{est}(f = 3) = 27.5 \tag{36}$$

and estimate the value of the N^3LO coefficient of R_τ: $(r_3^\tau)^{est} \approx 105.5$.

As follows from Eq. (12) the results given in Tables 1-4 correspond to the ECH-inspired variant of the $\overline{MS}$ scheme. However, we hope that there is some meaning in the relation $c_N = c_N^{ECH}$ used to obtain these numbers. Indeed, it is known from the explicite QED calculations of Refs. [34] that the difference between the numerical values of the higher order coefficients of the QED β-function in different schemes is decreasing at the four-loop level. So, we do not exclude the situation that the difference between scheme-invariants c_N^{ECH} of different QCD quantities, which probably have the factorial growth $c_N^{ECH} \sim AN!$, will also decrease in the higher orders of perturbation theory, and that it might happen that $c_N \approx c_N^{ECH}$. If so, the estimates of Tables 1-4 reveal the structure of the perturbative series for physical quantities in the fixed scheme.

In order to address the problem of fixing the values of the $O(a^5)$ QCD corrections to the considered Euclidean quantities we apply Eqs. (8), (11) with the explicitely calculated NLO and $NNLO$ coefficients d_1, d_2, c_1, c_2 in the $\overline{MS}$ scheme and use the determined from Eqs. (7), (10) estimates of the N^3LO coefficients d_3. In order to obtain the estimates of the next-to-next-to-next- to-leading order (N^4LO) coefficients of $R(s)$ and R_τ, which are related to the N^4LO coefficient d_4 of the D-function, the explicitly calculated terms in Eq.(24) and Eq. (29) are taken into account.

However, the expressions for Ω_4 in Eq. (8) and g_4 in Eq. (29) depend also on the four-loop coefficient c_3 of the QCD β-function, which is unknown at present. Therefore, in Tables 1-4 we present the estimates of the combinations $d_4 - d_1c_3$ and $r_4 - r_1c_3$. The existing uncertainty in the value of N^3LO coefficients d_3 is fixed by the assumption that the real values of these coefficients do not significantly differ from the N^3LO estimates, obtained by us. However, in the case of the D-function with $f = 3$ numbers of flavours we present also the more detailed expression of d_4, which follows from Eqs.(11),(8):

$$(d_4)^{est}_{ECH}(f = 3) = 1.64c_3(f = 3) + 5.97d_3(f = 3) - 52.8. \tag{37}$$

Taking into account the negative contributions into the coefficient g_4 in Eq. (28) we get the following estimate of the N^4LO coefficient of R_τ :

$$(r^\tau_4)^{est}_{ECH} = 5.2c_3(f = 3) + 20.22d_3(f = 3) - 519.5 . \tag{38}$$

Of course, the best way of fixing the value of c_3 would be its explicite calculation. However, at the current level of art one is free to invent his own way of fixing this part of the ambiguities of the less substantiated than N^3LO estimates of the N^4LO terms [5]. We will use here the bold guess estimate $c_3(f = 3) \approx c_2(f = 3)^2/c_1(f = 3) \approx 11$, which is motivated by the good agreement of our results of Eq. (36) with the approximation $d_3 \approx d_2^2/d_1 \approx 25$, previously used in Ref. [12] to fix the value of the N^3LO coefficient of the D-function for $f = 3$. Combining it with the estimate of Eq. (36) we get the following estimate of the N^4LO coefficients of R_τ: $(r^\tau_4)^{est} \approx 93$.

V.   We are now ready to discuss the main otcomes of our analysis.

1.   The estimates of the $NNLO$ corrections to the D-function, BjnSR and BjpSR/GLSSR are in qualitative agreement with the results of the explicit calculations of Refs. [1, 2], [5] and [6] and respect the tendency of the corresponding coefficients to decrease with increasing number of flavours.

2.   The best agreement of the $NNLO$ estimates with the exact results is obtained for the case $f = 3$. This fact supports the application of the method used for estimating the $NNLO$ and N^3LO corrections to R_τ.

3.   Notice that since the methods used correctly reproduce the renormalization-group-controllable terms [28], [16], the transformation of the coefficients d_1, d_2 from the $\overline{MS}$ scheme

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[5]Another kind of the uncertainties is related to the possible deviations of the real values of the N^3LO corrections from the presented by us estimates.

to other variants of the MS-like scheme will not spoil the qualitative agreement with the results of the explicit calculations.

4.   The comparison of the results of Table 1 with those of Table 2 demonstrate that the π^2 effects give dominating contributions to the coefficients of $R(s)$.

5.   Amongst other outcomes are the estimates of the N^3LO and N^4LO corrections to R_τ and $R(s)$ for $f = 5$ numbers of flavours. Taking $\alpha_s(M_Z) \approx 0.12$, we get the estimate of the corresponding N^3LO contribution to both $\Gamma(Z^0 \to hadrons)$ and $\Gamma(Z^0 \to \bar{q}q)$:

$$(\delta\Gamma_{Z^0})_{N^3LO} \approx -97(a(M_Z))^4 \approx -2 \times 10^{-4} \ . \tag{39}$$

It is of the order of magnitude of other corrections included in the current analysis of LEP data (see, e.g., [35]). Using the estimates presented in Table 2 and assuming that $c_3(f = 5) = c_2(f = 5)^2/c_1(f = 5) \approx 1.715$ one can also estimate the corresponding N^4LO contributions, namely

$$(\delta\Gamma_{Z^0})_{N^4LO} \approx 70(a(M_Z))^5 \approx 5 \times 10^{-6} \tag{40}$$

The way of fixation of the value of c_3 used above is not applicable for the case of $f = 6$ since we expect that in this case its real value is negative.

6.   Taking $\alpha_s(M_\tau) \approx 0.36$ [3], we get the numerical estimate of the N^3LO contribution to R_τ:

$$(\delta R_\tau)_{N^3LO} \approx 105.5 \, a_\tau^4 \approx 1.8 \times 10^{-2} \ . \tag{41}$$

It is larger than the recently calculated power-suppressed perturbative [36] and non-perturbative [37] contributions to R_τ. The possible contributions of the N^4LO corrections are significantly smaller:

$$(\delta R_\tau)_{N^4LO} \approx 93 a_\tau^5 \approx 1.8 \times 10^{-3} \ . \tag{42}$$

7.   Notice, that in spite of the possible $N!$ growth of the coefficients of the D-function, the additianal π^2-dependent contributions are shadowing down this possible growth in the Minkowskian region. In order to consider this effect in higher order levels we applied the RG-technique and derived the corresponding relations between the order $O(a^6)$ coefficients of the D-function and $R(s)$ and R_τ. In the case of $R(s)$ the relation reads

$$
\begin{aligned}
r_5 \ = \ & d_5 - \frac{\pi^2}{3}\beta_0^2(10d_3 + \frac{27}{2}c_1d_2 + 4c_1^2d_1 + \frac{7}{2}c_1c_2 + 8c_2d_1 + \frac{7}{2}c_3) \\
& + \frac{\pi^4}{5}\beta_0^4(5d_1 + \frac{77}{12}c_1).
\end{aligned}
\tag{43}
$$

In the case of R_τ the correction term to the relation $r_5^\tau = d_5(f = 3) + g_5(f = 3)$ is more complicated

$$
\begin{aligned}
g_5 \ = \ & -[5d_4 + 4c_1d_3 + 3c_2d_2 + 2c_3d_1 + c_4]\beta_0 I_1 \\
& + [10d_3 + \frac{27}{2}c_1d_2 + 4c_1^2d_1 + 8c_2d_1 + \frac{7}{2}c_1c_2 + \frac{7}{2}c_3]\beta_0^2 I_2 \\
& - [10d_2 + \frac{47}{3}c_1d_1 + \frac{35}{6}c_1^2 + 6c_2]\beta_0^3 I_3 + [5d_1 + \frac{77}{12}c_1]\beta_0^4 I_4 - \beta_0^5 I_5
\end{aligned}
\tag{44}
$$

10

where $I_5 = -2479295/10368 + 16775\pi^2/432 - 19\pi^4/12 \approx -10.113$. Keeping in the expression for $g_5(f = 3)$ all explicitly unknown coefficients we get

$$g_5(f = 3) = 17.8d_4(f = 3) + 45.07d_3(f = 3) + 3.56c_4(f = 3) + 18.6c_3(f = 3) - 8455. \quad (45)$$

Notice the appearence in the expression for $g_5(f = 3)$ of the huge negative coefficient. The derived expressions can mean that the asymptotic structure of the perturbative expansion in the Minkowskian region differs essentially from the one in the Euclidean region.

8.   The qualitative agreement of the results of Tables 3 and 4 for BjnSR and BjpSR with the corresponding Padé estimates of Ref. [33] can be considered as the argument in favour of the applicability of both theoretical methods in the Euclidean region for the concrete physical applications. Let us stress again that in the process of these applications the light-by-light-type structures, contributing to the GLSSR, should be treated seperately.

9.   Our estimate for $d_3(f = 3)$ of Eq. (36) is more definite than the one presented in Ref. [38], namely $d_3(f = 3) = 55^{+60}_{-24}$, and than the bold guess estimate $d_3(f = 3) = \pm25$, given in Ref. [12]. The result is in good agreement with the "geometric progression" assumption of Ref. [12]. The related estimate of the N^3LO contribution to R_τ (see Eq. (41)) is more precise than those presented in Refs. [3] and [12], namely $\delta R_\tau = \pm130\,a_\tau^4$ [3] and $\delta R_\tau = (78 \pm 25)a_\tau^4$ [12], and is smaller than the result of applying the Padé resummation technique directly to R_τ, namely $\delta R_\tau = 133a_\tau^4$ [38], [39].

10.   The application of the Padé resummation technique to $R(s)$ [38], stimulated by the previous similar studies of Ref. [40], gives less definite estimates than the results of applications of our methods (compare the estimate $\delta R(s) = \left(-49^{+54}_{-40}\right)a^4$ [38] with the result $\delta R(s) = -97a^4$ from Table 2 and Eq. (39)). Probably, this fact is related with the problems of applicability of the Padé resummation technique to the sign variation perturbative series for $R(s)$ in the $\overline{MS}$-scheme.

11.   Let us emphasize again, that the used by us method was applyed directly to the Euclidean quantities, namely to the D-function and deep-inelastic scattering sum rules. The results for $R(s)$ and R_τ were obtained from the ones of the D-function after taking into account explicitly calculable terms of the analytic continuation to the Minkowskian region. In principle, one can try to study the application of the procedure used by us directly to $R(s)$ and R_τ in the Minkowskian region. We checked that in the case of applications of Eqs. (6)-(8) with the substitution of $d_i = r_i$ or $d_i = r_i^\tau$ one can reproduce the same values of the Euclidean coefficients d_i but the structure of the high order π^2-terms will not agree with the explicitly known results. A similar problem will also definitely arise in the case of more rigorous studies of the applicability of the Padé resummation methods for the Minkowskian quantities $R(s)$ and R_τ (for the studies existing at present see Refs. [38, 39]). We think that these analyses should be supplemented by the construction of the Padé approximants directly to the D-function. In our case the solution of this problem lies in the necessity of modifications of Eqs. (6)-(8) in the Minkowskian region by adding concrete calculable π^2-dependent and scheme-independent factors. After these modifications it is possible to reproduce our results for $R(s)$ and R_τ by means of application of a similar technique directly in the Minkowskian region.

12. Our estimate for $d_3(f = 3)$ was recently supported by the phenomenological analysis of the ALEPH data for R_τ [41]. The above presented considerations of the diffence between the structure of the perturbative series for the D-funcion and $R(s)$ already stimulated the reconsiderations of the applications of the Padé approximants for the estimates of the higher-order coefficients of $R(s)$ and R_τ [42]. Note, however, that we are realizing that at the present level of understanding both methods suffer from the lack of more rigorous dynamical information about the analytical structure of the expanded in the perturbative series functions both in the Euclidean and Minkowskian regions. We hope that the considerations presented above will stimulate future more detailed studies of the related problems.

13. The interesting fact is that the presented $NNLO$ estimates are in better agreement with the explicit results for $f = 3$ numbers of flavours than say for $f = 5$ numbers of flavours. It should be stressed again that the basic assumption of our approach is the relation $c_N^{ECH} = c_N$. However, it is known that on the contary to the coefficients c_N the scheme-invariants c_N^{ECH} contain additional factors of order f^{N+1} [43] , though with small coefficients. For larger values of f these terms become more and more important. This might lead to the additional sourse of the violation of the basic relation $c_N^{ECH} = c_N$ of the re-expansion procedure used by us.

Acknowledgements

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f	d_2^{ex}	$(d_2^{est})_{ECH}$	$(d_2^{est})_{PMS}$	d_3^{est}	$(d_4^{est})_{ECH} - c_3 d_1$
1	14.11	7.54	7.70	75	476
2	10.16	6.57	7.55	50	260
3	6.37	5.61	6.40	27.5	111
4	2.76	4.68	5.27	8.4	22.7
5	-0.69	3.77	4.16	-7.7	-13.2
6	-3.96	2.88	3.08	-21	-2.76

Table 1 : The results of estimates of the $NNLO$, N^3LO and N^4LO corrections in the series for the D-functions.

f	r_2^{ex}	$(r_2^{est})_{ECH}$	$(r_2^{est})_{PMS}$	r_3^{est}	$(r_4^{est})_{ECH} - c_3 d_1$
1	-7.84	-14.41	-14.25	-166	-1750
2	-9.04	-12.65	-11.67	-147	-1161
3	-10.27	-11.04	-10.25	-128	-668
4	-11.52	-9.59	-9	-112	-263
5	-12.76	-8.32	-7.93	-97	67
6	-14.01	-7.19	-6.99	-83	330

Table 2: The results of estimates of the $NNLO$, N^3LO and N^4LO corrections in the series for $R(s)$.

f	d_2^{ex}	$(d_2^{est})_{ECH}$	$(d_2^{est})_{PMS}$	d_3^{est}	$(d_4^{est})_{ECH} - c_3 d_1$
1	44.97	39.62	40.78	424	4127
2	36.19	33.28	34.26	303	2613
3	27.89	27.37	28.16	200	1474
4	20.07	21.91	22.50	114	664
5	12.72	16.91	17.30	44	138
6	5.85	12.39	12.59	-10	-145

Table 3: The results of estimates of the $NNLO$, N^3LO and N^4LO corrections in the series for BjnSR.

f	d_2^{ex}	$(d_2^{est})_{ECH}$	$(d_2^{est})_{PMS}$	d_3^{est}	$(d_4^{est})_{ECH} - c_3 d_1$
1	34.01	27.25	28.41	290	2561
2	26.93	23.11	24.09	203	1580
3	20.21	19.22	20.01	130	852
4	13.84	15.57	16.16	68	343
5	7.83	12.19	12.59	18	25
6	2.17	9.08	9.29	-22	-130

Table 4: The results of estimates of the $NNLO$, N^3LO and N^4LO corrections in the series for BjpSR and GLSSR.