CERN-TH. 7505/94
UG-FT-49/94

$(I,\, q)$-graded Superspace Formalism for a $\mathbb{Z}_2 \times (\mathbb{Z}_4 \times \mathbb{Z}_4)$-graded Extension of the Poincaré Algebra

Luis Alberto **Wills Toro** *†

Theoretical Physics Division, CERN
CH-1211 Geneva 23, SWITZERLAND

and

Departamento de Física Teórica y del Cosmos,
Universidad de Granada,
E-18071 Granada, SPAIN

Abstract

Consistent generalized commutation relations between the (with each-other commuting) space-time parameters and some further para-Grassmann parameters have been recently obtained. We develop the superspace formalism involving para-Grassmann parameters associated with an $(I,\, q)$-graded Lie algebraic extension over $\mathbb{C}$ of the Poincaré algebra. We construct representations of group elements and obtain representations of the algebra generators as derivative operators with respect to the $(I,\, q)$-graded superspace parameters. We determine the covariant derivatives, the symmetry generators and the transformation properties of superfields in several equivalent representations. We initiate the study of the field content of superfield representations, covariant constraints and invariant models. The equation of motion for a spin-$\frac{1}{2}$ para-fermionic field is obtained. Severe limitations on the construction of physically meaningful models are related to the usage of non-fundamental spin-$\frac{1}{2}$ representations for the symmetry charges.

PACS numbers: 11.30.Cp, 11.30.Pb, 02.20.Sv.
MCS numbers: 17B70, 17B90.

CERN-TH. 7505/94
UG-FT-49/94
January 1995.

Work supported by the European Community under the contract No. CI1-CT-93-0248

†Present address: Depto. de F´ısica Teórica y del Cosmos, Facultad de Ciencias, Universidad de Granada, E-18071 Granada, SPAIN. E-Mail: LAW@UGR.ES

1 Introduction

The each-other commuting space-time parameters can have unexpected generalized commutation relations with further para-Grassmann parameters. We can consider for instance space-time parameters x^μ and parameters $\Theta^{js}, \bar{\Theta}^{\bar{j}\dot{s}}$ fulfilling

$$\begin{array}{ll} [x^\mu, x^\nu] = 0 \quad , & \mu, \nu \in \{0,1,2,3\}, \\ [x^0, \Theta^{js}] = 0 \quad , & [x^0, \bar{\Theta}^{\bar{j}\dot{s}}] = 0\,;\, j \in \{1,2,3\}\,;\, s \in \{0,1,2,3\}, \end{array} \tag{1. 1}$$

$$\begin{array}{ll} [x^j, \Theta^{js}] = 0 \quad , & [x^j, \bar{\Theta}^{\bar{j}\dot{s}}] = 0\,;\, j \in \{1,2,3\}\,;\, s \in \{0,1,2,3\} \\ \{x^i, \Theta^{js}\} = 0 \quad , & \{x^i, \bar{\Theta}^{\bar{j}\dot{s}}\} = 0\,;\, i,j \in \{1,2,3\}\,;\, i \neq j\,;\, s \in \{0,1,2,3\}. \end{array} \tag{1. 2}$$

The parameters $\Theta^{js}, \bar{\Theta}^{\bar{j}\dot{s}}\,;\, j \in \{1,2,3\}\,;\, s \in \{0,1,2,3\}$ can fulfil between them generalized commutation relations beyond commutation and anticommutation. Several consistent realizations of this idea have been constructed in [1]. On this observation lies a recent attempt to construct graded extensions of the Poincaré algebra beyond supersymmetry [2].

This possible nontrivial behaviour of the space-time parameters was not envisaged when studying the most general form of the quantization relations between quantum fields [3], and the corresponding connection between spin and statistic [4]. The so-called Klein transformations and further equivalences among graded Lie algebras [5] should be further developed in order to determine if the recently obtained graded extensions are actually inequivalent to supersymmetric extensions [6]. This might constitute the basis for a new generalization [7] of the no-go theorems of S. Coleman & J. Mandula [8] and of R. Haag, J. T. Łopuszański & M. F. Sohnius [9] about the symmetries of the S-matrix.

We study here the construction of concrete models that are invariant under the $\mathbb{Z}_2 \times (\mathbb{Z}_4 \times \mathbb{Z}_4)$-graded extension of the Poincaré algebra introduced in [2]. Our results are of relevance in order to determine which are the adequate algebraic structures allowing for a nontrivial behaviour of the space-time parameters, and for meaningful physical models.

For a review of the definition of $(I,\, q)$-graded Lie algebras over an arbitrary commutative (numeric) field $\mathbb{K}$, see Appendix A. In Appendix B we describe the single-grading model used in [1] to determine concrete grading Abelian groups for $\mathbb{K} = \mathbb{C}$. In section **2** we review the assumptions used to obtain particular graded extensions, the form of the corresponding multiplets of generators, and the algebraic relations of a particular $\mathbb{Z}_2 \times (\mathbb{Z}_4 \times \mathbb{Z}_4)$-graded extension of the Poincaré algebra. In section **3** we study the $(I,\, q)$-graded superfields and superspace formalism. In section **4** we briefly discuss the field content of superfield representations. In section **5** we briefly discuss the covariant constraints. In section **6** we study spin-$\frac{1}{2}$ para-fermionic fields. In section **7** the conclusions and open questions of the approach are presented. Appendices C, D, E provide, respectively, the used momentum representations, the operations with the superspace parameters, and some useful identities for spin-$\frac{1}{2}$ and superspace calculations which are needed to reproduce the presented results.

2 Particular $(I,\, q)$-graded Lie algebraic extensions of the Poincaré Lie algebra

In a recent publication [1] some constraints on the grading groups suitable for the construction of $(I,\, q)$-graded algebraic extensions of the Poincaré Lie algebra have been determined. In another publication [2], concrete $(I,\, q)$-graded Lie algebraic extensions of the Poincaré Lie algebra have been obtained for which the following chain of assumptions have been adopted (some assumptions imply the previous ones):

$\mathcal{A}_{-3}$ All the gradings are done with respect to an Abelian group $I\!I$.

$\mathcal{A}_{-2}$ The structure constants associated with the algebraic extension are parameters with trivial index $\tilde{o}$ (i.e. they are universally commutative, they are numbers), where $\tilde{o}$ is the additive neutral element of $I\!I$.

$\mathcal{A}_{-1}$ The algebraic extensions are formulated in terms of $(I,\, q)$-graded Lie algebras over a commutative field $I\!K$ (see appendix A for an introduction), which contain as a subalgebra the Poincaré Lie algebra and for which $I \subset I\!I$.

$\mathcal{A}$ The assignment of indices is made in such a way that the Hamel basis $\{M_{\mu\nu} : \mu, \nu \in \{0,1,2,3\}$ and $\mu < \nu\}$ of $so(3,1)$ can be locally adopted for the proper Lorentz Lie subalgebra.

The restricted Poincaré Lie subalgebra $I\!P$ can thus be expressed in terms of the algebraic relations

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= \mathrm{i}(g_{\mu\sigma}M_{\nu\rho} + g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho}),\\
[M_{\mu\nu}, P_\rho] &= \mathrm{i}(g_{\rho\nu}P_\mu - g_{\mu\rho}P_\nu),\\
[P_\mu, P_\rho] &= 0,\\
I\!P &= \{P, M\} = \{P\} \supset\!\!\!\!+ \{M\},
\end{aligned} \tag{2. 1}$$

where $\mu, \nu, \rho, \sigma \in \{0,1,2,3\}$, $\{P\} := \mathrm{Gen}\{P_\mu : \mu \in \{0,1,2,3\}\}$ denotes the translation algebra, $\{M\} := \mathrm{Gen}\{M_{\mu\nu} : \mu, \nu \in \{0,1,2,3\}$ and $\mu < \nu\}$ denotes the proper orthochronous Lorentz algebra $\subset so(3,1)$, and the metric tensor is given by $g := \mathrm{diag}(1,-1,-1,-1)$.

Equivalently, we can write the above algebraic relations in the following form:

$$\begin{aligned}
[J_i, J_j] &= \mathrm{i}\delta^{k\ell}\epsilon_{ijk}J_\ell, & [\hat{J}_i, J_j] &= \mathrm{i}\delta^{k\ell}\epsilon_{ijk}\hat{J}_\ell,\\
& & [\hat{J}_i, \hat{J}_j] &= -\mathrm{i}\delta^{k\ell}\epsilon_{ijk}J_\ell,\\
[J_i, P_j] &= \mathrm{i}\delta^{k\ell}\epsilon_{ijk}P_\ell, & [\hat{J}_i, P_j] &= -\mathrm{i}\delta_{ij}P_0,\\
[J_i, P_0] &= 0, & [\hat{J}_i, P_0] &= -\mathrm{i}P_i,\\
& & [P_i, P_j] &= [P_i, P_0] = [P_0, P_0] = 0,
\end{aligned} \tag{2. 2}$$

where $i, j, k \in \{1,2,3\}$, and ϵ_{ijk} is the totally antisymmetric Levi-Civita tensor with $\epsilon_{123} = 1$, and

$$J_i := \tfrac{1}{2}\epsilon_{ijk}M^{jk}, \quad \hat{J}_i := M^{0i}. \tag{2. 3}$$

$\mathcal{B}$ The assignment of indices is done in such a way that the following Hamel basis $\{T_i : i \in \{1,2,3\}\} \cup \{\bar{T}_i : i \in \{1,2,3\}\}$ can be locally adopted for the restricted Lorentz Lie subalgebra $\{M\}$:

$$\{M\} \cong \mathrm{Gen}\,(\{T_i : i \in \{1,2,3\}\} \cup \{\bar{T}_i : i \in \{1,2,3\}\}), \tag{2. 4}$$

where

$$T_i := \tfrac{1}{2}(J_i + \mathrm{i}\hat{J}_i), \quad \bar{T}_i := \tfrac{1}{2}(J_i - \mathrm{i}\hat{J}_i), \tag{2. 5}$$

Hence, the restricted Lorentz Lie algebra is isomorphic to a direct sum of two $su(2)$ algebras:

$$[T_i, T_j] = \mathrm{i}\delta^{k\ell}\epsilon_{ijk}T_\ell,$$

$$\begin{aligned} [T_i, \bar{T}_j] &= 0, \\ [\bar{T}_i, \bar{T}_j] &= \mathrm{i}\delta^{k\ell}\epsilon_{ijk}\bar{T}_\ell, \\ \{M\} &\cong su(2)\oplus su(2). \end{aligned} \tag{2. 6}$$

$\mathcal{C}$ The indices assigned to the generators of the Lorentz Lie algebra are not all $\tilde{\boldsymbol{o}}$.

$\mathcal{D}$ The index set $I \subset \mathbb{I}$ and the function q of the (I, q)-graded Lie algebraic extension are obtained from a single-grading model with $\mathbb{K} = \mathbb{C}$ (see Appendix B for an introduction).

$\mathcal{E}$ None of the indices assigned to the generators M_{12}, M_{23}, M_{13} of the rotation Lie subalgebra are trivial.

$\mathcal{F}$ The index assigned to the generator P_0 is trivial. We can associate just numbers to its eigenvalues. Furthermore, the Casimir operators of the Poincaré Lie algebra for massive representations turn out to have also trivial indices.

$\mathcal{G}$ There exist involution operations $\overline{(\cdot)}$, $(\cdot)^\star$, and $(\cdot)^*$ in $\mathbb{L}$, I, and $\mathbb{C}$ respectively, such that they act simultaneously producing

$$\overline{(\cdot)}: \quad \mathbb{L} \longrightarrow \mathbb{L}\,; \qquad \mathcal{O}_{\tilde{a}} \mapsto \overline{(\mathcal{O}_{\tilde{a}})} =: \bar{\mathcal{O}}_{\tilde{a}^\star}, \tag{2. 7}$$

$$(\cdot)^\star: \quad I \longrightarrow I\,; \qquad \tilde{a} = (a, \mathbf{a}) \mapsto \tilde{a}^\star = -\tilde{a} = (a, -\mathbf{a}), \tag{2. 8}$$

$$(\cdot)^*: \quad \mathbb{K} \longrightarrow \mathbb{K}\,; \qquad y \mapsto y^* \;\text{ complex conjugation,} \tag{2. 9}$$

and the operators of the Poincaré Lie algebra transform under involution in the following way

$$\overline{(T_i)} = \bar{T}_i, \,, \; \overline{(P_\mu)} = P_\mu. \tag{2. 10}$$

The involution corresponds to a Hermitic conjugation with respect to a Lorentz invariant bilinear form (completation is understood).

Notice that the definition of $(\cdot)^\star$ in (2. 8) does not contradict the choice in (2. 10) since $(0, \mathbf{a}^\mu) = -(0, \mathbf{a}^\mu)\,;\, \mu \in \{0,1,2,3\}$.

Notice also that

$$\varsigma_{\rm st}(\mathcal{O}_{\tilde{a}}) = (a, \mathbf{a}) \Longrightarrow \varsigma_{\rm st}(\mathcal{O}_{\tilde{a}}\bar{\mathcal{O}}_{\tilde{a}^\star}) = (a, \mathbf{a}) + (a, -\mathbf{a}) = (0, \mathbf{a}^0) = \tilde{\boldsymbol{o}}, \tag{2. 11}$$

and hence, we can associate real eigenvalues to the self-adjoint operators $\mathcal{O}_{\tilde{a}}\bar{\mathcal{O}}_{\tilde{a}^\star}$.

$\mathcal{H}$ The elements (generators) of the considered extensions $\mathbb{L}$ can be arranged into multiplets $Z^a_z, \bar{Z}^a_{\bar{z}}$ that transform linearly under the action of the generators of the Lorentz Lie subalgebra:

$$[\![M^{\mu\nu}, Z^a_{zs}]\!] = -\tfrac{1}{2}(\sigma^{a\,\mu\nu}_{z})_s{}^t Z^a_{zt}\,;\, \mu,\nu \in \{0,1,2,3\}, \tag{2. 12}$$

$$[\![M^{\mu\nu}, \bar{Z}^a_{\bar{z}\dot{s}}]\!] = +\tfrac{1}{2}(\bar{\sigma}^{a\,\mu\nu}_{\bar{z}})_{\dot{s}}{}^{\dot{t}} \bar{Z}^a_{\bar{z}\dot{t}}\,;\, \mu,\nu \in \{0,1,2,3\}, \tag{2. 13}$$

where $\boldsymbol{\sigma}^{a\,\mu\nu}_{z}$ and $\bar{\boldsymbol{\sigma}}^{a\,\mu\nu}_{\bar{z}}$ are constant complex square matrices.

$\mathcal{I}$ The grading Abelian group which turns out to have the general form $\mathbb{I} = \mathbb{Z}_2 \times (\mathbb{Z}_{4\Lambda} \times \mathbb{Z}_{4\Lambda}) \times \mathbb{G}_{re}$, with $\Lambda \in$ and $\mathbb{G}_{re}$ an Abelian group, is constrained to the particular case in which $\Lambda = 1$ and the $\mathbb{G}_{re}$-grading factor is trivial, i.e.

$$\mathbb{I} = \mathbb{Z}_2 \times (\mathbb{Z}_4 \times \mathbb{Z}_4). \tag{2. 14}$$

$+$	0	1
0	0	1
1	1	0

Table 1: *Addition table of the group* $(\mathbb{Z}_2\,;+)$.

$+$	$\mathbf{a}^0$	$\mathbf{a}^1$	$\mathbf{a}^2$	$\mathbf{a}^3$	$\mathbf{a}^{11+}$	$\mathbf{a}^{11-}$	$\mathbf{a}^{12+}$	$\mathbf{a}^{12-}$	$\mathbf{a}^{21+}$	$\mathbf{a}^{21-}$	$\mathbf{a}^{22+}$	$\mathbf{a}^{22-}$	$\mathbf{a}^{31+}$	$\mathbf{a}^{31-}$	$\mathbf{a}^{32+}$	$\mathbf{a}^{32-}$
$\mathbf{a}^0$	$\mathbf{a}^0$	$\mathbf{a}^1$	$\mathbf{a}^2$	$\mathbf{a}^3$	$\mathbf{a}^{11+}$	$\mathbf{a}^{11-}$	$\mathbf{a}^{12+}$	$\mathbf{a}^{12-}$	$\mathbf{a}^{21+}$	$\mathbf{a}^{21-}$	$\mathbf{a}^{22+}$	$\mathbf{a}^{22-}$	$\mathbf{a}^{31+}$	$\mathbf{a}^{31-}$	$\mathbf{a}^{32+}$	$\mathbf{a}^{32-}$
$\mathbf{a}^1$	$\mathbf{a}^1$	$\mathbf{a}^0$	$\mathbf{a}^3$	$\mathbf{a}^2$	$\mathbf{a}^{11-}$	$\mathbf{a}^{11+}$	$\mathbf{a}^{12-}$	$\mathbf{a}^{12+}$	$\mathbf{a}^{22-}$	$\mathbf{a}^{22+}$	$\mathbf{a}^{21-}$	$\mathbf{a}^{21+}$	$\mathbf{a}^{32+}$	$\mathbf{a}^{32-}$	$\mathbf{a}^{31+}$	$\mathbf{a}^{31-}$
$\mathbf{a}^2$	$\mathbf{a}^2$	$\mathbf{a}^3$	$\mathbf{a}^0$	$\mathbf{a}^1$	$\mathbf{a}^{12+}$	$\mathbf{a}^{12-}$	$\mathbf{a}^{11+}$	$\mathbf{a}^{11-}$	$\mathbf{a}^{21-}$	$\mathbf{a}^{21+}$	$\mathbf{a}^{22-}$	$\mathbf{a}^{22+}$	$\mathbf{a}^{32-}$	$\mathbf{a}^{32+}$	$\mathbf{a}^{31-}$	$\mathbf{a}^{31+}$
$\mathbf{a}^3$	$\mathbf{a}^3$	$\mathbf{a}^2$	$\mathbf{a}^1$	$\mathbf{a}^0$	$\mathbf{a}^{12-}$	$\mathbf{a}^{12+}$	$\mathbf{a}^{11-}$	$\mathbf{a}^{11+}$	$\mathbf{a}^{22+}$	$\mathbf{a}^{22-}$	$\mathbf{a}^{21+}$	$\mathbf{a}^{21-}$	$\mathbf{a}^{31-}$	$\mathbf{a}^{31+}$	$\mathbf{a}^{32-}$	$\mathbf{a}^{32+}$
$\mathbf{a}^{11+}$	$\mathbf{a}^{11+}$	$\mathbf{a}^{11-}$	$\mathbf{a}^{12+}$	$\mathbf{a}^{12-}$	$\mathbf{a}^1$	$\mathbf{a}^0$	$\mathbf{a}^3$	$\mathbf{a}^2$	$\mathbf{a}^{31-}$	$\mathbf{a}^{32+}$	$\mathbf{a}^{31+}$	$\mathbf{a}^{32-}$	$\mathbf{a}^{21-}$	$\mathbf{a}^{22-}$	$\mathbf{a}^{22+}$	$\mathbf{a}^{21+}$
$\mathbf{a}^{11-}$	$\mathbf{a}^{11-}$	$\mathbf{a}^{11+}$	$\mathbf{a}^{12-}$	$\mathbf{a}^{12+}$	$\mathbf{a}^0$	$\mathbf{a}^1$	$\mathbf{a}^2$	$\mathbf{a}^3$	$\mathbf{a}^{32-}$	$\mathbf{a}^{31+}$	$\mathbf{a}^{32+}$	$\mathbf{a}^{31-}$	$\mathbf{a}^{22+}$	$\mathbf{a}^{21+}$	$\mathbf{a}^{21-}$	$\mathbf{a}^{22-}$
$\mathbf{a}^{12+}$	$\mathbf{a}^{12+}$	$\mathbf{a}^{12-}$	$\mathbf{a}^{11+}$	$\mathbf{a}^{11-}$	$\mathbf{a}^3$	$\mathbf{a}^2$	$\mathbf{a}^1$	$\mathbf{a}^0$	$\mathbf{a}^{32+}$	$\mathbf{a}^{31-}$	$\mathbf{a}^{32-}$	$\mathbf{a}^{31+}$	$\mathbf{a}^{21+}$	$\mathbf{a}^{22+}$	$\mathbf{a}^{22-}$	$\mathbf{a}^{21-}$
$\mathbf{a}^{12-}$	$\mathbf{a}^{12-}$	$\mathbf{a}^{12+}$	$\mathbf{a}^{11-}$	$\mathbf{a}^{11+}$	$\mathbf{a}^2$	$\mathbf{a}^3$	$\mathbf{a}^0$	$\mathbf{a}^1$	$\mathbf{a}^{31+}$	$\mathbf{a}^{32-}$	$\mathbf{a}^{31-}$	$\mathbf{a}^{32+}$	$\mathbf{a}^{22-}$	$\mathbf{a}^{21-}$	$\mathbf{a}^{21+}$	$\mathbf{a}^{22+}$
$\mathbf{a}^{21+}$	$\mathbf{a}^{21+}$	$\mathbf{a}^{22-}$	$\mathbf{a}^{21-}$	$\mathbf{a}^{22+}$	$\mathbf{a}^{31-}$	$\mathbf{a}^{32-}$	$\mathbf{a}^{32+}$	$\mathbf{a}^{31+}$	$\mathbf{a}^2$	$\mathbf{a}^0$	$\mathbf{a}^1$	$\mathbf{a}^3$	$\mathbf{a}^{11-}$	$\mathbf{a}^{12+}$	$\mathbf{a}^{11+}$	$\mathbf{a}^{12-}$
$\mathbf{a}^{21-}$	$\mathbf{a}^{21-}$	$\mathbf{a}^{22+}$	$\mathbf{a}^{21+}$	$\mathbf{a}^{22-}$	$\mathbf{a}^{32+}$	$\mathbf{a}^{31+}$	$\mathbf{a}^{31-}$	$\mathbf{a}^{32-}$	$\mathbf{a}^0$	$\mathbf{a}^2$	$\mathbf{a}^3$	$\mathbf{a}^1$	$\mathbf{a}^{12-}$	$\mathbf{a}^{11+}$	$\mathbf{a}^{12+}$	$\mathbf{a}^{11-}$
$\mathbf{a}^{22+}$	$\mathbf{a}^{22+}$	$\mathbf{a}^{21-}$	$\mathbf{a}^{22-}$	$\mathbf{a}^{21+}$	$\mathbf{a}^{31+}$	$\mathbf{a}^{32+}$	$\mathbf{a}^{32-}$	$\mathbf{a}^{31-}$	$\mathbf{a}^1$	$\mathbf{a}^3$	$\mathbf{a}^2$	$\mathbf{a}^0$	$\mathbf{a}^{12+}$	$\mathbf{a}^{11-}$	$\mathbf{a}^{12-}$	$\mathbf{a}^{11+}$
$\mathbf{a}^{22-}$	$\mathbf{a}^{22-}$	$\mathbf{a}^{21+}$	$\mathbf{a}^{22+}$	$\mathbf{a}^{21-}$	$\mathbf{a}^{32-}$	$\mathbf{a}^{31-}$	$\mathbf{a}^{31+}$	$\mathbf{a}^{32+}$	$\mathbf{a}^3$	$\mathbf{a}^1$	$\mathbf{a}^0$	$\mathbf{a}^2$	$\mathbf{a}^{11+}$	$\mathbf{a}^{12-}$	$\mathbf{a}^{11-}$	$\mathbf{a}^{12+}$
$\mathbf{a}^{31+}$	$\mathbf{a}^{31+}$	$\mathbf{a}^{32+}$	$\mathbf{a}^{32-}$	$\mathbf{a}^{31-}$	$\mathbf{a}^{21-}$	$\mathbf{a}^{22+}$	$\mathbf{a}^{21+}$	$\mathbf{a}^{22-}$	$\mathbf{a}^{11-}$	$\mathbf{a}^{12-}$	$\mathbf{a}^{12+}$	$\mathbf{a}^{11+}$	$\mathbf{a}^3$	$\mathbf{a}^0$	$\mathbf{a}^2$	$\mathbf{a}^1$
$\mathbf{a}^{31-}$	$\mathbf{a}^{31-}$	$\mathbf{a}^{32-}$	$\mathbf{a}^{32+}$	$\mathbf{a}^{31+}$	$\mathbf{a}^{22-}$	$\mathbf{a}^{21+}$	$\mathbf{a}^{22+}$	$\mathbf{a}^{21-}$	$\mathbf{a}^{12+}$	$\mathbf{a}^{11+}$	$\mathbf{a}^{11-}$	$\mathbf{a}^{12-}$	$\mathbf{a}^0$	$\mathbf{a}^3$	$\mathbf{a}^1$	$\mathbf{a}^2$
$\mathbf{a}^{32+}$	$\mathbf{a}^{32+}$	$\mathbf{a}^{31+}$	$\mathbf{a}^{31-}$	$\mathbf{a}^{32-}$	$\mathbf{a}^{22+}$	$\mathbf{a}^{21-}$	$\mathbf{a}^{22-}$	$\mathbf{a}^{21+}$	$\mathbf{a}^{11+}$	$\mathbf{a}^{12+}$	$\mathbf{a}^{12-}$	$\mathbf{a}^{11-}$	$\mathbf{a}^2$	$\mathbf{a}^1$	$\mathbf{a}^3$	$\mathbf{a}^0$
$\mathbf{a}^{32-}$	$\mathbf{a}^{32-}$	$\mathbf{a}^{31-}$	$\mathbf{a}^{31+}$	$\mathbf{a}^{32+}$	$\mathbf{a}^{21+}$	$\mathbf{a}^{22-}$	$\mathbf{a}^{21-}$	$\mathbf{a}^{22+}$	$\mathbf{a}^{12-}$	$\mathbf{a}^{11-}$	$\mathbf{a}^{11+}$	$\mathbf{a}^{12+}$	$\mathbf{a}^1$	$\mathbf{a}^2$	$\mathbf{a}^0$	$\mathbf{a}^3$

Table 2: *Addition table of the group* $(\mathbb{Z}_4\times\mathbb{Z}_4\,;+)$.

We now discuss some of the results obtained from these assumptions [1] [2].

The (induced) addition associated to the grading group $\mathbb{I}$

$$+ : \mathbb{I}\times\mathbb{I} \longrightarrow \mathbb{I}\ ;\ ((a,\mathbf{a}),(b,\mathbf{b}))\mapsto(a+b,\mathbf{a}+\mathbf{b}), \tag{2. 15}$$

is given according to tables 1 and 2.

The corresponding grading contributions to the function q,

$$\begin{aligned} q : \mathbb{I}\times\mathbb{I} &\longrightarrow C_1, \\ ((a,\mathbf{a}),(b,\mathbf{b})) &\mapsto q_{(a,\mathbf{a}),(b,\mathbf{b})} = q^{\mathbb{Z}_2}_{a,b}\, q^{\mathbb{Z}_4\times\mathbb{Z}_4}_{\mathbf{a},\mathbf{b}}, \end{aligned} \tag{2. 16}$$

are shown in tables 3 and 4.

We naturally adopt the following multiplets for the generators of the Poincaré Lie subalgebra $\mathbb{P}$:

$$T := \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix},\ \bar{T} := \begin{bmatrix} \bar{T}_1 \\ \bar{T}_2 \\ \bar{T}_3 \end{bmatrix},\ P := \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}. \tag{2. 17}$$

$q^{\mathbb{Z}_2}$	0	1
0	1	1
1	1	-1

Table 3: *Function $q^{\mathbb{Z}_2}$: Symmetric phase contribution to the function q.*

$q^{\mathbb{Z}_4\times\mathbb{Z}_4}$	$\mathbf{a}^0$	$\mathbf{a}^1$	$\mathbf{a}^2$	$\mathbf{a}^3$	$\mathbf{a}^{11^+}$	$\mathbf{a}^{11^-}$	$\mathbf{a}^{12^+}$	$\mathbf{a}^{12^-}$	$\mathbf{a}^{21^+}$	$\mathbf{a}^{21^-}$	$\mathbf{a}^{22^+}$	$\mathbf{a}^{22^-}$	$\mathbf{a}^{31^+}$	$\mathbf{a}^{31^-}$	$\mathbf{a}^{32^+}$	$\mathbf{a}^{32^-}$
$(0,0)=\mathbf{a}^0$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$(2,0)=\mathbf{a}^1$	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1
$(0,2)=\mathbf{a}^2$	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
$(2,2)=\mathbf{a}^3$	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1
$(1,0)=\mathbf{a}^{11^+}$	1	1	-1	-1	1	1	-1	-1	i	$-$i	$-$i	i	$-$i	i	$-$i	i
$(3,0)=\mathbf{a}^{11^-}$	1	1	-1	-1	1	1	-1	-1	$-$i	i	i	$-$i	i	$-$i	i	$-$i
$(1,2)=\mathbf{a}^{12^+}$	1	1	-1	-1	-1	-1	1	1	i	$-$i	$-$i	i	i	$-$i	i	$-$i
$(3,2)=\mathbf{a}^{12^-}$	1	1	-1	-1	-1	-1	1	1	$-$i	i	i	$-$i	$-$i	i	$-$i	i
$(0,1)=\mathbf{a}^{21^+}$	1	-1	1	-1	$-$i	i	$-$i	i	1	1	-1	-1	i	$-$i	$-$i	i
$(0,3)=\mathbf{a}^{21^-}$	1	-1	1	-1	i	$-$i	i	$-$i	1	1	-1	-1	$-$i	i	i	$-$i
$(2,3)=\mathbf{a}^{22^+}$	1	-1	1	-1	i	$-$i	i	$-$i	-1	-1	1	1	i	$-$i	$-$i	i
$(2,1)=\mathbf{a}^{22^-}$	1	-1	1	-1	$-$i	i	$-$i	i	-1	-1	1	1	$-$i	i	i	$-$i
$(3,3)=\mathbf{a}^{31^+}$	1	-1	-1	1	i	$-$i	$-$i	i	$-$i	i	$-$i	i	1	1	-1	-1
$(1,1)=\mathbf{a}^{31^-}$	1	-1	-1	1	$-$i	i	i	$-$i	i	$-$i	i	$-$i	1	1	-1	-1
$(1,3)=\mathbf{a}^{32^+}$	1	-1	-1	1	i	$-$i	$-$i	i	i	$-$i	i	$-$i	-1	-1	1	1
$(3,1)=\mathbf{a}^{32^-}$	1	-1	-1	1	$-$i	i	i	$-$i	$-$i	i	$-$i	i	-1	-1	1	1

Table 4: *Function $q^{\mathbb{Z}_4\times\mathbb{Z}_4}$, the non-symmetric phase contribution to the function q.*

According to the assumptions $\mathcal{B}, \mathcal{F}$, and $\mathcal{I}$ the assignment of indices to the generators of the restricted Poincaré Lie algebra is given by

$$\varsigma_{\rm st}(J_i) = \varsigma_{\rm st}(\hat{J}_i) = \varsigma_{\rm st}(T_i) = \varsigma_{\rm st}(\bar{T}_i) = \varsigma_{\rm st}(P_i) = (0, \mathbf{a}^i)\,;\, i \in \{1,2,3\}, \qquad (2.\ 18)$$

$$\varsigma_{\rm st}(P_0) = (0, \mathbf{a}^0) = \tilde{o}. \qquad (2.\ 19)$$

From the addition table 2 we recognize that there are at least four main classes of non-scalar multiplets under Lorentz transformations:

- The 0-class have multiplets whose elements have indices with $(\mathbb{Z}_4\times\mathbb{Z}_4)$-factors of the set $\{\mathbf{a}^0, \mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3\}$.
- The i-class; $i \in \{1,2,3\}$ have multiplets whose elements have indices with $(\mathbb{Z}_4\times\mathbb{Z}_4)$-factors of the set $\{\mathbf{a}^{i1^+}, \mathbf{a}^{i1^-}, \mathbf{a}^{i2^+}, \mathbf{a}^{i2^-}\}$; $i \in \{1,2,3\}$ respectively.

In each one of these four classes we can distinguish two types of multiplets: multiplets with self-bosonic components and multiplets with self-fermionic components. Accordingly, the multiplets of generators of the algebraic extension $\mathbb{L}$ build up multiplets of the following types:

- Multiplet types of the i-classes, $i \in \{1,2,3\}$: **para-bosonic** multiplets W_i^ω and **para-fermionic** multiplets V_i^υ,

$$W_{is}^\omega = \begin{bmatrix} W_{i0}^\omega \\ W_{i1}^\omega \\ W_{i2}^\omega \\ W_{i3}^\omega \end{bmatrix}, \ V_{is}^\upsilon = \begin{bmatrix} V_{i0}^\upsilon \\ V_{i1}^\upsilon \\ V_{i2}^\upsilon \\ V_{i3}^\upsilon \end{bmatrix}, \tag{2. 20}$$

where $\omega \in I_W$, $\upsilon \in I_V$, $i \in \{1,2,3\}$.

After involution in $\mathcal{G}$ the multiplets of W- and V-type become:

$$\bar{W}_{\bar{\imath}\dot{s}}^\omega = \begin{bmatrix} \bar{W}_{\bar{\imath}\dot{0}}^\omega \\ \bar{W}_{\bar{\imath}\dot{1}}^\omega \\ \bar{W}_{\bar{\imath}\dot{2}}^\omega \\ \bar{W}_{\bar{\imath}\dot{3}}^\omega \end{bmatrix}, \ \bar{V}_{\bar{\imath}\dot{s}}^\upsilon = \begin{bmatrix} \bar{V}_{\bar{\imath}\dot{0}}^\upsilon \\ \bar{V}_{\bar{\imath}\dot{1}}^\upsilon \\ \bar{V}_{\bar{\imath}\dot{2}}^\upsilon \\ \bar{V}_{\bar{\imath}\dot{3}}^\upsilon \end{bmatrix}, \tag{2. 21}$$

where $\omega \in I_W$, $\upsilon \in I_V$, $\bar{\imath} \in \{\bar{1},\bar{2},\bar{3}\}$.

According to the assumption $\mathcal{G}$, the indices assigned to the multiplet components are given by

$$\begin{aligned}
\varsigma_{\rm st}(W_{i0}^\omega) = \varsigma_{\rm st}(\bar{W}_{\bar{\imath}\dot{1}}^\omega) = (0, \mathbf{a}^{i1^+}) \quad &, \quad \varsigma_{\rm st}(V_{i0}^\upsilon) = \varsigma_{\rm st}(\bar{V}_{\bar{\imath}\dot{1}}^\upsilon) = (1, \mathbf{a}^{i1^+}), \\
\varsigma_{\rm st}(W_{i1}^\omega) = \varsigma_{\rm st}(\bar{W}_{\bar{\imath}\dot{0}}^\omega) = (0, \mathbf{a}^{i1^-}) \quad &, \quad \varsigma_{\rm st}(V_{i1}^\upsilon) = \varsigma_{\rm st}(\bar{V}_{\bar{\imath}\dot{0}}^\upsilon) = (1, \mathbf{a}^{i1^-}), \\
\varsigma_{\rm st}(W_{i2}^\omega) = \varsigma_{\rm st}(\bar{W}_{\bar{\imath}\dot{3}}^\omega) = (0, \mathbf{a}^{i2^+}) \quad &, \quad \varsigma_{\rm st}(V_{i2}^\upsilon) = \varsigma_{\rm st}(\bar{V}_{\bar{\imath}\dot{3}}^\upsilon) = (1, \mathbf{a}^{i2^+}), \\
\varsigma_{\rm st}(W_{i3}^\omega) = \varsigma_{\rm st}(\bar{W}_{\bar{\imath}\dot{2}}^\omega) = (0, \mathbf{a}^{i2^-}) \quad &, \quad \varsigma_{\rm st}(V_{i3}^\upsilon) = \varsigma_{\rm st}(\bar{V}_{\bar{\imath}\dot{2}}^\upsilon) = (1, \mathbf{a}^{i2^-}).
\end{aligned} \tag{2. 22}$$

Observe that the multiplets of W-type, i.e. W_i^ω, $\bar{W}_{\bar{\imath}}^\omega$, $\cdots$, have self-bosonic components, while the multiplets of V-type, i.e. V_i^υ, $\bar{V}_{\bar{\imath}}^\upsilon$, $\cdots$, have self-fermionic components.

- Multiplet types of the 0-class: **bosonic** multiplets B_o^b and **fermionic** multiplets F_o^f,

$$B_{os}^b = \begin{bmatrix} B_{o0}^b \\ B_{o1}^b \\ B_{o2}^b \\ B_{o3}^b \end{bmatrix}, \ F_{os}^f = \begin{bmatrix} F_{o0}^f \\ F_{o1}^f \\ F_{o2}^f \\ F_{o3}^f \end{bmatrix}, \tag{2. 23}$$

where $b \in I_B$, $f \in I_F$.

After involution the multiplets of B- and F-type become:

$$\bar{B}_{\bar{o}\dot{s}}^b = \begin{bmatrix} \bar{B}_{\bar{o}\dot{0}}^b \\ \bar{B}_{\bar{o}\dot{1}}^b \\ \bar{B}_{\bar{o}\dot{2}}^b \\ \bar{B}_{\bar{o}\dot{3}}^b \end{bmatrix}, \ \bar{F}_{\bar{o}\dot{s}}^f = \begin{bmatrix} \bar{F}_{\bar{o}\dot{0}}^f \\ \bar{F}_{\bar{o}\dot{1}}^f \\ \bar{F}_{\bar{o}\dot{2}}^f \\ \bar{F}_{\bar{o}\dot{3}}^f \end{bmatrix}, \tag{2. 24}$$

where $b \in I_B$, $f \in I_F$.

The indices assigned to the multiplet components are given by

$$\begin{aligned} \varsigma_{\rm st}(B^b_{os}) &= \varsigma_{\rm st}(\bar{B}^b_{\bar{o}\dot{s}}) = (0, \mathbf{a}^s)\ ;\ s \in \{0,1,2,3\} \\ \varsigma_{\rm st}(F^f_{os}) &= \varsigma_{\rm st}(\bar{F}^f_{\bar{o}\dot{s}}) = (1, \mathbf{a}^s)\ ;\ s \in \{0,1,2,3\}. \end{aligned} \tag{2. 25}$$

Observe that the multiplets of B-type have self-bosonic components, while the multiplets of F-type have self-fermionic components. Observe also that the multiplets T, $\bar{T}$, and P can be seen as particular cases of multiplets of B-type.

We adopt from now on the following convention about indices:

- The following indices belong to particular index sets:

$$\begin{aligned} i,j,k,\ell,m &\in \{1,2,3\}\ ;\ \text{space indices, class indices,} \\ \bar{\imath},\bar{\jmath},\bar{k},\bar{\ell},\bar{m} &\in \{\bar{1},\bar{2},\bar{3}\}\ ;\ \text{class indices after involution in } \mathcal{G}, \\ \mu,\nu,\tau,\sigma,\rho &\in \{0,1,2,3\}\ ;\ \text{space-time indices,} \\ r,s,t,u &\in \{0,1,2,3\}\ ;\ \text{multiplet-component indices,} \\ \dot{r},\dot{s},\dot{t},\dot{u} &\in \{\dot{0},\dot{1},\dot{2},\dot{3}\}\ ;\ \text{adjoint-multiplet-component indices,} \\ \omega,\omega',\omega'' \in I_W &\quad,\quad v,v',v'' \in I_V, \\ b,b',b'' \in I_B &\quad,\quad f,f',f'' \in I_F.\ ;\ \text{dummy class indices.} \end{aligned} \tag{2. 26}$$

- If in the same monomial an index appears both as subindex and as superindex, then summation over its corresponding index set is understood.

We call a momentum representation graded-irreducible if the corresponding matrices are not reducible to a common multiple block-diagonal texture by a unitary transformation **allowed** by the graded symmetry.

Since the $(I,\ q)$-graded Lie algebraic structure fixes the texture of momentum representations, there are graded-irreducible representations which are reducible (by breaking the graded symmetry) through a unitary transformation. This is the case for the spin-$\frac{1}{2}$ graded-irreducible representations of extensions we are considering. The graded-irreducible representations of spin-$\frac{1}{2}$ are quadruplets, instead of the irreducible spin-$\frac{1}{2}$ doublets.

The classification of the graded-irreducible representations is given according to the eigenvalues with respect to the Casimir operators of the $su(2) \oplus su(2)$ algebra

$$C_T := \delta^{ij} T_i T_j, \quad C_{\bar{T}} := \delta^{ij} \bar{T}_i \bar{T}_j. \tag{2. 27}$$

These graded-irreducible representations are thus characterized by the spin function $\varsigma_{\rm sp}$:

$$\varsigma_{\rm sp} : I_{rep}(M) \longrightarrow \tfrac{1}{2}I\!N^* \times \tfrac{1}{2}I\!N^*\ ;\ \mathcal{O} \mapsto \varsigma_{\rm sp}(\mathcal{O}) = (n_{\mathcal{O}}, m_{\mathcal{O}}), \tag{2. 28}$$

where $n_{\mathcal{O}}(n_{\mathcal{O}}+1)$ and $m_{\mathcal{O}}(m_{\mathcal{O}}+1)$ are respectively the eigenvalues of $\mathcal{O}$ under C_T and $C_{\bar{T}}$, and $\frac{1}{2}I\!N^* := \{0, \frac{1}{2}, 1, \frac{3}{2}, \cdots\}$.

Observe that under involution $\overline{(\cdot)}$ in $I\!L$ we have:

$$\varsigma_{\rm sp}(\mathcal{O}) = (n_{\mathcal{O}}, m_{\mathcal{O}}) \Longrightarrow \varsigma_{\rm sp}(\bar{\mathcal{O}}) = (m_{\mathcal{O}}, n_{\mathcal{O}}). \tag{2. 29}$$

We now adopt two further assumptions:

$\mathcal{R}$ The Hilbert space has a positive-definite metric and the involution adopted for the generators of the extended algebra corresponds to the adjunction when considered as operators acting on the Hilbert space.

$\mathcal{S}$ The multiplets transforming under half-integer spin representations have self-fermionic components, the multiplets transforming under integer spin representations have self-bosonic components.

Using all the above assumptions, it has been determined in [2] the most general non-trivial extension $\mathbb{L}$ of the Poincaré algebra $\mathbb{P}$ involving only spin-$\frac{1}{2}$ multiplets of F- and V-type besides the multiplets of generators of $\mathbb{P}$. The algebraic relations of this algebraic extension are:

$$[M_{\mu\nu}, M_{\rho\sigma}] = \mathrm{i}(g_{\mu\sigma}M_{\nu\rho} + g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho}), \tag{2. 30}$$
$$[M_{\mu\nu}, P_\rho] = \mathrm{i}(g_{\rho\nu}P_\mu - g_{\mu\rho}P_\nu), \tag{2. 31}$$
$$[P_\mu, P_\rho] = 0, \tag{2. 32}$$

$$[\![F^f_{os}, \bar{F}^{f'}_{\bar{o}\dot{t}}]\!] = 2\hat{\Upsilon}^{ff'}(\tfrac{o\mu}{o\bar{o}})_{s\dot{t}}P_\mu, \tag{2. 33}$$
$$[\![F^f_{os}, F^{f'}_{ot}]\!] = 0, \qquad [\![\bar{F}^{f'}_{\bar{o}\dot{t}}, \bar{F}^f_{\bar{o}\dot{s}}]\!] = 0, \tag{2. 34}$$
$$[\![F^f_{os}, P_\mu]\!] = 0, \qquad [\![P_\mu, \bar{F}^f_{\bar{o}\dot{s}}]\!] = 0, \tag{2. 35}$$
$$[\![M^{\mu\nu}, F^f_{os}]\!] = -\tfrac{1}{2}(\sigma^{f\,\mu\nu}_{o})_s{}^t F^f_{ot}, \qquad [\![M^{\mu\nu}, \bar{F}^f_{\bar{o}\dot{s}}]\!] = +\tfrac{1}{2}(\bar{\sigma}^{f\,\mu\nu}_{\bar{o}})_{\dot{s}}{}^{\dot{t}}\bar{F}^f_{\bar{o}\dot{t}}, \tag{2. 36}$$
$$\varsigma_{\mathrm{sp}}(F^f_o) = (\tfrac{1}{2}, 0), \qquad \varsigma_{\mathrm{sp}}(\bar{F}^f_o) = (0, \tfrac{1}{2}), \tag{2. 37}$$

$$[\![V^\upsilon_{is}, \bar{V}^{\upsilon'}_{\bar{\imath}\dot{t}}]\!] = 2\hat{\gamma}^{\upsilon\upsilon'}(\tfrac{o\mu}{i\bar{\imath}})_{s\dot{t}}P_\mu, \tag{2. 38}$$
$$[\![V^\upsilon_{is}, V^{\upsilon'}_{it}]\!] = 0, \qquad [\![\bar{V}^{\upsilon'}_{\bar{\imath}\dot{t}}, \bar{V}^\upsilon_{\bar{\imath}\dot{s}}]\!] = 0, \tag{2. 39}$$
$$[\![P_\mu, V^\upsilon_{is}]\!] = 0, \qquad [\![P_\mu, \bar{V}^\upsilon_{\bar{\imath}\dot{s}}]\!] = 0, \tag{2. 40}$$
$$[\![M^{\mu\nu}, V^\upsilon_{is}]\!] = -\tfrac{1}{2}(\sigma^{\upsilon\,\mu\nu}_{i})_s{}^t V^\upsilon_{it}, \qquad [\![M^{\mu\nu}, \bar{V}^\upsilon_{\bar{\imath}\dot{s}}]\!] = +\tfrac{1}{2}(\bar{\sigma}^{\upsilon\,\mu\nu}_{\bar{\imath}})_{\dot{s}}{}^{\dot{t}}\bar{V}^\upsilon_{\bar{\imath}\dot{t}}, \tag{2. 41}$$
$$\varsigma_{\mathrm{sp}}(V^\upsilon_i) = (\tfrac{1}{2}, 0), \qquad \varsigma_{\mathrm{sp}}(\bar{V}^\upsilon_{\bar{\imath}}) = (0, \tfrac{1}{2}). \tag{2. 42}$$

$$[\![F^f_{os}, V^\upsilon_{it}]\!] = 0, \qquad [\![\bar{V}^\upsilon_{\bar{\imath}\dot{t}}, \bar{F}^f_{o\dot{s}}]\!] = 0, \tag{2. 43}$$
$$[\![F^f_{os}, \bar{V}^\upsilon_{\bar{\imath}\dot{t}}]\!] = 0, \qquad [\![V^\upsilon_{it}, \bar{F}^f_{\bar{o}\dot{s}}]\!] = 0, \tag{2. 44}$$

$$[\![V^\upsilon_{is}, V^{\upsilon'}_{jt}]\!] = 0, \qquad [\![\bar{V}^{\upsilon'}_{\bar{\jmath}\dot{t}}, \bar{V}^\upsilon_{\bar{\imath}\dot{s}}]\!] = 0\,;\, i \neq j, \tag{2. 45}$$
$$[\![V^\upsilon_{is}, \bar{V}^{\upsilon'}_{\bar{\jmath}\dot{t}}]\!] = 0, \qquad [\![V^{\upsilon'}_{jt}, \bar{V}^\upsilon_{\bar{\imath}\dot{s}}]\!] = 0\,;\, i \neq j, \tag{2. 46}$$

where the conventions in (2. 26) hold. The spin-$\frac{1}{2}$ graded-irreducible representations to be used here are presented in Appendix C. The $\hat{\Upsilon}$- and $\hat{\gamma}$-matrices are presented in Appendix E.

The algebra in (2. 30)-(2. 46) provides an extension of the Poincaré Lie algebra that seems to widen the possibilities given by supersymmetry: Both relations (2. 33) and (2. 38) can assure positive-definite energy eigenvalues.

If all the multiplets of V-type belong to one single i-class, then the extension might be more properly understood [2] as an $(I,\, q)$-graded Lie algebra with $I \subset \mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_4)$. If the extension has multiplets of V-type of several classes, then the smallest group generated by the addition operation among the elements of the set I is $\mathbb{Z}_2 \times (\mathbb{Z}_4 {\times} \mathbb{Z}_4)$.

A remarkable feature of the algebraic extensions $\mathbb{L}$ thus obtained is that the generators of the Lorentz Lie algebra can act with commutators **or** with anticommutators on further generators of the algebraic extension. Accordingly, the parameters associated with the Lorentz transformations can commute **or** anticommute with the parameters of further transformations. The parameters associated to the actual algebraic extension can have among them generalized commutative behaviour **beyond** commutativity and anticommutativity.

3 $(I,\, q)$-graded superfields and superspace formalism

We want to represent the algebra in (2. 30)-(2. 46) on superfields Φ. This means that we look for differential operators δ_G for each generator G of the algebraic extension $\mathbb{L}$, such that

$$\mathrm{i}[\![G, \Phi]\!] = \delta_G \Phi. \tag{3. 1}$$

The reiterated application of these transformations corresponds to the application of a single transformation. For an algebraic relation of the form

$$[\![G_1, G_2]\!] = \mathrm{i} G_3 \tag{3. 2}$$

we obtain the relation

$$\delta_{G_3} = [\![\delta_{G_2}, \delta_{G_2}]\!], \tag{3. 3}$$

independently of the superfield Φ on which the operators act.

In order to obtain the desired representation of the action of the algebra generators on the superfields, we use the Lie algebra and the Lie group associated with the $(I,\, q)$-graded Lie algebra [13]. In fact, the usage of a suitable set of parameters "compensating" the statistic behaviour of the algebra generators allows for the construction of Lie group elements in a way quite analogous to what is done for $\mathbb{Z}_2$-graded algebras in [11] (See [12] for an introduction to this subject). We associate parameters Θ^r to each generator G_r of the algebra, in such a way that the product $\Theta^r G_r$ has trivial (statistic) index. Hence,

$$\varsigma_{\mathrm{st}}(\Theta^r) = -\varsigma_{\mathrm{st}}(G_r). \tag{3. 4}$$

We introduce then parameters x^μ, Θ_f^{os}, $\bar{\Theta}_f^{\bar{o}\dot{s}}$, Θ_υ^{is}, $\bar{\Theta}_\upsilon^{\bar{\imath}\dot{s}}$:

$$\begin{aligned}
\varsigma_{\mathrm{st}}(x^\mu) \quad &= (0, \mathbf{a}^\mu) \quad &&; \mu \in \{0,1,2,3\},\\
\varsigma_{\mathrm{st}}(\Theta_f^{os}) = \varsigma_{\mathrm{st}}(\bar{\Theta}_f^{\bar{o}\dot{s}}) \quad &= (1, \mathbf{a}^s) \quad &&; s \in \{0,1,2,3\}\,;\, \dot{s} \in \{\dot{0},\dot{1},\dot{2},\dot{3}\},\\
\varsigma_{\mathrm{st}}(\Theta_f^{i0}) = \varsigma_{\mathrm{st}}(\bar{\Theta}_f^{\bar{\imath}\bar{1}}) \quad &= (1, \mathbf{a}^{i1^-}) \quad &&; i \in \{1,2,3\}\,;\, \bar{\imath} \in \{\bar{1},\bar{2},\bar{3}\},\\
\varsigma_{\mathrm{st}}(\Theta_f^{i1}) = \varsigma_{\mathrm{st}}(\bar{\Theta}_f^{\bar{\imath}\bar{0}}) \quad &= (1, \mathbf{a}^{i1^+}) \quad &&; i \in \{1,2,3\}\,;\, \bar{\imath} \in \{\bar{1},\bar{2},\bar{3}\},\\
\varsigma_{\mathrm{st}}(\Theta_f^{i2}) = \varsigma_{\mathrm{st}}(\bar{\Theta}_f^{\bar{\imath}\bar{3}}) \quad &= (1, \mathbf{a}^{i2^-}) \quad &&; i \in \{1,2,3\}\,;\, \bar{\imath} \in \{\bar{1},\bar{2},\bar{3}\},\\
\varsigma_{\mathrm{st}}(\Theta_f^{i3}) = \varsigma_{\mathrm{st}}(\bar{\Theta}_f^{\bar{\imath}\bar{2}}) \quad &= (1, \mathbf{a}^{i2^+}) \quad &&; i \in \{1,2,3\}\,;\, \bar{\imath} \in \{\bar{1},\bar{2},\bar{3}\}.
\end{aligned} \tag{3. 5}$$

where x^μ are the space-time (vector) parameters, Θ_f^{os} and $\bar{\Theta}_f^{\bar{o}\dot{s}}$ are spinorial (Grassmann) parameters, Θ_υ^{is} and $\bar{\Theta}_\upsilon^{\bar{\imath}\dot{s}}$ are para-spinorial (para-Grassmann) parameters.

We remain close to the little group approach and study first group elements associated with generators of translations (i.e. inhomogeneous transformations in the corresponding superspace). Such group elements are obtained through the exponential map:

$$g(x', \Theta'^{o}_{f}, \bar\Theta'^{\bar o}_{f}, \Theta'^{i}_{\upsilon}, \bar\Theta'^{\bar\imath}_{\upsilon}) = \\ = \exp \mathrm{i}\{x'P + \Theta'^{o}_{f} F^{f}_{o} + \bar F^{f}_{\bar o}\bar\Theta'^{\bar o}_{f} + \Theta'^{i}_{\upsilon} V^{\upsilon}_{i} + \bar V^{\upsilon}_{\bar\imath}\bar\Theta'^{\bar\imath}_{\upsilon}\}. \quad (3.\ 6)$$

In Appendix C we show the corresponding spin-$\frac{1}{2}$ representations, the rules for the lowering and rising of component indices and their summation conventions.

The action from the left of a group element $g(x', \Theta'^{o}_{f}, \bar\Theta'^{\bar o}_{f}, \Theta'^{i}_{\upsilon}, \bar\Theta'^{\bar\imath}_{\upsilon})$ on $g(x, \Theta^{o}_{f}, \bar\Theta^{\bar o}_{f}, \Theta^{i}_{\upsilon}, \bar\Theta^{\bar\imath}_{\upsilon})$ can be calculated using the Hausdorf formula (since the compounds $\Theta^r G_r$ fulfil a Lie algebra),

$$\exp\{A\}\exp\{B\} = \exp\{A + B + \tfrac{1}{2}[A, B]\}, \quad (3.\ 7)$$

as well as the generalized Jacobi associativity, the algebraic relations (2. 30)-(2. 46), and the definition

$$[\Theta^r G_r, \Theta^u G_u] := \Theta^u \Theta^r [\![G_r, G_u]\!]. \quad (3.\ 8)$$

We obtain:

$$g(x', \Theta'^{o}_{f}, \bar\Theta'^{\bar o}_{f}, \Theta'^{i}_{\upsilon}, \bar\Theta'^{\bar\imath}_{\upsilon})\, g(x, \Theta^{o}_{f}, \bar\Theta^{\bar o}_{f}, \Theta^{i}_{\upsilon}, \bar\Theta^{\bar\imath}_{\upsilon}) = \\ = g(x^\mu + x'^\mu + \mathrm{i}\Theta'^{o}_{f'}\hat\Upsilon^{f'f}(\tfrac{o\mu}{o\bar o})\bar\Theta^{\bar o}_{f} - \mathrm{i}\Theta^{o}_{f}\hat\Upsilon^{ff'}(\tfrac{o\mu}{o\bar o})\bar\Theta'^{\bar o}_{f'} + \\ + \mathrm{i}(2\delta^\mu_o + 2\delta^\mu_i - 1)\Theta'^{i}_{\upsilon'}\hat\gamma^{\upsilon'\upsilon}(\tfrac{o\mu}{i\bar\imath})\bar\Theta^{\bar\imath}_{\upsilon} - \mathrm{i}(2\delta^\mu_o + 2\delta^\mu_i - 1)\Theta^{i}_{\upsilon}\hat\gamma^{\upsilon\upsilon'}(\tfrac{o\mu}{i\bar\imath})\bar\Theta'^{\bar\imath}_{\upsilon'}, \\ \Theta^{o}_{f} + \Theta'^{o}_{f}, \bar\Theta^{\bar o}_{f} + \Theta'^{\bar o}_{f}, \Theta^{i}_{\upsilon} + \Theta'^{i}_{\upsilon}, \bar\Theta^{\bar\imath}_{\upsilon} + \bar\Theta'^{\bar\imath}_{\upsilon}). \quad (3.\ 9)$$

Hence, the multiplication by a group element from the left has caused a translation in the parameter space:

$$\begin{aligned} x^\mu &\mapsto x^\mu + x'^\mu + \mathrm{i}\Theta'^{o}_{f'}\hat\Upsilon^{f'f}(\tfrac{o\mu}{o\bar o})\bar\Theta^{\bar o}_{f} - \mathrm{i}\Theta^{o}_{f}\hat\Upsilon^{ff'}(\tfrac{o\mu}{o\bar o})\bar\Theta'^{\bar o}_{f'} + \\ &\quad + \mathrm{i}(2\delta^\mu_o + 2\delta^\mu_i - 1)\Theta'^{i}_{\upsilon'}\hat\gamma^{\upsilon'\upsilon}(\tfrac{o\mu}{i\bar\imath})\bar\Theta^{\bar\imath}_{\upsilon} - \mathrm{i}(2\delta^\mu_o + 2\delta^\mu_i - 1)\Theta^{i}_{\upsilon}\hat\gamma^{\upsilon\upsilon'}(\tfrac{o\mu}{i\bar\imath})\bar\Theta'^{\bar\imath}_{\upsilon'}, \\ \Theta^{o}_{f} &\mapsto \Theta^{o}_{f} + \Theta'^{o}_{f}, \\ \bar\Theta^{\bar o}_{f} &\mapsto \bar\Theta^{\bar o}_{f} + \Theta'^{\bar o}_{f}, \\ \Theta^{i}_{\upsilon} &\mapsto \Theta^{i}_{\upsilon} + \Theta'^{i}_{\upsilon}, \\ \bar\Theta^{\bar\imath}_{\upsilon} &\mapsto \bar\Theta^{\bar\imath}_{\upsilon} + TB'^{\bar\imath}_{\upsilon}. \end{aligned} \quad (3.\ 10)$$

This motion can be reproduced by differential operators acting from the left on the functions Φ defined in the parameter space,

$$\Phi = \Phi(x, \Theta^{o}_{f}, \bar\Theta^{\bar o}_{f}, \Theta^{i}_{\upsilon}, \bar\Theta^{\bar\imath}_{\upsilon}). \quad (3.\ 11)$$

These differential operators have the form:

$$\mathrm{i}[\![P_\mu, \Phi]\!] := \delta_{P_\mu}\Phi = \partial_\mu\Phi, \quad (3.\ 12)$$

$$\mathrm{i}[\![F^{f}_{os}, \Phi]\!] := \delta_{F^{f}_{os}}\Phi = (\partial_{\Theta^{os}_{f}} + \mathrm{i}\hat\Upsilon^{ff'}(\tfrac{o\mu}{o\bar o})_{si}\bar\Theta^{\bar o i}_{f'}\partial_\mu)\Phi, \quad (3.\ 13)$$

$$\mathrm{i}[\![\bar F^{f}_{\bar o t}, \Phi]\!] := \delta_{\bar F^{f}_{\bar o t}}\Phi = (-\partial_{\bar\Theta^{\bar o t}_{f}} - \mathrm{i}\Theta^{os}_{f'}\hat\Upsilon^{f'f}(\tfrac{o\mu}{o\bar o})_{st}\partial_\mu)\Phi, \quad (3.\ 14)$$

$$\mathrm{i}[\![V^{\upsilon}_{is}, \Phi]\!] := \delta_{V^{\upsilon}_{is}}\Phi = (\partial_{\Theta^{is}_{\upsilon}} + \mathrm{i}(2\delta^\mu_o + \delta^\mu_i - 1)\hat\gamma^{\upsilon\upsilon'}(\tfrac{o\mu}{i\bar\imath})_{st}\bar\Theta^{\bar\imath t}_{\upsilon'}\partial_\mu)\Phi, \quad (3.\ 15)$$

$$\mathrm{i}[\![\bar V^{\upsilon}_{\bar\imath t}, \Phi]\!] := \delta_{\bar V^{\upsilon}_{\bar\imath t}}\Phi = (-\partial_{\bar\Theta^{\bar\imath t}_{\upsilon}} - \mathrm{i}\Theta^{is}_{\upsilon'}\hat\gamma^{\upsilon'\upsilon}(\tfrac{o\mu}{i\bar\imath})_{st}\partial_\mu)\Phi. \quad (3.\ 16)$$

In Appendix D we describe the differential operations for Grassmann and para-Grassmann variables. The above differential operations generate the motion:

$$\begin{aligned} & g(x', \Theta'^{o}_{f}, \bar\Theta'^{\bar o}_{f}, \Theta'^{i}_{\upsilon}, \bar\Theta'^{\bar\imath}_{\upsilon})\, \Phi(x, \Theta^{o}_{f}, \bar\Theta^{\bar o}_{f}, \Theta^{i}_{\upsilon}, \bar\Theta^{\bar\imath}_{\upsilon})\, g^{-1}(x', \Theta'^{o}_{f}, \bar\Theta'^{\bar o}_{f}, \Theta'^{i}_{\upsilon}, \bar\Theta'^{\bar\imath}_{\upsilon}) = \\ & = \Phi(x^{\mu} + x'^{\mu} + \mathrm{i}\Theta'^{o}_{f'}\hat\Upsilon^{f'f}(\tfrac{o\mu}{o\bar o})\bar\Theta^{\bar o}_{f} - \mathrm{i}\Theta^{o}_{f'}\hat\Upsilon^{ff'}(\tfrac{o\mu}{o\bar o})\bar\Theta'^{\bar o}_{f'} + \\ & \quad + \mathrm{i}(2\delta^{\mu}_{o} + \delta^{\mu}_{i} - 1)\Theta'^{i}_{\upsilon'}\hat\gamma^{\upsilon'\upsilon}(\tfrac{o\mu}{i\bar\imath})\bar\Theta^{\bar\imath}_{\upsilon} - \mathrm{i}(2\delta^{\mu}_{o} + \delta^{\mu}_{i} - 1)\Theta^{i}_{\upsilon}\hat\gamma^{\upsilon\upsilon'}(\tfrac{o\mu}{i\bar\imath})\bar\Theta'^{\bar\imath}_{\upsilon'}, \\ & \quad \Theta^{o}_{f} + \Theta'^{o}_{f}, \bar\Theta^{\bar o}_{f} + \Theta'^{\bar o}_{f}, \Theta^{i}_{\upsilon} + \Theta'^{i}_{\upsilon}, \bar\Theta^{\bar\imath}_{\upsilon} + \bar\Theta'^{\bar\imath}_{\upsilon}) \qquad , \end{aligned} \tag{3. 17}$$

and provide a representation of the algebra on the superfields,

$$\begin{aligned} [\![\delta_{F^{f}_{os}}, \delta_{\bar F^{f'}_{\bar o t}}]\!] &= -2\mathrm{i}\hat\Upsilon^{ff'}(\tfrac{o\mu}{o\bar o})_{st}\partial_{\mu}, \\ [\![\delta_{V^{\upsilon}_{is}}, \delta_{\bar V^{\upsilon'}_{\bar\imath t}}]\!] &= -2\mathrm{i}\hat\gamma^{\upsilon\upsilon'}(\tfrac{o\mu}{i\bar\imath})_{st}\partial_{\mu}. \end{aligned} \tag{3. 18}$$

The further generalized commutators among $\delta_{P_{\mu}}, \delta_{F^{f}_{os}}, \delta_{\bar F^{f}_{\bar o t}}, \delta_{V^{\upsilon}_{is}}, \delta_{\bar V^{\upsilon}_{\bar\imath t}}$ vanish.

The Lorentz transformations will be carried out by differential operators:

$$\begin{aligned} \mathrm{i}[\![M^{\mu\nu}, \Phi]\!] &= \delta_{M^{\mu\nu}}\Phi = \\ &= (X^{\mu}\partial^{\nu} - X^{\nu}\partial^{\mu} + \\ & -\frac{\mathrm{i}}{2}\Theta^{os}_{f}(\sigma^{f\ \mu\nu}_{o})_{s}{}^{t}\partial_{\Theta^{os}_{f}} + \frac{\mathrm{i}}{2}\bar\Theta^{\bar o}_{f\dot s}(\bar\sigma^{f\ \mu\nu}_{\bar o})^{\dot s}{}_{\dot t}\partial_{\bar\Theta^{\bar o}_{f\dot t}} + \\ & -\frac{\mathrm{i}}{2}(2\delta^{\mu}_{i} + 2\delta^{\mu}_{o} - 1)(2\delta^{\nu}_{i} + 2\delta^{\nu}_{o} - 1)\Theta^{is}_{\upsilon}(\sigma^{\upsilon\ \mu\nu}_{i})_{s}{}^{t}\partial_{\Theta^{is}_{\upsilon}} + \\ & +\frac{\mathrm{i}}{2}(2\delta^{\mu}_{i} + 2\delta^{\mu}_{o} - 1)(2\delta^{\nu}_{i} + 2\delta^{\nu}_{o} - 1)\bar\Theta^{\bar\imath}_{\upsilon\dot s}(\bar\sigma^{\upsilon\ \mu\nu}_{\bar\imath})^{\dot s}{}_{\dot t}\partial_{\bar\Theta^{\bar\imath}_{\upsilon\dot t}})\Phi, \end{aligned} \tag{3. 19}$$

for which we verify

$$\begin{aligned} [\![\delta_{M^{\mu\nu}}, \delta_{F^{f}_{os}}]\!] &= +\frac{\mathrm{i}}{2}(\sigma^{f\ \mu\nu}_{o})_{s}{}^{t}\delta_{F^{f}_{ot}}, \\ [\![\delta_{M^{\mu\nu}}, \delta_{\bar F^{f}_{\bar o\dot s}}]\!] &= -\frac{\mathrm{i}}{2}(\bar\sigma^{f\ \mu\nu}_{\bar o})_{\dot s}{}^{\dot t}\delta_{\bar F^{f}_{\bar o\dot t}}, \\ [\![\delta_{M^{\mu\nu}}, \delta_{V^{\upsilon}_{is}}]\!] &= +\frac{\mathrm{i}}{2}(\sigma^{\upsilon\ \mu\nu}_{i})_{s}{}^{t}\delta_{V^{\upsilon}_{it}}, \\ [\![\delta_{M^{\mu\nu}}, \delta_{\bar V^{\upsilon}_{\bar\imath\dot s}}]\!] &= -\frac{\mathrm{i}}{2}(\bar\sigma^{\upsilon\ \mu\nu}_{\bar\imath})_{\dot s}{}^{\dot t}\delta_{\bar V^{\upsilon}_{\bar\imath\dot t}}. \end{aligned} \tag{3. 20}$$

The application of the transformation $g(x', \Theta'^{o}_{f}, \bar\Theta'^{\bar o}_{f}, \Theta'^{i}_{\upsilon}, \bar\Theta'^{\bar\imath}_{\upsilon})$ from the left has lead to the construction of the differential operators providing a representation of the algebra on the superfields. The action of the transformation $g(x', \Theta'^{o}_{f}, \bar\Theta'^{\bar o}_{f}, \Theta'^{i}_{\upsilon}, \bar\Theta'^{\bar\imath}_{\upsilon})$ from the right leads to the construction of differential operators with a good covariance as well. We define the **covariant derivatives:**

$$D^{f}_{os} := (\partial_{\Theta^{os}_{f}} - \mathrm{i}\hat\Upsilon^{ff'}(\tfrac{o\mu}{o\bar o})_{st}\bar\Theta^{\bar o\dot t}_{f'}\partial_{\mu}), \tag{3. 21}$$

$$\bar D^{f}_{\bar o\dot t} := (-\partial_{\bar\Theta^{\bar o\dot t}_{f}} + \mathrm{i}\Theta^{os}_{f'}\hat\Upsilon^{f'f}(\tfrac{o\mu}{o\bar o})_{st}\partial_{\mu}), \tag{3. 22}$$

$$D^{\upsilon}_{is} := (\partial_{\Theta^{is}_{\upsilon}} - \mathrm{i}(2\delta^{\mu}_{o} + \delta^{\mu}_{i} - 1)\hat\gamma^{\upsilon\upsilon'}(\tfrac{o\mu}{i\bar\imath})_{st}\bar\Theta^{\bar\imath\dot t}_{\upsilon'}\partial_{\mu}), \tag{3. 23}$$

$$\bar D^{\upsilon}_{\bar\imath\dot t} := (-\partial_{\bar\Theta^{\bar\imath\dot t}_{\upsilon}} + \mathrm{i}\Theta^{is}_{\upsilon'}\hat\gamma^{\upsilon'\upsilon}(\tfrac{o\mu}{i\bar\imath})_{st}\partial_{\mu}), \tag{3. 24}$$

where

$$\varsigma_{\rm st}(D^f_{os}) = \varsigma_{\rm st}(F^f_{os}) \quad , \quad \varsigma_{\rm st}(\bar{D}^f_{\bar{o}\dot{s}}) = \varsigma_{\rm st}(\bar{F}^f_{\bar{o}\dot{s}}), \tag{3. 25}$$

$$\varsigma_{\rm st}(D^\upsilon_{is}) = \varsigma_{\rm st}(V^\upsilon_{is}) \quad , \quad \varsigma_{\rm st}(\bar{D}^\upsilon_{\bar{i}\dot{s}}) = \varsigma_{\rm st}(\bar{V}^\upsilon_{\bar{i}\dot{s}}). \tag{3. 26}$$

These differential operators fulfil

$$\begin{aligned} [\![D^f_{os}, \bar{D}^{f'}_{\bar{o}\dot{t}}]\!] &= 2{\rm i}\hat{\Upsilon}^{ff'}(\tfrac{o\mu}{o\bar{o}})_{st}\partial_\mu, \\ [\![D^\upsilon_{is}, \bar{D}^{\upsilon'}_{\bar{i}\dot{t}}]\!] &= 2{\rm i}\hat{\gamma}^{\upsilon\upsilon'}(\tfrac{o\mu}{i\bar{i}})_{st}\partial_\mu. \end{aligned} \tag{3. 27}$$

The further generalized commutators among $D^f_{os}, \bar{D}^f_{\bar{o}\dot{t}}, D^\upsilon_{is}, \bar{D}^\upsilon_{\bar{i}\dot{t}}$ vanish. We also find

$$\begin{aligned} [\![D^f_{os}, \delta_G]\!] = 0 \quad &, \quad [\![\bar{D}^f_{\bar{o}\dot{s}}, \delta_G]\!] = 0, \\ [\![D^\upsilon_{is}, \delta_G]\!] = 0 \quad &, \quad [\![\bar{D}^\upsilon_{\bar{i}\dot{s}}, \delta_G]\!] = 0, \end{aligned} \tag{3. 28}$$

for all $\delta_G \in \{\delta_{P_\mu}, \delta_{F^f_{os}}, \delta_{\bar{F}^f_{\bar{o}\dot{t}}}, \delta_{V^\upsilon_{is}}, \delta_{\bar{V}^\upsilon_{\bar{i}\dot{t}}}\}$,

$$\begin{aligned} [\![\delta_{M^{\mu\nu}}, D^f_{os}]\!] &= +\frac{\rm i}{2}(\sigma^{f}_{o}{}^{\mu\nu})_s{}^t D^f_{ot}, \\ [\![\delta_{M^{\mu\nu}}, \bar{D}^f_{\bar{o}\dot{s}}]\!] &= -\frac{\rm i}{2}(\bar{\sigma}^{f}_{\bar{o}}{}^{\mu\nu})_{\dot{s}}{}^{\dot{t}} \bar{D}^f_{\bar{o}\dot{t}}, \\ [\![\delta_{M^{\mu\nu}}, D^\upsilon_{is}]\!] &= +\frac{\rm i}{2}(\sigma^{\upsilon}_{i}{}^{\mu\nu})_s{}^t D^\upsilon_{it}, \\ [\![\delta_{M^{\mu\nu}}, \bar{D}^\upsilon_{\bar{i}\dot{s}}]\!] &= -\frac{\rm i}{2}(\bar{\sigma}^{\upsilon}_{\bar{i}}{}^{\mu\nu})_{\dot{s}}{}^{\dot{t}} \bar{D}^\upsilon_{\bar{i}\dot{t}}. \end{aligned} \tag{3. 29}$$

The representation associated with the superfield Φ and the relations (3. 11)-(3. 29) is said to be given **in the real basis**. We now consider further representations. In fact, the group elements can be represented as well by $g_{(1)}$ or by $g_{(2)}$ of the generic forms

$$g_{(1)}(x, \Theta^o_f, \bar{\Theta}^{\bar{o}}_f, \Theta^i_\upsilon, \bar{\Theta}^{\bar{i}}_\upsilon) = \exp{\rm i}\{xP + \Theta^o_f F^f_o + \Theta^i_\upsilon V^\upsilon_i\}\cdot \exp{\rm i}\{\bar{F}^f_{\bar{o}}\bar{\Theta}^{\bar{o}}_f + \bar{V}^\upsilon_{\bar{i}}\bar{\Theta}^{\bar{i}}_\upsilon\}, \tag{3. 30}$$

$$g_{(2)}(x, \Theta^o_f, \bar{\Theta}^{\bar{o}}_f, \Theta^i_\upsilon, \bar{\Theta}^{\bar{i}}_\upsilon) = \exp{\rm i}\{xP + \bar{F}^f_{\bar{o}}\bar{\Theta}^{\bar{o}}_f + \bar{V}^\upsilon_{\bar{i}}\bar{\Theta}^{\bar{i}}_\upsilon\}\cdot \exp{\rm i}\{\Theta^o_f F^f_o + \Theta^i_\upsilon V^\upsilon_i\}. \tag{3. 31}$$

The left action of $g(x', \Theta'^o_f, \bar{\Theta}'^{\bar{o}}_f, \Theta'^i_\upsilon, \bar{\Theta}'^{\bar{i}}_\upsilon)$ on the group elements $g_{(1)}$ and $g_{(2)}$ above yields

$$\begin{aligned} & g(x', \Theta'^o_f, \bar{\Theta}'^{\bar{o}}_f, \Theta'^i_\upsilon, \bar{\Theta}'^{\bar{i}}_\upsilon) g_{(1)}(x, \Theta^o_f, \bar{\Theta}^{\bar{o}}_f, \Theta^i_\upsilon, \bar{\Theta}^{\bar{i}}_\upsilon) = \\ & \quad = g_{(1)}(x^\mu + x'^\mu - 2{\rm i}\Theta^o_f\hat{\Upsilon}^{ff'}(\tfrac{o\mu}{o\bar{o}})\bar{\Theta}'^{\bar{o}}_{f'} - 2{\rm i}(2\delta^\mu_o + 2\delta^\mu_i - 1)\Theta^i_\upsilon\hat{\gamma}^{\upsilon\upsilon'}(\tfrac{o\mu}{i\bar{i}})\bar{\Theta}'^{\bar{i}}_{\upsilon'}, \\ & \qquad \Theta^o_f + \Theta'^o_f, \bar{\Theta}^{\bar{o}}_f + \Theta'^{\bar{o}}_f, \Theta^i_\upsilon + \Theta'^i_\upsilon, \bar{\Theta}^{\bar{i}}_\upsilon + \bar{\Theta}'^{\bar{i}}_\upsilon), \end{aligned} \tag{3. 32}$$

$$\begin{aligned} & g(x', \Theta'^o_f, \bar{\Theta}'^{\bar{o}}_f, \Theta'^i_\upsilon, \bar{\Theta}'^{\bar{i}}_\upsilon) g_{(2)}(x, \Theta^o_f, \bar{\Theta}^{\bar{o}}_f, \Theta^i_\upsilon, \bar{\Theta}^{\bar{i}}_\upsilon) = \\ & \quad = g_{(2)}(x^\mu + x'^\mu + 2{\rm i}\Theta'^o_{f'}\hat{\Upsilon}^{f'f}(\tfrac{o\mu}{o\bar{o}})\bar{\Theta}^{\bar{o}}_f + 2{\rm i}(2\delta^\mu_o + 2\delta^\mu_i - 1)\Theta'^i_{\upsilon'}\hat{\gamma}^{\upsilon'\upsilon}(\tfrac{o\mu}{i\bar{i}})\bar{\Theta}^{\bar{i}}_\upsilon, \\ & \qquad \Theta^o_f + \Theta'^o_f, \bar{\Theta}^{\bar{o}}_f + \Theta'^{\bar{o}}_f, \Theta^i_\upsilon + \Theta'^i_\upsilon, \bar{\Theta}^{\bar{i}}_\upsilon + \bar{\Theta}'^{\bar{i}}_\upsilon). \end{aligned} \tag{3. 33}$$

We can associate again the transformation properties of the group elements $g_{(1)}$ and $g_{(2)}$ with superfield representations $\Phi_{(1)}$ and $\Phi_{(2)}$ respectively. The considered representations turn out to be related to one another by space-time shifts:

$$\Phi(x^\mu, \Theta^{os}_f, \bar{\Theta}^{\bar{o}\dot{s}}_f, \Theta^{it}_\upsilon, \bar{\Theta}^{\bar{i}\dot{t}}_\upsilon) = \tag{3. 34}$$

$$= \Phi_{(1)}(x^\mu + \mathrm{i}\Theta^o_f \hat{\Upsilon}^{ff'}(\tfrac{o\mu}{o\bar{o}})\bar{\Theta}^{\bar{o}}_{f'} + \mathrm{i}(2\delta^\mu_o + 2\delta^\mu_i - 1)\Theta^i_\upsilon \hat{\gamma}^{\upsilon\upsilon'}(\tfrac{o\mu}{i\bar{\imath}})\bar{\Theta}^{\bar{\imath}}_{\upsilon'}, \Theta^{os}_f, \bar{\Theta}^{\bar{o}\dot{s}}_f, \Theta^{it}_\upsilon, \bar{\Theta}^{\bar{\imath}\dot{t}}_\upsilon) =$$
$$= \exp(\mathrm{i}\{\Theta^o_f \hat{\Upsilon}^{ff'}(\tfrac{o\mu}{o\bar{o}})\bar{\Theta}^{\bar{o}}_{f'} + (2\delta^\mu_o + 2\delta^\mu_i - 1)\Theta^i_\upsilon \hat{\gamma}^{\upsilon\upsilon'}(\tfrac{o\mu}{i\bar{\imath}})\bar{\Theta}^{\bar{\imath}}_{\upsilon'}\}\partial_\mu)\Phi_{(1)}(x^\mu, \Theta^{os}_f, \bar{\Theta}^{\bar{o}\dot{s}}_f, \Theta^{it}_\upsilon, \bar{\Theta}^{\bar{\imath}\dot{t}}_\upsilon) =$$
$$= \Phi_{(2)}(x^\mu - \mathrm{i}\Theta^o_f \hat{\Upsilon}^{ff'}(\tfrac{o\mu}{o\bar{o}})\bar{\Theta}^{\bar{o}}_{f'} - \mathrm{i}(2\delta^\mu_o + 2\delta^\mu_i - 1)\Theta^i_\upsilon \hat{\gamma}^{\upsilon\upsilon'}(\tfrac{o\mu}{i\bar{\imath}})\bar{\Theta}^{\bar{\imath}}_{\upsilon'}, \Theta^{os}_f, \bar{\Theta}^{\bar{o}\dot{s}}_f, \Theta^{it}_\upsilon, \bar{\Theta}^{\bar{\imath}\dot{t}}_\upsilon) =$$
$$= \exp(-\mathrm{i}\{\Theta^o_f \hat{\Upsilon}^{ff'}(\tfrac{o\mu}{o\bar{o}})\bar{\Theta}^{\bar{o}}_{f'} + (2\delta^\mu_o + 2\delta^\mu_i - 1)\Theta^i_\upsilon \hat{\gamma}^{\upsilon\upsilon'}(\tfrac{o\mu}{i\bar{\imath}})\bar{\Theta}^{\bar{\imath}}_{\upsilon'}\}\partial_\mu)\Phi_{(2)}(x^\mu, \Theta^{os}_f, \bar{\Theta}^{\bar{o}\dot{s}}_f, \Theta^{it}_\upsilon, \bar{\Theta}^{\bar{\imath}\dot{t}}_\upsilon),$$

and have the transformation properties:

$$g(x', \Theta'^o_f, \bar{\Theta}'^{\bar{o}}_f, \Theta'^i_\upsilon, \bar{\Theta}'^{\bar{\imath}}_\upsilon)\Phi_{(1)}(x, \Theta^o_f, \bar{\Theta}^{\bar{o}}_f, \Theta^i_\upsilon, \bar{\Theta}^{\bar{\imath}}_\upsilon)g^{-1}(x', \Theta'^o_f, \bar{\Theta}'^{\bar{o}}_f, \Theta'^i_\upsilon, \bar{\Theta}'^{\bar{\imath}}_\upsilon) =$$
$$= \Phi_{(1)}(x^\mu + x'^\mu - 2\mathrm{i}\Theta^o_f \hat{\Upsilon}^{ff'}(\tfrac{o\mu}{o\bar{o}})\bar{\Theta}'^{\bar{o}}_{f'} - 2\mathrm{i}(2\delta^\mu_o + 2\delta^\mu_i - 1)\Theta^i_\upsilon \hat{\gamma}^{\upsilon\upsilon'}(\tfrac{o\mu}{i\bar{\imath}})\bar{\Theta}'^{\bar{\imath}}_{\upsilon'},$$
$$\Theta^o_f + \Theta'^o_f, \bar{\Theta}^{\bar{o}}_f + \Theta'^{\bar{o}}_f, \Theta^i_\upsilon + \Theta'^i_\upsilon, \bar{\Theta}^{\bar{\imath}}_\upsilon + \bar{\Theta}'^{\bar{\imath}}_\upsilon), \quad (3.\ 35)$$
$$g(x', \Theta'^o_f, \bar{\Theta}'^{\bar{o}}_f, \Theta'^i_\upsilon, \bar{\Theta}'^{\bar{\imath}}_\upsilon)\Phi_{(2)}(x, \Theta^o_f, \bar{\Theta}^{\bar{o}}_f, \Theta^i_\upsilon, \bar{\Theta}^{\bar{\imath}}_\upsilon)g^{-1}(x', \Theta'^o_f, \bar{\Theta}'^{\bar{o}}_f, \Theta'^i_\upsilon, \bar{\Theta}'^{\bar{\imath}}_\upsilon) =$$
$$= \Phi_{(2)}(x^\mu + x'^\mu + 2\mathrm{i}\Theta'^o_{f'} \hat{\Upsilon}^{f'f}(\tfrac{o\mu}{o\bar{o}})\bar{\Theta}^{\bar{o}}_f + 2\mathrm{i}(2\delta^\mu_o + 2\delta^\mu_i - 1)\Theta'^i_{\upsilon'} \hat{\gamma}^{\upsilon'\upsilon}(\tfrac{o\mu}{i\bar{\imath}})\bar{\Theta}^{\bar{\imath}}_\upsilon,$$
$$\Theta^o_f + \Theta'^o_f, \bar{\Theta}^{\bar{o}}_f + \Theta'^{\bar{o}}_f, \Theta^i_\upsilon + \Theta'^i_\upsilon, \bar{\Theta}^{\bar{\imath}}_\upsilon + \bar{\Theta}'^{\bar{\imath}}_\upsilon). \quad (3.\ 36)$$

The representations associated with $\Phi_{(1)}$ and $\Phi_{(2)}$ are said to be given **in the chiral** and **in the anti-chiral basis**, respectively. The corresponding representations of the symmetry generators and covariant derivatives as differential operators acting on $\Phi_{(1)}$ and $\Phi_{(2)}$ turn out to be:

$$\begin{aligned}
(\delta_{F^f_{os}}\Phi)_{(1)} &:= (\partial_{\Theta^{os}_f})\Phi_{(1)},\\
(\delta_{\bar{F}^f_{\bar{o}\dot{t}}}\Phi)_{(1)} &:= (-\partial_{\bar{\Theta}^{\bar{o}\dot{t}}_f} - 2\mathrm{i}\Theta^{os}_{f'}\hat{\Upsilon}^{f'f}(\tfrac{o\mu}{o\bar{o}})_{s\dot{t}}\partial_\mu)\Phi_{(1)},\\
(\delta_{V^\upsilon_{is}}\Phi)_{(1)} &:= (\partial_{\Theta^{is}_\upsilon})\Phi_{(1)},\\
(\delta_{\bar{V}^\upsilon_{\bar{\imath}\dot{t}}}\Phi)_{(1)} &:= (-\partial_{\bar{\Theta}^{\bar{\imath}\dot{t}}_\upsilon} - 2\mathrm{i}\Theta^{is}_{\upsilon'}\hat{\gamma}^{\upsilon'\upsilon}(\tfrac{o\mu}{i\bar{\imath}})_{s\dot{t}}\partial_\mu)\Phi_{(1)}\\
(D^f_{os}\Phi)_{(1)} &:= (\partial_{\Theta^{os}_f} - 2\mathrm{i}\hat{\Upsilon}^{ff'}(\tfrac{o\mu}{o\bar{o}})_{s\dot{t}}\bar{\Theta}^{\bar{o}\dot{t}}_{f'}\partial_\mu)\Phi_{(1)},\\
(\bar{D}^f_{\bar{o}\dot{t}}\Phi)_{(1)} &:= (-\partial_{\bar{\Theta}^{\bar{o}\dot{t}}_f})\Phi_{(1)},\\
(D^\upsilon_{is}\Phi)_{(1)} &:= (\partial_{\Theta^{is}_\upsilon} - 2\mathrm{i}(2\delta^\mu_o + \delta^\mu_i - 1)\hat{\gamma}^{\upsilon\upsilon'}(\tfrac{o\mu}{i\bar{\imath}})_{s\dot{t}}\bar{\Theta}^{\bar{\imath}\dot{t}}_{\upsilon'}\partial_\mu)\Phi_{(1)},\\
(\bar{D}^\upsilon_{\bar{\imath}\dot{t}}\Phi)_{(1)} &:= (-\partial_{\bar{\Theta}^{\bar{\imath}\dot{t}}_\upsilon})\Phi_{(1)}.
\end{aligned} \quad (3.\ 37)$$

$$\begin{aligned}
(\delta_{F^f_{os}}\Phi)_{(2)} &:= (\partial_{\Theta^{os}_f} + 2\mathrm{i}\hat{\Upsilon}^{ff'}(\tfrac{o\mu}{o\bar{o}})_{s\dot{t}}\bar{\Theta}^{\bar{o}\dot{t}}_{f'}\partial_\mu)\Phi_{(2)},\\
(\delta_{\bar{F}^f_{\bar{o}\dot{t}}}\Phi)_{(2)} &:= (-\partial_{\bar{\Theta}^{\bar{o}\dot{t}}_f})\Phi_{(2)},\\
(\delta_{V^\upsilon_{is}}\Phi)_{(2)} &:= (\partial_{\Theta^{is}_\upsilon} + 2\mathrm{i}(2\delta^\mu_o + \delta^\mu_i - 1)\hat{\gamma}^{\upsilon\upsilon'}(\tfrac{o\mu}{i\bar{\imath}})_{s\dot{t}}\bar{\Theta}^{\bar{\imath}\dot{t}}_{\upsilon'}\partial_\mu)\Phi_{(2)},\\
(\delta_{\bar{V}^\upsilon_{\bar{\imath}\dot{t}}}\Phi)_{(2)} &:= (-\partial_{\bar{\Theta}^{\bar{\imath}\dot{t}}_\upsilon})\Phi_{(2)}\\
(D^f_{os}\Phi)_{(2)} &:= (\partial_{\Theta^{os}_f})\Phi_{(2)},\\
(\bar{D}^f_{\bar{o}\dot{t}}\Phi)_{(2)} &:= (-\partial_{\bar{\Theta}^{\bar{o}\dot{t}}_f} + 2\mathrm{i}\Theta^{os}_{f'}\hat{\Upsilon}^{f'f}(\tfrac{o\mu}{o\bar{o}})_{s\dot{t}}\partial_\mu)\Phi_{(2)},\\
(D^\upsilon_{is}\Phi)_{(2)} &:= (\partial_{\Theta^{is}_\upsilon})\Phi_{(2)},\\
(\bar{D}^\upsilon_{\bar{\imath}\dot{t}}\Phi)_{(2)} &:= (-\partial_{\bar{\Theta}^{\bar{\imath}\dot{t}}_\upsilon} + 2\mathrm{i}\Theta^{is}_{\upsilon'}\hat{\gamma}^{\upsilon'\upsilon}(\tfrac{o\mu}{i\bar{\imath}})_{s\dot{t}}\partial_\mu)\Phi_{(2)}.
\end{aligned} \quad (3.\ 38)$$

Analogously to the superspace formalism of supersymmetric models [11], we can consider the space-time, the fermionic and the para-fermionic parameters on the same vein. Hence,

$$(x^\mu, \Theta^{os}_f, \bar{\Theta}^{\bar{o}\dot{t}}_f, (\Theta^{1s}_\upsilon, \Theta^{2s}_\upsilon, \Theta^{3s}_\upsilon), (\bar{\Theta}^{\bar{1}\dot{t}}_\upsilon, \bar{\Theta}^{\bar{2}\dot{t}}_\upsilon, \bar{\Theta}^{\bar{3}\dot{t}}_\upsilon)) \quad (3.\ 39)$$

provides the coordinates of the $(I,\,q)$-graded superspace associated to the algebraic extension in (2. 30)-(2. 46). The operator-valued distribution $\Phi(x, \Theta^{o}_{f}, \bar{\Theta}^{\bar{o}}_{f}, \Theta^{i}_{v}, \bar{\Theta}^{\bar{i}}_{v})$ transforming according to (3. 17), (3. 35) or (3. 36) is called an $(I,\,q)$-graded superfield representation of this algebraic extension. The quantization relations for the corresponding component fields will be discussed in [6].

4 Field content of superfield representations of $\mathbb{Z}_2 \times (\mathbb{Z}_4 \times \mathbb{Z}_4)$-graded extensions

We want to start a preliminary discussion about the possible structure of supermultiplet representations of the graded extension in (2. 30)-(2. 46).

We observe first that the space-time translation generators P_μ have vanishing generalized commutators with the symmetry charges F^{f}_{o}, $\bar{F}^{f}_{\bar{o}}$, V^{v}_{i}, $\bar{V}^{v}_{\bar{i}}$. This might indicate that irreducible representations are of equal mass. Nevertheless, the consequences of having reducible momentum representations for these spin-$\frac{1}{2}$ charges should be studied carefully. Remember that the structure of $(I,\,q)$-graded Lie algebras over $\mathbb{C}$ sets conditions on the texture and dimensions of momentum representations that lead to the usage of spin-$\frac{1}{2}$ quadruplets [2].

Much as has been done for supersymmetric extensions, we would like to have counting operators for bosonic, para-bosonic, fermionic and para-fermionic one-particle states. For the standard (anticommuting) supersymmetry charges we can define an operator $(-1)^{N_f}$ that simply reproduces the properties of a constant fermionic parameter Θ^{oo}. In this case, we can find a relation between the number of one-particle states associated with fields of self-fermionic components and one-particle states associated with fields of self-bosonic components. For finite-dimensional representations (in order to have a naive definition of the trace) we have:

$$\mathrm{Tr}\{(-1)^{N_f}[\![F^{f}_{os}, \bar{F}^{f'}_{\bar{o}\dot{t}}]\!]\} = 2\hat{\Upsilon}^{ff'}(\tfrac{o\mu}{o\bar{o}})_{s\dot{t}}\ \mathrm{Tr}\{(-1)^{N_f}P_\mu\} = 0. \tag{4. 1}$$

Hence, for degenerate masses, an equal number of one-particle states for self-fermionic fields and self-bosonic fields would be expected. The task of constructing counting operators for para-fermionic states is not trivial, since in this case the generalized commutator (see table 4)

$$[\![V^{v}_{is}, \bar{V}^{v'}_{\bar{i}\dot{t}}]\!] \tag{4. 2}$$

involves simultaneously commuting and anticommuting parts. We have to determine as well a suitable counting procedure when several couples of extending symmetry charges are present. Although these particularities have been identified, we could not exclude so far the existence of the searched counting operators.

These problems might be circumvented by considering only fundamental spin-$\frac{1}{2}$ representations for the para-fermionic symmetry charges, and that the parameters associated to each para-fermionic symmetry charge anticommute among them. These conditions would make it trivial to construct operators counting para-fermionic states, but goes beyond the possibities of the kind of extensions considered here.

The study of the component field spectrum in a supermultiplet thus merits further discussion.

5 Covariant constraints

Since the Θ-parameters introduced above have self-fermionic behaviour, they are nilpotent and thus generate only polynomial expansions of bounded degree.

We will constraint our considerations –for simplicity– to the case in which there are no fermionic symmetry charges, and there are only two para-fermionic symmetry charges: V_i and $\bar{V}_{\bar{\imath}}$ for i fixed (which are actually the novelty of the present formalism). A generic superfield might be expanded in the following way:

$$\Phi(x^\mu, \Theta^{is}, \bar{\Theta}^{\bar{\imath}\dot{s}}) = \\ = \phi^{(0,0)}(x) + \Theta^{is}\phi_s^{(1,0)}(x) + \phi_{\dot{s}}^{(0,1)}(x)\bar{\Theta}^{\bar{\imath}\dot{s}} + \cdots + (\Theta^i\Theta^i)^2\phi^{(4,4)}(x)\,(\bar{\Theta}^{\bar{\imath}}\bar{\Theta}^{\bar{\imath}})^2, \quad (5.\ 1)$$

where the $\phi^{(n,m)}(x)$ are ordinary fields (including para-fermionic representations), the so-called **component fields** of the given superfield.

If we act with a covariant derivative D_{is} or $\bar{D}_{\bar{\imath}\dot{s}}$ on the superfield, then we obtain again a superfield. This nice property allows for the definition of covariant constraints. These constraints must not yield differential equations in space-time coordinates. We can mention the following examples of constrained superfields:

- $\bar{A}$ is called an anti-chiral superfield, if

$$\varsigma_{\rm st}(\bar{A}) = \tilde{o}, \quad D_{is}\bar{A} = 0\,;\ s \in \{0,1,2,3\}. \quad (5.\ 2)$$

- A is called a chiral superfield, if

$$\varsigma_{\rm st}(A) = \tilde{o}, \quad \bar{D}_{\bar{\imath}\dot{s}}A = 0\,;\ \dot{s} \in \{\dot{0},\dot{1},\dot{2},\dot{3}\}. \quad (5.\ 3)$$

- Φ is called a real superfield, if

$$\varsigma_{\rm st}(\Phi) = \tilde{o}, \quad \bar{\Phi} = \Phi. \quad (5.\ 4)$$

The summation and product of superfields of the same type and representations preserve the latter constraints.

The symmetry transformations in the chiral basis have been listed in (3. 37). According to them, a chiral superfield A in chiral basis fulfils

$$\partial_{\bar{\Theta}^{\bar{\imath}\dot{s}}}A_{(1)} = 0, \quad A_{(1)} = A_{(1)}(x, \Theta^i) \ \text{ chiral superfield.} \quad (5.\ 5)$$

In an analogous way, in the anti-chiral basis, an anti-chiral superfield $\bar{A}$ fulfils

$$\partial_{\Theta^{is}}\bar{A}_{(2)} = 0, \quad \bar{A}_{(2)} = \bar{A}_{(2)}(x, \bar{\Theta}^{\bar{\imath}}) \ \text{ anti-chiral superfield.} \quad (5.\ 6)$$

Observe that, if A is a chiral superfield, then $\overline{(A)} = \bar{A}$ is an anti-chiral superfield. Observe as well that

$$A \ \text{ chiral superfield} \Longrightarrow D_iD_i\ D_iD_i\ A \ \text{ anti-chiral superfield,} \quad (5.\ 7)$$

$$\bar{A} \ \text{ anti-chiral superfield} \Longrightarrow \bar{D}_{\bar{\imath}}\bar{D}_{\bar{\imath}}\ \bar{D}_{\bar{\imath}}\bar{D}_{\bar{\imath}}\ \bar{A} \ \text{ chiral superfield.} \quad (5.\ 8)$$

We might try to construct a kinetic term for chiral fields using an expression of the form

$$\int d^4x\, D_iD_i\ D_iD_i\ A\ \bar{D}_{\bar{\imath}}\bar{D}_{\bar{\imath}}\ \bar{D}_{\bar{\imath}}\bar{D}_{\bar{\imath}}\ \bar{A}. \quad (5.\ 9)$$

We observe, nevertheless, that this adoption seems to produce bilinear terms involving cubic and quartic derivatives. We can test the effect of the symmetry charges on the component fields using a kinetic term of the form given in (5. 9). We then arrive to the conclusion that, with the exclusion of the component fields which might include higher derivatives the validity of the generalized Jacobi associativity cannot be verified. It should be determined if these components have to be included as well in order to verify the Ward-identities associated with the extended symmetry.

We then expect, that either a very smart choice of the field components is allowed in order to compensate the terms with higher derivatives without breaking the external symmetry, or we shall be confronted with unitarity problems. We do not know either if further superfield types could provide more appropriate models for matter fields.

Observe that the possible presence of higher derivative terms in (5. 9) comes from the integration over eight different Θ-parameters, instead of integration over four as in the standard supersymmetric chiral models. This fact is then directly associated with the adoption of quadruplets for spin-$\frac{1}{2}$ charges instead of fundamental spin-$\frac{1}{2}$ doublets.

6 Para-fermionic fields

We can make use of the discussed candidate for the kinetic term in (5. 9) to obtain which might be the Lagrangean and the equations of motion for a para-fermionic component field. We consider a chiral superfield A and its anti-chiral adjoint $\bar{A}$ of the form:

$$\begin{aligned} A_{(1)}(x,\Theta^i) &= \cdots + (\Theta^i\Theta^i)\mathcal{A}(x) + (\Theta^i\Theta^i)\Theta^{is}2\psi_{is}(x) + (\Theta^i\Theta^i)^2F(x), \\ \bar{A}_{(2)}(x,\bar{\Theta}^{\bar{\imath}}) &= \cdots + \bar{\mathcal{A}}(x)(\bar{\Theta}^{\bar{\imath}}\bar{\Theta}^{\bar{\imath}}) + 2\bar{\psi}_{\bar{\imath}\dot{s}}(x)\bar{\Theta}^{\bar{\imath}\dot{s}}(\bar{\Theta}^{\bar{\imath}}\bar{\Theta}^{\bar{\imath}}) + \bar{F}(x)\,(\bar{\Theta}^{\bar{\imath}}\bar{\Theta}^{\bar{\imath}})^2. \end{aligned} \tag{6. 1}$$

We construct a chiral superfield out of the anti-chiral superfield $\bar{A}$ using (5. 8):

$$\begin{aligned} &(\bar{D}_{\bar{\imath}}\bar{D}_{\bar{\imath}}\ \bar{D}_{\bar{\imath}}\bar{D}_{\bar{\imath}}\ \bar{A})_{(2)} = \\ &= \{1 - 2\mathrm{i}(2\delta^\mu_i + 2\delta^\mu_o - 1)(\Theta^i\hat{\gamma}(\tfrac{o\mu}{i\bar{\imath}})\bar{\Theta}^{\bar{\imath}})\partial_\mu - 2(\Theta^i\hat{\gamma}(\tfrac{o\mu}{i\bar{\imath}})(\Theta^i\hat{\gamma}(\tfrac{o\nu}{i\bar{\imath}})\partial_\mu\partial_\nu + \\ &+ 2\mathrm{i}(2\delta^\mu_i + 2\delta^\mu_o - 1)(\Theta^i\Theta^i)(\bar{\Theta}^{\bar{\imath}}\bar{\Theta}^{\bar{\imath}})(\Theta^i\hat{\gamma}(\tfrac{o\mu}{i\bar{\imath}})\bar{\Theta}^{\bar{\imath}})\partial_\mu\Box + \tfrac{1}{4}(\Theta^i\Theta^i)^2(\bar{\Theta}^{\bar{\imath}}\bar{\Theta}^{\bar{\imath}})^2\Box^2\}\cdot \\ &\cdot\{64\bar{F}(x) - \Theta^{ir}64\mathrm{i}\hat{\gamma}(\tfrac{o\mu}{i\bar{\imath}})_{rt}\partial_\mu\bar{\psi}_{\bar{\imath}}^{\ \dot{t}}(x) - (\Theta^i\Theta^i)32\Box\bar{\mathcal{A}}(x) + \cdots\} = \\ &= \exp\{-2\mathrm{i}(2\delta^\mu_i + 2\delta^\mu_o - 1)(\Theta^i\hat{\gamma}(\tfrac{o\mu}{i\bar{\imath}})\bar{\Theta}^{\bar{\imath}})\partial_\mu\}\cdot \\ &\cdot\{64\bar{F}(x) - \Theta^{ir}64\mathrm{i}\hat{\gamma}(\tfrac{o\mu}{i\bar{\imath}})_{rt}\partial_\mu\bar{\psi}_{\bar{\imath}}^{\ \dot{t}}(x) - (\Theta^i\Theta^i)32\Box\bar{\mathcal{A}}(x) + \cdots\}. \end{aligned} \tag{6. 2}$$

We can use the relation among the chiral and the anti-chiral bases in (3. 34) to present the same result in chiral basis:

$$\begin{aligned} &(\bar{D}_{\bar{\imath}}\bar{D}_{\bar{\imath}}\ \bar{D}_{\bar{\imath}}\bar{D}_{\bar{\imath}}\ \bar{A})_{(1)} = \\ &= \{64\bar{F}(x) - \Theta^{is}64\mathrm{i}\hat{\gamma}(\tfrac{o\mu}{i\bar{\imath}})_{st}\partial_\mu\bar{\psi}_{\bar{\imath}}^{\ \dot{t}}(x) - (\Theta^i\Theta^i)32\Box\bar{\mathcal{A}}(x) + \cdots\}. \end{aligned} \tag{6. 3}$$

Hence, we can write

$$\begin{aligned} \Gamma_{kin} = &\ \frac{1}{2^{11}}\int d^4x\ D_iD_i\ D_iD_i\ A\ \bar{D}_{\bar{\imath}}\bar{D}_{\bar{\imath}}\ \bar{D}_{\bar{\imath}}\bar{D}_{\bar{\imath}}\ \bar{A} = \\ &= \int d^4x\ \{2F\bar{F} - \mathrm{i}\psi_i^{\ s}\hat{\gamma}(\tfrac{o\mu}{i\bar{\imath}})_{st}\partial_\mu\bar{\psi}_{\bar{\imath}}^{\ \dot{t}} + \partial_\mu\mathcal{A}\partial^\mu\bar{\mathcal{A}} + \cdots\} = \\ &= \int d^4x\ \{2\bar{F}F - \mathrm{i}\bar{\psi}_{\bar{\imath}\dot{t}}\hat{\gamma}(\tfrac{o\mu}{\bar{\imath}i})^{\dot{t}s}\partial_\mu\psi_{is} + \partial_\mu\bar{\mathcal{A}}\partial^\mu\mathcal{A} + \cdots\}. \end{aligned} \tag{6. 4}$$

We have obtained a kinetic term for a complex massless scalar field $\mathcal{A}(x)$ and what might correspond to a kinetic term for a massless para-fermionic field $\psi_i(x)$. The component F turns out to be an auxiliary field.

The above construction provides an action for a para-fermionic Field. We can now consider an action for a massive para-fermionic field:

$$\Gamma_M = \int d^4x \ \{-\mathrm{i}\bar{\psi}_{\bar{\imath}\dot{t}}\hat{\gamma}(\tfrac{o\mu}{\bar{\imath}i})^{\dot{t}s}\partial_\mu\psi_{is} - \tfrac{1}{2}m(\psi_i\psi_i + \bar{\psi}_{\bar{\imath}}\bar{\psi}_{\bar{\imath}})\}. \tag{6. 5}$$

The corresponding equations of motion can be written in the following way

$$\left\{\mathrm{i}\begin{bmatrix} 0 & \hat{\gamma}(\frac{o\mu}{i\bar{\imath}})_{r\dot{u}} \\ \hat{\gamma}(\frac{o\mu}{\bar{\imath}i})^{\dot{t}s} & 0 \end{bmatrix}\partial_\mu + \begin{bmatrix} \delta_r{}^s & 0 \\ 0 & \delta^{\dot{t}}{}_{\dot{u}} \end{bmatrix}\right\}\begin{bmatrix} \psi_{is} \\ \bar{\psi}_{\bar{\imath}}{}^{\dot{u}} \end{bmatrix} = 0. \tag{6. 6}$$

The latter constitutes a system of coupled differential equations, which leads to the conditions:

$$(\Box + m^2)\psi_{ir} = 0, \ \ (\Box + m^2)\bar{\psi}_{\bar{\imath}}{}^{\dot{t}} = 0. \tag{6. 7}$$

Hence, each component of the massive para-fermionic multiplet fulfils a Klein-Gordon equation. Observe that the adjunction defined by

$$\bar{\Psi}_M \equiv \overline{\begin{bmatrix} \psi_{is} \\ \bar{\psi}_{\bar{\imath}}{}^{\dot{u}} \end{bmatrix}} := [\psi_i{}^t \ \ \bar{\psi}_{\bar{\imath}\dot{r}}] = [\psi_{is} \ \ \bar{\psi}_{\bar{\imath}}{}^{\dot{s}}]\begin{bmatrix} \epsilon^{ts} & 0 \\ 0 & \bar{\epsilon}_{\dot{s}\dot{r}} \end{bmatrix} \tag{6. 8}$$

allows for the construction of a Lorentz-invariant bilinear form (already used in the mass term)

$$\bar{\Psi}_M\Psi_M = \psi_i\psi_i + \bar{\psi}_{\bar{\imath}}\bar{\psi}_{\bar{\imath}}. \tag{6. 9}$$

According to the definition (6. 8), the column vector Ψ_M is linearly related to its adjoint $\bar{\Psi}_M$. We can thus consider the expression in (6. 6) as the equation of motion of a Majorana-like para-fermionic field.

We can easily construct an action for the Dirac-like para-fermionic field:

$$\Gamma_D = \int d^4x \ \{-\mathrm{i}\bar{\psi}_{\bar{\imath}\dot{t}}\hat{\gamma}(\tfrac{o\mu}{\bar{\imath}i})^{\dot{t}s}\partial_\mu\psi_{is} - \mathrm{i}\bar{\chi}_{\bar{\imath}\dot{t}}\hat{\gamma}(\tfrac{o\mu}{\bar{\imath}i})^{\dot{t}s}\partial_\mu\chi_{is} - m(\chi_i\psi_i + \bar{\chi}_{\bar{\imath}}\bar{\psi}_{\bar{\imath}})\}. \tag{6. 10}$$

The equations of motions are accordingly

$$\left\{\mathrm{i}\begin{bmatrix} 0 & \hat{\gamma}(\frac{o\mu}{i\bar{\imath}})_{r\dot{u}} \\ \hat{\gamma}(\frac{o\mu}{\bar{\imath}i})^{\dot{t}s} & 0 \end{bmatrix}\partial_\mu + \begin{bmatrix} \delta_r{}^s & 0 \\ 0 & \delta^{\dot{t}}{}_{\dot{u}} \end{bmatrix}\right\}\begin{bmatrix} \psi_{is} \\ \bar{\chi}_{\bar{\imath}}{}^{\dot{u}} \end{bmatrix} = 0. \tag{6. 11}$$

We obtain again suitable on-shell conditions for the field components

$$(\Box + m^2)\psi_{ir} = 0, \ \ (\Box + m^2)\bar{\chi}_{\bar{\imath}}{}^{\dot{t}} = 0. \tag{6. 12}$$

In the present case, the column vector in (6. 11) is not linearly related to its adjoint:

$$\bar{\Psi}_D \equiv \overline{\begin{bmatrix} \psi_{is} \\ \bar{\chi}_{\bar{\imath}}{}^{\dot{u}} \end{bmatrix}} := [\chi_i{}^t \ \ \bar{\psi}_{\bar{\imath}\dot{r}}] = [\chi_{is} \ \ \bar{\psi}_{\bar{\imath}}{}^{\dot{s}}]\begin{bmatrix} \epsilon^{ts} & 0 \\ 0 & \bar{\epsilon}_{\dot{s}\dot{r}} \end{bmatrix}. \tag{6. 13}$$

Equation (6. 11) doubled the degrees of freedom of the Majorana-like equation (6. 6).

If we want to associate any probabilistic interpretation to the para-fermionic fields, we have to find a non-negative norm, conserved by the time evolution.

The adjoint form of eq. (6. 11) is given by

$$\mathrm{i}(2\delta_{\mu i} + 2\delta_{\mu o} - 1)[\partial_\mu\chi_i{}^t \ \ \partial_\mu\bar{\psi}_{\bar{\imath}\dot{r}}]\begin{bmatrix} 0 & \hat{\gamma}(\frac{o\mu}{i\bar{\imath}})_{t\dot{u}} \\ \hat{\gamma}(\frac{o\mu}{\bar{\imath}i})^{\dot{r}s} & 0 \end{bmatrix} - m[\chi_i{}^s \ \ \bar{\psi}_{\bar{\imath}\dot{u}}] = 0. \tag{6. 14}$$

By multiplying eq. (6. 11) by $[\chi_i{}^s \;\; \bar{\psi}_{\bar{\imath}\dot{u}}]$ from the left and eq. (6. 14) by $[\psi_{is} \;\; \bar{\psi}_{\bar{\imath}}{}^{\dot{u}}]^T$ from the right, and adding the results we obtain:

$$\partial_\mu \left\{ (2\delta_i^\mu + 2\delta_o^\mu - 1)[\chi_i{}^t \;\; \bar{\psi}_{\bar{\imath}\dot{r}}] \begin{bmatrix} 0 & \hat{\gamma}(\tfrac{o\mu}{\bar{\imath}i})_{t\dot{u}} \\ \hat{\gamma}(\tfrac{o\mu}{\bar{\imath}i})^{\dot{r}s} & 0 \end{bmatrix} \begin{bmatrix} \psi_{is} \\ \bar{\chi}_{\bar{\imath}}{}^{\dot{u}} \end{bmatrix} \right\} =$$
$$= \partial_\mu \{\chi_i{}^t \hat{\gamma}(\tfrac{o\mu}{\bar{\imath}i})_{t\dot{u}} \bar{\chi}_{\bar{\imath}}{}^{\dot{u}} + \bar{\psi}_{\bar{\imath}\dot{r}} \hat{\gamma}(\tfrac{o\mu}{\bar{\imath}i})^{\dot{r}s} \psi_{is}\}(2\delta_i^\mu + 2\delta_o^\mu - 1) = \partial_\mu J^\mu = 0. \qquad (6.\ 15)$$

There is thus a conserved current

$$J^\mu := (2\delta_i^\mu + 2\delta_o^\mu - 1)\{\chi_i{}^t \hat{\gamma}(\tfrac{o\mu}{\bar{\imath}i})_{t\dot{u}} \bar{\chi}_{\bar{\imath}}{}^{\dot{u}} + \bar{\psi}_{\bar{\imath}\dot{r}} \hat{\gamma}(\tfrac{o\mu}{\bar{\imath}i})^{\dot{r}s} \psi_{is}\}, \qquad (6.\ 16)$$

whose time-component is a positive density

$$\rho := \sum_s \overline{(\psi_{is})} \psi_{is} + \sum_{\dot{t}} \overline{(\bar{\chi}_{\bar{\imath}}{}^{\dot{t}})} \bar{\chi}_{\bar{\imath}}{}^{\dot{t}}. \qquad (6.\ 17)$$

The current J^μ transforms as a Lorentz four-vector.

7 Conclusions and open questions

We have obtained here the following results:

- We introduced a $\mathbb{Z}_2 \times (\mathbb{Z}_4 \times \mathbb{Z}_4)$-graded extension of the Poincaré algebra using the standard Hamel basis $\{M^{\mu\nu}\}$ for the Lorentz subalgebra. We determined the form of the bilinear invariants for spin $(\frac{1}{2}, 0)$ and for spin $(0, \frac{1}{2})$ fermionic and para-fermionic representations. We determined the metric matrices associated with the lowering and rising of multiplet-component indices.

- We developed the superspace formalism which involves para-Grassmann parameters. We defined the superspace associated with a $\mathbb{Z}_2 \times (\mathbb{Z}_4 \times \mathbb{Z}_4)$-graded extension of the Poincaré algebra. We constructed representations of the group elements with the help of exponential mappings and the usage of superspace (or group) parameters. We studied the product (from the left and from the right) of group elements and determined their transformation effect on the group parameters.

- We obtained representations of the generators of the considered graded algebra as differential operators acting on functions of the superspace parameters. We obtained covariant derivatives as well. The superfields are defined to transform in the same way as the group elements (under group transformations). We determined three particular representations for group elements and superfields, which are the so-called real, chiral and anti-chiral bases. We determined the relations among them.

- We addressed the question about the classification of the superfield representations. We determined severe difficulties in providing a definition of an operator counting para-fermionic one-particle states. This problem appears to be related with the mixed (commutator-anticommutator) nature of the generalized commutator between the components of each para-Fermionic symmetry charge. This feature is in turn related with the usage of non-fundamental spin-$\frac{1}{2}$ representations for the symmetry charges.

- We studied covariant constraints on superfields using covariant derivatives. We observed the origin of a further drawback of the considered algebraic extensions since higher derivative terms can appear in constructing the simplest invariant actions for chiral models.

  We have not excluded, in any case, the possibility that a very careful choice of field components or the use of further superfield types could get rid of the undesired cubic and quartic derivatives in the kinetic terms without breaking the graded external symmetry.

  The possible appearance of higher derivative terms seems to be related with the usage of non-fundamental spin-$\frac{1}{2}$ representations for symmetry charges. We associated four independent (para-) Grassmann parameters with each non-fundamental spin-$\frac{1}{2}$ (para-) fermionic multiplet of symmetry generators. In contrast, the usage of fundamental spin-$\frac{1}{2}$ multiplets would require only two independent parameters for each (para-) fermionic symmetry charge.

- We obtained, as a by-product of the presented superspace formalism, the action and the equation of motion for a free para-fermionic field. We verified that the multiplet components fulfil adequate (Klein-Gordon) on-shell conditions. We derived Majorana- and Dirac-like equations of motion for para-fermions. We verified the existence of a positive probability density, together with a continuity equation for such para-fermionic fields. This construction might be the analogue of what is expected when using fundamental spin-$\frac{1}{2}$ representations for para-fermions.

The construction of a superspace formalism involving parameters which do not necessarily commute with all the space-time parameters has been developed here and seems to present no mathematical obstacle. From the physical point of view, nevertheless, we determined several indications that the physical meaningful graded extensions might be more properly related with graded extensions using fundamental spin-$\frac{1}{2}$ representations for fermionic and para-fermionic symmetry charges. This would avoid, on the one hand, the appearance of higher derivatives in the kinetic terms. On the other hand, it would allow for the definition of counting operators for para-fermionic one-particle states [6]. In a deeper sense, the results of this paper indicate several limits for usage of the $(I,\, q)$-graded extensions over $\mathbb{C}$ of the Poincaré algebra. It furthermore indicates which might be the adequate structure of the graded algebraic extensions that leads to meaningful physical models [10].

It should be investigated as well, if there is a symmetry structure among the fermionic and para-fermionic charges, i.e. among the different multiplet classes. The self-fermionic parameters might be written suggestively in the form of quadrupets of quadruplets $(\Theta^{os}, \Theta^{1s}, \Theta^{2s}, \Theta^{3s})$, $(\bar\Theta^{\bar o t}, \bar\Theta^{\bar 1 t}, \bar\Theta^{\bar 2 t}, \bar\Theta^{\bar 3 t})$, to which we might associate novel symmetry properties.

Acknowledgements

I would like to thank the Theory Division of CERN for the kind hospitality, and Prof. G. Veneziano for the invitation.

Appendices

A $(I,\, q)$-graded Lie algebra over a commutative field $\mathbb{K}$

The study of parameters with generalized commutative behaviour $\beta_{\tilde a}, \beta'_{\tilde e}, \cdots$ fulfilling

$$\beta_{\tilde a}\beta'_{\tilde e} = q_{\tilde a, \tilde e}\beta'_{\tilde e}\beta_{\tilde a}, \tag{A.1}$$

where $q_{\tilde{a},\tilde{e}} \in I\!\!K \setminus \{0\}$, leads to the determination [13] of the following graded algebraic structure: The set $I\!\!L$ of operators with generalized commutative behaviour is called an (I, q)-graded Lie algebra over a commutative field $I\!\!K$ if besides $I\!\!L$ we have

- a "statistic" function $\varsigma_{\rm st}|_{I\!\!L}$,
- an index set $I \subset I\!\!I$, $I\!\!I$ an Abelian additive group,
- a function $q|_{I\times I}$,
- a rule of composition $[\![\cdot,\cdot]\!]$,

fulfilling the axioms IQ0-IQ4:

IQ0 The function $\varsigma_{\rm st}|_{I\!\!L}$ is a surjective map from $I\!\!L$ into I, and the set of pre-images of each $\tilde{a} \in I$, i.e. $\varsigma_{\rm st}|_{I\!\!L}^{-1}(\tilde{a})$, is a vector space over $I\!\!K$:

$$\varsigma_{\rm st}|_{I\!\!L} : I\!\!L \longrightarrow I\,;\, \mathcal{O}_{\tilde{a}} \mapsto \varsigma_{\rm st}(\mathcal{O}_{\tilde{a}}) = \tilde{a}, \tag{A.2}$$

$$I\!\!L_{\tilde{a}} := \varsigma_{\rm st}|_{I\!\!L}^{-1}(\tilde{a})\,;\, \tilde{a} \in I, \tag{A.3}$$

$$(I\!\!L_{\tilde{a}}; +, .)\,;\, \tilde{a} \in I \text{ vector spaces over } I\!\!K. \tag{A.4}$$

IQ1 The $[\![\cdot,\cdot]\!]$-product is an internal and I-graded operation in $I\!\!L$:

$$[\![\cdot,\cdot]\!]\; I\!\!L \times I\!\!L \longrightarrow I\!\!L\,;\, (\mathcal{O}_{\tilde{a}}, \mathcal{O}'_{\tilde{e}}) \mapsto [\![\mathcal{O}_{\tilde{a}}, \mathcal{O}'_{\tilde{e}}]\!] \in I\!\!L, \tag{A.5}$$

$$(\tilde{a}, \tilde{e} \in I) \text{ and } (\tilde{a}+\tilde{e} \notin I) \Longrightarrow [\![I\!\!L_{\tilde{a}}, I\!\!L_{\tilde{e}}]\!] = \{0\}, \tag{A.6}$$

$$(\tilde{a}, \tilde{e} \in I) \text{ and } (\tilde{a}+\tilde{e} \in I) \Longrightarrow [\![I\!\!L_{\tilde{a}}, I\!\!L_{\tilde{e}}]\!] \subset I\!\!L_{\tilde{a}+\tilde{e}}, \tag{A.7}$$

where

$$+|_{I\times I} : I \times I \longrightarrow I\!\!I \supset I\,;\, (\tilde{a}, \tilde{e}) \mapsto \tilde{a}+\tilde{e} = \tilde{e}+\tilde{a}, \tag{A.8}$$

$$(I\!\!I; +) \text{ Abelian group .} \tag{A.9}$$

IQ2 The $[\![\cdot,\cdot]\!]$-product is bilinear with respect to the addition operation defined in each vector space $(I\!\!L_{\tilde{a}}; +, \cdot)\,;\, \tilde{a} \in I$:

$$\begin{aligned} [\![y\mathcal{O}_{\tilde{a}} + y''\mathcal{O}''_{\tilde{a}}, \mathcal{O}'_{\tilde{e}}]\!] &= y[\![\mathcal{O}_{\tilde{a}}, \mathcal{O}'_{\tilde{e}}]\!] + y''[\![\mathcal{O}''_{\tilde{a}}, \mathcal{O}'_{\tilde{e}}]\!], \\ [\![\mathcal{O}'_{\tilde{e}}, y\mathcal{O}_{\tilde{a}} + y''\mathcal{O}''_{\tilde{a}}]\!] &= y[\![\mathcal{O}'_{\tilde{e}}, \mathcal{O}_{\tilde{a}}]\!] + y''[\![\mathcal{O}'_{\tilde{e}}, \mathcal{O}''_{\tilde{a}}]\!], \end{aligned} \tag{A.10}$$

for all $(y, y'') \in I\!\!K \times I\!\!K$ and $(\tilde{a}, \tilde{e}) \in I \times I$ and $(\mathcal{O}_{\tilde{a}}, \mathcal{O}''_{\tilde{a}}, \mathcal{O}'_{\tilde{e}}) \in I\!\!L_{\tilde{a}} \times I\!\!L_{\tilde{a}} \times I\!\!L_{\tilde{e}}$.

IQ3 The $[\![\cdot,\cdot]\!]$-product is generalized antisymmetric:

$$[\![\mathcal{O}_{\tilde{a}}, \mathcal{O}'_{\tilde{e}}]\!] = -q_{\tilde{a},\tilde{e}}[\![\mathcal{O}'_{\tilde{e}}, \mathcal{O}_{\tilde{a}}]\!], \tag{A.11}$$

for all $(\tilde{a}, \tilde{e}) \in I \times I$ and $(\mathcal{O}_{\tilde{a}}, \mathcal{O}'_{\tilde{e}}) \in I\!\!L_{\tilde{a}} \times I\!\!L_{\tilde{e}}$, and where

$$q|_{I\times I} : I \times I \longrightarrow I\!\!K \setminus \{0\}\,;\, (\tilde{a}, \tilde{e}) \mapsto q(\tilde{a}, \tilde{e}) =: q_{\tilde{a},\tilde{e}}, \tag{A.12}$$

fulfils

$$q_{\tilde{a},\tilde{e}} = q^{-1}_{\tilde{e},\tilde{a}}, \tag{A.13}$$

$$q_{\tilde{a}+\tilde{e},\tilde{c}} = q_{\tilde{a},\tilde{c}}\, q_{\tilde{e},\tilde{c}}. \tag{A.14}$$

IQ4 The $[\![\cdot,\cdot]\!]$-product is generalized Jacobi-associative:

$$q_{\tilde{c},\tilde{a}}[\![\mathcal{O}_{\tilde{a}},[\![\mathcal{O}'_{\tilde{e}},\mathcal{O}''_{\tilde{c}}]\!]]\!]+q_{\tilde{a},\tilde{e}}[\![\mathcal{O}'_{\tilde{e}},[\![\mathcal{O}''_{\tilde{c}},\mathcal{O}_{\tilde{a}}]\!]]\!]+q_{\tilde{e},\tilde{c}}[\![\mathcal{O}''_{\tilde{c}},[\![\mathcal{O}_{\tilde{a}},\mathcal{O}'_{\tilde{e}}]\!]]\!]=0, \tag{A.15}$$

for all $(\tilde{a},\tilde{e},\tilde{c})\in I\times I\times I$; $(\mathcal{O}_{\tilde{a}},\mathcal{O}'_{\tilde{e}},\mathcal{O}''_{\tilde{c}})\in I\!\!L_{\tilde{a}}\times I\!\!L_{\tilde{e}}\times I\!\!L_{\tilde{c}}$.

We can easily verify that the particular definition of the $[\![\cdot,\cdot]\!]$-product provided by

$$(\mathcal{O}_{\tilde{a}},\mathcal{O}'_{\tilde{e}})\mapsto[\![\mathcal{O}_{\tilde{a}},\mathcal{O}'_{\tilde{e}}]\!]:=\mathcal{O}_{\tilde{a}}\mathcal{O}'_{\tilde{e}}-q_{\tilde{a},\tilde{e}}\mathcal{O}'_{\tilde{e}}\mathcal{O}_{\tilde{a}} \tag{A.16}$$

fulfils the requirements (A.10), (A.11) and (A.15). This is the product naturally associated to the adjoint representations and the derivations on the $(I,\,q)$-graded Lie algebra [13]. We call the product (A.16) the generalized commutator.

If $I\!\!L$ is an $(I,\,q)$-graded Lie algebra over $I\!\!K$ and fulfils additionally the requirement that there exist involutions acting simultaneously in $I\!\!L, I, I\!\!K$

$$\begin{aligned}
&\exists\overline{(\cdot)},(\cdot)^\star,(\cdot)^* \text{ involutions in } \; I\!\!L,I,I\!\!K \text{ respectively } \;: \\
&\overline{(\cdot)}:\quad I\!\!L\longrightarrow I\!\!L\,;\qquad \mathcal{O}_{\tilde{a}}\mapsto\overline{(\mathcal{O}_{\tilde{a}})}=:\bar{\mathcal{O}}_{\tilde{a}^\star}, &\text{(A.17)}\\
&(\cdot)^\star:\quad I\longrightarrow I\,;\qquad \tilde{a}\mapsto(\tilde{a})^\star=:\tilde{a}^\star, &\text{(A.18)}\\
&(\cdot)^*:\quad I\!\!K\longrightarrow I\!\!K\,;\qquad y\mapsto(y)^*=:y^*, &\text{(A.19)}
\end{aligned}$$

such that

$$q_{\tilde{a}^\star,\tilde{e}^\star}=(q^*_{\tilde{a},\tilde{e}})^{-1}, \tag{A.20}$$

then $I\!\!L$ is called an (I,q)-graded Lie algebra over $I\!\!K$ with involution . If $I\!\!K=\mathbb{C}$, then we adopt "$(\cdot)^*$" to be the complex conjugation.

We consider now a maximal set $\{G_n\}$ of linearly independent elements of $I\!\!L$. The set $\{G_n\}$ is called **a Hamel basis** of $I\!\!L$. We call $\{G_n\}_{\tilde{a}}$ the subset of elements of $\{G_n\}$ which have index $\tilde{a}$,

$$\{G_n\}_{\tilde{a}}:=\{G_{n_i}\in\{G_n\}:\varsigma_{\rm st}(G_{n_i})=\tilde{a}\}. \tag{A.21}$$

Hence, $\{G_n\}_{\tilde{a}}$ is a Hamel basis of the vector space $I\!\!L_{\tilde{a}}$:

$$I\!\!L_{\tilde{a}}=\mathrm{Gen}\{G_n\}_{\tilde{a}}. \tag{A.22}$$

Accordingly,

$$I\!\!L=\bigcup_{\tilde{a}\in I}\mathrm{Gen}\{G_n\}_{\tilde{a}}. \tag{A.23}$$

In terms of the Hamel basis $\{G_n\}$, the algebraic relations of $I\!\!L$ take the form:

$$[\![G_1,G_2]\!]=C^{G_x}_{G_1G_2}G_x, \tag{A.24}$$

where summation over $G_x\in\{G_n\}$ is understood, and $G_1,G_2\in\{G_n\}$. The coefficients $C^{G_x}_{G_1G_2}\in I\!\!K$ are called the structure constants of $I\!\!L$ using the Hamel basis $\{G_n\}$. These structure constants fulfil:

$$\begin{aligned}
C^{G_x}_{G_1G_2}&=0 &&\text{if } G_x\notin\{G_n\}_{\varsigma_{\rm st}(G_1)+\varsigma_{\rm st}(G_2)},\\
C^{G_x}_{G_1G_2}&\in I\!\!K &&\text{if } G_x\in\{G_n\}_{\varsigma_{\rm st}(G_1)+\varsigma_{\rm st}(G_2)}. &\text{(A.25)}
\end{aligned}$$

Using the generalized Jacobi associativity (A.15), we can write

$$\begin{aligned}
q_{\varsigma_{\rm st}(G_3),\varsigma_{\rm st}(G_2)}C^{G_y}_{G_2G_x}C^{G_x}_{G_1G_3}\;&+\;q_{\varsigma_{\rm st}(G_2),\varsigma_{\rm st}(G_1)}C^{G_y}_{G_1G_x}C^{G_x}_{G_3G_2}\;+\\
&+\;q_{\varsigma_{\rm st}(G_1),\varsigma_{\rm st}(G_3)}C^{G_y}_{G_3G_x}C^{G_x}_{G_2G_1}\;=\;0. &\text{(A.26)}
\end{aligned}$$

The generalized antisymmetry condition becomes

$$C^{G_z}_{G_x G_y} = -q_{\varsigma_{\rm st}(G_x),\varsigma_{\rm st}(G_y)} C^{G_z}_{G_y G_x}, \tag{A.27}$$

Consider an $(I,\, q)$-graded Lie algebra $\mathbb{L}$ with involution. If we act with the involution operations on both sides of the algebraic relations in (A.24) and use the property (A.20) we obtain

$$\overline{([\![G_1, G_2]\!])} = [\![\bar{G}_2, \bar{G}_1]\!] = -q^*_{\varsigma_{\rm st}(G_1),\varsigma_{\rm st}(G_2)}\, [\![\bar{G}_1, \bar{G}_2]\!]. \tag{A.28}$$

Hence,

$$[\![\bar{G}_1, \bar{G}_2]\!] = C^{\bar{G}_x}_{\bar{G}_1 \bar{G}_2} \bar{G}_x, \tag{A.29}$$

$$C^{\bar{G}_x}_{\bar{G}_1 \bar{G}_2} = -q^*_{\varsigma_{\rm st}(G_2),\varsigma_{\rm st}(G_1)} (C^{G_x}_{G_1 G_2})^* = (C^{G_x}_{G_2 G_1})^*. \tag{A.30}$$

The graded generalizations of Lie algebras have been studied first by P. Cartier [14] in 1955 and by R. Ree [15] in 1960. A concrete family of graded Lie algebras beyond the superalgebras have been introduced into physics by V. Rittenberg and D. Wyler [5] in 1978. A formal presentation of this subject with further developments has been accomplished by M. Scheunert [16] in 1983. Novel developments in graded Lie algebras can be found in [13] and [10].

B Single-grading model for the index set I and the function q when $\mathbb{K} = \mathbb{C}$

We now construct a particular model, the so-called single-grading model, for the index set I and the function q of an $(I,\, q)$-graded Lie algebra over $\mathbb{C}$. For more general models, multi-grading models, see [13].

We consider an Abelian group $(\check{E}; +)$ and a complex function $\check{q}$:

$$\begin{aligned} \check{E} &:= \mathbb{Z}_2 \times \mathbb{R}^2 \times \mathbb{R}^2, \\ \check{E} &\ni \vec{a} \equiv (a_0, \vec{\eta}_{\vec{a}}, \vec{\varrho}_{\vec{a}}) \equiv (a_0, (a_1, a_2), (a_3, a_4)), \end{aligned} \tag{B.1}$$

$$\begin{aligned} + :\ & \check{E} \times \check{E} \longrightarrow \check{E}\,; \\ & (\vec{a}, \vec{e}) \mapsto \vec{a} + \vec{e} := (a_0 + e_0, \vec{\eta}_{\vec{a}} + \vec{\eta}_{\vec{e}}, \vec{\varrho}_{\vec{a}} + \vec{\varrho}_{\vec{e}}) = \\ & \qquad = (a_0 + e_0, (a_1 + e_1, a_2 + e_2), (a_3 + e_3, a_4 + e4)), \end{aligned} \tag{B.2}$$

$$\begin{aligned} \check{q} :\ & \check{E} \times \check{E} \longrightarrow \mathbb{C}\,; \\ & (\vec{a}, \vec{e}) \mapsto \check{q}_{\vec{a},\vec{e}} := \exp\{{\rm i}\pi a_0 e_0 + {\rm i}\pi(a_1 e_2 - e_1 a_2) + \pi(a_3 e_4 - e_3 a_4)\}. \end{aligned} \tag{B.3}$$

We consider now that the index set I is a set of disjoint subsets of $\check{E}$, i.e.

$$\tilde{a} \in I \implies \tilde{a} \subset \hat{E}, \tag{B.4}$$

$$\tilde{a}, \tilde{e} \in I \text{ and } \tilde{a} \neq \tilde{e} \implies \tilde{a} \cap \tilde{e} = \emptyset. \tag{B.5}$$

The addition of indices will then have the form:

$$+|_{I \times I} : I \times I \longrightarrow \mathbb{I} \supset I\,; (\tilde{a}, \tilde{e}) \mapsto \tilde{a} + \tilde{e} := \{\vec{a} + \vec{e} : (\vec{a}, \vec{e}) \in \tilde{a} \times \tilde{e}\}. \tag{B.6}$$

The equivalence relation $\overset{I}{\simeq}$ associated to the partition of $\cup_{\tilde{a}\in I}(\tilde{a})$ into I should have the following property:

$$\vec{a} \overset{I}{\simeq} \vec{e} \Longleftrightarrow (\forall \tilde{c} \in I \text{ and } \forall \vec{c} \in \tilde{c} : \check{q}_{\vec{a},\vec{c}} = \check{q}_{\vec{e},\vec{c}}). \tag{B.7}$$

The connection between $\check{q}$ and q is given by the definition

$$q|_{I\times I} : I \times I \longrightarrow \mathbb{C} \; ; \; (\tilde{a}, \tilde{e}) \mapsto q_{\tilde{a},\tilde{e}} \in \{\check{q}_{\vec{a},\vec{e}} : (\vec{a}, \vec{e}) \in \tilde{a} \times \tilde{e}\}. \tag{B.8}$$

This definition of the function q given $\check{q}$ is not ambiguous since according to condition (B.7) the set $\{\check{q}_{\vec{a},\vec{e}} : (\vec{a}, \vec{e}) \in \tilde{a} \times \tilde{e}\}$ has only one single element. Observe that there might be multiple admissible choices of the Abelian set $\mathbb{I} \supset I$. This is a remarkable fact when considering the (I, q)-graded extensions of a given algebra.

It is easy to verify that the defined function q fulfils the properties (A.13) and (A.14) required by the (I, q)-graded Lie algebra over $\mathbb{C}$.

If we want an (I, q)-graded extension with involution, with the complex conjugation as involution in $\mathbb{C}$, then there should exist a map $(\cdot)^\star$ fulfilling (A.20). From this we obtain:

$$\vec{a} \in \tilde{a} \, ; \, \vec{e} \in \tilde{e} \, ; \, \vec{a}^{\,\star} \in \tilde{a}^\star \, ; \, \vec{e}^{\,\star} \in \tilde{e}^{\,\star} \Longrightarrow$$

$$\text{i)} \quad a_0^\star = a_0, \tag{B.9}$$

$$\text{ii)} \quad (a_1 a_2^\star - a_1^\star a_2) \bmod 2 \in \{0, 1\}, \tag{B.10}$$

$$\text{iii)} \quad (a_1^\star e_2^\star - e_1^\star a_2^\star) \bmod 2 = (a_1 e_2 - e_1 a_2) \bmod 2, \tag{B.11}$$

$$\text{iv)} \quad (a_3^\star e_4^\star - e_3^\star a_4^\star) = -(a_3 e_4 - e_3 a_4). \tag{B.12}$$

Every choice of the $(\cdot)^\star$-involution fulfilling the requirements (B.9)-(B.12) will be consistent with the condition (A.20).

Observe that

$$q_{\tilde{a},\tilde{a}} = q_{\tilde{a}^\star,\tilde{a}^\star} = \exp\{\mathrm{i}\pi a_0 a_0\} \; ; \; \vec{a} \in \tilde{a}. \tag{B.13}$$

Hence, we shall call a_0 the **intrinsic commutative behaviour** of the objects of index $\tilde{a} \ni \vec{a}$. The two allowed values of the intrinsic commutative behaviour are 0 and 1, and we call them **self-bosonic** and **self-fermionic** respectively.

C Spin-$\frac{1}{2}$ representations in terms of the Hamel basis $\{M^{\mu\nu}\}$

The connection between the momentum representations in terms of the Hamel basis $\{T_i\} \cup \{\bar{T}_i\}$ used in reference [2] and the standard basis $\{M^{\mu\nu}\}$ follows from the relations

$$\begin{aligned} M^{oi} &= \quad \hat{J}_i = \quad -\mathrm{i}(T_i - \bar{T}_i), \\ M^{jk} &= \epsilon^{ijk} J_i = \quad \epsilon^{ijk}(T_i + \bar{T}_i). \end{aligned} \tag{C.1}$$

There are some matrix arrays that are so frequently used in the following appendices that we assign to them the following names for short:

$$\sigma_0 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \sigma_1 := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{bmatrix}, \tag{C.2}$$

$$\sigma_2 := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ 1 & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{bmatrix}, \quad \sigma_3 := \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{C.3}$$

These matrices correspond to the Pauli-like matrices of the present formalism.

We will now list some graded-irreducible spin-$\frac{1}{2}$ representations as well the relations between them.

C.1 Spin $(\frac{1}{2},0)$ self-representation

The spin $(\frac{1}{2},0)$ representations to be considered here are those associated to the multiplets of generators F^f_{os} and V^υ_{is}, transforming according to:

$$[\![M^{\mu\nu},F^f_{os}]\!] = -\tfrac{1}{2}(\sigma^{f\mu\nu}_o)_s{}^t F^f_{ot}, \tag{C.4}$$
$$[\![M^{\mu\nu},V^\upsilon_{is}]\!] = -\tfrac{1}{2}(\sigma^{\upsilon\mu\nu}_i)_s{}^t V^\upsilon_{it}. \tag{C.5}$$

The corresponding structure constants arrays are given by

$$\sigma^{fo1}_o = \sigma^{\upsilon oi}_i = -\mathrm{i}\sigma_1,\quad \sigma^{fo2}_o = \sigma^{\upsilon oj}_i = -\mathrm{i}\sigma_2,\quad \sigma^{fo3}_o = \sigma^{\upsilon ok}_i = -\mathrm{i}\sigma_3,$$
$$\sigma^{f23}_o = \sigma^{\upsilon jk}_i = \sigma_1,\quad \sigma^{f31}_o = \sigma^{\upsilon ki}_i = \sigma_2,\quad \sigma^{f12}_o = \sigma^{\upsilon ij}_i = \sigma_3, \tag{C.6}$$

where $(i,j,k)\in\{(1,2,3),(2,3,1),(3,1,2)\}$, and the Pauli-like matrices σ_i ; $i\in\{1,2,3\}$ are those defined in (C.2)-(C.3).

C.2 Spin $(\frac{1}{2},0)$ dual self-representation

The multiplets F^{fs}_o and $V^{\upsilon s}_i$ transform under spin $(\frac{1}{2},0)$ dual self-representations:

$$[\![M^{\mu\nu},F^{fs}_o]\!] = \tfrac{1}{2}(\sigma^{f\mu\nu}_o)^s{}_t F^{ft}_o, \tag{C.7}$$
$$[\![M^{\mu\nu},V^{\upsilon s}_i]\!] = \tfrac{1}{2}(\sigma^{\upsilon\mu\nu}_i)^s{}_t V^{\upsilon t}_i. \tag{C.8}$$

The bilinear products of the form $F^{fs}_o F^{f'}_{os}$ and $V^{\upsilon s}_i V^{\upsilon'}_{is}$ are Lorentz-invariant. Hence,

$$(\sigma^{f\mu\nu}_o)^s{}_t = (\sigma^{f\mu\nu}_o)_t{}^s, \tag{C.9}$$
$$(\sigma^{\upsilon\mu\nu}_i)^s{}_t = (2\delta^\mu_i+2\delta^\mu_o-1)(2\delta^\nu_i+2\delta^\nu_o-1)(\sigma^{\upsilon\mu\nu}_i)_t{}^s. \tag{C.10}$$

The metric matrices relating the spin $(\frac{1}{2},0)$ self-representation and its dual self-representation are given by

$$\varepsilon^{us} := \begin{bmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & -\mathrm{i}\\ 0 & 0 & \mathrm{i} & 0\end{bmatrix}^{us},\quad \varepsilon_{st} := \begin{bmatrix} 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -\mathrm{i}\\ 0 & 0 & \mathrm{i} & 0\end{bmatrix}_{st}, \tag{C.11}$$

$$\epsilon^{us} := \begin{bmatrix} 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -\mathrm{i}\\ 0 & 0 & \mathrm{i} & 0\end{bmatrix}^{us},\quad \epsilon_{st} := \begin{bmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & -\mathrm{i}\\ 0 & 0 & \mathrm{i} & 0\end{bmatrix}_{st}. \tag{C.12}$$

The lowering and rising of the component indices of spin $(\frac{1}{2},0)$ multiplets is obtained as follows:

$$F^{fs}_o = \varepsilon^{st}F^f_{ot} \quad,\quad F^f_{os} = \varepsilon_{st}F^{ft}_o, \tag{C.13}$$
$$V^{\upsilon s}_i = \epsilon^{st}V^\upsilon_{it} \quad,\quad V^\upsilon_{is} = \epsilon_{st}V^{\upsilon t}_i. \tag{C.14}$$

Accordingly,

$$(\sigma^{f\mu\nu}_o)^s{}_t = \varepsilon^{su}\varepsilon_{tr}(\sigma^{f\mu\nu}_o)_u{}^r, \tag{C.15}$$
$$(\sigma^{\upsilon\mu\nu}_i)^s{}_t = \epsilon^{su}\epsilon_{tr}(\sigma^{\upsilon\mu\nu}_i)_u{}^r. \tag{C.16}$$

C.3 Spin $(0,\frac{1}{2})$ adjoint self-representation

The spin $(0,\frac{1}{2})$ representations to be considered here are those associated to the multiplets of generators $\bar{F}^{f}_{\bar{o}\dot{s}}$ and $\bar{V}^{\upsilon}_{\bar{\imath}\dot{s}}$, transforming according to:

$$\begin{aligned} [\![M^{\mu\nu},\bar{F}^{f}_{\bar{o}\dot{s}}]\!] &= \tfrac{1}{2}(\bar{\sigma}^{f\mu\nu}_{\bar{o}})_{\dot{s}}{}^{\dot{t}}\bar{F}^{f}_{\bar{o}\dot{t}}, &\text{(C.17)}\\ [\![M^{\mu\nu},\bar{V}^{\upsilon}_{\bar{\imath}\dot{s}}]\!] &= \tfrac{1}{2}(\bar{\sigma}^{\upsilon\mu\nu}_{\bar{\imath}})_{\dot{s}}{}^{\dot{t}}\bar{V}^{\upsilon}_{\bar{\imath}\dot{t}}. &\text{(C.18)}\end{aligned}$$

The corresponding structure constants arrays are given by

$$\begin{aligned} &\bar{\sigma}^{fo1}_{\bar{o}} = \bar{\sigma}^{\upsilon oi}_{\bar{\imath}} = -\mathrm{i}(\sigma_1)^*, \quad \bar{\sigma}^{fo2}_{\bar{o}} = -\bar{\sigma}^{\upsilon oj}_{\bar{\imath}} = -\mathrm{i}(\sigma_2)^*, \quad \bar{\sigma}^{fo3}_{\bar{o}} = -\sigma^{\upsilon ok}_{\bar{\imath}} = -\mathrm{i}(\sigma_3)^*,\\ &\bar{\sigma}^{f23}_{\bar{o}} = \bar{\sigma}^{\upsilon jk}_{\bar{\imath}} = (\sigma_1)^*, \quad \bar{\sigma}^{f31}_{\bar{o}} = -\bar{\sigma}^{\upsilon ki}_{\bar{\imath}} = (\sigma_2)^*, \quad \bar{\sigma}^{f12}_{\bar{o}} = -\bar{\sigma}^{\upsilon ij}_{\bar{\imath}} = (\sigma_3)^*, \qquad \text{(C.19)}\end{aligned}$$

where $(i,j,k)\in\{(1,2,3),(2,3,1),(3,1,2)\}$, and the Pauli-like matrices σ_i ; $i\in\{1,2,3\}$ are those defined in (C.2) and (C.3).

Observe that the spin $(\frac{1}{2},0)$ self-representation and the spin $(0,\frac{1}{2})$ adjoint self-representation are related by

$$\begin{aligned} \bar{\sigma}^{f\mu\nu}_{\bar{o}} &= (\sigma^{f\mu\nu}_{o})^*, &\text{(C.20)}\\ \bar{\sigma}^{\upsilon\mu\nu}_{\bar{\imath}} &= (2\delta^{\mu}_{i}+2\delta^{\mu}_{o}-1)(2\delta^{\nu}_{i}+2\delta^{\nu}_{o}-1)(\sigma^{\upsilon\mu\nu}_{i})^*. &\text{(C.21)}\end{aligned}$$

C.4 Spin $(0,\frac{1}{2})$ dual adjoint self-representation

The multiplets $\bar{F}^{f\dot{s}}_{\bar{o}}$ and $\bar{V}^{\upsilon\dot{s}}_{\bar{\imath}}$ transform under spin $(0,\frac{1}{2})$ dual adjoint self-representations:

$$\begin{aligned} [\![M^{\mu\nu},\bar{F}^{f\dot{s}}_{\bar{o}}]\!] &= -\tfrac{1}{2}(\bar{\sigma}^{f\mu\nu}_{\bar{o}})^{\dot{s}}{}_{\dot{t}}\bar{F}^{f\dot{t}}_{\bar{o}}, &\text{(C.22)}\\ [\![M^{\mu\nu},\bar{V}^{\upsilon\dot{s}}_{\bar{\imath}}]\!] &= -\tfrac{1}{2}(\bar{\sigma}^{\upsilon\mu\nu}_{\bar{\imath}})^{\dot{s}}{}_{\dot{t}}\bar{V}^{\upsilon\dot{t}}_{\bar{\imath}}. &\text{(C.23)}\end{aligned}$$

The bilinear products of the form $\bar{F}^{f}_{\bar{o}\dot{s}}\bar{F}^{f'\dot{s}}_{\bar{o}}$ and $\bar{V}^{\upsilon}_{\bar{\imath}\dot{s}}\bar{V}^{\upsilon'\dot{s}}_{\bar{\imath}}$ are Lorentz-invariant. Hence,

$$\begin{aligned} (\bar{\sigma}^{f\mu\nu}_{\bar{o}})^{\dot{s}}{}_{\dot{t}} &= (\bar{\sigma}^{f\mu\nu}_{\bar{o}})_{\dot{t}}{}^{\dot{s}}, &\text{(C.24)}\\ (\bar{\sigma}^{\upsilon\mu\nu}_{\bar{\imath}})^{\dot{s}}{}_{\dot{t}} &= (2\delta^{\mu}_{i}+2\delta^{\mu}_{o}-1)(2\delta^{\nu}_{i}+2\delta^{\nu}_{o}-1)(\bar{\sigma}^{\upsilon\mu\nu}_{\bar{\imath}})_{\dot{t}}{}^{\dot{s}}. &\text{(C.25)}\end{aligned}$$

The metric matrices relating the spin $(0,\frac{1}{2})$ adjoint self-representation and its dual adjoint self-representation are given by

$$\bar{\varepsilon}^{\dot{u}\dot{s}} := \begin{bmatrix} 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -\mathrm{i}\\ 0 & 0 & \mathrm{i} & 0 \end{bmatrix}^{\dot{u}\dot{s}}, \quad \bar{\varepsilon}_{\dot{s}\dot{t}} := \begin{bmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & -\mathrm{i}\\ 0 & 0 & \mathrm{i} & 0 \end{bmatrix}_{\dot{s}\dot{t}}, \qquad \text{(C.26)}$$

$$\bar{\epsilon}^{\dot{u}\dot{s}} := \begin{bmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & -\mathrm{i}\\ 0 & 0 & \mathrm{i} & 0 \end{bmatrix}^{\dot{u}\dot{s}}, \quad \bar{\epsilon}_{\dot{s}\dot{t}} := \begin{bmatrix} 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -\mathrm{i}\\ 0 & 0 & \mathrm{i} & 0 \end{bmatrix}_{\dot{s}\dot{t}}. \qquad \text{(C.27)}$$

The lowering and rising of the component indices of spin $(0,\frac{1}{2})$ multiplets is obtained as follows:

$$\begin{aligned} &\bar{F}^{f\dot{s}}_{\bar{o}} = \bar{F}^{f}_{\bar{o}\dot{t}}\bar{\varepsilon}^{\dot{t}\dot{s}} \quad , \quad \bar{F}^{f}_{\bar{o}\dot{s}} = \bar{F}^{f\dot{t}}_{\bar{o}}\bar{\varepsilon}_{\dot{t}\dot{s}}, &\text{(C.28)}\\ &\bar{V}^{\upsilon\dot{s}}_{\bar{\imath}} = \bar{V}^{\upsilon}_{\bar{\imath}\dot{t}}\bar{\epsilon}^{\dot{t}\dot{s}} \quad , \quad \bar{V}^{\upsilon}_{\bar{\imath}\dot{s}} = \bar{V}^{\upsilon\dot{t}}_{\bar{\imath}}\bar{\epsilon}_{\dot{t}\dot{s}}. &\text{(C.29)}\end{aligned}$$

Accordingly,

$$(\bar{\sigma}^{f\mu\nu}_{\bar{o}})^{\dot{s}}{}_{\dot{t}} = \bar{\varepsilon}^{\dot{u}\dot{s}}\bar{\varepsilon}_{\dot{r}\dot{t}}(\bar{\sigma}^{f\mu\nu}_{\bar{o}})_{\dot{u}}{}^{\dot{r}}, \qquad (C.30)$$

$$(\bar{\sigma}^{\upsilon\mu\nu}_{\bar{\imath}})^{\dot{s}}{}_{\dot{t}} = \bar{\epsilon}^{\dot{u}\dot{s}}\bar{\epsilon}_{\dot{r}\dot{t}}(\bar{\sigma}^{\upsilon\mu\nu}_{\bar{\imath}})_{\dot{u}}{}^{\dot{r}}. \qquad (C.31)$$

D Superspace parameters, summation conventions, and differentiation

The **summation convention:** The summation over a repeated multiplet component index in the same monomial is understood if this index appears as subindex as well as superindex. If a summed index is omitted, then the following convention is understood:

$$xy := x^\mu y_\mu = x_\mu y^\mu, \qquad (D.1)$$

$$\Theta^o_f\Theta^o_{f'} := \Theta^{os}_f\Theta^o_{f's} \quad ;\ (\text{NW-SE summation convention}), \qquad (D.2)$$

$$\bar{\Theta}^{\bar{o}}_f\bar{\Theta}^{\bar{o}}_{f'} := \bar{\Theta}^{\bar{o}}_{f\dot{s}}\bar{\Theta}^{\bar{o}\dot{s}}_{f'} \quad ;\ (\text{SW-NE summation convention}), \qquad (D.3)$$

$$\Theta^i_\upsilon\Theta^i_{\upsilon'} := \Theta^{is}_\upsilon\Theta^i_{\upsilon's} \quad ;\ (\text{NW-SE summation convention}), \qquad (D.4)$$

$$\bar{\Theta}^{\bar{\imath}}_\upsilon\bar{\Theta}^{\bar{\imath}}_{\upsilon'} := \bar{\Theta}^{\bar{\imath}}_{\upsilon\dot{s}}\bar{\Theta}^{\bar{\imath}\dot{s}}_{\upsilon'} \quad ;\ (\text{SW-NE summation convention}). \qquad (D.5)$$

The **lowering and rising** of the multiplet-component indices is given by

$$x^\mu = g^{\mu\nu}x_\nu \quad , \quad x_\mu = g_{\mu\nu}x^\nu, \qquad (D.6)$$

$$\Theta^{os}_f = \varepsilon^{su}\Theta^o_{fu} \quad , \quad \Theta^o_{fs} = \varepsilon_{su}\Theta^{ou}_f, \qquad (D.7)$$

$$\bar{\Theta}^{\bar{o}\dot{s}}_f = \bar{\Theta}^{\bar{o}}_{f\dot{u}}\bar{\varepsilon}^{\dot{u}\dot{s}} \quad , \quad \bar{\Theta}^{\bar{o}}_{f\dot{s}} = \bar{\Theta}^{\bar{o}\dot{u}}_f\bar{\varepsilon}_{\dot{u}\dot{s}}, \qquad (D.8)$$

$$\Theta^{is}_\upsilon = \epsilon^{su}\Theta^i_{\upsilon u} \quad , \quad \Theta^i_{\upsilon s} = \epsilon_{su}\Theta^{iu}_\upsilon, \qquad (D.9)$$

$$\bar{\Theta}^{\bar{\imath}\dot{s}}_\upsilon = \bar{\Theta}^{\bar{\imath}}_{\upsilon\dot{u}}\bar{\epsilon}^{\dot{u}\dot{s}} \quad , \quad \bar{\Theta}^{\bar{\imath}}_{\upsilon\dot{s}} = \bar{\Theta}^{\bar{\imath}\dot{u}}_\upsilon\bar{\epsilon}_{\dot{u}\dot{s}}, \qquad (D.10)$$

where the corresponding **metric matrices** are given by

$$g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1), \qquad (D.11)$$

$$\varepsilon^{su} = \bar{\varepsilon}_{\dot{s}\dot{u}} = \bar{\epsilon}^{\dot{s}\dot{u}} = \epsilon_{su} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{bmatrix}, \qquad (D.12)$$

$$\bar{\varepsilon}^{\dot{s}\dot{u}} = \varepsilon_{su} = \epsilon^{su} = \bar{\epsilon}_{\dot{s}\dot{u}} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{bmatrix}. \qquad (D.13)$$

The **differentiation with respect to superspace parameters** is defined by

$$(\partial_\mu x^\nu) \equiv [\![\partial_\mu, x^\nu]\!] := \delta^\nu_\mu, \qquad (D.14)$$

$$(\partial_{\Theta^{os}_f}\Theta^{ou}_{f'}) \equiv [\![\partial_{\Theta^{os}_f}, \Theta^{ou}_{f'}]\!] := \delta^u_s\delta^f_{f'}, \qquad (D.15)$$

$$(\partial_{\bar{\Theta}^{\bar{o}\dot{s}}_f}\bar{\Theta}^{\bar{o}\dot{u}}_{f'}) \equiv [\![\partial_{\bar{\Theta}^{\bar{o}\dot{s}}_f}, \bar{\Theta}^{\bar{o}\dot{u}}_{f'}]\!] := \delta^{\dot{u}}_{\dot{s}}\delta^f_{f'}, \qquad (D.16)$$

$$(\partial_{\Theta^{is}_\upsilon}\Theta^{ju}_{\upsilon'}) \equiv [\![\partial_{\Theta^{is}_\upsilon}, \Theta^{ju}_{\upsilon'}]\!] := \delta^u_s\delta^\upsilon_{\upsilon'}\delta j_i, \qquad (D.17)$$

$$(\partial_{\bar{\Theta}^{\bar{\imath}\dot{s}}_\upsilon}\bar{\Theta}^{\bar{\jmath}\dot{u}}_{\upsilon'}) \equiv [\![\partial_{\bar{\Theta}^{\bar{\imath}\dot{s}}_\upsilon}, \bar{\Theta}^{\bar{\jmath}\dot{u}}_{\upsilon'}]\!] := \delta^{\dot{u}}_{\dot{s}}\delta^\upsilon_{\upsilon'}\delta^{\bar{\jmath}}_{\bar{\imath}}. \qquad (D.18)$$

Hence, the (statistic) indices assigned to the differentiation operators are given by

$$\begin{aligned}
\varsigma_{\rm st}(\partial^{\mu}) = \quad & \varsigma_{\rm st}(\partial_{\mu}) = && (0, \mathbf{a}^{\mu})\,;\, \mu \in \{0,1,2,3\},\\
\varsigma_{\rm st}(\partial_{\Theta^{os}_{f}}) = \quad & \varsigma_{\rm st}(\partial_{\bar\Theta^{\bar o\dot s}_{f}}) = && (1, \mathbf{a}^{s})\,;\, s \in \{0,1,2,3\}\,;\, \dot s \in \{\dot 0,\dot 1,\dot 2,\dot 3\},\\
\varsigma_{\rm st}(\partial_{\Theta^{i1}_{\upsilon}}) = \quad & \varsigma_{\rm st}(\partial_{\bar\Theta^{\bar\imath\bar 0}_{\upsilon}}) = && (1, \mathbf{a}^{i1^{+}})\,;\, i \in \{1,2,3\}\,;\, \bar\imath \in \{\bar 1,\bar 2,\bar 3\},\\
\varsigma_{\rm st}(\partial_{\Theta^{i0}_{\upsilon}}) = \quad & \varsigma_{\rm st}(\partial_{\bar\Theta^{\bar\imath\bar 1}_{\upsilon}}) = && (1, \mathbf{a}^{i1^{-}})\,;\, i \in \{1,2,3\}\,;\, \bar\imath \in \{\bar 1,\bar 2,\bar 3\},\\
\varsigma_{\rm st}(\partial_{\Theta^{i3}_{\upsilon}}) = \quad & \varsigma_{\rm st}(\partial_{\bar\Theta^{\bar\imath\bar 2}_{\upsilon}}) = && (1, \mathbf{a}^{i2^{+}})\,;\, i \in \{1,2,3\}\,;\, \bar\imath \in \{\bar 1,\bar 2,\bar 3\},\\
\varsigma_{\rm st}(\partial_{\Theta^{i2}_{\upsilon}}) = \quad & \varsigma_{\rm st}(\partial_{\bar\Theta^{\bar\imath\bar 3}_{\upsilon}}) = && (1, \mathbf{a}^{i2^{-}})\,;\, i \in \{1,2,3\}\,;\, \bar\imath \in \{\bar 1,\bar 2,\bar 3\}.
\end{aligned} \tag{D.19}$$

According to the above definitions, we have

$$\overline{(\partial_{\Theta^{os}_{f}}\Phi)} \;=\; -q^{*}_{\varsigma_{\rm st}(\partial_{\Theta^{os}_{f}}),\varsigma_{\rm st}(\Phi)}(\partial_{\bar\Theta^{\bar o\dot s}_{f}}\bar\Phi), \tag{D.20}$$

$$\overline{(\partial_{\Theta^{is}_{\upsilon}}\Phi)} \;=\; -q^{*}_{\varsigma_{\rm st}(\partial_{\Theta^{is}_{\upsilon}}),\varsigma_{\rm st}(\Phi)}(\partial_{\bar\Theta^{\bar\imath\dot s}_{\upsilon}}\bar\Phi). \tag{D.21}$$

Using the metric matrices for the lowering or rising of multiplet-component indices, we obtain as well

$$(\partial_{\mu}x_{\nu}) = g_{\mu\nu}, \qquad (\partial^{\mu}x^{\nu}) = g^{\mu\nu}, \qquad (\partial^{\mu}x_{\nu}) = \delta^{\mu}_{\nu}, \tag{D.22}$$

$$(\partial_{\Theta^{os}_{f}}\Theta^{o}_{f'u}) = \varepsilon_{us}\delta^{f}_{f'}, \quad (\partial_{\Theta^{o}_{fs}}\Theta^{ou}_{f'}) = \varepsilon^{su}\delta^{f}_{f'}, \quad (\partial_{\Theta^{o}_{fs}}\Theta^{o}_{f'u}) = -\delta^{s}_{u}\delta^{f}_{f'}, \tag{D.23}$$

$$(\partial_{\bar\Theta^{\bar o\dot s}_{f}}\bar\Theta^{\bar o}_{f'\dot u}) = \bar\varepsilon_{\dot s\dot u}\delta^{f}_{f'}, \quad (\partial_{\bar\Theta^{\bar o}_{f\dot s}}\bar\Theta^{\bar o\dot u}_{f'}) = \bar\varepsilon^{\dot u\dot s}\delta^{f}_{f'}, \quad (\partial_{\bar\Theta^{\bar o}_{f\dot s}}\bar\Theta^{\bar o}_{f'\dot u}) = -\delta^{\dot s}_{\dot u}\delta^{f}_{f'}, \tag{D.24}$$

$$(\partial_{\Theta^{is}_{\upsilon}}\Theta^{j}_{\upsilon' u}) = \epsilon_{us}\delta^{\upsilon}_{\upsilon'}\delta^{j}_{i}, \quad (\partial_{\Theta^{i}_{\upsilon s}}\Theta^{ju}_{\upsilon'}) = \epsilon^{su}\delta^{\upsilon}_{\upsilon'}\delta^{j}_{i}, \quad (\partial_{\Theta^{i}_{\upsilon s}}\Theta^{j}_{\upsilon' u}) = -\delta^{s}_{u}\delta^{\upsilon}_{\upsilon'}\delta^{j}_{i}, \tag{D.25}$$

$$(\partial_{\bar\Theta^{\bar\imath\dot s}_{\upsilon}}\bar\Theta^{\bar\jmath}_{\upsilon'\dot u}) = \bar\epsilon_{\dot s\dot u}\delta^{\upsilon}_{\upsilon'}\delta^{\bar\jmath}_{\bar\imath}, \quad (\partial_{\bar\Theta^{\bar\imath}_{\upsilon\dot s}}\bar\Theta^{\bar\jmath\dot u}_{\upsilon'}) = \bar\epsilon^{\dot u\dot s}\delta^{\upsilon}_{\upsilon'}\delta^{\bar\jmath}_{\bar\imath}, \quad (\partial_{\bar\Theta^{\bar\imath}_{\upsilon\dot s}}\bar\Theta^{\bar\jmath}_{\upsilon'\dot u}) = -\delta^{\dot s}_{\dot u}\delta^{\upsilon}_{\upsilon'}\delta^{\bar\jmath}_{\bar\imath}. \tag{D.26}$$

E Useful Identities for spin-$\frac{1}{2}$ and superspace calculations

We collect here some identities useful to reproduce the results presented in the main text.

E.1 The upsilon and gamma matrices

The $\hat\Upsilon$- and $\hat\gamma$-matrices are structure constants arrays in the expressions

$$[\![F^{f}_{os}, \bar F^{f'}_{\bar o\dot t}]\!] = \quad 2\hat\Upsilon^{ff'}(\tfrac{o\mu}{o\bar o})_{s\dot t}P_{\mu} = \quad 2\delta^{ff'}\hat\Upsilon(\tfrac{o\mu}{o\bar o})_{s\dot t}P_{\mu}, \tag{E.1}$$

$$[\![V^{\upsilon}_{is}, \bar V^{\upsilon'}_{\bar\imath\dot t}]\!] = \quad 2\hat\gamma^{\upsilon\upsilon'}(\tfrac{o\mu}{i\bar\imath})_{s\dot t}P_{\mu} = \quad 2\delta^{\upsilon\upsilon'}\hat\gamma(\tfrac{o\mu}{i\bar\imath})_{s\dot t}P_{\mu}. \tag{E.2}$$

These matrix arrays are constrained by the generalized Jacobi associativity conditions, and can be chosen to have the form:

$$\hat\Upsilon(\tfrac{o0}{o\bar o})_{s\dot t} \;= \hat\gamma(\tfrac{o0}{i\bar\imath})_{s\dot t} \;= (\sigma_0)_{s\dot t}, \tag{E.3}$$

$$\begin{aligned}
\hat\Upsilon(\tfrac{o1}{o\bar o})_{s\dot t} \;&= \hat\gamma(\tfrac{oi}{i\bar\imath})_{s\dot t} \;= (\sigma_1)_{s\dot t},\\
\hat\Upsilon(\tfrac{o2}{o\bar o})_{s\dot t} \;&= \hat\gamma(\tfrac{oj}{i\bar\imath})_{s\dot t} \;= (\sigma_2)_{s\dot t},\\
\hat\Upsilon(\tfrac{o3}{o\bar o})_{s\dot t} \;&= \hat\gamma(\tfrac{ok}{i\bar\imath})_{s\dot t} \;= (\sigma_3)_{s\dot t},
\end{aligned} \tag{E.4}$$

where $(i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\}$, and the Pauli-like matrices $\sigma_\mu\,;\,\mu \in \{0,1,2,3\}$ are those defined in (C.2) and (C.3).

It is easy to verify that they fulfil:

$$\hat{\gamma}(\tfrac{o\mu}{i\bar{i}})_{s\dot{t}} = (2\delta^{\mu}_{i} + 2\delta^{\mu}_{o} - 1)(-[\star]_{s\dot{t}}\hat{\gamma}(\tfrac{o\mu}{i\bar{i}})_{s\dot{t}}), \tag{E.5}$$

where

$$[\star] := \begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}. \tag{E.6}$$

We can define further arrays:

$$\hat{\Upsilon}(\tfrac{o\mu}{o\bar{o}})^{s\dot{t}} := \varepsilon^{su}\hat{\Upsilon}(\tfrac{o\mu}{o\bar{o}})_{u\dot{r}}\bar{\varepsilon}^{\dot{r}\dot{t}}, \tag{E.7}$$

$$\hat{\Upsilon}(\tfrac{o\mu}{\bar{o}o})^{\dot{t}s} := \hat{\Upsilon}(\tfrac{o\mu}{o\bar{o}})^{s\dot{t}}, \tag{E.8}$$

$$\hat{\gamma}(\tfrac{o\mu}{i\bar{i}})^{s\dot{t}} := \epsilon^{su}\hat{\gamma}(\tfrac{o\mu}{i\bar{i}})_{u\dot{r}}\bar{\epsilon}^{\dot{r}\dot{t}}, \tag{E.9}$$

$$\hat{\gamma}(\tfrac{o\mu}{\bar{i}i})^{\dot{t}s} := -[\star]^{s\dot{t}}\hat{\gamma}(\tfrac{o\mu}{i\bar{i}})^{s\dot{t}}. \tag{E.10}$$

We easily verify

$$\hat{\Upsilon}(\tfrac{o0}{\bar{o}o})^{\dot{t}s} = \hat{\gamma}(\tfrac{o0}{\bar{i}i})^{\dot{t}s} = (\sigma_0)^{\dot{t}s}, \tag{E.11}$$

$$\hat{\Upsilon}(\tfrac{o1}{\bar{o}o})^{\dot{t}s} = \hat{\gamma}(\tfrac{oi}{\bar{i}i})^{\dot{t}s} = -(\sigma_1)^{\dot{t}s}, \tag{E.12}$$

$$\hat{\Upsilon}(\tfrac{o2}{\bar{o}o})^{\dot{t}s} = \hat{\gamma}(\tfrac{oj}{\bar{i}i})^{\dot{t}s} = -(\sigma_2)^{\dot{t}s}, \tag{E.13}$$

$$\hat{\Upsilon}(\tfrac{o3}{\bar{o}o})^{\dot{t}s} = \hat{\gamma}(\tfrac{ok}{\bar{i}i})^{\dot{t}s} = -(\sigma_3)^{\dot{t}s}, \tag{E.14}$$

where $(i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$.

E.2 The sigma matrices

We recall some useful identities relating the different spin-$\frac{1}{2}$ representations presented in Appendix C.

$$\bar{\sigma}^{f\mu\nu}_{\bar{o}} = (\sigma^{f\mu\nu}_{o})^{*}, \tag{E.15}$$

$$(\sigma^{f\mu\nu}_{o})^{s}{}_{t} = \varepsilon^{su}\varepsilon_{tr}(\sigma^{f\mu\nu}_{o})_{u}{}^{r} = (\sigma^{f\mu\nu}_{o})_{t}{}^{s}, \tag{E.16}$$

$$(\bar{\sigma}^{f\mu\nu}_{\bar{o}})^{\dot{s}}{}_{\dot{t}} = \bar{\varepsilon}^{\dot{u}\dot{s}}\bar{\varepsilon}_{\dot{r}\dot{t}}(\bar{\sigma}^{f\mu\nu}_{\bar{o}})_{\dot{u}}{}^{\dot{r}} = (\bar{\sigma}^{f\mu\nu}_{\bar{o}})_{\dot{t}}{}^{\dot{s}}. \tag{E.17}$$

$$\bar{\sigma}^{\upsilon\mu\nu}_{\bar{i}} = (2\delta^{\mu}_{i} + \delta^{\mu}_{o} - 1)(2\delta^{\nu}_{i} + \delta^{\nu}_{o} - 1)(\sigma^{\upsilon\mu\nu}_{i})^{*}, \tag{E.18}$$

$$(\sigma^{\upsilon\mu\nu}_{i})^{s}{}_{t} = \epsilon^{su}\epsilon_{tr}(\sigma^{\upsilon\mu\nu}_{i})_{u}{}^{r} = (2\delta^{\mu}_{i} + \delta^{\mu}_{o} - 1)(2\delta^{\nu}_{i} + \delta^{\nu}_{o} - 1)(\sigma^{\upsilon\mu\nu}_{i})_{t}{}^{s}, \tag{E.19}$$

$$(\bar{\sigma}^{\upsilon\mu\nu}_{\bar{i}})^{\dot{s}}{}_{\dot{t}} = \bar{\epsilon}^{\dot{u}\dot{s}}\bar{\epsilon}_{\dot{r}\dot{t}}(\bar{\sigma}^{\upsilon\mu\nu}_{\bar{i}})_{\dot{u}}{}^{\dot{r}} = (2\delta^{\mu}_{i} + \delta^{\mu}_{o} - 1)(2\delta^{\nu}_{i} + \delta^{\nu}_{o} - 1)(\bar{\sigma}^{\upsilon\mu\nu}_{\bar{i}})_{\dot{t}}{}^{\dot{s}}. \tag{E.20}$$

E.3 Useful identities involving upsilon, gamma, and sigma matrices

$$(\sigma^{f\mu\nu}_{o})_{s}{}^{t} = \tfrac{\mathrm{i}}{2}\{\hat{\Upsilon}(\tfrac{o\mu}{o\bar{o}})_{s\dot{r}}\hat{\Upsilon}(\tfrac{o\nu}{\bar{o}o})^{\dot{r}t} - \hat{\Upsilon}(\tfrac{o\nu}{o\bar{o}})_{s\dot{r}}\hat{\Upsilon}(\tfrac{o\mu}{\bar{o}o})^{\dot{r}t}\}, \tag{E.21}$$

$$(\bar{\sigma}^{f\mu\nu}_{\bar{o}})^{\dot{s}}{}_{\dot{t}} = \tfrac{\mathrm{i}}{2}\{\hat{\Upsilon}(\tfrac{o\mu}{\bar{o}o})^{\dot{s}r}\hat{\Upsilon}(\tfrac{o\nu}{o\bar{o}})_{r\dot{t}} - \hat{\Upsilon}(\tfrac{o\nu}{\bar{o}o})^{\dot{s}r}\hat{\Upsilon}(\tfrac{o\mu}{o\bar{o}})_{r\dot{t}}\}. \tag{E.22}$$

$$(\sigma^{\upsilon\mu\nu}_{i})_{s}{}^{t} = \tfrac{\mathrm{i}}{2}\{\hat{\gamma}(\tfrac{o\mu}{i\bar{i}})_{s\dot{r}}\hat{\gamma}(\tfrac{o\nu}{\bar{i}i})^{\dot{r}t} - \hat{\gamma}(\tfrac{o\nu}{i\bar{i}})_{s\dot{r}}\hat{\gamma}(\tfrac{o\mu}{\bar{i}i})^{\dot{r}t}\}, \tag{E.23}$$

$$(\bar{\sigma}^{\upsilon\mu\nu}_{\bar{i}})^{\dot{s}}{}_{\dot{t}} = \tfrac{\mathrm{i}}{2}\{\hat{\gamma}(\tfrac{o\mu}{\bar{i}i})^{\dot{s}r}\hat{\gamma}(\tfrac{o\nu}{i\bar{i}})_{r\dot{t}} - \hat{\gamma}(\tfrac{o\nu}{\bar{i}i})^{\dot{s}r}\hat{\gamma}(\tfrac{o\mu}{i\bar{i}})_{r\dot{t}}\}. \tag{E.24}$$

$$i\{\hat\Upsilon(\tfrac{o\mu}{o\bar o})_{s\dot t}g^{\nu\rho} - \hat\Upsilon(\tfrac{o\nu}{o\bar o})_{s\dot t}g^{\mu\rho}\} = \tfrac{1}{2}\{(\sigma_o^{f\mu\nu})_s{}^r\hat\Upsilon(\tfrac{o\rho}{o\bar o})_{r\dot t} - \hat\Upsilon(\tfrac{o\rho}{o\bar o})_{s\dot u}(\bar\sigma_{\bar o}^{f\mu\nu})^{\dot u}{}_{\dot t}\}. \quad (E.25)$$

$$\begin{aligned}
i\{\hat\gamma(\tfrac{o\mu}{i\bar\imath})_{s\dot t}g^{\nu\rho} - \hat\gamma(\tfrac{o\nu}{i\bar\imath})_{s\dot t}g^{\mu\rho}\} &= \tfrac{1}{2}\{(\sigma_i^{\upsilon\mu\nu})_s{}^r\hat\gamma(\tfrac{o\rho}{i\bar\imath})_{r\dot t} - \hat\gamma(\tfrac{o\rho}{i\bar\imath})_{s\dot u}(\bar\sigma_{\bar\imath}^{\upsilon\mu\nu})^{\dot u}{}_{\dot t}\}, && (E.26)\\
i\{\hat\gamma(\tfrac{o\mu}{\bar\imath i})_{\dot t s}g^{\nu\rho} - \hat\gamma(\tfrac{o\nu}{\bar\imath i})_{\dot t s}g^{\mu\rho}\} &= \tfrac{1}{2}\{-(\bar\sigma_{\bar\imath}^{\upsilon\mu\nu})_{\dot t}{}^{\dot r}\hat\gamma(\tfrac{o\rho}{\bar\imath i})_{\dot r s} - \hat\gamma(\tfrac{o\rho}{\bar\imath i})_{\dot t r}(\sigma_i^{\upsilon\mu\nu})^r{}_s\}. && (E.27)
\end{aligned}$$

$$\begin{aligned}
\{\hat\Upsilon(\tfrac{o\mu}{o\bar o})_{s\dot r}\hat\Upsilon(\tfrac{o\nu}{\bar o o})^{\dot r t} + \hat\Upsilon(\tfrac{o\nu}{o\bar o})_{s\dot r}\hat\Upsilon(\tfrac{o\mu}{\bar o o})^{\dot r t}\} &= \\
\{\hat\gamma(\tfrac{o\mu}{i\bar\imath})_{s\dot r}\hat\gamma(\tfrac{o\nu}{\bar\imath i})^{\dot r t} + \hat\gamma(\tfrac{o\nu}{i\bar\imath})_{s\dot r}\hat\gamma(\tfrac{o\mu}{\bar\imath i})^{\dot r t}\} &= 2g^{\mu\nu}\delta_s^t, && (E.28)\\
\{\hat\Upsilon(\tfrac{o\mu}{\bar o o})^{\dot u s}\hat\Upsilon(\tfrac{o\nu}{o\bar o})_{s\dot t} + \hat\Upsilon(\tfrac{o\nu}{\bar o o})^{\dot u s}\hat\Upsilon(\tfrac{o\mu}{o\bar o})_{s\dot t}\} &= \\
\{\hat\gamma(\tfrac{o\mu}{\bar\imath i})^{\dot u s}\hat\gamma(\tfrac{o\nu}{i\bar\imath})_{s\dot t} + \hat\gamma(\tfrac{o\nu}{\bar\imath i})^{\dot u s}\hat\gamma(\tfrac{o\mu}{i\bar\imath})_{s\dot t}\} &= 2g^{\mu\nu}\delta_{\dot t}^{\dot u}. && (E.29)
\end{aligned}$$

E.4 Useful identities for superspace calculations

$$\begin{aligned}
\overline{(\Theta^i\Theta^i)} &= (\bar\Theta^{\bar\imath}\bar\Theta^{\bar\imath}), && (E.30)\\
(\Theta^i\Theta^i)^2 &= -8i\Theta^{i0}\Theta^{i1}\Theta^{i2}\Theta^{i3}, && (E.31)\\
(\bar\Theta^{\bar\imath}\bar\Theta^{\bar\imath})^2 &= 8i\bar\Theta^{\bar\imath\dot 0}\bar\Theta^{\bar\imath\dot 1}\bar\Theta^{\bar\imath\dot 2}\bar\Theta^{\bar\imath\dot 3}, && (E.32)\\
(\Theta^i\Theta^i)\Theta^{is}\Theta^{ir} &= -\tfrac{1}{4}\epsilon^{sr}(\Theta^i\Theta^i)^2, && (E.33)\\
(\bar\Theta^{\bar\imath}\bar\Theta^{\bar\imath})\bar\Theta^{\bar\imath\dot s}\bar\Theta^{\bar\imath\dot r} &= -\tfrac{1}{4}\bar\epsilon^{\dot s\dot r}(\bar\Theta^{\bar\imath}\bar\Theta^{\bar\imath})^2. && (E.34)
\end{aligned}$$

$$\begin{aligned}
(\partial_{\Theta^{is}}(\Theta^i\Theta^i)) &= 2\epsilon_{st}\Theta^{it}, && (E.35)\\
(\partial_{\bar\Theta^{\bar\imath\dot t}}(\bar\Theta^{\bar\imath}\bar\Theta^{\bar\imath})) &= 2\bar\epsilon_{\dot t\dot r}\bar\Theta^{\bar\imath\dot r}, && (E.36)\\
(\partial_{\Theta^i}\partial_{\Theta^i}\ (\Theta^i\Theta^i)) &= -8, && (E.37)\\
(\partial_{\bar\Theta^{\bar\imath}}\partial_{\bar\Theta^{\bar\imath}}\ (\bar\Theta^{\bar\imath}\bar\Theta^{\bar\imath})) &= -8, && (E.38)\\
(\partial_{\bar\Theta^{\bar\imath}}\partial_{\bar\Theta^{\bar\imath}}\ (\bar\Theta^{\bar\imath}\bar\Theta^{\bar\imath}\ \bar\Theta^{\bar\imath\dot s})) &= -4\bar\Theta^{\bar\imath\dot s}, && (E.39)\\
(\partial_{\bar\Theta^{\bar\imath\dot t}}\ (\bar\Theta^{\bar\imath}\bar\Theta^{\bar\imath})^2) &= 4\bar\epsilon_{\dot t\dot u}\bar\Theta^{\bar\imath\dot u}(\bar\Theta^{\bar\imath}\bar\Theta^{\bar\imath}), && (E.40)\\
(\partial_{\bar\Theta^{\bar\imath}}\partial_{\bar\Theta^{\bar\imath}}\ (\bar\Theta^{\bar\imath}\bar\Theta^{\bar\imath})^2) &= -8(\bar\Theta^{\bar\imath}\bar\Theta^{\bar\imath}), && (E.41)\\
(\partial_{\Theta^i}\partial_{\Theta^i}\ \partial_{\Theta^i}\partial_{\Theta^i}\ (\Theta^i\Theta^i)^2) &= 64, && (E.42)\\
(\partial_{\bar\Theta^{\bar\imath}}\partial_{\bar\Theta^{\bar\imath}}\ \partial_{\bar\Theta^{\bar\imath}}\partial_{\bar\Theta^{\bar\imath}}\ (\bar\Theta^{\bar\imath}\bar\Theta^{\bar\imath})^2) &= 64. && (E.43)
\end{aligned}$$

$$\begin{aligned}
(D_iD_i)_{(1)} &= \partial_{\Theta^i}\partial_{\Theta^i} + 4i\hat\gamma(\tfrac{o\mu}{i\bar\imath})_{t\dot r}\bar\Theta^{\bar\imath\dot r}\epsilon^{ts}\partial_{\Theta^{is}}\partial_\mu + 4(\bar\Theta^{\bar\imath}\bar\Theta^{\bar\imath})\Box, && (E.44)\\
(\bar D_{\bar\imath}\bar D_{\bar\imath})_{(2)} &= \partial_{\bar\Theta^{\bar\imath}}\partial_{\bar\Theta^{\bar\imath}} + 4i\Theta^{ir}\hat\gamma(\tfrac{o\mu}{i\bar\imath})_{r\dot u}\partial_\mu\bar\epsilon^{\dot u\dot t}\partial_{\bar\Theta^{\bar\imath\dot t}} + 4(\Theta^i\Theta^i)\Box. && (E.45)
\end{aligned}$$

$$\begin{aligned}
&\exp\{-2i(2\delta_i^\mu + 2\delta_o^\mu - 1)(\Theta^i\hat\gamma(\tfrac{o\mu}{i\bar\imath})\bar\Theta^{\bar\imath})\partial_\mu\} = \\
&= \{1 - 2i(2\delta_i^\mu + 2\delta_o^\mu - 1)(\Theta^i\hat\gamma(\tfrac{o\mu}{i\bar\imath})\bar\Theta^{\bar\imath})\partial_\mu - 2(\Theta^i\hat\gamma(\tfrac{o\mu}{i\bar\imath})(\Theta^i\hat\gamma(\tfrac{o\nu}{i\bar\imath})\partial_\mu\partial_\nu + \\
&+ 2i(2\delta_i^\mu + 2\delta_o^\mu - 1)(\Theta^i\Theta^i)(\bar\Theta^{\bar\imath}\bar\Theta^{\bar\imath})(\Theta^i\hat\gamma(\tfrac{o\mu}{i\bar\imath})\bar\Theta^{\bar\imath})\partial_\mu\Box + \tfrac{1}{4}(\Theta^i\Theta^i)^2(\bar\Theta^{\bar\imath}\bar\Theta^{\bar\imath})^2\Box^2\}. \quad (E.46)
\end{aligned}$$

For an anti-chiral field $\bar A$, in the anti-chiral basis

$$\bar A_{(2)}(x, \bar\Theta^{\bar\imath}) = \cdots + \bar{\mathcal{A}}(x)(\bar\Theta^{\bar\imath}\bar\Theta^{\bar\imath}) + 2\bar\psi_{\bar\imath\dot s}(x)\bar\Theta^{\bar\imath\dot s}(\bar\Theta^{\bar\imath}\bar\Theta^{\bar\imath}) + \bar F(x)(\bar\Theta^{\bar\imath}\bar\Theta^{\bar\imath})^2, \quad (E.47)$$

we obtain

$$\begin{aligned}
&(\bar D_{\bar\imath}\bar D_{\bar\imath}\ \bar D_{\bar\imath}\bar D_{\bar\imath}\ \bar A)_{(2)} = \\
&= \exp\{-2i(2\delta_i^\mu + 2\delta_o^\mu - 1)(\Theta^i\hat\gamma(\tfrac{o\mu}{i\bar\imath})\bar\Theta^{\bar\imath})\partial_\mu\}\cdot \\
&\cdot\{64\bar F(x) - \Theta^{ir}64i\hat\gamma(\tfrac{o\mu}{i\bar\imath})_{r\dot t}\partial_\mu\bar\psi_i{}^{\dot t}(x) - (\Theta^i\Theta^i)32\Box\bar{\mathcal{A}}(x) + \cdots\}. \quad (E.48)
\end{aligned}$$

References

[1] L. A. Wills Toro, *(I, q)-graded Lie Algebraic Extensions of the Poincaré Algebra, Constraints on I and q*, to appear in J. Math. Phys. (March or April 1995).

[2] L. A. Wills Toro, *(I, q)-graded Lie Algebraic Extensions of the Poincaré Algebra: Grading beyond Supersymmetry*, Univ. Granada preprint UG-FT-44/94 (November 1994).

[3] O. Klein, J. Phys. **9**,1 (1938);
G. Lüders, Z. Naturforschg. **13** a, 254-260 (1958);
T. Kinoshita, Phys. Rev. **110** Nr. 4, 978-981 (1958).

[4] W. Pauli, Phys. Rev. **58**, 716-722 (1940);
H. Araki, J. Math. Phys. **2** Nr. 3, 267-270 (1961).

[5] V. Rittenberg and D. Wyler, *Generalized Superalgebras*, Nucl. Phys. **B139**, 189 (1978).

[6] L. A. Wills Toro, *Generalized quantization relations and the relation among spin and statistic*, in preparation.

[7] L. A. Wills Toro, *All Possible Generators of* $\mathbb{Z}_2 \times (\mathbb{Z}_4 \times \mathbb{Z}_4)$*-graded symmetries of the S-Matrix*, in preparation.

[8] S. Coleman and J. Mandula, *All Possible Symmetries of the S-Matrix*, Phys. Rev. **159**, 1251 (1967).

[9] R. Haag, J.T. Łopuszański and M.F. Sohnius, *All Possible Generators of Supersymmetries of the S-Matrix*, Nucl. Phys. **B88**, 257 (1975).

[10] L. A. Wills Toro, *Reduced (I, q)-graded Lie algebras*, in preparation.

[11] A. Salam and J. Strathdee, Nucl. Phys. **B76**, 477 (1974);
S. Ferrara, J. Wess and B. Zumino, Phys. Lett. **51B**, 239 (1974).

[12] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, 2nd. rev. and expanded ed. (Princeton Univ. Press, Princeton, 1992);
O. Piguet and K. Sibold, *Renormalized Supersymmetry*, Progress in Physics V.**12** (Birkhäuser, Boston, 1986).

[13] L. A. Wills Toro, *(I, q)-graded Lie Algebras over a Commutative Field*, CBPF-preprint (1994).

[14] P. Cartier, Séminaire Bourbaki, (May 1955).

[15] R. Ree, Can. J. Math. **12**, 493 (1960).

[16] M. Scheunert, *Graded tensor calculus*, J. Math. Phys. **24** No. 11, 2658 (1983).
M. Scheunert, *Casimir elements of* ϵ *Lie algebras*, J. Math. Phys. **24** No. 11, 2671 (1983).
M. Scheunert, *The theory of Lie Superalgebras*, Lecture Notes in Mathematics **716** (Springer-Verlag, 1979).