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Scale Factor Duality: A Quantum Cosmological Approach

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Abstract

We consider the minisuperspace model arising from the lowest order string effective action containing the graviton and the dilaton and study solutions of the resulting Wheeler-DeWitt equation. The scale factor duality symmetry is discussed in the context of our quantum cosmological model.

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1. Introduction

Duality transformations relate different, but actually equivalent, conformal string backgrounds and, in particular, string theories [1]. An example of this duality is the $O(d,d)$ transformation connecting all toroidal compactifications in d-dimensions [2]. In this latter case, it is shown that the duality transformation holds to all orders in the string-loop expansion parameter by performing a suitable change in the dilaton field when transforming metric and torsion fields [3]. An important subset of this duality symmetries is the so-called scale-factor or abelian duality of string models embedded in flat homogeneous and isotropic spacetimes [4]. Scale-factor duality symmetry is present in the lowest order string effective action and means that the transformation of the scale factor of a homogeneous and isotropic target space metric, $a(t) \to a^{-1}(t)$, leaves the model invariant provided that, in d spatial dimensions, the string coupling, i.e. the dilaton, is properly transformed as well

$$\Phi(t) \to \varphi(t) = \Phi(t) - \frac{d}{2} \ln a(t). \tag{1}$$

Other transformations were also proposed to implement these dualities for backgrounds with non-abelian isometry groups which are, in principle, compatible with homogeneous Bianchi cosmological backgrounds [5] (see however [6, 7]).

Scale-factor duality is an important guidance to introduce genuine stringy features into an already known cosmological framework based on General Relativity. However, although in the context of the resulting models one is allowed to address important issues such as the problem of singularities, inflation, generation of primordial energy density fluctuations and show that a radiation dominated era naturally emerges from a string cosmological scenario (see e.g. Refs. [8] and [9]), these considerations and conclusions remain essentially classical and are, therefore, presumably not capable of capturing the deep quantum gravity features one expects to extract from string theories. Since a complete quantum string theory is not yet available, a possible way to implement the classical stringy cosmological scenarios discussed so far is to consider their quantum cosmological extension by performing the canonical quantization of the corresponding string model and solving the resulting Wheeler-DeWitt (WDW) equation in the minisuperspace approximation.

In this work, we shall consider the canonical quantization of the lowest order string effective action using the standard ADM formalism in a $R \times S^3$ topology. This choice allows us to remain within the standard formalism of quantum cosmology and use its interpretative framework [10], although scale factor duality will be lost as an exact symmetry of the resulting minisuperspace model.

2. The model and Wheeler-DeWitt equation

We consider the following lowest order string effective action

$$S = \frac{1}{2}\int_M d^D x\sqrt{-g}e^{-2\Phi}[R + 4(\partial\Phi)^2 - 8V(\Phi)], \tag{2}$$

where we have allowed for a potential for the dilaton field and we have set the Kalb-Ramond tensor to vanish. We shall consider a (D=4)–dimensional homogeneous and isotropic spacetime described by a closed Friedmann-Robertson-Walker metric.

In the canonical quantization formalism, the dynamical piece of the metric is the induced three-dimensional metric $h_{ij}(i,j = 1,2,3)$ on the boundary of the manifold M over which integration in (2) is performed and, furthermore, to (2) one has to add the following boundary action:

$$S_B = -\int_{\partial M} d^3x\sqrt{h}e^{-2\Phi}K, \tag{3}$$

where h is the determinant of the induced metric and K is the trace of the second fundamental form on ∂M.

One obtains, after some computation, the following minisuperspace Lagrangian density

$$\mathcal{L} = Na^3e^{-2\Phi}\left[\frac{3}{N^2}\left(\frac{\dot{a}}{a}\right)^2 + \frac{3}{a^2} - \frac{6}{N^2}\frac{\dot{a}}{a}\dot{\Phi} + 2\frac{\dot{\Phi}^2}{N^2} - 4V(\Phi)\right], \tag{4}$$

where $N(t)$ is the lapse function.

Introducing the variable $\mathtt{z}(t) = \ln a(t)$ and the transformation (1), one finds, in the N=1 gauge, that

$$\mathcal{L} = e^{-2\varphi}\left[-\frac{3}{2}\dot{\mathtt{z}}^2 + 3e^{-2\mathtt{z}} + 2\dot{\varphi}^2 - 4V(\varphi,\mathtt{z})\right]. \tag{5}$$

which, except for the second term, exhibits the scale factor duality symmetry under the transformation $\mathtt{z} \to -\mathtt{z}, \varphi \to \varphi$, provided $V(\varphi,\mathtt{z}) = V(\varphi,-\mathtt{z})$.

Aiming to obtain the WDW equation describing the quantum cosmological features of model (2), we compute the canonical conjugate momenta to $\mathtt{z}$ and φ:

$$\Pi_{\mathtt{z}} = \frac{\partial \mathcal{L}}{\partial \dot{\mathtt{z}}} = -3\dot{\mathtt{z}}e^{-2\varphi}, \tag{6}$$

$$\Pi_{\varphi} = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = 4\dot{\varphi}e^{-2\varphi}. \tag{7}$$

The Hamiltonian density is then given by

$$\begin{aligned} \mathcal{H} &= \Pi_{\mathtt{z}}\dot{\mathtt{z}} + \Pi_{\varphi}\dot{\varphi} - \mathcal{L} \\ &= e^{2\varphi}\left\{-\frac{1}{6}\Pi_{\mathtt{z}}^2 + \frac{1}{8}\Pi_{\varphi}^2 + e^{-4\varphi}\left[4V(\varphi,\mathtt{z}) - 3e^{-2\mathtt{z}}\right]\right\}, \end{aligned} \tag{8}$$

and vanishes, on account of invariance under time reparametrization. The WDW equation is obtained transforming this classical constraint, i.e. $\mathcal{H} = 0$, into the vanishing of the Hamiltonian operator acting on the wavefunction $\Psi(\varphi, \mathtt{z})$. For the latter step, one has to promote the canonical conjugate momenta into operators:

$$\begin{aligned} \Pi_{\mathtt{z}} &\to -i\frac{\partial}{\partial \mathtt{z}}, \\ \Pi_{\varphi} &\to -i\frac{\partial}{\partial \varphi}. \end{aligned} \tag{9}$$

Hence, one finds, after a proper operator ordering choice and trivial rescaling of variables

$$\left[\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial \phi^2} + U(\phi, z)\right]\Psi(\phi, z) = 0, \tag{10}$$

where

$$U(\phi, z) = \frac{1}{2}e^{-\phi}\left[4V(\phi, z) - 3e^{-\frac{1}{\sqrt{3}}z}\right], \tag{11}$$

and $\phi = 4\varphi$, $z = 2\sqrt{3}\mathtt{z}$.

3. Solutions of the Wheeler-DeWitt equation

We shall first study the case where the dilaton has no potential (a scenario favoured e.g. in ref. [11]) and then proceed to analyse the effect of the introduction of a potential; we shall consider the following simple cases:

$$
\begin{aligned}
V(\Phi) &= \Lambda, && (12)\\
&= \frac{m^2}{2}(\Phi - \Phi_o)^2, && (13)\\
&= V_o e^{-\alpha\Phi}. && (14)
\end{aligned}
$$

The case of a cosmological constant, (12), is the one which is compatible with scale factor duality in a flat 3-dimensional spacetime [12]. We shall see, however, that the wave function is fairly insensitive to the 3-dimensional curvature and, using the interpretational rules of quantum cosmology, that the most likely initial configuration for the universe is approximately compatible with scale factor duality. The second choice for the dilaton potential leads, in the classical case, to chaotic inflationary solutions provided $\Phi_i \gtrsim 4M_P$, where Φ_i is the initial dilaton configuration and M_P is the Planck mass [13,14]; our quantum analysis indicates that these configurations are actually favoured. Finally, choice (14) is of interest in connection with scenarios of supersymmetry breaking due to gaugino condensation in the hidden sector of the theory [15].

For the case where the dilaton has no potential, the corresponding WDW equation is given by

$$
\left[\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial \phi^2} - \frac{3}{2}\, e^{-\phi - \frac{1}{\sqrt{3}} z}\right] \Psi(\phi, z) = 0. \tag{15}
$$

Introducing the variables $x = \exp[-\frac{1}{2}(\phi + \frac{1}{\sqrt{3}} z)]$ and $y = \phi + z$, this equation becomes

$$
\left[x \frac{\partial^2}{\partial x^2} - \frac{1}{2}\frac{\partial}{\partial x} - \frac{A}{2}\frac{\partial^2}{\partial x \partial y} + 9x\right] \Psi(x, y) = 0, \tag{16}
$$

where $A = 4(3 - \sqrt{3})$. This equation is separable, i.e. writing the wave function as $\Psi(x, y) = F(x)G(y)$, one obtains

$$
\begin{aligned}
\frac{d^2F}{dx^2} - \frac{1}{2}(A\mu + 1)\frac{1}{x}\frac{dF}{dx} + 9F &= 0, && (17)\\
\frac{dG}{dy} - \mu G &= 0, && (18)
\end{aligned}
$$

where μ is a separation constant. The solution of (15) is then given by [16]:

$$\Psi(x,y) = c e^{\mu y} x^{\alpha} \mathcal{C}_{\alpha}(3x), \tag{19}$$

where c is an integration constant, $\alpha = \frac{1}{4}[3 + A\mu]$ and $\mathcal{C}_\alpha$ is a generic Bessel function of order α. To ensure that $\lim_{a\to 0}\Psi = 0$, i.e. a singularity free cosmological scenario, we require that $\mu > \mu_1 = \frac{1}{12(\sqrt{3}-1)}$ and, for $\mu > 3\mu_1$, the wave function is an increasing function in ϕ.

We shall now consider the situation where the dilaton potential is non-vanishing. Since the WDW equation becomes more involved in this case and, within the spirit of the quantum cosmology framework, we shall not look for exact solutions but rather for the striking features of the wave function. We start with the simple case of a constant dilaton potential, eq. (12). In this case, the minisuperspace potential is given by $U(\phi,z) = \frac{1}{2}e^{-\phi}\left[4\Lambda - 3e^{-\frac{1}{\sqrt{3}}z}\right]$, which we shall analyse by patching up solutions for $\phi > 0$ (the wave function clearly vanishes in the $\phi < 0$ region due to the steepness of the potential) in the following three regions: $z > 0$, $z < 0$ and around the origin. In fact, it is quite straightforward to show that Ψ essentially vanishes in the $z < 0$ region. For $z > 0$, assuming that $4\Lambda \gg 3e^{-\frac{1}{\sqrt{3}}z}$ and that the second term in $U(\phi,z)$ can therefore be neglected, the resulting WDW equation becomes separable, i.e. the wave function can be written as $\Psi(\phi,z) = F(z)G(\phi)$. It then yields the following ordinary differential equations

$$\frac{d^2F(z)}{dz^2} - \mu F(z) = 0, \tag{20}$$

$$\frac{d^2G(\phi)}{d\phi^2} - \left(2\Lambda e^{-\phi} + \mu\right)G(\phi) = 0, \tag{21}$$

where μ is the separation constant. The solution is given by [16]

$$\Psi(\phi,z) = c e^{\pm\sqrt{\mu}z}\mathcal{Z}_{i\sqrt{\mu}}\left(\sqrt{2\Lambda}e^{-\phi/2}\right), \qquad \text{for} \ \ \mu > 0, \tag{22}$$

$$= c\sin\left(\sqrt{|\mu|}\ z\right)\mathcal{Z}_{\sqrt{|\mu|}}\left(\sqrt{2\Lambda}e^{-\phi/2}\right), \qquad \text{for} \ \ \mu < 0, \tag{23}$$

for $\Lambda > 0$, $\mathcal{Z}_\nu$ is a generic modified Bessel function of order ν. If $\Lambda < 0$, $\mathcal{Z}_\nu$ should be replaced by $\mathcal{C}_\nu$ and Λ by $|\Lambda|$ in the above equations. Usual quantum cosmology interpretational formalism is suitable for the ground-state wave function of the universe [10] and we shall consider henceforth the case $\mu = 0$. It then follows that the wave function, in the original variables, is given by

$$\Psi(\varphi, \mathrm{z}) = c\mathrm{z}K_0\left(\sqrt{2\Lambda}e^{-2\varphi}\right), \tag{24}$$

for $\Lambda > 0$, where K_0 is the modified Bessel function of order zero; if $\Lambda < 0$, K_0 should be replaced by the Bessel function J_0 and Λ by $|\Lambda|$. Notice that, in both cases, the wave function is positive and has a regular behaviour at the origin. Moreover, (24) implies that

$$\Psi \ \sim 2\varphi \ \ln a, \qquad \text{for} \ \ \varphi \to +\infty; \tag{25}$$

hence Ψ increases for large φ and a.

Let us now study the region where ϕ and z are close to the origin. Expanding the exponentials up to second order and separating variables, i.e. setting $\Psi(\phi, z) = F(z)G(\phi)$, we obtain the following ordinary differential equations

$$\frac{d^2F(z)}{dz^2} + \left(\frac{\sqrt{3}}{2}z + \mu\right)F(z) = 0, \tag{26}$$

$$\frac{d^2G(\phi)}{d\phi^2} + [\beta(\phi - 1) - \mu]G(\phi) = 0, \tag{27}$$

where $\beta = 2\Lambda - 3/2$. The solutions of these equations can be given again in terms of Bessel functions, respectively

$$F(z) = c\hat{z}^{1/2}\mathcal{C}_{1/3}\left(2\gamma^2\hat{z}^{3/2}\right) \qquad \text{for} \ \ \hat{z} > 0, \tag{28}$$

$$= c\,|\hat{z}|^{1/2}\,\mathcal{Z}_{1/3}\left(2\gamma^2|\hat{z}|^{3/2}\right) \qquad \text{for} \ \ \hat{z} < 0, \tag{29}$$

where $\hat{z} = \frac{\sqrt{3}}{2}z - \mu$, $\gamma^2 = \frac{2\sqrt{3}}{9}$ and

$$G(\phi) = c\hat{\phi}^{1/2}\mathcal{C}_{1/3}\left(\frac{2}{3\beta}\hat{\phi}^{3/2}\right) \qquad \text{for} \ \ \hat{\phi} > 0, \tag{30}$$

$$= c|\hat{\phi}|^{1/2}\mathcal{Z}_{1/3}\left(\frac{2}{3\beta}|\hat{\phi}|^{3/2}\right) \qquad \text{for} \ \ \hat{\phi} < 0, \tag{31}$$

where $\hat{\phi} = \beta(\phi - 1) - \mu$. For the vacuum state, $\mu = 0$, we have

$$\Psi(\phi, z) = B(z\beta|\phi - 1|)^{1/2} J_{1/3}(\gamma z^{3/2}) I_{1/3}\left(\frac{2}{3}\beta^{1/2}|\phi - 1|^{3/2}\right) , \tag{32}$$

for $\phi < 1$, $\beta > 0$ and B a constant. If $\beta < 0$, $I_{1/3}$ should be interchanged by $J_{1/3}$ in the above equation and β replaced by $|\beta|$. Hence, the ground-state wave function $\Psi(\phi, z)$ around the origin is essentially an increasing function of ϕ and z [16].

We find that the salient features of the ground-state wave function are that it is an increasing function of ϕ and z and vanishing in the ϕ and z negative region. Hence, we conclude that the favoured initial conditions are the ones for which ϕ and z are large.

We turn our attention to the case where the dilaton potential is given by (13). As discussed in Refs. [13,14], conditions for successful chaotic inflation require $10^{-8} M_P < m < 10^{-6} M_P$, $\Phi_o \sim M_P$ and $\Phi_i \gtrsim 4 M_P$. For this choice of parameters, the potential $U(\phi, z)$ is controlled by the overall exponential factor, which implies that the previous results for the case of a constant dilaton potential remain essentially unaltered. This means that, as before, large values of ϕ and z are the favoured field configurations.

Let us now turn to the case where the dilaton potential is given by (14). The value of α depends of the gauge group of the hidden sector and for the known models $\alpha \gtrsim 24\pi^2/10$ [15]. For these values of α, the contribution of the dilaton potential to $U(\phi, z)$ is negligible and the ground-state wave function is essentialy the one for the case where there is no potential and given by (19). Thus the conclusions drawn for that case are equally valid here.

4. Conclusions

Let us summarize our results and comment on some of its implications. We have seen that for any of the potentials we have analysed, including the case of a vanishing potential, the ground-state wave-function vanishes for z negative, which is consistent with the expectation that $\Psi(a = 0, \Phi) = 0$ in order to solve the singularity problem. Another generic feature of our results is that for all cases we have analysed the ground-state wave function of the universe is an increasing function of ϕ. The is true for the variable z in the $z > 0$ region. We can then conclude that the striking features of the wave function of the universe are fairly independent on the dilaton potential, being also insensitive to the presence of the spatial curvature. Moreover, these conclusions imply that the favoured initial field configurations are the ones for which ϕ and z are large (in Planck units) and that the scale factor duality, although not a symmetry of the classical closed FRW metric-dilaton system, actually holds as an approximate symmetry. Thus, our results are compatible with conclusions drawn from classical cosmological scenarios in which string features are introduced through the scale factor duality [8] and, in particular, with a pre-big-bang era [9]; furthermore, our approach shows that quantization naturally solves the singularity problem.

This approximate scale factor duality symmetry displayed by the system we have analysed implies that the ground-state wave function of the universe will consist effectively of a superposition of $\Psi(\Phi, a)$ and $\Psi(\Phi, a^{-1})$. It is interesting to speculate on the possiblity that the transition from this superposed state to the classical state, where scale factor duality is lost, can be achieved through the process of decoherence, with the inhomogeneous modes of a massive dilaton field playing the role of the environment. For this latter purpose, one could also consider massless dilaton and Yang-Mills fields. For a fixed value of the dilaton field, the ground-state wave function for the Yang-Mills fields is essentially the one describing an anharmonic oscillator with a quartic potential [17]. Inhomogeneous modes of dilaton and Yang-Mills fields could together drive decoherence due to their coupling. Decoherence could be also driven by massive vector fields [18].

Interestingly, one finds that, as in the case where one considers only the dilaton and this field is endowed with a potential, there exist chaotic inflationary solutions if $\Phi_i \gtrsim 4 M_P$, a feature which holds even when including Yang-Mills fields [14]. One expects likewise that large initial values for the dilaton will also be favoured in the quantum cosmological analysis of the metric-Yang-Mills-dilaton system.

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