CERN-TH.7518/94

THE RENORMALIZATION GROUP INSPIRED APPROACHES AND ESTIMATES OF THE TENTH-ORDER CORRECTIONS TO THE MUON ANOMALY IN QED

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ABSTRACT

We present the estimates of the five-loop QED corrections to the muon anomaly using the scheme-invariant approaches and demonstrate that they are in good agreement with the results of exact calculations of the corresponding tenth-order diagrams supplemented by the additional guess about the values of the non-calculated contributions.

CERN-TH.7518/94
December 1994

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I. Introduction

The direct analytical or numerical calculations of the higher-order terms to the physical quantities in the concrete renormalization schemes provide the important information about the behaviour of the corresponding perturbative approximations. However, there are also some other approaches to treat the problem of the extraction of certain information from the truncated perturbative series. These approaches are the principle of minimal sensitivity (PMS) [1] and the effective charges (ECH) prescription [2], which is equivalent *a posteriori* to the scheme-invariant perturbation theory [3]. Of course, it is better to use these approaches directly in the concrete orders of the perturbation theory, as was done in QCD in Refs. [4]-[8]. However, if one adopts the point of view that these methods really pretend to the role of "optimal" procedures in the sense that they might provide better convergence of the corresponding approximations in the non-asymptotic regime, it is possible to try to go one step further and apply the procedure of re-expansion of the "optimized" expressions in the coupling constant of an initial scheme. One can consider the residual $(N{+}1)$-th order term as the estimate of the $(N{+}1)$-th order correction in the initial scheme [1].

The re-expansion procedure was already applied for the analysis of the perturbative predictions for $(g-2)_\mu$ in QED [1, 9] (for related considerations see Ref. [10]) and for the estimates of the QCD corrections to definite physical quantities. In these works, the quantities under study are the Drell-Yan cross-section at the $O(\alpha_s^2)$-level [11], $R(s) = \sigma_{tot}(e^+e^- \to hadrons)/\sigma(e^+e^- \to \mu^+\mu^-)$, $R_\tau = \Gamma(\tau \to \nu_\tau + hadrons)/\Gamma(\tau \to \nu_\tau \overline{\nu}_e e)$, non-polarized and polarized Bjorken sum rules at the $O(\alpha_s^4)$ and even $O(\alpha_s^5)$- levels [12, 13] and the singlet contribution to the Ellis-Jaffe sum rule at the $O(\alpha_s^3)$-order [14].

It is clear that the re-expansion formalism, which is similar to the procedure used in Ref. [15] to predict the RG-controlable $ln(m_\mu/m_e)$-terms from the expression for $(g-2)_\mu$ through the effective coupling constant $\overline{\alpha}(m_\mu/m_e)$, correctly reproduces the RG-controlable terms [1], [16]. One can also hope that it can give the impression about the possible values of the constant terms as well. This hope is based on the observation of the existence of a satisfactory agreement of the results of application of the re-expansion procedure in QED [9] and QCD [12, 13] with the results of the explicit calculations. It should be stressed that on the contrary to the RG considerations of Ref. [15], the "optimization methods" are dealing with the full RG-invariance of the quantities under consideration, which produce the additional equations, relevant to the freedom in the choice of the higher-order coefficients of the β-function. The solution of these equations gives the possibility to define the sets of scheme-invariants [1] which are the cornerstones of the "optimization" methods.

However, in the definite cases the procedure of re-expansion of the "optimized" results can run against some reef, which was overlooked in the process of some previous applications [1, 9, 10]. In the case of the analysis of the perturbative series for $(g-2)_\mu$ this problem grows out from the non-careful treatment of the light-by-light scattering graphs with the electron loop coupled to the external photon line.

In Sec.I of this work we repeat the decription of the basis of the formalism used by us. The exact expressions for the terms in the re-expansion formulas are derived. It is demonstrated that the estimates obtained using the re-expansion of the ECH expressions are identical to the results of calculations of the $(N+1)$-th order corrections in the special scheme, where all lower order coefficients of the physical quantities and the QCD β-function are defined in a certain fixed scheme (in the case of QED the on-shell (OS) scheme is usually used) and the $(N+1)$-th order coefficient of the β-function coincides with the $(N+1)$-th order scheme-invariant coefficient of the ECH β-function β_{eff}.

In Sec.II, using the information about the four-loop coefficient of the QED β-function in the OS scheme [17] we generalize the considerations of Refs. [1, 9, 10] to the five-loop level. We follow the proposals of Ref. [18] and consider the light-by-light scattering graphs mentioned above separately in our RG-inspired analysis. We show that this empirical improvement leads to more satisfactory and thus more reliable estimates of the five-loop contributions to $(g-2)_\mu$ than in the case of non-separation of the light-by-light scattering contributions. Finally in the Appendix we present the expressions for the six-loop RG-controllable contributions to the muon anomalous magnetic moment which follow from the analysis of Sec.II.

II. The Description of the Formalism

Consider first the order $O(a^N)$ approximation of a Euclidean renormalization group invariant quantity

$$D_N = d_0 a(1 + \sum_{i=1}^{N-1} d_i a^i) \tag{1}$$

with $a = \alpha_s/\pi$ being the solution of the corresponding renormalization group equation for the β-function which is defined as

$$\mu^2 \frac{\partial a}{\partial \mu^2} = \beta(a) = -\beta_0 a^2 (1 + \sum_{i=1}^{N-1} c_i a^i) \ . \tag{2}$$

The coefficients $d_i, i \geq 1$ and $c_i, i \geq 2$ are scheme-dependent. In order to calculate them in practice it is necessary to specify the scheme of subtractions of the ultraviolet divergences. In QED the OS scheme is commonly used. However, this scheme is not the unique prescription for fixing the RS ambiguities, which affect the values of these coefficients. In both phenomenological and theoretical studies other methods are also widely applied.

The PMS [1] and ECH [2] prescriptions stand out from various methods of treating scheme-dependence ambiguities. Indeed, they are based on the conceptions of the scheme-invariant quantities, which are defined as the combinations of the scheme-dependent coefficients in Eqs. (1) and (2). Both these methods pretend to the role of "optimal" prescriptions, in the sense that they might provide better convergence of the corresponding approximations in the non-asymptotic regime, and thus allow an estimation of the uncertainties of the perturbative series in the definite order of perturbation theory. Therefore, applying these "optimal" methods one can try to estimate the effects of the order $O(a^{N+1})$-corrections starting from the approximations $D_N^{opt}(a_{opt})$ calculated in a certain "optimal" approach [1], [9], [16].

Let us follow the considerations of Ref. [1] re-expand $D_N^{opt}(a_{opt})$ in terms of the coupling constant a of the particular scheme

$$D_N^{opt}(a_{opt}) = D_N(a) + \delta D_N^{opt} a^{N+1} \tag{3}$$

where

$$\delta D_N^{opt} = \Omega_N(d_i, c_i) - \Omega_N(d_i^{opt}, c_i^{opt}) \tag{4}$$

are the numbers which simulate the coefficients of the order $O(a^{N+1})$-corrections to the physical quantity, calculated in the particular initial scheme. The coefficients Ω_N can be obtained from the following system of equations:

$$\begin{aligned} \frac{\partial}{\partial \tau}(D_N + \Omega_N a^{N+1}) &= O(a^{N+2}), \\ \frac{\partial}{\partial c_i}(D_N + \Omega_N a^{N+1}) &= O(a^{N+2}), \;\; i \geq 2 \end{aligned} \tag{5}$$

where the parameter $\tau = \beta_0 \ell n(\mu^2/\Lambda^2)$ represents freedom in the choice of the renormalization point μ. The conventional scale parameter Λ will not explicitly appear in all our final formulas. The system of these equations can be solved following the lines of ref. [1]. Let us stress again that the difference between the "optimization" equations and the RG-approach of Ref. [15] lies in the fact that the latter one is dealing with the first equation from the the system of Eq.(5) only. The quantities Ω_l can be related to the scheme invariants ρ_l in the following way:

$$\rho_l = d_l + \frac{1}{l-1} c_l - \Omega_l(d_1, ..., d_{l-1}; c_1, ..., c_{l-1}). \tag{6}$$

Note that the general expressions of the scheme-invariants ρ_l and of the correction terms Ω_l can be defined in different ways. Various definitions differ by scheme-independent constant terms. We are choosing these correlated constant terms imposing the condition that the expressions for the scheme-invariants ρ_l are connected with the coefficients c_l^{ECH} of the ECH β-function

$$\beta_{eff}(a_{ECH}) = -\beta_0 a_{ECH}^2 \Big(1 + c_1 a_{ECH} + \sum_{i\geq 2} c_i^{ECH} a_{ECH}^i\Big) \tag{7}$$

as

$$\rho_l = \frac{c_l^{ECH}}{l-1} \tag{8}$$

where

$$D(a_{ECH}) = d_0 a_{ECH}(a) \; . \tag{9}$$

The concrete expressions for the invariants ρ_l and thus for the correction terms Ω_l can be derived from the following equation:

$$\beta_{eff}(a_{ECH}) = \frac{\partial a_{ECH}}{\partial a} \beta(a). \tag{10}$$

We present here the final expressions, which are already known [1]:

$$\Omega_2 = d_0 d_1 (c_1 + d_1), \tag{11}$$

$$\Omega_3 = d_0 d_1 (c_2 - \frac{1}{2} c_1 d_1 - 2d_1^2 + 3d_2) \ . \tag{12}$$

and the new terms which we evaluated

$$\Omega_4 = \frac{d_0}{3}(3c_3 d_1 + c_2 d_2 - 4c_2 d_1^2 + 2c_1 d_1 d_2 - c_1 d_3 + 14 d_1^4 - 28 d_1^2 d_2 + 5 d_2^2 + 12 d_1 d_3) \tag{13}$$

$$\begin{aligned} \Omega_5 &= \frac{d_0}{4}(4c_4 d_1 - 8c_3 d_1^2 + 2c_3 d_2 - 4c_2 d_1 d_2 + 8 c_2 d_1^3 - 2c_1 d_4 + 6c_1 d_1^4 - 16 c_1 d_1^2 d_2 + 3c_1 d_2^2 \\ &\quad + 8c_1 d_1 d_3 - 48 d_1^5 + 120 d_1^3 d_2 - 48 d_1 d_2^2 + 16 d_2 d_3 - 56 d_1^2 d_3 + 20 d_1 d_4). \end{aligned} \tag{14}$$

These terms reproduce the RG controllable logarithmic contributions. In the case of the five-loop level one can reobtain the QED results presented in Ref. [19]. We discuss this point in more detail in the next Section.

It should be stressed that in the ECH approach $d_i^{ECH} \equiv 0$ for all $i \geq 1$. Therefore one gets the following expressions for the higher-order corrections in Eq. (3):

$$\delta D_2^{ECH} = \Omega_2(d_1, c_1) \tag{15}$$

$$\delta D_3^{ECH} = \Omega_3(d_1, d_2, c_1, c_2) \tag{16}$$

$$\delta D_4^{ECH} = \Omega_4(d_1, d_2, d_3, c_1, c_2, c_3) \tag{17}$$

where Ω_2, Ω_3 and Ω_4 are defined in Eqs. (11), (12) and (13) respectively.

One can understand from Eqs. (6), (8) that the expressions for Ω_N and for the corrections δD_N^{ECH} in Eqs. (15)-(17) are the exact numbers which are related to the special scheme. This scheme is identical to the initial scheme at the lower order levels and is defined by the condition $c_N = c_N^{ECH}$ at the $(N{+}1)$-order, where c_N^{ECH} is considered as an unknown number. This means that the correction coefficients δD_N are related to the initial scheme only partly. However, it was shown in Refs. [12, 13] that in certain cases the numerical values of these coefficients are in satisfactory agreement with the results of the explicit calculations. *A posteriori* we consider this fact as an argument in favour of the possibility of the application of the re-expansion procedure in the cases discussed by us.

In order to find similar corrections to Eq. (3) in the N-th order of perturbation theory starting from the PMS approach [1], it is necessary to use the relations obtained in Ref. [20] between the coefficients d_i^{PMS} and c_i^{PMS} $(i \geq 1)$ in the expression for the order $O(a_{PMS}^N)$ approximation $D_N^{PMS}(a_{PMS})$ of the physical quantity under consideration:

$$d_i^{PMS} = \frac{1}{i+1}\Big(\frac{N-2i-1}{N-1}\Big) c_i^{PMS} + O(a_{PMS}) \tag{18}$$

where $c_1^{PMS} = c_1$. Using now Eq. (18) one can find the corresponding coefficients of the NLO approximation $D_2^{PMS}(a_{PMS})$

$$d_1^{PMS} = -\frac{1}{2} c_1 + O(a_{PMS}) \tag{19}$$

and the related expression for the NNLO correction δD_2^{PMS}

$$\delta D_2^{PMS} = \delta D_2^{ECH} + \frac{d_0 c_1^2}{4} \tag{20}$$

where δD_2^{ECH} is defined by Eqs. (15), (11).

Repeating now the similar considerations at the NNLO level we get from Eq. (18) the following expressions for the NLO and NNLO coefficients d_1^{PMS} and d_2^{PMS}

$$\begin{aligned} d_1^{PMS} &= 0 + O(a_{PMS}); \\ d_2^{PMS} &= -\frac{1}{3} c_2^{PMS} + O(a_{PMS}). \end{aligned} \tag{21}$$

Substituting now Eqs. (21) into Eq. (12) one can observe that the corresponding next-to-next-to-next-to-leading order (N^{3}LO) correction δD_3^{PMS} in the re-expansion formula of Eq. (3) identically coincide with δD_3^{ECH} defined by Eqs. (16), (12), namely that

$$\delta D_3^{PMS} = \delta D_3^{ECH}. \tag{22}$$

A similar observation was made in Ref. [9] using different (but related) considerations. In fact this expression means that the order $O(a_{PMS})$ correction to d_1^{PMS} is cancelling the leading order term in the expression for d_2^{PMS}. We have checked this feature explicitely.

In fourth order of the perturbation theory the additional contribution to δD_4^{PMS} has a more complicated structure. In order to get it, it is necessary to substitute the following expressions

$$\begin{aligned} d_1^{PMS} &= \frac{1}{6} c_1 + O(a) \\ d_2^{PMS} &= -\frac{1}{9} c_2^{PMS} + O(a) \\ d_3^{PMS} &= -\frac{1}{4} c_3^{PMS} + O(a) \end{aligned} \tag{23}$$

into the expressions for the scheme-invariants ρ_2 and ρ_3 and then into the analytical expression for Ω_4. The expression for $\Omega_4(d_i^{PMS}, c_i^{PMS})$ in Eq. (4), which results from this analysis, reads:

$$\Omega_4(d_i^{PMS}, c_i^{PMS}) = \frac{d_0}{3}[\frac{1}{4} c_1 c_3^{PMS} - \frac{4}{81}(c_2^{PMS})^2 - \frac{5}{81} c_1^2 c_2^{PMS} + \frac{7}{648} c_1^4] \tag{24}$$

where

$$\begin{aligned} c_2^{PMS} &= \frac{9}{8}(c_2^{ECH} + \frac{7}{36} c_1^2) \\ &= \frac{9}{8}(d_2 + c_2 - d_1^2 - c_1 d_1 + \frac{7}{36} c_1^2) + O(a) \end{aligned} \tag{25}$$

and

$$c_3^{PMS} = 4(d_3 + \frac{1}{2} c_3 - c_2 d_1 - 3 d_1 d_2 + 2 d_1^3) + \frac{1}{2} c_1 (d_2 + c_2 + 3 d_1^2 - c_1 d_1 + \frac{1}{108} c_1^2) + O(a). \tag{26}$$

The expressions for Eqs. (24) - (26) are the pure numbers, which do not depend on the choice of the initial scheme. We will show in the next Section that in the case of the consideration of perturbative series for $(g-2)_\mu$ the numerical values of $\Omega_4(d_i^{PMS}, c_i^{PMS})$ are small and thus the *a posteriori* approximate equivalence of the ECH and PMS approaches is preserved for the quantities under consideration at this level also.

In certain considerations we will need to use a generalization of the expression for Ω_2 to the case when the intitial perturbative series is starting from corrections of order $O(a^p)$ with $p > 1$

$$D_N^{(p)} = d_0 a^p (1 + \sum_{i\geq 1} d_i a^N). \tag{27}$$

In this case the expression for the corrections terms read

$$(\Omega_2^{(p)})_{ECH} = \frac{p+1}{2p} d_0 d_1^2 + d_0 d_1 c_1. \tag{28}$$

The corresponding correction related to the PMS-improved expression was originally obtained in Ref. [1]:

$$(\Omega_2^{(p)})_{PMS} = \frac{p+1}{2p} d_0 d_1^2 + d_0 d_1 c_1 + \frac{p}{2(p+1)} d_0 c_1^2. \tag{29}$$

III. Applications to $(g-2)_\mu$

It is well-known that the expressions for anomalous magnetic moments of the electron $a_e = (g-2)_e/2$ and muon $a_\mu = (g-2)_\mu/2$ are known at the four-loop order from the results of calculations of Ref. [21] and Refs. [22], [23] respectively. The three-loop correction to a_e is now known with more accuracy than previously [24]. Combining the currently available information about the coefficients of the perturbative series for a_e and a_μ we have the following expressions:

$$a_e = 0.5a - 0.3294789...a^2 + 1.17619(21)a^3 - 1.434(138)a^4 \tag{30}$$

$$a_\mu - a_e = 1.09433583(7)a^2 + 22.869265(4)a^3 + 127.55(41)a^4 \tag{31}$$

where the expansion parameter $a = \alpha/\pi$ is related to the fine structure constant α and the last term in Eq. (31) is the result of the most recent calculations of Ref. [23] stimulated by the work of Ref. [17]. Combining Eq. (30) with Eq. (31) we arrive at the following approximate expression for a_μ:

$$a_\mu = 0.5a + 0.76585a^2 + 24a^3 + 126a^4 + O(a^5). \tag{32}$$

The order $O(a^5)$ correction to a_μ is only partly known [22]. Our aim will be to try to touch the existing uncertainty due to the totally non-calculated order $O(a^5)$-contribution to Eq. (32) using the re-expansion procedure outlined in the previous section , which is compatible with the RG-formalism.

It is known that in the OS-scheme the coefficients of the corresponding perturbative series depend on the large $\ln(m_\mu/m_e)$-contributions starting from the two-loop level. The parts of these effects are governed by the RG-method [15, 25] (for a recent application of the RG method to a_μ, see Refs. [26, 19]). However, there are also certain $\ln(m_\mu/m_e)$-contributions, which are not governed by the RG-method. They are associated with the light-by-light-scattering electron loop insertions coupled to the external photon line. These contributions appear first in the three-loop graphs, which were subsequently calculated numerically in the works of Ref. [27, 22] and recently evaluated analytically in the work of Ref. [28].

In view of the different origin of the lower $\ln(m_\mu/m_e)$-contributions we divide all diagrams into two classes. The first class contains all diagrams with an external muon vertex and dressed interenal photon lines (see Fig. 1). As well as in Ref. [22] we will not include the diagrams with electron loops to which four internal photon lines are attached. However, we will include four-loop diagrams typical to a_e but with substitution of the external electron vertex to the muon one. The second class of the diagrams includes the diagrams with electron light-by-light scattering subgraph, to which three and four internal photon lines are attached (see Fig.2). Let us stress that all $\log(m_\mu/m_e)$-terms of the diagrams contributing to the first class are totally controlled by the RG-method, while in class (II) only the parts of these contributions are governed by the RG-technique.

In accordance with our classification we represent the expression for a_μ in the following form

$$a_\mu = a_\mu^{(I)} + a_\mu^{(II)}. \tag{33}$$

The concrete contributions to Eq.(33) read

$$\begin{aligned} a_\mu^{(I)} &= d_0^{(I)} a(1 + d_1^{(I)} a + d_2^{(I)} a^2 + d_3^{(I)} a^3 + ...) &(34)\\ a_\mu^{(II)} &= d_0^{(II)} a^3(1 + d_1^{(II)} a + d_2^{(II)} a^2 + ...). &(35)\end{aligned}$$

Note, that the coefficients d_i $(i \geq 1)$ contain the RG-controllable $\ln(x) = \ln(m_\mu/m_e)$-terms. Indeed, the corresponding contributions to a_μ are governed by the RG-equation

$$(m^2\frac{\partial}{\partial m^2} + \beta(a)\frac{\partial}{\partial a})a_\mu^{(I,II)} = 0 \tag{36}$$

where $\beta(a)$ is the QED β-function in the OS-scheme, which is defined as

$$m^2\frac{\partial a}{\partial m^2} = \beta(a) = \sum_{i\geq 0} \beta_i a^{i+2}. \tag{37}$$

To our point of view, the separation of all diagrams to the two classes mentioned above is respected by the property of the RG-invariance. At least we do not know the arguments why the sum of the diagrams which belong to the class (I) and to the class (II) should not obey the RG-equations seperately.

The coefficients of the β-function are known at the four-loop level [17]. They have the following form

$$
\begin{aligned}
\beta_0 &= \frac{1}{3} \\
\beta_1 &= \frac{1}{4} \\
\beta_2 &= -\frac{121}{288} = -0.42 \\
\beta_3 &= \Big(\frac{5561}{5184} - \frac{23}{9}\zeta(2) + \frac{8}{3}\zeta(2)\ln(2) - \frac{7}{8}\zeta(3)\Big)\frac{1}{2} \\
&= -0.571.
\end{aligned} \tag{38}
$$

Thus, the related coefficients $c_i = \beta_i/\beta_0$ $(i \geq 1)$ read $c_1 = 3/4$, $c_2 = -1.26$, $c_3 = -1.713$. Let us write down the asymptotic expansions of the coefficients of the contributions a_μ as

$$
\begin{aligned}
d_0^{(I)} &= B_1 \\
d_0^{(I)} d_1^{(I)} &= B_2 + C_2 \ln(x) \\
d_0^{(I)} d_2^{(I)} &= B_3 + C_3 \ln(x) + D_3 \ln^2(x) \\
d_0^{(I)} d_3^{(I)} &= B_4 + C_4 \ln(x) + D_4 \ln^2(x) + E_4 \ln^3(x) \\
d_0^{(I)} d_4^{(I)} &= B_5 + C_5 \ln(x) + D_5 \ln^2(x) + E_5 \ln^3(x) + F_5 \ln^4(x)
\end{aligned} \tag{39}
$$

and

$$
\begin{aligned}
d_0^{(II)} &= \overline{B}_1 \\
d_0^{(II)} d_1^{(II)} &= \overline{B}_2 + \overline{C}_2 \ln(x) \\
d_0^{(II)} d_2^{(II)} &= \overline{B}_3 + \overline{C}_3 \ln(x) + \overline{D}_3 \ln^2(x).
\end{aligned} \tag{40}
$$

The coefficients C_i, D_i, E_i, F_i and $\overline{C}_i$, $\overline{D}_i$ can be related to the coefficients of the β-function using either the RG-considerations of Refs. [15, 25, 19] or the explicit expressions for the coefficients Ω_i and $\Omega_i^{(p)}$ in the corresponding re-expansion formulas (see Eqs.(11)-(14) and Eq.(28)). The results of the corresponding analysis have the following form

$$C_2 = 2\beta_0 B_1 \tag{41}$$

$$
\begin{aligned}
C_3 &= 4\beta_0 B_2 + 2\beta_1 B_1 \\
D_3 &= 4\beta_0^2 B_1
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
C_4 &= 6\beta_0 B_3 + 4\beta_1 B_2 + 2\beta_2 B_1 \\
D_4 &= 12\beta_0^2 B_2 + 10\beta_0\beta_1 B_1 \\
E_4 &= 8\beta_0^3 B_1
\end{aligned} \tag{43}
$$

$$
\begin{aligned}
C_5 &= 8\beta_0 B_4 + 6\beta_1 B_3 + 4\beta_2 B_2 + 2\beta_3 B_1 \\
D_5 &= 24\beta_0^2 B_3 + 28\beta_0\beta_1 B_2 + 6\beta_1^2 B_1 + 12\beta_0\beta_2 B_1 \\
E_5 &= 32\beta_0^3 B_2 + \frac{104}{3}\beta_0^2\beta_1 B_1 \\
F_5 &= 16\beta_0^4 B_1
\end{aligned} \tag{44}
$$

$$\begin{aligned}\overline{C}_2 &= 6\beta_0\overline{B}_1\\ \overline{C}_3 &= 8\beta_0\overline{B}_2+6\beta_1\overline{B}_1\\ \overline{D}_3 &= 24\beta_0^2\overline{B}_1.\end{aligned}\qquad(45)$$

Note, that in the case of the diagrams of set (II) the corresponding coefficients $\overline{B}_1$, $\overline{B}_2$ and $\overline{B}_3$ contain the contributions of the non-controllable by the RG method $\ln(x)$-terms.

Let us first discuss the applications of the procedure of Sec.II to the diagrams of set (I). In this case the correction terms $\Omega_2-\Omega_4$ reproduce all $\ln(x)$-contributions presented in Eqs. (39). Moreover, one can get from re-expansion procedure the exact values of the constant terms B_i $(i\geq 3)$ which do not depend on the $\ln(x)$-terms. In the case of the application of the ECH-improved variant of the OS-scheme these constant terms are defined by the conditions

$$\begin{aligned}B_i &= \Omega_{i-1}(d_0^{OS},d_1^{OS},...,d_{i-2}^{OS},c_1,..,c_{i-2}^{OS})\\ &= \Omega_{i-1}(B_1^{OS},...,B_{i-1},c_1,...,c_{i-2}^{OS}).\end{aligned}\qquad(46)$$

Similar terms which arise from the PMS-improved expressions can be obtained after taking into account the additional scheme-independent contributions derived in Sec.II. We will demonstrate that the numerical values of these contributions in the cases considered by us are not large.

The concrete values of the coefficients B_1, B_2^{OS}, B_3^{OS} are known from a comparison of the results of the RG-inspired analysis with the results of the analytical and numerical calculations [22]. The coefficient $B_1=0.5$ is of course well known. The asymptotic expression of the coefficient B_2, derived in the limit $m_e/m_\mu\to 0$, can be found in Ref. [22]: $B_2^{OS}=-\frac{25}{36}+a_e^{(4)}=-1.022923$. The value of the coefficient $B_3^{OS}=2.741$ was obtained in Ref. [22] after subtracting the contributions of the light-by-light scattering graphs of the set (II) and of the RG-controllable contribution of Eq. (39) from the expression for the three-loop correction to a_μ.

The value of the coefficient B_4^{OS}, which will be used by us, is different from the one given in Ref. [22]. The difference comes from the fact that contrary to the classification of Ref. [22], we are including into the considered set (I) the four-loop diagrams typical to a_e but with substitution of the electron vertex and internal electron loops to the muon ones. Moreover, it is necessary to modify the value of B_4^{OS} presented in Ref. [22] in accordance with the results of the analytical [17] and numerical [23] re-calculations of the diagrams with three-loop insertion into the internal photon line of the lowest order contribution to a_μ.

In order to determine the value of the coefficient B_4^{OS} in our case we used the following expression

$$B_4^{OS}=a_\mu^{(8)}-A_\mu^{(8)}(\gamma\gamma)-C_4\ln(x)-D_4\ln^2(x)-E_4\ln^3(x)\qquad(47)$$

where C_4, D_4 and E_4 are determined by Eqs.(43) and the value of $A_\mu^{(8)}(\gamma\gamma)\approx -116.7$ is the sum of the eight-order contributions of the diagrams with electron light-by-light scattering subgraphs [22]. The numerical value of the coefficient B_4^{OS} is thus $B_4^{OS}=-7.74$.

In order to study the predictive abilities of the re-expansion procedure described in Sec. II we present in Table 1 the numerical results of our estimates of the coefficients B_i $(i\geq 3)$ and compare them with the exact results for B_3^{OS} and B_4^{OS} presented above.

Order	B_i^{OS}	$B_i(ECH)$	$B_i(PMS)$
i=1	0.5	—-	—
i=2	$-$ 1.022923	—	—
i=3	2.741	1.326	1.396
i=4	-7.74	-5.48	-5.48
i=5	—	41.6	41.7

Table 1: Estimated values of the coefficients B_i for the diagrams of set (I).

One can see that the re-expansion procedure used by us is reproducing well enough the values of the coefficients B_3 and B_4 (it gives the correct sign and predicts the order of magnitude of these coefficients). Therefore, we hope that the estimate of the five-loop constant term B_5 is also rather realistic. Notice also the sign-alternating character of the results of the estimates presented in Table 1. This feature has something in common with the expectation that the RG-improved QED series for the Euclidean physical quantities should have sign-alternating behaviour [29].

Taking now into account the numerical value of the RG-controllable terms in Eqs.(39), (44) we arrive at the following estimate of the five-loop contributions of the diagrams of set (I) into a_μ

$$\begin{aligned} a_\mu^{(10)}(I) &= B_5^{OS} + 8.55 \\ &= 50.1(ECH) \\ &= 50.2(PMS). \end{aligned} \tag{48}$$

This estimate is almost non-sensitive to the concrete realization of the method of optimization. Notice also the effect of reduction of the value of the RG-controllable five-loop contributions presented in Refs. [19, 17]. Let us stress again that this fact is explained by necessity of the modifications of the results used in Refs. [19, 17] for the constant term B_4^{OS} derived in Ref.[22]. These modifications come from two ingredients. First, it is necessary to use the corrected expressions obtained in Refs. [17, 23] of certain four-loop graphs contributing to a_μ and second to add to the values of B_4^{OS} cited in Ref. [22] the constant terms due to the four-loop graphs typical of a_e but with a substitution of the electron vertex and internal electon loops to the muon ones. As is known from the results of Ref. [30] the addition to the considerations of Ref. [19] of the diagrams with the internal muon loops leads to strong cancellations. The comparison of the RG-controllable expressions of Eq.(48) with the similar one derived in Refs. [19, 17] indicates the same pattern.

Let us now discuss the applications of the outlined procedure for the estimates of the five-loop contributions of the diagrams with the light-by-light scattering subgraphs of Fig.(2). The special feature of the application of the re-expansion procedure to the diagrams of set (II) is that the corresponding terms $\overline{B}_i$ depend from the $\ln(x)$-terms non-controllable by the RG method. The most precise value of the coefficient $d_0^{(II)} = \overline{B}_1 = 20.94792...$ is known from the results of the analytical calculations of Ref. [28]. The numerical result for the sum of the corresponding four-loop graphs reads [22]

$$d_0^{(II)} d_1^{(II)} = 116.7. \tag{49}$$

Using now Eqs. (28), (29) we arrive at the following numerical estimate of the sum of the corresponding five-loop graphs

$$\begin{aligned} a_\mu^{(10)}(II) = \Omega_2^{(3)} &= 520.8(ECH) \\ &= 525.2(PMS) \end{aligned} \tag{50}$$

which includes the contribution of both RG-controllable and RG-non-controllable $\ln(x)$-terms.

The estimates of Eqs.(50), (48) should be compared with the one given in Ref. [22]

$$a_\mu^{(10)} = 570(140) \tag{51}$$

where the central value comes from the exact calculation of the contributions of the diagrams of the set of the light-by-light-type diagrams with two one-loop electron loops inserted into the internal photon lines (see Fig.3) and the error bar ± 140 stands for the estimate of other contributions (mainly RG-controllable ones). Our estimate of Eq. (50) is in very good agreement with the central value of the estimate of Eq. (51), while the estimate of Eq.(48) lies within the range of the careful estimate ± 140 of other contributions.

Note, however, that our total estimate of the considered tenth-order terms

$$\begin{aligned} a_\mu^{(10)} &= 570.9(ECH) \\ &= 575.4(PMS). \end{aligned} \tag{52}$$

also includes the contribution of the tenth-order diagrams depicted in Fig.4 and not included in the estimate of Eq. (51). These diagrams are formed by the insertion of the two-loop electron loop into the internal photon line of the lower light-by-light-type diagram. The contribution of this set of tenth-order diagrams was estimated in Ref. [31]. In order to understand the uncertainties of this estimate better it is useful to write down a RG-relation analogous to Eqs. (43) for this set of diagrams separately. Notice, that this contribution should be proportional to the two-loop coefficient of the β-function (which is determined by the graphs inserted into the internal photon line). Using this observation we arrive at the following relation

$$a_\mu^{(10)}(Fig.4) = \overline{B}_3(Fig.4) + 6\overline{B}_1\beta_1 \ln(x). \tag{53}$$

The main contribution to the estimate of Ref. [31] comes from the $\ln(x)$-term. Indeed, it has the following numerical value $6\overline{B}_1\beta_1 \ln(x) = 167.47$. This expression should be compared with the estimate $a_\mu^{(10)}(Fig.4) = 176 \pm 35$ given in Ref. [31]. One can see that this estimate is relevant to the RG-controllable contribution only. However, from the re-expansion procedure we can see that the contributions non-controllable by the RG-methods might be non-negligible (see Eg. (48)) and might affect the final numerical value of the diagrams belonging to this set. In order to study this guess in detail it is of interest to calculate the diagrams of Fig.4 explicitly. This calculational project is rather realistic [32].

It is also interesting to understand deeper the uncertainties due to other diagrams which are included neither in the "optimized" estimates of Eqs. (48), (50) nor in the original estimates of Ref. [22]. These diagrams, depicted in Fig. 5, form a new class of diagrams, which cannot be touched by the RG-inspired analysis. Indeed, one can hardly expect that any resummation procedures dealing with light-by-light-type graphs with three internal photon lines will be able to give the estimate of the light-by-light-type graph with five internal photon lines. The expressions for the $\ln(x)$-terms the non-controllable by the RG-method for this type of graphs can be read from the considerations of Refs. [33]. The result was used in Ref. [31] where the following estimate of the diagrams of Fig.5 was presented

$$a_\mu^{(10)}(Fig.4) = 185 \pm 85. \tag{54}$$

Combining our estimates of Eq.(52) with the ones of Eq.(54) we get the final result of applications of the re-expansion procedure supplemented by the estimates of the diagrams of new structure which are non-touched by this method

$$a_\mu^{(10)} \approx 700. \tag{55}$$

Let us stress again that the new ingredient of our analysis, which distinguishes it from previous applications of the re-expansion procedures in QED [1, 9, 10], is the separation of the considered initial diagrams to two classes, one of which consists of the diagrams, relevant to the effects of "new physics", discussed in more detail in Refs. [34, 33]. This procedure finds its support in the theoretical considerations of Ref.[18].

Moreover, we checked that in spite of the good agreement of the application of the re-expansion procedure to the non-separated sixth-order expressions for a_μ with results of the eighth-order calculations [9], the straightforward application of Eq.(13) to the non-separated eighth-order approximation of Eq. (32) results in the non-confortably large tenth-order estimate $a_\mu^{(10)} \approx 2160$. It is possible to understand that the reason of the success of the application of the re-expansion procedure to the non-separated sixth-order approximation is connected with the fact that the use of Eq.(12) (and more definitely its last term) gives for the eight-order light-by-light-type term the following estimate $a_\mu^{(8)}(\gamma\gamma) = 3d_0d_1d_2(\gamma\gamma) = 6a_\mu^{(4)}a_\mu^{(6)}(\gamma\gamma)$ which is known to be in good agreement with the results of direct numerical calculations [22]. However, at the next level of perturbation theory the expression for the correction term Ω_4 of Eq.(13) has a more complicated structure and thus the resulting non-separated estimates turn out to be non-confortably large. Moreover, we consider the satisfactory agreement of the results of separated estimates with the estimates given in Ref. [22] as an argument in favour of treating the diagrams with the electron-loop light-by-light scattering graphs separately.

Another interesting question is connected with the problem of the comparison of our estimates with the results of the recent applications of the Padé resummation technique to the perturbative series for a_μ [35] and $a_\mu - a_e$ [35, 36]. It should be stressed that in their analysis the authors of Refs. [35, 36] did not consider the light-by-light scattering graphs separately. Note also that the coefficients of the corresponding Padé approximants depend from the $\ln(x)$-terms. In spite of the fact that our results for a_μ are in qualitative agreement with the results of the applications of the Padé resummation method [35, 36] it is interesting to try to understand the predictive abilities of the Padé resummation methods better. Clearly, this problem is connected with the necessity of more detailed understanding of the relations of the Padé results to the ones obtained using the RG-inspired analysis. Note, that the Padé resummation methods can face the problem in reproducing the structure of the RG-controlable $\ln(x)$-terms.

Acknowledgements

It is a pleasure to thank R.N. Faustov, T. Kinoshita and P.M. Stevenson for discussions.

Appendix

Using Eq.(14) it is possible to derive the six-loop RG-controllable contributions to the diagrams of set (I):

$$d_0^{(I)} d_5^{(I)} = B_6 + C_6 \ln(x) + D_6 \ln^2(x) + E_6 \ln^3(x) + F_6 \ln^4(x) + G_6 \ln^5(x). \qquad (56)$$

The expressions for the logarithmic coefficients are

$$\begin{aligned}
C_6 &= 10\beta_0 B_5 + 8\beta_1 B_4 + 6\beta_2 B_3 + 4\beta_3 B_2 + 2\beta_4 B_1 \\
D_6 &= 40\beta_0^2 B_4 + 54\beta_0\beta_1 B_3 + 16\beta_1^2 B_2 + 32\beta_0\beta_2 B_2 + 14\beta_1\beta_2 B_1 + 14\beta_0\beta_3 B_1 \\
E_6 &= 80\beta_0^3 B_3 + \frac{376}{3}\beta_0^2\beta_1 B_2 + \frac{140}{3}\beta_0\beta_1^2 B_1 + 48\beta_0^2\beta_2 B_1 \\
F_6 &= 80\beta_0^4 B_2 + \frac{308}{3}\beta_0^3\beta_1 B_1 \\
G_6 &= 32\beta_0^5 B_1.
\end{aligned} \qquad (57)$$

However, in order to use these expressions in concrete considerations it is necessary to fix somehow the value of the five-loop coefficient β_4 of the QED β-function in the OS scheme.

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