The Refined Elliptic Genus and Coulomb Gas Formulations * of $N = 2$ Superconformal Coset Models

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We describe, in some detail, a number of different Coulomb gas formulations of $N = 2$ superconformal coset models. We also give the mappings between these formulations. The ultimate purpose of this is to show how the Landau-Ginzburg structure of these models can be used to extract the W-generators, and to show how the computation of the elliptic genus can be refined so as to extract very detailed information about the characters of component parts of the model.

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1. Introduction

It has long been known that the $N = 2$ superconformal coset models based upon:

$$\mathcal{G}_n \;=\; \frac{SU_k(n+1) \times SO(2n)}{SU_{k+1}(n) \times U(1)} \tag{1.1}$$

have Landau-Ginzburg formulations and also have an underlying $N = 2$ super-$W_{(n+1)}$ chiral algebra. It is natural to ask if one can determine, solely from a Landau-Ginzburg formulation, whether or not the corresponding $N = 2$ superconformal model has such an extended chiral algebra. Moreover, given that a model has an $N = 2$ super-W algebra, one would like to know to what extent one can determine the spectrum of the zero-modes of the chiral algebra by using the Landau-Ginzburg structure alone.

Techniques by which one can answer these questions were introduced in [1,2]. In particular, a number of the Ramond sector characters can be extracted from the elliptic genus, and it was shown how the latter can easily be calculated from the Landau-Ginzburg formulation. Similar computations of Ramond characters, but refined by $N = 2$, $U(1)$ charge, were performed for more complex models in [3-5]. However for models with central charge $c \geq 3$, the elliptic genus, even when refined by the $U(1)$ charge, is too coarse to determine the complete structure of the Hilbert spaces constructed above the Ramond ground states. To completely characterize these Hilbert spaces, one needs to look for extended chiral algebras and then appropriately refine the elliptic genus as in [6].

Using methods of [2], it was shown in [8] that at the classical level, the form of the superpotential determines whether there is a super-W algebra acting upon the elliptic cohomology of the theory. Quantum versions of these results were obtained in [6,7], and in [6] it was also shown how the elliptic genus could be refined to yield much more complete information about the structure of the Ramond sector Hilbert space.

Our primary purpose in this paper is to expand and develop the results of our earlier letter [6]. In addition to doing this we also wish to discuss the relationships between the many fomulations of the coset model (1.1). This will be done in section 3. In doing this we will encounter an interesting feature of Coulomb gas formulations tensor products of conformal models with a special choice of modular invariant. Partly for its own interest, and partly as preparation for section 3, we will exhibit a simple example of this feature in section 2. Indeed, section 2 can be read independently of the rest of the paper.

In section 4, we will descibe, as simply as possible, the $N = 2$ super-W algebra by giving an explicit method for constructing the lowest components of each of the chiral algebra superfields. We then re-express this in terms of Landau-Ginzburg fields and show how it can be used to refine the elliptic genus completely with respect to the super-W algebra. In section 5 we expand the fully refined elliptic genus and obtain formulae for the branching functions that make up the model. Finally, we discuss some issues about fermionic screening currents in the Coulomb gas formulation, and indicate how this might possibly be used to extract information about the modular invariant partition function of the complete model.

2. Coulomb gas formulations of special tensor product models

The model that we wish to consider in this section is the coset theory $\frac{SU(2)_k \times SU(2)_1 \times SU(2)_1}{SU(2)_{k+2}}$. This can be written in terms of a tensor product of minimal models:

$$\mathcal{M}_1 \times \mathcal{M}_2 \;=\; \frac{SU(2)_k \times SU(2)_1}{SU(2)_{k+1}} \times \frac{SU(2)_{k+1} \times SU(2)_1}{SU(2)_{k+2}} \quad . \tag{2.1}$$

However, one has to remember that to recover the original model, the reperesentation of the denominator factor of $SU_{k+1}(2)$ in $\mathcal{M}_1$ must always be the same as that of the numerator factor of $SU_{k+1}(2)$ in $\mathcal{M}_2$. This "locking together" of representations defines a special modular invariant of the tensor product model.

The stress-tensors for the Coulomb gas formulation of $\mathcal{M}_1$ and $\mathcal{M}_2$ are

$$\begin{aligned} T_1(z) \;&=\; -\tfrac{1}{2}(\partial\phi_1)^2 + i(\alpha_+ - \alpha_-)\partial^2\phi_1 \\ T_2(z) \;&=\; -\tfrac{1}{2}(\partial\phi_2)^2 + i(\beta_+ - \beta_-)\partial^2\phi_2 \quad , \end{aligned} \tag{2.2}$$

where we define:

$$\alpha_\pm \;=\; \left(\frac{k+2}{k+3}\right)^{\pm\frac{1}{2}} \qquad \text{and} \qquad \beta_\pm \;=\; \left(\frac{k+3}{k+4}\right)^{\pm\frac{1}{2}} \quad . \tag{2.3}$$

The primary fields in the two models can be represented in terms of vertex operators:

$$\begin{aligned} V^{(1)}_{m,n} \;&=\; e^{-\frac{i}{\sqrt{2}}(m\alpha_+ - n\alpha_-)\phi_1} \quad , \\ V^{(2)}_{m,n} \;&=\; e^{-\frac{i}{\sqrt{2}}(m\beta_+ - n\beta_-)\phi_2} \quad , \end{aligned} \tag{2.4}$$

whose conformal dimensions are given by:

$$\begin{aligned} \Delta^{(1)}_{m,n} \;&=\; \tfrac{1}{4}[(m+1)\alpha_+ - (n+1)\alpha_-]^2 - \tfrac{1}{4}(\alpha_+ - \alpha_-)^2 \ , \\ \Delta^{(2)}_{m,n} \;&=\; \tfrac{1}{4}[(m+1)\beta_+ - (n+1)\beta_-]^2 - \tfrac{1}{4}(\beta_+ - \beta_-)^2 \quad . \end{aligned} \tag{2.5}$$

In particular, the vertex operators:

$$W^{(i)} \equiv V^{(i)}_{-2,-2} \tag{2.6}$$

are the dual representatives of the vacuum states. The screening currents in $\mathcal{M}_1$ and $\mathcal{M}_2$ have the form

$$R^{\pm} = e^{\pm i\sqrt{2}\alpha_{\pm}\phi_1} \qquad \text{and} \qquad S^{\pm} = e^{\pm i\sqrt{2}\beta_{\pm}\phi_2} \ . \tag{2.7}$$

The tensor product of the minimal models can be constructed using the screening currents $R^{\pm}$ and $S^{\pm}$ independently. The tensor product then has an obvious spin-2 element of the chiral algebra:

$$S(z) = c_2 T_1(z) - c_1 T_2(z) \ . \tag{2.8}$$

The coefficients, c_1 and c_2, are the central charges of the two minimal models, and the foregoing combination of T_1 and T_2 is a good conformal field with respect to the total stress-tensor.

The locking together of representations of $SU_{k+1}(2)$ in the tensor product means that the allowed vertex operators have the form:

$$V_{m,n,p} \equiv V^{(1)}_{m,n}\, V^{(2)}_{p,m} = e^{\frac{i}{\sqrt{2}}[(n\alpha_-\phi_1 - p\beta_+\phi_2) - m(\alpha_+\phi_1 - \beta_-\phi_2)]} \ . \tag{2.9}$$

Consider the operators:

$$\mathcal{X}_r \equiv V^{(1)}_{-2r,0}\, V^{(2)}_{0,-2r} = e^{i\sqrt{2}r(\alpha_+\phi_1 - \beta_-\phi_2)} \ . \tag{2.10}$$

Note that $\mathcal{X}_1 = R^+ S^-$. The operator $\mathcal{X}_r$ has dimension $2r^2$ and is local with respect to all of the vertex operators $V_{m,n,p}$. Thus the operators $\mathcal{X}_r$ can be thought of as elements of an extended chiral algebra in the free bosonic theory. They are also local with respect to R^- and S^+, but not with respect to R^+ and S^-.

Generally the operators $\mathcal{X}_r$ do not play any role after one has reduced to the simple tensor product of minimal conformal models. However, if one follows the spirit of locking the $SU_{k+1}(2)$ representations together, then it is much more natural to introduce the screening charges:

$$\begin{aligned}
Q^{(1)}_{+} &= \oint V_{-2,0,-2} = \oint R^+ W^{(2)} = \oint e^{i\sqrt{2}\alpha_+\phi_1} e^{i\sqrt{2}(\beta_+ - \beta_-)\phi_2} \\
Q^{(1)}_{-} &= \oint V_{0,-2,0} = \oint R^- = \oint e^{-i\sqrt{2}\alpha_-\phi_1} \\
Q^{(2)}_{+} &= \oint V_{0,0,-2} = \oint S^+ = \oint e^{i\sqrt{2}\beta_+\phi_2} \\
Q^{(2)}_{-} &= \oint V_{-2,-2,0} = \oint W^{(1)} S^- = \oint e^{i\sqrt{2}(\alpha_+ - \alpha_-)\phi_1} e^{-i\sqrt{2}\beta_-\phi_2}
\end{aligned} \tag{2.11}$$

Observe that the current in $Q_+^{(1)}$ is simply R^+ multiplied by the dual vacuum vector of $\mathcal{M}_2$, and similarly for $Q_-^{(2)}$. One then finds that (2.8) no longer commutes with the screening charges. However, define

$$S \ = \ c_2 T_1 \ - \ c_1 T_2 \ + \ \xi R^+ S^- \quad , \tag{2.12}$$

for some constant ξ. One then finds

$$\begin{aligned} &[\ Q_-^{(1)} \ + \ \zeta \ Q_-^{(2)} \ , \ S \ ] \ = 0 \\ &[\ \zeta \ Q_+^{(1)} \ + \ Q_+^{(2)} \ , \ S \ ] \ = 0 \quad , \end{aligned} \tag{2.13}$$

where $\zeta \ = \ \frac{\beta_+ \xi}{(c_1+c_2)(\beta_+ - \beta_-)} \ = \ \frac{\alpha_- \xi}{(c_1+c_2)(\alpha_+ - \alpha_-)}$. Thus in the locked model the screening currents involve the dual vacuum vectors, and the naive representations of non-trivial elements of the chiral algebra receive nilpotent vertex operator corrections.

In the next section we will encounter examples of the foregoing "locked" tensor product model. Moreover, they naturally come equipped with screening charges analogous to (2.11). We will find it convenient to convert this description into the simple version of the tensor product where one has to remember to lock the representations, but in which the screening currents are not mixed with dual vacua, and in which the chiral algebra contains no nilpotent vertex operator parts.

3. The multifarious formulations of the $N = 2$ super-W_{n+1} models

The first and most obvious formulation of the model (1.1) is as a coset model [14]. There is also a formulation that comes from Drinfel'd-Sokolov reduction [12,13]. There is the Landau-Ginzburg formulation [9,10], and a related Coulomb gas description [11,2]. There is also a Coulomb gas description of the model considered as a tensor product. We will consider all of these formulations here, and describe how they are related. We begin with the coset formulation.

3.1. The structure of the coset model

We will not review the details of [14], but simply wish to describe some of the general structure of the that can be seen from the coset formulation.

We begin by observing that the model (1.1) can be written as a tensor product:

$$\begin{aligned} \frac{SU_k(n+1)\times SO(2n)}{SU_{k+1}(n)\times U(1)} &= \mathcal{M}_1 \ \times \ \mathcal{M}_2 \ \times \ U(1) \\ &= \frac{SU_k(n+1)}{SU_k(n)\times U(1)} \ \times \ \frac{SU_k(n)\times SU_1(n)}{SU_{k+1}(n)} \ \times \ U(1) \ , \end{aligned} \tag{3.1}$$

where $\mathcal{M}_2$ is a non-supersymmetric W_n-model. However, exactly as in the last section, the representations of $SU_k(n)$ in $\mathcal{M}_1$ and $\mathcal{M}_2$ are locked together.

The chiral algebra of the $N = 2$ supersymmetric model contains a non-supersymmetric W_n subalgebra. We now wish to argue that for k sufficiently large, the generators of this W_n, along with the $U(1)$ current, provide a complete set of lowest spin (bottom) components of the $N = 2$ superfields that make up the full $N = 2$ chiral algebra. In doing this we will also elucidate a duality between the chiral algebra and chiral ring of (1.1). This has been well known for some time [15], and the classical version of it has been described in [8].

The first thing to observe is that, for k large, there are n independent supermultiplets in the $N = 2$ chiral algebra, and that the spins of the lowest components are $1, \ldots, n$. This matches the spins of the generators of the $W_n \times U(1)$ algebra. It also matches the $N = 2$, $U(1)$ charges of the generators of the chiral ring. Next, one considers an equivalent formulation of the coset model [14]:

$$\begin{aligned} \frac{G\times SO(dim(G/H)}{H} &\equiv \frac{SU_1(k+n+1)\times SO(2kn)}{SU_{n+1}(k)\times SU_{k+1}(n)\times U(1)} \\ &= \frac{SU_1(k)\times SU_n(k)}{SU_{n+1}(k)} \ \times \ \frac{SU_1(n)\times SU_k(n)}{SU_{k+1}(n)} \ \times \ U(1) \ . \end{aligned} \tag{3.2}$$

Note that the second factor is $\mathcal{M}_2$. One can define this model entirely in terms of free bosons [20]. The elements of the chiral algebra can be represented by those polynomials in derivatives of the bosons that are invariant (up to total derivatives) under the Weyl group, $W(H_0)$, of $H_0 = SU(k)\times SU(n)$ [16–20]. In [20] it was shown that the supercurrents could be represented by vertex operators that are related to screening currents via the action of the maximal cyclic generator of the Weyl group, $W(G)$, of G. Thus the top components of superfields are those polynomials in derivatives of bosons that are invariant (up to total derivatives) under the action of $W(G)$. This is because $W(G)$ invariance means that the polynomial will commute (up to total derivatives) with the supercharges.

There is now a natural finite ring structure that we can define upon the chiral algebra: consider all the $W(H_0)$ invariant polynomials, modulo the $W(G)$ invariant polynomials.

This set consists of all the polynomials in elements of the chiral algebra such that these polynomials are not top components of superfields. This is also the characterization, in terms of 'Cartan subalgebra variables,' of the chiral ring of the $N = 2$ superconformal model [9,21]. It is also a well know fact that the foregoing ring can be generated by restricting to the bosons that correspond to either the first or second factors in (3.2). (This fact was used in [9] to construct the Landau-Ginzburg potential for the model.) Thus, this finite quotient ring of the chiral algebra is isomorphic to the chiral ring. Moreover, the ring can be generated by the chiral algebra generators of the second factor of (3.2), that is, by the W_n algebra of $\mathcal{M}_2$. Therefore, the task of finding representatives of the $N = 2$ superconformal chiral algebra is complete once we have the supercharges and either the W_n algebra, or some free bosonic realization of $\mathcal{M}_2$.

The fact that a description of the chiral ring can be mapped onto the foregoing quotient ring of the chiral algebra will not be important to this paper, and we have included merely for interest's sake. We feel that one should be able to establish this relationship more directly within the superconformal model itself, and that one should be able to use it to understand the conserved charge structure discussed in [21] for solitons of the quantum integrable, off-critical models based upon (1.1).

3.2. Supersymmetric Drinfel'd – Sokolov reduction

The free superfield formulation of (1.1) can be obtained from the Lie superalgebra $A(n, n-1)$ through a Hamiltonian reduction [12,13]. Before describing the free field formulation we first review some basic properties of the super Lie algebra $A(n, n-1)$ that are relevant to our discussion [22].

The algebra $A(n, n-1)$ has a $\mathbb{Z}_2$-grading under which roots are viewed as either even or odd. If we denote the simple roots by $\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}, \alpha_{2n}$, then the even roots are:

$$\alpha_i + \alpha_{i+1} + \ldots + \alpha_{i+2k-2} + \alpha_{i+2k-1} \qquad k = 1, 2, \ldots, \left[\frac{2n+1-i}{2}\right] , \tag{3.3}$$

and the odd roots are:

$$\alpha_i + \alpha_{i+1} + \ldots + \alpha_{i+2k-1} + \alpha_{i+2k} \qquad k = 0, 1, \ldots, \left[\frac{2n-i}{2}\right] . \tag{3.4}$$

The simple roots of $A(n, n-1)$ satisfy the following relations:

$$\alpha_{2i-1} \cdot \alpha_{2i} \;=\; 1 \; ; \qquad \alpha_{2i+1} \cdot \alpha_{2i} \;=\; -1 \; . \tag{3.5}$$

All other inner products are zero (including $\alpha_i \cdot \alpha_i$). The fundamental weights $\lambda_1, \ldots, \lambda_{2n}$ are defined by:

$$\alpha_i \cdot \lambda_j \;=\; \delta_{ij} \quad . \tag{3.6}$$

It is easy to see from (3.5), that in terms of the simple roots, the fundamental weights are given by:

$$\begin{aligned} \lambda_{2i} \;&=\; \alpha_1 + \alpha_3 + \ldots + \alpha_{2i-3} + \alpha_{2i-1} \;, \\ \lambda_{2i-1} \;&=\; \alpha_{2i} + \alpha_{2i+2} + \ldots \alpha_{2n-2} + \alpha_{2n} \;. \end{aligned} \tag{3.7}$$

The super Lie algebra $A(n, n-1)$ contains the even subalgebras A_n and A_{n-1}. The simple roots of these two subalgebras are given respectively by:

$$\alpha_{2i-1} + \alpha_{2i} \;, \quad i = 1, \ldots, n; \quad \text{and} \quad \alpha_{2i} + \alpha_{2i+1} \;, \quad i = 1, \ldots, n-1 \;. \tag{3.8}$$

From (3.5) we see that the root system for A_n has a positive definite metric, whereas for A_{n-1}, the metric is negative definite.

To write down the free superfield description of (1.1) it is most convenient to use an $N = 1$ superfield formulation. We therefore introduce a single anti-commuting coordinate θ, and define the super-derivative, D by:

$$D \;=\; \frac{\partial}{\partial\theta} + \theta \frac{\partial}{\partial z} \;. \tag{3.9}$$

Consider $2n$ (real) superfields

$$\Phi^i(z, \theta) \;=\; \varphi^i(z) + \theta\; \chi^i(z) \;, \tag{3.10}$$

where $\varphi^i(z)$ is a free bosonic field and $\chi^i(z)$ is a free, real fermion. These superfields satisfy the operator product expansion:

$$\Phi^i(z_1, \theta_1)\; \Phi^j(z_2, \theta_2) \;=\; -\; \delta^{ij}\; log(z_{12}) \;, \tag{3.11}$$

where $z_{12} \;\equiv\; z_1 - z_2 - \theta_1\theta_2$. In terms of components, we have:

$$\varphi_i(z)\; \varphi_j(w) \;=\; -\; \delta^{ij}\; log(z - w) \;, \qquad \chi_i(z)\; \chi_j(w) \;=\; -\; \delta^{ij}\; \frac{1}{(z - w)} \;. \tag{3.12}$$

The generators of the extended chiral algebra are then obtained from the Lax operator [12,13]:

$$L \;=\; \prod_{j=1}^{2n+1} \Big[\; i\alpha_0 D \;-\; (-1)^j (\lambda_j - \lambda_{j-1}) \cdot D\Phi \;\Big] \;, \tag{3.13}$$

where $\lambda_0 \equiv \lambda_{2n+1} \equiv 0$.

The parameter α_0 is background charge of Feigin-Fuchs representation. In order to reproduce (1.1), whose central charge is $c = \frac{3kn}{k+n+1}$, we must set

$$\alpha_0 \ = \ \frac{1}{\sqrt{k+n+1}} \ . \tag{3.14}$$

In the $N=1$ superfield formulation the stress tensor $T(z)$ is the top component of an $N=1$ superfield $\mathcal{T}(z,\theta)$ with conformal dimension $3/2$,

$$\mathcal{T}(z,\theta) \ = \ \tfrac{1}{2}\big(\ G^{+}(z) \ + \ G^{-}(z) \ \big) \ + \ \theta \ T(z) \ . \tag{3.15}$$

The fields $G^{\pm}(z)$ in (3.15) are the two supersymmetry generators of the $N=2$ supersymmetry algebra. The $U(1)$ current, $J(z)$, of the $N=2$ algebra is the lowest component of the superfield $\mathcal{J}(z,\theta)$

$$\mathcal{J}(z,\theta) \ = \ J(z) \ + \ \theta \ \tfrac{1}{2}\big(\ G^{+}(z) \ - \ G^{-}(z) \ \big) \ . \tag{3.16}$$

The free field forms of these superfields are obtained from the quadratic and linear parts of the Lax operator. One finds:

$$\begin{aligned} \mathcal{T}(z,\theta) \ = \ & -\tfrac{1}{2} \ \sum_{i=1}^{n} \lambda_{2i}\cdot D\Phi^{i} \alpha_{2i}\cdot \partial\Phi^{i} \ - \ \tfrac{1}{2} \ \sum_{i=1}^{n} \alpha_{2i}\cdot D\Phi^{i} \lambda_{2i}\cdot \partial\Phi^{i} \\ & - \ \frac{i}{2\sqrt{k+n+1}} \ \sum_{i=1}^{2n} \lambda_i \cdot D^3\Phi \ . \end{aligned} \tag{3.17}$$

and

$$\mathcal{J}(z,\theta) \ = \ \sum_{i=1}^{n} \big(\lambda_{2i}\cdot D\Phi\big)\big(\alpha_{2i}\cdot D\Phi\big) \ - \ \frac{i}{\sqrt{k+n+1}} \ \sum_{i=1}^{n} (\lambda_{2i}-\lambda_{2i-1})\cdot\partial\Phi \ . \tag{3.18}$$

To define the conformal model fully, we need the screening operators. These are in one-to-one correspondence with the roots of the Lie superalgebra $A(n,n-1)$ and its even subalgebras A_n and A_{n-1}. The screening operators corresponding to the roots of A_n have the form:

$$Q_{\alpha_{2i-1}+\alpha_{2i}} \ = \ \oint dz d\theta \ (\alpha_{2i}-\alpha_{2i-1})\cdot D\Phi \ e^{-\frac{i}{\sqrt{k+n+1}}(\alpha_{2i-1}+\alpha_{2i})\cdot\Phi} \ , \tag{3.19}$$

while the screening operators corresponding to the roots of A_{n-1} have the form:

$$Q_{\alpha_{2i}+\alpha_{2i+1}} \;=\; \oint dz d\theta \; (\alpha_{2i}-\alpha_{2i+1})\cdot D\Phi \; e^{+\frac{i}{\sqrt{k+n+1}}(\alpha_{2i}+\alpha_{2i+1})\cdot\Phi} \; . \tag{3.20}$$

These screening operators are usually called D-type screeners. The screening operators associated the simple root of $A(n,n-1)$, are usually referred to as F-type, or "fermionic," screening operators, and these have the form:

$$Q_{\alpha_i} \;=\; \oint dz d\theta \; e^{i\sqrt{k+n+1}\;\alpha_i\cdot\Phi} \; . \tag{3.21}$$

It is relatively easy re-express the foregoing in a manifestly $N=2$ supersymmetric formalism. We will adopt the following $N=2$ superfield conventions. First introduce

$$D^{\pm} \;=\; \frac{\partial}{\partial\theta^{\mp}} + \theta^{\pm}\frac{\partial}{\partial z} \qquad \bar{D}^{\pm} \;=\; \frac{\partial}{\partial\bar{\theta}^{\mp}} + \bar{\theta}^{\pm}\frac{\partial}{\partial\bar{z}} \; . \tag{3.22}$$

These satisfy

$$\{D^+,D^-\} \;=\; 2\;\frac{\partial}{\partial z}\;; \qquad \{\bar{D}^+,\bar{D}^-\} \;=\; 2\;\frac{\partial}{\partial\bar{z}} \; . \tag{3.23}$$

Let $\Phi_i^+(z,\theta^+,\theta^-)$ denote a set of n holomorphic, chiral bosonic superfields. That is, they satisfy

$$D^-\;\Phi_i^+ \;=\; 0\;; \qquad \bar{D}^{\pm}\;\Phi_i^+ \;=\; 0\; . \tag{3.24}$$

Similarly, Φ_i^- will denote conjugate anti-chiral bosonic superfields, that satisfy

$$D^+\;\Phi_i^- \;=\; 0\;; \qquad \bar{D}^{\pm}\;\Phi_i^- \;=\; 0\; . \tag{3.25}$$

In terms of components, $\Phi_j^{\pm}$ can expanded as follows:

$$\begin{aligned} \Phi_j^+(z,\theta^+,\theta^-) \;&=\; \phi_j(z) \;+\; \sqrt{2}\,\theta^-\;\psi_j(z) \;+\; \theta^-\theta^+\;\partial\phi_j(z) \\ \Phi_j^-(z,\theta^+,\theta^-) \;&=\; \bar{\phi}_j(z) \;+\; \sqrt{2}\,\theta^+\;\bar{\psi}_j(z) \;-\; \theta^-\theta^+\;\partial\bar{\phi}_j(z)\; . \end{aligned} \tag{3.26}$$

We take the operator product to be

$$\Phi_i^{\pm}(z_1,\theta_1^+,\theta_1^-)\;\Phi_j^{\mp}(z_2,\theta_2^+,\theta_2^-) \;\sim\; -\,\delta_{ij}\;log(\tilde{z}_{12}\;\pm\;\theta_{12}^-\theta_{12}^+)\;, \tag{3.27}$$

where $\theta_{12}=\theta_1-\theta_2$ and $\tilde{z}_{12}=z_1-z_2-\theta_1^+\theta_2^- -\theta_1^-\theta_2^+$.

For the component fields this means that

$$\begin{aligned} \phi_i(z_1)\;\bar{\phi}_j(z_2) \;&\sim\; -\,\delta_{ij}\;log(z_1-z_2)\;, \\ \psi_i(z_1)\;\bar{\psi}_j(z_2) \;&\sim\; -\,\delta_{ij}\;\frac{1}{z_1-z_2}\;. \end{aligned} \tag{3.28}$$

To relate these to the $N = 1$ superfields, we first note that the weight space of $A(n, n-1)$ has a natural complex basis spanned by α_{2j} and λ_{2j}, $j = 1, \ldots, n$. These vectors satisfy:

$$\alpha_{2i} \cdot \alpha_{2j} \ = \ 0 \ ; \quad \lambda_{2i} \cdot \lambda_{2j} \ = \ 0 \ ; \quad \alpha_{2i} \cdot \lambda_{2j} \ = \ \delta_{ij} \ . \tag{3.29}$$

We can then identify the complex components of the $N = 2$ superfields with the $N = 1$ components according to:

$$\begin{aligned} \phi_j \ &= \ \lambda_{2j} \cdot \varphi \ ; \qquad & \bar{\phi}_j \ &= \ \alpha_{2j} \cdot \varphi \ , \\ \psi_j \ &= \ \lambda_{2j} \cdot \chi \ ; \qquad & \bar{\psi}_j \ &= \ \alpha_{2j} \cdot \chi \ . \end{aligned} \tag{3.30}$$

With these superfields, and using (3.18) and (3.7), one can write the complete energy momentum tensor as:

$$\mathcal{J} \ = \ +\tfrac{1}{4} \ \sum_{j=1}^{n} \ (D^+ \Phi_j^+) \ (D^- \Phi_j^-) \ - \ i \ \frac{\alpha_0}{2} \sum_{j=1}^{n} \left[\ \partial \Phi_j^+ \ - \ j \ \partial \Phi_j^- \ \right] . \tag{3.31}$$

The components of $\mathcal{J}$ are simply the $N = 2$ superconformal generators, and they can be read off from the expansion:

$$\mathcal{J} \ = \ \tfrac{1}{2} \ J \ + \ \tfrac{1}{\sqrt{2}} \ \theta^+ G^+ \ - \ \tfrac{1}{\sqrt{2}} \ \theta^- G^- \ + \ \theta^- \theta^+ T \ . \tag{3.32}$$

Explicitly, one has:

$$\begin{aligned} J(z) \ &= \ - \ \sum_{j=1}^{n} \left[\bar{\psi}_j(z) \ \psi_j(z) \ + \ i\alpha_0 \partial \phi_j(z) - ij\alpha_0 \partial \bar{\phi}_j(z) \right] \\ G^+(z) \ &= \ \sum_{j=1}^{n} \left[\bar{\psi}_j(z) \partial \phi_j(z) \ + \ ij\alpha_0 \ \partial \bar{\psi}_j(z) \right] \\ G^-(z) \ &= \ \sum_{j=1}^{n} \left[\psi_j(z) \partial \bar{\phi}_j(z) \ + \ i\alpha_0 \ \partial \psi_j(z) \right] \\ T(z) \ &= \ - \sum_{j=1}^{n} \left[(\partial \phi_j)(\partial \bar{\phi}_j) \ - \ \tfrac{1}{2}(\psi_j \partial \bar{\psi}_j + \bar{\psi}_j \partial \psi_j) \ + \ i\frac{\alpha_0}{2} \partial^2 \phi_j \ + \ ij\frac{\alpha_0}{2} \partial^2 \bar{\phi}_j \right] . \end{aligned} \tag{3.33}$$

The screening operators may also be similarly translated:

$$
\begin{aligned}
Q_{\alpha_{2i-1}+\alpha_{2i}} &= \oint dz\ \big[(\partial\bar{\phi}_i - \partial\phi_i + \partial\phi_{i-1}) + \\
&\qquad\qquad 2i\alpha_0(\bar{\psi}_i\psi_i - \bar{\psi}_i\psi_{i-1})\big]\ e^{-i\alpha_0(\bar{\phi}_i+\phi_i-\phi_{i-1})}\ , \\
Q_{\alpha_{2i}+\alpha_{2i+1}} &= \oint dz\ \big[(\partial\bar{\phi}_i + \partial\phi_i - \partial\phi_{i+1}) + \\
&\qquad\qquad 2i\alpha_0(\bar{\psi}_i\psi_i - \bar{\psi}_i\psi_{i+1})\big]\ e^{+i\alpha_0(\bar{\phi}_i-\phi_i+\phi_{i+1})}\ , \\
Q_{\alpha_{2i}} &= \oint dz\ \bar{\psi}_i\ e^{i\sqrt{k+n+1}\ \bar{\phi}_i}\ , \\
Q_{\alpha_{2i-1}} &= \oint dz\ (\psi_i\ -\ \psi_{i-1})\ e^{i\sqrt{k+n+1}\ (\phi_i-\phi_{i-1})}\ ,
\end{aligned}
\tag{3.34}
$$

with the convention that $\phi_0 \equiv \phi_{n+1} \equiv 0$, $\psi_0 \equiv \psi_{n+1} \equiv 0$.

The higher spin W-generators can, in principle, be extracted from (3.13) and rewritten in terms of the $N=2$ superfields. In practice, this can be algebraically very cumbersome, and has only been done for the model (1.1) with $n=2$. In this model, the spin-2 superfield may be written explicitly as:

$$
\begin{aligned}
\mathcal{W} \ =\ & \widetilde{W}(z,\theta^+,\theta^-)\ -\ \tfrac{1}{4}\ \alpha_0^2\partial\mathcal{J}(z,\theta^+,\theta_-)\ -\ \frac{3-8\alpha_0^2}{2(5-18\alpha_0^2)}\ \ :\mathcal{J}^2(z,\theta^+,\theta^-): \\
& +\ \frac{(1-3\alpha_0^2)(1+2\alpha_0^2)}{8(5-18\alpha_0^2)}\ (D^+D^- - D^-D^+)\ \mathcal{J}(z,\theta^+,\theta^-)\ ,
\end{aligned}
\tag{3.35}
$$

where

$$
\begin{aligned}
\widetilde{W}(z,\theta^+,\theta^-) \ =\ & -i\frac{\alpha_0^3}{4}\partial^2\Phi_1^+ + i\frac{\alpha_0^3}{4}\partial^2\Phi_1^- + i\frac{\alpha_0^3}{4}\partial^2\Phi_2^- + \frac{\alpha_0^2}{8}D^+\partial\Phi_1^+D^-\Phi_1^- \\
& + \frac{\alpha_0^2}{8}D^+\Phi_1^+\partial D^-\Phi_1^- + \frac{\alpha_0^2}{8}D^+\partial\Phi_1^+D^-\Phi_2^- + \frac{\alpha_0^2}{4}\partial\Phi_2^+\partial\Phi_1^- + \frac{\alpha_0^2}{4}\partial\Phi_2^+\partial\Phi_2^- \\
& - \frac{\alpha_0^2}{4}\partial\Phi_2^-\partial\Phi_1^- - \frac{\alpha_0^2}{4}\partial\Phi_2^-\partial\Phi_2^- - \frac{\alpha_0^2}{4}\partial\Phi_2^+\partial\Phi_1^+ + \frac{\alpha_0^2}{4}\partial\Phi_2^-\partial\Phi_1^+ \\
& - i\frac{\alpha_0}{8}\partial\Phi_2^+D^+\Phi_1^+D^-\Phi_1^- + i\frac{\alpha_0}{8}\partial\Phi_2^-D^+\Phi_1^+D^-\Phi_1^- + i\frac{\alpha_0}{8}\partial\Phi_1^-D^+\Phi_2^+D^-\Phi_2^- \\
& + i\frac{\alpha_0}{8}\partial\Phi_2^-D^+\Phi_2^+D^-\Phi_2^- + i\frac{\alpha_0}{8}\partial\Phi_1^-D^-\Phi_2^-D^+\Phi_1^+ - i\frac{\alpha_0}{8}\partial\Phi_1^+D^+\Phi_2^+D^-\Phi_2^- \\
& + \frac{1}{16}D^+\Phi_1D^-\Phi_1^-D^+\Phi_2^+D^-\Phi_2^-\ .
\end{aligned}
\tag{3.36}
$$

If we define

$$
\begin{aligned}
\hat{\mathcal{J}} \ =\ & +\ \tfrac{1}{\sqrt{2(k+2)}}\ (\partial\Phi_1^+ - \partial\Phi_2^+ - \partial\Phi_1^-) \\
& +\ \tfrac{i}{2\sqrt{2}}\sqrt{\tfrac{k+3}{k+2}}\ (D^+\Phi_1^+D^-\Phi_1^- - D^+\Phi_2^+D^-\Phi_2^-)
\end{aligned}
\tag{3.37}
$$

Then we can write (3.36) as

$$\begin{aligned} \widetilde{\mathcal{W}} \;=\; & \tfrac{1}{4}\mathcal{J}^2 \;+\; \tfrac{1}{8(1-\alpha_0^2)}\;\hat{\mathcal{J}}^2 \;+\; \tfrac{\alpha_0^2}{4}\Big(\partial\mathcal{J} \;+\; i\tfrac{1}{2(1-\alpha_0^2)}\;\hat{\partial}\mathcal{J}\Big) \;+ \\ & \tfrac{\alpha_0^2}{8}\; D^+\partial\Phi_1^+ D^-\Phi_2^- \;+\; \tfrac{i\alpha_0}{8}\;\partial\Phi_1^- D^-\Phi_2^- D^+\Phi_1^+ \;. \end{aligned} \tag{3.38}$$

Combining eq. (3.35) and (3.38) we have

$$\begin{aligned} \mathcal{W} \;=\; & \tfrac{c_2}{2}(D^+D^- - D^-D^+)\mathcal{J} \;-\; \tfrac{6c_2}{c}\;\mathcal{J}^2 \\ & -\; (c_1+c_2)\Big(-\tfrac{1}{2}\;\hat{\mathcal{J}}^2 \;+\; \tfrac{i\alpha_0^2}{\sqrt{2(1-\alpha_0^2)}}\;\partial\hat{\mathcal{J}}\Big) \\ & +\; \tfrac{\alpha_0^2}{2\sqrt{2(1-\alpha_0^2)}}\; D^+\partial\Phi_1^+ D^-\Phi_2^- \\ & +\; \tfrac{i\alpha_0^2}{2(1-\alpha_0^2)}\;\partial\Phi_1^- D^-\Phi_2^- D^+\Phi_2^+, \end{aligned} \tag{3.39}$$

where

$$c_2 \;=\; \frac{(1-3\alpha_0^2)(1+2\alpha_0^2)}{(1-\alpha_0^2)}\;, \qquad c_1+c_2 \;=\; 5-18\alpha_0^2 \quad \text{and} \quad c \;=\; 6(1-3\alpha_0^2) \tag{3.40}$$

3.3. The Landau-Ginzburg free field formulation

The idea in this formalism is to directly use the $N = 2$ supersymmetric Landau-Ginzburg model with action:

$$S \;=\; \int d^2x\; d^4\theta\; \sum_j \Phi^+{}_j\;\Phi_j^- \;-\; \int d^2x\; d^2\theta\; W(\Phi_j^+) \;-\; \int d^2x\; d^2\bar{\theta}\; W(\Phi_j^-)\;, \tag{3.41}$$

where $\Phi_j^\pm$, $j = 1, \ldots, n$ are $N = 2$ (anti)-chiral superfields. If W is quasihomogenous then the Landau-Ginzburg model (3.41) with its "trivial" kinetic term is superconformally invariant on the cohomology of the half of the supercharges [2]. This is in the same spirit as the work of [23–26] in that there is certainly a kinetic term that renders the model exactly superconformal, and such a kinetic term can be viewed as a cohomologically trivial correction to that of (3.41). It was also shown in [2] that the superconformal generators could be identified using the equations of motion of (3.41) alone. Indeed (3.41) implies that the fields Φ_j^+ and $\Phi^-{}_j$ have logarithmic short distance expansion, and the left-moving $N = 2$ superconformal stress energy tensor can be represented by:

$$\mathcal{J} \;=\; \sum_j \Big[\; \tfrac{1}{4}(1-\omega_j)D^+\Phi_j^+ D^-\Phi_j^- \;-\; \tfrac{1}{2}\omega_j\Phi_j^+\partial\Phi_j^- \;\Big]\;, \tag{3.42}$$

where the ω_j are the scaling dimensions of the Landau-Ginzburg fields Φ_j^+. For the model (1.1) they are given by $\omega_j = \frac{j}{k+n+1}$. The current $\mathcal{J}$ has been constructed so as to satisfy

$$\bar{D}^- \mathcal{J} \ = \ 0 \quad , \tag{3.43}$$

given the equations of motion of (3.41).

This is closely related to the free field approach of [27]. That is, one can describe the Landau-Ginzburg system in terms of twisted ghost and superghost fields. Introduce anti-commuting fields $\hat{b}_j(z)$ and $\hat{c}_j(z)$, and commuting fields $\hat{\beta}_j(z)$ and $\hat{\gamma}_j(z)$, with operator products:

$$\hat{b}_i(z)\ \hat{c}_j(w) \ \sim \ \frac{\delta_{ij}}{z-w} \qquad \hat{\beta}_i(z)\ \hat{\gamma}_j(w) \ \sim \ -\frac{\delta_{ij}}{z-w} \ . \tag{3.44}$$

The superconformal generators are then:

$$\begin{aligned}
J(z) \ &= \ - \sum_{j=1}^{n} \big[(1-\omega_j)\hat{b}_j\ \hat{c}_j \ - \ \omega_j \hat{\beta}_j \hat{\gamma}_j \big] \\
G^+(z) \ &= \ \sum_{j=1}^{n} \big[(1-\omega_j)\hat{c}_j\ \partial\hat{\beta}_j \ - \ \omega_j \hat{\beta}_j\ \partial\hat{c}_j \big] \ ; \quad G^-(z) \ = \ \sum_{j=1}^{n} \hat{b}_j\hat{\gamma}_j \\
T(z) \ &= \ -\tfrac{1}{2} \sum_{j=1}^{n} \big[(1+\omega_j)\hat{b}_j\ \partial\hat{c}_j \ + \ (1-\omega_j)\ \hat{c}_j\ \partial\hat{b}_j \ + \ \omega_j\hat{\beta}_j\partial\hat{\gamma}_j \\
&\qquad - \ (2-\omega_j)\hat{\gamma}_j\partial\hat{\beta}_j \big] \ .
\end{aligned} \tag{3.45}$$

The fields $\hat{\beta}_j(z)$ and $\hat{b}_j(z)$ can be identified with the bosonic and and fermionic components of the superfield Φ_j^+, while $\hat{\gamma}_j$ and $\hat{c}_j(z)$ can be identified with the components of Φ_j^-.

One can easily determine the relationship between the the Landau-Ginzburg fields and the free fields of the last subsection by using the dimensions and charges of the Landau-Ginzburg fields along with the fact that (3.45) must be the same as (3.33). From this we find

$$\begin{aligned}
\hat{\beta}_j \ &= \ e^{i\alpha_0\phi_j} \ ; \qquad \hat{\gamma}_j \ = \ (\psi_j\bar{\psi}_j \ + \ i\sqrt{k+n+1}\ \partial\bar{\phi}_j)\ e^{-i\alpha_0\phi_j} \\
\hat{b}_j \ &= \ -\ \tfrac{i}{\sqrt{k+n+1}}\ \psi_j\ e^{i\alpha_0\phi_j} \ ; \qquad \hat{c}_j \ = \ -i\sqrt{k+n+1}\ \bar{\psi}_j\ e^{-i\alpha_0\phi_j} \ .
\end{aligned} \tag{3.46}$$

The shortcoming of the Landau-Ginzburg motivated free field formulation is that the Landau-Ginzburg formulation provides one with very little information about the screening currents. From [27,2,6] it is evident that such knowledge is unnecessary if one wants to study the topological matter model or extract the elliptic genus. However, the screeners are essential in order to get the complete conformal theory. Using (3.46) one could, at least in principle, obtain the proper Landau-Ginzburg screeners from the screeners of the Drinfeld-Sokolov reduction.

It is also interesting to observe that in the complete Landau-Ginzburg theory, the holomorphic supercurrent $G^+(z)$, and its anti-holomorphic counterpart, $\bar{G}^+(\bar{z})$, receive corrections from the superpotential. Indeed the complete supercurrent with anti-holomorphic component $\bar{G}^+(\bar{z})$ has a holomorphic component that can be written:

$$\sum_{j=1}^{n} \frac{\partial W(\Phi_\ell^+)}{\partial \Phi_j^+} \, D_- \Phi_j^+ \Big|_{\theta=\bar{\theta}=0} \, . \tag{3.47}$$

These currents appear to be the Landau-Ginzburg analogue of the F-type screening currents in the Drinfeld-Sokolov reduction. In the $N = 2$ superconformal minimal model (with one superfield) the identification is exact, but the precise relationship is rather less clear for the more general models.

One can now use (3.46) to translate the W-algebra generators of the previous section into the Landau-Ginzburg formulation. Alternatively, one can obtain these W-generators by making an Ansatz, imposing chirality of the W-superfield and using the Landau-Ginzburg equations of motion as in [8,6]. In the appendix to this paper we give details of such a computation for the first W-superfield for the Landau-Ginzburg theory with two fields. This computation, along with the foregoing translation to the Drinfel'd-Sokolov formulation, lead us to believe that the process of imposing chirality and the operator equations of motion in the Landau-Ginzburg formulation is basically equivalent to imposing commutation with the fermionic screening charges in the Drinfel'd–Sokolov reduction, and so the chirality and the Landau-Ginzburg equations of motion are, in principle, a little less stringent than the requirements of the full Coulomb gas description. In practice, for the Ansätze that we have used, chirality and the Landau-Ginzburg equations of motion are sufficient to determine the W-generator. However, the process of constructing the quantum versions of the W-generators using the Landau-Ginzburg formulation is operationally more difficult to implement, and it is easier to use the Drinfel'd–Sokolov reduction (along with the simplifications to be discussed in the next section).

3.4. Coulomb Gas formulations of related coset models

The Coulomb gas formulation that we will discuss here can only be properly justified by the results of the next section, and we include it here for completeness. The idea is to find free bosonic descriptions of the factors in the tensor product (3.1).

There is a well known, standard Coulomb gas description of $\mathcal{M}_2$ in terms of free bosons [17–28]. Let $\sigma(z)$ denote a vector of $n-1$ canonically normalized free bosons with energy-momentum tensor:

$$T_2(z) \ = \ -\frac{1}{2}(\partial\sigma(z))^2 \ + \ i \ (\beta_+ \ - \ \beta_-) \ \rho \cdot \partial^2\sigma(z) \ , \tag{3.48}$$

where ρ is the Weyl vector of $SU(n)$ and

$$\beta_\pm \ \equiv \ \left[\sqrt{\frac{k+n+1}{k+n}} \ \right]^{\pm 1} \ . \tag{3.49}$$

The screening currents are then

$$S^\pm_{\gamma_j} \ = \ e^{\pm i \beta_\pm \gamma_j \cdot \sigma(z)} \ , \tag{3.50}$$

where the γ_j are the simple roots of $SU(n)$.

The highest weight fields of $\mathcal{M}_2$ can be represented as:

$$V_{\lambda_+,\lambda_-}(z) \ = \ e^{-i(\beta_+\lambda_+ - \beta_-\lambda_-)\cdot\sigma(z)} \ . \tag{3.51}$$

This has conformal weight

$$\begin{aligned} \Delta_{\lambda_+,\lambda_-} \ &= \frac{1}{2}(\beta_+\lambda_+ - \beta_-\lambda_-)^2 \ + \ (\beta_+ - \beta_-)\rho\cdot(\beta_+\lambda_+ \ - \ \beta_-\lambda_-) \\ &= \frac{\lambda_+\cdot(\lambda_+ + 2\rho)}{2(k+n)} \ + \ \frac{1}{2}(\lambda_+ - \lambda_-)^2 \ - \ \frac{\lambda_-\cdot(\lambda_- + 2\rho)}{2(k+n+1)} \ . \end{aligned} \tag{3.52}$$

Thus λ_+ and λ_- can be thought of as corresponding to the weights of the $SU_k(n)$ and $SU_{k+1}(n)$ factors of $\mathcal{M}_2$.

The model $\mathcal{M}_1$ can be realized in terms of $2n$ free bosons. Let χ and ξ be vectors of n canonically normalized free bosons, and take

$$T_1(z) \ = \ -\frac{1}{2}(\partial\xi)^2 \ - \ \frac{1}{2}(\partial\chi)^2 \ - \ \frac{i}{\sqrt{k+n+1}} \ \tilde{\rho}\cdot\partial^2\xi \ - \ \frac{1}{\sqrt{k+n}} \ \rho\cdot\partial^2\chi \ \ , \tag{3.53}$$

where $\tilde{\rho}$ is the Weyl vector of $SU(n+1)$. The natural choice for representations of the highest weight fields are

$$U_{\lambda_+,\lambda_-} \;=\; exp\Big[+i\; \frac{\lambda_+\cdot\xi}{\sqrt{k+n+1}} \;+\; \frac{\lambda_-\cdot\chi}{\sqrt{k+n}}\Big]\;. \tag{3.54}$$

This has conformal weight

$$\frac{\lambda_+\cdot(\lambda_+ + 2\tilde{\rho})}{2(k+n+1)} - \frac{\lambda_-\cdot(\lambda_- + 2\rho)}{2(k+n)}\quad , \tag{3.55}$$

which is consistent with with identifying λ_+ and λ_- with heighest weights of the numerator and denominator factors respectively of $\mathcal{M}_1$. The screening currents are somewhat more difficult to determine, and will be given in the next section.

4. The $N=2$ super-W structure and factorizing the Coulomb gas description

It was observed in section 3.1 that the simplest way to get at the generators of the $N=2$ super-W algebra is to find the supercharges, and the W-generators of the model $\mathcal{M}_2$ in (3.1). We will therefore show explicitly how the Coulomb gas descriptions of the last section decompose into a tensor product. We will also have to handle the subtleties described in section 2.

The key to extracting the bosonic formulations of the factor models in (3.1) from the Drinfeld-Sokolov reduction is to use the screening charges. Modulo the subtleties of section 2, the screening charges (3.19) and (3.20) must be sums of screening charges for the factor models. (We will discuss the role of the fermionic screeners, (3.21), later.) Moreover, the roots of the A_n and A_{n-1} subalgebras of the superalgebra $A(n,n-1)$ should coincide with the roots of the factors of A_n and A_{n-1} in (3.1). This leads to the following fairly unambiguous identification:

$$\gamma_i\cdot\xi \;\equiv\; (\alpha_{2i-1}+\alpha_{2i})\cdot\varphi \;\equiv\; \phi_i + \bar{\phi}_i - \phi_{i-1}\;,\quad i=1,\ldots,n\;; \tag{4.1}$$

where the γ_i are the simple roots of $SU(n+1)$, the α_j are the simple roots of $A(n,n-1)$, and the bosons ξ are those of section 3.4.

The simple roots of an A_{n-1} subalgebra are given by $\alpha_{2i}+\alpha_{2i+1}$, and one would expect this to coincide with a linear combination of the bosons, σ, of $\mathcal{M}_2$, and the bosons χ of the denominator of $\mathcal{M}_1$. To isolate bosons corresponding $\mathcal{M}_2$ one seeks the screening currents corresponding to the denominator of $\mathcal{M}_2$, that is, those screening currents that have monodromy involving $(k+n+1)^{\rm th}$ roots of unity. From (2.11), and its obvious generalizations, one sees that the screeners that can be modified (as in section 2) are those associated with the locked numerator factor of $\mathcal{M}_2$, while the other screeners ar unchanged. Noting that the screening charges in (3.34) involves the $(k+n+1)^{\rm th}$ roots of unity, it is natural to look for the pure vertex operator screeners, $S^-_{\gamma_j}$, of (3.50) in the $SU(n)$ screener, $Q_{\alpha_{2i}+\alpha_{2i+1}}$, of (3.34). From this it is not hard to identify the second fermion bilinear term as the one we want.

Bosonize the fermions according to:

$$
\begin{aligned}
\psi_j(z) \ &= \ e^{iH_j(z)} \ , \qquad \bar\psi_j(z) \ = \ -e^{-iH_j(z)} \ , \\
\bar\psi_j(z)\ \psi_j(z) \ &= \ i\partial H_j(z) \ , \qquad \psi_j(z)\ \bar\psi_\ell(z) \ = \ e^{i(H_j(z)-H_\ell(z))} \ , \ j\neq\ell \ ;
\end{aligned}
\tag{4.2}
$$

where $H_j(z)\ H_\ell(w) \sim -\delta_{j\ell}\ log(z-w)$. Writing the second fermion bilinear of $Q_{\alpha_{2i}+\alpha_{2i+1}}$ as a pure vertex operator, we can then identify the free bosons of $\mathcal{M}_2$:

$$
\gamma_j\cdot\sigma \ \equiv \ \sqrt{\tfrac{k+n+1}{k+n}}\ (H_j - H_{j+1}) \ + \ \tfrac{1}{\sqrt{k+n}}\ (\phi_j - \bar\phi_j - \phi_{j+1}) \ , \quad j=1,\ldots,n-1 \ . \tag{4.3}
$$

The last $U(1)$ factor in (3.1) is the $N=2$, $U(1)$ current, which can be written:

$$
J(z) \ = \ i\ \sum_{j=1}^{n} \Big[\ \partial H_j(z) \ + \ \tfrac{1}{\sqrt{k+n+1}}\ (\ \partial\phi_j \ - \ j\ \partial\bar\phi_j\ )\Big] \ . \tag{4.4}
$$

The remaining bosons of $\mathcal{M}_1$ are the natural orthogonal combinations to (4.1), (4.3) and (4.4). This yields the identifications:

$$
\begin{aligned}
\gamma_j\cdot\chi \ &\equiv \ \tfrac{i}{\sqrt{k+n}}\ (H_j - H_{j+1}) \ + \ i\ \sqrt{\tfrac{k+n+1}{k+n}}\ (\phi_j - \bar\phi_j - \phi_{j+1}) \ , \quad j=1,\ldots,n-1 \\
K(z) \ &\equiv \ 2i\ \sqrt{\tfrac{k}{n(n+1)}}\ (\tilde\rho - \rho)\cdot\partial\chi \\
&\equiv \ \sqrt{\tfrac{n+1}{n}}\ \sum_{j=1}^{n} \Big[\ \partial H_j(z) \ + \ \tfrac{\sqrt{k+n+1}}{n+1}\ (\ \partial\phi_j \ - \ j\ \partial\bar\phi_j\ )\Big] \ .
\end{aligned}
\tag{4.5}
$$

The current, $K(z)$, corresponds to the $U(1)$ factor in $\mathcal{M}_1$ and has been normalized according to $K(z)\ K(w) \sim \frac{k}{(z-w)^2}$.

One can now rewrite the entire model in terms of these free bosonic fields. The W-generators of $\mathcal{M}_2$ can be written in the usual manner as Weyl invariant combinations of the derivatives of the bosons σ [16–19]. This enables one to write down rather explicit expressions for the bottom components of the super-multiplets. There are, however, the subtleties discussed in section 2.

One can easily express the D-type screening currents in the new free bosonic basis. To do this, it is convenient to introduce the standard basis, e_j, $j = 1, \dots, n+1$ for the weight space of $SU(n+1)$. The vectors e_j satisfy: $\gamma_j = e_j - e_{j+1}$, $\sum_{j=1}^{n+1} e_j = 0$ and $e_i \cdot e_j = \delta_{ij} - \frac{1}{(n+1)}$. Introduce a similar basis $\hat{e}_j$ for $SU(n)$. The vectors $\hat{e}_j$ are orthogonal to e_{n+1} and are given by $\hat{e}_j = e_j - \frac{1}{n} e_{n+1}$, $j = 1, \dots, n$. Using these vectors the screening currents corresponding to $Q_{\alpha_{2j-1}+\alpha_{2j}}$ and $Q_{\alpha_{2j}+\alpha_{2j+1}}$ can be respectively written as:

$$
\begin{aligned}
U_j(z) \;=\; & \frac{2i}{\sqrt{k+n+1}}\, exp\Big[- \frac{i}{\sqrt{k+n+1}}\, (\gamma_j \cdot \xi) - \frac{1}{\sqrt{k+n}}\, (\gamma_{j-1} \cdot \chi) \Big] \cdot \\
& exp\Big[+ i \sqrt{\tfrac{k+n+1}{k+n}}\, (\gamma_j \cdot \sigma) \Big] \quad - \; i\sqrt{k+n+1}\; \partial \left(e^{-\frac{i}{\sqrt{k+n+1}}\, \gamma_j \cdot \xi} \right) \\
& - \; 2 \Big[\, e_{j+1} \cdot \partial\xi(z) \quad - \; i \sqrt{\tfrac{k+n}{k+n+1}}\, \big(\, \hat{e}_j \cdot \partial\chi(z) \, \big) \\
& + \; \frac{1}{\sqrt{(k+n+1)n(n+1)}}\, K(z) \, \Big] \cdot e^{-\frac{i}{\sqrt{k+n+1}}\, \gamma_j \cdot \xi} \, , \quad j \;=\; 1, \dots, n \; ;
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
V_j(z) \;=\; & \frac{2i}{\sqrt{k+n+1}}\, exp\Big[\, - i \sqrt{\tfrac{k+n}{k+n+1}}\, (\gamma_j \cdot \sigma) \, \Big] \\
& + \quad i \sqrt{k+n+1}\; \partial \Big(\, exp\Big[\, - \, \frac{1}{\sqrt{k+n}}\, (\gamma_j \cdot \chi) \\
& + \; \frac{i}{\sqrt{(k+n+1)(k+n)}}\, (\gamma_j \cdot \sigma) \, \Big] \Big) \\
& - \; 2 \Big[\, e_{j+1} \cdot \partial\xi(z) \; - \; i \sqrt{\tfrac{k+n}{k+n+1}}\, \big(\, \hat{e}_j \cdot \partial\chi(z) \, \big) \\
& + \; \frac{1}{\sqrt{(k+n+1)n(n+1)}}\, K(z) \, \Big] \, exp\Big[- \frac{1}{\sqrt{k+n}}\, (\gamma_j \cdot \chi) \\
& + \frac{i}{\sqrt{(k+n+1)(k+n)}}\, (\gamma_j \cdot \sigma) \, \Big] \, , \quad j \;=\; 1, \dots, n-1
\end{aligned}
\tag{4.7}
$$

.

These screening currents have precisely the kind of structure that was described in section 2. That is, the screening currents of one factor of the tensor product (3.1) have been mixed with dual representatives of the vacuum of the other factor in the tensor product. The operators analogous to (2.10), that extend the chiral algebra of the bosonized theory, are nothing other than combinations of derivatives of ϕ_ℓ and $\bar{\phi}_\ell$ with nilpotent fermion bilinears like $\psi_i \bar{\psi}_j$ and $(\partial\psi_i)\bar{\psi}_j$. We should therefore expect such corrections to the standard forms of the W-generators. Indeed, the corrections to $S(z)$ analogous to $R^+ S^-$ in (2.12) can be read off from (4.6) and (4.7). These terms are of the form:

$$
\begin{aligned}
N_j(z) \;\equiv\; & \Big[\, e_{j+1} \cdot \partial\xi(z) \; - \; i \sqrt{\tfrac{k+n}{k+n+1}}\, \big(\, \hat{e}_j \cdot \partial\chi(z) \, \big) \; - \; \frac{1}{\sqrt{(k+n+1)n(n+1)}}\, K(z) \, \Big] \\
& \times \; exp\Big[\; - \; \frac{1}{\sqrt{k+n}}\, (\gamma_j \cdot \chi) \; + \; i \sqrt{\tfrac{k+n+1}{k+n}}\, (\gamma_j \cdot \sigma) \, \Big] \\
\equiv\; & \Big[\; (\partial\bar{\phi}_j) \; \psi_j \; \bar{\psi}_{j+1} \; + \; i\alpha_0 \; (\partial\psi_j) \; \bar{\psi}_{j+1} \, \Big] \, .
\end{aligned}
\tag{4.8}
$$

These terms must be added (with appropriate coefficients) to the naive form for $S(z)$, and for $n \geq 3$ there will be further terms of the form

$$\left[\ (\partial\bar{\phi}_j)\ \psi_j\ \bar{\psi}_{j+\ell}\ +\ i\alpha_0\ (\partial\psi_j)\ \bar{\psi}_{j+\ell}\ \right]\ . \tag{4.9}$$

These terms are necessary to cancel other terms that result from commuting the screening charges with the $N_j(z)$. Alternatively, they have to be present for $S(z)$ to have the proper operator product with itself.

Rather than get too deeply involved in the technical details of the general problem, we will specialize to the model (1.1) with $n = 2$. The model, $\mathcal{M}_2$, is then an ordinary minimal ($c < 1$) model, and is realized by a single free boson σ. The energy momentum tensor of the complete $N = 2$ supersymmetric model is, of course, the simple sum of all the component energy momentum tensors:

$$T_1(z)\ +\ T_2(z)\ +\ \tfrac{k+3}{12k}\ J^2(z)\ , \tag{4.10}$$

where T_1 and T_2 are given by (3.53) and (3.48). The bottom component, $S(z)$, of the W_3-supermultiplet is a spin-2 current, and the naive guess for its form is (2.8). As explained earlier, even though T_2 is not a good conformal field, it can be viewed as defining the extension of the chiral algebra, and once one has it, one can easily construct $S(z)$. The proper representative of T_2 in the tensor product model will involve a correction of the form (4.8). Indeed, we find that the complete free-field expression for $T_2(z)$ is:

$$\begin{aligned} \widehat{T}_2(z)\ &=\ -\frac{1}{2}(\partial\sigma(z))^2\ +\ \tfrac{i}{\sqrt{2(k+2)(k+3)}}\ \partial^2\sigma(z) \\ &\qquad +\ \tfrac{k+3}{2(k+2)}\left[\ (\partial\bar{\phi}_1)\ \psi_1\ \bar{\psi}_2\ +\ \tfrac{i}{\sqrt{k+3}}\ (\partial\psi_1)\ \bar{\psi}_2\ \right]\ . \end{aligned} \tag{4.11}$$

The coefficient of the fermion bilinear terms is determined by requiring that $\widehat{T}_2(z)$ commute with the D-type screeners. The extra nilpotent fermion bilinears are, of course, present in (3.36): the two relevant terms are the bottom components of $\frac{\alpha_0^2}{8}D^+\partial\Phi_1^+D^-\Phi_2^-$ and $i\frac{\alpha_0}{8}\partial\Phi_1^-D^-\Phi_2^-D^+\Phi_1^+$.

To summarize, the key to extracting the W-algebra generators is in the identification of the free bosons, σ, given in (4.3). The lowest component of each W-supermultiplet can then easily be constructed from them. Since we will need it later, we conclude by giving the form of these bosons in terms of the Landau-Ginzburg free fields. Indeed, from (4.3) and (3.46) one easily obtains:

$$\gamma_j\cdot\sigma\ =\ \sqrt{\tfrac{k+n+1}{k+n}}\ \Big[\big(1-\tfrac{1}{k+n+1}\big)\ \hat{b}_j\hat{c}_j\ -\ \hat{b}_{j+1}\hat{c}_{j+1}\Big]\ -\ \tfrac{1}{\sqrt{(k+n+1)(k+n)}}\ \hat{\beta}_j\hat{\gamma}_j\ . \tag{4.12}$$

5. The elliptic genus and other characters

One of the beautiful features of the Landau-Ginzburg model is that the elliptic genus of the model can be easily computed solely from the knowledge of the field content and scaling dimensions [2]. This computation can be refined so as to determine the the $U(1)$ eigenvalues of the states contributing to the elliptic index [3–5]. One of the basic ideas of [8,6,7] was that the Landau-Ginzburg potential contains the information about when the conformal model has an extended chiral algebra. This fact was further employed in [6] to show how, at least for the model (1.1) with $n = 2$, the elliptic genus could be further refined so as to extract exactly how the different eigenstates of the extended chiral algebra contribute to the elliptic index of the model. The result, for (1.1) with general n, was also conjectured in [6], and in the last section we have developed enough information to now show that this conjecture is correct.

5.1. The refined elliptic genus

The idea is to introduce the function:

$$\mathcal{F}(q,\mu,\nu) \ = \ Tr_{\mathcal{H}} \ \big((-1)^F q^{H_L} \ \bar{q}^{H_R} \ exp(i\mu\cdot j_0) \ exp(i\nu J_0)\big) \ . \tag{5.1}$$

In this expression $\mathcal{H}$ is the complete Hilbert space of the model in the Ramond sector, $H_L = L_0$ and $H_R = \bar{L}_0$ are the hamiltonians of the left-movers and right-movers, F is the total fermion number, J_0 is the left-moving $N = 2$, $U(1)$ charge, and j_0 is the vector of zero modes of the left-moving bosons, σ, defined in (4.12). The standard index argument can be used to show that in the right-moving sector, only the ground-states contribute to the trace. As a result, the function $\mathcal{F}$ is a function of q alone (and not a function of $\bar{q}$), and consists of a sum of the (left-moving) Ramond ground-state characters. Unless one sets $\mu \equiv 0$, the result will *not* be characters of (1.1). This is because the charges j_0 do not commute with the screening charges that reduce the free field Hilbert space down to that of the coset model.

However, it was argued in [6] that one can obtain a character of the coset model by the simple expedient of symmetrizing with respect to the Weyl group of $SU(n)$. That is, one defines

$$\mathcal{F}_s(q,\mu,\nu) \ = \sum_{w\in W(SU(n))} \mathcal{F}(q,w(\mu),\nu) \ . \tag{5.2}$$

To see why this is so, one first evades all the subtleties of section 2 by simply deciding to describe the model as a naive tensor product (3.1), and not as a locked tensor product with non-standard screening charges. That is, one uses the same set of free fields, but simply chooses the naive set of screening charges for a tensor product model. The cost of doing this is that one must remember to lock the Hilbert spaces together by hand once one has constructed them from the free fields. The advantage of taking the naive tensor product is, of course, that the W-generators have the simple polynomial form in derivatives of the free bosons.

If one temporarily ignores the oscillator contributions to the bosonic Hilbert space of $\mathcal{M}_2$ of (3.1), one can see, by performing integral transforms as in [6], that $\mathcal{F}_s$ contains the same information as refining the elliptic genus with respect to the zero-modes of the W-algebra of $\mathcal{M}_2$. This is simply a version of the theorem that a weight of a Lie algebra is uniquely specified, *up to Weyl rotations*, by the values of all the Casimirs on that weight. The problem is with the oscillator contributions. The zero-modes of W-generators are notorious for only really being diagonalizable on pure momentum states. We do not know how to make a compelling argument solely from the perspective of the Landau-Ginzburg formulation. However, based on the results of the last section, we know that the bosons σ are precisely those of the standard Coulomb gas formulation of $\mathcal{M}_2$. These characters consist of trivial oscillator η-function factors multiplying sums over pure σ-momentum states. Thus the null states introduced by the full screening charges of (3.1) only involve the pure momentum states and are thus correctly reproduced in (5.2). As a result, the Weyl symmetrized $\mathcal{F}$ will suffice to produce a function on the Hilbert space of $\mathcal{M}_2$ and hence on the Hilbert space of (1.1).

The argument can be made rather more directly if one merely concentrates upon the zero mode, S_0, of $S(z)$. Since this is a linear combination of the energy momentum tensors in the tensor product, this grades the elliptic genus according to the energies associated with the factors in (3.1). (It is the higher spin generators of the W-algebra that cause the problems with simultaneous diagonalization.) As was observed in [6], this refinement of the elliptic genus is related fo $\mathcal{F}_s$ by a Laplace transform. The oscillator parts are dealt with by multiplying by ratios of η-functions in such a manner as to reflect the fact the $n-1$ of the bosons have their energies measured in $\mathcal{M}_2$. Thus the component parts of the tensor product can be factored out easily from $\mathcal{F}_s$.

To be more specific, the $\mathcal{M}_2$ components of (5.2) will consist of functions of the form:

$$
\begin{aligned}
\chi_{\lambda_+}^{\lambda_-} \ &= \frac{1}{\eta(\tau)^\ell} \sum_{w \in W(SU(n))} \sum_{\gamma \in M(G)} \epsilon(w) \\
&\quad \times q^{\frac{1}{2}\left[\beta_+ w(\lambda_+ + \rho) \ - \ \beta_-(\lambda_- + \rho) \ + \ \sqrt{(k+n)(k+n+1)} \ \gamma\right]^2} \\
&\quad \times e^{i\mu \cdot \left[\beta_+ w(\lambda_+ + \rho) \ - \ \beta_-(\lambda_- + \rho) \ + \ \sqrt{(k+n)(k+n+1)}\gamma\right]} \ .
\end{aligned}
\tag{5.3}
$$

For $\mu \equiv 0$ these functions are characters of the model $\mathcal{M}_2$. Therefore, if we extract the coefficient of

$$
e^{i\mu \cdot \left[\beta_+ w(\lambda_+ + \rho) \ - \ \beta_-(\lambda_- + \rho) \ + \ \sqrt{(k+n)(k+n+1)}\gamma\right]} \ \times \ e^{i\nu a} \ , \tag{5.4}
$$

in $\mathcal{F}_s$, then we will obtain the character of the model $\mathcal{M}_1$ that is paired with the states in the Hilbert space of $\mathcal{M}_2$ labelled by $\chi_{\lambda_+}^{\lambda_-}$, and which also have $N = 2$, $U(1)$ charge equal to a. While we have not rigorously proved the foregoing statement, we think it is emminently plausible, and in the next sub-section we will confirm our results by computing branching functions in the factors of (3.1).

Thus the refined elliptic genus, $\mathcal{F}_s$, enables us to completely decompose and isolate the component parts of the partition function of (1.1).

5.2. Explicit formulae and a simple example

Following the arguments of Witten, we know that the elliptic genus can be expressed very simply in terms of the free fields in the Ramond sector. That is, it can be expressed as a simple product of ratios of theta functions. The refined "elliptic character," $\mathcal{F}$, is obtained by grading this product of theta functions with the bosonic zero-modes J_0 and j_0. Using (3.45) and (4.12) we obtain the following formula for $\mathcal{F}$:

$$
\mathcal{F}(\tau, \mu, \nu) \ = \ \prod_{j=1}^{n} \frac{\theta_1(a_j|\tau)}{\theta_1(b_j|\tau)} \ , \tag{5.5}
$$

where

$$
\begin{aligned}
a_j \ &= \ \left(1 - \tfrac{1}{k+n+1}\right) \mu_j \ - \ \mu_{j-1} \ + \ \left(1 - \tfrac{j}{k+n+1}\right) \nu \ , \\
b_j \ &= \ -\tfrac{1}{k+n+1} \ \mu_j \ - \ \tfrac{j}{k+n+1} \ \nu \ ,
\end{aligned}
\tag{5.6}
$$

with the convention that $\mu_0 = \mu_n \equiv 0$. The parameters, μ_j are defined by writing $\mu \cdot j(z) = \sum_j \sqrt{\frac{k+n}{k+n+1}} \ \mu_j \gamma_j \cdot \partial\sigma(z)$. Recall that we may write $\gamma_j = e_j - e_{j+1}$, $j = 1, \ldots, n-1$. The Weyl group of $SU(n)$ is the permutation group on n objects, acting in the obvious manner on the e_j. From this it is trivial to determine the Weyl action on the μ_j, and hence obtain the function $\mathcal{F}_s$ from $\mathcal{F}$.

For example, taking $n = 2$, one obtains:

$$\mathcal{F}(q,y,z) \;=\; y^{-1} z^{k} \prod_{p=1}^{\infty} \left\{ \frac{(1-q^{p-1}y^{-(k+2)}z^{(k+2)})\,(1-q^{p}y^{(k+2)}z^{-(k+2)})}{(1-q^{p-1}y^{-1}z)\,(1-q^{p}yz^{-1})} \right. \left. \frac{(1-q^{p-1}y^{(k+3)}z^{(k+1)})\,(1-q^{p}y^{-(k+3)}z^{-(k+1)})}{(1-q^{p-1}z^{2})\,(1-q^{p}z^{-2})} \right\} . \tag{5.7}$$

where $y = exp\big[-\frac{i\mu}{\sqrt{(k+2)(k+3)}}\big]$ and $z = exp[-\frac{i\nu}{k+3}]$. One can immediately see that this function is singular at $z = 1$, and thus it cannot be a character of a unitary coset model. However, to Weyl symmetrize, one simply replaces μ by $-\mu$, and obtains:

$$\mathcal{F}_s(q,y,z) \;=\; \mathcal{F}(q,y,z) \;+\; \mathcal{F}(q,y^{-1},z) \;=\; \mathcal{F}(q,y,z) \;+\; \mathcal{F}(q,y,z^{-1}) \;. \tag{5.8}$$

This function is regular at $z = 1$, and extensive expansion using *Mathematica*^{TM} confirms that it generates the proper characters of the factors in (3.1).

5.3. Decomposing the refined elliptic character

To complete the process of isolating the component characters of the model we need to extract the coefficient of terms of the form (5.4) in the Weyl symmetrized form of (5.5). To do this, one needs to expand the theta functions in the denominator of (5.5) using the identity [29]:

$$\begin{aligned} \frac{1}{\theta_1(\nu|\tau)} \;&=\; \Big[\, (e^{i\pi\nu} - e^{-i\pi\nu}) \prod_{p=1}^{\infty} \,(1 - q^p e^{2\pi i\nu})\,(1-q^p e^{-2\pi i\nu})\,\Big]^{-1} \\ &=\; i\; q^{-\frac{1}{8}} \Big[\prod_{n=1}^{\infty} (1-q^n)^{-3}\Big] \times \\ &\qquad \sum_{\ell=-\infty}^{\infty} \sum_{p=0}^{\infty} (-1)^p \; e^{2\pi i(\ell-\frac{1}{2})\nu} \; q^{\frac{1}{2}(p\pm\ell+\frac{1}{2})^2 - \frac{1}{2}(\ell-\frac{1}{2})^2} \;. \end{aligned} \tag{5.9}$$

In this formula one can choose the $\pm$ sign in any manner one pleases because of the identity:

$$\sum_{p=0}^{2m-1} (-1)^p \; q^{\frac{1}{2}(p-m+\frac{1}{2})^2 - \frac{1}{2}(m-\frac{1}{2})^2} \;\equiv\; 0 \;. \tag{5.10}$$

To give the expressions for the branching functions of $\mathcal{M}_1$ as they emerge from the elliptic genus, we need to introduce some notation. We will need another basis for the roots of $SU(n+1)$:

$$\bar{\alpha}_j \ \equiv \ e_{n+1-j} \ - \ e_{n+1} \ , \quad j = 1, \ldots, n \ ; \tag{5.11}$$

along with the corresponding dual weight basis (satisfying $\bar{\lambda}_i \cdot \bar{\alpha}_j = \delta_{ij}$):

$$\bar{\lambda}_j \ \equiv \ e_{n+1-j} \ - \ \frac{1}{n+1}(e_1 + \ldots + e_{n+1}) \ , \quad j = 1, \ldots, n \ . \tag{5.12}$$

Given any vector, ζ, define vector and scalar projections, ζ_0 and $\hat{\zeta}$, via:

$$\begin{aligned} \zeta \ &= \ \sum_{j=1}^{n} \zeta_j \ \bar{\alpha}_j \ ; \qquad \hat{\zeta} \ = \ 2(\tilde{\rho} - \rho) \cdot \zeta \ = \ (n+1) \sum_{j=1}^{n} \zeta_j \ ; \\ \zeta_0 \ &= \ \zeta \ - \ \tfrac{2\hat{\zeta}}{n(n+1)} \ (\tilde{\rho} - \rho) \ = \ \sum_{j=1}^{n} \zeta_j \ \Big[e_{n+1-j} \ - \ \frac{1}{n}(e_1 + \ldots + e_n) \ \Big] \ . \end{aligned} \tag{5.13}$$

Recall that $\tilde{\rho}$ and ρ are the Weyl vectors of $SU(n+1)$ and $SU(n)$ respectively, and that $\tilde{\rho} - \rho$ defines the $U(1)$ direction in $\mathcal{M}_1$. Thus ζ_0 and $\hat{\zeta}$ are the components of ζ parallel and perpendicular to the $U(1)$. Introduce two vectors:

$$\begin{aligned} v \ &\equiv \ \tfrac{1}{k+n+1} \ \Big[\ \nu \ \tilde{\rho} \ + \ \sum_{j=1}^{n-1} \ \mu_j \ \bar{\lambda}_j \ \Big] \ , \\ u \ &\equiv \ \tfrac{2\nu}{n+1} \ (\tilde{\rho} \ - \ \rho) \ - \ v \ - \ \sum_{j=1}^{n-1} \ \mu_j \ (\bar{\lambda}_{j+1} - \bar{\lambda}_j) \ . \end{aligned} \tag{5.14}$$

The whole point of these vectors is that $\bar{\alpha}_j \cdot u \ = \ a_j$ and $\bar{\alpha}_j \cdot v \ = \ -b_j$, where a_j and b_j are given by (5.6). Finally, introduce a vector ξ defined by

$$\xi \ \equiv \ \sum_{j=1}^{n} \ p_j \ \bar{\lambda}_j \ , \quad \text{with} \quad p_j \geq 0 \ . \tag{5.15}$$

This vector will generate all the sums over $p_j \geq 0$ when we invert the denominators of (5.5) using (5.9).

The function, $\mathcal{F}(q, \mu, \nu)$, can then expanded according to:

$$\begin{aligned} \mathcal{F}(q, \mu, \nu) \ = \ \frac{1}{\eta(q)^n} \sum_{\beta \in \Gamma} \ \sum_{\lambda \in \Gamma^*} \ & e^{i\pi(\hat{u} + \hat{v} + \frac{1}{(n+1)}\hat{\beta})} \ e^{-\frac{2\pi i}{(k+n+1)(n+1)}(\hat{u}+\hat{v})\hat{\lambda}} \\ & e^{-2\pi i(u+v)\cdot(\beta + \frac{\lambda}{k+n+1})} \ q^{\Delta(\beta,\lambda)} \ G_{\beta,\lambda}(q) \ , \end{aligned} \tag{5.16}$$

where Γ is the root lattice of $SU(n+1)$, Γ^* is the weight lattice of $SU(n+1)$, and

$$
\begin{aligned}
\Delta(\beta,\lambda) \;\equiv\; & \tfrac{1}{2(k+n)(k+n+1)}\Big[\ \lambda_0 \ +\ (k+n+1)\beta_0\ \Big]^2 \\
& +\ \tfrac{1}{2kn(n+1)^2(k+n+1)}\Big[\ (n+1)\hat{\lambda}\ +\ (k+n+1)\hat{\beta} \\
& -\ \tfrac{1}{2}n(n+1)(k+n+1)\ \Big]^2 \ .
\end{aligned}
\tag{5.17}
$$

The quadratic form, $\Delta(\beta,\lambda)$, is the energy in $\mathcal{M}_2\times U(1)$ of the momentum state labelled by $\beta+\frac{\lambda}{k+n+1}$. After a considerable amount of work, the functions $G_{\beta,\lambda}$ can be written:

$$
G_{\beta,\lambda} \ \equiv\ \frac{1}{\eta(q)^{2n}} \sum_{\eta\in\Gamma^*}\ \sum_{\xi}\ e^{i\pi(\hat{\xi}+\hat{\eta})}\ q^{\Omega(\beta,\lambda;\xi,\eta)}\ , \tag{5.18}
$$

where ξ is defined by (5.15) and the sum is, of course, only over $p_j\ \geq 0$. Because of the conditional convergence of the double sum in (5.9), one must perform the sum over ξ before performing the sum over η in (5.18). One must also be very careful in performing any reordering of this sum, and in shifting summation variables. The quadratic form $\Omega(\beta,\lambda;\xi,\eta)$ is defined by:

$$
\begin{aligned}
\Omega(\beta,\lambda;\xi,\eta) \;\equiv\; & \tfrac{1}{2(k+n+1)}\Big[\ (k+n+1)(\xi+w_c(\eta))\ +\ w_c(\lambda)\ \Big]^2 \\
& -\ \tfrac{1}{2(k+n)}\Big[\ (k+n)(\xi_0+\eta_0)\ +\ (\beta_0+\lambda_0)\ \Big]^2 \\
& -\ \tfrac{1}{2kn(n+1)}\Big[\ k(\hat{\xi}+\hat{\eta})\ +\ (\hat{\lambda}\ +\ \hat{\beta})\ -\ \tfrac{1}{2}n(n+1)\ \Big]^2 \ .
\end{aligned}
\tag{5.19}
$$

The function, w_c, is a cyclic Weyl rotation of $SU(n+1)$ that takes $e_1 \to e_2 \to \ldots \to e_{n+1} \to e_1$. Note that one does not sum over w_c in any manner, it is simply used to transform the vectors λ and η in (5.19).

Finally, to get the branching functions, $G^{(s)}_{\beta,\lambda}$, of $\mathcal{M}_1$, we must Weyl symmetrize. That is,

$$
G^{(s)}_{\beta,\lambda} \ \equiv \sum_{w\in W(SU(n))} G_{w(\beta),w(\lambda)}\ . \tag{5.20}
$$

5.4. Branching functions of $\frac{SU_k(n+1)}{SU_k(n)\times U(1)}$

The direct way of obtaining the branching functions of $\mathcal{M}_1$ is from the Weyl-Kac character formula. We will now do this so as to obtain expressions that can be compared with those for $G^{(s)}_{\beta,\lambda}$. One expands the appropriate parts of the denominator of the character formula using (5.9) and then factors out the characters of $SU_k(n)\times U(1)$ from the resulting expression. The coefficient functions of the $SU_k(n)\times U(1)$ characters are then the branching functions. If Λ is a highest weight of $SU_k(n+1)$, and χ is a weight of $SU_k(n) \times U(1)$, then the branching functions are only non-zero if $\chi = \Lambda + \beta$ for some root β of $SU(n+1)$, and then one has

$$b^{\Lambda}{}_{\chi} \;=\; \frac{1}{\eta(q)^{2n}} \sum_{w \in W(SU(n+1))} \; \sum_{\gamma \in \Gamma} \; \sum_{\xi} \; \epsilon(w) \; (-1)^{\hat{\xi}} \; q^{\widetilde{\Omega}_w(\Lambda,\chi;\xi,\gamma)} \; . \tag{5.21}$$

The sum over ξ is exactly as above, and $\epsilon(w)$ is, as usual, the determinant of w. Note that unlike above, the sum over w is over the Weyl group of $SU(n+1)$ and not just that of $SU(n)$. The quadratic form $\widetilde{\Omega}_w(\Lambda, \chi; \xi, \gamma)$ is closely related to Ω in (5.19):

$$\begin{aligned} \widetilde{\Omega}_w(\Lambda, \chi; \xi, \gamma) \;\equiv\; & \tfrac{1}{2(k+n+1)} \big[\; (k+n+1)(\xi+\gamma) \;+\; w(\Lambda+\tilde{\rho}) \; \big]^2 \\ & -\; \tfrac{1}{2(k+n)} \big[\; (k+n)\xi_0 \;+\; (\chi_0+\rho) \; \big]^2 \\ & -\; \tfrac{1}{2kn(n+1)} \big[\; k\hat{\xi} \;+\; \hat{\chi} \; \big] \; . \end{aligned} \tag{5.22}$$

These branching functions have the following symmetries:

$$b^{\Lambda}{}_{\chi} \;=\; b^{\Lambda}{}_{\chi+(k+n)\beta_0} \;=\; b^{\Lambda}{}_{\chi+2k(\tilde{\rho}-\rho)} \; , \tag{5.23}$$

where β_0 is a root of $SU(n)$. There are also the spectral flow identifications: The vector $\lambda_{n+1} = \frac{2}{n+1}(\tilde{\rho} - \rho)$ is a weight of $SU(n+1)$. For any given Λ, there is a root, β, of $SU(n+1)$, and an element, w', of the Weyl group of $SU(n+1)$, such that the vector, Λ', defined by:

$$\Lambda' \;\equiv\; w'(\Lambda+\tilde{\rho}) \;+\; (k+n+1)(\lambda_{n+1}+\beta) \;-\; \tilde{\rho} \; , \tag{5.24}$$

is once again an affine label of $SU_k(n+1)$. For such vectors Λ and Λ' one has:

$$b^{\Lambda}{}_{\chi} \;=\; \epsilon(w') \; b^{\Lambda'}{}_{\chi+k\lambda_{n+1}} \; . \tag{5.25}$$

From the Weyl-Kac character formula, it is natural to extend the label Λ, of the branching function, to any vector Λ' on the weight lattice of $SU(n+1)$. That is, one takes $b^{\Lambda'}{}_{\chi} = 0$ if $(\Lambda' + \tilde{\rho}) \cdot \beta \equiv 0 \bmod k+n+1$ for any root β of $SU(n+1)$. For any other weight, Λ', there is a root β, and a Weyl rotation, w, such that $\Lambda = w(\Lambda' + \tilde{\rho}) - \tilde{\rho} + (k+n+1)\beta$ is a highest weight of affine $SU_k(n+1)$. We therefore take:

$$b^{\Lambda'}{}_{\chi} \;=\; \epsilon(w) \; b^{\Lambda}{}_{\chi} \qquad \text{where} \quad \Lambda \;=\; w(\Lambda'+\tilde{\rho}) - \tilde{\rho} + (k+n+1)\beta \; . \tag{5.26}$$

From our general arguments about the properties of the refined elliptic genus, the foregoing branching functions are related to the functions $G^{(s)}_{\beta,\lambda}$ by:

$$b^{\Lambda}{}_{\chi} \;=\; G^{(s)}_{\beta,\lambda} \quad \text{with} \quad \lambda = w(\Lambda+\tilde{\rho}) \; ; \quad \beta + \lambda = \chi + \tilde{\rho} \; . \tag{5.27}$$

The Weyl element w in this relation can be chosen at will.

One can easily see that the following replacements:

$$\xi \to \xi - \eta \; ; \quad w_c(\eta) - \eta \to \gamma \; ; \quad \lambda \to w(\Lambda+\tilde{\rho}) \; ; \quad \beta \to \chi - [w(\Lambda+\tilde{\rho}) - \tilde{\rho}] \; , \tag{5.28}$$

transform Ω of (5.19) directly into $\tilde{\Omega}_{w'}$ with $w' = w_c w$. The problem with going further and directly establishing the identity (5.27), independently of the elliptic genus, is that the sums in (5.18) and (5.21) are conditionally convergent, and thus must be handled with considerable care. We have thus only been able to prove (5.27) directly for $n = 1$ and $n = 2$, and based upon this, we believe that a general direct proof will require breaking sums over the root or weight lattice into many sums over different cones on the lattice and then making extensive rearrangements and use of the identity (5.10). Since we have the general argument based on the elliptic genus, the direct proof for $n = 1$ and $n = 2$, as well as extensive checks using $Mathematica^{TM}$, we have not pursued a general direct proof any further.

5.5. *A simple example*

For $n = 1$ the forgoing functions, $G^{(s)}$, are labelled with by two integers (a root and a weight of $SU(2)$) and are the branching functions of $SU(2)/U(1)$. That is, we will recover the string functions, c^{ℓ}_{m}, that are the partition functions for parafermions. Taking $\xi = \frac{p}{2}(e_1 - e_2)$, $\eta = -\frac{n}{2}(e_1 - e_2)$, $\lambda = -\frac{(\ell+1)}{2}(e_1 - e_2)$ and $\beta = (s+1)(e_1 - e_2)$ in (5.18), one arrives at:

$$G^{(s)}_{s,\ell}(\tau) \;=\; \frac{1}{\eta(q)^2} \sum_{n=-\infty}^{\infty} \sum_{p=0}^{\infty} (-1)^{n+p} \, q^{\frac{1}{4(k+2)}[(k+2)(n+p)+(\ell+1)]^2 \; - \; \frac{1}{4k}[k(p-n)+2s-\ell]^2} \; . \tag{5.29}$$

The standard form of the string functions of parafermionic models is [30,31]:

$$\begin{aligned} c^{\ell}_{m}(\tau) \;&=\; \sum_{-|x|<y\leq|x|} sign(x) \; q^{(k+2)x^2 - ky^2} \; ; \\ &\qquad \text{with } (x,y) \text{ or } (\tfrac{1}{2}-x, \tfrac{1}{2}+y) \in \left(\tfrac{\ell+1}{2(k+2)}, \tfrac{m}{2k}\right) + \mathbb{Z}^2 \; . \end{aligned} \tag{5.30}$$

An equivalent form for the string function can also be obtained from (5.21). Taking $\xi = \frac{p}{2}(e_1 - e_2)$, $\gamma = n(e_1 - e_2)$, $\Lambda = \frac{\ell}{2}(e_1 - e_2)$ and $\chi = \frac{m}{2}(e_1 - e_2)$, one obtains:

$$c^\ell_m(\tau) \;=\; \frac{1}{\eta(q)^2} \sum_{\epsilon=\pm 1} \sum_{n=-\infty}^{\infty} \sum_{p=0}^{\infty} \epsilon\,(-1)^p\, q^{\frac{1}{4(k+2)}[(k+2)(2n+p)+\epsilon(\ell+1)]\;-\;\frac{1}{4k}[kp+m]^2}\;, \quad (5.31)$$

with the selection rule $m \equiv \ell \bmod 2$.

It turns out to be a little involved to establish directly that these three forms for c^ℓ_m of are equivalent. The easiest is to show that $G^{(s)}_{s,\ell}(\tau) = c^\ell_{2s-\ell}(\tau)$, where the latter is given by (5.30). One simply has to parametrize the sums in (5.30) over the four sectors of the (x, y) plane, make modest use of (5.10), and then regroup the sums into the form of (5.29). The equivalence with (5.31) requires that one start by first breaking the sum into $n \geq 0$ and $n < 0$, and then breaking one of the two resulting sums into sums with $p \geq n$ and $p < n$, while the other sum is broken into sums with $p > n$ and $p \leq n$. One then appropriately relabels the summation variables, makes use of (5.10), and regroups the terms. The result is (5.31).

Thus one sees that the refined elliptic genus provides us with precisely the proper branching functions.

6. Fermionic screening

Before making some general comments about our results, there is one minor, and perhaps interesting, loose end that needs to be addressed.

So far we have accounted for all of the D-type screening that is involved in the $N = 2$ supersymmetric Coulomb gas description, but we have, as yet, said very little about the fermionic screening. This is most easily understood by looking at the simplest model, with $n = 1$. This model is based upon $SU_k(2)$, which has a Kac-Wakimoto realization in terms of a bosonic β–γ, or superghost, system and a single free boson. There is a single screening current for this model, and it can be written as a product of $\beta(z)$ and a bosonic vertex operator. To get to the Coulomb gas realization of the $N = 2$ model, one first bosonizes the superghost system according to [32]:

$$\begin{aligned} \beta &= (\partial\xi)\, e^{-\phi} = i(\partial\chi)\, e^{i\chi-\phi} \\ \gamma &= \;\;\eta\, e^{\phi} \;\;= e^{-i\chi+\phi}\;. \end{aligned} \quad (6.1)$$

Next, one tensors in a new $U(1)$, and factors out the appropriate diagonal $U(1)$ factor to arrive at the coset model $SU(2) \times U(1)/U(1)$. In this process, the $SU_k(2)$ screening current maps directly onto the D-type screener of the $N = 2$ supersymmetric model.

The necessity of having a fermionic screener creeps in at the point where one bosonizes the β–γ system. To recover the Hilbert space of the superghosts from the Hilbert space of the ξ–η, and ϕ system, or from the ϕ–χ Fock space, of (6.1), one has to fix the momenta $p_\phi - p_\chi$ and exclude all states involving the zero mode, ξ_0 [32]. An equivalent way of accomplishing the same thing is that one can allow states with $p_\phi - p_\chi \geq 0$, and then compute the cohomology of the fermionic charge $Q = \oint \eta$ [28]. Once again, if one translates this across to the $N = 2$ supersymmetric Coulomb gas language, one finds that this is precisely what is done by the fermionic screener. If one does not employ the fermionic screeners, one obtains infinitely many copies of the Hilbert space of the $N = 2$ supersymmetric model. These infinitely many copies are related by shifts in the momenta $p_\phi - p_\chi$.

The foregoing observations generalize in a fairly obvious way to the $N = 2$ supersymmetric Coulomb gas description of (1.1) for arbitrary n. Indeed in [33] it was shown how the copies of the physical Hilbert space can be reinterpreted as gravitational descendants of the matter sector. In terms of the characters derived in the last section, the fermionic screening can be seen rather explicitly as being responsible for the sums over the vector ξ in (5.18) and (5.21). As a consequence of this we see that the elliptic genus has implicitly taken care of this fermionic screening as well. The moral reason for why this happens is probably related to the fact that the elliptic genus originates from the Landau-Ginzburg formulation which can be intrinsically expressed in terms of superghosts as in sect. 3.3. It is only when the superghosts are bosonized that one needs to worry about the fermionic screeners explicitly.

The Landau-Ginzburg formulation, along with the work of [33], suggests another interesting possibility for the fermionic screening charges. Specifically, they can also be incorporated as parts of the supercharges, as in (3.47). The immediately apparent obstacle to doing this in the Coulomb gas language is that by definition, the screening charges commute with the chiral algebra, whereas the supercharges have non-zero $U(1)$ charge. To rectify this, one makes a very simple change to the fermionic screeners. Introduce the operators:

$$
\begin{aligned}
\widetilde{G}^-(z,\bar{z}) &= \sum_{i=1}^{n} \bar{\psi}_i \, e^{i\sqrt{k+n+1}\ \bar{\phi}_i(z,\bar{z})} \ , \\
\widetilde{G}^+(z,\bar{z}) &= \sum_{i=1}^{n} (\psi_i \ - \ \psi_{i-1}) \, e^{i\sqrt{k+n+1}\ (\phi_i(z,\bar{z}) - \phi_{i-1}(z,\bar{z}))} \ .
\end{aligned}
\qquad (6.2)
$$

At first sight, these operators appear to be nothing other than sums of the fermionic screeners in (3.34). The crucial difference is that the bosons in the vertex operator part are to be taken as the complete boson, *i.e.* as a function of z and $\bar{z}$, and not just the holomorphic, or left-moving, part. The effect of doing this is to give the operators, $\widetilde{G}^{\pm}(z,\bar{z})$ a right-moving $U(1)$ charge of ± 1. The idea is to now view $\widetilde{G}^{\mp}(z,\bar{z})$ as anti-holomorphic components of a conserved current whose holomorphic components are $G^{\pm}(z)$. (There will be similar holomorphic components to the anti-holomorphic, or right moving supercurrents $\bar{G}^{\mp}(\bar{z})$.) The motivation for doing this comes from the Landau-Ginzburg formulation and the corresponding corrections to the supercurrent due to the presence of a non-trivial superpotential. In the Coulomb gas language, the corrections to the supercharge become essential if one imagines perturbing the model by the conformal weight $(1,1)$ operators $\sum_i S_i(z)\bar{S}_i(\bar{z})$, where the S_i are the fermionic screening currents. In correlators, such perturbations yield the proper screening prescriptions in the conformal blocks. Hence if one does not perform the fermionic screening, then the supercharges receive the foregoing corrections. This is analogous to viewing the Landau-Ginzburg potential as a perturbation of the free theory.

One might naturally ask what one learns from this apparently somewhat perverse perspective. First, the Landau-Ginzburg potential does generate the foregoing modifications to the conserved supercurrents, and so connecting these modifications with screening currents yields a rather better understanding of the infra-red limit of the renormalization group flow of the Landau-Ginzburg theory. On a much more practical level, it was very much part of the original thinking in the Landau-Ginzburg program [23] that the Landau-Ginzburg potential should encode the structure of the modular invariant partition function. Thus one would hope that the same is true for the foregoing modifications of the supercurrents. In particular, if one looks at the right-moving vertex operator parts of (6.2) then these will map one between different representations of the extended $N=2$ super-chiral algebra, whereas the left-moving screening charge part will map into effectively equivalent representations. Some preliminary investigations for the simplest model indicate that this is true. If one considers the model (1.1) with $n=2$, then the partition function contains combinations of string functions and $U(1)$ characters for both the left and right moving sector. The foregoing vertex operator shifts the $N=2$, $U(1)$ charge by one, *and* shifts the m quantum number on c^{ℓ}_m by two units. This suggests that in the modular invariant partition function, a given left-moving character c^{ℓ}_m will be paired with all the right moving characters c^{ℓ}_{m+2p} for all p. It also indicates a particular correlation of these quantum numbers and $N = 2$ $U(1)$ charges. This is in fact what one finds when one decomposes the partition function into its component parts. This particular example is a little trivial since we have deduced something that the we could have easily inferred from the presence of the supercharge itself. However, for models with more than one Landau-Ginzburg variable the foregoing procedure will generate more than one vertex operator, and we will get more than one set of correlations between left-moving and right-moving characters. The situation is a little reminscent of the lattice structure that underlies non-trival modular invariants of affine Lie algebras [34].

Thus we suspect that the Landau-Ginzburg potential, and its Coulomb gas concomitants, contain the information about how left-moving and right-moving representations are locked together, and that the foregoing might provide a method of explicitly extracting this information.

7. Final Comments

We have shown in some considerable detail how the various formulations of the $N = 2$ super-W minimal models are interrelated and have shown how the elliptic genus and the Landau-Ginzburg potential can be used to get very detailed information about the partition functions of these models. There also remains the interesting question as to how to decode from the Landau-Ginzburg potential the content of the modular invariant. One would also like to know if one can get information from the elliptic genus about the complete partition function of the model, and not just about the characters above the Ramond ground states.

There are also natural questions about the underlying exactly solvable lattice models. Given that these models have now been constructed [35,36], one might hope to adapt some of the topological index results to the lattice model. One possible hope might be to extract the elliptic index from the lattice formulation without having to resort to detailed computations involving Bethe Ansatz or the corner transfer matrix. The fact that the free energy of these lattice models vanishes for topological reasons [36] gives us hope that the lattice models may contain other pieces of topological information.

From the point of view of the field theory alone, we think it remarkable that so much of the structure of the theory can be deduced from the Landau-Ginzburg potential alone. It compelling to see if yet more information can be obtained about other related, and perhaps even massive, models using Landau-Ginzburg methods.

Appendix A. Determining W_3 in the Landau-Ginzburg formulation

In this appendix we will consider the Landau-Ginzburg formulation of the model (1.1), with $n=2$, and we will obtain the W_3-current by writing down the most general Ansatz in superspace, imposing chirality and using the Landau-Ginzburg equations of motion. The current we are looking for has dimension two and therefore the top component of the current has dimension three. The possible terms in the ansatz can be reduced by realizing that current has to be neutral. This means that it contains an equal number of Φ_i^+ and Φ_i^- fields. The most general Ansatz contains eighteen terms and has the form:

$$\begin{aligned}\mathcal{W} \ = & \ a_1 D^+\Phi_1^+ D^-\Phi_1^- D^+\Phi_2^+ D^-\Phi_2^- + a_2\Phi_1^+\partial\Phi_1^- D^+\Phi_2^+ D^-\Phi_2^- + \\ & a_3\Phi_1^+\partial\Phi_1^- D^+\Phi_1^+ D^-\Phi_1^- + a_4\Phi_2^+\partial\Phi_2^- D^+\Phi_2^+ D^-\Phi_2^- + \\ & a_5\Phi_2^+\partial\Phi_2^- D^+\Phi_1^+ D^-\Phi_1^- + a_6\Phi_1^+\partial\Phi_2^- D^+\Phi_2^+ D^-\Phi_1^- + \\ & a_7\Phi_2^+\partial\Phi_1^- D^+\Phi_1^+ D^-\Phi_2^- + a_8\partial\Phi_2^+\partial\Phi_2^- + a_9\partial\Phi_1^+\partial\Phi_1^- + \\ & a_{10}D^+\Phi_1^+\partial D^-\Phi_1^- + a_{11}D^-\Phi_1^-\partial D^+\Phi_1^+ + a_{12}D^+\Phi_2^+\partial D^-\Phi_2^- + \\ & a_{13}D^-\Phi_2^-\partial D^+\Phi_2^+ + a_{14}\Phi_1^+\partial^2\Phi_1^- + a_{15}\Phi_2^+\partial^2\Phi_2^- + \\ & a_{16}\Phi_1^+\Phi_1^+\partial\Phi_1^-\partial\Phi_1^- + a_{17}\Phi_2^+\Phi_2^+\partial\Phi_2^-\partial\Phi_2^- + a_{18}\Phi_1^+\Phi_2^+\partial\Phi_1^-\partial\Phi_2^- \ ,\end{aligned} \tag{A.1}$$

where a_i are unknown coefficients. Most of these coefficients are determined by requiring that W satisfy:

$$\bar{D}^-\mathcal{W} \ = \ 0 \ . \tag{A.2}$$

In simplifying the expression that results from (A.2) one uses the Landau-Ginzburg equations of motion:

$$\bar{D}^- D^-\Phi_i^- \ = \ \tfrac{1}{2}\frac{\partial W}{\partial\Phi_i^+} \ , \tag{A.3}$$

along with the fact that the superpotential $W(\Phi_1^+,\Phi_1^-)$ has a very specific form. First, one needs to use the fact that W is quasihomogeneous, and hence:

$$W \ = \ \frac{1}{k+3}\left[\Phi_1^+ \ \frac{\partial W}{\partial\Phi_1^+} \ + \ 2 \ \Phi_2^+ \ \frac{\partial W}{\partial\Phi_2^+}\right] . \tag{A.4}$$

Secondly, the fact that the potential comes from a coset model determines its form uniquely. Indeed, the exact form is given implicitly by [9]:

$$W \ = \ \xi_1^{k+3} \ + \ \xi_2^{k+3} \ , \tag{A.5}$$

where

$$\Phi_1^+ \ = \ \xi_1 + \xi_2 \quad \text{and} \quad \Phi_2^+ \ = \ \xi_1 \ \xi_2 \ . \tag{A.6}$$

This form of the potential is uniquely characterized (up to trivial scaling) by the equation $\partial^2 W/\partial\xi_1\partial\xi_2 = 0$, which may be rewritten as:

$$\frac{\partial^2 W}{(\partial\Phi_1^+)^2} \ + \ \Phi_1^+ \ \frac{\partial^2 W}{\partial\Phi_1^+\Phi_2^+} \ + \ \Phi_2^+ \ \frac{\partial^2 W}{(\partial\Phi_2^+)^2} \ + \ \frac{\partial W}{\partial\Phi_2^+} \ = \ 0 \ . \tag{A.7}$$

This, along with quasihomogeneity, implies numerous relationships between partial derivatives of W. In particular, one finds:

$$\begin{aligned} \frac{\partial^2 W}{(\partial\Phi_1^+)^2} \ &= \ -\frac{k+2}{k+1} \ \Phi_1^+ \ \frac{\partial^2 W}{\partial\Phi_1^+\partial\Phi_2^+} \ - \ \frac{k+3}{k+1} \ \Phi_2^+ \ \frac{\partial^2 W}{(\partial\Phi_2^+)^2} \\ \frac{\partial W}{\partial\Phi_1^+} \ &= \ \frac{\Phi_1^+}{k+2}\frac{\partial^2 W}{(\partial\Phi_1^+)^2} \ + \ \frac{2\Phi_2^+}{k+2} \ \frac{\partial^2 W}{\partial\Phi_1^+\partial\Phi_2^+} \\ \frac{\partial W}{\partial\Phi_2^+} \ &= \ \frac{\Phi_1^+}{k+1}\frac{\partial^2 W}{\partial\Phi_1^+\Phi_2^+} \ + \ \frac{2\Phi_2^+}{k+1}\frac{\partial^2 W}{(\partial\Phi_2^+)^2} \ . \end{aligned} \tag{A.8}$$

Conversely, it is only when these relations (A.8) are satisfied that one can find a non-trivial solution to (A.2) for some choice of the coefficients a_i. After some algebra we found the following solution:

$$\begin{aligned} \mathcal{W} \ = & \ b_1 \ \mathcal{J}^2 \ + \ b_2 \ \partial\mathcal{J} \ + \ b_3 \ (D^+D^- - D^-D^+)\mathcal{J} \\ & + \ b_4 \left(-\tfrac{1}{2}\hat{\mathcal{J}}^2 \ + \ \tfrac{i}{\sqrt{2(k+2)(k+3)}} \ \partial\hat{\mathcal{J}} \ - \ \tfrac{1}{2(k+2)}\Phi_1^-\Phi_2^+D^+\partial\Phi_1^+D^-\Phi_2^- \right. \\ & \left. - \ \tfrac{i\sqrt{k+3}}{2(k+2)} \ \Phi_1^+\partial\Phi_1^-D^-\partial\Phi_2^-D^+\Phi_1^+ \right) \ , \end{aligned} \tag{A.9}$$

where

$$\hat{\mathcal{J}} \ \equiv \ \tfrac{i}{\sqrt{2(k+3)(k+2)}} \ \left(\Phi_1^+\partial\Phi_1^- - \tfrac{1}{2}(k+2)D^+\Phi_1^+D^-\Phi_1^- + \tfrac{1}{2}(k+3)D^+\Phi_2^+D^-\Phi_2^- \right) \tag{A.10}$$

The terms $\mathcal{J}^2$, $\partial\mathcal{J}$ and $(D^+D^- - D^-D^+)\mathcal{J}$ correspond to trivial solutions since $\mathcal{J}$ is the only dimension one supercurrent that is conserved (*i.e.* satisfies the chirality condition (A.2)). We can fix the coefficients in $\mathcal{W}$ up to an overall normalization by demanding $\mathcal{W}$ has the appropriate operator product expansion with $\mathcal{J}$. This gives:

$$\begin{aligned} \mathcal{W} = & \tfrac{k(k+5)}{2(k+2)(k+3)}(D^+D^- - D^-D^+)\mathcal{J} \ - \ \tfrac{(k+5)}{k+2} \ :\mathcal{J}^2: \\ & - \ \tfrac{(5k-3)}{k+3}\left(-\tfrac{1}{2}\hat{\mathcal{J}}^2 \ + \ \tfrac{i}{\sqrt{2(k+2)(k+3)}} \ \partial\hat{\mathcal{J}} \right. \\ & \left. - \ \tfrac{1}{2(k+2)} \ \Phi_1^-\Phi_2^+D^+\partial\Phi_1^+D^-\Phi_2^- \ - \ \tfrac{i\sqrt{k+3}}{2(k+2)} \ \Phi_1^+\partial\Phi_1^-D^-\partial\Phi_2^-D^+\Phi_1^+ \right) \end{aligned} \tag{A.11}$$

Note that (A.11) has a form that is identical to the one we obtained from the Drinfel'd-Sokolov reduction (3.38). The only apparent difference is in the definitions of the currents $\mathcal{J}$ and $\hat{\mathcal{J}}$. However, these currents can be mapped onto one another using the translation table (3.46).

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