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# Exactly solvable string models of curved space-time backgrounds

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**Abstract**

We consider a new 3-parameter class of exact 4-dimensional solutions in closed string theory and solve the corresponding string model, determining the physical spectrum and the partition function. The background fields (4-metric, antisymmetric tensor, two Kaluza-Klein vector fields, dilaton and modulus) generically describe axially symmetric stationary rotating (electro)magnetic flux-tube type universes. Backgrounds of this class include both the 'dilatonic' ($a = 1$) and 'Kaluza-Klein' ($a = \sqrt{3}$) Melvin solutions and the uniform magnetic field solution, as well as some singular space-times. Solvability of the string $\sigma$-model is related to its connection via duality to a simpler model which is a "twisted" product of a flat 2-space and a space dual to 2-plane. We discuss some physical properties of this model (tachyonic instabilities in the spectrum, gyromagnetic ratio, issue of singularities, etc.). It provides one of the first examples of a consistent solvable conformal string model with explicit $D = 4$ curved space-time interpretation.

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## 1. Introduction

Conformal $\sigma$-models describing the propagation of closed strings in curved space-times are, unfortunately, so complicated that the spectrum of the physical string excitations is known only in a few special cases. In this paper we will introduce a new class of exact conformal string models representing non-trivial space-time backgrounds, for which the corresponding CFT's can be described in terms of free oscillators. The resulting world-sheet Hamiltonian is quartic in the free creation/annihilation operators and is diagonal in Fock space. This makes it possible to calculate the spectrum and the partition function. The reason for solvability of these models is in their formal connection through world-sheet (or target space) duality to a flat theory. Because of non-trivial boundary conditions the duality does not leave us within flat CFT but leads to a new conformal theory.

The corresponding string backgrounds interpolate continuously between very different configurations, e.g. between a homogeneous "rotating" universe with a uniform magnetic field [1,2]

$$ds_4^2 = -(dt + \tfrac{1}{2}\beta\rho^2 d\varphi)^2 + d\rho^2 + \rho^2 d\varphi^2 + dx_3^2 \ , \tag{1.1}$$

$$\mathcal{A} = -\mathcal{B} = \tfrac{1}{2}\beta\rho^2 d\varphi \ , \quad B = \tfrac{1}{2}\beta\rho^2 d\varphi \wedge dt \ , \quad \phi = \phi_0 = \text{const} \ , \quad \sigma = 0 \ ,$$

the dilatonic "$a = 1$" Melvin-type [3] static magnetic flux-tube universe [4,5]

$$ds_4^2 = -dt^2 + d\rho^2 + F^2(\rho)\rho^2 d\varphi^2 + dx_3^2 \ , \tag{1.2}$$

$$\mathcal{A} = -\mathcal{B} = \beta F(\rho)\rho^2 d\varphi \ , \quad B = 0 \ ,$$

$$e^{2(\phi-\phi_0)} = F \ , \quad \sigma = 0 \ , \quad F^{-1} = 1 + \beta^2\rho^2 \ ,$$

the "$a = \sqrt{3}$" (Kaluza-Klein) Melvin solution [4]

$$ds_4^2 = -dt^2 + d\rho^2 + \tilde{F}(\rho)\rho^2 d\varphi^2 + dx_3^2 \ , \tag{1.3}$$

$$\mathcal{A} = q_+\tilde{F}(\rho)\rho^2 d\varphi \ , \quad \mathcal{B} = 0 \ , \quad B = 0 \ ,$$

$$\phi = \phi_0 \ , \quad e^{2\sigma} = \tilde{F}^{-1} = 1 + q_+^2\rho^2 \ ,$$

and singular "rotating" space-times with no gauge fields. Here $\mathcal{A}$ and $\mathcal{B}$ are the abelian vector and axial vector 1-forms, $B$ is the antisymmetric tensor 2-form, $\phi$ is the dilaton and $\sigma$ is the modulus scalar corresponding to the compact ($x^5 \equiv y \in (0, 2\pi R)$) Kaluza-Klein dimension.

Explicitly, our 3-parameter ($\alpha, \beta, q_+$ or $a_+, c_+, c_-$) class of $D = 4$ axially symmetric exact string solutions is represented by the following background fields:

$$ds_4^2 = -f_1(\rho)dt^2 + f_2(\rho)d\varphi dt + f_3(\rho)d\varphi^2 + d\rho^2 + dx_3^2 \ , \tag{1.4}$$

$$f_1 = 1 + \tfrac{1}{4}a_+^2c_-^2\rho^4 F(\rho)\tilde{F}(\rho) \ , \quad f_2 = c_-[1 + \tfrac{1}{4}(c_+^2 - a_+^2 - c_-^2)\rho^2]F(\rho)\tilde{F}(\rho)\rho^2,$$

$$f_3 = (1 - \tfrac{1}{4}c_-^2\rho^2)F(\rho)\tilde{F}(\rho)\rho^2 \ ,$$

$$\mathcal{A} = \tfrac{1}{2}\tilde{F}(\rho)\rho^2[(a_+ + c_+)d\varphi + a_+c_-dt] \ , \tag{1.5}$$

$$\mathcal{B} = \tfrac{1}{2}F(\rho)\rho^2[(a_+ - c_+)d\varphi + a_+c_-dt] \ , \quad B = -\tfrac{1}{2}c_-F(\rho)\rho^2 d\varphi \wedge dt \ ,$$

$$e^{2(\phi-\phi_0)} = F(\rho) \ , \qquad e^{2\sigma} = \frac{F(\rho)}{\tilde{F}(\rho)} \ , \tag{1.6}$$

$$F(\rho) \equiv \frac{1}{1+\rho^2/\rho_0^2} \ , \quad \tilde{F}(\rho) \equiv \frac{1}{1+\rho^2/\tilde{\rho}_0^2} \ , \tag{1.7}$$

$$\rho_0^{-2} \equiv \tfrac{1}{4}[(a_+ - c_+)^2 - c_-^2] = \alpha\beta \ ,$$

$$\tilde{\rho}_0^{-2} \equiv \tfrac{1}{4}[(a_+ + c_+)^2 - c_-^2] = q_+(q_+ + \beta - \alpha) \ ,$$

$$a_+ = q_+ - \alpha \ , \quad c_+ = q_+ + \beta \ , \quad c_- = \alpha - \beta \ .$$

The previously known magnetic solutions (1.1), (1.2) and (1.3) are obtained in the special cases: $a_+ = 0, \ c_+ = -c_- = \beta$; $\ a_+ = 0, \ c_+ = 2\beta, \ c_- = 0$ and $a_+ = c_+ = q_+, \ c_- = 0$.

In general, the metric is stationary and describes a rotating electro-magnetic flux tube universe. Asymptotically the space-time is that of a product of a flat space and a rotating cylinder with radius going to zero at large $\rho$. These models can be interpreted, in particular, as describing the geometry induced by the two magnetic fields (in $d\mathcal{A}$ and $d\mathcal{B}$) associated with the Kaluza-Klein gauge groups $U(1)_{\rm v}$ and $U(1)_{\rm a}$,

$$\mathbf{B}_{\rm v} = \frac{\mathbf{B}_{\rm v0}}{\left(1+\rho^2/\tilde{\rho}_0^2\right)^2} \ , \quad \mathbf{B}_{\rm a} = \frac{\mathbf{B}_{\rm a0}}{\left(1+\rho^2/\rho_0^2\right)^2} \ , \tag{1.8}$$

$$\mathbf{B}_{\rm v0} = a_+ + c_+ \ , \qquad \mathbf{B}_{\rm a0} = a_+ - c_+ \ .$$

For generic values of the parameters there are also non-vanishing electric fields. When $\rho_0^2$ or $\tilde{\rho}_0^2$ are negative, the curvature has singularities at $\rho^2 = -\rho_0^2$ or $\rho^2 = -\tilde{\rho}_0^2$.

The important special cases correspond to

$$\alpha\beta q_+(q_+ + \beta - \alpha) = 0 \ , \tag{1.9}$$

i.e. to $\rho_0^2 = \infty$ or $\tilde{\rho}_0^2 = \infty$, when at least one of the magnetic fields uniformly extends to infinity. For these values of the parameters the corresponding CFT simplifies; in particular, the quantum Hamiltonian becomes quadratic in oscillators and the partition function takes the form of a modular integral and a sum over winding sectors of a left-right factorized expression.

These string solutions represent typical flux tube type uniform electromagnetic backgrounds in closed string theory. Such backgrounds are interesting for several reasons. Strings in magnetic fields are expected to undergo phase transitions with a possible symmetry restoration in a way analogous to gauge theories [6]. This is suggested by the emergence of tachyons in the spectrum for critical values of the magnetic field where the partition function develops new divergences [2]. Similar electromagnetic backgrounds are also related to a description of pair creation of charged black holes [7,8,9]. Knowing the solution of the string model (i.e. of the conformal field theory) corresponding, e.g., to the Melvin solution of the low-energy effective field theory, is a necessary step towards the analysis of such processes at the level of string theory.

The present construction illustrates, in particular, how one can find complicated exact string solutions without actually solving the equations of low-energy effective field theory but instead looking directly for conformal 2d $\sigma$-models with the desired properties. The world-sheet duality transformations relating complicated $\sigma$-models to simpler ones provide a useful tool. For example, starting with a dimension $D \geq d+n$ flat model with a number

$d$ of periodic coordinates and making formal $O(d+n, d+n; R)$ world-sheet duality transformations (see e.g. [10,11,12]) with *continuous* parameters, one obtains new *inequivalent* conformal theories (with $O(d,d;Z)$ dualities as symmetries), corresponding to complicated space-time backgrounds which solve the low-energy effective equations.

The contents of this paper, which is a sequel to [2,5], is the following.

We shall start in Section 2.1 with a construction of our string model by applying coordinate shifts and simple duality (or, equivalently, a special $O(3,3;R)$ duality transformation) to a (dual to) flat model. This $O(3,3;R)$ duality leads to a non-trivial CFT since two of our coordinates are compact. We shall also consider a simple duality transformation in the Kaluza-Klein coordinate which will act on our class of models relating members with certain different values of the parameters. Next, in Section 2.2 we shall show that the resulting $\sigma$-model is locally (ignoring topology) equivalent to a particular model with two null Killing symmetries considered in [1] and, as such, is conformal to all orders in $\alpha'$. In Section 2.3 we shall discuss some special cases, namely plane-wave type models (corresponding, e.g., to the solution (1.1)) and the model behind the Melvin solution [5]. Some generalizations, involving, in particular, the replacement of the (dual to) 2-plane by a $D=2$ black hole space and applying a similar ($O(3,3;R)$) duality transformation, will be briefly considered in Section 2.4.

Section 3 will be devoted to a space-time/low energy field theory interpretation of our conformal string model. Rearranging the $\sigma$-model action in a Kaluza-Klein way (i.e. separating terms with compact $x^5$-coordinate) we shall determine the background fields (1.4)–(1.7) which solve the equations of the corresponding $D=4$ effective field theory. We shall then discuss in turn various special cases, emphasizing, in particular, that differently looking backgrounds which originate from $\sigma$-models related by simple duality in a compact coordinate represent different "sides" of the same string solution, i.e. of the same CFT. We shall also mention (in Section 3.7) some closely related exact string solutions which may have an interpretation of (3+1)-dimensional black holes in external electromagnetic fields (but for which we are unable to solve the corresponding string model).

In Section 4 we shall start the discussion of the solution of this string model. We shall first explain (on a simple special case) how the theory can be effectively expressed in terms of free fields and then proceed with a computation of the partition function $Z$ on the torus defined by the string path integral. Although the model is not gaussian, we will show that all path integrals can be evaluated explicitly and obtain the expression for $Z$ in terms of the standard modular integral and sum over winding numbers and also of two additional ordinary integrals. The latter integrals can be easily computed in the special cases (1.9). We shall also discuss peculiar target space duality invariance properties of $Z$.

In Section 5 we shall systematically describe a solution of the string model using the canonical operator quantization approach. We shall first (in Section 5.1) derive the expression for the general solution of the classical equations of motion on the cylinder (free string propagation) in terms of constant zero-mode parameters and free oscillators. We shall then canonically quantize the theory (Section 5.2) using a light-cone type gauge and derive the quantum Hamiltonian. The latter, in general, will be fourth order in creation/annihilation operators (becoming quadratic only in the special case of (1.9)) but will be diagonal in Fock space, enabling a straightforward determination of the string spectrum. In Section 5.3 we shall illustrate this construction by considering its point-particle limit in the Melvin model case, i.e. derive the expression for the zero level scalar (tachyon) spectrum directly from the Klein-Gordon operator in the Melvin background (1.2). In Section 5.4 we shall show that the operator approach leads to the same expression for the partition function that was earlier obtained in the path integral approach.

Some physical properties of our class of string models will be considered in Section 6. In Section 6.1 we shall discuss the string spectrum in the special case when $\alpha = \beta$

(which includes the Melvin model). Such models turn out to have tachyonic instabilities in the charged state sector of the spectrum. Implications for the value of the gyromagnetic factor in closed string theory will be mentioned in Section 6.2. In Section 6.3 we shall look at the string models with $\alpha\beta q_+(q_+ + \beta - \alpha) < 0$ corresponding to singular space-time backgrounds and point out the existence of new tachyonic states related to the presence of the quartic term in the string Hamiltonian. We will find that in these cases even the state which is the counterpart of the usual graviton becomes tachyonic, what reflects a strong instability of these backgrounds. We shall also comment on the relation of the corresponding backgrounds to $D = 3$ black string and black hole geometries.

Section 7 contains some concluding remarks. In the Appendix we give the expression for the curvature scalar for the metric (1.4).

## 2. String model

We shall start with a construction of our class of exact conformal $D = 4$ string solutions directly at the string $\sigma$-model level, using duality transformations to relate simple $\sigma$-models to a more complicated one. This will provide a clue to why the resulting string models, describing complicated curved space-time backgrounds are actually exactly solvable.

### *2.1. Duality transformations*

We shall consider axially-symmetric $D = 4$ string backgrounds which are direct products of a non-trivial three-dimensional $(t, x_1, x_2)$ part and a line ($x_3$-direction). We shall also introduce a coupling of a closed string to an external gauge field background by using a stringy version of the Kaluza-Klein approach, i.e. by adding an extra compact internal direction $x^5 = y$ (with period $2\pi R$). We shall often use the "light-cone" coordinates

$$u = y - t \ , \qquad v = y + t \ , \qquad y \in (0, 2\pi R) \ . \tag{2.1}$$

Let us start with an auxiliary $\sigma$-model describing a string in $D = 4$ space-time which is a direct product of a $D = 2$ space-time cylinder $(t, y)$ and a space dual to a 2-plane[1]

$$I = \frac{1}{\pi\alpha'} \int d^2\sigma \big[\partial u \bar{\partial} v + \partial \rho \bar{\partial} \rho + \rho^{-2} \partial \tilde{\varphi} \bar{\partial} \tilde{\varphi} + \mathcal{R}(\phi_0 - \tfrac{1}{2} \ln \rho^2)\big] \ , \tag{2.2}$$

[1] The construction that follows can be repeated in the case when the dual plane is replaced by a plane itself (with a trivial dilaton). The resulting backgrounds will also represent exact string solutions. Their form can be obtained from the expressions that follow by replacing $\rho$ by $1/\rho$ in the functions (but not in derivatives or differentials). This class of backgrounds will generically be singular at $\rho = 0$ (and will not contain solutions (1.1),(1.2),(1.3)). Moreover, in contrast to the model (2.3) discussed below, the corresponding string model will *not* be exactly solvable by our methods (for example, the partition function will not be explicitly computable). One may also consider a generalization of (2.2) (or of a similar model with the 2-plane part) by adding an arbitrary constant in front of the $\rho^{\mp 2} \partial \tilde{\varphi} \bar{\partial} \tilde{\varphi}$ term. The resulting models will have conical singularities and do not seem to be well defined (for a discussion in a particular case see [5]).

where $\mathcal{R} \equiv \frac{1}{4}\alpha'\sqrt{g}R^{(2)}$ and $\tilde{\varphi}$ should have period $2\pi\alpha'$ to preserve equivalence of the "dual plane" model to the flat 2-plane CFT [13]. A more general model is obtained from (2.2) by making a coordinate shift and adding constant antisymmetric tensor terms

$$I = \frac{1}{\pi\alpha'}\int d^2\sigma\big[(\partial u + \alpha\partial\tilde{\varphi})(\bar{\partial}v + \beta\bar{\partial}\tilde{\varphi}) + \partial\rho\bar{\partial}\rho + \rho^{-2}\partial\tilde{\varphi}\bar{\partial}\tilde{\varphi} \tag{2.3}$$

$$+ q_1(\partial u\bar{\partial}\tilde{\varphi} - \bar{\partial}u\partial\tilde{\varphi}) + q_2(\partial v\bar{\partial}\tilde{\varphi} - \bar{\partial}v\partial\tilde{\varphi}) + \mathcal{R}(\phi_0 - \tfrac{1}{2}\ln\rho^2)\big] \ .$$

Here $\alpha, \beta, q_i$ $(i = 1, 2)$ are constant free parameters of dimension $cm^{-1}$, i.e. the dimensionless parameters of our model are $\sqrt{\alpha'}\alpha,\ \sqrt{\alpha'}\beta,\ \sqrt{\alpha'}q_i$ and $r = R/\sqrt{\alpha'}$. Ignoring target space topology, the two models (2.2) and (2.3) are of course "locally-equivalent"; in particular, (2.3) also solves the conformal invariance equations. The corresponding conformal field theories will not, however, be equivalent because of the compactness of $y = \frac{1}{2}(u + v)$ and $\tilde{\varphi}$ (if $R \neq \infty$ the redefined coordinates $u + \alpha\tilde{\varphi}$ and $v + \beta\tilde{\varphi}$ will be well-defined, i.e. periodic, only for special values of $\alpha'\alpha/R$ and $\alpha'\beta/R$).

The string models we are going to discuss can be obtained from (2.3) by making the duality transformation in the $\tilde{\varphi}$ direction. Gauging the corresponding isometry and introducing the zero gauge field constraint with the Lagrange multiplier $\varphi$ [14,13] we find the dual action (we add also the flat $x_3$-direction):

$$I = \frac{1}{\pi\alpha'}\int d^2\sigma\big[\partial u\bar{\partial}v + \partial\rho\bar{\partial}\rho$$

$$+ F(\rho)\rho^2[\partial\varphi + (q_1 + \beta)\partial u + q_2\partial v][\bar{\partial}\varphi + q_1\bar{\partial}u + (q_2 - \alpha)\bar{\partial}v] \tag{2.4}$$

$$+ \partial x_3\bar{\partial}x_3 + \mathcal{R}(\phi_0 + \tfrac{1}{2}\ln F)\big] \ , \qquad F \equiv (1 + \alpha\beta\rho^2)^{-1} \ ,$$

or, equivalently (dropping the total derivative term $\partial y\bar{\partial}t - \partial t\bar{\partial}y$)

$$I = \frac{1}{\pi\alpha'}\int d^2\sigma\big[ - \partial t\bar{\partial}t + \partial y\bar{\partial}y + \partial\rho\bar{\partial}\rho$$

$$+ F(\rho)\rho^2[\partial\varphi + c_+\partial y + c_-\partial t][\bar{\partial}\varphi + a_+\bar{\partial}y + a_-\bar{\partial}t] \tag{2.5}$$

$$+ \partial x_3\bar{\partial}x_3 + \mathcal{R}(\phi_0 + \tfrac{1}{2}\ln F)\big] \ ,$$

$$F^{-1} = 1 + \tfrac{1}{4}[(a_+ - c_+)^2 - (a_- - c_-)^2]\rho^2 \ .$$

We have introduced the following linear combinations of the parameters which will be convenient to use below along with $\alpha, \beta, q_i$

$$a_+ \equiv q_+ - \alpha \ , \quad c_+ \equiv q_+ + \beta \ , \quad a_- \equiv -q_- - \alpha \ , \quad c_- \equiv -q_- - \beta \ , \quad q_\pm \equiv q_1 \pm q_2 \ ,$$

$$\alpha = \tfrac{1}{2}(c_+ - a_+ + c_- - a_-) \ , \quad \beta = \tfrac{1}{2}(c_+ - a_+ - c_- + a_-) \ , \tag{2.6}$$

$$q_+ = \tfrac{1}{2}(c_+ + a_+ + c_- - a_-) \ , \quad q_- = \tfrac{1}{2}(-c_+ + a_+ - c_- - a_-) \ .$$

The physical meaning of these parameters will become clear in the next section. The angle $\varphi$ in (2.5) has period $2\pi$ so that, in the trivial case of $\alpha, \beta, q_i = 0$, we get a flat 5-space with coordinates $t,\ y,\ x_1 = \rho\cos\varphi,\ x_2 = \rho\sin\varphi,\ x_3$.

The $\sigma$-model (2.4) can be interpreted as a particular $O(3,3;R)$ $(t,y,\tilde{\varphi})$ duality transformation (making shifts of $t$ and $y$ by $\tilde{\varphi}$, adding torsion terms, and performing the simple duality in $\tilde{\varphi}$) of the direct product model (2.2) $R_t \times S^1_y \times$ (dual 2-plane)$_{\rho,\varphi}$ or, since the latter itself is a duality rotation of a plane, directly of the flat model $R_t \times S^1_y \times R^2$. Since $y$ and $\tilde{\varphi}$ are compact, this $O(3,3;R)$ duality transformation with continuous values of the parameters is *not* a symmetry of one conformal theory (cf. [15]) but maps a trivial flat-space CFT into a non-trivial one we shall discuss below. Members of the resulting class of conformal theories parametrized by $\alpha,\beta,q_\pm$ will be invariant only under some special $O(2,2;Z)$ duality transformations.

Eq. (2.4) or (2.5) is the action of the string model investigated in this paper (another form of the action is (2.13)). It contains four parameters but one of them, $q_- = q_1 - q_2$, can be removed by a coordinate transformation. In fact, the dependence on $q_i$ can be formally eliminated by introducing the new coordinate

$$\varphi' \equiv \varphi + q_1 u + q_2 v = \varphi + q_+ y - q_- t \ . \tag{2.7}$$

However, this transformation does not, in general, preserve the equivalence of the corresponding conformal models because of the different periods of the compact coordinates $\varphi$ and $y = \frac{1}{2}(u+v)$ (the equivalence is preserved only in the special cases when $q_+ R = m = \pm 1, \pm 2, \ldots$). At the same time, assuming $t$ is non-compact, we can, in fact, shift $\varphi \to \varphi + \lambda t$, eliminating the dependence on $q_-$ by choosing $\lambda = q_-$. As a result, the models we shall discuss below will essentially depend only on the *three* parameters $\alpha,\beta$ and $q_+$ (as well as on $R$).

Different choices of $q_-$ correspond to different definitions of the coordinate $\varphi$ (or different choices of "frames" or "gauges"). For example, one may set $q_- = 0$, i.e. (see (2.6))

$$a_+ = q_+ - \alpha \ , \quad a_- = -\alpha \ , \quad c_+ = q_+ + \beta \ , \quad c_- = -\beta \ , \tag{2.8}$$

or $q_- = -\alpha$, i.e.

$$a_- = 0 \ , \quad c_- = \alpha - \beta \ , \quad a_+ = q_+ - \alpha, \quad c_+ = q_+ + \beta \ . \tag{2.9}$$

This second gauge may be called a "chiral gauge" since when $a_- = 0$ there is no $\bar{\partial} y$ coupling term in (2.5). Below it will be convenient to present some of the results in a general form without specifying a particular gauge.

Since the models (2.3) and (2.5) are related by the simple $\varphi$ - duality transformation they should represent the same string solution, i.e. the corresponding CFT's should be completely equivalent [13] (see also [16,12]). We will explicitly demonstrate this in Sections 4,5 by solving the theory (computing the quantum Hamiltonian and the partition function) starting directly with the action (2.5) and with the simpler dual action (2.3).

Let us now perform the duality transformation of the $\sigma$-model (2.5) in the compact $y$-direction. We get[2]

$$\tilde{I} = \frac{1}{\pi\alpha'} \int d^2\sigma \big[ - \partial t \bar{\partial} t + \partial \tilde{y} \bar{\partial} \tilde{y} + \partial\rho\bar{\partial}\rho$$

$$+ \ \tilde{F}(\rho)\rho^2 [\partial\varphi + c_+ \partial\tilde{y} + c_- \partial t][\bar{\partial}\varphi - a_+ \bar{\partial}\tilde{y} + a_- \bar{\partial} t] \tag{2.10}$$

[2] If one adds to (2.5) the total derivative term $C(\partial y \bar{\partial} t - \partial t \bar{\partial} y)$, $C = \text{const}$, one gets $\tilde{I}$ with $\tilde{y}$ replaced by $\tilde{y}' = \tilde{y} - Ct$. This leads to a redefinition of parameters $a_-$, $c_-$.

$$+\,\partial x_3\bar\partial x_3+\mathcal{R}(\phi_0+\tfrac12\ln\tilde F)\big]\ ,$$

$$\tilde F^{-1}(\rho)\equiv F^{-1}(\rho)[1+a_+c_+F(\rho)\rho^2]=1+\tfrac14[(a_++c_+)^2-(a_--c_-)^2]\rho^2\ .$$

This action is related to (2.5) by any of the two transformations

$$y\to\tilde y\ ,\qquad a_+\to -a_+\ ,\qquad or\qquad y\to-\tilde y\ ,\qquad c_+\to -c_+\ . \tag{2.11}$$

The same conclusion is reached by making the $y$-duality directly in our starting model (2.3) (the duality transformations in $\tilde\varphi$ and $y$ directions are independent). Note that the $\sigma$-models with $a_+=0$ or $c_+=0$ are "self-dual", i.e. preserve their form under $y$-duality.

Again, the two $y$-dual models (2.5) and (2.10) should represent the same conformal theory (provided the period of the dual coordinate $\tilde y$ is taken to be $2\pi\alpha'/R$): the CFT (Hamiltonian, spectrum, etc.) will be invariant under replacing $R$ by $\alpha'/R$, interchanging the winding $w$ and momentum $m$ quantum numbers in the $y$-direction *and* transforming $a_+\to -a_+$ (or $w\leftrightarrow -m,\ \ c_+\to -c_+$). The additional transformation of a parameter $a_+$ (or $c_+$) is an interesting feature of this model. Naively, it may seem that the model has just *one* non-trivial compact direction ($y$), so that the CFT should be invariant just under the transformation $R\to\alpha'/R$ and interchanging winding and momentum numbers ($a_+\to -a_+$ should then be an additional symmetry of CFT). As we shall see in Sections 4,5 this is not the case; for example, the partition function $Z(R,a_+,c_+,q_+)$ is *not* invariant just under $R\to\alpha'/R$ (as happened in the "self-dual" $a_+=0, q_i=0$ model of [2]) but satisfies

$$Z(R,a_+,c_+,q_+)=Z(\alpha'/R,-a_+,c_+,q_+)=Z(\alpha'/R,a_+,-c_+,q_+)\ . \tag{2.12}$$

This may be compared to the case of 2-torus model, where one has 4 parameters transforming under $O(2,2;Z)$ duality. The need to transform an extra parameter ($a_+$ or $c_+$) under the $y$-duality is related to the presence in (2.5) (or (2.3)) of the couplings of $y$ to another angular variable $\varphi$ (or $\tilde\varphi$).

*2.2. Conformal invariance*

Using the combination $\varphi'$ (2.7) the model (2.4) can be represented also as

$$I=\frac{1}{\pi\alpha'}\int d^2\sigma\big[F(\rho)(\partial u-2\alpha A)(\bar\partial v+2\beta\bar A)+\partial\rho\bar\partial\rho+\rho^2\partial\varphi'\bar\partial\varphi' \tag{2.13}$$

$$+\,\partial x_3\bar\partial x_3+\mathcal{R}(\phi_0+\tfrac12\ln F)\big]\ ,$$

$$A=\tfrac12\rho^2\partial\varphi'\ ,\qquad \bar A=\tfrac12\rho^2\bar\partial\varphi'\ . \tag{2.14}$$

This form of the action is useful in order to demonstrate its exact conformal invariance.[3] In order to check the $\sigma$-model conformal invariance conditions (which are local and covariant

[3] The argument about the duality to the simple model (2.3), which is itself formally (ignoring topology) related to a flat space by a combination of a coordinate transformation and duality, guarantees only that the leading-order $\beta$-function equations are satisfied, i.e. is not by itself sufficient in order to prove that there exists a local scheme in which (2.4) is conformal to all orders in $\alpha'$.

target space equations) one may ignore the difference between $\varphi$ and $\varphi'$, or simply set $q_i = 0$. Then the Lagrangian in (2.5) becomes ($I = \int d^2\sigma L/\pi\alpha'$)

$$L(q_i = 0) \ = F(x)[\partial u - 2\alpha A_i(x)\partial x^i][\bar\partial v + 2\beta A_i(x)\bar\partial x^i] \tag{2.15}$$

$$+ \ \partial x_a \bar\partial x^a + \mathcal{R}(\phi_0 + \tfrac{1}{2}\ln F) \ ,$$

$$A_i = -\tfrac{1}{2}\epsilon_{ij}x^j \ , \qquad F^{-1} = 1 + \alpha\beta x^i x_i \ , \quad x^a = (x^i, x^3) \ ,$$

i.e. it corresponds to a special case of a generalized "$F$-model" considered in Section 5 of ref. [1]

$$L = F(x)\big[\partial u + 2B_a(x)\partial x^a\big][\bar\partial v + 2\bar B_a(x)\bar\partial x^a\big] + \partial x_a\bar\partial x^a + \mathcal{R}\phi(x) \ . \tag{2.16}$$

When the vectors $B_a$ and $\bar B_a$ have constant field strengths $\mathcal{F}_{ab}$ and $\bar{\mathcal{F}}_{ab}$ this model was shown to be conformally invariant to all loop orders provided [1]

$$\partial^a\partial_a F^{-1} + 2\mathcal{F}^{ab}\bar{\mathcal{F}}_{ab} = 0 \ , \qquad \phi = \phi_0 + \tfrac{1}{2}\ln F \ . \tag{2.17}$$

In the case of (2.5)

$$\mathcal{F}_{ij} = -\alpha\epsilon_{ij} \ , \quad \bar{\mathcal{F}}_{ij} = \beta\epsilon_{ij} \ , \quad F^{-1} = 1 + \alpha\beta x^i x_i \ , \quad i = 1,2 \ , \tag{2.18}$$

so that the condition (2.17) is indeed satisfied.

*2.3. Special cases*

The $R = \infty$ limit of our model (2.5) can also be represented by the Lagrangian (2.15) corresponding to the choice of $q_i = 0$. Indeed, when both $y$ and $t$ are non-compact the transformation $\varphi \to \varphi'$ (2.7) is completely legitimate and thus the resulting CFT should depend only on the two parameters $\alpha, \beta$.[4] If we drop the trivial $x_3$-direction, (2.15) describes an interesting exact $D = 4$ string background (see Section 3.6) on which, as on flat space, the string theory can be explicitly solved as discussed in Sections 4 and 5.

The simplicity of the $q_i = 0$ model (2.15) is also reflected in the fact that the Lagrangian related to (2.15) by the $y$-duality transformation is quadratic in $x^i$ and thus represents a straightforward generalization of the model of [1,2]. In fact, setting $q_i = 0$ ($\tilde F(\rho) = 1$) in (2.10) we get a generalized "non-chiral" plane-wave type model (see [17,1] and references there)[5]

$$\tilde I = \frac{1}{\pi\alpha'}\int d^2\sigma\big(\partial\tilde u\bar\partial\tilde v + \alpha\beta\rho^2\partial\tilde u\bar\partial\tilde u + \alpha\rho^2\partial\varphi\bar\partial\tilde u + \beta\rho^2\bar\partial\varphi\partial\tilde u \tag{2.19}$$

$$+ \ \partial\rho\bar\partial\rho + \rho^2\partial\varphi\bar\partial\varphi + \partial x_3\bar\partial x_3 + \mathcal{R}\phi_0\big) \ , \qquad \tilde u \equiv \tilde y - t \ , \quad \tilde v \equiv \tilde y + t \ ,$$

---

[4] The finite $R$ theory with $q_i = 0$ and the $R = \infty$ theory are of course inequivalent as conformal field theories.

[5] To obtain this dual action directly from (2.15) one needs to add to (2.15) the total derivative term $(\partial y\bar\partial t - \partial t\bar\partial y)$ which was dropped in going from (2.4) to (2.5). If one starts directly with (2.15) one obtains $\tilde I$ with $\tilde y$ replaced by $\tilde y + t$.

or, equivalently,

$$\tilde{L} = \partial\tilde{u}\bar{\partial}\tilde{v} + \alpha\beta x^i x_i \partial\tilde{u}\bar{\partial}\tilde{u} + \alpha\epsilon_{ij}x^i\partial x^j\bar{\partial}\tilde{u} + \beta\epsilon_{ij}x^i\bar{\partial}x^j\partial\tilde{u} + \partial x_i\bar{\partial}x^i + \partial x_3\bar{\partial}x_3 + \mathcal{R}\phi_0 \ . \quad (2.20)$$

Solving for $\tilde{v}$ one finds that $\tilde{u}$ satisfies the free equation of motion and then the linear equation for $x^i$ is also easily solvable. Thus the operator quantization of this model can be carried out essentially in the same way as was done for its special case of $\alpha = 0$ in [2]. Equivalently, in the path integral approach, the integral over $\tilde{v}$ implies the constraint that $\tilde{u}$ is given by the zero (winding) mode term only, so that, e.g., the partition function on the torus is readily computable. Since the resulting CFT is duality-invariant, in this way we get also the partition function of the equivalent $q_i = 0$ model (2.15) (see Section 4.2).

Let us now consider some other special cases of models (2.4) or (2.5). When $\alpha\beta = 0$ the dilaton field is constant. For $q_i = 0$ and $\alpha = 0$ ($\beta = 0$ gives a similar model with $\partial \to \bar{\partial}$) we get back to the constant magnetic field model of [1,2]

$$\begin{aligned} L(q_i = 0, \alpha = 0) \ &= \ \partial u\bar{\partial}v + \beta\rho^2\bar{\partial}\varphi\partial u + \partial\rho\bar{\partial}\rho + \rho^2\partial\varphi\bar{\partial}\varphi + \partial x_3\bar{\partial}x_3 + \mathcal{R}\phi_0 \quad (2.21) \\ &= \ \partial u\bar{\partial}v + \beta\epsilon_{ij}x^i\bar{\partial}x^j\partial u + \partial x_i\bar{\partial}x^i + \partial x_3\bar{\partial}x_3 + \mathcal{R}\phi_0 \ . \end{aligned}$$

More generally, let us choose the gauge $a_- = 0$ (2.9) and consider the 2-parameter subclass of models defined by the condition $a_+ = 0$ (i.e. $q_1 = 0, \ q_2 = \alpha$)

$$\begin{aligned} L(a_\pm = 0) \ &= \ \partial u\bar{\partial}v + \partial\rho\bar{\partial}\rho + F(\rho)\rho^2(\partial\varphi + \beta\partial u + \alpha\partial v)\bar{\partial}\varphi \quad (2.22) \\ &+ \ \partial x_3\bar{\partial}x_3 + \ \mathcal{R}[\phi_0 + \tfrac{1}{2}\ln F(\rho)] \ , \qquad F^{-1} = 1 + \alpha\beta\rho^2 \ , \end{aligned}$$

or, up to total derivative,

$$\begin{aligned} L(a_\pm = 0) \ &= \ -\partial t\bar{\partial}t + \partial y\bar{\partial}y + \partial\rho\bar{\partial}\rho + \frac{\rho^2}{1+\alpha\beta\rho^2}(\partial\varphi + c_+\partial y + c_-\partial t)\bar{\partial}\varphi \quad (2.23) \\ &+ \ \partial x_3\bar{\partial}x_3 + \mathcal{R}[\phi_0 - \tfrac{1}{2}\ln(1+\alpha\beta\rho^2)] \ . \end{aligned}$$

These models with $a_+ = 0$ (or $c_+ = 0$) are "chiral" (or "heterotic") in the sense that the background gauge field only couples to the left or to the right sector. The special case of (2.23) with $c_- = 0$ (i.e. with $\alpha = \beta$) is the string model [5] corresponding to the dilatonic Melvin solution [4] which describes a magnetic flux-tube background (1.2)

$$\begin{aligned} L(a_\pm = c_- = 0) \ &= \ -\partial t\bar{\partial}t + \partial\rho\bar{\partial}\rho + F(\rho)\rho^2\partial\varphi\bar{\partial}\varphi + \partial y\bar{\partial}y + 2\beta F(\rho)\rho^2\partial y\bar{\partial}\varphi \\ &+ \partial x_3\bar{\partial}x_3 + \ \mathcal{R}[\phi_0 + \tfrac{1}{2}\ln F(\rho)] \ , \quad F^{-1} = 1 + \beta^2\rho^2 \ , \quad \beta = \alpha = \tfrac{1}{2}c_+ \ . \quad (2.24) \end{aligned}$$

In the non-compact case $R = \infty$ the non-trivial 3-dimensional $(y, \rho, \varphi)$-part of this model can be considered [5] as a special singular limit of the $SL(2,R) \times R/R$ "charged black string" coset model [18].[6]

Changing the parameter $\alpha$ in eq. (2.23) from 0 to $\beta$ we thus interpolate between the constant magnetic field model (2.21) and the flux-tube Melvin model (2.24). The parameter $1/\sqrt{\alpha\beta}$ represents an effective width of the flux tube inside which the magnetic field is approximately constant. As (2.21) and (2.24) all the members of the 2-parameter class of models (2.23) are invariant with respect to the duality transformation in the Kaluza-Klein $y$-direction (cf. (2.11)).

[6] Correspondingly, the $(\rho, \varphi)$-part of the Melvin background (1.2) can be interpreted as a limit of the Euclidean version of the charged $D = 2$ black hole. A.T. is grateful to A. Strominger for this remark.

*2.4. Generalizations*

The idea of constructing non-trivial $D=4$ string backgrounds by starting with simple $\sigma$-models such as (2.3) and applying duality transformations in angular coordinates can be generalized in several ways. The model (2.3) admits the following generalization which preserves the number of its isometries: if we shift $v \to v+\kappa u$, $\kappa=\mathrm{const}$ in (2.3) then the new model will not be equivalent to the $\kappa=0$ one as long as $y$ is compact. The corresponding dual $\sigma$-model action (2.4) will take the form

$$I=\frac{1}{\pi\alpha'}\int d^2\sigma\big[\partial u\bar\partial v+\kappa\partial u\bar\partial u+\partial\rho\bar\partial\rho$$

$$+\ F(\rho)\rho^2[\partial\varphi+(q_1'+\beta')\partial u+q_2\partial v][\bar\partial\varphi+q_1'\bar\partial u+(q_2-\alpha)\bar\partial v] \tag{2.25}$$

$$+\ \partial x_3\bar\partial x_3+\mathcal{R}(\phi_0+\tfrac{1}{2}\ln F)\big]\ .$$

$\kappa$ will be a new parameter of the corresponding CFT. For example, in computing the partition function on the torus $u$ and $v'=v+\kappa u=(1+\kappa)y+(1-\kappa)t$ will now have different radii of their winding zero mode parts, namely, $R$ and $R'=(1+\kappa)R$. As a result, $Z$ will depend on $R$ *and* $R'$ (as well as on $\alpha,\beta,q_+$). In the rest of this paper we shall assume for simplicity that $\kappa=0$.

One may also replace the "dual 2-plane" part of (2.3) by the $\sigma$-model representing the (dual to the) Euclidean $D=2$ black hole (i.e. the gauged WZW model $SL(2,R)/U(1)$ [19,20,21]),

$$I=\frac{1}{\pi\alpha'}\int d^2\sigma\big[(\partial u+\alpha\partial\tilde\varphi)(\bar\partial v+\beta\bar\partial\tilde\varphi)+\partial\rho\bar\partial\rho+f^2(\rho)\partial\tilde\varphi\bar\partial\tilde\varphi \tag{2.26}$$

$$+\ q_1(\partial u\bar\partial\tilde\varphi-\bar\partial u\partial\tilde\varphi)+q_2(\partial v\bar\partial\tilde\varphi-\bar\partial v\partial\tilde\varphi)+\mathcal{R}(\phi_0-\ln g(\rho))\big]\ ,$$

with

$$f(\rho)=b\coth b\rho\ ,\qquad g(\rho)=b^{-1}\sinh b\rho\ , \tag{2.27}$$

or

$$f(\rho)=b^{-1}\tanh b\rho\ ,\qquad g(\rho)=\cosh b\rho\ . \tag{2.28}$$

The resulting $\sigma$-model is conformal to all orders in the "leading-order" scheme [22]. The constant $b$ ($\alpha' b^2=1/k$) is fixed by the condition of the vanishing of the total central charge, $2+3k/(k-2)-1+N-26=0$, where $N$ is a number of extra free bosonic dimensions. The value of $b$ can be made continuous by introducing a linear dilaton coupling in $t$ direction. The action (2.26) with (2.27) reduces to (2.3) in the limit $b\to 0$ (i.e. $N\to 22$). The $\sigma$-model dual to (2.26),(2.27) is given by the following generalization of (2.5)

$$I=\frac{1}{\pi\alpha'}\int d^2\sigma\big[\partial u\bar\partial v+\partial\rho\bar\partial\rho$$

$$+\ b^{-2}\tanh^2 b\rho\ F(\rho)[\partial\varphi+(q_1+\beta)\partial u+q_2\partial v][\bar\partial\varphi+q_1\bar\partial u+(q_2-\alpha)\bar\partial v] \tag{2.29}$$

$$+\ \partial x_3\bar\partial x_3+\mathcal{R}(\phi_0+\tfrac{1}{2}\ln F'(\rho))\big]\ ,$$

$$F^{-1}=1+\frac{\alpha\beta}{b^2}\tanh^2 b\rho\ ,\qquad F'=F(\rho)\cosh^{-2}b\rho\ . \tag{2.30}$$

The parameter $b$ "regularizes" the large $\rho$ form of the models (2.26) and (2.29) (cf. (2.3), (2.4)): the target space becomes a product of a Minkowski 3-space $(t,\rho,x_3)$ and a "twisted" 2-torus $(y,\varphi)$. The "mixing" of the Kaluza-Klein and angular $\varphi$ coordinates makes the corresponding conformal theory and its space-time interpretation quite non-trivial.

The $\sigma$-model (2.29) can be described as a particular $O(3,3;R)$ duality transformation (shifting of $t$ and $y$ by $\tilde\varphi$, adding torsion, and performing duality in $\tilde\varphi$) of the direct product model $R_t \times S^1_y\times$ $[SL(2,R)/U(1)\mathrm{WZW}]_{\rho,\varphi} \times R_{x_3}$. As was already mentioned in Section 2, for general values of the parameters such $O(3,3;R)$ duality transformation is not a symmetry of the original CFT, i.e. it gives a new conformal theory.

For special values of the parameters the 3-dimensional $(y,\rho,\varphi)$ part of this model is equivalent to the $[SL(2,R)_k \times U(1)_{k'}]/U(1)$ gauged WZW model (or, for non-compact $y$ and $\varphi$, to the "charged black string" model of [18]). For $b\neq 0$ the conformal theory corresponding to (2.26) or (2.29) is much more complicated than the $b=0$ one discussed in the main part of the present paper.[7]

## 3. Space-time/low energy field theory interpretation

To give a space-time interpretation to the string models (2.5), i.e. to determine the corresponding string background geometry, we shall make the Kaluza-Klein–type rearrangement of terms in the above $\sigma$-model actions (see, e.g., [23,1])

$$I_5 = \frac{1}{\pi\alpha'}\int d^2\sigma\big[(G_{MN}+B_{MN})(X)\partial X^M\bar\partial X^N + \mathcal{R}\phi(X)\big]$$

$$= \frac{1}{\pi\alpha'}\int d^2\sigma\big[(\hat G_{\mu\nu}+B_{\mu\nu})(x)\partial x^\mu\bar\partial x^\nu + \ e^{2\sigma(x)}[\partial y + \mathcal{A}_\mu(x)\partial x^\mu][\bar\partial y + \mathcal{A}_\nu(x)\bar\partial x^\nu]$$

$$+ \mathcal{B}_\mu(x)(\partial x^\mu\bar\partial y - \bar\partial x^\mu\partial y) + \mathcal{R}\phi(x)\big] \ , \qquad (3.1)$$

where $X^M=(x^\mu,x^5),\ x^\mu=(t,x^i,x^3),\ x^5\equiv y$ and [8]

$$\hat G_{\mu\nu}\equiv G_{\mu\nu} - G_{55}\mathcal{A}_\mu\mathcal{A}_\nu \ , \quad G_{55}\equiv e^{2\sigma} \ , \quad \mathcal{A}_\mu\equiv G^{55}G_{\mu5} \ , \ \ \mathcal{B}_\mu\equiv B_{\mu5} \ . \qquad (3.2)$$

From the point of view of the low-energy effective field theory, this decomposition corresponds to starting with the $D=5$ bosonic string effective action and assuming that one

[7] Another generalization which will not be explored here is to consider the coordinate $t$ in (2.3) to be periodic. It may be of interest in connection with a finite temperature description as well as for construction of related (e.g. by analytic continuation) models where the role of time can be assigned to one of the other coordinates. This may permit, in particular, to obtain models with expressions for the electric and magnetic fields being interchanged. One more possibility is to start with the Euclidean $D=4$ model which is a combination of the two "dual 2-planes", make coordinate shifts preserving the number of isometries, apply the duality in two angular coordinates and then look for the range of parameters (or analytic continuations) for which the resulting $\sigma$-model has Minkowski signature.

[8] The modulus field $\sigma$ should not be confused with the world-sheet coordinates $\sigma^\alpha$.

spatial dimension $x^5$ is compactified on a small circle. Dropping the massive Kaluza-Klein modes one finds the following dimensionally reduced $D=4$ action (see, e.g., [24])

$$S_4 = \int d^4x\sqrt{\hat{G}}\ e^{-2\phi+\sigma}\ \big[\ \hat{R}\ + 4(\partial_\mu\phi)^2 - 4\partial_\mu\phi\partial^\mu\sigma \tag{3.3}$$

$$-\frac{1}{12}(\hat{H}_{\mu\nu\lambda})^2\ -\frac{1}{4}e^{2\sigma}(F_{\mu\nu}(\mathcal{A}))^2 - \frac{1}{4}e^{-2\sigma}(F_{\mu\nu}(\mathcal{B}))^2 + O(\alpha')\big]\ ,$$

where, in addition to (3.2), we have defined

$$F_{\mu\nu}(\mathcal{A}) = 2\partial_{[\mu}\mathcal{A}_{\nu]}\ ,\quad F_{\mu\nu}(\mathcal{B}) = 2\partial_{[\mu}\mathcal{B}_{\nu]}\ ,\quad \hat{H}_{\lambda\mu\nu} = 3\partial_{[\lambda}B_{\mu\nu]} - 3\mathcal{A}_{[\lambda}F_{\mu\nu]}(\mathcal{B})\ . \tag{3.4}$$

As follows from (3.1), the $\sigma$-model duality transformation in $y$ induces the following target space transformation

$$\mathcal{A}\to\pm\mathcal{B}\ ,\quad \mathcal{B}\to\pm\mathcal{A}\ ,\quad \sigma\to-\sigma\ ,\quad \phi\to\phi-\sigma\ , \tag{3.5}$$

$$\hat{G}_{\mu\nu}\to\hat{G}_{\mu\nu}\ ,\quad B_{\mu\nu}\to B_{\mu\nu}+\mathcal{A}_\mu\mathcal{B}_\nu-\mathcal{B}_\mu\mathcal{A}_\nu\ ,\quad \hat{H}_{\mu\nu\lambda}\to\hat{H}_{\mu\nu\lambda}\ ,$$

which is obviously the invariance of (3.3). Dual backgrounds related by (3.5) correspond to the same string solution (CFT).

### *3.1. 3-parameter class of new string solutions*

Let us define the two functions $F$ and $\tilde{F}$ (which already appeared in (2.5), (2.10))

$$F(\rho)\equiv(1+\frac{\rho^2}{\rho_0^2})^{-1}\ ,\qquad \tilde{F}(\rho)\equiv(1+\frac{\rho^2}{\tilde{\rho}_0^2})^{-1}\ , \tag{3.6}$$

$$\rho_0^{-2} = \tfrac{1}{4}[(a_+-c_+)^2-(a_--c_-)^2] = \alpha\beta\ , \tag{3.7}$$

$$\tilde{\rho}_0^{-2} = \tfrac{1}{4}[(a_++c_+)^2-(a_--c_-)^2] = q_+(q_++\beta-\alpha)\ ,$$

$$\tilde{F}(\rho) = [F(\rho)]_{a_+\to-a_+} = [F(\rho)]_{c_+\to-c_+}\ .$$

Note that $\rho_0^2$ and $\tilde{\rho}_0^2$ can take both positive and negative values. Starting with (2.5) we find the following expressions for the $D=4$ background fields:
dilaton and modulus scalars

$$e^{2(\phi-\phi_0)} = F(\rho)\ ,\qquad e^{2\sigma} = \frac{F(\rho)}{\tilde{F}(\rho)}\ \ , \tag{3.8}$$

abelian vector and axial-vector potentials

$$\mathcal{A} = \mathcal{A}_\mu dx^\mu = \tfrac{1}{2}\tilde{F}(\rho)\rho^2[(a_++c_+)d\varphi + (a_+c_-+c_+a_-)dt]\ , \tag{3.9}$$

$$\mathcal{B} = \mathcal{B}_\mu dx^\mu = \tfrac{1}{2}F(\rho)\rho^2[(a_+-c_+)d\varphi + (a_+c_--c_+a_-)dt]\ ,$$

with field strengths

$$d\mathcal{A} = \tilde{F}^2(\rho)\rho d\rho \wedge [(a_+ + c_+)d\varphi + (a_+c_- + c_+a_-)dt] \ , \tag{3.10}$$

$$d\mathcal{B} = F^2(\rho)\rho d\rho \wedge [(a_+ - c_+)d\varphi + (a_+c_- - c_+a_-)dt] \ ,$$

the effective $D = 4$ metric

$$ds_4^2 \equiv \hat{G}_{\mu\nu}dx^\mu dx^\nu = -dt^2 + F(\rho)\rho^2(d\varphi + c_-dt)(d\varphi + a_-dt) \tag{3.11}$$

$$- \tfrac{1}{4}\tilde{F}(\rho)F(\rho)\rho^4\big[(a_+ + c_+)d\varphi + (a_+c_- + c_+a_-)dt\big]^2 + d\rho^2 + dx_3^2 \ ,$$

$$\det \hat{G} = -\rho^2 F(\rho)\tilde{F}(\rho) \ , \tag{3.12}$$

and the antisymmetric tensor

$$B = \tfrac{1}{2}B_{\mu\nu}dx^\mu \wedge dx^\nu = \tfrac{1}{2}(a_- - c_-)F(\rho)\rho^2 d\varphi \wedge dt \ . \tag{3.13}$$

Note that the invariant antisymmetric tensor field strength $\hat{H}_{\mu\nu\lambda}$ in (3.4), is, like $B$ itself, proportional to $a_- - c_- = \beta - \alpha$. The duality transformation in the $y$ direction (3.5) is indeed induced by $a_+ \to -a_+$ (i.e. $F \to \tilde{F}$, etc.) in agreement with (2.10),(2.11). The duality invariant 'shifted' $D = 4$ dilaton which determines the effective coupling in the $D = 4$ action (3.3) is given by

$$\Phi \equiv \phi - \tfrac{1}{2}\sigma = \phi_0 + \tfrac{1}{4}\ln[F(\rho)\tilde{F}(\rho)] \ . \tag{3.14}$$

In general, we get a stationary ("rotating") metric of the form

$$ds_4^2 = -f_1(\rho)dt^2 + f_2(\rho)d\varphi dt + f_3(\rho)d\varphi^2 + d\rho^2 + dx_3^2 \ , \tag{3.15}$$

and the gauge field strengths which have both magnetic and electric components. There are pure magnetic solutions but no purely electric ones, except in special limits.[9]

Using the expressions for $F$, $\tilde{F}$ in (3.6) and (3.7) one finds that for generic values of the parameters the metric becomes degenerate at $\rho \to \infty$ and the asymptotic $\rho \to 0$ and $\rho \to \infty$ forms of the metric are

$$(ds_4^2)_{\rho\to 0} = -dt^2 + (a_- + c_-)\rho^2 d\varphi dt + d\rho^2 + \rho^2 d\varphi^2 + dx_3^2 \ , \tag{3.16}$$

$$(ds_4^2)_{\rho\to\infty} = -k_0(dt + k_1 d\varphi)^2 + \rho^{-2}k_2 d\varphi^2 + d\rho^2 + dx_3^2 \ , \quad k_s = \text{const} \ , \tag{3.17}$$

i.e. the 2-space $(\rho, \varphi)$ which looks like a 2-plane near the origin $\rho = 0$ at large $\rho$ becomes a (rotating) cylinder with radius going to zero as $\rho \to \infty$. This is similar to the topology of the Melvin solution [4].

[9] It is possible to formally exchange the roles of the electric and magnetic fields by assigning the role of time to the $\varphi$ coordinate (e.g. by the rotation $\varphi \to i\varphi$, or by specifying to the regions where $\varphi$ has a time-like signature).

In the special cases when $F=1$ (or $\tilde{F}=1$) the determinant of the metric (3.12) at large $\rho$ approaches a finite value and, for $\alpha \neq \beta$, we get instead a rotating Rindler-type asymptotic space

$$(ds_4^2)_{\rho\to\infty} = -\rho^2(dt + n_1 d\varphi)^2 + \rho^{-2} n_2 d\varphi^2 + d\rho^2 + dx_3^2 \ , \quad n_s = \text{const} \ . \tag{3.18}$$

For $\alpha = \beta$ the asymptotics is that of a cylinder, $(ds_4^2)_{\rho\to\infty} = -dt^2 + n' d\varphi^2 + d\rho^2 + dx_3^2$. $F$ or $\tilde{F}$ are equal to 1 when either $\alpha\beta = 0$ or $q_+(q_+ + \beta - \alpha) = 0$ (see (2.6)), i.e. when either of the axial or vector field strengths (cf. (3.9)) is constant (uniformly extends to infinity). When both $F=1$ and $\tilde{F}=1$ the asymptotic form of the metric is that of the plane.

For $\alpha\beta < 0$ or $q_+(q_+ + \beta - \alpha) < 0$ the determinant (3.12) has singularities at $\rho^2 = -\rho_0^2$ or $\rho^2 = -\tilde{\rho}_0^2$, respectively. These are, in fact, curvature singularities, as can be seen from the general expression for the curvature scalar of metric (3.11) which is given in the Appendix.

Excluding the singular cases (and considering, e.g., the case when the electric fields vanish, i.e. $a_- = c_- = 0$) the resulting geometry admits a straightforward physical interpretation as the one induced by the two Kaluza-Klein ($U(1)_{\rm v}$ and $U(1)_{\rm a}$) magnetic fields of strengths $\mathbf{B}_{\rm v0} = a_+ + c_+$ and $\mathbf{B}_{\rm a0} = a_+ - c_+$ which are non-vanishing inside the regions of characteristic scales $\rho_0$ and $\tilde{\rho}_0$ (their profile functions are $\tilde{F}^2$ and $F^2$, see (3.10)). As will be discussed in Section 6.3, the singular configurations can be related to black or naked strings.

We have mentioned in Section 2 that one combination of the parameters $a_-$ and $c_-$ can be fixed by a coordinate transformation (cf. (2.7)). Below we shall use the "chiral" gauge $a_- = 0$ (2.9) in which the metric (3.11) can be put into the following form:

$$ds_4^2 = -[1 + \tfrac{1}{4}a_+^2 c_-^2 \rho^4 F(\rho)\tilde{F}(\rho)]dt^2 + c_-[1 + \tfrac{1}{4}(c_+^2 - a_+^2 - c_-^2)\rho^2]F(\rho)\tilde{F}(\rho)\rho^2 d\varphi dt \tag{3.19}$$

$$+ \ (1 - \tfrac{1}{4}c_-^2\rho^2)F(\rho)\tilde{F}(\rho)\rho^2 d\varphi^2 + d\rho^2 + dx_3^2 \ .$$

To clarify the properties of this 3-parameter class of solutions (3.8)–(3.13) corresponding to the action (3.3) let us now consider some special choices of the free parameters $a_+, c_+, c_-$.

*3.2. Chiral magnetic backgrounds with constant modulus $\sigma$ ($a_+ = 0$)*

The solutions with $a_+ = 0$ (or $c_+ = 0$) are precisely the ones which are self-dual, i.e. invariant under (2.11), (3.5). They have zero modulus, $\sigma = 0$, and are "chiral" (cf. (2.23)): the two vector fields $\mathcal{A}$ and $\mathcal{B}$ in (3.9) are equal up to sign (and have only magnetic components). In the $a_- = 0$ gauge

$$\mathcal{A} = -\mathcal{B} = \tfrac{1}{2}c_+ F(\rho)\rho^2 d\varphi \ , \quad d\mathcal{A} = c_+ F^2(\rho)\rho d\rho \wedge d\varphi \ , \quad c_\pm = \alpha \pm \beta \ , \tag{3.20}$$

$$e^{2(\phi-\phi_0)} = F = \tilde{F} = \frac{1}{1 + \frac{1}{4}(c_+^2 - c_-^2)\rho^2} \ , \quad \sigma = 0 \ , \quad B = -\tfrac{1}{2}c_- F(\rho)\rho^2 d\varphi \wedge dt \ , \tag{3.21}$$

$$ds_4^2 = -[dt - \tfrac{1}{2}c_- F(\rho)\rho^2 d\varphi]^2 + d\rho^2 + F^2(\rho)\rho^2 d\varphi^2 + dx_3^2 \ . \tag{3.22}$$

The Melvin solution (1.2) corresponds to $c_- = 0$. Then the metric is static and the antisymmetric tensor vanishes. The solution of [1] (eq. (1.1)) is obtained for $c_+ = c_- = \beta$:

then $F=1$ so that the magnetic field strength and the dilaton become constant everywhere in space, the metric is of "rotating" type and $B_{\mu\nu}\neq 0$.

Among other members of this $a_+=0$ subclass there is a special solution with $c_+=0,\ c_-\neq 0$ (i.e. with $\alpha=-\beta$). In this case both gauge fields vanish identically, but the metric, dilaton and $B_{\mu\nu}$ remain non-trivial,

$$ds_4^2=-dt^2+F(\rho)\rho^2(d\varphi+c_-dt)d\varphi+d\rho^2+dx_3^2\ , \tag{3.23}$$

$$B=-\tfrac{1}{2}c_-F(\rho)\rho^2 d\varphi\wedge dt\ ,\quad e^{2(\phi-\phi_0)}=F=(1-\tfrac{1}{4}c_-^2\rho^2)^{-1}\ .$$

Dropping the trivial $x_3$-direction we get a $D=3$ string solution which (like the $\alpha=\beta$ Melvin model [5]) can be formally (ignoring the issue of different periodicities of coordinates) identified with a special singular limit of the charged black string background of [18]. The corresponding string model (given by (2.23) with $c_+=0, c_-\neq 0$) is an "antipode" of the Melvin model (which has $c_-=0,\ c_+\neq 0$) being connected to it by a change of sign of $\alpha$ and formal replacement of $y$ by $t$. Thus, like the Melvin model [5], it can be related (ignoring the global issues) to a special limit of the $SL(2,R)\times R/R$ gauged WZW model. Equivalently, if all the coordinates are formally taken to be non-compact, this model is the same as the $D=3$ $\sigma$-model corresponding to $E_2^c/U(1)$ gauged WZW theory [25].

Solutions with $|c_-|>|c_+|$ (i.e. $\alpha\beta<0$ ) correspond to singular geometries. In fact, the derivatives of $\phi$ and $B_{\mu\nu}$ and the curvature blow up on a $(x_3,\varphi)$-cylinder with radius $\rho^2=4/(c_-^2-c_+^2)$ where $F\to\infty$. Computing the curvature for the metric (3.22) in the obvious vierbein basis, one finds the following non-vanishing vierbein components

$$R_{0101}=R_{0202}=\frac{1}{4}c_-^2F^2(\rho)\ ,\quad R_{0112}=R_{1201}=-\frac{1}{4}c_-(c_+^2-c_-^2)\rho F^2(\rho)\ , \tag{3.24}$$

$$R_{1212}=\frac{1}{8}[12c_+^2-6c_-^2-(c_+^2-c_-^2)^2\rho^4]F^2(\rho)\ .$$

Note that the duality-invariant shifted dilaton (3.14) is also singular at the points where $F=\infty$, i.e. where thus the effective string coupling diverges.

*3.3. Static magnetic backgrounds with vanishing antisymmetric tensor field*

According to (3.9) the subclass of solutions for which the vector field strengths have only magnetic components is determined (in the gauge $a_-=0$) by the condition $c_-a_+=0$. We have already discussed the case of $a_+=0$ so let us now assume that $c_-=0$. As follows from (2.5), for $c_-=a_-=0$ (i.e. for $\alpha=\beta$) the time direction decouples and one gets a static $D=4$ metric and zero antisymmetric tensor (in fact, $\alpha=\beta$ is the only case when the metric (3.19) becomes of non-rotating type). Indeed, in this case we get from (3.8),(3.19),(3.9)

$$e^{2(\phi-\phi_0)}=F(\rho)\ ,\quad e^{2\sigma}=\frac{F(\rho)}{\tilde F(\rho)}\ , \tag{3.25}$$

$$F^{-1}=1+\tfrac{1}{4}(a_+-c_+)^2\rho^2\ ,\quad \tilde F^{-1}=1+\tfrac{1}{4}(a_++c_+)^2\rho^2\ ,$$

$$\mathcal{A}=\tfrac{1}{2}(a_++c_+)\tilde F(\rho)\rho^2d\varphi\ ,\quad \mathcal{B}=\tfrac{1}{2}(a_+-c_+)F(\rho)\rho^2d\varphi\ ,\quad B=0\ , \tag{3.26}$$

$$ds_4^2=-dt^2+d\rho^2+F(\rho)\tilde F(\rho)\rho^2d\varphi^2+dx_3^2\ . \tag{3.27}$$

The metric (3.27) is flat near $\rho = 0$ and, for $\alpha \neq \beta$, it becomes effectively 3-dimensional at large $\rho$, $(ds_4^2)_{\rho\to\infty} = -dt^2 + d\rho^2 + k\rho^{-2}d\varphi^2 + dx_3^2$. An interesting simple special case is $\alpha = \beta = 0,\ q_+ \neq 0$, i.e.

$$e^{2(\phi-\phi_0)} = 1 \ , \qquad e^{2\sigma} = \tilde F^{-1}(\rho) = 1 + q_+^2\rho^2 \ , \tag{3.28}$$

$$\mathcal{A} = q_+ \tilde F(\rho)\rho^2 d\varphi \ , \quad \mathcal{B} = 0 \ , \quad B = 0 \ , \tag{3.29}$$

$$ds_4^2 = -dt^2 + d\rho^2 + \tilde F(\rho)\rho^2 d\varphi^2 + dx_3^2 \ , \tag{3.30}$$

which corresponds to the "$a = \sqrt{3}$" (or "Kaluza-Klein") Melvin solution [4].[10]

*3.4. Backgrounds with uniform magnetic field*

We have seen in Section 2 that the string model with $q_i = 0$ (2.15) and its dual (2.20) have a particularly simple form. More generally, a simplification occurs for $F = 1$ or $\tilde F = 1$ when one of the two parameters $\rho_0$ or $\tilde\rho_0$ in (3.7) is infinite, i.e. when $\alpha\beta = 0$ or $q_+(q_+ + \beta - \alpha) = 0$.

Let us first consider the case of $\tilde F = 1$, i.e. $q_+(q_+ + \beta - \alpha) = 0$, or, in particular, $q_+ = 0$, corresponding to the case when the magnetic field associated with the vector $U(1)_{\rm v}$ is constant. To present the expressions for the corresponding background fields we shall use the gauge $q_1 - q_2 = 0$ (2.8) (as discussed in Section 2, different gauges are related by the coordinate transformation $\varphi \to \varphi + \lambda t$). For $q_i = 0$ we have $a_+ = a_- = -\alpha\ ,\ c_+ = -c_- = \beta$ and thus find from (3.8)–(3.13) that the modulus is equal to the (non-constant part of the) dilaton

$$e^{2\sigma} = e^{2(\phi-\phi_0)} = F(\rho) = (1 + \alpha\beta\rho^2)^{-1} \ , \quad \tilde F(\rho) = 1 \ , \tag{3.31}$$

the vector $\mathcal{A}$ has a constant magnetic field strength

$$\mathcal{A} = \tfrac{1}{2}(\beta - \alpha)\rho^2 d\varphi \ , \quad \mathcal{B} = -\tfrac{1}{2}F(\rho)\rho^2[(\alpha+\beta)d\varphi - 2\alpha\beta dt] \ , \tag{3.32}$$

and the metric and the antisymmetric tensor have the following simple form

$$ds_4^2 = -F(\rho)[dt + \tfrac{1}{2}(\alpha+\beta)\rho^2 d\varphi]^2 + d\rho^2 + \rho^2 d\varphi^2 + dx_3^2 \ , \tag{3.33}$$

$$B = \tfrac{1}{2}(\beta - \alpha)F(\rho)\rho^2 d\varphi \wedge dt \ .$$

This class of backgrounds is another generalization of the constant magnetic field solution (1.1) different from the Melvin-type generalization (3.20)–(3.22): here the deformation is in the $a_+ = -\alpha$ direction. It induces nontrivial scalars $\sigma$ and $\phi$, and makes $\mathcal{B}$ different from $-\mathcal{A}$ (while still preserving the "rotational" structure of the metric).

[10] As was pointed out in [9], it is possible to obtain the dilatonic Melvin solution (with arbitrary value of the coupling parameter $a$) by applying the duality transformation in the angular coordinate to the flat space solution of the dilaton-Einstein-Maxwell theory. For $a = 1$ and $a = \sqrt{3}$ (which are the only cases that can be embedded into the string effective action (3.3)) this is the effective field theory analogue of our construction of the corresponding string theory in Section 2.

The curvature components corresponding to (3.33) are proportional (as in (3.24)) to the powers of $F$ so that for $\alpha\beta < 0$ they are singular on a cylinder corresponding to the special value of $\rho^2 = -\tilde{\rho}_0^2$ where $F = \infty$, and where $\phi$ and $\Phi$ (3.14) also blow up.

An interesting special case of this subclass is obtained for $\alpha = -\beta$. Then $\mathcal{A}$ has only magnetic component while $\mathcal{B}$ has only the electric one,

$$e^{2\sigma} = e^{2(\phi-\phi_0)} = F(\rho) = (1-\beta^2\rho^2)^{-1} \ , \quad B = \beta F(\rho)\rho^2 d\varphi \wedge dt \ , \tag{3.34}$$

$$\mathcal{A} = \beta\rho^2 d\varphi \ , \quad \mathcal{B} = -\beta^2 F(\rho)\rho^2 dt = -F(\rho)dt + dt \ , \tag{3.35}$$

$$ds_4^2 = -F(\rho)dt^2 + d\rho^2 + \rho^2 d\varphi^2 + dx_3^2 \ . \tag{3.36}$$

The vierbein components of the corresponding Riemann tensor and curvature scalar (see the Appendix) are

$$R_{0101} = \beta(2\beta^4\rho^4 - \beta^2\rho^2 - 1)F^3(\rho), \quad R_{0202} = \beta^2 F(\rho), \quad R = -2\beta^2(2+\beta^2\rho^2)F^2(\rho), \tag{3.37}$$

so that $\rho^2 = 1/\beta^2$ is the singularity.

The solutions with $\alpha\beta = 0$ are $y$-dual (3.5) to (3.31)–(3.33). These are the solutions with constant dilaton,

$$e^{2(\phi-\phi_0)} = F(\rho) = 1 \ , \quad e^{2\sigma} = \tilde{F}^{-1}(\rho) = 1 + \tilde{\alpha}\tilde{\beta}\rho \ , \tag{3.38}$$

$$\mathcal{B} = \tfrac{1}{2}(\tilde{\alpha} - \tilde{\beta})\rho^2 d\varphi \ , \quad \mathcal{A} = \tfrac{1}{2}\tilde{F}(\rho)\rho^2[(\tilde{\alpha}+\tilde{\beta})d\varphi - 2\tilde{\alpha}\tilde{\beta}dt] \ , \tag{3.39}$$

$$ds_4^2 = -\tilde{F}(\rho)[dt + \tfrac{1}{2}(\tilde{\alpha}+\tilde{\beta})\rho^2 d\varphi]^2 + d\rho^2 + \rho^2 d\varphi^2 + dx_3^2 \ , \tag{3.40}$$

$$B = \tfrac{1}{2}(\tilde{\beta} - \tilde{\alpha})\rho^2 d\varphi \wedge dt \ , \quad \tilde{\alpha} \equiv q_+ \ , \quad \tilde{\beta} \equiv q_+ + \beta - \alpha \ .$$

Indeed, this background corresponds to the "plane-wave-type" string model (2.20) which is dual to (2.15). Thus the two backgrounds (3.31)–(3.33) and (3.38)–(3.40) are different "faces" of the same string solution (they correspond to the same CFT).

The metric (3.33) or (3.40) is that of a homogeneous space only when both $F$ and $\tilde{F}$ are equal to 1, i.e. when both magnetic fields are uniform (both $\rho_0$ and $\tilde{\rho}_0$ are infinite). This requires that $\alpha\beta = 0$ and $q_+(q_+ + \beta - \alpha) = 0$, i.e. that $c_+ = \pm a_+$, leaving only one free parameter (representing the magnetic field strength). This is essentially the solution (1.1) of [1] investigated in [2] with the metric representing the space of the Heisenberg group.

*3.5. Backgrounds corresponding to the $\varphi$-dual model (2.3)*

In addition to $y$-dual backgrounds related by (3.5) (i.e. by $a_+ \to -a_+$ or $c_+ \to -c_+$) which represent two different "sides" of the same string solution as "seen" by point-like modes with linear or winding Kaluza-Klein momentum (charge) there is another "face" of our string solution (CFT) which is described by the $\varphi$-dual $\sigma$-model (2.3). Representing (2.3) in the form (3.1) we find another $D = 4$ solution of the equations for the action (3.3) (we drop the exact pure-gauge $O(dt)$ and $O(d\tilde{\varphi} \wedge dt)$ parts of the 1-forms $\mathcal{A}, \mathcal{B}$ and the 2-form $B$)

$$e^{2(\phi-\phi_0)} = \rho^{-2} \ , \quad e^{2\sigma} = 1 \ , \tag{3.41}$$

$$\mathcal{A} = -\tfrac{1}{2}(a_+ - c_+)d\tilde{\varphi} \ , \quad \mathcal{B} = \tfrac{1}{2}(a_+ + c_+)d\tilde{\varphi} \ , \quad B = 0 \ , \tag{3.42}$$

$$ds_4^2 = -[dt + \tfrac{1}{2}(c_- - a_-)d\tilde{\varphi}]^2 + d\rho^2 + \rho^{-2}d\tilde{\varphi}^2 + dx_3^2 \ . \tag{3.43}$$

The metric (3.43) can be called a "rotating dual 2-plane". While $\mathcal{A}$ and $\mathcal{B}$ are locally trivial and have zero field strengths, their configuration is globally non-trivial: $d\tilde{\varphi}$ is closed but not exact since $\tilde{\varphi}$ is periodic and $\rho = 0$ is the curvature singularity.[11] It is remarkable that the two such different $D = 4$ backgrounds as (3.8)–(3.13) and (3.41)–(3.43) correspond to the same string solution.

*3.6. $D = 4$ solution associated with non-compact $R = \infty$ model (2.15)*

When the coordinate $y$ is non-compact (i.e. $R = \infty$) the model is characterised by the two parameters $\alpha, \beta$. Dropping the trivial $x_3$ direction we then get from (2.15) the following $D = 4$ $(u = y - t, \ v = t + y, \ x_1, x_2)$ exact string solution:

$$ds^2 = F(\rho)[dudv + \rho^2 d\varphi(\beta du - \alpha dv) + \rho^2 d\varphi^2] + d\rho^2 \ , \tag{3.44}$$

$$B = -\tfrac{1}{2}\rho^2 F(\rho) d\varphi \wedge (\beta du + \alpha dv) \ , \quad e^{2(\phi - \phi_0)} = F(\rho) = (1 + \alpha\beta\rho^2)^{-1} \ .$$

This is a special case of the class of solutions represented by (2.16),(2.17) which were found in [1]. The metric (3.44) has two null Killing vectors which become covariantly constant if $\alpha\beta = 0$. In the case of $\alpha = 0$ this background corresponds to the $E_2^c$ WZW model of [26] with (3.44) representing (in proper coordinates) the metric of a monochromatic left-moving plane wave. Similarly, for $\beta = 0$ (3.44) describes a right-moving plane wave. For $\alpha\beta \neq 0$ the metric is static, as can be seen by the coordinate transformation $u' = 2\beta u, \ v' = -2\alpha v$, i.e. $y' = \beta u - \alpha v$. It may be possible to interpret this background as describing a "superposition" of two interacting gravitational waves moving in opposite directions. The corresponding curvature asymptotically (in transverse space) goes to zero (and is singular at finite $\rho$ if $\alpha\beta < 0$).

As in the compact ($R < \infty$) case the corresponding conformal field theory can be studied explicitly. We shall show in Sections 4 and 5 that the Hamiltonian of the $R = \infty$ model is quadratic in oscillators (but still non-trivial) and the partition function is equivalent to the partition function of the free closed string theory.

*3.7. Generalizations*

Starting with the generalizations (2.25) and (2.29) of our model (2.4) we obtain two different one-parameter extensions of the class of solutions (3.8)–(3.13). The second one (2.29) may be of particular interest since the introduction of the parameter $b$ changes the large $\rho$ behaviour of the background fields (while for small $\rho$ the form of the background fields remains the same as for $b = 0$). In particular, we obtain generalizations of the Melvin (1.2) and the constant magnetic field (1.1) solutions. For example, in the latter case:

$$ds_4^2 = -(dt + \tfrac{1}{2}\beta b^{-2}\tanh^2 b\rho \ d\varphi)^2 + d\rho^2 + b^{-2}\tanh^2 b\rho \ d\varphi^2 + dx_3^2 \ , \tag{3.45}$$

$$\mathcal{A} = -\mathcal{B} = \tfrac{1}{2}\beta b^{-2}\tanh^2 b\rho \ d\varphi \ ,$$

[11] The "regularized" version of the dual 2-plane – the dual $D = 2$ black hole – has a "trumpet" topology (with the radius of the $\tilde{\varphi}$-circle being nonvanishing everywhere), suggesting that $\tilde{\varphi}$ should be considered as a "true" angular coordinate.

$$B = \tfrac{1}{2}\beta b^{-2}\tanh^2 b\rho \; d\varphi \wedge dt \ , \quad e^{\phi-\phi_0} = \cosh^{-1} b\rho \ , \quad \sigma = 0 \ .$$

The magnetic field is no longer uniform everywhere but decays asymptotically with a characteristic scale $b^{-1}$, the space is not homogeneous and the dilaton is non-constant.

It is possible also to construct new exact string solutions by generalizing the $q_i = 0$ model (2.15). Indeed, as was pointed out in [1], the model (2.16) is conformal for any $F$ satisfying (2.17). Thus we may consider solutions of (2.17) more general than (2.18). In particular, we may add to $F^{-1} = 1 + \alpha\beta\rho^2$ a solution of the homogeneous $D = 3$ $(\rho, \varphi, x_3)$ Laplace equation, obtaining, e.g.,

$$F^{-1}(\rho) = 1 + \mu \ln \frac{\rho}{\rho_0} + \alpha\beta\rho^2 \ . \tag{3.46}$$

Another possibility is

$$F^{-1}(\rho, x_3) = 1 + \frac{M}{r} + \alpha\beta\rho^2 \ , \qquad r^2 \equiv \rho^2 + x_3^2 \ . \tag{3.47}$$

Dimensionally reducing the resulting string model along the $y$-direction we obtain again an exact solution similar to (3.31)–(3.33) with $F$ now being given by (3.46) or (3.47). These backgrounds seem to represent (3+1)-dimensional string and black-hole type configurations in external electromagnetic fields (cf. [1]). The corresponding string model, however, is no longer solvable by our methods.

For example, the $q_i = 0$ background (3.31),(3.32),(3.33) with $F$ given by (3.47) represents a generalisation to the case of $\alpha \neq 0$ of the solution in [1] which was an extension ($\beta \neq 0$) of the extremal Kaluza-Klein ($a = \sqrt{3}$) black hole [4].[12]

## 4. Solution of the string model: path integral approach

Looking at the world-sheet action of our model (2.4) or (2.5) it may seem unlikely that such a complicated interacting 2d theory may have explicitly solvable classical equations and computable path integral. The reason for the solvability of this model is that it is $\varphi$-dual to a much simpler theory (2.3) which, in turn, is locally (ignoring topology) related (by $\varphi$-duality and coordinate transformation) to a flat free-field model. This explains, in particular, why the classical equations corresponding to (2.4) can be solved in terms of the free fields: the classical solutions of the two dual $\sigma$-models are related in a simple way. It is the topology (periodicity of $y$ in (2.1) and $\varphi$) that makes the model nontrivial, and it turns out to be possible to take the boundary conditions into account in a rather straightforward way.

Below we shall first discuss the reduction to free fields (on a simple example of the $q_i = 0$ model) and then present the computation of the general expression for the partition function in the path integral approach.

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[12] At the same time, the solutions in our class with $F$ given by (3.47) do not include a generalization of the extremal $a = 1$ dilatonic black hole [4,27] since the model (2.13) does not contain the term $K\partial u\bar{\partial}u$ (or $K\partial v\bar{\partial}v$) which is necessary in order to obtain the $a = 1$ extremal black hole by dimensional reduction [28,1]. In fact, for $\alpha\beta \neq 0$ such a term (or, e.g., its "gauge-invariant" generalization $K(\partial u - 2\alpha A)(\bar{\partial}u - 2\alpha\bar{A})$) cannot be added to (2.13) whithout spoiling its conformal invariance (this can be shown following the discussion in [1]).

*4.1. Reduction to free fields*

To clarify why this model can be effectively transformed (up to zero modes) into a free-field one, it is helpful to consider first a particularly simple case of the $q_i = 0$ model.[13] As was already noted in Section 2.2, the $q_i = 0$ model (2.15) is $y$-dual to (2.20) which is quadratic in $x^i$ and in which integrating out $\tilde{v}$ restricts $\tilde{u}$ to the free classical solution. As a result, this model is essentially "gaussian", like the constant magnetic field model (2.21) solved in [2]. A similar conclusion can be reached by starting directly with (2.15). In Minkowski world-sheet notation ($\sigma_\pm \equiv \tau \pm \sigma$; in this section we shall ignore the trivial direction $x_3$)

$$L(q_i = 0) \ = \ F(x)[\partial_+ u - 2\alpha A_+(x)][\partial_- v + 2\beta A_-(x)] + \partial_+ x \partial_- x^* + \alpha' \mathcal{R}\phi(x) \ , \qquad (4.1)$$

$$A_\pm = \frac{1}{4} i(x\partial_\pm x^* - x^* \partial_\pm x) \ , \quad F = e^{2(\phi-\phi_0)} = (1+\alpha\beta x x^*)^{-1} \ ,$$
$$x = x_1 + i x_2 = \rho e^{i\varphi} \ , \quad x^* = x_1 - i x_2 = \rho e^{-i\varphi} \ . \qquad (4.2)$$

The classical equations for $u$ and $v$ can be integrated once, giving

$$F(x)[\partial_+ u - 2\alpha A_+(x)] = h_+(\sigma_+) \ , \quad F(x)[\partial_- v + 2\beta A_-(x)] = h_-(\sigma_-) \ , \qquad (4.3)$$

where $h_\pm$ are arbitrary functions. Then the equation for $x$ becomes *linear*

$$\partial_+\partial_- x + i\beta h_+ \partial_- x - i\alpha h_- \partial_+ x + \alpha\beta h_+ h_- x = 0 \ , \qquad (4.4)$$

and is readily solved

$$x = e^{i\alpha g_- - i\beta g_+} X \ , \quad \partial_\pm g_\pm \equiv h_\pm \ , \quad X = X_+ + X_- \ , \quad X_\pm = X_\pm(\sigma_\pm) \ , \qquad (4.5)$$

where $X$ satisfies the free wave equation, $\partial_+\partial_- X = 0$.[14] The functions $h_\pm(\sigma_\pm)$ can be fixed to be constants by using the remaining freedom of conformal transformations ($\sigma_\pm \to f_\pm(\sigma_\pm)$). This is a natural light-cone type gauge in this model, in which

$$x = e^{i\alpha h_- \sigma_- - i\beta h_+ \sigma_+} X \ , \qquad h_\pm = \text{const} \ . \qquad (4.6)$$

As in the special case of the $\alpha = 0$ model (2.21) discussed in [2] the transformation $x \to X$ in (4.5) or (4.6) makes the theory effectively a free one and is a key to its solution. Using (4.5) one finds that the equations for $u$ and $v$ (4.3) take a very simple form (because of the special quadratic form of $F^{-1}$ all $XX^*$-terms cancel out)

$$\partial_+ u = h_+ + \tfrac{1}{2} i\alpha(X\partial_+ X^* - X^* \partial_+ X) \ , \quad \partial_- v = h_- - \tfrac{1}{2} i\beta(X\partial_- X^* - X^*\partial_- X). \qquad (4.7)$$

To proceed, one needs to specify the boundary conditions. Let us first consider the case of the cylindrical world sheet. The closed string periodicity condition $x(\sigma+\pi,\tau) = x(\sigma,\tau)$ is solved if $X$ satisfies the "twisted" boundary condition (see also [2])

$$X(\sigma+\pi,\tau) = e^{i\gamma\pi} X(\sigma,\tau) \ , \quad \gamma \equiv \beta h_+ + \alpha h_- \ , \qquad (4.8)$$

[13] Similar simplifications occur also in the models with either $\alpha$, $\beta$ or $q_+ + \beta - \alpha$ equal to zero.

[14] Note that in the general case of $\alpha\beta q_+(q_+ + \beta - \alpha) \neq 0$ the equation for $x$ will no longer be linear but will still be solvable, see Section 5.1.

implying

$$X_+ = e^{i\gamma\sigma_+}\mathcal{X}_+ \ , \qquad X_- = e^{-i\gamma\sigma_-}\mathcal{X}_- \ , \tag{4.9}$$

where $\mathcal{X}_\pm = \mathcal{X}_\pm(\sigma_\pm)$ are single-valued free fields

$$\mathcal{X}_+ = i\sqrt{\alpha'/2}\sum_{n=-\infty}^{\infty} \tilde{a}_n \exp(-2in\sigma_+) \ , \quad \mathcal{X}_- = i\sqrt{\alpha'/2}\sum_{n=-\infty}^{\infty} a_n \exp(-2in\sigma_-) \ . \tag{4.10}$$

One can then solve (4.7) expressing $u$ and $v$ in terms of momentum and winding $y$-modes and oscillators in (4.10) (see Section 5.1).

The solution of the general model (2.5) or (2.13) with $q_i \neq 0$ can be essentially reduced to that of the $q_i = 0$ case. Since the only difference between (2.13) and the $q_i = 0$ model (2.15) is in the substitution (2.7), i.e. $\varphi \to \varphi' = \varphi + q_1 u + q_2 v$, starting with (2.13) we get (4.1),(4.3),(4.5), etc., with $x = \rho e^{i\varphi}$ replaced by

$$x' = e^{i(q_1 u + q_2 v)} x \ . \tag{4.11}$$

If $y$ were non-compact, $x'$ would be single-valued like $x$ and the $q_i \neq 0$ theory would be equivalent to the $q_i = 0$ one. The only subtlety is thus to take into account the winding mode part of $y$, which should be treated separately, while the single-valued part of $y$ and $q_-t$-term in (2.7) can be eliminated by the transformation (4.11) with $u, v$ replaced by $u', v'$, which do not contain the winding part of $y$. A systematic way of doing this will be discussed below in Section 5.1 using angular coordinates.

*4.2. Path integral computation of the partition function on the torus*

Let us now illustrate the solvability of the model by computing the partition function on the torus $Z$ using the path integral approach. It turns out to be possible to compute all the path integrals explicitly expressing $Z(R, \alpha, \beta, q_+)$ in terms of sums over winding numbers and two extra (in addition to modular) ordinary integrals. The latter will be absent (easily computable) in the special cases when $\alpha\beta q_+(q_+ + \beta - \alpha) = 0$.

The fields $x$ and $t$ are single-valued, i.e., on the torus, $x(\sigma_1 + n, \sigma_2 + m) = x(\sigma_1, \sigma_2)$, $t(\sigma_1 + n, \sigma_2 + m) = t(\sigma_1, \sigma_2)$ ($n, m$ are integers).[15] Since $y = \frac{1}{2}(u + v)$ has period $2\pi R$ it should satisfy the condition

$$y(\sigma_1 + n, \sigma_2 + m) = y(\sigma_1, \sigma_2) + 2\pi R(nw + mw') \ , \tag{4.12}$$

where $w, w'$ are two integer winding numbers. Then

$$y = y_* + y' \ , \quad y_* = y_0 + 2\pi R(w\sigma_1 + w'\sigma_2) \ , \quad y' = \sum_{n,n'} y_{nn'} e^{2\pi i(n\sigma_1 + n'\sigma_2)} \ , \tag{4.13}$$

[15] We shall follow the notation of [2] with the following exceptions: we use $y$ and not $\phi$ for the compact coordinate, and $c_+$ instead of $f$ for the ("left") magnetic strength parameter. In particular, for the torus $ds^2 = |d\sigma_1 + \tau d\sigma_2|^2$, $0 < \sigma_\alpha \leq 1$ , $\tau = \tau_1 + i\tau_2$, $g_{\alpha\beta} = \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}$, $\sqrt{g}g^{\alpha\beta} = \tau_2^{-1}\begin{pmatrix} |\tau|^2 & -\tau_1 \\ -\tau_1 & 1 \end{pmatrix}$. Also, $\partial = \frac{1}{2}(\partial_2 - \tau\partial_1)$, $\bar{\partial} = \frac{1}{2}(\partial_2 - \bar{\tau}\partial_1)$.

where $y'$ is the single-valued part of $y$. The computation in the general case of $q_+ \neq 0$ turns out to be a simple generalization of the $q_+ = 0$ case. If one starts with the $q_i = 0$ action (2.15) and separates the single-valued parts in $u, v$ ($u = y_* + u', \ v = y_* + v'$) one obtains

$$I = \frac{1}{\pi\alpha'\tau_2} \int d^2\sigma \big[F(x)(\partial u' + A_1)(\bar\partial v' + A_2) + \partial x \bar\partial x^* + \tau_2 \mathcal{R}(\phi_0 + \tfrac{1}{2}\ln F)\big], \tag{4.14}$$

$$A_1 \equiv \partial y_* - \tfrac{1}{2} i\alpha(x\partial x^* - x^*\partial x) \ , \tag{4.15}$$

$$A_2 \equiv \bar\partial y_* + \tfrac{1}{2} i\beta(x\bar\partial x^* - x^*\bar\partial x) \ , \quad F^{-1} = 1 + \alpha\beta x x^* \ ,$$

$$\partial y_* = \pi R(w' - \tau w) \ , \quad \bar\partial y_* = \pi R(w' - \bar\tau w) \ . \tag{4.16}$$

For $A_{1,2} = 0$ the integral over $u', v'$ would lead to the conclusion that the dilaton term is cancelled out [29] and that the partition function is thus given by the free-theory one [30]. For $A_{1,2} \neq 0$ the integral over $u', v'$ gives also the product of the "zero-mode" parts of $A_i$, i.e. the term $\sim \langle F\rangle\langle A_1\rangle\langle A_2\rangle$, $(\langle ...\rangle \equiv \int d^2\sigma ...)$ which is non-gaussian in $x, x^*$. To be able to then integrate over $x, x^*$, it is convenient to "split" this term into quadratic parts using an ordinary integral over two auxiliary parameters. Equivalently, one may introduce from the very beginning an auxiliary vector field $(C, \bar C)$ representing $e^{-I}$ as[16]

$$e^{-I} = \int [dC d\bar C] \exp\big( - \frac{1}{\pi\alpha'\tau_2} \int d^2\sigma \big[F^{-1}(x) C\bar C \tag{4.17}$$

$$+ \bar C(\partial u' + A_1) - C(\bar\partial v' + A_2) + \partial x\bar\partial x^*\big]\big) \ .$$

Integrating over $u', v'$ we find that (on the torus) $C$ and $\bar C$ are constrained to be equal to constants. We shall denote these constants as $C_0$ and $\bar C_0$ (the integral over $C_0, \bar C_0$ will contain the factor of $\tau_2^{-1}$ in the measure). The remaining action is then quadratic in $x, x^*$ so that the expression for the partition function takes the following form[17]

$$Z(r, \alpha, \beta, q_+) = \int d^2\tau \ \tau_2^{-12} e^{11\pi\tau_2/3} |f_0(e^{2\pi i\tau})|^{-44} \ \mathcal{Z}(\tau, \bar\tau) \ ,$$

$$\mathcal{Z} = r \sum_{w,w'=-\infty}^{\infty} \int dC_0 d\bar C_0 \ \tau_2^{-1} \ \exp[-\frac{1}{\pi\alpha'\tau_2}(C_0\bar C_0 + \bar C_0 \partial y_* - C_0\bar\partial y_*)] \ \mathcal{Z}_x \ , \tag{4.18}$$

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[16] We omit the dilaton term which is cancelled out at the end after one integrates over $u, v, C, \bar C$. Note also that the measure of integration over $u, v$, which originally contained the $\sqrt{-G} = F(x)$ factor, becomes trivial after the introduction of $(C, \bar C)$ (the $F$-factor is "exponentiated").

[17] The modular measure contains the contribution of the 22 extra free scalar degrees of freedom added to satisfy the zero central charge condition. In general, the integrand of the partition function is modular invariant (as it should be, being derived from a reparametrisation invariant world-sheet theory): the transformations $\tau \to \tau + 1$ and $\tau \to -1/\tau$ are "undone" by the redefinitions of other integration and summation parameters.

$$\mathcal{Z}_x = \int [dxdx^*] \exp(-I'[x,x^*;C_0,\bar{C}_0,w,w',\tau,\bar{\tau}]) \ ,$$

$$I' = \frac{1}{\pi\alpha'\tau_2} \int d^2\sigma \big[\alpha\beta C_0\bar{C}_0 xx^* - \tfrac{1}{2}i\alpha\bar{C}_0(x\partial x^* - x^*\partial x) \qquad (4.19)$$

$$- \tfrac{1}{2}i\beta C_0(x\bar{\partial}x^* - x^*\bar{\partial}x) + \partial x\bar{\partial}x^*\big] \ .$$

The factor $r = R/\sqrt{\alpha'}$ in (4.18) as usual comes from the integral over the compact constant mode $y_0$ (we drop out an infinite integral over $t_0$). Expanding

$$x = x_0 + x' = x_0 + \sum_{(n,n')\neq 0} a_{nn'} e^{2\pi i(n\sigma_1 + n'\sigma_2)} \ , \qquad (4.20)$$

one finds that the gaussian integral over $x', x'^*$ leads to the following simple result (cf. ref. [2])

$$\mathcal{Z}'_x = c_0[\det{}'\Delta_0]^{-1} \ Y^{-1}(\tau,\bar{\tau},\chi,\tilde{\chi}) \ , \qquad (4.21)$$

$$\det{}'\Delta_0 = \tau_2^2\eta^2\bar{\eta}^2 \ , \quad \eta = e^{i\pi\tau/12}\prod_{n=1}^{\infty}(1-e^{2\pi i n\tau}) \equiv e^{i\pi\tau/12} f_0(e^{2\pi i\tau}) \ , \qquad (4.22)$$

$$Y(\tau,\bar{\tau},\chi,\tilde{\chi}) = \prod_{(n,n')\neq(0,0)} \big(1 + \frac{\chi}{n'-\tau n}\big) \prod_{(n,n')\neq(0,0)} \big(1 + \frac{\tilde{\chi}}{n'-\bar{\tau}n}\big) \ , \qquad (4.23)$$

$$\chi \equiv \frac{1}{\pi}\beta C_0 \ , \qquad \tilde{\chi} \equiv \frac{1}{\pi}\alpha\bar{C}_0 \ . \qquad (4.24)$$

The factorized "chiral" form of $G$ is spoiled by a diffeomorphism (modular) invariant regularisation. To define explicitly the formal expression (4.23) consider

$$U(\tau,\bar{\tau},\chi,\tilde{\chi}) \equiv \prod_{(n,n')\neq(0,0)} (n'-\tau n+\chi)(n'-\bar{\tau}n+\tilde{\chi}) \qquad (4.25)$$

$$= \prod_{k\neq 0}(k+\chi)(k+\tilde{\chi}) \prod_{n\neq 0,n'} (n'-\tau n+\chi)(n'-\bar{\tau}n+\tilde{\chi}) \ .$$

Computing first the product over $k$ and $n'$ using

$$\prod_{n=-\infty}^{\infty}(n+\chi) = \chi\prod_{n=1}^{\infty}(-n^2)(1-\frac{\chi^2}{n^2}) = 2i\sin\pi\chi \ ,$$

one gets the product of sin-functions. Separating the factor

$$\prod_{n\neq 0} \exp(-i\pi\tau n + i\pi\chi) \ \exp(i\pi\bar{\tau}n - i\pi\tilde{\chi}) \ ,$$

and defining it as

$$\exp\big[2\pi\tau_2\sum_{n\neq 0}\big(n + i\frac{\chi-\tilde{\chi}}{2\tau_2}\big)\big] \ ,$$

one should compute the sum using the generalised $\zeta$-function regularisation

$$\sum_{n=1}^{\infty}(n+c) = \lim_{s\to -1}\sum_{n=1}^{\infty}(n+c)^{-s} = -\frac{1}{12} + \frac{1}{2}c(1-c) \ . \tag{4.26}$$

As a result,

$$Y(\tau,\bar\tau,\chi,\tilde\chi) \equiv \frac{U(\tau,\bar\tau,\chi,\tilde\chi)}{U(\tau,\bar\tau,0,0)} = \ \exp[\frac{\pi(\chi-\tilde\chi)^2}{2\tau_2}] \tag{4.27}$$

$$\times[\frac{\sin\pi\chi}{\pi\chi}\prod_{n=1}^{\infty}\frac{(1-\rho^{-1}q^n)(1-\rho q^n)}{(1-q^n)^2}]\ [\frac{\sin\pi\tilde\chi}{\pi\tilde\chi}\prod_{n=1}^{\infty}\frac{(1-\tilde\rho^{-1}\bar q^n)(1-\tilde\rho\bar q^n)}{(1-\bar q^n)^2}] \ ,$$

$$\rho\equiv\exp(2\pi i\chi) \ , \quad \tilde\rho\equiv\exp(2\pi i\tilde\chi) \ , \quad q=\exp(2\pi i\tau) \ ,$$

or, finally,

$$Y(\tau,\bar\tau,\chi,\tilde\chi) = \exp[\frac{\pi(\chi-\tilde\chi)^2}{2\tau_2}]\ \frac{\theta_1(\chi|\tau)}{\chi\theta_1'(0|\tau)}\ \frac{\theta_1(\tilde\chi|\bar\tau)}{\tilde\chi\theta_1'(0|\bar\tau)} \ . \tag{4.28}$$

The contribution of the integral over the constant parts $x_0, x_0^*$ is

$$\mathcal{Z}_{x0} = \int dx_0 dx_0^* \exp[-(\pi\alpha'\tau_2)^{-1}\alpha\beta C_0\bar C_0 x_0 x_0^*] = \frac{\alpha'\pi^2\tau_2}{\alpha\beta C_0\bar C_0} = \frac{\alpha'\tau_2}{\chi\tilde\chi} \ . \tag{4.29}$$

For $\alpha\beta=0$ this expression gives a divergent factor corresponding to the area of the $x_1, x_2$ plane.[18] If one defines $Z$ using the factor (4.29) then the free-theory limit becomes singular. Alternatively, one may leave the integral over $x_0, x_0^*$ to the end so that the limit $\alpha\beta\to 0$ is regular in the integrand.[19] Another possibility to obtain $Z$ with a regular free-theory limit (equal to the standard partition function of a free string compactified on a circle) is to project out the constant mode factor (4.29) (e.g. by inserting the $\delta$-functions $\delta(x_0)\delta(x_0^*)$). Using the latter prescription we find

$$Z(r,\alpha,\beta,q_+) = c_1\int [d^2\tau]_1\ W(r,\alpha,\beta,q_+|\tau,\bar\tau) \ , \tag{4.30}$$

$$[d^2\tau]_1 \equiv d^2\tau\ \tau_2^{-14} e^{4\pi\tau_2}|f_0(e^{2\pi i\tau})|^{-48} \ , \tag{4.31}$$

[18] For $\alpha\beta\neq 0$ the external fields break down the translational invariance on the $x_1, x_2$-plane (in particular, $x_{cl}=x_0$ is no longer a classical solution for the $\sigma$-model action (4.14) when $u_{cl}=v_{cl}=y_*$ and $w,w'\neq 0$) and thus "regularize" the divergent area factor $\int dx_0 dx_0^*$ in $Z$.

[19] In general, the $\sigma$-model partition function on the torus is $Z=\int d^D X_0 Z_1(X_0)$, $Z_1 = \sqrt{-G(X_0)}Z_1'(X_0)$, where $X_0^M$ are the constant parts of all $\sigma$-model coordinates. The $\sigma$-model path integral $\int[dX]\exp(-\int G_{MN}(X)\partial X^M\bar\partial X^N)$ is defined using the measure $|\delta X|^2 = \tau_2\int d^2\sigma G_{MN}(X)\delta X^M\delta X^N$. In the present case of (4.14) (before one introduces $C,\bar C$) the volume $\int d^D X_0\sqrt{-G(X_0)} = 2\pi R\int dt_0\int dx_0 dx_0^*(1+\alpha\beta x_0 x_0^*)^{-1}$ is divergent in the $x_0, x_0^*$ direction: logarithmically if $\alpha\beta\neq 0$ and quadratically – if $\alpha\beta=0$.

$$W(r,\alpha,\beta,q_+=0) \ = \ r(\alpha'\alpha\beta\tau_2)^{-1} \sum_{w,w'=-\infty}^{\infty} \int d\chi d\tilde\chi \tag{4.32}$$

$$\times \ \exp\big(-\pi(\alpha'\alpha\beta\tau_2)^{-1}[\chi\tilde\chi + \sqrt{\alpha'}r\beta(w'-\tau w)\tilde\chi - \sqrt{\alpha'}r\alpha(w'-\bar\tau w)\chi]\big)$$

$$\times \ \exp\big[-\frac{\pi(\chi-\tilde\chi)^2}{2\tau_2}\big] \ \frac{\chi\tilde\chi|\theta_1'(0|\tau)|^2}{\theta_1(\chi|\tau)\theta_1(\tilde\chi|\bar\tau)} \ .$$

When $\alpha=0$ this expression reduces to the partition function for the "chiral" $\alpha=q_+=0$ theory (2.21) found in [2]: for $\alpha\to 0$ the integral over $\bar C_0=\pi\tilde\chi/\alpha$ produces the $\delta$-function constraint $\chi=-\chi_0=-\sqrt{\alpha'}r\beta(w'-\tau w)$ so that

$$W(r,\alpha=0,\beta,q_+=0) = \sum_{w,w'=-\infty}^{\infty} \exp[-I_0(r)] \ \exp[-\frac{\pi\chi_0^2}{2\tau_2}] \ \frac{\chi_0\theta_1'(0|\tau)}{\theta_1(\chi_0|\tau)} \ , \tag{4.33}$$

$$I_0(r) \equiv \pi r^2\tau_2^{-1}(w'-\tau w)(w'-\bar\tau w) \ , \tag{4.34}$$

$$\chi_0 = \sqrt{\alpha'}\beta r(w'-\tau w) \ . \tag{4.35}$$

The representation (4.32) is not the simplest one possible for $Z$ in the case when $q_+=0$ (the sums over $w,w'$ give $\delta$-functions, and thus the integrals over $\chi,\tilde\chi$ in (4.32) can be computed explicitly, see below) but its advantage is that it has a straightforward generalization to the case of $q_+\neq 0$. Indeed, according to (2.13) to include the dependence on $q_+$ one is to make the transformation (4.11) in the action (4.14). This transformation can be represented as $x\to\exp(iq_+y_*)\hat x$, $\hat x=\exp(iq_1u'+iq_2v')x$, where $u',v'$ are single-valued parts of $u,v$. Then $\hat x$ is also single-valued and can be used as a new integration variable instead of $x$. The $q_+$-dependent analogues of (4.14), (4.17),(4.19) are thus obtained by the formal substitution $x\to\exp(iq_+y_*)x$. One finds that (4.19), and thus (4.21)–(4.24), have the same form with

$$\beta C_0 \to \beta C_0 + q_+\partial y_* \ , \qquad \alpha\bar C_0 \to \alpha\bar C_0 + q_+\bar\partial y_* \ , \tag{4.36}$$

$$\chi \to \chi + q_+R(w'-\tau w) \ , \qquad \tilde\chi \to \tilde\chi + q_+R(w'-\bar\tau w) \ ,$$

$$\chi \equiv \frac{1}{\pi}\beta C_0 + q_+R(w'-\tau w), \ \ \tilde\chi \equiv \frac{1}{\pi}\alpha\bar C_0 + q_+R(w'-\bar\tau w) \ . \tag{4.37}$$

Then the general expression for the partition function is given by (4.30) with (4.32) replaced by[20]

$$W(r,\alpha,\beta,q_+) \ = \ r(\alpha'\alpha\beta\tau_2)^{-1} \sum_{w,w'=-\infty}^{\infty} \int d\chi d\tilde\chi \tag{4.38}$$

$$\times \exp\big(-\pi(\alpha'\alpha\beta\tau_2)^{-1}[\chi\tilde\chi + \sqrt{\alpha'}r(q_++\beta)(w'-\tau w)\tilde\chi + \sqrt{\alpha'}r(q_+-\alpha)(w'-\bar\tau w)\chi$$

$$+ \ \alpha' r^2 q_+(q_++\beta-\alpha)(w'-\tau w)(w'-\bar\tau w)] \ \big)$$

[20] In considering formal singular limits of the partition function (like $R=\infty$, see below) it is more convenient to use the original integration parameters $C_0, \ \bar C_0$ instead of $\chi,\tilde\chi$ (cf. eq. (4.37)).

$$\times \ \exp[-\frac{\pi(\chi-\tilde{\chi})^2}{2\tau_2}] \ \frac{\chi\tilde{\chi}|\theta_1'(0|\tau)|^2}{\theta_1(\chi|\tau)\theta_1(\tilde{\chi}|\bar{\tau})} \ .$$

Like the measure in (4.30) this expression is $SL(2,Z)$ modular invariant (to show this one needs to shift $w, w'$ and redefine $\chi, \tilde{\chi}$). The expression in the approach where the integrals over the constant parts $x_0, x_0^*$ are left until the very end is obtained by an obvious modification: the $\chi\tilde{\chi}$ term in the exponential in (4.38) is replaced by $\chi\tilde{\chi}(1+\alpha\beta x_0 x_0^*)$. If one explicitly integrates over $x_0, x_0^*$, then the integrand in (4.38) is multiplied by the factor in (4.29). In all cases the integrands of $Z$ are modular invariant.

Given that the model (2.5) is $\varphi$-dual to the model (2.3), the two should lead to the same partition function. Starting with (2.3) one should thus be able to reproduce the same expression (4.30),(4.38) in a simpler way. Separating $u$ and $v$ into the "winding" and single-valued parts ($u = y_* + u', \ v = y_* + v'$) and integrating over $u', v'$ in (2.3) one again must introduce the integral over the auxiliary constant parameters $C_0, \bar{C}_0$ in order to "split" the zero mode factor $\langle\partial\tilde{\varphi}\rangle\langle\bar{\partial}\tilde{\varphi}\rangle$. Then (2.3) is transformed into

$$L = C_0\bar{C}_0 + \bar{C}_0(\partial y_* + \alpha\partial\tilde{\varphi}) - C_0(\bar{\partial} y_* + \beta\bar{\partial}\tilde{\varphi}) \tag{4.39}$$

$$-q_+\partial\tilde{\varphi}\bar{\partial}y^* + q_+\bar{\partial}\tilde{\varphi}\partial y^* + \partial\rho\bar{\partial}\rho + \rho^{-2}\partial\tilde{\varphi}\bar{\partial}\tilde{\varphi} + \tau_2\mathcal{R}(\phi_0 - \tfrac{1}{2}\ln\rho^2) \ .$$

Making *now* the path integral duality transformation $\tilde{\varphi} \to \varphi$ one obtains the same action as in (4.19),(4.36) which is quadratic in $x, x^*$. The resulting expression for $Z$ is thus equivalent to (4.30),(4.38).

To conclude, we have found the explicit representation for the partition function in terms of the two auxiliary *ordinary* integrals. In general, $Z(r,\alpha,\beta,q_+)$ depends on four real dimensionless parameters ($R/\sqrt{\alpha'}, \ R\alpha, \ R\beta, \ Rq_+$) and has several symmetry properties which follow from (4.38). $Z$ is symmetric under $\alpha \leftrightarrow -\beta$ as well as under the simultaneous changing of the signs of $\alpha$, $\beta$ and $q_+$

$$Z(r,\alpha,\beta,q_+) = Z(r,-\beta,-\alpha,q_+) = Z(r,-\alpha,-\beta,-q_+) = Z(r,\beta,\alpha,-q_+) \ . \tag{4.40}$$

It is also invariant under the $y$-duality which transforms the theory with $y$-period $2\pi R$ and parameters $a_+ = q_+ - \alpha, \ c_+ = q_+ + \beta, \ c_- - a_- = \alpha - \beta$ into the theory with $y$-period $2\pi\alpha'/R$ and parameters $-a_+, \ c_+, \ c_- - a_-$ or parameters $a_+, \ -c_+, \ c_- - a_-$ (see (2.10),(2.11),(2.6)). This is seen explicitly from the representation (4.38): doing the Poisson resummation (see (5.55)) in $w$ or in $w'$ (which is equivalent to performing the duality transformation in $y$) one obtains the same expression for the exponential in (4.38) with the parameters interchanged according to the above relations, namely,

$$Z(r,\alpha,\beta,q_+) = Z(r^{-1},q_+,\beta-\alpha+q_+,\alpha) = Z(r^{-1},\alpha-\beta-q_+,-q_+,-\beta) \ . \tag{4.41}$$

Combining (4.40) and (4.41) we also learn that

$$Z(r,\alpha,\beta,q_+) = Z(r,-\beta,-\alpha,\alpha-\beta-q_+) = Z(r,\alpha,\beta,\alpha-\beta-q_+) \ . \tag{4.42}$$

When $\alpha = q_+$ ($a_+ = 0$) or $\beta = -q_+$ ($c_+ = 0$) the duality relations (4.41) retain their standard "circle" form

$$Z(r,\alpha,\beta,\alpha) = Z(r^{-1},\alpha,\beta,\alpha) \ , \quad Z(r,\alpha,\beta,-\beta) = Z(r^{-1},\alpha,\beta,-\beta) \ . \tag{4.43}$$

These duality relations will be manifest also in the representation for $Z$ derived in the operator approach in Section 5.3.

As follows from (4.38), the expression for $Z$ (4.30),(4.38) simplifies substantially when $\alpha\beta q_+(q_+ + \beta - \alpha) = 0$, i.e. when one of the parameters $\alpha, \beta, q_+$ or $q_+ + \beta - \alpha$ vanishes (i.e. when at least one of the two magnetic fields in (3.10),(3.6),(3.7) is uniform). Then the integrals over $C_0, \bar{C}_0$ can be computed explicitly and one obtains a direct generalization of (4.33). In view of the relations (4.40),(4.41),(4.42) these four cases are equivalent. For example, when $q_+ = 0$ the integrals over $\chi, \tilde{\chi}$ in (4.32) can be easily computed if one notes that the sums over $w, w'$ produce $\delta$-functions when there is no quadratic term in $w, w'$. The result, of course, is the same as the one found by starting directly with the model (2.20) which is $y$-dual to the $q_i = 0$ model (2.15). The path integral for (2.20) can be computed without need to introduce the auxiliary fields $C, \bar{C}$ in (4.17): as in the $\alpha = 0, q_i = 0$ model [2] the integral over $\tilde{v}$ restricts $\tilde{u}$ to the zero-mode value $\tilde{u}_* = \tilde{y}_* = 2\pi\tilde{R}(w\sigma_1 + w'\sigma_2)$, $\tilde{R} = \alpha'/R$, and one finishes with the partition function (4.30) with (cf. (4.32),(4.33),(4.43))[21]

$$W(r, \alpha, \beta, q_+ = 0) \ = \ r^{-1} \sum_{w,w'=-\infty}^{\infty} \exp[-I_0(r^{-1})] \tag{4.44}$$

$$\times \exp[-\frac{\pi(\chi_0 - \tilde{\chi}_0)^2}{2\tau_2}] \ \frac{\chi_0\tilde{\chi}_0|\theta_1'(0|\tau)|^2}{\theta_1(\chi_0|\tau)\theta_1(\tilde{\chi}_0|\bar{\tau})} \ ,$$

$$\chi_0 = \sqrt{\alpha'}\beta r^{-1}(w' - \tau w) \ , \qquad \tilde{\chi}_0 = \sqrt{\alpha'}\alpha r^{-1}(w' - \bar{\tau} w) \ . \tag{4.45}$$

Starting directly with (4.38) or using (4.44) and (4.41) (i.e. $Z(r, \alpha, \beta, 0) = Z(r^{-1}, 0, \beta - \alpha, \alpha)$) we can find also a simple form of $Z$ in the case when $\alpha = 0$ or when $\beta = 0$, i.e. when $\alpha\beta = 0$ (cf. (4.44),(4.38))

$$W(r, \alpha, \beta, q_+)|_{\alpha\beta=0} \ = \ r \sum_{w,w'=-\infty}^{\infty} \exp[-I_0(r)] \tag{4.46}$$

$$\times \exp[-\frac{\pi(\chi_0 - \tilde{\chi}_0)^2}{2\tau_2}] \ \frac{\chi_0\tilde{\chi}_0|\theta_1'(0|\tau)|^2}{\theta_1(\chi_0|\tau)\theta_1(\tilde{\chi}_0|\bar{\tau})} \ ,$$

$$\chi_0 = \sqrt{\alpha'}(q_+ + \beta) r (w' - \tau w) \ , \qquad \tilde{\chi}_0 = \sqrt{\alpha'}(q_+ - \alpha) r (w' - \bar{\tau} w) \ , \quad \alpha\beta = 0 \ .$$

Another interesting special case is $a_+ = c_+ = 0$ (i.e. $\alpha = q_+ = -\beta$) when the modular integrand $W$ (4.38) formally factorises into the $r$-dependent part $W_0(r)$ and an $\alpha$-dependent part $W_1(\alpha)$.[22] As follows from (4.38),

$$Z(r, \alpha, -\alpha, \alpha) = c_1 \int [d^2\tau]_1 \ W_0(r) \ W_1(\alpha) \ , \tag{4.47}$$

[21] As before in (4.38), we have projected out the contribution $\sim \tau_2(\chi_0\tilde{\chi}_0)^{-1}$ (which is present in the winding $(w, w')$ sector) of the integral over $x_0, x_0^*$.

[22] This factorization is valid under certain analytic continuation assumption since the contour of integration over $\chi, \tilde{\chi}$ depends on $w, w'$ according to (4.37).

$$W_0 = r \sum_{w,w'=-\infty}^{\infty} \exp[-I_0(r)] \ , \qquad (4.48)$$

$$W_1 = -(\alpha'\alpha^2\tau_2)^{-1} \int d\chi d\tilde{\chi} \ \exp\big(\frac{\pi\chi\tilde{\chi}}{\alpha'\alpha^2\tau_2}\big) \qquad (4.49)$$

$$\times \exp[-\frac{\pi(\chi-\tilde{\chi})^2}{2\tau_2}] \ \frac{\chi\tilde{\chi}|\theta_1'(0|\tau)|^2}{\theta_1(\chi|\tau)\theta_1(\tilde{\chi}|\bar{\tau})} \ .$$

$W_0$ is the same as the partition function of a free boson on a circle. Clearly, in this case $Z(r,\alpha) = Z(r^{-1},\alpha)$.

Finally, let us note that in the limit of non-compact $y$-dimension ($R \to \infty$) $Z$ (4.30),(4.38) reduces to the partition function of the free bosonic closed string theory. A simple way to see this is to note that for the non-compact $y$ the parameters $q_i$ can be set equal to zero by a coordinate transformation (equivalently, in the case of $R = \infty$ the winding sector becomes trivial ($w = w' = 0$) and thus according to (4.36) $Z$ does not depend on $q_+$). Then taking $R \to \infty$ in (4.44) one finds that $Z$ takes the flat space expression. To show this in general starting directly with (4.30),(4.38) one should first use (4.37) to return to the integral over the variables $C_0, \bar{C}_0$, rescale the latter by $R$ and then take the limit $R = \infty$ (in (4.47) this effectively corresponds to shrinking the contour of integration over $\chi, \tilde{\chi}$ to zero so that $W_1 \to 1$).

$Z(R \to \infty) = Z_{free}$ is also clear directly from the form of the path integral for the $y$-dual to $q_i = 0$ theory (2.20): in the non-compact case the zero mode of $\tilde{u}$ is constant and after integrating out $\tilde{v}$ one gets a free $x, x^*$-theory. This generalizes a similar observation for the $\alpha = q_+ = 0$ model [2].[23]

## 5. Solution of the string model: canonical operator approach

In this section we shall first derive the expression for the solution of the classical equations of motion for the general values of parameters of our model (2.5) in terms of constant zero-mode parameters and free oscillators. We shall then canonically quantize the model using a light-cone type gauge and derive the quantum Hamiltonian (which will be fourth order in oscillators but diagonal in Fock space). The possibility to choose the light-cone gauge combined with conformal invariance guarantees the unitarity of the model. Finally, we will show that the operator approach leads to the same expression for the partition function that was found above in the path integral approach.

### *5.1. General solution of the classical equations of motion and light-cone gauge*

Let us now return to the discussion of the solution of the classical equations of motion corresponding to our model (2.4) (we shall consider the flat cylindrical Minkowski world sheet with $\sigma \in [0,\pi)$, $-\infty < \tau < \infty$, $\sigma_\pm = \tau \pm \sigma$)

$$L = \partial_+ u \partial_- v + \partial_+ \rho \partial_- \rho \qquad (5.1)$$

$$+ \ F(\rho)\rho^2[\partial_+\varphi + (\beta + q_1)\partial_+ u + q_2\partial_+ v][\partial_-\varphi + q_1\partial_- u + (q_2 - \alpha)\partial_- v] \ ,$$

[23] In the limit $R = \infty$ the $\alpha = q_+ = 0$ model (2.21) is equivalent to the model of [26] which has trivial (free) partition function [31].

$$F^{-1}(\rho) = 1 + \alpha\beta\rho^2 \ .$$

To solve the corresponding equations of motion

$$\partial_+[F(\rho)\rho^2\partial_-(\varphi + (q_2-\alpha)v + q_1 u)] + \partial_-[F(\rho)\rho^2\partial_+(\varphi + q_2 v + (q_1+\beta)u)] = 0 \ , \qquad (5.2)$$

$$\partial_+\partial_-\rho - \rho F^2(\rho)\partial_+[\varphi + (\beta+q_1)u + q_2 v]\partial_-[\varphi + (q_2-\alpha)v + q_1 u] = 0 \ , \qquad (5.3)$$

$$\partial_+\partial_- v = -\beta\partial_+[F(\rho)\rho^2\partial_-(\varphi + q_1 u + (q_2-\alpha)v)] \ , \qquad (5.4)$$

$$\partial_+\partial_- u = \alpha\partial_-[F(\rho)\rho^2\partial_+(\varphi + (q_1+\beta)u + q_2 v)] \ , \qquad (5.5)$$

we shall utilize the $\varphi$-duality relation between (5.1) and the model (2.3) or

$$\tilde{L} = \partial_+(u+\alpha\tilde{\varphi})\partial_-(v+\beta\tilde{\varphi}) + \partial_+\rho\partial_-\rho + \rho^{-2}\partial_+\tilde{\varphi}\partial_-\tilde{\varphi} \qquad (5.6)$$

$$+\partial_+(q_1 u + q_2 v)\partial_-\tilde{\varphi} - \partial_-(q_1 u + q_2 v)\partial_+\tilde{\varphi} \ ,$$

which has the equations of motion

$$\partial_+\partial_-(u+\alpha\tilde{\varphi}) = 0 \ , \qquad \partial_+\partial_-(v+\beta\tilde{\varphi}) = 0 \ , \qquad (5.7)$$

$$\partial_+\partial_-\rho + \rho^{-3}\partial_+\tilde{\varphi}\partial_-\tilde{\varphi} = 0 \ , \qquad \partial_+(\rho^{-2}\partial_-\tilde{\varphi}) + \partial_-(\rho^{-2}\partial_+\tilde{\varphi}) = 0 \ . \qquad (5.8)$$

Eqs. (5.8) are the equations of motion for the "dual 2-plane" model

$$\tilde{L}_0 = \partial_+\rho\partial_-\rho + \rho^{-2}\partial_+\tilde{\varphi}\partial_-\tilde{\varphi} \ . \qquad (5.9)$$

Since the solutions of the equations of motion for two dual $\sigma$-models are in general related (locally) by $(G_{\mu\nu} + B_{\mu\nu})\partial_a x^\nu = \epsilon_{ab}\partial^b\tilde{x}_\mu$, we can express the solution of (5.8) in terms of the solution of the free model dual to (5.9)

$$L_0 = \partial_+\rho\partial_-\rho + \rho^2\partial_+\hat{\varphi}\partial_-\hat{\varphi} = \partial_+ X\partial_- X^* \ , \qquad X \equiv \rho e^{i\hat{\varphi}} \ . \qquad (5.10)$$

We get[24]

$$\rho^2 = XX^* \ , \quad \hat{\varphi} = \frac{1}{2i}\ln\frac{X}{X^*} \ , \quad X = X_+(\sigma_+) + X_-(\sigma_-) \ , \qquad (5.11)$$

$$\partial_\pm\tilde{\varphi} = \mp\rho^2\partial_\pm\hat{\varphi} = \pm\frac{i}{2}(X^*\partial_\pm X - X\partial_\pm X^*) \ ,$$

and thus

$$\tilde{\varphi}(\sigma,\tau) = 2\pi\alpha'[J_-(\sigma_-) - J_+(\sigma_+)] + \frac{i}{2}\big(X_+X_-^* - X_+^*X_-\big) \ , \qquad (5.12)$$

$$J_\pm(\sigma_\pm) \equiv \frac{i}{4\pi\alpha'}\int_0^{\sigma_\pm} d\sigma_\pm\big(X_\pm\partial_\pm X_\pm^* - X_\pm^*\partial_\pm X_\pm\big) \ . \qquad (5.13)$$

The solution of (5.7) is then

$$u = U_+ + U_- - \alpha\tilde{\varphi} \ , \qquad v = V_+ + V_- - \beta\tilde{\varphi} \ , \qquad (5.14)$$

[24] For a discussion of a relation between 2-plane and dual 2-plane models see [13,20,32].

where $U_\pm$ and $V_\pm$ are arbitrary functions of $\sigma_\pm$. Returning now to the system (5.2)–(5.5) we conclude that since it is $\varphi$-dual to (5.7)–(5.8), $u, v, \rho$ have the same expressions (5.11),(5.14) while $\varphi$ is found to be given by

$$\varphi + q_1 u + q_2 v = -\beta U_+ + \alpha V_- + \hat{\varphi} \ . \tag{5.15}$$

Thus

$$x \equiv \rho e^{i\varphi} = \exp[-i(q_1 u + q_2 v)] \exp(i\alpha V_- - i\beta U_+) X \ , \tag{5.16}$$

in agreement with our previous discussion (4.5),(4.11).

Let us now take into account the boundary conditions. The physical coordinate $x = \rho e^{i\varphi}$ is single-valued, i.e. $x(\sigma + \pi, \tau) = x(\sigma, \tau)$. This implies that the free field $X = X_+ + X_-$ must satisfy the "twisted" condition as in (4.8),(4.9) (see also [2])

$$X(\sigma + \pi, \tau) = e^{i\gamma\pi} X(\sigma, \tau) \ , \quad X_\pm = e^{\pm i\gamma\sigma_\pm} \mathcal{X}_\pm \ , \quad \mathcal{X}_\pm(\sigma_\pm \pm \pi) = \mathcal{X}_\pm(\sigma_\pm) \ , \tag{5.17}$$

where $\mathcal{X}_\pm = \mathcal{X}_\pm(\sigma_\pm)$ are as defined in eq. (4.10). Since the scalar field $y = \frac{1}{2}(u + v)$ is compactified on a circle of radius $R$,

$$u(\sigma + \pi, \tau) = u(\sigma, \tau) + 2\pi w R \ , \quad v(\sigma + \pi, \tau) = v(\sigma, \tau) + 2\pi w R \ , \tag{5.18}$$

where $w$ is an integer winding number. Let us now determine $\gamma$. Eqs. (5.12),(5.13) and (5.17) imply

$$\tilde{\varphi}(\sigma + \pi, \tau) = \tilde{\varphi}(\sigma, \tau) - 2\pi\alpha' J \ , \quad J = J_L + J_R \ , \quad J_L \equiv J_+(\pi) \ , \quad J_R \equiv J_-(\pi) \ . \tag{5.19}$$

We shall see below that after the quantization $J$ becomes the total angular momentum operator and has integer eigenvalues. This is consistent with the fact that $\tilde{\varphi}$ has period $2\pi\alpha'$, as implied by its duality to $\varphi$ or $\hat{\varphi}$ which have periods $2\pi$. As follows from (5.14),(5.19) the boundary conditions (5.18) are satisfied by setting

$$U_\pm = \sigma_\pm p^u_\pm + U'_\pm \ , \quad V_\pm = \sigma_\pm p^v_\pm + V'_\pm \ , \tag{5.20}$$

$$p^u_\pm = \pm(wR - \alpha\alpha' J) + p_u \ , \quad p^v_\pm = \pm(wR - \beta\alpha' J) + p_v \ , \tag{5.21}$$

$$p_u \equiv \tfrac{1}{2}(s - p) \ , \qquad p_v \equiv \tfrac{1}{2}(s + p) \ , \tag{5.22}$$

where $U'_\pm$ and $V'_\pm$ are single-valued functions of $\sigma_\pm$ and $s$ and $p$ are arbitrary parameters (later they will be expressed in terms of the Kaluza-Klein momentum and the energy of the string). Then it follows from (5.16) that (5.17) is satisfied provided $\gamma$ (which is defined modulo 2) is given by

$$\gamma = 2[q_1 + q_2 + \tfrac{1}{2}(\beta - \alpha)]wR + \beta p_u + \alpha p_v$$

$$= (c_+ + a_+)wR + \tfrac{1}{2}(\beta + \alpha)s + \tfrac{1}{2}(\alpha - \beta)p \ , \tag{5.23}$$

where $a_+ = q_+ - \alpha, \ c_+ = q_+ + \beta$ (see (2.6)).

Starting from the general expression for the classical stress-energy tensor of the theory (5.1) and evaluating it on the general solution (5.11), (5.16) one finds that it takes the "free-theory" form[25]

$$T_{\pm\pm} = \partial_\pm U_\pm \partial_\pm V_\pm + \partial_\pm X \partial_\pm X^* \ . \tag{5.24}$$

This of course is not surprising since the on-shell values of the stress-energy tensors in the two dual $\sigma$-models should be the same ((5.24) is precisely $T_{\pm\pm}$ corresponding to the theory (5.6), see (5.11), (5.14)). It is convenient to fix the light-cone gauge, using the residual symmetry to gauge away, e.g., $U'_\pm$. Then the classical constraints $T_{--} = T_{++} = 0$ can be solved as usual and determine the remaining oscillators of $V'_\pm$ in terms of the free fields $X_\pm$. It is also straightforward to quantize the model in the covariant formalism (which is more suitable, e.g., for a study of scattering amplitudes in the operator approach) but in order to determine the physical spectrum the light-cone gauge is as usual more convenient.

After using (5.20),(5.17) $T_{\pm\pm}$ takes the form

$$T_{\pm\pm} = p^u_\pm p^v_\pm + p^u_\pm \partial_\pm V'_\pm \pm i\gamma(\mathcal{X}_\pm \partial_\pm \mathcal{X}^*_\pm - \mathcal{X}^*_\pm \partial_\pm \mathcal{X}_\pm) + \gamma^2 \mathcal{X}_\pm \mathcal{X}^*_\pm + \partial_\pm \mathcal{X} \partial_\pm \mathcal{X}^* \ , \tag{5.25}$$

where $\mathcal{X}_\pm$ have the standard mode expansions (4.10). The classical expressions for the Virasoro operators $L_0, \tilde{L}_0$ are obtained by integrating over $\sigma$

$$L_0 \equiv \frac{1}{4\pi\alpha'} \int_0^\pi d\sigma \ T_{--} = \frac{p^u_- p^v_-}{4\alpha'} + \tfrac{1}{2} \sum_n \left(n + \tfrac{1}{2}\gamma\right)^2 a^*_n a_n, \tag{5.26}$$

$$\tilde{L}_0 \equiv \frac{1}{4\pi\alpha'} \int_0^\pi d\sigma \ T_{++} = \frac{p^u_+ p^v_+}{4\alpha'} + \tfrac{1}{2} \sum_n \left(n - \tfrac{1}{2}\gamma\right)^2 \tilde{a}^*_n \tilde{a}_n \ . \tag{5.27}$$

Hence the Hamiltonian is given by

$$H = L_0 + \tilde{L}_0 = \tfrac{1}{8\alpha'}(4w^2R^2 + s^2 - p^2) \tag{5.28}$$

$$+\tfrac{1}{2} \sum_n \left(n + \tfrac{1}{2}\gamma\right)^2 a^*_n a_n + \tfrac{1}{2} \sum_n \left(n - \tfrac{1}{2}\gamma\right)^2 \tilde{a}^*_n \tilde{a}_n - \tfrac{1}{2} wR(\alpha + \beta)J + \tfrac{1}{2}\alpha'\alpha\beta J^2 \ ,$$

where we have used (5.21). $J$ is the angular momentum defined in (5.19),(5.13) which has the following mode expansion:

$$J = J_R + J_L \ , \quad J_R = -\tfrac{1}{2} \sum_n (n + \tfrac{1}{2}\gamma) a^*_n a_n \ , \quad J_L = -\tfrac{1}{2} \sum_n (n - \tfrac{1}{2}\gamma) \tilde{a}^*_n \tilde{a}_n \ . \tag{5.29}$$

*5.2. Operator quantization*

We can now quantize the theory using the light-cone operator approach by imposing the canonical commutation relations, in particular,

$$[P_x(\sigma, \tau), x^*(\sigma', \tau)] = [P^*_x(\sigma, \tau), x(\sigma', \tau)] = -i\delta(\sigma - \sigma') \ , \tag{5.30}$$

[25] The dilaton term is ignored in this section since its role is only to maintain the conformal invariance of the quantum theory. This term cancels out anyway once one performs the transformation to the free-theory variables.

$$[x^i(\sigma,\tau), \partial_\sigma x^j(\sigma',\tau)] = 0 \ ,$$

where $P_x = \frac{1}{2}(P_1 + iP_2)$, $P_x^* = \frac{1}{2}(P_1 - iP_2)$ are the canonical momenta corresponding to $x$ and $x^*$ in (5.1). As a result, $s$, $p$ in (5.22) and the Fourier modes $a_n, \tilde{a}_n$ will become operators acting in a Hilbert space. Again, the duality between (5.1) and (5.6) and between (5.9) and (5.10) implies that imposing (5.30) is equivalent to demanding the canonical commutation relations for the fields $X, X^*$ of the free (but globally non-trivial, cf. (5.17)) theory

$$[P_X(\sigma,\tau), X^*(\sigma',\tau)] = [P_X^*(\sigma,\tau), X(\sigma',\tau)] = -i\delta(\sigma - \sigma') \ , \tag{5.31}$$
$$[X^i(\sigma,\tau), \partial_\sigma X^j(\sigma',\tau)] = 0 \ ,$$

where $P_X(\sigma,\tau) = \frac{1}{4\pi\alpha'}\partial_\tau X$. Using (5.17), eq.(5.31) implies

$$[a_n, a_m^*] = 2(n + \tfrac{1}{2}\gamma)^{-1}\delta_{nm} \ , \quad [\tilde{a}_n, \tilde{a}_m^*] = 2(n - \tfrac{1}{2}\gamma)^{-1}\delta_{nm} \ . \tag{5.32}$$

One also finds that $s$ and $p$ in (5.21) and thus $\gamma$ in (5.23) commute with the mode operators. It is then easy to check directly that (5.30) are indeed satisfied.

The string energy and the Kaluza-Klein linear momentum operators are given by[26]

$$E = \int_0^\pi d\sigma P_t \ , \quad p_y = \int_0^\pi d\sigma P_y = \frac{m}{R} \ , \quad m = 0, \pm 1, \pm 2, ... \ . \tag{5.33}$$

$P_t, P_y$ are the canonical momenta which correspond to (5.1), or, equivalently (on the solution of the equations of motion), to (5.6) (we use again $a_\pm, c_\pm$ defined in (2.6))[27]

$$P_t = \frac{1}{2\pi\alpha'}(-\partial_\tau t - a_-\partial_+\tilde{\varphi} + c_-\partial_-\tilde{\varphi}) \ , \quad P_y = \frac{1}{2\pi\alpha'}(\partial_\tau y - a_+\partial_+\tilde{\varphi} + c_+\partial_-\tilde{\varphi}) \ . \tag{5.34}$$

Using (5.12),(5.14),(5.20),(5.21) and integrating over $\sigma$ we get

$$E = -\frac{1}{2\alpha'}[p - \alpha'(c_- + a_-)\hat{J}] \ , \quad p_y = \frac{1}{2\alpha'}[s + \alpha'(c_+ + a_+)\hat{J}] \ . \tag{5.35}$$

Here $\hat{J}$ is the angular momentum operator obtained by symmetrizing the classical expression $J = J_R + J_L$ in (5.29)

$$\hat{J} = \hat{J}_R + \hat{J}_L = -\tfrac{1}{4}\sum_n (n + \tfrac{1}{2}\gamma)(a_n^* a_n + a_n a_n^*) - \tfrac{1}{4}\sum_n (n - \tfrac{1}{2}\gamma)(\tilde{a}_n^*\tilde{a}_n + \tilde{a}_n\tilde{a}_n^*) \ . \tag{5.36}$$

Expressing $s$ and $p$ in terms of $E, p_y$ and $\hat{J}$

$$p = -2\alpha'[E - \tfrac{1}{2}(c_- + a_-)\hat{J}] \ , \quad s = 2\alpha'[p_y - \tfrac{1}{2}(c_+ + a_+)\hat{J}] \ , \tag{5.37}$$

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[26] As usual, the eigenvalue of $p_y$ is quantized since $[p_y, y_0] = -i$, where $y_0$ is the compact zero mode of $y$.

[27] We assume that the free $\partial u \partial v$ term in the Lagrangian is taken in the "symmetrized" form, i.e. the total derivative term $\frac{1}{2}(\partial_- u \partial_+ v - \partial_+ u \partial_- v)$ is added to (5.1),(5.6). If one does not add such term the expression for $E$ is shifted by $wR$-term.

we can represent $\gamma$ (5.23) in the form

$$\gamma = (a_+ + c_+)wR + \alpha'[(c_+ - a_+)p_y + (a_- - c_-)E] \qquad (5.38)$$
$$+ \ \tfrac{1}{2}\alpha'(a_+^2 - a_-^2 - c_+^2 + c_-^2)\hat{J} \ ,$$

which is consistent with the property that $\gamma$ commutes with the mode operators in (5.32).

The Virasoro operators $\hat{L}_0$ and $\hat{\tilde{L}}_0$ are obtained by symmetrizing the mode operator products in (5.26),(5.27). In particular, starting from (5.28) and using (5.37) we get the quantum Hamiltonian

$$\hat{H} = \hat{L}_0 + \hat{\tilde{L}}_0 = -\tfrac{1}{2}\alpha'[E - \tfrac{1}{2}(c_- + a_-)\hat{J}]^2 + \tfrac{1}{2}\alpha'[p_y - \tfrac{1}{2}(c_+ + a_+)\hat{J}]^2 \qquad (5.39)$$
$$+\tfrac{1}{2}\alpha'[\alpha'^{-1}wR - \tfrac{1}{2}(c_+ - a_+)\hat{J}]^2 - \tfrac{1}{8}\alpha'(c_- - a_-)^2\hat{J}^2$$
$$+ \ \tfrac{1}{4}\sum_n \left(n + \tfrac{1}{2}\gamma\right)^2(a_n^* a_n + a_n a_n^*) + \tfrac{1}{4}\sum_n \left(n - \tfrac{1}{2}\gamma\right)^2(\tilde{a}_n^*\tilde{a}_n + \tilde{a}_n\tilde{a}_n^*) \ .$$

The sectors of states of the model can be labeled by conserved quantum numbers: the energy $E$, the angular momentum $\hat{J}$ in the $x_1, x_2$ plane, the linear $p_y = m/R$ and winding $wR$ Kaluza-Klein momenta or "charges" (and also by momenta in additional spatial dimensions). The value of $\gamma$ (5.38) in a given sector depends on $E,\ \hat{J},\ m,\ w$ as well as on the parameters $a_\pm,\ c_\pm,\ R$ which determine the strength of the corresponding background fields.

In agreement with the defining relations in (5.17) the expressions for $\hat{H}$, $\hat{J}$ and the commutation relations (5.32) are invariant under $\gamma \to \gamma+2$ combined with the corresponding renaming of the mode operators $a_n \to a_{n+1},\ \tilde{a}_n \to \tilde{a}_{n-1}$.[28]

The states belonging to the $\gamma = 0$ (in general, $|\gamma| = 2n,\ n = 0, 1, ...,$) "hyperplane" in the $(m, w, E, J)$ space are special. For these states the translational invariance on the plane is restored: the zero-mode oscillators $a_0, a_0^*, \tilde{a}_0, \tilde{a}_0^*$ are replaced by the zero mode coordinate and conjugate linear momentum.[29]

Restricting for the moment the consideration to the sector of states with $2 > \gamma > 0$, where $\gamma$ is defined by eq. (5.38), one can introduce (as in [2]) the normalized creation and annihilation operators which will be used to define the Fock space of our model (the subindices $\pm$ correspond to the components with spin "up" and "down" respectively)

$$[b_{n\pm}, b_{m\pm}^\dagger] = \delta_{nm} \ , \quad [\tilde{b}_{n\pm}, \tilde{b}_{m\pm}^\dagger] = \delta_{nm} \ , \quad [b_0, b_0^\dagger] = 1 \ , \quad [\tilde{b}_0, \tilde{b}_0^\dagger] = 1 \ , \qquad (5.40)$$

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[28] Let us note that the region of $|\gamma| \approx 2$ corresponds to values of field strengths or quantum numbers which are of Planck order. The fact that the mass spectrum depends on $\gamma$ only modulo 2 suggests that physics at strong fields with $\gamma = 2 + \epsilon$ is equivalent to the weak field regime with $\gamma = \epsilon$, i.e. implies certain periodicity in dependence on the field strengths.

[29] Strictly speaking, this is true provided $\gamma = 0$ is satisfied with no constraint imposed on the orbital part of the angular momentum $J$. Otherwise one gets just one continuous (radial) quantum number in the "plane" part of the spectrum (see Section 5.3).

$$b^\dagger_{n+} = a_{-n}\omega_- \ , \quad b_{n+} = a^*_{-n}\omega_- \ , \quad b^\dagger_{n-} = a^*_n\omega_+ \ , \quad b_{n-} = a_n\omega_+ \ , \tag{5.41}$$

$$\tilde b^\dagger_{n+} = \tilde a_{-n}\omega_+ \ , \quad \tilde b_{n+} = \tilde a^*_{-n}\omega_+ \ , \quad \tilde b^\dagger_{n-} = \tilde a^*_n\omega_- \ , \quad \tilde b_{n-} = \tilde a_n\omega_- \ , \tag{5.42}$$

$$b^\dagger_0 = \tfrac12\sqrt{\gamma}a^*_0, \quad b_0 = \tfrac12\sqrt{\gamma}a_0, \quad \tilde b^\dagger_0 = \tfrac12\sqrt{\gamma}\tilde a_0, \quad \tilde b_0 = \tfrac12\sqrt{\gamma}\tilde a^*_0 \ , \tag{5.43}$$

$$\omega_\pm \equiv \sqrt{\tfrac12\big(n \pm \tfrac12\gamma\big)}, \qquad n = 1,2,..., \qquad 0 < \gamma < 2 \ . \tag{5.44}$$

Then the angular momenta operators (5.36) become (after normal ordering)

$$\hat J = \hat J_R + \hat J_L = \tilde b^\dagger_0\tilde b_0 - b^\dagger_0 b_0 + S_R + S_L = J \ , \tag{5.45}$$

$$\hat J_R = -b^\dagger_0 b_0 - \tfrac12 + \sum_{n=1}^\infty \big(b^\dagger_{n+}b_{n+} - b^\dagger_{n-}b_{n-}\big) \equiv J_R - \tfrac12 \ , \tag{5.46}$$

$$\hat J_L = \tilde b^\dagger_0\tilde b_0 + \tfrac12 + \sum_{n=1}^\infty \big(\tilde b^\dagger_{n+}\tilde b_{n+} - \tilde b^\dagger_{n-}\tilde b_{n-}\big) \equiv J_L + \tfrac12 \ .$$

The operators $L_0$ and $\tilde L_0$ will be normal ordered with the ordering constant being fixed by the Virasoro algebra. The Virasoro operators $L_n, \tilde L_n$, $n \neq 0$ can be obtained in the standard way as the Fourier components of the stress-energy tensor, e.g., $L_n = (4\pi\alpha')^{-1}\int d\sigma e^{-i2\sigma n}T_{--} = p^u_- V_n + \mathcal{L}_n$. In the light-cone gauge $L_n$ are required to vanish so that their "transverse" parts $\mathcal{L}_n$ are proportional to the modes of the operator $V'_\pm$,

$$V'_+ = 2i\alpha' \sum_{n\neq 0}\frac1n \tilde V_n \exp(-2in\sigma_+) \ , \qquad V'_- = 2i\alpha' \sum_{n\neq 0}\frac1n V_n \exp(-2in\sigma_-) \ . \tag{5.47}$$

One finds for the "transverse" parts of $L_n$

$$\mathcal{L}_n \equiv -p^u_- V_n = \tfrac12 \sum_k (k + \tfrac12\gamma)(k - n + \tfrac12\gamma) a_k a^*_{k-n} \tag{5.48}$$

$$= \sum_{k=1}^\infty [(k - \tfrac12\gamma)(n + k - \tfrac12\gamma)]^{1/2} b^\dagger_{k+} b_{(n+k)+} + \sum_{k=0}^\infty [(k + \tfrac12\gamma)(n + k + \tfrac12\gamma)]^{1/2} b^\dagger_{k-} b_{(n+k)-}$$

$$- \sum_{k=0}^{n-1} [(k + \tfrac12\gamma)(n - k - \tfrac12\gamma)]^{1/2} b_{k-} b_{(n-k)+} \ ,$$

$$\mathcal{L}_{-n} \equiv -p^u_- V_{-n} = \mathcal{L}^\dagger_n \ , \quad n = 1,2,3,... \ ,$$

and similarly for $\tilde{\mathcal{L}}_n$. It is an easy exercise to verify that after adding the contribution of the remaining 22 free scalar fields, the operators $\mathcal{L}_n$ satisfy the Virasoro algebra with the central charge equal to 24 and a trivial cocycle which shifts the free-theory normal ordering constant in $L_0$ from 1 to $1 - \frac14\gamma(1 - \frac12\gamma)$ (or to $1 - \frac14\gamma'(1 - \frac12\gamma')$, where $\gamma' = \gamma - 2k$ and $k$ is integer, in the case when $2k < \gamma < 2k + 2$). This corresponds to regularizing the infinite sums arising in the normal ordering process by the using the generalised $\zeta$-function regularisation (4.26). Similar result is found in the open string theory in a constant magnetic field [33] and is typical to the case of a free scalar field with twisted boundary conditions (which appear also in orbifold or "cone" models, see, e.g., [34,35,36]). We will see that this shift is also consistent with the modular invariant path integral approach discussed in Section 4.

Inserting (5.38),(5.35) into (5.26)–(5.28) and replacing the zero mode operators $E$ and $p_y$ by their eigen-values we obtain the expressions for the Virasoro operators and the Hamiltonian

$$\hat{L}_0 = \tfrac{1}{4}\alpha'\big( - E^2 + p_a^2 + Q_-^2\big) + \tfrac{1}{2}\mathcal{H} + N - c_0 \ , \tag{5.49}$$

$$\hat{\tilde{L}}_0 = \tfrac{1}{4}\alpha'\big( - E^2 + p_a^2 + Q_+^2\big) + \tfrac{1}{2}\mathcal{H} + \tilde{N} - c_0 \ , \tag{5.50}$$

$$\hat{H} = \tfrac{1}{2}\alpha'\big( - E^2 + p_a^2 + \tfrac{1}{2}Q_+^2 + \tfrac{1}{2}Q_-^2\big) + \mathcal{H} + N + \tilde{N} - 2c_0 \ , \tag{5.51}$$

$$\mathcal{H} \equiv -\alpha'[(c_+Q_+ - c_-E)J_R + (a_+Q_- - a_-E)J_L] \tag{5.52}$$
$$+\tfrac{1}{2}\alpha'\big[(c_+^2 - c_-^2)J_R^2 + (a_+^2 - a_-^2)J_L^2 + (a_+^2 + c_+^2 - c_-^2 - a_-^2)J_RJ_L\big] \ .$$

Here $Q_\pm$ are the left and right combinations of the Kaluza-Klein linear and winding momenta (which play the role of charges in the present context)

$$Q_\pm \equiv \frac{1}{\sqrt{\alpha'}}(\frac{m}{r} \pm wr) = p_y \pm \frac{wR}{\alpha'} \ , \qquad r \equiv \frac{R}{\sqrt{\alpha'}} \ , \tag{5.53}$$

$$c_0 \equiv 1 - \tfrac{1}{4}\gamma(1 - \tfrac{1}{2}\gamma), \tag{5.54}$$

and $N$ and $\tilde{N}$ are the standard operators[30]

$$N = \sum_{n=1}^{\infty} n(b_{n+}^\dagger b_{n+} + b_{n-}^\dagger b_{n-} + a_{na}^\dagger a_{na}) \ , \tag{5.55}$$

$$\tilde{N} = \sum_{n=1}^{\infty} n(\tilde{b}_{n+}^\dagger \tilde{b}_{n+} + \tilde{b}_{n-}^\dagger \tilde{b}_{n-} + \tilde{a}_{na}^\dagger \tilde{a}_{na}) \ . \tag{5.56}$$

The Hamiltonian for the case of $\gamma = 0$ is obtained by adding $\frac{1}{2}\alpha'(p_1^2 + p_2^2)$ and replacing $-b_0^\dagger b_0 - \frac{1}{2}$ and $\tilde{b}_0^\dagger \tilde{b}_0 + \frac{1}{2}$ in $\hat{J}_R$ and $\hat{J}_L$ in (5.45),(5.46) by one half of the center of mass orbital momentum $(x_1p_2 - x_2p_1)$.

As explained above, the mass formula for states with $2k < \gamma < 2k+2, \ k =$integer, can be found in a similar way by renaming the creation and annihilation operators. The result is the same as in (5.49)-(5.51) with the replacement $\gamma \to \gamma' = \gamma - 2k$ in $c_0$.

Note that the continuous momenta $p_{1,2}$ corresponding to the zero modes of the coordinates $x_{1,2}$ of the plane are effectively replaced in the $0 < \gamma < 2$ sector by the integer eigenvalues $l_R, l_L = 0, 1, 2, ...$ of the zero-mode parts $b_0^\dagger b_0$ and $\tilde{b}_0^\dagger \tilde{b}_0$ of $\hat{J}_R$ and $\hat{J}_L$. Thus the "2-plane" part of the spectrum is discrete (but, as mentioned above, it is continuous when

[30] We have introduced extra 22 spatial free dimensions with momenta $p_a$ and oscillators $a_{na}, \tilde{a}_{na}$ to ensure that the total space-time dimension is 26.

$\gamma = 0$). For example, in the Melvin model case (most of) the string states are "trapped" by the flux tube (see also Section 5.3). This result is consistent with a picture of a closed string moving in a magnetic field orthogonal to the plane (see [2] and below).

The Hamiltonian (5.51) is non-trivial: because of the angular momenta squared terms it is, in general, of *fourth* order in creation/annihilation operators. It is clear from our construction that (5.51) is, at the same time, also the Hamiltonian for the $\varphi$-dual theory (5.6), i.e. it can be derived by starting directly with (5.6) (the origin of the quartic terms in $\hat{H}$ can be traced, in particular, to the presence of the $\alpha\beta\partial\tilde{\varphi}\bar{\partial}\tilde{\varphi}$ term in (5.6)).[31] The two $\varphi$-dual $\sigma$-models (5.1) and (5.6) thus represent the same CFT. The quartic terms are absent only when $a_+ = \pm a_-$, $c_+ = \pm c_-$ (i.e. when $\alpha\beta q_+(q_+ + \beta - \alpha) = 0$). In the case of $a_+ = a_- = 0$, $c_+ = -c_- = \beta$ corresponding to the model (2.21) $\hat{H}$ in (5.51) reduces to the quadratic Hamiltonian found in [2]. As for general values of the parameters, the spectrum can still be computed explicitly, just as in the special case considered in [2], since the Hamiltonian (5.51) is in a *diagonal* form (note that $N, \tilde{N}, J_L$ and $J_R$ are diagonal in Fock space).

It is easy to see that (in agreement with a discussion in Section 2) the coordinate-invariant physical quantities depend only on the *three* parameters $\alpha, \beta, q_+$ (as well as on $r$) even though $\hat{H}$ contains also $q_-$. The dependence of on $q_-$ reflects a choice of the frame that was made, and translates into the expected frame dependence of the energy eigenvalues.

The Hamiltonian (5.51) is invariant under the following transformations

$$r \to r^{-1}\ , \quad m \leftrightarrow w\ , \qquad a_+ \to -a_+\ , \tag{5.57}$$

$$r \to r^{-1}\ , \quad m \leftrightarrow -w\ , \quad c_+ \to -c_+\ . \tag{5.58}$$

This is consistent with the observation made in Section 2 that the $y$-dual of our $\sigma$-model (2.5) is an equivalent $\sigma$-model (2.10) with either $y \to \tilde{y}$ and $a_+ \to -a_+$ or $y \to -\tilde{y}$ and $c_+ \to -c_+$ (2.11). As was already mentioned after (2.11), an interesting feature of our model is that a nontrivial structure of the Lagrangian (2.4) or (5.6) involving a coupling between the two angular coordinates $y$ and $\varphi$ implies that the $y$-duality transformation ($r \to r^{-1}$), in general, must be accompanied by a change in one extra parameter.

The presence of the $O(\gamma^2)$ normal ordering term in (5.51),(5.54) implies (see (5.38)) that the quantum Hamiltonian contains $O(\alpha'^2)$ terms of one order higher in $\alpha'$. This is a consequence of the regularisation (normal ordering) prescription we used which is consistent with the reparametrisation invariance of the theory (in particular, the Virasoro algebra and modular invariance of the partition function). Such higher order term is also consistent with current algebra approaches. Indeed, there are two special cases when our model becomes equivalent to a special WZW or coset model:

(i) the non-compact ($R = \infty$) limit of the constant magnetic field model (2.21) is equivalent (see [2]) to the $E_2^c$ WZW model [26] for which the quantum stress tensor contains one order $1/k \sim \alpha'$ correction term [31,37] (which is, indeed, equivalent to the term appearing in (5.51) in this limit);

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[31] In the $\varphi$-dual theory the expression (5.23) for $\gamma$ in (5.17) follows from the condition that the zero mode of the momentum conjugate to $\tilde{\varphi}$ should take integer eigenvalues to make $\exp(i\tilde{\varphi}p_{\tilde{\varphi}})$ single-valued.

(ii) the non-compact limit of the Melvin model (2.24) (when all coordinates are formally taken to be non-compact) can be related [5] to a special limit of the $SL(2,R)\times R/R$ gauged WZW model, or, equivalently, to the $E_2^c/U(1)$ coset theory [25], the quantum Hamiltonian of which (obtained, e.g., by a taking a limit in the standard semisimple coset model Hamiltonian) also contains a $1/k$ correction term [25].

As follows from (5.49),(5.50), the Virasoro conditions $\hat{L}_0 = \hat{\tilde{L}}_0 = 0$ become

$$\hat{H} \equiv H - 2 + \tfrac{1}{2}\gamma - \tfrac{1}{4}\gamma^2 = 0 \ , \qquad N - \tilde{N} = mw \ . \tag{5.59}$$

The analysis of the spectrum in the special case $q_1 = q_2 = 0$, $\alpha = 0$, was performed in [2]. The spectrum for the general class of models considered here displays similar qualitative features, in particular, tachyonic instabilities. We shall discuss some of its properties in Section 6.

*5.3. Point-particle limit: zero level scalar (tachyon) spectrum in the Melvin model*

To illustrate our construction of the string Hamiltonian and the solvability of the model it is useful to discuss the point-particle limit. Let us consider, for example, the case of the Melvin model (2.24) ($a_+ = 0,\ c_+ = 2\beta,\ a_- = c_- = 0$).[32] The point-particle limit of its action is (we omit the trivial $x_3$ direction)

$$S = \frac{1}{4\alpha'} \int d\tau \big[ - \dot{t}^2 + \dot{\rho}^2 + F(\rho)\rho^2 \dot{\varphi}(\dot{\varphi} + 2\beta\dot{y}) + \dot{y}^2 \big] \ , \quad F^{-1} = 1 + \beta^2\rho^2 \ , \tag{5.60}$$

with the Hamiltonian being

$$H = \alpha' \big[ - p_t^2 + p_\rho^2 + \rho^{-2} F^{-2}(\rho) p_\varphi^2 + F^{-1}(\rho) p_y^2 - 2\beta F^{-1}(\rho) p_\varphi p_y \big] \tag{5.61}$$

$$= \alpha' \big[ - p_t^2 + (p_y - \beta p_\varphi)^2 + \beta^2 p_\varphi^2 + p_\rho^2 + \rho^{-2} p_\varphi^2 + \beta^2 (p_y - \beta p_\varphi)^2 \rho^2 \big] \ .$$

The scalar product is defined with respect to the measure $\sqrt{-G}e^{-2\phi} = \rho$. The Hamiltonian commutes with the Kaluza-Klein charge operator ($p_y$) and the angular momentum operator ($p_\varphi$) and the quantum mechanical problem reduces to that of a free two-dimensional oscillator.

The corresponding Klein-Gordon operator is the Laplacian which appears in the (zero winding sector) tachyon equation

$$\Delta = - \frac{1}{\sqrt{-G}e^{-2\phi}} \partial_\mu (\sqrt{-G}e^{-2\phi} G^{\mu\nu} \partial_\nu) \ , \qquad \alpha'(\Delta + O(\alpha'))T = 4T \ , \tag{5.62}$$

where the Melvin model "$D=5$ metric" (with $x_3$ direction omitted) and the dilaton are

$$ds^2 = -dt^2 + d\rho^2 + F(\rho)\rho^2 d\varphi(d\varphi + 2\beta dy) + dy^2 \tag{5.63}$$

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[32] The case of the constant magnetic field model (2.21) was discussed in detail in [2]. Other cases can be considered in a similar way (with special care taken in comparing the definitions of the string and particle momenta because of the presence of the antisymmetric tensor.

$$= -dt^2 + d\rho^2 + F(\rho)\rho^2(d\varphi + 2\beta dy)^2 + F(\rho)dy^2 \ , \quad e^{2(\phi-\phi_0)} = F(\rho) = (1+\beta^2\rho^2)^{-1} \ .$$

Let us first ignore possible higher order $\alpha'$-terms in the tachyon equation. Then we get

$$\big[ - \partial_t^2 + \rho^{-1}\partial_\rho(\rho\partial_\rho) + (\rho^{-2} + 2\beta^2 + \beta^4\rho^2)\partial_\varphi^2 \tag{5.64}$$

$$+ \ (1+\beta^2\rho^2)\partial_y^2 - 2\beta(1+\beta^2\rho^2)\partial_\varphi\partial_y\big]T = -4{\alpha'}^{-1}T \ .$$

If we consider a particle state with a given energy $E = p_t$, charge $p_y = m/R$ $(m = 0, \pm 1, ...)$ and orbital momentum $p_\varphi = l$ $\ (l = 0, \pm 1, ...)$

$$T = \exp(iEt + ip_y y + il\varphi)\ \tilde{T}(\rho) \ ,$$

then

$$\big[ - \rho^{-1}\partial_\rho(\rho\partial_\rho) + l^2\rho^{-2} + \gamma_0^2\rho^2\big]\tilde{T} = \mu^2\tilde{T} \ , \tag{5.65}$$

$$\gamma_0 = \beta(p_y - \beta l) \ , \qquad \mu^2 \equiv E^2 + 4{\alpha'}^{-1} - (p_y - \beta l)^2 - \beta^2 l^2 \ .$$

This is the Schrödinger equation for a two-dimensional oscillator with the frequency $\gamma_0$. In what follows we assume that $\gamma_0 > 0$ (the spectrum for $\gamma_0 < 0$ is the same since the Hamiltonian depends on $\gamma_0^2$). Provided $\gamma_0 \neq 0$ the spectrum is discrete, i.e. (5.65) has normalizable solutions only if

$$\mu^2 = 2\gamma_0(2k + |l| + 1) \ , \qquad k = 0, 1, 2, ... \ . \tag{5.66}$$

To make the analogy with the solution of the string problem more explicit let us derive (5.66) by introducing the creation/annihilation operators

$$C \equiv -i(\partial_x^* + \tfrac{1}{2}\gamma_0 x) \ , \qquad \tilde{C} \equiv -i(\partial_x + \tfrac{1}{2}\gamma_0 x^*) \ , \tag{5.67}$$

$$C^\dagger \equiv -i(\partial_x - \tfrac{1}{2}\gamma_0 x^*) \ , \qquad \tilde{C}^\dagger \equiv -i(\partial_x^* - \tfrac{1}{2}\gamma_0 x) \ , \qquad x = \rho e^{i\varphi} \ , \ \ x^* = \rho e^{-i\varphi} \ ,$$

$$[C, C^\dagger] = \gamma_0 \ , \qquad [\tilde{C}, \tilde{C}^\dagger] = \gamma_0 \ , \quad [C, \tilde{C}] = [C, \tilde{C}^\dagger] = 0 \ ,$$

$$p_\rho^2 + \rho^{-2}p_\varphi^2 = -4\partial_x\partial_x^* = (C^\dagger + \tilde{C})(\tilde{C}^\dagger + C) \ , \quad \gamma_0^2\rho^2 = (C - \tilde{C}^\dagger)(C^\dagger - \tilde{C}) \ ,$$

$$p_\varphi = -i\partial_\varphi = x\partial_x - x^*\partial_x^* = \gamma_0^{-1}(\tilde{C}^\dagger\tilde{C} - C^\dagger C) \ . \tag{5.68}$$

As a result, (5.65) is equivalent to

$$2(CC^\dagger + \tilde{C}^\dagger\tilde{C})\tilde{T} = 2\gamma_0(b^\dagger b + \tilde{b}^\dagger\tilde{b} + 1)T = \mu^2\tilde{T} \ , \tag{5.69}$$

$$b = \gamma_0^{-1/2}C \ , \qquad \tilde{b} = \gamma_0^{-1/2}\tilde{C} \ , \qquad [b, b^\dagger] = 1 \ , \quad [\tilde{b}, \tilde{b}^\dagger] = 1 \ .$$

The eigen-functions are thus given by

$$T_{E,p_y,l_R,l_L} = e^{iEt + ip_y y}(b^\dagger)^{l_R}(\tilde{b}^\dagger)^{l_L}\exp(-\tfrac{1}{2}\gamma_0 x x^*) \ , \tag{5.70}$$

with the eigen-values

$$\mu^2 = 2\gamma_0(l_R + l_L + 1) \ , \qquad l_R, l_L = 0, 1, 2, ... \ . \tag{5.71}$$

This is the same condition as in (5.66) since according to (5.68) the orbital momentum eigen-value is

$$l = l_L - l_R \ , \qquad l_R + l_L = 2k + |l| \ . \tag{5.72}$$

The resulting tachyon spectrum is the same as the semiclassical (leading order in $\alpha'$) spectrum that follows from the string constraints (5.59) with the Hamiltonian (5.51) depending on the "angular momenta" operators (5.45),(5.46). At the zero string excitation level $S_L = S_R = N = \tilde{N} = 0$ and the eigen-values of $\hat{J}_R$ and $\hat{J}_L$ are $-l_R - \frac{1}{2}$ and $l_L + \frac{1}{2}$ so that $\hat{H} = 0$ reduces to ($p_a, \ a = 3, ...,$ are momenta of additional dimensions)

$$M^2 \equiv E^2 - p_a^2 = -4\alpha'^{-1} + p_y^2 \tag{5.73}$$

$$-4\beta p_y \hat{J}_R + 4\beta^2(\hat{J}_L + \hat{J}_R)\hat{J}_R - 2\alpha'\beta^2(p_y - \beta\hat{J})^2$$

$$= -4\alpha'^{-1} + p_y^2 + 2\beta p_y(2l_R + 1) - 2\beta^2(l_L - l_R)(2l_R + 1) - 2\alpha'\beta^2[p_y - \beta(l_L - l_R)]^2 \ ,$$

where it is assumed that the eigen-value $2\alpha'\beta[p_y - \beta(l_L - l_R)]$ of the operator $\gamma$ in (5.38) is positive, i.e $\gamma = 2\alpha'\gamma_0 > 0$. The $O(\alpha')$ correction comes from the $\gamma^2$ term in $c_0$ in (5.51). It is easy to see that the two expressions (5.71) and (5.73) for the point-like tachyon spectrum indeed agree up to the $O(\alpha')$-term in (5.73). A similar correspondence between the point-particle spectrum and the zero level string spectrum can be established also for other choices of background parameters (the solution of the tachyon equation in the case of the constant magnetic field model (1.1), (2.21), which is similar to the Landau spectrum, was already given in [2]).

We see that for $\gamma = 2\alpha'\gamma_0 = 2\alpha'\beta[p_y - \beta(l_L - l_R)] \neq 0$ the scalar particles are "localized" near the core of the flux tube, i.e. oscillate near $\rho = 0$ (they can, of course, move freely along the flux tube direction $x_3$). Note that even the neutral tachyon states with nonzero orbital momentum are trapped by the flux tube. Similar "bound state" interpretation should apply for higher excitations in the string spectrum. Another interesting feature is that, in contrast to the Landau spectrum or the spectrum of the standard 2d oscillator, here there is no usual degeneracy in the energy since the frequency $\gamma_0$ itself depends on $l$ (there is still smaller degeneracy $\beta(p_y - \beta l) \to -\beta(p_y - \beta l)$). When $\gamma_0 = 0$, i.e. $p_y = \beta l \neq 0$, the spectrum becomes continuous, i.e. $\mu^2$ in (5.65) can take arbitrary positive values (yet the translational invariance on the plane is not fully restored since the eigen-value of the orbital momentum is subject to the constraint $l = p_y/\beta$). This is possible only for special values of the magnetic field strength, $\beta R = m/l$. The spectrum is also continuous for $p_y = 0, \ l = 0$ when the solution of (5.65) which decays at infinity and is bounded at zero is given by the zeroth Bessel function $\tilde{T} = J_0(\mu\rho)$ with arbitrary $\mu$.[33]

In general, the tachyon equation (5.62) contains higher order $O(\alpha'^2)$ terms. Such terms are scheme-dependent and may be non-vanishing in the scheme in which the exact expressions for the $\sigma$-model couplings (metric, dilaton and antisymmetric tensor) do not depend on $\alpha'$ (for a discussion of $\alpha'$-corrections to the tachyon equation see [39,40] and refs. there). The exact form of the tachyon equation which is usually hard to determine at the $\sigma$-model level is determined by the underlying CFT. In the present model it can be fixed

[33] The solution of a similar (uncharged) Klein-Gordon equation in the $a = 0$ Melvin background was discussed in [38], where it was also found that the spectrum contains both discrete ($l \neq 0$) and continuous ($l = 0$) branches.

by using directly the Hamiltonian (5.51) or the relation to the coset model (for a similar discussion in the case of the constant magnetic field model see [2]). The appearence of the $\alpha'$-correction term in the point-particle limit of the string Hamiltonian (5.51) or in (5.73) is consistent with the presence of the "quantum" $1/k$ correction term in the Hamiltonian of the special limit of the $SL(2,R) \times R/R$ WZW model [5] or in the Hamiltonian of the $E_2^c/U(1)$ coset model [25] which, as was mentioned above, is related to the "non-compact" limit of the Melvin model.[34] To find the $\alpha'$-correction term in the tachyon equation which produces the $O(\alpha')$-term in (5.73) one may start, e.g., with the quantum action of the $SL(2,R) \times R/R$ gauged WZW model with the full $1/k$-dependence included (see [40]) and take the special limit $1/k = \alpha'\epsilon^2$, $q_0 = -1+\beta^{-2}\epsilon^2$, $r = 2\epsilon\rho$, $\theta = \varphi+\beta y$, $\tilde{\theta} = \beta^{-1}\epsilon^2 y$, $\epsilon \to 0$ ($q_0$ is the parameter of embedding of the subgroup, $r, \theta, \tilde{\theta}$ are $SL(2,R)$ coordinates). As a result, one finds the exact ($\alpha'$-corrected) form of the Melvin model action from which one reads off the following metric and dilaton (cf. (5.63))

$$ds^2 = -dt^2 + d\rho^2 + F(\rho)\rho^2(d\varphi + 2\beta dy)^2 + F'(\rho)dy^2 \ , \tag{5.74}$$

$$e^{2(\phi-\phi_0)} = [F(\rho)F'(\rho)]^{1/2} = \rho^{-1}\sqrt{-G} \ , \qquad F' = (1 + \beta^2\rho^2 - 2\alpha'\beta^2)^{-1} \ .$$

These expressions correspond to the "CFT scheme" where the exact tachyon equation is the Klein-Gordon one for the exact metric and dilaton. As a result, the differential operator in (5.64) gets an extra term $-2\alpha'\beta^2(\partial_y - \beta\partial_\varphi)^2$ so that $\mu^2$ in (5.65) is shifted by $2\alpha'\beta^2(p_y - \beta l)^2$ and the spectrum (5.66) becomes exactly equivalent to (5.73).

### *5.4. Partition function from the operator quantization*

In the operator formalism, the one-loop partition function is obtained by using the Hamiltonian to propagate the states along the cylinder and taking the trace to identify its ends (and imposing the Virasoro constraint with a Lagrange multiplier). Then

$$Z = \int \frac{d^2\tau}{\tau_2} \int dE \prod_{a=1}^{22} dp_a \sum_{m,w=-\infty}^{\infty} \mathrm{Tr} \exp \left[ 2\pi i (\tau \hat{L}_0 - \bar{\tau}\hat{\bar{L}}_0) \right] \ , \tag{5.75}$$

where $\hat{L}_0$ and $\hat{\bar{L}}_0$ are the Virasoro operators (5.49) and (5.50) constructed above. Our aim is to compute $Z$ defined by (5.75) and to show its equivalence to the expression obtained in the path integral approach (4.30),(4.38). In order to integrate over $E$ and perform the Poisson resummation, it is convenient to represent (5.49) and (5.50) in the form (cf. (5.51))

$$\hat{L}_0 = \tfrac{1}{4}w^2r^2 - \tfrac{1}{2}mw + N - 1 - \tfrac{1}{4}\alpha'(E - c_-\hat{J}_R - a_-\hat{J}_L)^2 \tag{5.76}$$

$$+\tfrac{1}{4}[mr^{-1} - \sqrt{\alpha'}(c_+\hat{J}_R + a_+\hat{J}_L)]^2 - \tfrac{1}{2}wr\sqrt{\alpha'}(c_+\hat{J}_R - a_+\hat{J}_L) + \alpha'\alpha\beta\hat{J}_R\hat{J}_L - \tfrac{1}{8}\gamma^2,$$

[34] As usual in the gauged WZW models, there are two possible interpretations of the $\alpha'$-correction terms in the tachyon spectrum: in the "CFT scheme" in which the tachyon equation retains its leading-order Klein-Gordon form they come from the $\alpha'$-corrections in the background $\sigma$-model fields [20,41]; in the "leading-order scheme" in which the $\sigma$-model fields have semiclassical values they originate from the corrections to the tachyon equation [39].

$$\hat{\tilde{L}}_0 = \hat{L}_0 + mw + \tilde{N} - N \ . \tag{5.77}$$

It is convenient to express the part of the exponential factor in (5.75) containing $\gamma^2$ in the following way:

$$\exp\left(\tfrac{1}{2}\pi\tau_2\gamma^2\right) = \sqrt{\tau_2}\int d\nu \ \exp(-\tfrac{1}{2}\pi\tau_2\nu^2 - \pi\tau_2\gamma\nu) \ ,$$

where $\nu$ is an auxiliary variable. The term $\pi\tau_2\gamma\nu$ can be absorbed into a redefinition of $\hat{J}_R \to \hat{J}'_R \equiv \hat{J}_R - \frac{1}{2}\nu$, $\hat{J}_L \to \hat{J}'_L \equiv \hat{J}_L + \frac{1}{2}\nu$ as can be easily verified.

The gaussian integrals over $E, p_a$ give an extra factor of $\tau_2^{-23/2}$. By using the Poisson resummation formula, one can trade the sum over the discrete loop momentum $m$ for the sum over the conjugate winding number $w'$:

$$\sum_{m=-\infty}^{\infty} \mathcal{F}(m) = \sum_{w'=-\infty}^{\infty} \int d\mu \ e^{2\pi i\mu w'} \mathcal{F}(\mu) \ , \tag{5.78}$$

$$\mathcal{F}(m) \equiv \exp\big( - \pi\tau_2[mr^{-1} - \sqrt{\alpha'}(c_+\hat{J}'_R + a_+\hat{J}'_L)]^2 - 2\pi i m w\tau_1\big) \ .$$

Integrating over $\mu$ we get

$$Z = \int d^2\tau \ \tau_2^{-13} \ \sqrt{\tau_2}\int dx \ \exp(-\tfrac{1}{2}\pi\tau_2\nu^2) \ \mathrm{Tr} \ \mathcal{Z} \ ,$$

$$\mathcal{Z} = r \sum_{w,w'=-\infty}^{\infty} \exp[-\pi r^2\tau_2^{-1}(w' - \tau w)(w' - \bar{\tau}w)] \ \exp\big[2\pi i(\tau N - \bar{\tau}\tilde{N})\big] \tag{5.79}$$

$$\times \exp[2\pi i(w' - \tau w)r\sqrt{\alpha'}c_+\hat{J}'_R] \ \exp[2\pi i(w' - \bar{\tau}w)r\sqrt{\alpha'}a_+\hat{J}'_L]$$

$$\times \ \exp(-4\pi\tau_2\alpha'\alpha\beta\hat{J}'_R\hat{J}'_L) \ .$$

In order to compute the trace in (5.75) and relate the result to the path integral expression (4.18) it is convenient to "split" the $\hat{J}_R\hat{J}_L$-term by introducing the auxiliary variables, inserting the following identity:

$$1 = 4\tau_2^{-1}\int d\lambda d\bar{\lambda} \tag{5.80}$$

$$\times \exp\big( - 4\pi\tau_2^{-1}[\lambda - \tfrac{1}{2}r(w' - \tau w) - i\tau_2\sqrt{\alpha'}\alpha\hat{J}'_L][\bar{\lambda} + \tfrac{1}{2}r(w' - \bar{\tau}w) - i\tau_2\sqrt{\alpha'}\beta\hat{J}'_R]\big) \ .$$

Then the first and the last exponential factors in (5.79) are cancelled out and we are left with the following expression

$$Z = c_1\int d^2\tau \ \tau_2^{-14}e^{4\pi\tau_2}\sqrt{\tau_2}\int d\nu \ \exp(-\tfrac{1}{2}\pi\tau_2\nu^2) \sum_{w,w'=-\infty}^{\infty}\int d\lambda d\bar{\lambda} \tag{5.81}$$

$$\times \exp\left(-4\pi\tau_2^{-1}\big[\lambda\bar{\lambda} - \tfrac{1}{2}r(w' - \tau w)\bar{\lambda} + \tfrac{1}{2}r(w' - \bar{\tau}w)\lambda\big]\right) \mathcal{W}(\nu, \lambda, \bar{\lambda}, w, w'; \tau, \bar{\tau}) \ ,$$

where

$$\mathcal{W} \equiv \mathrm{Tr}\exp\big[2\pi i(\tau N - \chi\hat{J}'_R)\big] \ \mathrm{Tr}\exp\big[ - 2\pi i(\bar{\tau}\tilde{N} + \tilde{\chi}\hat{J}'_L)\big] \ , \tag{5.82}$$

$$\chi \equiv -\sqrt{\alpha'}[2\beta\lambda + q_+ r(w' - \tau w)] \ , \qquad \tilde{\chi} \equiv -\sqrt{\alpha'}[2\alpha\bar{\lambda} + q_+ r(w' - \bar{\tau} w)] \ . \tag{5.83}$$

The traces can now be easily computed by using that

$$\mathrm{Tr}\exp\left[2\pi i(\tau N - \chi \hat{J}'_R)\right] = \frac{1}{\sqrt{\tau_2}}\frac{\pi\chi}{\sin\pi\chi}\exp(\pi i\chi\nu) \tag{5.84}$$

$$\times \prod_{n=1}^{\infty}(1-e^{2\pi i n\tau})^{-22}\left[1-e^{2\pi i(n\tau+\chi)}\right]^{-1}\left[1-e^{2\pi i(n\tau-\chi)}\right]^{-1} \ ,$$

and a similar expression for the "left" part. The factor $\pi\chi/\sqrt{\tau_2}$ comes from the normalization of the zero mode so that (5.84) has a regular $\chi \to 0$ limit (see also [2]). As a result, $\mathcal{W}$ (5.82) can be expressed in terms of the Jacobi $\theta_1$-function of the torus (cf. (4.23), (4.38)). After integrating over $\nu$ we obtain

$$\mathcal{W} = {\tau_2}^{-1}|f_0(e^{2\pi i\tau})|^{-48} \ \exp\big[-\frac{\pi(\chi-\tilde{\chi})^2}{2\tau_2}\big] \ \frac{\chi\tilde{\chi}|\theta'_1(0|\tau)|^2}{\theta_1(\chi|\tau)\theta_1(\tilde{\chi}|\bar{\tau})} \ . \tag{5.85}$$

The final expression for $Z$ in (5.81),(5.85) becomes the same as found in the path integral approach (4.38) if we transform the integration variables from $\lambda, \bar{\lambda}$ to $\chi, \tilde{\chi}$ (or, equivalently, identify the auxiliary variables $C_0$ and $\bar{C}_0$ in (4.19),(4.36) with $-2\pi\sqrt{\alpha'}\lambda$ and $-2\pi\sqrt{\alpha'}\bar{\lambda}$ in (5.81)). The zero-mode normalization used in (5.84) directly corresponds to the prescription of projecting out the constant mode factor (4.29) we used in (4.38) (if one does not use this normalization and directly computes the trace in (5.84) one finds the expression for $\mathcal{W}$ (5.85) without the factor $\chi\tilde{\chi}/\tau_2$ which is singular in the free-theory limit).

The operator formalism representation for $Z$ (5.75), (5.76) makes its duality invariance properties manifest. It is also clear why, e.g., the case of $\alpha\beta = 0$ is special: here there is no $\hat{J}_R\hat{J}_L$ term in (5.77) and thus the traces can be computed without introducing the auxiliary integrals (5.80). Then one finds again (4.46). To reproduce the expression for $Z$ in the $q_+ = 0$ case in the operator approach one is to do the Poisson resummation not in $m$ but in the winding number $w$. Then the $\hat{J}_L\hat{J}_R$ term disappears again and we obtain (4.44). Equivalently, one may note that when $q_+ = 0$ the sums over $w', w$ in (5.81) give $\delta$-functions and the integrals over $\lambda, \bar{\lambda}$ are easily computable (one may still apply (5.78) to arrive at (4.44)).

## 6. Some physical properties of particular models

Having obtained a diagonal string Hamiltonian (5.51) and constraints (5.59) it is straightforward to determine the spectrum of this string model. This was already done in the special case of the uniform magnetic field model ($\alpha\beta = 0$, $q_i = 0$) in ref. [2]. Its space-time background (1.1) is represented by a homogeneous space metric and non-trivial antisymmetric tensor. Below, in Section 6.1, we shall examine the spectrum in a complementary special case, $\alpha = \beta$ (or $c_- = a_-$), which includes, in particular, the Melvin model ($\alpha = \beta = q_+$). The corresponding non-singular ($\alpha\beta > 0$) space-time backgrounds are no longer homogeneous but have vanishing antisymmetric tensor (see (3.13) and Section 3.3).

The general property of the spectrum observed in [2] was the appearance of tachyonic instabilities, typically associated with states with angular momentum aligned along the magnetic field. Similar instabilities are present in point-particle field theories in external magnetic fields (and may lead to a phase transition with restoration of some symmetries, see [6]). In the context of open string theory they were observed in ref. [33] and further investigated in ref. [42]. The new feature of the closed string theory [2,43] is the existence of states with arbitrarily large charges. Since the critical magnetic field at which a given state of a charge $Q$ becomes tachyonic is of order of $(\alpha' Q)^{-1}$, the generic pattern is that there is an infinite number of tachyonic instabilities for any given finite value of the magnetic field.[35] As we shall discuss below, in the general case of $\alpha\beta q_+(q_+ + \beta - \alpha) \neq 0$ there exist also other types of tachyonic instabilities associated with the presence of non-trivial geometrical background.

Below we shall consider only the sector with $0 < \gamma < 2$, where $\gamma$ is defined in (5.38). Other sectors can be analysed in a similar way.

### *6.1. Spectrum of models with vanishing antisymmetric tensor ($\alpha = \beta$)*

It is easy to show that the term in $\hat{H}$ (5.51) which is linear in the energy of a string state $E$ is directly related with the presence of an antisymmetric tensor background. When $\alpha = \beta$ the antisymmetric tensor is absent and this term can be eliminated with choosing the frame with $a_- = 0$, i.e. $q_- = -\alpha$. Then the corresponding background gauge fields (3.9) have no electric components, so that the models with $\alpha = \beta$ can be characterized by the two magnetic field strength parameters $\mathbf{B}_{\mathrm{v}0} = a_+ + c_+ = 2q_+$ and $\mathbf{B}_{\mathrm{a}0} = a_+ - c_+ = -2\beta$. The particular (self-dual) case of $a_+ = 0$ corresponds to the dilatonic Melvin model, while the case of $a_+ = c_+$ (i.e. $\alpha = \beta = 0$) corresponds to the "Kaluza-Klein" ($a = \sqrt{3}$) Melvin model (1.3). The spectrum is determined by the conditions (5.59), i.e. $\hat{H} = 0$ and $N - \tilde{N} = mw$, where (see (5.51),(5.45),(5.46))

$$\hat{H} = \tfrac{1}{2}\big( - \alpha' M^2 + w^2 r^2 + \frac{m^2}{r^2}\big) + N + \tilde{N} - 2 - \tfrac{1}{4}\gamma^2 \tag{6.1}$$

$$- \alpha'\big(c_+ Q_+ \hat{J}_R + a_+ Q_- \hat{J}_L\big) + \tfrac{1}{2}\alpha'(\hat{J}_R + \hat{J}_L)\big(c_+^2 \hat{J}_R + a_+^2 \hat{J}_L\big) \ , \quad M^2 = E^2 - p_a^2 \ .$$

$M^2$ represents the mass, invariant with respect to the residual Lorentz group in the hyperplane orthogonal to the plane $x_1, x_2$ ($p_a$ are momenta in possible additional spatial dimensions). We recall that in presence of a non-trivial background including the magnetic field the momenta $p_{1,2}$ of the transverse coordinates $x_{1,2}$ of a closed string are traded for the two (right and left) Landau-type quantum numbers $l_R, l_L = 0, 1, 2, ...,$ associated with the zero-mode operators $b_0^\dagger, b_0$ and $\tilde{b}_0^\dagger, \tilde{b}_0$. The center of mass orbital momentum is $l = l_L - l_R = 0, \pm 1, \pm 2, ...$ (see, e.g., (5.72)). The Hamiltonian (6.1) defines the spectrum of states with quantum numbers such that $0 < \gamma < 2$ where $\gamma$ is given by (see (5.38))

$$\gamma = (a_+ + c_+) w R + \alpha' [(c_+ - a_+) m R^{-1} + (a_- - c_-) E] \tag{6.2}$$

[35] One should keep in mind that these are tree-level results. At the one-loop level the mass of a given mode with charge $Q$ is expected to receive corrections of order $O(Q^2 g^2)$ ($g$ is the string coupling) which can be neglected for $Q \ll 1/g$. This suggests that the minimum magnetic field at which tachyonic instabilities first appear is $\mathbf{B}_{\mathrm{cr}} \sim (\alpha' Q)^{-1} \sim O(g/\alpha')$ which is essentially of Planck order for a reasonable value of $g$.

$$+\ \tfrac{1}{2}\alpha'(a_+^2 - a_-^2 - c_+^2 + c_-^2)(l + S_R + S_L)\ .$$

As discussed in Sections 5.2, 5.3, for the special states with $\gamma = 0$ (mod 2) the mass formula takes the form

$$M^2 = E^2 - p_a^2 - p_1^2 - p_2^2 = \alpha'^{-1}\big( - 4 - \tfrac{1}{2}\gamma^2 + 2N + 2\tilde N + w^2 r^2 + \frac{m^2}{r^2}\big) \tag{6.3}$$

$$-2\big[c_+ Q_+ (\tfrac{1}{2}l + S_R) + a_+ Q_-(\tfrac{1}{2}l + S_L)\big] + (l + S_R + S_L)\big[c_+^2(\tfrac{1}{2}l + S_R) + a_+^2(\tfrac{1}{2}l + S_L)\big]\ ,$$

where $l$ is the eigen-value of the center of mass orbital momentum $(x_1 p_2 - x_2 p_1)$ and the quantum numbers are subject to the constraint $\gamma = 0$ (see (6.2)).

A novel feature of the $\alpha = \beta$ model as compared to the special $\alpha\beta = 0$ model studied in [2] is the presence in $\hat H$ of the additional terms *quartic* in the oscillators. Let us consider the spectrum of lowest-level neutral states ($N = \tilde N = 0$, $m = w = 0$). For these states $\gamma$ (6.2) takes the form: $\gamma = \frac{1}{2}\alpha'(c_+^2 - a_+^2)(l_R - l_L)$, $l_{L,R} = 0, 1, 2, \ldots$. Then the generalization (to the case of $a_+ \neq 0$) of the expression (5.73) for the tachyonic spectrum for the Melvin model is

$$M^2 = -4\alpha'^{-1} + \tfrac{1}{2}(l_R - l_L)[c_+^2(2l_R + 1) - a_+^2(2l_L + 1)] - \tfrac{1}{8}\alpha'(a_+^2 - c_+^2)^2 l^2\ . \tag{6.4}$$

Let us take for definiteness $c_+^2 > a_+^2$. A closer inspection of eq.(6.4) shows that, in the range $0 < \gamma < 2$, the new contribution modifying the free-theory value $\alpha' M^2 = -4$ is positive definite, and that the usual tachyon disappears for sufficiently large values of the magnetic fields $a_+, c_+$. This can be attributed to the effect of curvature of the corresponding background which produces a mass gap in the Laplace operator. Similar conclusion is true for the "massless" level ($N = \tilde N = 1$) neutral ($m = w = 0$) states with $0 < \gamma < 2$: they receive positive corrections to their $M^2$.[36]

The situation is different in the charged sector. Let us recall that in the case of the constant magnetic field model ($\alpha = q_i = 0$) it was shown in [2] that the term in $H$ proportional to $c_+ Q_+ \hat J_R$ gives rise to the tachyonic instabilities which are similar to magnetic instabilities in gauge theories. For a fixed value of the parameter $c_+$ one finds an infinite number of tachyonic states in the spectrum.

Instabilities caused by the linear in $\hat J_{L,R}$ terms in $H$ are present also in the $\alpha = \beta$ models (in particular, in the Melvin model). There are, in fact, infinitely many tachyonic charged states at higher levels. Let us first consider the level one state with $w = 0$, $m > 0$, $N = \tilde N = 1$, $l_R = l_L = 0$, $S_R = 1, S_L = -1$ (i.e. $\hat J_{R,L} = \pm\frac{1}{2}$) which corresponds to a "massless" scalar state with a Kaluza-Klein charge. We may assume without loss of generality that $R > \sqrt{\alpha'}$ (if $R < \sqrt{\alpha'}$ a similar discussion applies with $m$ replaced by $w$). The expression for the mass that follows from (6.1) is ($2\beta = c_+ - a_+$)

$$M^2 = p_y(p_y - 2\beta - 2\alpha'\beta^2 p_y)\ , \qquad p_y = mR^{-1}\ . \tag{6.5}$$

For $R >> 1$, $M^2$ becomes negative when $\beta > \beta_{\rm cr} \cong \frac{1}{2}p_y$. For these states $\gamma = 2\alpha'\beta p_y$ and thus $\gamma < 2$ if $\beta > \beta_{\rm cr}$ and $\alpha' p_y^2 < 2$ (i.e. $R^2 > \frac{1}{2}\alpha' m^2$). The critical value of the magnetic

[36] For the massless level ($N = \tilde N = 1$) perturbations this implies, in particular, the stability of the corresponding field-theoretic Melvin solutions (see [4,38] and refs. there).

field goes to zero as $R \to \infty$. In the noncompact $R = \infty$ theory $p_y$ becomes a continuous parameter representing the momentum of the "massless" state in the $y$-direction. Thus the "massless" state with an infinitesimal momentum $p_y$ becomes tachyonic for an infinitesimal value of $\beta$ (corresponding to an infinitesimal deformation of the metric and antisymmetric tensor backgrounds).

To illustrate the appearance of an infinite number of tachyons, let us consider the $a = 1$ Melvin model with $r = 1$ where

$$\alpha' M^2 = -4 + 4\tilde{N} + \alpha'(Q_+ - c_+ J_R)^2 + \alpha' c_+^2 J_R J_L + \eta \ , \qquad \eta \equiv \gamma(1 - \tfrac{1}{2}\gamma) \leq \tfrac{1}{2} \ . \tag{6.6}$$

Consider states with $\tilde{N} = J_L = 0, \ m = w$. Then a given state becomes tachyonic for

$$(\sqrt{\alpha'} Q_+ - \sqrt{4 - \eta})/J_R < \sqrt{\alpha'} c_+ < (\sqrt{\alpha'} Q_+ + \sqrt{4 - \eta})/J_R \ , \tag{6.7}$$

with $\gamma = -2J_R + 1 + \sqrt{(2J_R - 1)^2 + 2\alpha' Q_+^2 - 8}$. All states with $J_R \cong Q_+/c_+ = (2/c_+)\sqrt{N}, \ N >> 1$, will become massless in the infinitesimal interval $c_+ = Q_+/J_R - \epsilon$ and $c_+ = Q_+/J_R$. The emergence of an infinite number of massless particles suggests an enhancement of gauge symmetries of the theory (for a further discussion see [43]).

Next, let us consider the $a = \sqrt{3}$ ("Kaluza-Klein") Melvin model where the expression for $M^2$ in (6.1) (for $0 < \gamma < 2$) and for $\gamma$ in (6.2) are given by ($a_+ = c_+ = q_+, \ a_- = c_- = 0$)

$$M^2 = 2\alpha'^{-1}(-2 + N + \tilde{N}) + (p_y - q_+ \hat{J})^2 + [\alpha'^{-1} w R - q_+(\hat{J}_R - \hat{J}_L)]^2 \tag{6.8}$$

$$- \ q_+^2 (\hat{J}_R - \hat{J}_L)^2 - 2\alpha'^{-1} q_+^2 w^2 R^2 \ , \qquad \gamma = 2 q_+ w R \ .$$

It follows from (6.3) that in the non-winding (i.e. $\gamma = 0$) sector this model is stable, i.e. has no new instabilities (except the usual flat space tachyon). As for the winding sector, there exists a range of parameters $q_+, \ R$ for which we find again the same linear instability as in the $a = 1$ Melvin model (6.5).

*6.2. Gyromagnetic factors*

In the case when $B_{\mu\nu} = 0$, the standard definition for the gyromagnetic factor in terms of the non-relativistic expansion gives a frame independent result. In the weak field limit the mass formula takes the form (see (5.51),(6.1); $S_{L,R}$ are the spin parts of the left and right angular momenta (5.45),(5.46))

$$M^2 = M_0^2 - 2(c_+ Q_+ S_R + a_+ Q_- S_L) \tag{6.9}$$

$$+ \ [(2l_R + 1)c_+ Q_+ - (2l_L + 1)a_+ Q_-] + O(c_+^2, a_+^2) \ ,$$

$$\alpha' M_0^2 = -4 + 2N + 2\tilde{N} + \tfrac{1}{2} Q_+^2 + \tfrac{1}{2} Q_-^2 \ .$$

Note that in the case of the self-dual model $a_+ = 0$ (or $c_+ = 0$) we get the familiar field-theory expression in the non-winding sector ($w = 0$)

$$M^2 = M_0^2 - 2e\mathcal{B} S_R + e\mathcal{B}(2l_R + 1) + O(R^2 \mathcal{B}^2) \ , \tag{6.10}$$

$$e = Q_+ R \ , \quad \mathcal{B} = c_+ R^{-1} \ , \quad l_R = 0, 1, ..., \ .$$

In the models under consideration we have two $U(1)$ gauge fields with the magnetic strengths determined by $c_+ = \frac{1}{2}(\mathbf{B}_{\rm v0} - \mathbf{B}_{\rm a0})$ and $a_+ = \frac{1}{2}(\mathbf{B}_{\rm v0} + \mathbf{B}_{\rm a0})$. The associated (tree-level) gyromagnetic factors can be easily read off from the Hamiltonian. In the $B_{\mu\nu} = 0$ case, i.e. when $a_- = c_- = 0$, the $g$-factors reduce to the form derived in ref. [44] (see (6.9))

$$g_R = \frac{2S_R}{S} \ , \qquad g_L = \frac{2S_L}{S} \ , \qquad S = S_R + S_L \ . \tag{6.11}$$

As was found in [2], in the case of the uniform magnetic field model (which has $B_{\mu\nu} \neq 0$) there exists another contribution to the $g$-factor coming from the term in $H$ which is linear in energy. Since the magnetic dipole moment is expected to be a background-independent property of the state, this seems puzzling.

The resolution of this puzzle is the following. In closed string theory a magnetic field background necessarily induces other fields (curved metric, dilaton, antisymmetric tensor, etc.) and their consistent configuration is not unique. If one insists on having strictly uniform magnetic field background, like the one with $c_+ = -c_- = \beta$ considered in [2], this implies the presence of an antisymmetric tensor. Then the term linear in $E$ in the Hamiltonian ($\alpha' c_- E \hat{J}_R$) produces an additional contribution to the gyromagnetic coupling ($-\mu\beta$) and thus the $g$-factor gets an extra term proportional to the energy. Note, however, that in order to determine the magnetic dipole moment it is sufficient to consider a magnetic field which is approximately constant in some finite region. For generic values of our parameters the magnetic fields in (3.10) are indeed constant in the region near $\rho = 0$ (and are equal to $a_+ \pm c_+$, i.e. do not depend on $c_-$ and $a_-$). That means that, in general (even when $B_{\mu\nu} \neq 0$), the linear term in $H$ does not, in fact, produce a contribution to the gyromagnetic ratio. This suggests that $g_{R,L}$ in (6.11) are the correct, background independent values of the gyromagnetic factors, in accordance with the suggestion of ref. [44] (see also [45]). These values are not inconsistent with the values characteristic to charged rotating black holes (see [44,45]), i.e. do not contradict the conjecture [46] about the correspondence between string states and black holes.

Another interesting special case (including, in particular, the Kaluza-Klein Melvin solution (1.3)) is when $a_+ = c_+$. As follows from (6.1), the magnetic dipole moment is then given by

$$\mu = \frac{\alpha'}{2M}[r^{-1} m S + w r (S_R - S_L)] \ . \tag{6.12}$$

As a consequence, the ordinary Kaluza-Klein states with $w = 0$ (i.e. with charges of "non-winding" origin) have the standard Kaluza-Klein field-theory value of the gyromagnetic factor, $g = 1$ (see also [44]).

### *6.3. Singular backgrounds*

Given a solvable model one is able, in principle, to address the important question of how space-time singularities of a given background are reflected in the properties of the string solution (CFT) it represents, for example, if there is something special happening with the spectrum of physical string states and the partition function for the values of parameters when curvature may have singularities.

As was mentioned in Section 3 (see also (A.1)), the singularities appear in the cases when

$$\alpha\beta < 0 \ , \quad \text{or} \quad q_+(q_+ + \beta - \alpha) < 0 \ . \tag{6.13}$$

These conditions imply that $\beta \neq \alpha$, i.e. require the presence of an antisymmetric tensor background. There is an additional instability in the spectrum which occurs only for singular geometries and leads to a tachyonic mass even for the usual (transverse) graviton state. It is related to the presence of the $O(\hat{J}^2)$ terms in the Hamiltonian and is present even in the absence of a magnetic field background.[37] Consider, for example, the model with $a_+ = c_+ = 0$ or $\alpha = -\beta = q_+$ (see (3.10)). The corresponding singular background (3.23) was mentioned in Section 3.2. In this case the energy formula for the graviton state, $N = \tilde{N} = 1, m = w = 0$, is given by (see (5.51); we choose the frame with $p_a = 0$ and $c_- = -a_-$)

$$[E - c_-(\hat{J}_R - \hat{J}_L)]^2 + 2\alpha' c_-^2 E^2 = -4c_-^2 \hat{J}_R \hat{J}_L \ . \tag{6.14}$$

Thus all the states with $\hat{J}_R \hat{J}_L > 0$ have complex energy for infinitesimal values of the parameter $\beta = \frac{1}{2}(a_- - c_-)$.[38] When the energy gets an imaginary part, the partition function develops a new divergence coming from the modular region $\tau_2 \to \infty$ (see (5.75) and ref. [2]).

The backgrounds with (6.13) generically have singularities which may be related to naked and black string type ones (in what follows we drop the trivial $x_3$-direction). Consider, for example, the case of $a_+ = c_+$ or $\alpha = -\beta$ in the frame with $c_- = -a_-$. The corresponding metric is (see (3.11))

$$ds^2 = -\frac{dt^2}{1 - c_-^2 \rho^2} + d\rho^2 + \frac{\rho^2}{1 + (c_+^2 - c_-^2)\rho^2} d\varphi^2 \ . \tag{6.15}$$

For $c_+ = 0$ this metric is related by a coordinate transformation to (3.23). Changing $\rho = 1/\tilde{\rho}$ and replacing $\varphi$ by a noncompact coordinate one can see that near the singularity (6.15) coincides with the metric of the extreme $D = 3$ black string in [18]. For $c_+ = a_+ \neq 0$ let us make the analytic continuation $c_\pm \to ic_\pm$ and at the same time replace $\varphi \to i\varphi$ ($\varphi$ will still be an angular variable). Then the background fields and the corresponding $\sigma$-model remain real. Introducing the coordinate $x = -1/\rho^2 + c_-^2$ one can represent the metric (6.15) in the form

$$ds^2 = -(1 - \frac{\mu}{x})dt^2 + \frac{dx^2}{4(x-\mu)^3} + \frac{d\varphi^2}{c_+^2 - x} \ , \qquad \mu \equiv c_-^2 \ . \tag{6.16}$$

The corresponding geometry has two singularities, at $x = 0$ and $x = c_+^2$ (this is clear, e.g. from the expression for the curvature scalar in (A.1)). For $x \to \infty$ the curvature

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[37] The quartic terms in the Hamiltonian (5.51) (which can be eliminated by a proper choice of $q_-$ when $\alpha\beta q_+(q_+ + \beta - \alpha) = 0$), are not necessarily associated with singularities. Indeed, there are models with singular geometry as, e.g., the one with $q_+ = 0$, $\alpha\beta < 0$, where the quartic term is absent (conversely, there are regular geometries with a quartic term in $H$ which were discussed in Section 6.1).

[38] A pathology of such states is reflected also in the fact that the value of $\gamma$ in (6.2) $\gamma = \alpha'(a_- - c_-)[E - \frac{1}{2}(a_- + c_-)\hat{J}]$ which depends on $E$ also becomes complex.

approaches a constant value. If $c_+^2 \gg c_-^2 = \mu$ one can consider the region $x \ll c_+^2$ where the metric takes the form

$$ds^2 = -(1 - \frac{\mu}{x})dt^2 + \frac{dx^2}{4(x-\mu)^3} + c_+^{-2} d\varphi^2 \ . \qquad (6.17)$$

This is recognized as the metric of a (2+1)–dimensional black hole with an event horizon at $x = \mu$.

## 7. Concluding remarks

In this paper we have presented an explicit solution of a class of non-trivial string models describing curved space-time backgrounds with uniform magnetic fields. The corresponding CFT's are simpler than generic coset models (having "free" central charge and just one $\alpha'$-correction term in the Hamiltonian which is diagonal in the free-field Fock space), are unitary (as follows, e.g., from the possibility to choose a light-cone gauge) and thus (along with the model of ref.[26] studied in [31] and its "compact" generalisation discussed in [2]) provide first examples of consistent solvable conformal string models with explicit $D = 4$ curved space-time interpretation.

The method of constructing exact string solutions and solving the corresponding 2-dimensional conformal theories used in this paper can be applied also to a number of related models. Solvable cases include, e.g., various analytic continuations of (5.6), (2.5) and their duality transforms, a generalization (2.25) from Section 2.4, as well as superstring and heterotic string generalizations. In addition, there are extensions of our class of backgrounds (mentioned in Sections 2.4 and 3.7) which are also exact string solutions, but they correspond to more complicated CFT's which cannot be analysed using the methods of Sections 4 and 5.

There are also possible interesting applications. In particular, the family of string backgrounds described above provides a laboratory to study the issue of tachyonic instabilities in string theories. We have seen that some of the members of this family are, in a certain sense, "more unstable" than others. For example, the $a = \sqrt{3}$ Kaluza-Klein theory does not have the infinite number of instabilities associated with large charge states discussed in Section 6.1. It may happen that some related (superstring) models may be stable for some special values of parameters. This might suggest a mechanism to select a stable string vacuum.


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## Appendix A. Curvature scalar for the metric (3.11)

The curvature scalar corresponding to the metric (3.11) is given by

$$R = \tfrac{1}{2}(A_0 + A_2\rho^2 + A_4\rho^4 + A_6\rho^6)F^2\tilde{F}^2 \ , \qquad (A.1)$$

where

$$F=(1+\alpha\beta\rho^2)^{-1}\ ,\quad \tilde{F}=[1+q_+(q_++\beta-\alpha)\rho^2]^{-1}\ , \tag{A.2}$$

and

$$\begin{gathered}
A_0=\alpha^2+10\alpha\beta+\beta^2+12q_+(q_++\beta-\alpha)\ ,\\
A_2=\alpha\beta(\alpha-\beta)^2+q_+(q_++\beta-\alpha)[36\alpha\beta+(\alpha+\beta)^2]\ ,\\
A_4=\alpha\beta q_+(q_++\beta-\alpha)[4q_+(q_++\beta-\alpha)+(\alpha+\beta)^2]\ ,\\
A_6=-8[\alpha\beta q_+(q_++\beta-\alpha)]^2\ .
\end{gathered} \tag{A.3}$$

For $\alpha\beta<0$ or $q_+(q_++\beta-\alpha)<0$ it is singular at the points where $F$ or $\tilde{F}$ diverge, and generically it goes to zero as $\rho\to\infty$ (special cases are e.g. $\alpha=0$, $q_+=0$ where $R$=constant). At $\rho=0$ it has a finite value given by $R=\frac{1}{2}A_0$.

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