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# Mirror Symmetry for Calabi–Yau Hypersurfaces in Weighted $\mathbb{P}_4$ and Extensions of Landau–Ginzburg Theory *

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**ABSTRACT**

Recently two groups have listed all sets of weights $\mathbf{k} = (k_1, \ldots, k_5)$ such that the weighted projective space $\mathbb{P}_4^{\mathbf{k}}$ admits a transverse Calabi–Yau hypersurface. It was noticed that the corresponding Calabi–Yau manifolds do not form a mirror symmetric set since some 850 of the 7555 manifolds have Hodge numbers $(b_{11}, b_{21})$ whose mirrors do not occur in the list. By means of Batyrev's construction we have checked that each of the 7555 manifolds does indeed have a mirror. The 'missing mirrors' are constructed as hypersurfaces in toric varieties. We show that many of these manifolds may be interpreted as non-transverse hypersurfaces in weighted $\mathbb{P}_4$'s, *i.e.* , hypersurfaces for which $dp$ vanishes at a point other than the origin. This falls outside the usual range of Landau–Ginzburg theory. Nevertheless Batyrev's procedure provides a way of making sense of these theories.

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## Contents

## 1. **Introduction**

The considerations of this article arose in relation to the construction by Klemm and Schimmrigk, and Kreuzer and Skarke [1,2] of a complete list, comprising 7555 cases, of all sets of weights $\mathbf{k} = (k_1, k_2, k_3, k_4, k_5)$ such that the weighted projective space $\mathbb{P}_4^{\mathbf{k}}$ admits a transverse polynomial $p$ of degree $d = \sum_{j=1}^{5} k_j$. That is the equations $dp = 0$, taken to hold in $\mathbb{C}^5$, are satisfied only when all five of the coordinates $x_j$ vanish. In this case it is known that the singularities of the weighted space can be resolved and that the resulting hypersurface, specified by the equation $p = 0$, is a Calabi–Yau manifold[3,4]. One reason for being interested in this list is that it manifests a compelling mirror symmetry. If one lists the Hodge numbers $(b_{11}, b_{21})$ corresponding to these manifolds then in some 90% of cases where some value of $(b_{11}, b_{21})$ occurs in the list the reflected numbers $(b_{21}, b_{11})$ also occur. The list however does not manifest a complete symmetry leading to the question "Where are the missing mirrors?"[2,5]. The context in which this should be discussed is toric geometry since it is the methods of toric geometry that permit the singularities of the ambient weighted projective spaces to be resolved. Within toric geometry there is a powerful method, due to Batyrev, for constructing the mirrors of a certain class of manifolds[6]. We will outline this construction in the following but for the present it suffices to remark that there is a natural way to associate a four-dimensional polyhedron to a Calabi–Yau hypersurface in certain toric varieties. In many cases the polyhedra associated to Calabi–Yau hypersurfaces have a property termed reflexivity by Batyrev. Batyrev shows that a Calabi–Yau manifold can be constructed from each reflexive polyhedron, $\Delta$, and observes that if $\Delta$ is reflexive then the dual polyhedron, $\nabla$, is also reflexive. Hence a Calabi–Yau manifold $\mathcal{W}$ may be constructed from $\nabla$. The new manifold has its Hodge numbers $(b_{11}, b_{21})$ reflected relative to those of $\mathcal{M}$. It is generally assumed that the $\mathcal{W}$ so constructed is the mirror of $\mathcal{M}$ although this has not been checked at the level of superconformal theories. It has not been shown that every Calabi–Yau manifold gives rise to a reflexive polyhedron. However if a given Calabi–Yau manifold is associated to a reflexive polyhedron then the mirror may be constructed from the dual polyhedron. We have checked by means of a computer program that all 7555 manifolds of the list are indeed associated to reflexive polyhedra and so have mirrors in virtue of Batyrev's construction[1].

Mirror symmetry was discovered 'empirically' by the generation of many models $\mathbb{P}_4^{\mathbf{k}}[d]$ admitting transverse polynomials[3] and, contemporaneously, by the work of Greene and Plesser[8] who explicitly constructed a class of mirror pairs and showed that the mirror pairs corresponded to the same superconformal theory. Although manifesting a striking mirror symmetry the list produced in [3] was not perfectly symmetric and this was thought

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[1] We note that A. Klemm has independently checked the reflexivity of the polyhedra corresponding to the members of the list whose reflected Hodge numbers do not appear[7].

to be due, in part, to the fact that the list of weights admitting transverse hypersurfaces was incomplete. It was intriguing therefore when the complete list of Refs.[1,2] manifested an asymmetry that was greater rather than less than the earlier list. The fact that we report here is that each manifold of the list nevertheless has a mirror. The cases that were missing correspond to manifolds that cannot be realized as transverse hypersurfaces in a weighted $\mathbb{P}_4$ but are to be understood as hypersurfaces in a toric variety. In at least some cases it is possible to think of the 'missing mirrors' as hypersurfaces in a weighted $\mathbb{P}_4$ for which the defining polynomial is not transverse, that is, the equations $dp = 0$ are satisfied at some point(s) apart from $x_j = 0$. The condition of transversality that was used to construct the list was employed because it was known to guarantee that the singularities of the hypersurface $p = 0$ could be resolved. This criterion is overly strong since it can happen that a zero of $dp$ lies on a coordinate plane where the embedding $\mathbb{P}_4^{\mathbf{k}}$ is singular. In some cases the singularity of the embedding space can be repaired in such a way as to produce a smooth Calabi–Yau manifold.

In the context of Witten's linear sigma models[9], the 'missing mirrors' do not have a Landau–Ginzburg phase (because of the non–transversality of of $p$) but instead have interesting new phases which may be considered as extensions of Landau–Ginzburg theories. An interesting feature is that the description of these models requires the introduction of extra coordinates and extra gauge symmetries associated to the blowing up of the singularities of the ambient space. This is in fact a general feature of toric geometry (following D. Cox[10]) which can be naturally implemented in Witten's linear sigma model. In many cases it is possible to eliminate these extra fields and present the model as a hypersurface in a $\mathbb{P}_4^{\mathbf{k}}$. The 'missing mirrors' are cases for which this is not possible. Strictly speaking, in the great majority of cases these extra fields should be retained in order to obtain a full description of the phases of the model. This is true even for models which can be represented as hypersurfaces in a $\mathbb{P}_4^{\mathbf{k}}$.

To underline the point that toric geometry and Batyrev's construction are the correct way to understand mirror symmetry we show for the manifolds of the list how the Berglund–Hübsch transposition rule for finding the mirror of a given manifold is a special case of Batyrev's method. It is perhaps worth remarking also that this procedure provides a useful way of computing the Hodge numbers of a Calabi–Yau hypersurface in a weighted $\mathbb{P}_4$, numbers that were previously calculated via Landau–Ginzburg theory.

The layout of the paper is the following: in Section 2 we recall Batyrev's procedure and describe its application to the list of weights. In Section 3 we show that the transposition rule of Berglund and Hübsch follows as a special case of Batyrev's construction. In Section 4 we study a manifold whose mirror does not appear in the list. If the mirror is interpreted as a hypersurface in a weighted $\mathbb{P}_4$ then the weights associated with the mirror are such that the hypersurface cannot be transverse (this is the reason that the mirror was not listed) thus, in this case, the mirror does not have a Landau–Ginzburg phase. The methods of toric geometry however afford a good description of this manifold. We describe in detail the chiral ring of this manifold and the exotic phases of the corresponding theory. Section 5 is concerned with an illustration of the application of toric methods to manifolds that are related by birational transformations. The existence of such transformations between manifolds is a pervasive phenomenon and the reason we include this here is that these transformations tend to relate manifolds whose Newton Polyhedra are similar and we wish to illustrate the fact that toric geometry provides a natural framework in which this can be discussed.

We present, in an appendix, a plot of the Hodge numbers of the manifolds of the list and of their mirrors which is now (by construction) symmetric.

## 2. **Toric Considerations**

*2.1. Newton polyhedra and Batyrev's construction*

Consider a weighted projective space $\mathbb{P}_r^{(k_1,\ldots,k_{r+1})}$, and let $d = k_1 + \ldots + k_{r+1}$. To understand hypersurfaces of degree $d$ as Calabi–Yau manifolds, we apply the ideas of Batyrev[6] and Aspinwall, Greene and Morrison[11], which we shall briefly review below. The basic idea is to construct the Newton polyhedron associated to degree $d$ monomials, and note that this is often a reflexive polyhedron.

Let $\mathbf{m} = (m_1, \ldots, m_{r+1})$ be a degree vector and let $(x_1, \ldots, x_{r+1})$ be the homogeneous coordinates of the weighted projective space. We denote by $x^{\mathbf{m}}$ the monomial $x_1^{m_1} x_2^{m_2} \ldots x_{r+1}^{m_{r+1}}$ and, as previously, we denote the weight vector by $\mathbf{k}$. The general polynomial of degree $d$ is then a linear combination

$$p \;=\; \sum_{\mathbf{m}} c_{\mathbf{m}} x^{\mathbf{m}}$$

of monomials $x^{\mathbf{m}}$ for which $\mathbf{m}.\mathbf{k} = d$. We shall sometimes speak of a monomial $\mathbf{m}$ as an abbreviation for the monomial $x^{\mathbf{m}}$. The convex hull of all $\mathbf{m}$'s of degree $d$ forms the *Newton polyhedron*, $\Delta$, of $p$.

If we naively formed the Newton polyhedron as the convex hull of the set of exponents of all degree $d$ monomials, we would typically get the point $\mathbf{1} = (1, \ldots, 1)$ (corresponding to the monomial $x_1 \cdots x_{r+1}$) as an interior point. We therefore translate this vector to the origin by subtracting $\mathbf{1}$. So given the degree $d$ monomial $x^{\mathbf{m}}$ (which satisfies $\mathbf{k}.\mathbf{m} = d$), we associate the vector $(a_1, \ldots, a_{r+1}) = (m_1 - 1, \ldots, m_{r+1} - 1)$. Since we have $\mathbf{k}.\mathbf{a} = 0$, we define the lattice

$$\Lambda \;=\; \{\, \mathbf{a} \in \mathbb{Z}^{r+1} \;|\; \mathbf{k}.\mathbf{a} = 0 \,\}.$$

There is correspondingly the dual lattice

$$\mathrm{V} \;=\; \mathbb{Z}^{r+1}/(\mathbb{Z} \cdot \mathbf{k}).$$

We put $\Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}$ and $\mathrm{V}_{\mathbb{R}} = \mathrm{V} \otimes \mathbb{R}$; these are the vector spaces in which the lattices are embedded.

The Newton polyhedron is therefore identified with

$$\Delta = \text{the convex hull of } \{\, \mathbf{a} \in \Lambda \mid a_i \geq -1 \;\forall i \,\} \;.$$

Note that $\Lambda$ is a lattice of rank $r$. If one of the weights $k_i$ ($k_1$ say) has the value unity then we can use the equation $\mathbf{k}.\mathbf{a} = 0$ to solve for $a_1$ and take $(a_2, \ldots, a_{r+1})$ as coordinates for $\Lambda$ (in this case, our computer program will make this choice of coordinates up to a sign change). If none of the weights is unity then we may of course still find coordinates for $\Lambda$ though the procedure is more involved.

A polyhedron, $\Delta$, is *reflexive* if the following three conditions obtain:

i. The vertices of $\Delta$ are integral, *i.e.* correspond to vectors $\mathbf{m}$ whose components are integers.
ii. There is precisely one integral point interior to $\Delta$.
iii. The 'distance' of any facet (a codimension 1 face) of $\Delta$ from the interior point is 1.

By 'distance' in (iii) is meant the following: We may choose the unique interior point as the origin of coordinates. Let $(y_2, y_3, \ldots, y_{r+1})$ be coordinates for $\Lambda_{\mathbb{R}}$ (if $k_1 = 1$ these can be taken to be the quantities $(a_2, \ldots, a_{r+1})$ ). The equation of a facet of $\Delta$ has the form

$$l_2 y_2 + l_3 y_3 + \ldots + l_{r+1} y_{r+1} \;=\; \delta \;.$$

Since the vectors $\mathbf{m}$ lie on an integral lattice the quantities $(l_2, \ldots, l_{r+1}, \delta)$ are rational and hence, by multiplying through by a suitable integer if necessary, can be taken to be integers with no common factor. Also, $\delta$ may be taken positive. With this understanding the 'distance' of this face from the origin is $\delta$.

In the remainder of this paper, we will always assume that $\Delta$ has been translated if necessary so that the origin becomes its unique integral interior point.

As a simple example consider the quintic threefold $\mathbb{P}_4[5]$. Here all the weights are unity and $\Delta$ is the set of integral points $(m_2, m_3, m_4, m_5)$ such that

$$0 \le m_i \;, \quad i = 2, \ldots, 5 \quad \text{and} \quad \sum_{i=2}^{5} m_i \le 5$$

which is a simplex. An interior point is such that these inequalities are satisfied with strict inequality. When this is the case we have

$$1 \le m_i \;, \quad i = 2, \ldots, 5 \quad \text{and} \quad \sum_{i=2}^{5} m_i \le 4$$

which has the unique solution $m_2 = m_3 = m_4 = m_5 = 1$. If we take the interior point as the origin and write $a_i = m_i - 1$ then we see that the five facets of the simplex are given by the five equations

$$\begin{aligned} -\,a_2 &= 1 \\ -\,a_3 &= 1 \\ -\,a_4 &= 1 \\ -\,a_5 &= 1 \\ a_2 + a_3 + a_4 + a_5 &= 1 \;. \end{aligned}$$

The unique interior point corresponds to the monomial $x_1x_2x_3x_4x_5$ and it is the case in general that the unique interior point corresponds to the product of the homogeneous coordinates when $\Delta$ is reflexive.

One of the key points of Batyrev's construction is that to a convex polyhedron $\Delta$ which has the origin as an interior point we may associate a dual, or polar polyhedron $\nabla$:

$$\nabla = \left\{ \mathbf{y} \;\middle|\; \mathbf{x}.\mathbf{y} \geq -1,\ \forall\, \mathbf{x} \in \Delta \right\} .$$

If $\Delta$ is reflexive then so is $\nabla$ and Batyrev has shown that we may associate a family of Calabi–Yau manifolds to $\nabla$. These Calabi–Yau manifolds are hypersurfaces in a toric variety $X_\nabla$ whose fan consists of the set of cones over the faces of $\nabla$. The hypersurfaces are associated to sections of the anticanonical bundle of $X_\nabla$. While $X_\nabla$ need not be smooth, it is Gorenstein, which means that the canonical bundle (which is a priori only defined on the smooth locus of $X_\nabla$) extends to a bundle on all of $X_\nabla$. Thus sections of the anticanonical bundle will still give Calabi–Yau manifolds.

The Hodge numbers $(b_{11}, b_{21})$ of a hypersurface $\mathcal{M}$ of this family may be calculated directly in terms of data derived from the Newton polyhedron. Let $\mathrm{pts}(\Delta)$ denote the number of integral points of $\Delta$ and let $\Delta_r$ denote the set of $r$-dimensional faces of $\Delta$. Write also $\mathrm{int}(\theta)$ for the number of integral points interior to a face, $\theta$, of $\Delta$ and define similar quantities with $\Delta$ and $\nabla$ interchanged. Duality provides a unique correspondence between an $r$-dimensional face, $\theta$, of $\Delta$ and a $(3-r)$-dimensional face $\tilde{\theta}$ of $\nabla$. With this notation the formulae[12,13] for the Hodge numbers are

$$b_{21}(\Delta) = \mathrm{pts}(\Delta) - \sum_{\theta\in\Delta_3} \mathrm{int}(\theta) + \sum_{\theta\in\Delta_2} \mathrm{int}(\theta)\,\mathrm{int}(\tilde{\theta}) - 5 \ ,$$

$$b_{11}(\Delta) = \mathrm{pts}(\nabla) - \sum_{\tilde{\theta}\in\nabla_3} \mathrm{int}(\tilde{\theta}) + \sum_{\tilde{\theta}\in\nabla_2} \mathrm{int}(\tilde{\theta})\,\mathrm{int}(\theta) - 5 \ .$$

Here, the expressions $b_{i1}(\Delta)$ denote the appropriate Hodge number of the Calabi–Yau hypersurfaces of $X_\nabla$. The notation emphasizes the role of $\Delta$ as the Newton polyhedron of the Calabi–Yau manifold (in the toric context, $\Delta$ arises as the Newton polyhedron associated to sections of the anticanonical bundle on $X_\nabla$). From these expressions it is clear that $b_{11}$ and $b_{21}$ are exchanged under the operation $\Delta \leftrightarrow \nabla$.

### *2.2. Application to weighted projective spaces*

With a computer program, we can check that Newton polyhedron is reflexive in all 7555 cases corresponding to transverse hypersurfaces in a weighted $\mathbb{P}_4$. For each weight vector $\mathbf{k}$ of the list the program makes a list of all possible monomials, constructs the corresponding Newton polyhedron and checks that it is reflexive. We insist here that this check was highly nontrivial since, as mentioned in the introduction, there was no theorem that the polyhedra associated to these examples had to be reflexive, except for the few cases for which the weights admit a polynomial of Fermat type.

Let $w_1, \dots w_5$ be the elements of the dual lattice V (with $r = 4$) induced by the standard coordinate vectors of $\mathbb{Z}^5$. Recall that the fan for $\mathbb{P}_4{}^{\mathbf{k}}$ is the simplicial fan with edges spanned by $w_1, \dots, w_5$. Since $w_i \in \nabla \cap \mathrm{V}$[14], the edges of the cones of the fan of $\mathbb{P}_4{}^{\mathbf{k}}$ are a subset of the edges of the fan of $X_\nabla$; hence $X_\nabla$ is birational to $\mathbb{P}_4{}^{\mathbf{k}}$; it follows the the Calabi–Yau hypersurfaces in $X_\nabla$ are birational to the original Calabi–Yau hypersurfaces in $\mathbb{P}_4{}^{\mathbf{k}}$. So Batyrev's construction is indeed an appropriate one to use. It would not have made geometric sense to work directly with $\mathbb{P}_4{}^{\mathbf{k}}$, since the hypersurfaces would have had unacceptable singularities.

We note that this construction generalizes examples that have appeared previously in the literature[15-17]. We illustrate the procedure with two examples the first corresponding to weights that admit a transverse polynomial and the second to weights that do not.

$\underline{\mathbf{k} = (1,1,1,2,2)}$

Consider first an example taken from the list: the weighted projective space $\mathbb{P}_4^{(1,1,1,2,2)}[7]$. This is not of Fermat type and was not, prior to this analysis, known to correspond to a reflexive polyhedron. The program lists 120 monomials and finds among them the 9 vertices

$$\begin{array}{rl}
v_1 : & (-6,\ \ 1,\ \ 1,\ \ 1) \\
v_2 : & (\ \ 1,\ \ 0, -2,\ \ 1) \\
v_3 : & (\ \ 1,\ \ 0,\ \ 1, -2) \\
v_4 : & (\ \ 1,\ \ 1, -2,\ \ 1) \\
v_5 : & (\ \ 1,\ \ 1,\ \ 1,\ \ 1) \\
v_6 : & (\ \ 1,\ \ 1,\ \ 1, -2) \\
v_7 : & (\ \ 0,\ \ 1,\ \ 1, -2) \\
v_8 : & (\ \ 1, -6,\ \ 1,\ \ 1) \\
v_9 : & (\ \ 0,\ \ 1, -2,\ \ 1)\ .
\end{array}$$

These, as discussed previously, are expressed in terms of the coordinates $(a_2, a_3, a_4, a_5)$ for $\Lambda$. These vertices define a polyhedron, $\Delta$, with the six facets

$$\begin{array}{lrl}
f_1 : & x_1 = 1\ , & \{2,3,4,5,6,8\}\ , \\
f_2 : & x_2 = 1\ , & \{1,4,5,6,7,9\}\ , \\
f_3 : & x_3 = 1\ , & \{1,3,5,6,7,8\}\ , \\
f_4 : & x_4 = 1\ , & \{1,2,4,5,8,9\}\ , \\
f_5 : & -x_3 - x_4 = 1\ , & \{2,3,4,6,7,9\}\ , \\
f_6 : & -x_1 - x_2 - 2\,x_3 - 2\,x_4 = 1\ , & \{1,2,3,7,8,9\}\ ,
\end{array}$$

where the lists on the right correspond to the vertices that are incident on each facet. The origin is the only integral interior point so we see that the polyhedron is reflexive. The dual polyhedron has vertices corresponding to the facets of $\Delta$

$$\begin{aligned}
\tilde{f}_1 &: (\ \ 1,\ \ 0,\ \ 0,\ \ 0)\ , \\
\tilde{f}_2 &: (\ \ 0,\ \ 1,\ \ 0,\ \ 0)\ , \\
\tilde{f}_3 &: (\ \ 0,\ \ 0,\ \ 1,\ \ 0)\ , \\
\tilde{f}_4 &: (\ \ 0,\ \ 0,\ \ 0,\ \ 1)\ , \\
\tilde{f}_5 &: (\ \ 0,\ \ 0,-1,-1)\ , \\
\tilde{f}_6 &: (-1,-1,-2,-2)\ .
\end{aligned}$$

The coordinates of each vertex being given by the coefficients in the equation of the corresponding facet of $\Delta$. The equations of the facets of the dual likewise correspond to the vertices of $\Delta$

$$\begin{array}{lrl}
\tilde{v}_1 : & 6\,y_1 - y_2 - y_3 - y_4 = 1\ , & \{2,3,4,6\}\ , \\
\tilde{v}_2 : & -y_1 + 2\,y_3 - y_4 = 1\ , & \{1,4,5,6\}\ , \\
\tilde{v}_3 : & -y_1 - y_3 + 2\,y_4 = 1\ , & \{1,3,5,6\}\ , \\
\tilde{v}_4 : & -y_1 - y_2 + 2\,y_3 - y_4 = 1\ , & \{1,2,4,5\}\ , \\
\tilde{v}_5 : & -y_1 - y_2 - y_3 - y_4 = 1\ , & \{1,2,3,4\}\ , \\
\tilde{v}_6 : & -y_1 - y_2 - y_3 + 2\,y_4 = 1\ , & \{1,2,3,5\}\ , \\
\tilde{v}_7 : & -y_2 - y_3 + 2\,y_4 = 1\ , & \{2,3,5,6\}\ , \\
\tilde{v}_8 : & -y_1 + 6\,y_2 - y_3 - y_4 = 1\ , & \{1,3,4,6\}\ , \\
\tilde{v}_9 : & -y_2 + 2\,y_3 - y_4 = 1\ , & \{2,4,5,6\}\ .
\end{array}$$

Note that the vertices $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{f}_4, \tilde{f}_6$ of $\nabla$ determine the simplicial fan of $\mathbb{P}_4^{(1,1,1,2,2)}$. The remaining vertex $\tilde{f}_5$ lies in the interior of the cone spanned by $\tilde{f}_1, \tilde{f}_2, \tilde{f}_6$, so the fan for $X_\nabla$ is obtained by subdividing this cone and all cones which contain it. This geometrically corresponds to blowing up the curve $x_1 = x_2 = x_3 = 0$, which is the singular locus of of $\mathbb{P}_4^{(1,1,1,2,2)}$. This illustrates the general point that $X_\nabla$ is birational to $\mathbb{P}_4^{\mathbf{k}}$ and is less singular. The reflexivity of $\Delta$ is in fact the starting point for the calculation of the instanton numbers for this model[18]. Many of the toric calculations in that work were done using our program as well as a similar program written later by A. Klemm.

$\underline{\mathbf{k} = (1,1,1,1,5)}$

An example of a space that does not admit any transverse polynomial is $\mathbb{P}_4^{(1,1,1,1,5)}$[9]. Since the homogeneous coordinate $X_5$ has weight 5 a polynomial of degree 9 must have the form

$$p = F_9 + X_5 G_4$$

with $F_9$ a polynomial of degree 9 and $G_4$ a polynomial of degree 4 in the variables $(X_1, X_2, X_3, X_4)$. It is clear that all the derivatives of such a $p$ vanish at the point $(0,0,0,0,1)$. For this space the program lists 255 monomials and finds among them the vertices

$$\begin{aligned}
v_1 &: \ (-8, \ \ 1, \ \ 1, \ \ 1)\ , \\
v_2 &: \ (-3, \ \ 1, \ \ 1, \ \ 0)\ , \\
v_3 &: \ (\ \ 1, -3, \ \ 1, \ \ 0)\ , \\
v_4 &: \ (\ \ 1, \ \ 1, -8, \ \ 1)\ , \\
v_5 &: \ (\ \ 1, \ \ 1, \ \ 1, \ \ 0)\ , \\
v_6 &: \ (\ \ 1, -8, \ \ 1, \ \ 1)\ , \\
v_7 &: \ (\ \ 1, \ \ 1, -3, \ \ 0)\ , \\
v_8 &: \ (\ \ 1, \ \ 1, \ \ 1, \ \ 1)\ .
\end{aligned}$$

However if we examine the facets of the polyhedron we find

$$\begin{array}{lrl}
f_1 : & x_1 = 1\ , & \{3,4,5,6,7,8\}\ , \\
f_2 : & x_2 = 1\ , & \{1,2,4,5,7,8\}\ , \\
f_3 : & x_3 = 1\ , & \{1,2,3,5,6,8\}\ , \\
f_4 : & x_4 = 1\ , & \{1,4,6,8\}\ , \\
f_5 : & x_4 = 0\ , & \{2,3,5,7\}\ , \\
f_6 : & -x_1 - x_2 - x_3 - 5\,x_4 = 1\ , & \{1,2,3,4,6,7\}\ .
\end{array}$$

The polyhedron is not reflexive owing to the fact that there is no interior point (an interior point would have to have $0 < x_4 < 1$, which is impossible). The origin now lies in the facet $f_5$.

## 3. **A Generalized Transposition Rule**

*3.1. Generalization of the Berglund–Hübsch rule*

In this section we generalize the transposition rule of Berglund and Hübsch[19]. For a review and examples see [20].

Suppose that, as previously, one starts with a weighted projective space $\mathbb{P}_r^{\mathbf{k}}$ whose Newton polyhedron $\Delta$ is reflexive. Suppose that one is also given $r+1$ monomials $\mathbf{m}_1, \ldots, \mathbf{m}_{r+1}$ of degree $d$. Let $\mathbf{a}_i = \mathbf{m}_i - \mathbf{1}$, so that $\mathbf{a}_i \in \Lambda$. Suppose in addition that the $\mathbf{a}_i$ span $\Lambda_{\mathbb{R}}$. Note that we do not require that the general polynomial formed from these $r+1$ monomials be transverse.

Form the matrix $M = (\mathbf{m}_1^T, \ldots, \mathbf{m}_{r+1}^T)$ of exponents of the terms of the polynomial $p = \sum_{\mathbf{m}} c_{\mathbf{m}} x^{\mathbf{m}}$, this is an $(r+1) \times (r+1)$ matrix (we think of $\mathbf{m}$ and $\mathbf{k}$ as row vectors). Then $\mathbf{k}M = d\mathbf{1}$. Equivalently, consider the matrix $A$ obtained from $M$ by subtracting 1 from each entry, to correspond to the translated polyhedron. Then $\mathbf{k}A = \mathbf{0}$, the zero vector.

Our assumptions imply that $A$ has rank $r$, since $\Lambda$ has rank $r$ and the $\mathbf{a}_i$ span $\Lambda_{\mathbb{R}}$. This implies that there are uniquely determined (up to an overall sign) relatively prime integers $\hat{k}_i$ such that

$$\sum_{i=1}^{r+1} \hat{k}_i \mathbf{a}_i = \mathbf{0} \ . \tag{3.1}$$

In other words, we have $\hat{\mathbf{k}}A^T = \mathbf{0}$, where $\hat{\mathbf{k}}$ is the vector $(\hat{k}_1, \ldots, \hat{k}_{r+1})$. This can of course be rephrased as

$$\sum_{i=1}^{r+1} \hat{k}_i \mathbf{m}_i = \hat{d}\mathbf{1} \ , \tag{3.2}$$

where $\hat{d} = \sum_i \hat{k}_i$. We make the final assumption that the $\hat{k}_i$ all have the same sign, and in particular may be chosen to be all positive.

With these assumptions, our assertion is that the mirror manifold is obtained from the original equation by the transposition rule. That is, one transposes $M$ to get $r+1$ new monomials in $\mathbb{P}_r^{\hat{\mathbf{k}}}$, forms their sum to get the transposed polynomial $\hat{p}$, takes an appropriate orbifold, and resolves singularities to get the mirror manifold.

More precisely, we are asserting that the conformal field theory derived from the superpotential corresponding to $p$ is identified via mirror symmetry with an orbifold of the theory derived from $\hat{p}$. While we do not have a field-theoretic proof of this assertion (see however[21]), our confidence is based on two observations: we can identify the symmetries of these theories, and the respective theories are associated with a pair of polar polyhedra.

Recall that the fan for the toric $r+1$-fold determined by the polar polyhedron $\nabla$ is just the normal fan of $\Delta$, which is the collection of cones over the proper faces of $\Delta$. To find the mirror family, this fan must further be subdivided, using all of the lattice points of $\Delta$ to span new edges. Note that this fan is a refinement of the fan $F$ obtained from coning the proper faces of the simplex spanned by the $r+1$ chosen lattice points.

Now the fan $\Sigma$ for $\mathbb{P}_r^{\hat{\mathbf{k}}}$ naturally lives inside

$$\mathrm{V}' = \mathbb{R}^{r+1}/\mathbb{R}\cdot\hat{\mathbf{k}}\ . \tag{3.3}$$

This $\Sigma$ is determined by coning the proper faces of the simplex spanned by the vertices $\mathbf{w}_i'$, where $\mathbf{w}_i'$ is the element of $\mathrm{V}'$ represented by the standard basis vector $e_i = (0,\ldots,0,1,0,\ldots,0)$ of $\mathbb{C}^{r+1}$. There is clearly a map from $\Sigma$ to $F$ induced by the linear map sending $\mathbf{w}_i'$ to $\mathbf{a}_i$. By simple considerations of toric geometry this corresponds to a finite quotient mapping[22]. The process of refinement of $F$ to get the subdivided normal fan corresponds to a birational transformation. In summary, the mirror family sits inside a partially desingularized orbifold of $\mathbb{P}_r^{\hat{\mathbf{k}}}$.

We now recall from [14] that to the points $e_i$ of V correspond monomials in the toric variety determined by $\nabla$, and one obtains a polynomial from adding up these terms. We can now observe that when referred back to $\mathbb{P}_r^{\hat{\mathbf{k}}}$ as described above, this coincides with the transposed polynomial $\hat{p}$. In other words, we must take the toric hypersurface given by Batyrev's procedure, then pull the equations back to $\mathbb{P}_r^{\hat{\mathbf{k}}}$, and check that the transposed monomials occur among the monomials so obtained. This can be done directly using the toric description, since for a weighted projective space, the exponent of a monomial belonging to a particular variable can be calculated by taking the inner product of the lattice point corresponding to the monomial with the standard basis vector corresponding to the variable (the one for which the position of the "1" is determined by the subscript of the variable). This immediately gives the desired result. (More precisely, we obtain the columns of $A^T$ by this procedure, then add $\mathbf{1}$ to get $M^T$.) Examples appear in [20].

The final thing to do is to verify that the group of geometric symmetries has the claimed order. Of course, the toric method gives the group explicitly, so we have given more information than noticed by Berglund and Hübsch (but see [23]). To do this, we must show that mirror symmetry exchanges the groups of geometric and quantum symmetries. This follows from several observations.

1. The order of the group of symmetries of the theory is just the determinant of $M$. Equivalently, this is also the index of the sublattice $K$ of $\mathbb{Z}^{r+1}$ spanned by the $\mathbf{m}_i$.
2. Let $e$ be the index (in $\Lambda$) of the sublattice $L$ spanned by the $\mathbf{a}_i$. Then $\det(M) = d\hat{d}e$.
3. The group of quantum symmetries of the manifold corresponding to $p$ is $\mathbb{Z}_d$. The group of geometric symmetries of the manifold corresponding to $\hat{p}$ has order $\det(M)/\hat{d} = de$.
4. The order of the orbifold given by the toric procedure is just $e$.

To establish these facts: for 1, we observe that the group of symmetries is just $\mathbb{Z}^{r+1}/K$ (the roots of unity needed to define the symmetries arise from describing the homomorphisms from $\mathbb{Z}^{r+1}/K$ to $\mathbb{C}^*$ in coordinates); 3 then follows immediately from 1. Observation 2 is established by exhibiting coset representatives for $K$ as follows.

Choose vectors $\vec{\alpha}_i$ for $1 \leq i \leq d$ such that $\mathbf{k}.\vec{\alpha}_i = i$. Put $\vec{\beta}_j = j\mathbf{1}$ for $1 \leq j \leq \hat{d}$. Pick coset representatives $\vec{\gamma}_k$ for $1 \leq k \leq e$ of $L$ in $\Lambda$. Then the set of all vectors $\vec{\alpha}_i + \vec{\beta}_j + \vec{\gamma}_k$ has the desired cardinality, and is seen to be a set of coset representatives of $K$ as follows. To see that these vectors span all of $\mathbb{Z}^{r+1}/K$, we pick an arbitrary vector $\mathbf{v} \in \mathbb{Z}^{r+1}$, and write $\mathbf{k} \cdot \mathbf{v} = qd + i$ with $1 \leq i \leq d$ and $q$ integral. Then $\mathbf{v} - \vec{\alpha}_i - q\mathbf{m}_1 \in \Lambda$. So for some $k$ we have that

$$\mathbf{v} - \vec{\alpha}_i - q\mathbf{m}_1 - \vec{\gamma}_k = \sum_l r_l \mathbf{a}_l = \sum_l r_l(\mathbf{m}_l - \mathbf{1}) \tag{3.4}$$

for some integers $r_l$. Equation (3.2) says that $\hat{d}\mathbf{1} \in K$; so can multiply out the right hand side of (3.4), and see that modulo $K$ it must be equal to $\vec{\beta}_j$ for some $j$. Thus $\mathbf{v}$ is congruent to $\vec{\alpha}_i + \vec{\beta}_j + \vec{\gamma}_k$ modulo $K$. On the other hand, suppose that

$$\vec{\alpha}_i + \vec{\beta}_j + \vec{\gamma}_k \equiv \vec{\alpha}_{i'} + \vec{\beta}_{j'} + \vec{\gamma}_{k'} \qquad \text{modulo } K. \tag{3.5}$$

Since $\mathbf{k} \cdot K = d\mathbb{Z}$, premultiplying (3.5) by $\mathbf{k}$ is well-defined modulo $d$. This gives $i \equiv i'$ mod $d$, which implies that $i = i'$. This in turn implies that

$$\vec{\beta}_j + \vec{\gamma}_k \equiv \vec{\beta}_{j'} + \vec{\gamma}_{k'} \qquad \text{modulo } K. \tag{3.6}$$

Furthermore, $K$ is a sublattice of $L + \mathbb{Z} \cdot \mathbf{1}$; so the congruence in (3.6) holds modulo $L + \mathbb{Z} \cdot \mathbf{1}$. This implies that $\vec{\gamma}_k \equiv \vec{\gamma}_{k'}$ modulo $L + \mathbb{Z} \cdot \mathbf{1}$. But $\vec{\gamma}_k$ and $\vec{\gamma}_{k'}$ lie in $\Lambda$, and $(L + \mathbb{Z} \cdot \mathbf{1}) \cap \Lambda = L$; hence $\vec{\gamma}_k \equiv \vec{\gamma}_{k'}$ modulo $L$. Thus $k = k'$ as desired, establishing 2.

Observation 4 follows from toric generalities [22].

### *3.2. The Berglund-Hübsch cases*

In this subsection, we show that the results of Berglund and Hübsch follow immediately from the previous considerations.

Recall that the polynomials under consideration are sums of expressions of the following type:

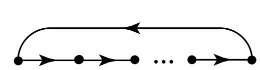
$$x^\beta =$$



$$x_1^{\beta_1}x_2 + x_2^{\beta_2}x_3 + \ldots + x_{n-1}^{\beta_{n-1}}x_n + x_n^{\beta_n} =$$



$$x_1^{\beta_1}x_2 + x_2^{\beta_2}x_3 + \ldots + x_{n-1}^{\beta_{n-1}}x_n + x_n^{\beta_n}x_1 =$$



(3.7)

The polynomial $p$ is transverse for these cases. Berglund and Hübsch assume further that $r = 4$. Since we now know by the computer program that $\Delta$ is reflexive in this case, we can use Batyrev's construction of the mirror.

In these cases, the matrix $M$ has a simple block diagonal form, and each of the blocks is easily seen to be nonsingular. This implies that the monomials $m_1, \ldots, m_5$ are linearly independent, and so span all of $\mathbb{R}^5$. Hence their translates $a_1, \ldots, a_5$ span all of $\Lambda_{\mathbb{R}}$.

It only remains to check that the weights $\hat{k}_i$ may all be taken to be positive. This is simplified by the following claim: if $\mathbf{v}$ satisfies $\mathbf{v}M^T = c\mathbf{1}$ for any constant $c$, then $\mathbf{v}A^T = \mathbf{0}$. As a consequence, this says that we must merely find positive weights $\hat{\mathbf{k}}$ which result in identical degrees for the five transposed monomials, and these weights are in fact the desired weights.

To see the claim, we note that $M^T$ is also invertible, hence the equation $\mathbf{v}M^T = c\mathbf{1}$ has a unique solution for $\mathbf{v}$. But since $A^T$ has rank 4 by the discussion at the beginning of this section, there is a unique solution of $\hat{\mathbf{k}}A^T = \mathbf{0}$ for $\hat{\mathbf{k}}$ (up to multiple). For such a $\hat{\mathbf{k}}$, we have $(c/\hat{d})\hat{\mathbf{k}}M^T = c\mathbf{1}$; the uniqueness noted above shows that $\mathbf{v} = (c/\hat{d})\hat{\mathbf{k}}$, which implies that $\mathbf{v}A^T = \mathbf{0}$.

Now $M^T$ also has a block diagonal form. Suppose that we can find positive weights for the variables in each block such that each of the transposed monomials in a block acquires the same weight. Then we can rescale the weights in each block relative to the other blocks to ensure that the weights of the monomials from all blocks agree with each other. In other words, we reduce the problem to consideration of each of the three types given in (3.7).

We finally check that we can find such weights for each of the three types of blocks. The first case is trivial, and the second is similarly straightforward. The third case results from calculation. We illustrate the calculation for the most difficult case, the "loop" case with $n = 5$.

Here we have to solve the system of equations

$$\hat{k}_1\beta_1 + \hat{k}_2 = \hat{k}_2\beta_2 + \hat{k}_3 = \hat{k}_3\beta_3 + \hat{k}_4 = \hat{k}_4\beta_4 + \hat{k}_5 = \hat{k}_5\beta_5 + \hat{k}_1\ . \tag{3.8}$$

The solution is given by

$$\hat{k}_i = \lambda^{-1}\Big[1 + \beta_{i-1}(\beta_{i-2} - 1) + \beta_{i-1}\beta_{i-2}\beta_{i-3}(\beta_{i-4} - 1)\Big]\ , \tag{3.9}$$

where we think of the subscripts in $\beta_i$ as indexed by $\mathbb{Z}_5$ and $\lambda$ is chosen so as to render the $\hat{k}_i$ mutually prime. Now each $\beta_i > 1$, since otherwise the degree of each monomial in (3.7) could not be equal to $\sum_{i=1}^5 k_i$, so it is clear the form of (3.9) that all of the $\hat{k}_i$ are positive.

Finally, we would like to remark that the examples in the list that had no known mirror did not fall within the Berglund–Hübsch cases. At the time, the transversality of the polynomial was required to have a well defined Landau–Ginzburg theory and transversal polynomials for these examples always have more than five monomials. These models then seemed to necessitate a non-square matrix $M$ and therefore the transposition rule could not be applied. Of course, we now understand that it is not necessary to insist on the transversality of the polynomial and that we only need to choose monomials that span the lattice $\Lambda_{\mathbb{R}}$.

*3.3. A non-transverse example*

Consider the example $\mathbb{P}_4[5]$, with polynomial

$$p = x_1^3 x_2 x_3 + x_2^5 + x_3^5 + x_4^5 + x_5^5 \ .$$

This polynomial is not transverse at $(1,0,0,0,0)$. The transposed polynomial is

$$\hat{p} = y_1^3 + y_1 y_2^5 + y_1 y_3^5 + y_4^5 + y_5^5$$

in $\mathbb{P}_4^{(5,2,2,3,3)}[15]$. The matrix $M$ has determinant 1875, while $d = 5$ and $\hat{d} = 15$. Thus the group of geometric symmetries of the transposed polynomial has order $1875/15 = 125$. To reduce this to the group of order 5 which is the group of quantum symmetries of the original manifold, we have to take an orbifold by a group of order 25, and this group can easily be written down explicitly if desired. This is seen to coincide with the toric description.

*3.4. A cautionary note*

We have given above a proof of the Berglund–Hübsch rule which shows that the rule is applicable outside the domain in which it was originally stated. Berglund and Hübsch required that it be possible to write a transverse polynomial with five monomials. We have seen that it is sufficient to choose five monomials that form a basis for $\Lambda$. There is however a catch: which is that it is important to keep in mind that a set of weights $\hat{\mathbf{k}}$ may arise which does not permit the existence of any transverse polynomial of degree $\hat{d} = \sum \hat{k}_i$. More generally, the transposed polynomial of a non-transverse polynomial need not be transverse, thus defining a singular hypersurface in the class $\mathbb{P}^{\hat{\mathbf{k}}}[\hat{d}]$ which should be resolved to become a smooth (or at least less singular) hypersurface in a toric variety. The point we want to make here is that it might happen that this singular hypersurface corresponds to a point in the moduli space which lies in the common boundary of the moduli spaces for two (or more) inequivalent resolutions of the singularity. Said differently a set of weights does not necessarily specify a unique family of Calabi–Yau manifolds if the weights do not admit a transverse polynomial. Moreover a given hypersurface in a toric variety need not correspond to a hypersurface in a weighted projective space for any set of weights. The difficulty arises only if we insist on thinking in terms of hypersurfaces in weighted $\mathbb{P}_4$. As hypersurfaces in toric varieties specified by polyhedra the varieties are well defined.

Perhaps this can be clarified by the following example. Consider the manifolds

$$\begin{aligned} \mathcal{M}_1 &= \mathbb{P}^{(7,41,247,590,885)}[1770] \ , \qquad (b_{11}, b_{21}) = (294, 36) \\ \mathcal{M}_2 &= \mathbb{P}^{(4,41,147,343,494)}[1029] \ , \qquad (b_{11}, b_{21}) = (293, 38) \end{aligned}$$

which have reflexive Newton polyhedra and different Hodge numbers. (These examples were also missing a mirror previously.) An indiscriminate application of the Berglund–Hübsch rule to the polynomials

$$\begin{aligned} p_1 &= x_1^{247}x_2 + x_2^{43}x_1 + x_3^7x_2 + x_4^3 + x_5^2 \\ p_2 &= x_1^{247}x_2 + x_2^{25}x_1 + x_3^7 \quad + x_4^3 + x_5^2x_2 \end{aligned}$$

would seem to show that both of these manifolds correspond to a mirror with weights $\hat{\mathbf{k}} = (1, 5, 36, 84, 126)$. Neither $p_1$ nor $p_2$ are transverse but can be made so by adding a suitable monomial (one is enough in these examples). Now

$$\mathcal{W} = \mathbb{P}^{(1,5,36,84,126)}[252] \quad \text{has} \quad (b_{11}, b_{21}) = (36, 294) \ ,$$

the Hodge numbers being calculated from the monomials and polyhedron corresponding to weights $\hat{\mathbf{k}}$. Note that the weights $(1, 5, 36, 84, 126)$ are such that they do not admit any transverse polynomial of degree $\hat{d} = 252$. By studying the polyhedra for $\mathcal{M}_1$ and $\mathcal{W}$, it is possible to show that the mirror class $\mathcal{W}_1$ for $\mathcal{M}_1$ actually coincides with $\mathcal{W}$. The singular space defined by the transposed polynomial of $p_2$ in the class $\mathcal{W} = \mathbb{P}^{(1,5,36,84,126)}[252]$ defines a point in the moduli space of $\mathcal{W}_2$ that resides in the common boundary of the moduli spaces of each of $\mathcal{W}_2$ and $\mathcal{W}_1$ and there is a resolution of this singularity which produces the class $\mathcal{W}_2$. To our knowledge, the class $\mathcal{W}_2$ cannot be described as a hypersurface in a weighted projective space but is a more general hypersurface in a toric variety.

## 4. **Manifolds with No Landau–Ginzburg Phase**

*4.1. A manifold whose mirror does not appear in the list*

Consider the manifold $\mathcal{M} = \mathbb{P}_4{}^{(21,37,108,295,424)}[885]$ which has $b_{11} = 295$ and $b_{21} = 7$. This is a manifold that appears in the lists of Klemm and Schimmrigk, and Kreuzer and Skarke. No mirror of this manifold appears in the list. It is shown in[24] that the periods of a given Calabi–Yau manifold $\mathcal{M}$ are most easily written as hypergeometric functions in terms of the weights $\hat{\mathbf{k}}$ of the mirror of $\mathcal{M}$. Since we are able to write the periods directly we may read off the weights of the mirror. Alternatively we may apply the Berglund–Hübsch procedure. Both procedures give the same result and suggest that the mirror, $\mathcal{W}$, is, in some sense, the manifold $\mathcal{W} = \mathbb{P}_4{}^{(1,1,5,14,21)}[42]$. The problem is that the coordinate $x_3$ in $\mathcal{W}$ has weight 5 so we cannot write down a transverse polynomial. (This is why $\mathcal{W}$ was not listed.) The best we can do is to write down a polynomial such as

$$p \;=\; x_1^{42} + x_2^{42} + x_3^8 x_1 x_2 + x_4^3 + x_5^2 \;.$$

This polynomial fails to be transverse at the point $(0, 0, 1, 0, 0)$. Since this point does not lie in the algebraic torus we might hope to be able to proceed via the Newton polyhedron. We find that the Newton polyhedron is reflexive and that $\mathcal{W}$ has its Hodge numbers exchanged relative to $\mathcal{M}$.

In the choices made by the computer program, the vertices of $\Delta$ have coordinates

$$\begin{array}{lll} (\;\;1,\;\;0,\;\;0,\;\;0), & (-1,\;\;3,\;\;4,\;\;5), & (\;\;0,-2,\;\;2,-1), \\ (\;\;0,-1,-1,\;\;0), & (\;\;0,\;\;1,\;\;0,\;\;0), & (\;\;0,\;\;2,\;\;2,\;\;3). \end{array}$$

We claim the this coincides with the polar of the Newton polyhedron of $\mathbb{P}_4{}^{(1,1,5,14,21)}$ after a coordinate change. The polar polyhedron has vertices

$$\begin{array}{lll} (-1,-5,-14,-21), & (\;\;1,\;\;0,\;\;0,\;\;0), & (\;\;0,\;\;1,\;\;0,\;\;\;0), \\ (\;\;0,\;\;0,\;\;\;1,\;\;\;0), & (\;\;0,\;\;0,\;\;0,\;\;1), & (\;\;0-3,-8,-12). \end{array}$$

Since the set of vertices consisting of all except the first and last vertices are linearly independent for each of these two polyhedra, there is a unique integral linear transformation taking one set to the other set. We need only notice that the first and last vertices also correspond under this transformation, thereby identifying the polyhedra as claimed.

Note that our polyhedron contains numerous other lattice points; in other words, $\mathbb{P}_4^{(1,1,5,14,21)}$ needs more blowups than the one determined by the insertion of the generator $(\;\;0,-3,-8,-12)$ into the fan in order to get the full 7 parameter theory. We will simplify our discussion of this example by not performing these further blowups, thereby constraining ourselves to a 2 parameter subfamily of the boundary of the moduli space.

We describe chiral rings by the procedure of Batyrev and Cox[25]. We must first identify the homogeneous coordinate rings of our toric varieties. We associate coordinates $X_1, \ldots, X_6$ to the 6 edges of the fan (in the order written), and denote by $S$ the polynomial ring that they generate. We need to weight them. While they are weighted by divisor classes according to Ref.[10], we do not need this geometry for our purposes and content ourselves to describe the weights as certain special multidegrees. To do this, we note the relations

$$\begin{array}{ccccccccccccc} 1\,v_1 & + & 1\,v_2 & + & 5\,v_3 & + & 14\,v_4 & + & 21\,v_5 & + & 0\,v_6 & = & 0 \\ 0\,v_1 & + & 0\,v_2 & + & 3\,v_3 & + & 8\,v_4 & + & 12\,v_5 & + & 1\,v_6 & = & 0 \end{array}$$

among the vertices of the polyhedron, numbered in the order listed above. These relations tell us that the weights are as follows.

$$\begin{array}{cc} X_1 & (1,0) \\ X_2 & (1,0) \\ X_3 & (5,3) \\ X_4 & (14,8) \\ X_5 & (21,12) \\ X_6 & (0,1) \end{array} \tag{4.1}$$

The anticanonical class as always has weight equal to the sum of the weights of all of the edges, in this case $(42, 24)$. The equation of $\mathcal{W}$ becomes

$$f = X_1^{42} X_6^{24} + X_2^{42} X_6^{24} + X_3^8 X_1 X_2 + X_4^3 + X_5^2 \ .$$

Note that we needed extra factors of $X_6$ to make the equation homogeneous. The chiral ring consists of the parts of the quotient ring $S/J_f$ of weights

$$(0,0)\ , \qquad (42,24)\ , \qquad (84,48)\ , \qquad (126,72)\ .$$

The monomial of top degree may be taken to be $X_1^{41} X_2^{41} X_3^6 X_4 X_6^{46}$.

*4.2. Phases of the model*

The phases for the model[2] $\mathbb{P}_4^{(1,1,5,14,21)}[42]$ can be obtained by requiring the vanishing of scalar potential $U$ of the corresponding linear sigma model[9]. For the present model this is given by

$$U = -\frac{1}{2} \sum_a \frac{1}{e_a^2} D_a^2 + |f|^2 + |X_0|^2 \sum_{i=1}^{6} \left| \frac{\partial f}{\partial X_i} \right|^2 \ ,$$

[2] Much of the analysis of the phases of this model emerged in discussions with R. Plesser.

where $X_0$ is the fiber coordinate on the canonical bundle and the $D_a$ are the $D$-components of vector superfields $V_a$. The gauge symmetry in our case is $U(1) \times U(1)$ with charges for the chiral fields $X_0, X_1, \ldots, X_6$ given in Eq. (4.1). Using their equations of motion in the linear sigma model action, the $D_a$'s are given by

$$D_1 = |X_1|^2 + |X_2|^2 + 5|X_3|^2 + 14|X_4|^2 + 21|X_5|^2 - 42|X_0|^2 - r_1$$

$$D_2 = 3|X_3|^2 + 8|X_4|^2 + 12|X_5|^2 + |X_6|^2 - 24|X_0|^2 - r_2 \ .$$

The scalar potential vanishes only when the $D$-terms vanish and when $X_0 = 0$ and either $f = 0$ or $df = 0$. Thus the minima of $U$ correspond to three possible branches

$$\begin{array}{ll} a) & X_0 = 0 \text{ and } f = 0 \\ b) & X_1 = X_2 = X_4 = X_5 = 0 \text{ and } X_0 \text{ is not zero} \\ c) & X_3 = X_6 = X_4 = X_5 = 0 \text{ and } X_0 \text{ is not zero.} \end{array}$$

From these, we find that the phases are

$$\begin{array}{ll} I & 0 < \frac{4r_1}{7} < r_2 < \frac{3r_1}{5} \\ II & 0 < r_2 < \frac{4r_1}{7} \\ III & r_2 < 0 \text{ and } r_2 < \frac{4r_1}{7} \\ IV & r_1 < 0 \text{ and } r_2 > \frac{4r_1}{7} \\ V & 0 < \frac{3r_1}{5} < r_2 \ . \end{array}$$

It is easy to see that branch $a$ covers phases $I$, $II$ and $V$, branch $b$ covers phase $IV$ and $V$ and branch $c$ covers phase $III$. This gives the phase diagram shown in Figure 1. The phases have the following interpretation:

<u>Phases $I$ and $II$</u>

$X_0 = 0$ and $f = 0$, and

$$\begin{array}{ll} \text{for phase } I & 0 < \frac{4r_1}{7} < r_2 < \frac{3r_1}{5} \quad \text{or} \\ \text{for phase } II & 0 < r_2 < \frac{4r_1}{7} \ . \end{array}$$

The sets of coordinates that do not vanish simultaneously in phase $I$ are $(X_1, X_2, X_4, X_5)$ and $(X_3, X_6)$. For phase $II$ we just interchange (1,2) with (3,6). Both of these phases

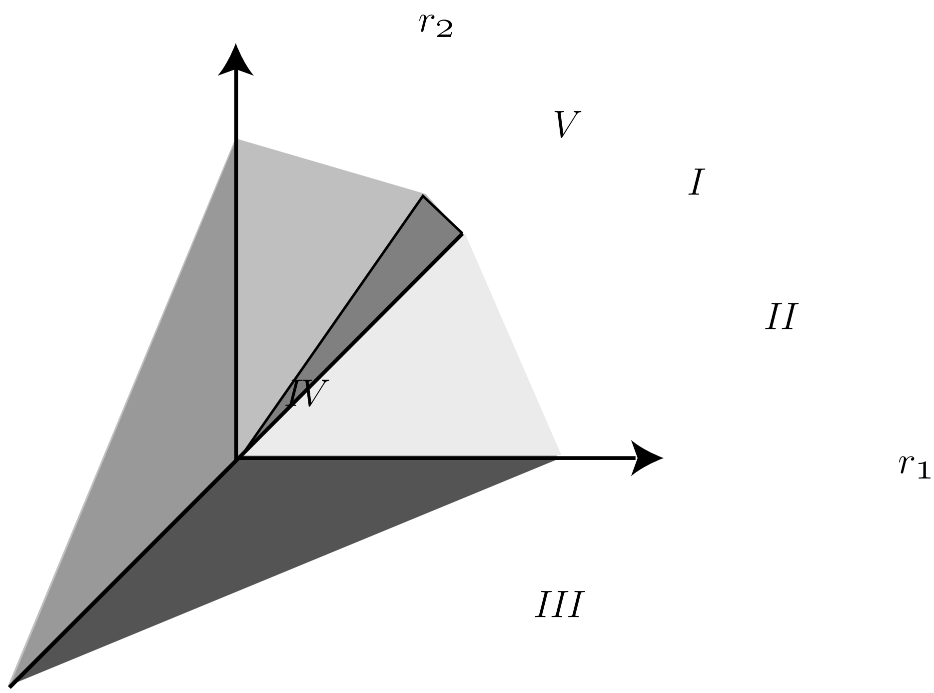



**Figure 4.1:** The phases of the theory

correspond to Calabi–Yau hypersurfaces which are related to each other by a (non-simple) flop. Note that the point $X_1 = X_2 = X_4 = X_5 = 0$ at which the polynomial fails to be transverse is forbidden in both phases, *i.e.* the singularity of $p$ does not belong to the toric variety in which the Calabi–Yau hypersurface is embedded. In fact, it is precisely in this sense that the Calabi–Yau hypersurface $\mathbb{P}_4^{(1,1,5,14,21)}[42]$ makes sense.

The boundary between phases $I$ and $II$ ($r_1 > 0$ and $r_2 = \frac{4r_1}{7}$) corresponds to a singular Calabi–Yau manifold with a conifold singularity at $X_1 = X_2 = X_3 = X_6 = 0$, which gives phase $I$ or phase $II$ depending on the choice of blowup.

Phase $III$

$$X_3 = X_6 = X_4 = X_5 = 0\ , \quad X_0 \neq 0 \qquad \text{and} \qquad r_2 < 0\ , \quad r_2 < \frac{4r_1}{7}\ .$$

This phase corresponds to a hybrid of a $\mathbb{P}_3^{(1,3,8,12)}[24]$ (= K3) Landau–Ginzburg orbifold fibered over the $\mathbb{P}_1$ defined by the coordinates $(X_1, X_2)$. The effective potential for the

Landau–Ginzburg orbifold is

$$W_{\text{eff}} = \sqrt{\frac{|r_2|}{24}}(c_1 X_6^{24} + c_2 X_3^8 + X_4^3 + X_5^2)$$

and, obviously, the quantum symmetry is $\mathbb{Z}_{24}$.

In the boundary between phases $II$ and $III$, the $\mathbb{Z}_{24}$ quantum symmetry is promoted to a U(1) gauge symmetry so the hybrid corresponds to a *gauged* Landau–Ginzburg model fibered over the $\mathbb{P}_1$ and with effective potential

$$W_{\text{eff}} = X_0(c_1 X_6^{24} + c_2 X_3^8 + X_4^3 + X_5^2)$$

where the fields $X_0, X_6, X_3, X_4, X_5$ have $U(1)$ charges $(-24, 1, 3, 8, 12)$. We can also describe the boundary by the mapping to $\mathbb{P}^1$ given by $(X_1, X_2)$; we approach the boundary by letting the size of the fibers approach zero.

Phase $IV$

$$X_1 = X_2 = X_4 = X_5 = 0\ , \quad X_0 \neq 0 \qquad \text{and} \qquad r_1 < 0\ , r_2 > \frac{4r_1}{7}\ .$$

This phase is completely new and does not admit as nice an interpretation as the phases already described.

The boundary between this phase and phase $III$ is the closest we can get to a Landau–Ginzburg orbifold. Since for this boundary $r_1 < 0$ and $r_2 = \frac{4r_1}{7}$ and all the fields vanish except $X_0$, we get a singular Landau–Ginzburg orbifold with $\mathbb{Z}_{42} \times \mathbb{Z}_{24}$ quantum symmetry.

Phase $V$

$$0 < \frac{3r_1}{5} < r_2$$

This phase is very strange too. First notice that branches $a$ and $c$ overlap over this phase. The interpretation here is that we get a singular Calabi–Yau $\mathbb{P}_4^{(1,1,5,14,21)}$ (from branch $a$) with the point singularity at $X_1 = X_2 = X_4 = X_5 = 0$ replaced by the strange model of phase $IV$ (from branch $c$).

The boundary between phases $IV$ and $V$ corresponds to the Calabi–Yau in phase $V$ shrinking to a point.

The boundary between phases $V$ and $I$ is a singular Calabi–Yau with singularity at $X_1 = X_2 = X_4 = X_5 = 0$. Aspinwall, Greene and Morrison[11] have called an "exoflop" the process of crossing this type of boundary between a smooth Calabi–Yau phase like phase $I$ and a phase like $V$ which consists of the Calabi–Yau with the point singularity replaced by a hybrid model. Along the locus $X_1 = X_2 = X_4 = X_5 = 0$, we note that $X_3$ cannot also vanish in this phase. There remains a $\mathbb{P}^1$ determined by the variables $X_0, X_6$. This $\mathbb{P}^1$ gets flopped *outside* the original Calabi–Yau, hence the term exoflop.

## 5. **Some Observations on Fractional Transformations**

### *5.1. A simple identification*

We wish to discuss, in the context of another example, some issues associated with fractional transformations. Consider the following pair of Calabi–Yau manifolds which we have taken from Ref.[26]

$$\begin{aligned}
\mathcal{M}_1 \in \mathbb{P}_4{}^{(1,1,1,1,3)}[7] \quad &: \qquad p_1 = x_1^7 + x_2^7 + x_3^7 + x_4^7 + x_1 x_5^2 \ . \\
\mathcal{M}_2 \in \mathbb{P}_4{}^{(1,2,2,2,7)}[14] \quad &: \qquad p_2 = y_1^{14} + y_2^7 + y_3^7 + y_4^7 + y_5^2 \ .
\end{aligned}$$

Both $\mathcal{M}_1$ and $\mathcal{M}_2$ have $b_{11} = 2$ and $b_{21} = 122$ and it is tempting to identify the two manifolds via the transformation

$$\begin{aligned}
x_1 &= y_1^2 \\
x_i &= y_i \ , \quad i = 2,3,4, \\
x_5 &= \frac{y_5}{y_1}
\end{aligned} \tag{5.1}$$

It is easy to check that while all 122 complex structure deformations of $\mathcal{M}_1$ can be realised as polynomial deformations of $p_1$ the same is not true of $\mathcal{M}_2$. Only 107 of the parameters of $\mathcal{M}_2$ can be realised as polynomial deformations of $p_2$. This is not surprising in virtue of the identification (5.1) since the 15 missing deformations are of the form

$$q x_5 = q \frac{y_5}{y_1} \tag{5.2}$$

with $q$ a quartic in the variables $(x_2, x_3, x_4)$. Note that in this case the $\mathbb{Z}_2$ ambiguity of the transformation is part of the projective equivalence.

Below we shall give a brief toric description of this birational relation. The point is that when the weighted hypersurfaces are desingularized as required by the general procedure, that the manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ are indeed isomorphic. (This is the case despite the fact that $\mathcal{M}_1$ and $\mathcal{M}_2$ have distinct Newton polyhedra.) The relation of this example to that of the previous section is that, apart from the fact that we can treat the present case by the same methods as the previous one, is that one of the questions that we are asking is how to represent the non-polynomial deformations of $p_2$. We are motivated in part by the following question. Suppose that we had started from $\mathcal{M}_2$ and that in virtue of the Landau-Ginzburg formalism, or the results of Berglund and Hübsch[27], we had learnt that the missing deformations were of the form (5.2). How would we see that the "correct" way to represent the deformations is by making the change of variables (5.1)? The example of the previous section was an extreme example for which the polynomial was not transverse for any choice of the parameters so that none of the elements of the chiral ring could be represented as deformations of a transverse polynomial.

*5.2. Isomorphism of $\mathcal{M}_1$ and $\mathcal{M}_2$*

The vertices of the Newton polyhedron $\Delta_1$ for $\mathbb{P}_4{}^{(1,1,1,1,3)}[7]$ are

$$\begin{array}{llll} (-1,-1,-1,-1), & (-1,-1,\ \ 0,\ \ 1), & (-1,-1,\ \ 6,-1), & (-1,\ \ 6,-1,-1), \\ (\ \ 6,-1,-1,-1), & (-1,-1,-1,\ \ 1), & (\ \ 0,-1,-1,\ \ 1), & (-1,\ \ 0,-1,\ \ 1). \end{array}$$

which yield the vertices of the polar polyhedron $\nabla_1$ as

$$\begin{array}{lll} (-1,-1,-1,-3), & (\ \ 1,\ \ 0,\ \ 0,\ \ 0), & (\ \ 0,\ \ 1,\ \ 0,\ \ 0), \\ (\ \ 0,\ \ 0,\ \ 1,\ \ 0), & (\ \ 0,\ \ 0,\ \ 0,\ \ 1), & (\ \ 0,\ \ 0,\ \ 0,-1). \end{array}$$

We note that $\nabla_1$ has no other lattice points besides the origin. It is clear from looking at the first 5 vertices that the normal fan of $\nabla_1$ describes a toric variety $X_1$ birational to $\mathbb{P}_4{}^{(1,1,1,1,3)}$. In fact, $X_1$ is just a blowup of $\mathbb{P}_4{}^{(1,1,1,1,3)}$ at the point $(0,0,0,0,1)$; the exceptional set is a $\mathbb{P}_3$. Numbering the edges from 1 to 6, we note first the fan of $\mathbb{P}_4{}^{(1,1,1,1,3)}$ has maximal cones spanned by the set of edges numbered

$$\{2,3,4,5\}\ , \quad \{1,3,4,5\}\ , \quad \{1,2,4,5\}\ , \quad \{1,2,3,5\}\ , \quad \{1,2,3,4\}\ .$$

Since the edge $(\ \ 0,\ \ 0,\ \ 0,-1)$ lies in the interior of the last cone, to get the fan for $X_1$ from the fan for $\mathbb{P}_4{}^{(1,1,1,1,3)}$, the last cone is replaced by the cones spanned by edges numbered

$$\{2,3,4,6\}\ , \quad \{1,3,4,6\}\ , \quad \{1,2,4,6\}\ , \quad \{1,2,3,6\}\ .$$

Turning next to $\mathbb{P}_4{}^{(1,2,2,2,7)}$, the vertices of the Newton polyhedron $\Delta_2$ are

$$\begin{array}{c} (-1,-1,-1,-1), \quad (-1,-1,-1,\ \ 1), \quad (-1,-1,\ \ 6,-1), \\ (-1,\ \ 6,-1,-1), \quad (\ \ 6,-1,-1,-1). \end{array}$$

so those of $\nabla_2$ are

$$\begin{array}{c} (-2,-2,-2,-7), \quad (\ \ 1,\ \ 0,\ \ 0,\ \ 0), \quad (\ \ 0,\ \ 1,\ \ 0,\ \ 0), \\ (\ \ 0,\ \ 0,\ \ 1,\ \ 0), \quad (\ \ 0,\ \ 0,\ \ 0,\ \ 1). \end{array}$$

($\Delta_2$ and $\nabla_2$ are both simplicial; this corresponds to the existence of a Fermat hypersurface). In this case, $\nabla_2$ has 3 more lattice points in addition to vertices and the origin: $\{(\ \ 0,\ \ 0,\ \ 0,-1),\ (-1,-1,-1,-4),\ (-1,-1,-1,-3)\}$. The first two of these lie in the interior of the facet of $\nabla_2$ dual to the vertex $(-1,-1,-1,\ \ 1)$; the last lies on the edge spanned by the last two vertices of $\nabla_2$. To get Calabi-Yau hypersurfaces, we take a subdivision of the normal fan of $\nabla_2$. A subdivision yielding a toric variety $X_2$ can in fact be obtained by further subdividing the fan for $X_1$. We first insert the edge spanned by

$(-1,-1,-1,-4)$. Since this is just $(-1,-1,-1,-3)+(\ \ 0,\ \ 0,\ \ 0,-1)$, we see that we are blowing up the $\mathbb{P}_2$ where the proper transform of $x_1=0$ meets the exceptional divisor. On cones, we replace each cone containing $(-1,-1,-1,-3)$ and $(\ \ 0,\ \ 0,\ \ 0,-1)$ by two cones, the first one containing instead $(-1,-1,-1,-3)$ and $(-1,-1,-1,-4)$, while the second cone replaces them by $(-1,-1,-1,-4)$ and $(\ \ 0,\ \ 0,\ \ 0,-1)$. A similar procedure applies for the insertion of $(-2,-2,-2,-7)$ (since this is $(-1,-1,-1,-3)+(-1,-1,-1,-4)$)— here we are blowing up the new proper transform of $x_1=0$ with the newest exceptional divisor.

In a similar fashion, one sees that the same fan can be obtained from the fan for $\mathbb{P}_4^{(1,2,2,2,7)}$ by first blowing up the the locus $x_1=x_5=0$ (this inserts the edge $(-1,-1,-1,-3)$ between $(-2,-2,-2,-7)$ and $(\ \ 0,\ \ 0,\ \ 0,-1)$), then resolving the point $(0,0,0,0,1)$ by placing the edge $(\ \ 0,\ \ 0,\ \ 0,-1)$ inside the cone spanned by

$$(-2,-2,-2,-7)\ ,\quad (\ \ 1,\ \ 0,\ \ 0,\ \ 0)\ ,\quad (\ \ 0,\ \ 1,\ \ 0,\ \ 0)\ ,\quad (\ \ 0,\ \ 0,\ \ 1,\ \ 0)\ ,$$

then blowing up the intersection of the proper transform of $x_1=0$ with the last exceptional divisor (corresponding to the insertion the edge $(-1,-1,-1,-4)$ between $(-2,-2,-2,-7)$ and $(0,\ \ 0,\ \ 0,-1)$). Either way, the result is a fan with 14 maximal cones.

The upshot of all this is that there naturally results an everywhere defined map $X_2\to X_1$. The equations given above describe these in terms of the coordinates of the weighted projective space.

While the $\Lambda$ lattices used in the two examples are a priori different, our choice of coordinates gives a natural way to identify them. A similar assertion holds for the V lattices. With these identifications, we note the inclusions $\Delta_2\subset\Delta_1$ and $\nabla_1\subset\nabla_2$.

So the points of $\Delta_2\cap\Lambda$ correspond not only to monomials for $\mathcal{M}_2$, but also a subset of the monomials for $\mathcal{M}_1$. A closer investigation of the geometry reveals that these monomials are precisely the ones with the property that when a polynomial is formed from them yielding a hypersurface $\mathcal{M}_1\subset X_1$, the pullback of $\mathcal{M}_1$ to $X_2$ contains both of the expectional divisors of the map $X_2\to X_1$. This implies that if the proper transform $\mathcal{M}_2\subset X_2$ of $\mathcal{M}_1$ is smooth, then $\mathcal{M}_2$ is a Calabi-Yau hypersurface. In fact more is true: for the generic hypersurface $\mathcal{M}_1$ formed from these monomials, the manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ are actually isomorphic[3]. In this way, we can be certain that we can identify chiral rings.

[3] This has been strikingly underscored by a calculation in [18], where the instanton numbers of the two models are computed, and are seen to coincide.

*5.3. Chiral rings*

We can again describe the chiral rings by the procedure of [25]. Actually, this will give only the part of the chiral ring corresponding to the polynomial deformations. The reason why [25] does not apply to give the entire chiral ring is that the hypersurface in the blown-up toric variety is not ample (this was pointed out to us by Batyrev). Nevertheless, we continue to refer to this subring as the chiral ring.

We start with $X_1$. We note the relations

$$\begin{array}{ccccccccccccc} 1\,v_1 & + & 1\,v_2 & + & 1\,v_3 & + & 1\,v_4 & + & 3\,v_5 & + & 0\,v_6 & = & 0 \\ 0\,v_1 & + & 0\,v_2 & + & 0\,v_3 & + & 0\,v_4 & + & 1\,v_5 & + & 1\,v_6 & = & 0 \end{array}$$

These relations tell us that $X_1, \ldots, X_4$ has weight $(1,0)$, that $X_5$ has weight $(3,1)$, and that $X_6$ has weight $(0,1)$.

The anticanonical class has weight $(7,2)$. We write the equation of $\mathcal{M}_1$ in homogeneous coordinates and get

$$f = X_1^7 X_6^2 + X_2^7 X_6^2 + X_3^7 X_6^2 + X_4^7 X_6^2 + X_1 X_5^2.$$

The Jacobian ideal $J_f$ is the ideal of partial derivatives of $f$ with respect to the $X_i$. The results of [25] imply that the chiral ring consists of the parts of the quotient ring $S/J_f$ of weights $(0,0)$, $(7,2)$, $(14,4)$, $(21,6)$.

The situation for $\mathcal{M}_2$ is easier, since $\nabla_2$ is simplicial. The chiral ring is contained in $S'/J_{p_2}$, where $S'$ is the polynomial ring in $y_1, \ldots, y_5$ and $J_{p_2}$ is the Jacobian ideal of $p_2$. The chiral ring is given by the parts of $S'/J_{p_2}$ of degrees 0, 14, 28, and 42.

The reason why this simpler description of the chiral ring (let us for the moment call this the "naive" chiral ring) suffices rests on two points. First of all, if we had put in the extra 3 vertices (as the one extra vertex was added for $\mathcal{M}_1$), we would have obtained 3 new variables, and modified $p_2$ to get a polynomial $g$ involving the new variables. The polynomials on the toric variety "restrict" to polynomials on the weighted projective space by setting the new variables to 1 (in particular, $g$ restricts to $p_2$). The chain rule shows that the restriction of $J_g$ is contained in $J_{p_2}$. In other words, restriction gives a mapping from the chiral ring to the naive chiral ring. Secondly, Since $p_2$ is transverse, the chiral ring as we have written it down automatically satisfies Poincaré duality. The parts of these rings corresponding to $H^{2,1}$ are isomorphic, as they each have $\Delta_2$ as a basis by construction. Poincaré duality then shows that the restriction map is an isomorphism between the chiral ring and the naive chiral ring, justifying our identification.

By the geometric reasoning, the natural maps should induce an inclusion of chiral rings $S'/J_{p_2} \hookrightarrow S/J_f$ (more precisely, after restricting to the parts of the relevant (multi)degrees). This can be checked directly. So if we want to incorporate the non-polynomial deformations of $\mathbb{P}_4{}^{(1,2,2,2,7)}$ into $S'/J_{p_2}$, we know the answer explicitly— it is just $S/J_f$.

It may help the reader to observe that when the inclusion $\Delta_2 \subset \Delta_1$ is interpreted via monomials on the respective toric varieties, the monomial $y_1^* y_2^a y_3^b y_4^c y_5^d$ is identified with $X_1^* X_2^a X_3^b X_4^c X_5^d X_6^*$, where some exponents (denoted with a $*$) are intentionally supressed to emphasize the coincidence of the remaining exponents; the supressed exponents can be recovered by considering (multi)degrees.

We note also that the 15 'missing' deformations of $p_2$ arise in this approach because of the blowup of the $\mathbb{P}_2$ with equations $x_1 = x_5 = 0$ given by insertion of $(-1, -1, -1, -3)$. The generic weight 14 polynomial intersects $\mathbb{P}_2$ in a degree 7 curve which has genus 15. The blowup can be seen by general considerations to add 15 to the dimension of $H^{2,1}$, thereby inducing 15 new deformations. In toric language, this can be seen in [6].

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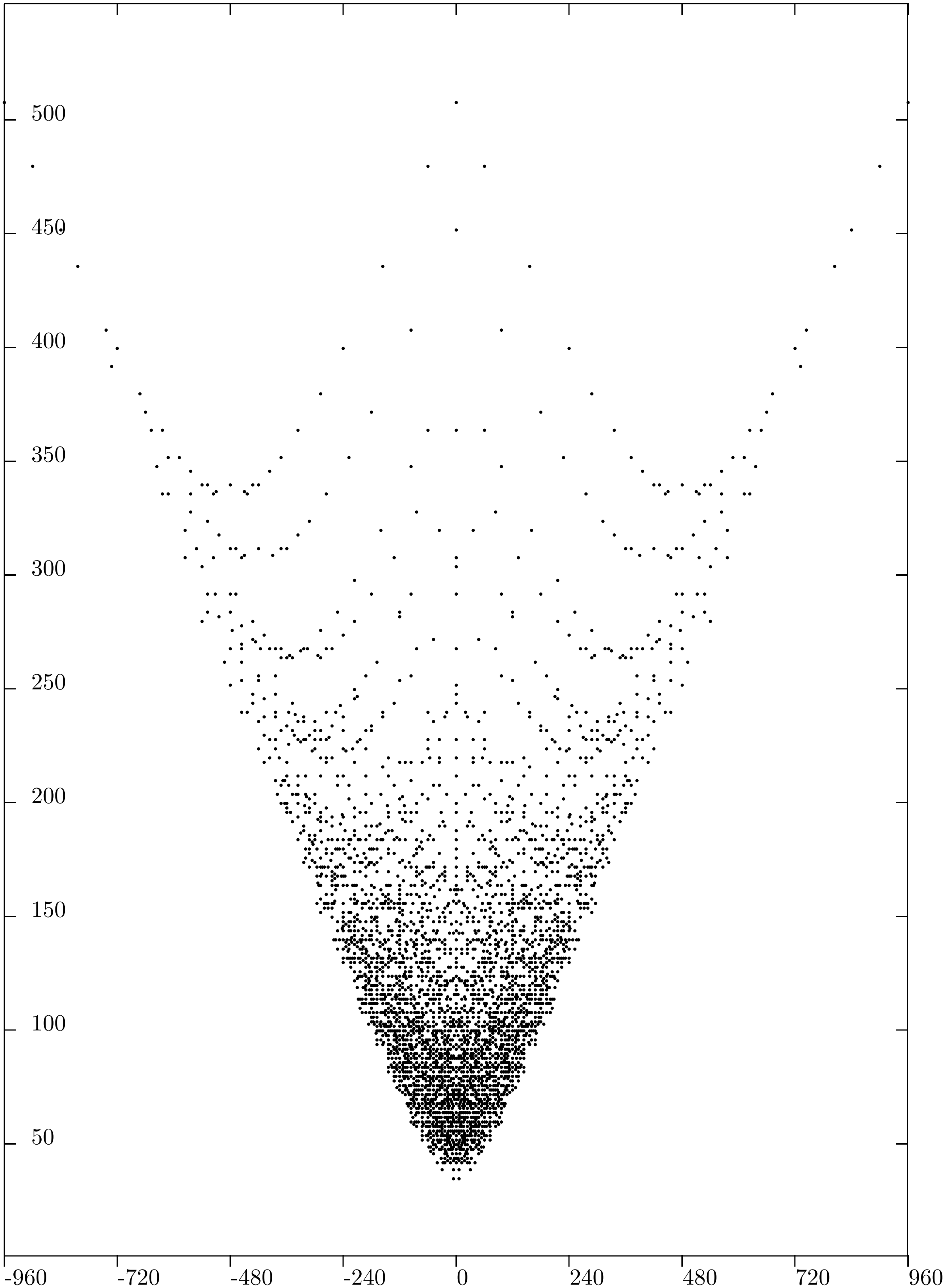
## A. **Plot of the Hodge Numbers**

On the following page we plot the Hodge numbers of each manifold of the list together with the Hodge numbers of the mirrors. The Euler number, $\chi = 2(b_{1,1} - b_{2,1})$, is plotted horizontally and $b_{1,1} + b_{2,1}$ is plotted vertically. The plot, which is similar to the plots of [3] and [1], is now symmetric by construction.


500
450
400
350
300
250
200
150
100
50
-960
-720
-480
-240
0
240
480
720
960

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