# Two-loop electroweak top corrections: are they under control? *

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**Abstract**

The assumption that two-loop top corrections are well approximated by the $O(G_\mu^2 m_t^4)$ contribution is investigated. It is shown that in the case of the ratio neutral-to-charged current amplitudes at zero momentum transfer the $O(G_\mu^2 m_t^2 M_Z^2)$ terms are numerically comparable to the $m_t^4$ contribution for realistic values of the top mass. An estimate of the theoretical error due to unknown two-loop top effect is presented for a few observables of LEP interest.

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*To appear in "Reports of the Working Group on Precision Calculations for the $Z$-resonance", CERN.

This research is partially supported by EU under contract No. CHRX-CT92-0004.

# 1 Introduction

The constant improvement of the experimental precision on line shape and asymmetry parameters at LEP has stimulated the evaluation of two-loop corrections of a purely electroweak nature in order to assess the reliability of the theoretical predictions. Although the latter seem to be affected mainly by the uncertainty of the hadronic contribution on $\Delta\alpha$, it is not yet clear which error may be attributed to the ignorance of higher orders in the electroweak perturbative expansion. The first attempt made in this direction was the computation of the Higgs contribution to the $\rho$ parameter in the limit of large $M_H$ [1]. Subsequently, top effects were also investigated [2]. Concerning the top, we only have at the moment two-loop results obtained from the SM in the limit of vanishing gauge coupling constants [3–5]. Such contributions are of $O(G_\mu^2 m_t^4)$ and formally leading in the limit of large top mass. They should be considered as the present best estimate of the top influence on higher-order corrections. This note deals with the next-to-leading corrections of $O(G_\mu^2 m_t^2 M_Z^2)$. Such terms are suppressed by a power $M_Z^2/m_t^2$ with respect to the leading ones, but the present range of values for $m_t$ [6, 7] does not exclude that these corrections may be numerically important. Our computation can be regarded as an attempt to check the validity of such an expansion, until the full two-loop results are available. At the same time we should be able to provide a more realistic estimate of the error associated with the two-loop electroweak effects.

To keep the computation as simple as possible we have focused on neutrino scattering on a leptonic target, of which we will compute the electroweak corrections of $O(G_\mu^2 m_t^2 M_Z^2)$ to the $\rho$ parameter, defined as the ratio of neutral-to-charged current amplitudes, at zero momentum transfer. To be more precise, we identify $\rho$ with the cofactor, expressed in units of $G_\mu$, the $\mu$-decay constant, of the $J_Z\, J_Z$ interaction in neutral current amplitudes. It is well known that radiative effects also lead to a modification of the mixing angle, described by a parameter usually called $\kappa$. These effects will not be discussed in the present paper.

For the processes under examination, we found large subleading corrections of the same sign and of about the same magnitude as the leading one. Therefore, at least for the case we have investigated, the use of the first term of an expansion in inverse power of $m_t$ to approximate the full two-loop result appears to be doubtful. Our result, being obtained at $q^2=0$, cannot be directly applied to LEP physics, but can give us a flavour of the size of subleading effects that are due to one-particle irreducible contributions. In the concluding Section, we will elaborate this point, analysing the consequences of a naïve extrapolation of our result to some LEP observables.

## 2 $O(G_\mu^2 m_t^2 M_Z^2)$ corrections to the $\rho$ parameter.

In this Section we outline the computation of the electroweak corrections of $O(G_\mu^2 m_t^2 M_Z^2)$ to the $\rho$ parameter. We begin by writing the relation between the $\mu$-decay constant and the charged current amplitude expressed in terms of bare quantities. At the two-loop level, neglecting contributions that will not give $O(G_\mu^2 m_t^2 M_Z^2)$ terms, we have

$$\frac{G_\mu}{\sqrt{2}} = \frac{g_0^2}{8M_{W_0}^2}\left\{1-\frac{A_{WW}}{M_{W_0}^2}+V_W+M_{W_0}^2 B_W+\frac{{A_{WW}}^2}{M_W^4}-\frac{A_{WW}V_W}{M_W^2}\right\} \quad , \tag{1}$$

where $g_0$ and $M_{W_0}$ are the bare $SU(2)_L$ coupling and $W$ mass, respectively, $A_{WW}$ is the transverse part of the $W$ self-energy at zero momentum transfer, and the quantities $V_W$ and $B_W$ represent the relevant vertex and box corrections. At the bare level, using the fact that $M_{Z_0}^2 c_0^2 = M_{W_0}^2$, where $c_0 \equiv \cos\theta_{W_0}$ with $\theta_{W_0}$ the weak mixing angle and $M_{Z_0}$ the bare $Z$ mass, the $\rho$ parameter can be written as:

$$\rho = \frac{\left(1-\frac{A_{ZZ}}{M_{Z_0}^2}+V_Z+M_{Z_0}^2 c_0^2 B_Z+\frac{{A_{ZZ}}^2}{M_Z^4}-\frac{A_{ZZ}V_Z}{M_Z^2}\right)}{\left(1-\frac{A_{WW}}{M_{W_0}^2}+V_W+M_{W_0}^2 B_W+\frac{{A_{WW}}^2}{M_W^4}-\frac{A_{WW}V_W}{M_W^2}\right)} \quad , \tag{2}$$

where $A_{ZZ}$, $V_Z$ and $B_Z$ are the corresponding self-energy, vertex, and box contribution in the neutral current amplitude. To the order we are interested in, Eq. (2) reduces to:

$$\begin{aligned}\rho = 1 &+ \left(\frac{A_{WW}}{M_{W_0}^2}-\frac{A_{ZZ}}{M_{Z_0}^2}\right)+(V_Z-V_W)+(M_{W_0}^2+A_{WW})(B_Z-B_W)\\ &+\left(\frac{A_{WW}}{M_W^2}-\frac{A_{ZZ}}{M_Z^2}\right)\left(-\frac{A_{ZZ}}{M_Z^2}+(V_Z-V_W)-M_W^2 B_W\right) \quad .\end{aligned} \tag{3}$$

We proceed by separating the self-energies into one-loop and two-loop contributions:

$$A_{ZZ} = A^{(1)}_{ZZ} + A^{(2)}_{ZZ} \; ; \qquad\qquad A_{WW} = A^{(1)}_{WW} + A^{(2)}_{WW} \quad , \tag{4}$$

on the understanding that the one-loop term is still expressed in terms of bare parameters. The one-loop part can be decomposed further into pure bosonic ($b$) and fermionic ($f$) terms:

$$A^{(1)}_{ZZ} = A^{b(1)}_{ZZ} + A^{f(1)}_{ZZ} \; ; \qquad\qquad A^{(1)}_{WW} = A^{b(1)}_{WW} + A^{f(1)}_{WW} \quad , \tag{5}$$

and the one-loop fermionic contribution to the $\rho$ parameter, assuming a vanishing bottom mass, can be expressed as follows:

$$X^0_d = \left( \frac{A^f_{WW}}{M^2_{W_0}} - \frac{A^f_{ZZ}}{M^2_{Z_0}} \right)^{(1)} = \frac{g_0^2}{8 M^2_{W_0}} f(m^2_{t_0}, \epsilon) \tag{6a}$$

$$f(m_t^2, \epsilon) \equiv \frac{3}{2\pi^2} \, \frac{1}{(4-2\epsilon)} \, m_t^2 \; \epsilon \; \Gamma(\epsilon) \left( \frac{4\pi\mu^2}{m_t^2} \right)^{\epsilon} \quad . \tag{6b}$$

where $\epsilon$ is related to the dimension $d$ of the space–time by $\epsilon = (4-d)/2$ and $\mu$ is the 't-Hooft mass scale.

We want to express our final result in terms of the physical $Z$ mass, therefore we perform the shift $M^2_{Z_0} = M^2_Z - \mathrm{Re}\;\Pi_{ZZ}(M^2_Z)$, where $\Pi_{ZZ}(M^2_Z)$ is the transverse part of the $Z$ self-energy at $q^2 = M^2_Z$. Using the decompositions given in Eqs. (4) and (5), and keeping only terms up to $O(G^2_\mu m^2_t M^2_Z)$, we obtain after simple algebra:

$$\begin{aligned}
\rho = {} & 1 + X^0_d + X_d \left( - \frac{A_{WW}}{M^2_W} + V_W + M^2_W B_W \right) \\
& + \left( \frac{A^b_{WW}/c_0^2 - A^b_{ZZ}}{M^2_Z} \right)^{(1)} + \left( \frac{A_{WW}}{M^2_W} - \frac{A_{ZZ}}{M^2_Z} \right)^{(2)} \\
& + (V_Z - V_W) + M^2_Z c_0^2 (B_Z - B_W) - X_d (V_W + 2\; M^2_W B_W) \\
& + X_d \left[ \left( \frac{A_{WW}}{M^2_W} - \frac{A_{ZZ}}{M^2_Z} \right) + (V_Z - V_W) + M^2_W (B_Z - B_W) \right] \quad ,
\end{aligned} \tag{7}$$

where $X_d$ is the same quantity introduced in Eq. (6), but expressed in terms of renormalized parameters.

We observe that Eq. (7) further simplifies if we express the one-loop fermionic contribution in terms of the Fermi constant $G_\mu$. Indeed, as can be seen from Eq. (1), the first line of Eq. (7) reproduces the effective coupling in the charged sector:

$$X_d^0\left(1-\frac{A_{WW}}{M_W^2}+V_W+M_W^2 B_W\right)=\frac{g_0^2}{8M_{W_0}^2}\left(1-\frac{A_{WW}}{M_W^2}+V_W+M_W^2 B_W\right)f(m_{t_0}^2,\epsilon)$$
$$\simeq\frac{G_\mu}{\sqrt{2}}\,f(m_{t_0}^2,\epsilon)\quad . \tag{8}$$

Until now, apart from the use of the physical $Z$ mass, we have not specified any particular renormalization condition. In order to simplify the structure of the counterterms, we have found it convenient to perform the calculation using the $\overline{MS}$ parameter $\sin^2\hat{\theta}_W(M_Z)$ (henceforth abbreviated as $\hat{s}^2$). Indeed, while in the on-shell (OS) scheme the counterterm associated with the quantity $s^2 = 1 - M_W^2/M_Z^2$ contains terms proportional to $m_t^2$ and gives rise to $O(G_\mu^2 m_t^2 M_Z^2)$ contributions to $\rho$, the counterterm related to $\hat{s}^2$ does not exhibit any $m_t^2$ dependence and this greatly simplifies our task. Therefore, to the order we are interested in, we can directly replace $c_0^2$ with $\hat{c}^2$ in Eq. (7) ($\hat{c}^2 \equiv 1-\hat{s}^2$). It will always be possible to recover the result in the pure OS scheme, by appropriately shifting $\hat{s}^2$ in the one-loop expression for $\rho$.

We now notice that the one-loop contribution is still written in terms of bare quantities. To put $\rho$ in its final form, we split it into the usual $O(\alpha)$ result, $\delta\rho^{(1)}$, plus the counterterm part, $\delta\rho_C$, namely

$$\frac{G_\mu}{\sqrt{2}}\,f(m_{t_0}^2,\epsilon)+\left(\frac{A^b_{WW}/\hat{c}^2-A^b_{ZZ}}{M_Z^2}\right)^{(1)}+(V_Z-V_W)^{(1)}+M_Z^2\hat{c}^2(B_Z-B_W)^{(1)}\equiv\delta\rho^{(1)}+\delta\rho_C \tag{9}$$

with

$$\delta\rho^{(1)}=\delta\rho^{f(1)}+\delta\rho^{b(1)} \tag{10a}$$

$$\delta\rho^{f(1)}=N_c x_t\equiv N_c\frac{G_\mu m_t^2}{8\pi^2\sqrt{2}} \tag{10b}$$

$$\delta\rho^{b(1)}=\frac{\hat{\alpha}}{4\pi\hat{s}^2}\left[\frac{3}{4\hat{s}^2}\ln\hat{c}^2-\frac{7}{4}+\frac{2\,c_Z}{\hat{c}^2}+\hat{s}^2\;G(\xi,\hat{c}^2)\right]\quad , \tag{10c}$$

where $N_c$ is the colour factor, and $\hat{\alpha}=\alpha/(1+2\delta e/e)_{\overline{MS}}$ is the $\overline{MS}$ coupling as defined in [8]. In Eqs. (10)

$$c_Z = \frac{\hat{c}^2}{4}(5-3I_3) - 3\left(\frac{I_3}{8} - \frac{\hat{s}^2}{2}Q + \hat{s}^4\ I_3\ Q^2\right) \quad , \tag{11a}$$

where $I_3$ and $Q$ are the isospin and electric charge of the target ($I_3 = -1$ for electrons) and

$$G(\xi, \hat{c}^2) = \frac{3}{4}\frac{\xi}{\hat{s}^2}\left[\frac{\ln\hat{c}^2 - \ln\xi}{\hat{c}^2 - \xi} + \frac{1}{\hat{c}^2}\frac{\ln\xi}{1-\xi}\right] \quad , \tag{11b}$$

with $\xi \equiv M_H^2/M_Z^2$. Using eqs. (7), (8), and (9) we can express $\rho$ as follows:

$$\rho = 1 + \delta\rho^{(1)} + N_c x_t \delta\rho^{(1)} + \delta\rho^{(2)} \quad , \tag{12}$$

where the previous relation defines the two-loop contribution, $\delta\rho^{(2)}$, as:

$$\delta\rho^{(2)} = \delta\rho_C + \left(\frac{A_{WW}}{M_W^2} - \frac{A_{ZZ}}{M_Z^2}\right)^{(2)} + (V_Z - V_W)^{(2)} + M_Z^2\hat{c}^2(B_Z - B_W)^{(2)} \\ - X_d(V_W + 2\ M_W^2 B_W) \tag{13}$$

Eq. (12) suggests that a possible way to take into account higher-order effects is to write $\rho$ as

$$\rho = \frac{1}{(1-\delta\rho^{f(1)})}(1 + \delta\rho^{b(1)} + \delta\rho^{(2)}) \quad , \tag{14}$$

where the resummation of $\delta\rho^{f(1)}$ can be justified theoretically on the basis of $1/N_c$ expansion arguments [9]. Explicitly we find, in units $N_c\,[\hat{\alpha}/(16\pi\hat{s}^2\hat{c}^2)\,m_t^2/M_Z^2]^2 \simeq N_c x_t^2$:

$$\begin{aligned}\delta\rho^{(2)} = {} & 25 - 4\,ht + \left(\frac{1}{2} - \frac{1}{ht}\right)\pi^2 + \frac{(-4+ht)\,\sqrt{ht}\,g(ht)}{2} + \left(-6 - 6\,ht + \frac{ht^2}{2}\right)\ln ht \\ & + \left(-15 + \frac{6}{ht} + 12\,ht - 3\,ht^2\right) Li_2(1-ht) + \left(-15 + 9\,ht - \frac{3\,ht^2}{2}\right)\phi\left(\frac{ht}{4}\right) \\ & + zt\ \left[\frac{25}{2} + \frac{4}{ht} - 10\,\hat{c}^2 + \frac{3}{\hat{s}^2} + \frac{277\,\hat{s}^2}{9} - \frac{4\,\hat{s}^2}{ht}\right. \\ & + \left(9 + \frac{3}{\hat{s}^4} - \frac{6}{\hat{s}^2} - 6\,\hat{s}^2\right)\ln\hat{c}^2 + 3\left(5 - 6\,\hat{s}^2\right)\ln zt\ + 6\,I_3\,\hat{c}^2\end{aligned} \tag{15a}$$

$$+\left(2-\frac{4}{ht}-8\,\hat{s}^2+\frac{28\,\hat{s}^2}{ht}\right)\ln ht+\pi^2\left(-\frac{7}{3}-\frac{2}{3\,ht^2}+\frac{1}{ht}-\frac{56\,\hat{s}^2}{27}+\frac{2\,\hat{s}^2}{3\,ht^2}-\frac{\hat{s}^2}{ht}\right)$$
$$+\frac{12\,(-4+ht)\,\hat{s}^2}{ht}\Lambda\left(-1+\frac{4}{ht}\right)+\left(2\,ht\;\hat{c}^2-\frac{2\,(-2+3\,ht)\,\hat{c}^2}{ht^2}\right)Li_2(1-ht)$$
$$+\left(-2-\frac{8}{ht}+5\,\hat{s}^2+\frac{24\,\hat{s}^2}{ht^2}-\frac{10\,\hat{s}^2}{ht}+ht\,\hat{c}^2\right)\phi\left(\frac{ht}{4}\right)\Bigg], \tag{15b}$$

for $M_H \gg M_Z$, whilst in the region $M_H \ll M_Z$,

$$\delta\rho^{(2)}=19-2\,\pi^2-4\,\pi\sqrt{ht}+ht\left(-\frac{27}{2}+2\,\pi^2-6\,\ln ht-5\,\ln\hat{c}^2+3\,\ln zt\right)$$
$$+zt\left[-\frac{11}{2}+\frac{3}{\hat{s}^2}+\frac{319\,\hat{s}^2}{9}+6\;I_3\;\hat{c}^2+\pi^2\left(-\frac{7}{3}-\frac{56\,\hat{s}^2}{27}\right)\right.$$
$$\left.+\left(7+\frac{3}{\hat{s}^4}-\frac{6}{\hat{s}^2}-4\,\hat{s}^2\right)\ln\hat{c}^2+\left(21-16\,\hat{s}^2\right)\ln zt\right]. \tag{15c}$$

In Eqs. (15) $ht \equiv (M_H/m_t)^2$, $zt \equiv (M_Z/m_t)^2$,

$$g(x)=\begin{cases}\sqrt{4-x}\,\left(\pi-2\arcsin\sqrt{x/4}\right) & 0<x\le 4\\[2ex] 2\sqrt{x/4-1}\,\ln\left(\frac{1-\sqrt{1-4/x}}{1+\sqrt{1-4/x}}\right) & x>4\;,\end{cases} \tag{16a}$$

$$\Lambda(-1+\frac{4}{x})=\begin{cases}-\frac{1}{2\sqrt{x}}\,g(x)+\frac{\pi}{2}\,\sqrt{4/x-1} & 0<x\le 4\\[2ex] -\frac{1}{2\sqrt{x}}\,g(x) & x>4\;,\end{cases} \tag{16b}$$

$$Li_2(x)=-\int_0^x dt\frac{\ln(1-t)}{t}\quad, \tag{16c}$$

and

$$\phi(z)=\begin{cases}4\sqrt{\frac{z}{1-z}}\;Cl_2(2\arcsin\sqrt{z}) & 0<z\le 1\\ \frac{1}{\lambda}\left[-4Li_2(\frac{1-\lambda}{2})+2\ln^2(\frac{1-\lambda}{2})-\ln^2(4z)+\pi^2/3\right] & z>1\,,\end{cases} \tag{16d}$$

where $Cl_2(x) = \mathrm{Im}\, Li_2(e^{ix})$ is the Clausen function with

$$\lambda = \sqrt{1 - \frac{1}{z}} \,. \tag{16e}$$

The first two lines of eq. (15b) represent the leading $O(G_\mu^2 m_t^4)$ result [3], which is completely independent of the gauge sector of the theory. Indeed this part can be computed in the framework of a pure Yukawa theory, obtained from the SM in the limit of vanishing gauge coupling constants. The rest of eq. (15b) is proportional to $zt = M_Z^2/m_t^2$ and represents the first correction to the Yukawa limit. Eqs. (15) show a process-dependent contribution, i.e. $6\, zt\, I_3\, \hat{c}^2$ that comes from ${B_Z}^{(2)}$. This reflects the fact that, already at one-loop, the box diagrams in neutral current depend on the process under consideration [10] [cf. Eq. (11a)].

# 3 Numerical results

In the previous Section we derived the expression for the $\rho$ parameter up to $O(G_\mu^2 m_t^2 M_Z^2)$ in the $\overline{MS}$ scheme. We expressed our result in terms of the $\overline{MS}$ quantities $\hat{\alpha}$, $\hat{s}^2$, and the physical mass of the $Z$ boson. To obtain the corresponding expressions in terms of $G_\mu$ and the on-shell (OS) parameter $c^2 \equiv M_W^2/M_Z^2$ , we use the relations [8]

$$\frac{\hat{\alpha}}{4\pi \hat{s}^2} = \frac{G_\mu M_W^2}{2\sqrt{2}\pi^2} \, \frac{1 - \Delta \hat{r}_W}{1 + (\frac{2\delta e}{e})_{\overline{MS}}} \simeq \frac{G_\mu M_Z^2 c^2}{2\sqrt{2}\pi^2} \tag{17a}$$

$$\hat{c}^2 = c^2(1 - Y_{\overline{MS}}) \simeq c^2(1 - N_c x_t) \quad . \tag{17b}$$

Eq. (17b) will create additional contributions to $\delta\rho^{(2)}$. The one-loop result is then given by Eqs. (10) with the substitutions $\hat{\alpha}/(4\pi\hat{s}^2) \to (G_\mu M_Z^2 c^2)/(2\sqrt{2}\pi^2)$, $\quad \hat{s}^2,\, \hat{c}^2 \to s^2,\, c^2$, while for the two-loop contribution we have

$$\delta\rho^{(2)}_{OS} = \delta\rho^{(2)}(\hat{s}^2,\, \hat{c}^2 \to s^2,\, c^2) \\ + N_c x_t^2 zt \left[ -\frac{3c^4}{s^4} \ln c^2 - \frac{3c^2}{s^2} - 3I_3 + 12Q - 24 s^2(1+c^2) I_3 Q^2 + 4c^2 G'(\xi, c^2) \right] \tag{18a}$$

**Table 1**

$\delta\rho^{(2)}$ ($\overline{MS}$) and $\delta\rho^{(2)}_{OS}$ ($OS$) relevant to $\nu_\mu\, e$ scattering for $zt \equiv M_Z^2/m_t^2 = 0.2, 0.3$, in units $N_c x_t^2$ as a function of $r = M_H/m_t$. The column $zt = 0$ is the result of the Yukawa theory.

| | | $\overline{MS}$ | | $OS$ | |
|---|---|---|---|---|---|
| $r = \frac{M_H}{m_t}$ | $zt = 0$ | $zt = 0.2$ | $zt = 0.3$ | $zt = 0.2$ | $zt = 0.3$ |
| 0.1 | − 1.8 | −12.6 | −15.8 | −12.7 | −16.0 |
| 0.2 | − 2.7 | −13.3 | −16.5 | −13.5 | −16.8 |
| 0.3 | − 3.5 | −13.9 | −17.0 | −14.2 | −17.4 |
| 0.4 | − 4.1 | −14.5 | −17.6 | −14.9 | −18.1 |
| 0.5 | − 4.7 | −15.2 | −18.3 | −15.7 | −18.9 |
| 0.6 | − 5.2 | −16.1 | −20.2 | −16.7 | −20.9 |
| 0.7 | − 5.7 | −16.2 | −20.1 | −16.9 | −20.9 |
| 0.8 | − 6.2 | −16.4 | −20.1 | −17.1 | −21.0 |
| 0.9 | − 6.6 | −16.5 | −20.1 | −17.4 | −21.2 |
| 1.0 | − 6.9 | −16.6 | −20.1 | −17.6 | −21.3 |
| 1.1 | − 7.3 | −16.8 | −20.2 | −17.8 | −21.4 |
| 1.2 | − 7.6 | −16.9 | −20.2 | −18.0 | −21.6 |
| 1.3 | − 7.9 | −17.0 | −20.2 | −18.2 | −21.7 |
| 1.4 | − 8.2 | −17.2 | −20.3 | −18.4 | −21.9 |
| 1.5 | − 8.4 | −17.3 | −20.3 | −18.6 | −22.0 |
| 1.6 | − 8.7 | −17.4 | −20 4 | −18.7 | −22.1 |
| 1.7 | − 8.9 | −17.5 | −20.5 | −18.9 | −22.3 |
| 1.8 | − 9.1 | −17.6 | −20.5 | −19.1 | −22.4 |
| 1.9 | − 9.3 | −17.7 | −20.6 | −19.2 | −22.6 |
| 2.0 | − 9.5 | −17.8 | −20.6 | −19.4 | −22.7 |
| 2.5 | −10.2 | −18.2 | −20.9 | −20.0 | −23.3 |
| 3.0 | −10.8 | −18.4 | −20.8 | −20.4 | −23.5 |
| 3.5 | −11.2 | −18.3 | −20.6 | −20.6 | −23.6 |
| 4.0 | −11.4 | −18.3 | −20.4 | −20.6 | −23.5 |
| 4.5 | −11.6 | −18.2 | −20.1 | −20.6 | −23.4 |
| 5.0 | −11.7 | −18.0 | −19.8 | −20.5 | −23.3 |
| 5.5 | −11.8 | −17.8 | −19.4 | −20.4 | −23.1 |
| 6.0 | −11.8 | −17.5 | −19.0 | −20.3 | −22.9 |

where

$$G'(\xi, c^2) = \frac{3}{4}\,\xi\left[c^2\frac{\ln(c^2/\xi)}{(c^2-\xi)^2} - \frac{1}{c^2-\xi} + \frac{1}{c^2}\frac{\ln\xi}{1-\xi}\right] \quad . \tag{18b}$$

In Eq. (18a) $\delta\rho^{(2)}(\hat{s}^2,\, \hat{c}^2 \to s^2,\, c^2)$ represents a term obtained from Eqs. (15) applying the same substitutions as in the one-loop case.

From Eq. (18a) we notice that the process-dependence is more pronounced in the OS framework. This is easily understood by noticing that the expansion of the bare couplings in the one-loop box diagrams gives rise, unlike the $\overline{MS}$ case, to $m_t^2$ contributions.

In Fig. 1 we plot $\delta\rho^{(2)}$ [Eqs. (15)] as a function of $m_t$ for few values of $M_H$. As a comparison we also show the values obtained including only the $O(G_\mu^2 m_t^4)$ contribution.

The process under consideration is $\nu_\mu e$ scattering. From Figure 1 it is evident that the inclusion of corrections suppressed by a factor $M_Z^2/m_t^2$ with respect to the leading term is quite substantial.

To have a better understanding of the size of these corrections in Table 1 we present the values of $\delta\rho^{(2)}$ and $\delta\rho^{(2)}_{OS}$ for $zt = 0, 0.2$, and 0.3 as a function of $r = M_H/m_t$. When preparing the Table we matched the values from (15b) and (15c) when the latter were very close ($r \simeq 0.5$). We see that in the region of light Higgs the $O(G_\mu^2 m_t^2 M_Z^2)$ corrections are much larger than the $m_t^4$ term that is actually suppressed by accidental cancellations, while for large Higgs mass, in the TeV region, their contribution is still 50% of the leading part. It is worth noticing that the numbers shown in Table 1 are very close to the corresponding ones obtained in Ref. [11] in the case of a model with $SU(2)$ symmetry. That is not surprising $\hat{s}$ being a relatively small number ($\hat{s}^2 \simeq 0.23$).

# 4 Conclusions

We have seen that the calculation of the difference of self-energies is not sufficient to compute the $O(G_\mu^2 m_t^2 M_Z^2)$ corrections to the $\rho$ parameter [cf. Eq. (13)] but one has to resort to physical processes and this introduces process-dependent quantities. Our result, being obtained at $q^2 = 0$, cannot be directly applied to LEP physics. However one can ask general questions about the two-loop electroweak corrections involving the top and use the answers coming from the calculation of $\delta\rho^{(2)}$ as a "ringing bell" for the estimation of the theoretical error in the present knowledge of these corrections.

It is natural to ask whether we can expect that the $O(G_\mu^2 m_t^4)$ term will approximate well the complete unknown result for values of $m_t$ not larger than 250 GeV. Table 1 shows that in the case of $\delta\rho^{(2)}$ the answer is negative. We have looked for the asymptotic regime of the top, namely for which value of $m_t$ $\delta\rho^{(2)}$ begins to be close to the $O(G_\mu^2 m_t^4)$ contribution. We found that, typically, $\delta\rho^{(2)}$ starts to be within 10% the leading $m_t^4$ value for $m_t \simeq 800$ GeV.

To consider the top as an asymptotically heavy particle can be an unrealistic assumption also for electroweak quantities of LEP interest, like $\Delta r$ [12] and $\Delta\hat{r}$ [8, 13]. It is then important to have a feeling of how large the theoretical error one is making can be when

**Table 2**

Calculated ratio ($R$), for few values of $m_t$ and $M_H$, between the $O(G_\mu^2 m_t^2 M_Z^2)$ and the $O(G_\mu^2 m_t^4)$ contributions in $\delta\rho^{(2)}$. The corresponding estimate of the shifts in the W mass and $\sin^2\theta_{\rm eff}^{\rm lep}$ are also presented (see text).

| $m_t$ (GeV) | $M_H$ (GeV) | $R$ % | $\Delta M_W$ (MeV) | $\Delta\sin^2\theta_{\rm eff}^{\rm lep}$ ($10^{-4}$) |
|---|---|---|---|---|
| 150 | 65 | 247 | –10 | 0.6 |
| | 250 | 100 | –8 | 0.5 |
| | 800 | 35 | –4 | 0.2 |
| 175 | 65 | 234 | –16 | 0.9 |
| | 250 | 94 | –14 | 0.8 |
| | 800 | 38 | –8 | 0.5 |
| 200 | 65 | 221 | –23 | 1.4 |
| | 250 | 88 | –20 | 1.2 |
| | 800 | 38 | –13 | 0.7 |

these quantities are computed including only the $O(G_\mu^2 m_t^4)$ correction. A possible way to obtain this is to assume that the ratio between the $O(G_\mu^2 m_t^2 M_Z^2)$ and the $O(G_\mu^2 m_t^4)$ contributions in $\delta\rho^{(2)}$ can be representative of the unknown two-loop top effects in $\Delta r$ and $\Delta\hat{r}$. We can then use this ratio to estimate the additional contributions to $\Delta r$ and $\Delta\hat{r}$ simply multiplying it by the known $O(G_\mu^2 m_t^4)$ terms of these quantities. The shifts in the W mass and the effective sinus, $\sin^2\theta_{\rm eff}^{\rm lep}$, due to these additional contributions can be estimated from the relations

$$\frac{\Delta M_W}{M_W} = -\frac{s^2}{2(c^2-s^2)}\,\delta(\Delta r)$$

$$\Delta\sin^2\theta_{\rm eff}^{\rm lep} = \frac{\hat{s}^2\hat{c}^2}{\hat{c}^2-\hat{s}^2}\,\delta(\Delta\hat{r}) + \hat{s}^2\delta\hat{k}_l(M_Z^2) \quad ,$$

where the correction $\hat{k}_l$ is defined in [14].

In Table 2 we show, for few values of $m_t$ and $M_H$, the effect of our estimate of the unknown top contributions on the W mass and $\sin^2\theta_{\rm eff}^{\rm lep}$ . In our estimate we have put $\delta\hat{k}_l = 0$. The ratio between subleading and leading terms in $\delta\rho^{(2)}$ has been computed using expressions slightly different from Eqs. (15). In fact, we decided to maximize the one-loop result of our $\overline{MS}$ calculation by writing it in terms of the physical masses of both W and Z. Such a procedure is frequently used in one-loop calculations [8], and in our case has the further advantage of eliminating the process-dependent terms. From the third column, it can immediately be seen that, for a fixed value of the top mass, the effect is more pronounced for light Higgs. This is not surprising, bearing in mind the fact that the $O(G_\mu^2 m_t^4)$ term is a monotonically increasing (in modulus) function of $M_H$.

We want to stress that the numbers presented in Table 2, more than a definite estimate of the shifts in $M_W$ and $\sin^2\theta_{\rm eff}^{\rm lep}$ should be taken as an indication that subleading two-loop $m_t$ effects could be larger than what is "naïvely" expected. Their size is probably comparable to, or may be larger than, the theoretical uncertainty due to the hadronic contribution to the photonic self-energy. The latter amounts to $\pm16$ MeV and $\pm3\times10^{-4}$ in $M_W$ and $\sin^2\theta_{\rm eff}^{\rm lep}$ , respectively.

To conclude, we think that our calculation of $\delta\rho^{(2)}$ shows that it is questionable to believe that two-loop electroweak top contributions are well approximated by the $O(G_\mu^2 m_t^4)$ term and therefore sufficiently under control. However, the possibility of establishing top effects of a purely electroweak nature at the two-loop level seems quite remote. The experimental accuracy envisaged for the W mass is $(\delta M_W)_{\rm exp} = \pm50$ MeV, whilst $\sin^2\theta_{\rm eff}^{\rm lep}$ is presently known with a precision $(\delta\sin^2\theta_{\rm eff}^{\rm lep})_{\rm exp} \equiv \pm4\times10^{-4}$. At this level of precision it is likely that only QCD corrections to one-loop top contribution can be relevant. However, if the experimental precision improves in the future to reach $(\delta\sin^2\theta_{\rm eff}^{\rm lep})_{\rm exp} = \pm2\times10^{-4}$, or $\pm1\times10^{-4}$, then a meaningful theoretical interpretation will require a complete study of two-loop top effect of electroweak nature.


## Acknowledgements

The authors are indebted to A. Sirlin and M. Tonin for valuable discussions. One of us (P.G.) would like to thank the University of Padova for financial support, and its Physics Department for its warm hospitality during his staying in Padua.


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