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On the Lagrangian Realization of Non-Critical $\mathcal{W}$-Strings

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Abstract

A large class of non-critical string theories with extended worldsheet gauge symmetry are described by two coupled, gauged Wess-Zumino-Witten Models. We give a detailed analysis of the gauge invariant action and in particular the gauge fixing procedure and the resulting BRST symmetries. The results are applied to the example of $\mathcal{W}_3$ strings.

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I. INTRODUCTION

Whereas the simplest models in string theory are based on the Virasoro algebra or supersymmetric extensions thereof, a lot of interest has been generated by extensions based on non-linear symmetry algebras [1], called $\mathcal{W}$ algebras. There are several lines of investigation for systems having an extended conformal symmetry. One possibility is to make use of the symmetry algebras only, trying to gain information about their representations, and in this way about the possible physical string models these non-linear algebras correspond to. This is an ambitious line, but probably still too difficult at the present time. More or less complete data about representations have till now only been obtained for some simple finite analogues of these $\mathcal{W}$ algebras [2], and for $\mathcal{W}_3$ [3]. A different approach has been to realize the Operator Product Expansions of the $\mathcal{W}$ algebras in terms of free fields — which are easily realized in Fock-space — and investigate physical consequences (BRST operators and their cohomology) in these realizations. In this paper we follow a third line, related to the previous one, and accord a central role to Lagrangian realizations of the symmetry algebras in terms of Wess-Zumino-Witten models. This is done first on a classical level, after which the theories described by these Lagrangians can be quantized. The transition to quantum theory is in practice very simple: it amounts to assuming the validity of affine Lie algebra OPEs for the symmetry currents of the theory. Moreover, these models are very malleable in that, by gauging and constraining, they allow the construction of (almost?) all extended conformal algebras.

One has to distinguish between critical and non-critical models. The critical models impose a cancellation between the central charges of the "matter" component of the model against the "ghost" particles (implying, for example, for the simplest bosonic string a central charge $c = 26$ and for a model based on the $\mathcal{W}_3$ algebra a value $c = 100$). The non-critical models achieve this cancellation by introducing another sector, the gravitational sector. This can be understood from the fact that integrating over matter and ghosts first induces, through a quantum anomaly, an action for classically non-existing degrees of freedom. For the simple bosonic string in the conformal gauge this induced action is the Liouville action, whence it is also called the "Liouville" sector. The induced action describes an extension of two dimensional gravitation theory. The subsequent integration over its degrees of freedom restores the non-linear symmetry of the theory.

Non-critical $\mathcal{W}$ string theories were first constructed "by hand" [4], meaning that the symmetry currents of both the matter and gravity sectors are realized in terms of free fields and the BRST operator is then constructed by trial and error. Though this is quite feasible for the simplest models, it turns out to be a formidable task for more complicated models. Obviously, a more systematic apprach is needed. Recently several possible approaches were discovered.

A most elegant way to solve extended non-critical string theories is by using the (suspected) equivalence of a large class of them, the so-called $(1, q)$ models, to topological stringtheories [5]. Using the matter picture [6,7] for these topological strings, choosing a Landau-Ginzburg type realization of the matter sector provides a very quick way to investigate several essential properties, such as the spectrum, of the non-critical string theory.

A related approach takes advantage of the hidden N=2 structure of any string theory. The BRST current and the Virasoro anti-ghost together provide the two supercharges [8–10].

Adopting this as the essential structure of any string theory, one then views the construction of string theories as the study of realizations and representations of extensions of the N=2 conformal algebra. This implies then that one should be able to construct a large class of non-critical string theories from Hamiltonian reduction. Indeed, many N=2 algebras can be constructed by the reduction of WZW models on supergroups, the reduction being determined by an embedding of $SU(2|1)$ in a supergroup. By an appropriate choice of the grading — which is necessary to determine the reduction completely — one obtains a certain free-field realization which can immediately be viewed as a non-critical string theory. Though this approach looks very elegant and promising, it has only been established in certain cases [10].

A last approach, again relying on gauged or reduced WZW models, takes reduced WZW models for both the matter and the gravity sector separately. Precisely this approach will be studied here.

In this paper we will exploit the versatility of the WZW models. First, in section II A, we will analyse a constrained WZW model, showing how, following the ideas of the Drinfeld-Sokolov reduction scheme, one can use them to realize $\mathcal{W}$ algebras. Our treatment here improves on the ones existing in the literature in that the auxiliary fields, necessary to save DS gauge invariance on the Lagrangian level, now arise as a natural part of the construction, based as it is on that gauge invariance from the start. This is shown with the help of a recursion method to perform the transition to the so-called highest weight gauge, in which the appearance of the $\mathcal{W}$ algebra is the most manifest. As a by-product, we also give an efficient recursive method to construct the gauge invariant polynomials that realize the $\mathcal{W}$ algebra. The constructions in this section are relevant for both critical and non-critical strings. Then, in section II B, we introduce the transformations of the $\mathcal{W}$ symmetry. We use the previous construction in section III both for the matter and the gravity sectors. We show how, already at the classical level, it is only through a cancellation of central charges of the sectors that the symmetry is achieved. As an application, we give in subsection III B the expression for the classical BRST charge for the combined matter-ghost-gravity system in the case of $\mathcal{W}_3$ that follows from our construction. Our method gives an expression for this charge that extends to the quantum theory by a simple renormalization of a single coefficient, without the need for any additional terms.

In section IV we give a more thorough treatment of the gauge fixing procedure, using the field-antifield formalism of Batalin and Vilkovisky. First (IV A) we use this method in the realization of a single sector to explicitize the fixing of the DS gauge invariance. This simplifies the derivation of the gauge fixed action in [11] as it avoids any explicit reference to open gauge algebras. Then (IV B) we apply the same method to the additional $\mathcal{W}$ symmetry that is present if one combines a matter and a gravity sector. We keep the discussion general, working out the $\mathcal{W}_3$ case explicitly at the end. This serves as a justification of the ghost Lagrangian used in that (relatively simple) case in section III B, and also points the way to extend the present treatment to arbitrary extensions that can be obtained from DS reduction.

A more detailed treatment of the results presented in this paper can be found in [12,13].

II. THE CLASSICAL ACTION OF $\mathcal{W}$ MATTER

A. The Drinfeld-Sokolov procedure revisited

In this first section, we realize a $\mathcal{W}$ system (matter or gravity) by constraining the currents of a WZW model. We will not review the method of hamiltonian reduction here — we only give a cursory description to establish notation — and refer the interested reader to e.g. [14,11] for a general introduction and references. We will supplement the standard treatment with some detailed recursion formulas to carry out this reduction in practice, since we need these for later use.

The starting point is the usual WZW action $\kappa\mathcal{S}^-[g]$ for some Lie (super)group G with generic element $g(z,\bar z)$. The $\mathcal{W}$ algebra is determined by choosing a particular $sl(2)$ embedding $\mathcal{S} = \{e_0, e_+, e_-\}$ in the algebra $\mathfrak{g}$. The first step is to constrain the current $J(z) = \frac{\kappa}{2}\partial g \cdot g^{-1}$ to the form

$$J \mapsto \tilde J = \frac{\kappa}{2}\partial\tilde g \cdot \tilde g^{-1} = \frac{\kappa}{2}e_- + \frac{\kappa}{2}[\tau, e_-] + J^{\geq 0}, \tag{1}$$

where $J^{\geq 0}$ denotes the positively graded components in the grading induced by e_0, and τ is a set of auxiliary fields with grading $1/2$ that are introduced to insure that all constraints are first class [15,11]. These constraints generate the Drinfeld-Sokolov (DS) gauge transformations. They can be used to put the current $\tilde J$ in the *highest weight gauge*:

$$\tilde J = \frac{\kappa}{2}\partial\tilde g \cdot \tilde g^{-1} \mapsto \frac{\kappa}{2}\partial e^{\gamma}\tilde g \cdot \tilde g^{-1}e^{-\gamma} = \frac{\kappa}{2}e_- + W(\tilde J), \tag{2}$$

where W contains only highest weight components. These components are *gauge invariant polynomials* of the original components of $\tilde J$ and their derivatives, and form a classical $\mathcal{W}$ algebra under Poisson Brackets. We will denote the gauge fixed group element by $e^{\gamma}\tilde g = w$. The existence and uniqueness of the algebra element γ defining the transition to the highest weight gauge has been proven long ago [14], but we present here an algorithmic procedure to calculate it exactly. For convenience we first introduce the notations $E_- \equiv \mathrm{ad}(e_-)$, $E_+ \equiv \mathrm{ad}(e_+)$ and furthermore we define the "inverse" L of E_- [16], which vanishes on highest weight generators and $LE_- = 1$ on $\tilde g/\ker(E_-)$. The highest weight gauge can now be defined by

$$L\left\{e^{\gamma}\left(e_- + 2J^{\geq 0}/\kappa + [\tau, e_-] - \partial\right)e^{-\gamma} - e_-\right\} = 0. \tag{3}$$

This equation can be solved order by order in $J^{\geq 0}$ and τ by writing $\gamma = \sum_{n\geq 1}\gamma_n$, $W = \sum_{n\geq 1} W_n$. Up to first order the equation becomes

$$L\left\{-E_-\gamma_1 + \partial\gamma_1 + 2J^{\geq 0}/\kappa + [\tau, e_-]\right\} = 0, \tag{4}$$

and since γ is positively graded (and thus $LE_-\gamma_1 = \gamma_1$) the solution is

$$\gamma_1 = \frac{L}{1 - L\partial}\left\{2J^{\geq 0}/\kappa + [\tau, e_-]\right\} . \tag{5}$$

At higher order one may easily construct the recursive algorithm

$$\gamma_n = \frac{L}{1 - L\partial}\mathcal{P}_n \Big\{ e^{\gamma_1 + \cdots + \gamma_{n-1}}$$
$$\Big(e_- + 2J^{\geq 0}/\kappa + [\tau, e_-] - \partial\Big)\, e^{-\gamma_1 - \cdots - \gamma_{n-1}} \Big\}$$
$$2W_n/\kappa = \Pi_{\rm hw}\frac{1}{1 - L\partial}\mathcal{P}_n \Big\{ e^{\gamma_1 + \cdots + \gamma_{n-1}} \tag{6}$$
$$\Big(e_- + 2J^{\geq 0}/\kappa + [\tau, e_-] - \partial\Big)\, e^{-\gamma_1 - \cdots - \gamma_{n-1}} \Big\}\ ,$$

where $\mathcal{P}_n$ indicates that we only retain the part of order n. Notice that the expansion of γ and W terminates after a finite number of steps, since the expression

$$\mathcal{P}_n \Big\{ e^{\gamma_1 + \cdots + \gamma_{n-1}} \Big(e_- + 2J^{\geq 0}/\kappa + [\tau, e_-] - \partial\Big)\, e^{-\gamma_1 - \cdots - \gamma_{n-1}} \Big\} \tag{7}$$

contains only components of grading $(\frac{n}{2} - 1)$ or higher.

The action $\mathcal{S}^-[w]$ is obviously invariant under DS gauge transformations, as it involves only the gauge invariant polynomials W. In addition, the WZW action $\mathcal{S}^-[g]$ has, from the start, an invariance under (left) multiplication of g with an arbitrary *holomorphic* group element. The constraints imposed in the DS reduction also reduce this additional invariance, namely to the transformations generated by the DS gauge invariant polynomials W. These are called $\mathcal{W}$ transformations. One may attempt to lift the restriction to holomorphic parameters by coupling the W to an extra external field μ. This will be discussed further in the next section. This same coupling can also be used to great effect to study the induced $\mathcal{W}$ gravity theory itself, see [17–19,16,11]. We therefore continue with the action $S = \mathcal{S}^-[w] + \int \mu \cdot W$. The recursion relations derived above can be used to rewrite it as follows, making explicit the dependences on the auxiliary field and the WZW currents J. Using $w(\tilde{J}) = e^{\gamma}\tilde{g}$, and splitting the WZW action $\kappa\mathcal{S}^-[w]$, with help of the Polyakov-Wiegmann identity [20],

$$\mathcal{S}^-[hg] = \mathcal{S}^-[h] + \mathcal{S}^-[g] - \frac{1}{2\pi x}\int {\rm str}\Big\{ h^{-1}\bar\partial h\, \partial g g^{-1} \Big\}\ , \tag{8}$$

we obtain

$$\kappa\mathcal{S}^-[w] = \kappa\mathcal{S}^-[\tilde{g}] + \kappa\mathcal{S}^-[e^{\gamma}] + \frac{1}{\pi x}\int {\rm str}\Big\{ \bar\partial e^{-\gamma} \cdot e^{\gamma}\tilde{J} \Big\}\ . \tag{9}$$

Since γ is strictly positively graded, the WZW action $\kappa\mathcal{S}^-[e^{\gamma}]$ vanishes identically. The local mixed term simplifies too, and we find that

$$\begin{aligned} S &= \kappa\mathcal{S}^-[\tilde{g}] + \frac{1}{\pi x}\int {\rm str}\Big\{ \frac{\kappa}{2}\bar\partial e^{-\gamma}\cdot e^{\gamma}\,(e_- + [\tau, e_-]) + \mu W(\tilde{J}) \Big\} \\ &= \kappa\mathcal{S}^-[\tilde{g}] + \frac{\kappa}{4\pi x}\int {\rm str}\Big\{ [\tau, e_-]\bar\partial\tau \Big\} + \frac{1}{\pi x}\int {\rm str}\Big\{ \mu W(\tilde{J}) \Big\}\,. \end{aligned} \tag{10}$$

To derive this last result we inserted the explicit expressions for $\tilde{J}$, γ_1 and γ_2 that can be read off from the equations (1) and (6). Higher order terms of γ do not contribute to the supertrace. In eq.(10) the DS gauge invariance is still present, and will have to be fixed eventually. This can be done in different ways, which allows one to derive all order expressions for the induced $\mathcal{W}$-action, see [16,11]. Remark that the kinetic term for the auxiliary field τ, added ad-hoc in [11] to preserve gauge invariance, emerges very naturally in the present formulation, which is based on that gauge invariance from the start. We will come back to the gauge fixing in section IV.

B. $\mathcal{W}$ transformations

In the previous subsection we introduced a constrained WZW action, where the (DS) gauge invariance could be used to bring the currents in a highest weight form. Here, we analyze the $\mathcal{W}$ transformations themselves.

Infinitesimally the $\mathcal{W}$ transformations are of the form $\delta w = X_w w$ where $X_w \in \mathfrak{g}$ should be determined such that the highest weight gauge (1) is preserved. This means that the transformation acts on the highest weight current W only, so we demand [1] that

$$L\delta\left(2W/\kappa\right) = L\left(D\left[2W/\kappa\right] - E_-\right) X_w = 0. \tag{11}$$

Defining, for any current j, the operator $I[j]$ by

$$I\left[j\right] \equiv 1 - LD\left[j\right], \tag{12}$$

and using the identity $1 - LE_- = \Pi_{\rm lw} =$ the projection operator on lowest weight components, the general solution for X_w can be written as

$$X_w = \frac{1}{I\left[2W/\kappa\right]}\eta \qquad \text{with} \qquad \eta \in \ker E_-. \tag{13}$$

Notice that the inverse operator $\frac{1}{I[2W/\kappa]} \equiv \sum_{i\geq 0}\left(LD\left[2W/\kappa\right]\right)^i$ is well-defined since each factor $LD\left[2W/\kappa\right]$ increases the $sl(2)$ grading with at least one unit, so that the sum, when applied to any current, terminates after a finite number of steps.

Once we have determined the form of the parameter X_w, we can derive the η transformation rules for the highest weight currents. They can be encoded in the matrix equation

$$\delta W = \frac{\kappa}{2}\Pi_{\rm hw} D\left[2W/\kappa\right]\frac{1}{I\left[2W/\kappa\right]}\eta\,. \tag{14}$$

These constraint preserving η transformations are nothing but the $\mathcal{W}$ transformations, which are generated by the W currents themselves through Dirac brackets [14]. These Dirac brackets are equivalent to the Poisson brackets of the gauge invariant polynomials discussed above, defining the classical $\mathcal{W}$ algebra.

In the previous section we introduced the action

$$S = \kappa\mathcal{S}^-[w] + \frac{1}{\pi x}\int \mathrm{str}\left\{\mu W\right\}. \tag{15}$$

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[1] We denote, for any current j, the covariant derivative as $D[j] \equiv \partial - \mathrm{ad}(j)$. Later we will also use $\bar{D}[A] \equiv \bar{\partial} - \mathrm{ad}(A)$.

It describes a fully constrained WZW model, of which the highest weight currents W are coupled to chiral $\mathcal{W}$ gravitational (lowest $sl(2)$ weight) field μ. The currents transform under $\mathcal{W}$ transformations as in eq.(14).

Consider the variation of the action S:

$$\delta_\eta S = \frac{1}{\pi x}\int \mathrm{str}\left\{-\bar{\partial}\eta \cdot W + \delta_\eta\mu \cdot W + \frac{\kappa}{2}\mu D\left[2W/\kappa\right]\frac{1}{I\left[2W/\kappa\right]}\eta\right\}. \tag{16}$$

It can be derived by using

$$\delta_X k\mathcal{S}^-[g] = \frac{-1}{\pi x}\int \mathrm{str}\left\{\bar{\partial}X \, . \, J\right\} \tag{17}$$

and the fact that η is of lowest weight. The W-independent part of the variation (16) reads

$$(\delta_\eta S)|_{W=0} = \frac{1}{\pi x}\int \mathrm{str}\left\{\frac{\kappa}{2}\mu\frac{\partial}{1-L\partial}\eta\right\}, \tag{18}$$

which can not be canceled by the $\delta_\eta\mu$ term. This shows that, already at the level of the classical realization, we have to face the central extension terms, which in some treatments appear only at the quantum level. Although this forces one to arrange for a cancellation also at this classical level, it is in fact a blessing in disguise, since exactly the same cancellation mechanism turns out to suffice for the quantum treatment.

III. NON-CRITICAL $\mathcal{W}$ STRING MODELS

In this section, we will lift the obstruction to the $\mathcal{W}$ invariance of the classical realization by introducing, besides the matter sector, also the Liouville sector. Then, adding ghosts, we show how this can be used to deduce the BRST charge of [4] for the combined system.

A. The $\mathcal{W}$ invariant action

The $\mathcal{W}$ transformations *can* be gauged if we introduce *two* WZW models, which we call "matter"(M) and "gravity"(G) respectively. For convenience we introduce the following notations

$$\begin{aligned} D_M &\equiv D\left[2W_M/\kappa_M\right] \\ I_M &\equiv I\left[2W_M/\kappa_M\right] = 1 - LD_M \, . \end{aligned} \tag{19}$$

Later on we will also need the conjugated operator I_M^+, which is defined by

$$I_M^+ \equiv 1 - D_M L. \tag{20}$$

All these definitions of course apply, mutatis mutandis, for the gravitational sector as well.

Our action at this stage is

$$S_{M+G} = \kappa_M\mathcal{S}^-[w_M] + \kappa_G\mathcal{S}^-[w_G] + \frac{1}{\pi x}\int \mathrm{str}\left\{\mu(W_M + W_G)\right\}. \tag{21}$$

From eq.(18) it is seen that the obstruction to invariance is lifted if the levels of the matter and gravity sector add up to zero

$$\kappa_M + \kappa_G = 0. \tag{22}$$

Using this relation, it remains to be checked that the last term in the resulting variation of the action eq.(21),

$$\begin{aligned}\delta_\eta S_{M+G} = \frac{1}{\pi x}\int \mathrm{str}\Big\{&-\bar\partial\eta\cdot(W_M+W_G)+\delta_\eta\mu\cdot(W_M+W_G)\\ &+\frac{\kappa_M}{2}\mu\left(D_M\frac{1}{I_M}-D_G\frac{1}{I_G}\right)\eta\Big\}\ ,\end{aligned} \tag{23}$$

is proportional to $W_M + W_G$. Indeed, we find that

$$\int \mathrm{str}\left\{\frac{\kappa_M}{2}\mu\left(\frac{1}{I_M^+}D_M - D_G\frac{1}{I_G}\right)\eta\right\} = -\int \mathrm{str}\left\{\mu\frac{1}{I_M^+}\mathrm{ad}(W_M+W_G)\frac{1}{I_G}\eta\right\}\ , \tag{24}$$

so that S_{M+G} is invariant under $\mathcal{W}$ transformations if we define

$$\delta_\eta\mu = \bar\partial\eta - \Pi_{\mathrm{lw}}\ \mathrm{ad}\left(\frac{1}{I_M}\mu\right)\frac{1}{I_G}\eta\Big|_{M,G}. \tag{25}$$

There is some arbitrariness in this choice. The symbol $|_{M,G}$ indicates that we have chosen an additive M-G symmetrization of the transformation law for μ. Explicitly, $F(M,G)|_{M,G} = \frac{1}{2}\{F(M,G)+F(G,M)\}$, and $D_{M,G} = \frac{1}{2}(D_M + D_G)$.

The gauge fixing of the action (21) is a non-trivial problem. It can for instance be checked that the $\mathcal{W}$ gauge algebra in general only closes modulo $W_M + W_G$ terms. This will cause higher ghost interaction terms in the gauge fixed theory. In section IV we will treat the derivation of these terms in some detail using the formalism of Batalin and Vilkovisky, which is eminently suited to master these complications. At the moment we only present the lowest order terms explicitly:

$$\begin{aligned}S_{\mathrm{gf}} = {}&\kappa_M\mathcal{S}^-[w_M]+\kappa_G\mathcal{S}^-[w_G]+\frac{1}{\pi x}\int \mathrm{str}\left\{b\bar\partial c\right\}\\ &+\frac{1}{\pi x}\int \mathrm{str}\left\{\widehat{\mu}\left(W_M+W_G+\frac{1}{I_M^+}ad(b)\frac{1}{I_G}c\Big|_{M,G}\right)+\text{more ghosts}\right\}.\end{aligned} \tag{26}$$

B. The BRST charge of non-critical $\mathcal{W}_3$ strings

The BRST charge for $\mathcal{W}_3$ gravity can be read off from the gauge fixed action (26). Let us explain why this is the case. The background field $\widehat{\mu}$ that was introduced during the gauge fixing of the $\mathcal{W}$ symmetry of our model, is in fact nothing but the antifield b^* for the antighost b. But this means that operator that couples to the field $\widehat{\mu}$ is nothing but the BRST variation of b. The BRST transformation of b splits into three distincts pieces

$$\delta_{\mathrm{BRS}}b \sim W_M + W_G + W_{\mathrm{gh}}, \tag{27}$$

where the ghost current $W_{\rm gh}$ is given by

$$W_{\rm gh} = \Pi_{\rm hw} \frac{1}{I_M^+} ad(b) \frac{1}{I_G} c\Big|_{M,G} + \cdots. \tag{28}$$

On the other hand we know that the BRST charge Q when acting on b, generates the BRST transformation (27), so Q can easily be constructed once the W currents are known.

For the case of $\mathcal{W}_3$ gravity we evaluate the ghost current $W_{\rm gh}$ explicitly. It contains terms quadratic in the ghosts only. If we parametrize

$$W_\alpha = \begin{pmatrix} 0 & \frac{1}{4}T_\alpha & \frac{1}{2}W_{3,\alpha} \\ 0 & 0 & \frac{1}{4}T_\alpha \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{for } \alpha = M, G, \text{gh} \tag{29}$$

and

$$b = \begin{pmatrix} 0 & \frac{1}{4}b_1 & \frac{1}{2}b_2 \\ 0 & 0 & \frac{1}{4}b_1 \\ 0 & 0 & 0 \end{pmatrix} \qquad c = \begin{pmatrix} 0 & 0 & 0 \\ c_1 & 0 & 0 \\ c_2 & c_1 & 0 \end{pmatrix} \tag{30}$$

we find that

$$\begin{aligned} T_{\rm gh} &= -2b_1\partial c_1 - \partial b_1 \cdot c_1 - 3b_2\partial c_2 - 2\partial b_2 \cdot c_2 \\ W_{3,\rm gh} &= -3b_2\partial c_1 - \partial b_2 \cdot c_1 - \frac{2}{3\kappa_M} b_1 \partial c_2 \cdot (T_M - T_G) \qquad (31) \\ &\quad -\frac{1}{3\kappa_M}\partial b_1 \cdot c_2 \left(T_M - T_G\right) - \frac{1}{3\kappa_M} b_1 c_2 \left(\partial T_M - \partial T_G\right) \\ &\quad +\frac{1}{12}\left\{10 b_1 \partial^3 c_2 + 15\partial b_1 \cdot \partial^2 c_2 + 9\partial^2 b_1 \cdot \partial c_2 + 2\partial^3 b_1 \cdot c_2\right\} \qquad (32) \end{aligned}$$

To compare our result with the currents $T_{\rm gh}$ and $W_{3,\rm gh}$ that were obtained in [4] we introduce rescaled spin 3 ghosts b_2' and c_2' :

$$b_2 = \frac{1}{\sqrt{\kappa_M}} b_2' \qquad c_2 = \sqrt{\kappa_M} c_2' \tag{33}$$

To make this rescaling into a canonical operation we also redefine the antifields of the ghosts. It is then very natural to rescale the background field $\widehat{\mu}_3$, and the W_3 currents as well

$$\widehat{\mu}_3 = \sqrt{\kappa_M}\widehat{\mu}_3' \qquad W_{3,\alpha} = \frac{1}{\sqrt{\kappa_M}} W'_{3,q\alpha}. \tag{34}$$

The rescaled ghost current $W'_{3,\rm gh}$ reads (dropping the primes)

$$\begin{aligned} W_{3,\rm gh} &= -3b_2\partial c_1 - \partial b_2 \cdot c_1 - \frac{2}{3} b_1 \partial c_2 \cdot (T_M - T_G) \qquad (35) \\ &\quad -\frac{1}{3}\partial b_1 \cdot c_2 \left(T_M - T_G\right) - \frac{1}{3} b_1 c_2 \left(\partial T_M - \partial T_G\right) \\ &\quad +\frac{\kappa_M}{12}\left\{10 b_1 \partial^3 c_2 + 15\partial b_1 \cdot \partial^2 c_2 + 9\partial^2 b_1 \cdot \partial c_2 + 2\partial^3 b_1 \cdot c_2\right\}. \end{aligned}$$

Now we comment on the transition to quantum theory. There is a general formula [11] for arbitrary DS reductions,

$$c = \frac{1}{2}c_{\rm crit} - \frac{(d_B - d_F)\tilde{h}}{\kappa + \tilde{h}} - 6y(\kappa + \tilde{h}). \tag{36}$$

where $c_{\rm crit}$ is the critical value of the central charge for the $\mathcal{W}$ algebra under consideration, d_B and d_F count the number of bosonic and fermionic generators in the Lie algebra $\bar{g}$, and y is the index of embedding of $sl(2)$ in $\bar{g}$. The values of these characteristic numbers can be computed with simple counting formulas [11]. For the case at hand, the DS reduction of the $\mathcal{W}_3$ algebra proceeds via the principal embedding of $sl(2)$ in $sl(3)$ (so $d_B = 8, d_F = 0$), and the $sl(3)$ algebra branches into an $sl(2)$ spin $j = 1$ and $j = 2$ representation. The values $c_{\rm crit} = 100$ and $y = 4$ follow. In the limit of large central charges (which in our case corresponds to the classical limit) $-24\kappa_M = c_M$, as is clear from (36). We may write the factor

$$\frac{\kappa_M}{12} = -\frac{1}{90}\frac{5c_M}{16} = -\frac{1}{90\beta_M^0}\,. \tag{37}$$

Upon quantization this factor, and only this factor, must be renormalized

$$-\frac{1}{90\beta_M^0} \mapsto \frac{17\beta_M - 1}{90\beta_M} \;\text{ with }\; \beta_M = \frac{16}{22 + 5c_M}\,, \tag{38}$$

leading immediately to the nilpotent BRST charge [9,4]

$$Q_{\text{non-crit},W_3} = \oint \frac{dz}{2\pi i} c_1 \left(T_M + T_G + \frac{1}{2}T_{\rm gh}\right) + c_2 \left(W_{3,M} + W_{3,G} + \frac{1}{2}W_{3,\rm gh}\right) \tag{39}$$

This may be compared with the procedure in [4], where the same final result was obtained only after adding additional terms to a classical charge. The reader wil have noticed that in the present treatment the BRST charge follows almost automatically from the equation (28). Once the classical ghost currents of eqs. (32) and (35) have been derived, one can obtain the quantum currents by a simple renormalization of one factor in front of the classical terms. In this respect the realization of the $\mathcal{W}_3$ algebra via WZW models, proves to be superior to the realization in terms of scalar fields which was used in [4]. In the classical analysis of [4] the term proportional to $\kappa_M \sim c_M$ in (35) was absent, and arose at the quantum level from counterterms. Clearly, using WZW models one already has a non-zero central charge at the classical level, so that the transition to the quantum theory can proceed in a very gentle way.

IV. GAUGE FIXING

In this section we treat more thoroughly the questions related to gauge fixing, both for the Drinfeld-Sokolov symmetry and for the $\mathcal{W}$ symmetries. For the DS symmetry we present a realization of the gauging that, at the expense of introducing extra Lagrange multipliers, succeeds in closing the algebra of the transformations. As a result the gauge fixing procedure simplifies, and although one could dispense with the full Batalin-Vilkovisky treatment[2], we nevertheless phrase it in that language for uniformity. For the $\mathcal{W}$ symmetries our treatment does not (at least not automatically) lead to such a simplified algebra. Because the symmetries close only modulo field equations, it is expedient to use the BV treatment to take this into account. We will not succeed in deducing all order (in antifields) expressions for arbitrary DS reductions, but at the end we will illustrate the general procedure by deriving the relevant expression for the $\mathcal{W}_3$ case.

A. The Drinfeld-Sokolov symmetry

The relevant information concerning the Drinfeld-Sokolov symmetries, which are quite conventional gauge symmetries, are encoded by adding the antifield-dependent terms

$$S_* = \frac{1}{\pi x}\int \mathrm{str}\left\{-\frac{\kappa}{2}\tilde{J}^* D[2\tilde{J}/\kappa]c_{\mathrm{DS}} + \frac{1}{2}c^*_{\mathrm{DS}}\mathrm{ad}\,(c_{\mathrm{DS}})\,c_{\mathrm{DS}}\right\}, \tag{40}$$

where $\tilde{J}$ is given in eq.(1) and $c_{\mathrm{DS}} \in \Pi_{>0}\mathfrak{g}$. The extended action $S_{1,\mathrm{ext}} = S + S_*$, with S from eq.(10), is a cornerstone of the Batalin-Vilkovisky treatment. Gauge invariance is expressed through the classical master equation $(S_{1,\mathrm{ext}}, S_{1,\mathrm{ext}}) = 0$. The term in the extended action proportional to c^*_{DS} expresses the closure of the DS gauge algebra. The particular form of this c^*_{DS} dependent term is typical for non-abelian gauge theories.

To proceed, we now add a (cohomologically) trivial system, with the extended action

$$\begin{aligned} S_{\mathrm{triv}} = \kappa\mathcal{S}^-[g] - \kappa\mathcal{S}^-[\tilde{g}] + \frac{1}{\pi x}\int \mathrm{str}\left\{A(J-\tilde{J})\right\} \\ -\frac{1}{\pi x}\int \mathrm{str}\left\{\frac{\kappa}{2}J^* D\,[2J/\kappa]\,c_{\mathrm{DS}} + A^*\bar{D}\,[A]\,c_{\mathrm{DS}}\right\}. \end{aligned} \tag{41}$$

The extra variables introduced here are a Lie algebra valued Lagrange multiplier A, and an extra current J which is completely unconstrained. The action is trivial in the antibracket sense. The addition of this extra trivial system allows us to "unconstrain" the currents on which the DS transformations are acting, achieving in this way a decoupling of the constraints and the gauge transformations. This is the basic reason why we succeed in obtaining a *closed* algebra, which, upon elimination of the trivial systems (by integrating out the Lagrange multipliers and putting their antifields to zero), goes over into the open algebra computed in [16,11]. This we now show. We split the full Lagrange multiplier A and its antifield into two parts:

$$\begin{aligned} A &= A_{\mathrm{DS}} + A_{\mathrm{ident}} \quad \text{with} \quad A_{\mathrm{DS}} \in \Pi_{>0}\mathfrak{g}\,; \quad A_{\mathrm{ident}} \in \Pi_{\leq 0}\mathfrak{g} \\ A^* &= A^*_{\mathrm{DS}} + A^*_{\mathrm{ident}} \quad \text{with} \quad A^*_{\mathrm{DS}} \in \Pi_{<0}\mathfrak{g}\,; \quad A^*_{\mathrm{ident}} \in \Pi_{\geq 0}\mathfrak{g}. \end{aligned} \tag{42}$$

The Lagrange multipliers in A_{DS} are precisely the ones that impose the Drinfeld-Sokolov constraints, bringing the current J into the $\tilde{J}$ form. We keep these Lagrange multipliers

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[2] A review of the Batalin-Vilkovisky formalism can be found in [21].

manifest in the action. The multipliers in $A_{\rm ident}$ identify the free components $J^{\geq 0}$ which are contained in $\tilde{J}$, with the corresponding components in J. We will implement this identification, by integrating explicitly over $A_{\rm ident}$ and over $J^{\geq 0}$. To this end we rewrite the extended action $S + S_* + S_{\rm triv}$ as

$$\begin{aligned} S_{2,\rm ext} = \kappa \mathcal{S}^-[g] &+ \frac{\kappa}{4\pi x} \int \mathrm{str} \left\{ [\tau, e_-] \bar{\partial} \tau \right\} + \frac{1}{\pi x} \int \mathrm{str} \left\{ \mu W(\tilde{J}) \right\} \\ &+ \frac{1}{\pi x} \int \mathrm{str} \left\{ (A_{\rm DS} + A_{\rm ident}) \left(J - \mathrm{ad}\,(A^*_{\rm DS} + A^*_{\rm ident})\, c_{\rm DS} - \tilde{J} \right) \right\} \\ &+ \frac{1}{\pi x} \int \mathrm{str} \left\{ -A^*_{\rm DS} \bar{\partial} c_{\rm DS} - \frac{\kappa}{2} \tilde{J}^* D[2\tilde{J}/\kappa] c_{\rm DS} - \frac{\kappa}{2} J^* D\,[2J/\kappa]\, c_{\rm DS} \right\} \\ &+ \frac{1}{\pi x} \int \mathrm{str} \left\{ \frac{1}{2} c^*_{\rm DS} \mathrm{ad}\,(c_{\rm DS})\, c_{\rm DS} \right\} . \end{aligned} \tag{43}$$

Next we introduce the shifted current

$$\check{J} = J - \mathrm{ad}\,(A^*_{\rm DS})\, c_{\rm DS} \tag{44}$$

and now eliminate the $A_{\rm ident}$ and $J^{\geq 0}$ fields, with their corresponding antifields. This leads to the extended action

$$\begin{aligned} S_{\rm ext} = \kappa \mathcal{S}^-[g] &+ \frac{\kappa}{4\pi x} \int \mathrm{str} \left\{ [\tau, e_-] \bar{\partial} \tau \right\} + \frac{1}{\pi x} \int \mathrm{str} \left\{ \mu W(\check{J}^{\geq 0}, \tau) \right\} \\ &+ \frac{1}{\pi x} \int \mathrm{str} \left\{ A_{\rm DS} \left(\check{J}^{<0} - \frac{\kappa}{2} e_- - \frac{\kappa}{2} [\tau, e_-] \right) - A^*_{\rm DS} \bar{\partial} c_{\rm DS} \right\} \\ &+ \frac{1}{\pi x} \int \mathrm{str} \left\{ -\frac{\kappa}{2} J^* D\,[2J/\kappa]\, c_{\rm DS} - \tau^* \Pi_{+1/2} c_{\rm DS} \right\} \\ &+ \frac{1}{\pi x} \int \mathrm{str} \left\{ \frac{1}{2} c^*_{\rm DS} \mathrm{ad}\,(c_{\rm DS})\, c_{\rm DS} \right\} . \end{aligned} \tag{45}$$

Notice that this action may contain terms with multiple antifields $A^*_{\rm DS}$, due to the appearance of shifted currents in the gauge invariant polynomials $W(\check{J}^{\geq 0}, \tau)$. If this happens, this is a manifestation of the non-closure of the gauge algebra, that belongs to the DS invariant classical action $S_{\rm cl} = S_{\rm ext}[A^* = J^* = c^* = \tau^* = 0]$. It is precisely this classical action $S_{\rm cl}$ that was used in [16] in the case of $\mathcal{W}_3$ gravity, and in [11] in the case of $SO(N)$ supergravities, as a starting point for a direct construction of the BV-extended action. The existence of non-closure terms made this construction rather cumbersome, but as we showed here, this can be avoided by introducing a redundant set of Lagrange multipliers $A = A_{\rm DS} + A_{\rm ident}$, which keeps the gauge algebra closed. The BV extended action can be constructed easily in this extended space of variables, and be reduced afterwards.

The gauge fixing of the DS symmetry in the extended action (45) can now be simply achieved by putting $A_{\rm DS} = \widehat{A}_{\rm DS} = b^*_{\rm DS}$ and $A^*_{\rm DS} = -b_{\rm DS}$, a transformation of variables canonical in the antibracket. This is one of the gauges used in [11]. If we keep the dependence on $\widehat{A}_{\rm DS}$, so that the reader may still transit to the other gauge used in [11] if (s)he wants, we find

$$
\begin{aligned}
S_{\mathrm{gf}} &= \kappa\mathcal{S}^{-}[g] + \frac{\kappa}{4\pi x}\int \mathrm{str}\left\{[\tau, e_{-}]\bar{\partial}\tau\right\} + \frac{1}{\pi x}\int \mathrm{str}\left\{b_{\mathrm{DS}}\bar{\partial}c_{\mathrm{DS}}\right\} \\
&\quad + \frac{1}{\pi x}\int \mathrm{str}\left\{\mu W(\check{J}^{\geq 0}, \tau) + \widehat{A}_{\mathrm{DS}}\left(\check{J}^{<0} - \frac{\kappa}{2}e_{-} - \frac{\kappa}{2}[\tau, e_{-}]\right)\right\}
\end{aligned}
\tag{46}
$$

where, apart from the kinetic term, the ghost dependence is through the shifted current

$$
\check{J} = J + \mathrm{ad}\,(b_{\mathrm{DS}})\,c_{\mathrm{DS}}. \tag{47}
$$

It should be remarked that the BRST transformation rules of the fields in the gauge fixed action do not depend on the sources μ. From this we learn that the DS invariant polynomials computed from eq.(6), have become *BRST invariant* polynomials through the replacement of the currents $J^{\geq 0}$ by the shifted $\check{J}^{\geq 0}$.

B. The $\mathcal{W}$ gauge symmetry

We propose to start from an unconstrained system of coupled WZW models, for which the extended action can be obtained more easily. Using only canonical methods (with respect to the antibracket) we then implement the various constraints, necessary to bring the WZW models in the highest weight form.

The starting point is

$$
\begin{aligned}
S_0 &= \kappa_M\mathcal{S}^{-}[g_M] + \kappa_G\mathcal{S}^{-}[g_G] + \frac{1}{\pi x}\int str\left\{A\ (J_M + J_G)\right\} \\
&\quad - \frac{1}{\pi x}\int str\left\{A^{*}\,\bar{D}[A]C - \frac{1}{2}C^{*}\mathrm{ad}\,(C)\,C\right\} \\
&\quad - \frac{1}{\pi x}\int str\left\{\frac{\kappa_M}{2}J_M^{*}D[2J_M/\kappa]C + \frac{\kappa_G}{2}J_G^{*}D[2J_G/\kappa]C\right\},
\end{aligned}
\tag{48}
$$

where all the fields take values in the entire Lie algebra, and the covariant derivatives involve at the moment unconstrained currents J_M and J_G. One may notice that we are treating the currents J as basic variables, rather than the group elements g: this simplifies the calculations, but should not influence the results. One can read off the gauge (or BRST) transformations from the terms with starred fields. The gauge invariance (i.e. the BV master equation) can be checked explicitly if $\kappa_M + \kappa_G = 0$. It can also be seen by parametrizing $A = h^{-1}\bar{\partial}h$, and rewriting the first line, with the help of the Polyakov-Wiegmann formula eq.(8), as the sum of two (separately invariant) WZW actions $\kappa_M\mathcal{S}^{-}[hg_M] + \kappa_G\mathcal{S}^{-}[hg_G]$: the condition $\kappa_M + \kappa_G = 0$ eliminates the additional $\mathcal{S}^{-}[h]$ terms. The gauge field A acts as a Lagrange multiplier imposing $J_M + J_G - \mathrm{ad}(A^{*})C = 0$. The antifield dependence of this constraint can be absorbed into a redefinition of the currents J_M, J_G. We implement this redefinition by performing the canonical transformation generated by

$$
F = \mathbf{1} - str\left\{\frac{1}{2}\left(J_M'^{*} + J_G'^{*}\right)\mathrm{ad}\left(A'^{*}\right)C\right\}. \tag{49}
$$

Dropping the primes, it leads to the following extended action:

$$
\begin{aligned}
S_1 = \kappa_M \mathcal{S}^-[h_M] + \kappa_G \mathcal{S}^-[h_G] + \frac{1}{\pi x}\int str\left\{A\ (J_M + J_G)\right\} \\
-\frac{1}{\pi x}\int str\left\{A^*\,\bar\partial C - \frac{1}{2}C^* \mathrm{ad}\,(C)\,C\right\} \\
-\frac{1}{\pi x}\int str\left\{\frac{\kappa_M}{2}J_M^* D_{J_{av}}C + \frac{\kappa_G}{2}J_G^* D_{J_{av}}C\right\},
\end{aligned}
\tag{50}
$$

where the covariant derivative $D_{J_{av}} = D[J_M/\kappa_M + J_G/\kappa_G]$ involves a current that averages over matter and gravitational sectors, and the group elements h_α, for $\alpha \in \{M, G\}$, are defined through

$$
\frac{\kappa_\alpha}{2}\partial h_\alpha h_\alpha^{-1} = J_\alpha + \frac{1}{2}\mathrm{ad}(A^*)C. \tag{51}
$$

The next step is to split the gauge field A into pieces, say $A = \overline{\Pi}_{\mathrm{lw}}A + \mu \equiv \overline{A} + \mu$, and accordingly $A^* = \overline{\Pi}_{\mathrm{hw}}A^* + \mu^* \equiv \overline{A}^* + \mu^*$.[3] It is clear that the $\overline{A}$ field imposes the condition $\overline{\Pi}_{\mathrm{hw}}\,(J_M + J_G) = 0$. To achieve our aim of constraining both currents in the Drinfeld-Sokolov way, we need an extra condition. The gauge freedom allows us to impose such a condition. We choose to impose it in a $M \leftrightarrow G$ symmetric way: the condition $\overline{\Pi}_{\mathrm{hw}}\left(\frac{J_M}{\kappa_M} + \frac{J_G}{\kappa_G} - e_-\right) = 0$ precisely brings the currents J_M and J_G in the desired highest weight form. In the Batalin-Vilkovisky scheme we may implement that constraint by first adding the following trivial system to the action:

$$
S_{\mathrm{triv}} = \frac{1}{\pi x}\int str\left\{\rho^*\,\lambda\right\}, \tag{52}
$$

where $\lambda, \rho \in \overline{\Pi}_{\mathrm{lw}}\mathfrak{g}$. Then we perform the canonical transformation with generator

$$
F = \mathbf{1} + str\left\{\rho\,\overline{\Pi}_{\mathrm{hw}}\left(\frac{J_M}{\kappa_M} + \frac{J_G}{\kappa_G} - e_-\right)\right\}. \tag{53}
$$

The resulting extended action reads

$$
\begin{aligned}
S_2 = \kappa_M \mathcal{S}^-[h_M] + \kappa_G \mathcal{S}^-[h_G] + \frac{1}{\pi x}\int str\left\{\mu\ (V_M + V_G)\right\} \\
-\frac{1}{\pi x}\int str\left\{\left(\overline{A}^* + \mu^*\right)\,\bar\partial C - \frac{1}{2}C^* ad\,(C)\,C - \overline{A}\,\overline{\Pi}_{\mathrm{hw}}\,(J_M + J_G)\right\} \\
-\frac{1}{\pi x}\int str\left\{\frac{\kappa_M}{2}J_M^* D_{J_{av}}C + \frac{\kappa_G}{2}J_G^* D_{J_{av}}C\right\} \\
-\frac{1}{\pi x}\int str\left\{\rho D_{J_{av}}C - \lambda\,\overline{\Pi}_{\mathrm{hw}}\left(\frac{J_M}{\kappa_M} + \frac{J_G}{\kappa_G} - e_- + \rho^*\right)\right\},
\end{aligned}
\tag{54}
$$

[3]By definition $\overline{\Pi}_{\mathrm{lw}} = 1 - \Pi_{\mathrm{lw}}$.

where the currents V_α are the highest weight components of the J_α's. Now we eliminate the variables $\left\{\overline{A}, \lambda, \overline{\Pi}_{\rm hw} J_M, \overline{\Pi}_{\rm hw} J_G\right\}$, and find that

$$
\begin{aligned}
S_3 = {} & \kappa_M \mathcal{S}^-[f_M] + \kappa_G \mathcal{S}^-[f_G] + \frac{1}{\pi x}\int str\left\{\mu\ (V_M + V_G)\right\} \\
& -\frac{1}{\pi x}\int str\left\{\mu^*\,\bar\partial C - \frac{1}{2}C^*\mathrm{ad}\,(C)\,C + \rho\,(D_{V_{av}} - E_- + \mathrm{ad}(\rho^*))\,C\right\} \\
& -\frac{1}{\pi x}\int str\left\{\kappa_M V_M^*\,(D_{V_{av}} - E_- + \mathrm{ad}(\rho^*))\,C|_{M,G}\right\}.
\end{aligned} \tag{55}
$$

The group elements f_α are given by

$$
\frac{\kappa_\alpha}{2}\partial f_\alpha f_\alpha^{-1} = \frac{\kappa_\alpha}{2}(e_- - \rho^*) + V_\alpha + \frac{1}{2}\mathrm{ad}(\mu^*)C. \tag{56}
$$

The fields $\left\{\rho, \overline{\Pi}_{\rm lw} C\right\}$ also form a "trivial" pair of variables, albeit in a more subtle way. Indeed, the equation of motion of ρ evaluated in the point $\rho^* = 0$ is equivalent to eq.(11). The structure of this equation is such that all the $\overline{\Pi}_{\rm lw} C$ fields can be exactly solved for, yielding

$$
C \rightarrow \frac{1}{I_{V_{av}}}c, \tag{57}
$$

where c denotes the lowest weight part of the original ghost field C and $I_{V_{av}}$ is defined in terms of the average current as $I_{V_{av}} = 1 - LD_{V_{av}}$. In doing so we find the action

$$
\begin{aligned}
S_4 = {} & \kappa_M \mathcal{S}^-[v_M] + \kappa_G \mathcal{S}^-[v_G] + \frac{1}{\pi x}\int str\left\{\mu\ (V_M + V_G)\right\} \\
& -\frac{1}{\pi x}\int str\left\{\mu^*\,\bar\partial c - \frac{1}{2}c^*\mathrm{ad}\left(\frac{1}{I_{V_{av}}}c\right)\frac{1}{I_{V_{av}}}c\right\} \\
& -\frac{1}{\pi x}\int str\left\{\frac{\kappa_M}{2}V_M^* D_{V_{av}}\frac{1}{I_{V_{av}}}c + \frac{\kappa_G}{2}V_G^* D_{V_{av}}\frac{1}{I_{V_{av}}}c\right\},
\end{aligned} \tag{58}
$$

with

$$
\frac{\kappa_\alpha}{2}\partial v_\alpha v_\alpha^{-1} = \frac{\kappa_\alpha}{2}e_- + V_\alpha + \frac{1}{2}\mathrm{ad}(\mu^*)\frac{1}{I_{V_{av}}}c. \tag{59}
$$

The last term in eq.(59) will be called the ghost current J_{gh}. We may replace μ^* by (minus) the antighost b, and put μ equal to a background value. The action (58) then becomes the gauge fixed action.

This expression, albeit not very transparant, is valid to all orders in the ghost fields. The ghost field dependence is partly explicit, but also implicit in the WZW functionals, where it enters through the definition of the group elements v_α in eq.(59). We now investigate how to make this dependence more explicit. Although at present we can not give the end result in general, the following constitutes a constructive procedure. We will explicitize the ghost dependence in a specific case, namely the reduction of $sl(3)$ to the $\mathcal{W}_3$ algebra, which also served as an example in section III B.

The first step is, to disentangle the dependence in the WZW actions. To this end, in the same spirit as in section II A, we factorise $v_\alpha = e^{-\gamma_\alpha} w_\alpha$. where w_α is such that the current $\frac{\kappa_\alpha}{2}\partial w_\alpha w_\alpha^{-1}$ is in the highest weight form $\frac{\kappa_\alpha}{2} e_- + W_\alpha$. To obtain this form, we follow the same method as in section II A. Note however that the right hand side of eq.(59) is not restricted to non-negative e_0-grading, due to the ghost contribution. It is not obvious from the group property that such a heighest weight gauge can be reached. Proceeding nevertheless in the same manner, we put $\gamma_\alpha = \sum_{n\geq 1} \gamma_\alpha^{(n)}$ and $W_\alpha = \sum_{n\geq 0} W_\alpha^{(n)}$, where the expansion now is not in the full current (as in section II A), but in the deviation from the highest weight form, namely the ghost current in eq.(59). Consequently, the successive terms in this expansion will be sums of products of two, four, six, etc. ghost fields. In addition we impose $\gamma_\alpha \in \bar{\Pi}_{\rm lw}\mathfrak{g}$, which guarantees that $LE_-\gamma_\alpha = \gamma_\alpha$. Now an algorithm can be given to construct γ_α and W_α iteratively. The recursive construction is:

$$
\begin{aligned}
\gamma^{(0)} &= 0 \quad ; \qquad W_\alpha^{(0)} = V_\alpha \\
g^{(n)} &= exp\left\{\gamma_\alpha^{(0)} + \ldots + \gamma_\alpha^{(n-1)}\right\} \\
X_\alpha^{(n)} &= \frac{1}{I_\alpha^+} \mathcal{P}^{(n)} \left[g^{(n)} \left(e_- - D_\alpha + \frac{2}{\kappa_\alpha} J_{gh}\right) (g^{(n)})^{-1}\right] , \\
\gamma_\alpha^{(n)} &= L\, X_\alpha^{(n)} , \\
W_\alpha^{(n)} &= \frac{\kappa_\alpha}{2} \Pi_{hw}\, X_\alpha^{(n)} , \qquad\qquad n \geq 1 .
\end{aligned} \tag{60}
$$

In these expressions, the derivatives are covariant derivatives, with V_α (either V_M or V_L) as gauge fields. These derivatives are also used to construct I_α via eq.(12). Finally, $\mathcal{P}^{(n)}$ now denotes that only terms with products of $2n$ ghost fields are kept. For concreteness, we list the first few terms

$$
\begin{aligned}
\gamma_\alpha &= \frac{2}{\kappa_\alpha} L \frac{1}{I_\alpha^+} J_{gh} - \frac{2}{\kappa_\alpha^2} L \frac{1}{I_\alpha^+} ad\left(J_{gh} + \Pi_{\rm hw} \frac{1}{I_\alpha^+} J_{gh}\right) L \frac{1}{I_\alpha^+} J_{gh} + \cdots \\
W_\alpha &= V_\alpha + \Pi_{\rm hw} \frac{1}{I_\alpha^+} J_{gh} - \frac{1}{\kappa_\alpha} \Pi_{\rm hw} \frac{1}{I_\alpha^+} ad\left(J_{gh} + \Pi_{\rm hw} \frac{1}{I_\alpha^+} J_{gh}\right) L \frac{1}{I_\alpha^+} J_{gh} + \cdots
\end{aligned} \tag{61}
$$

Explicitly, for $\mathcal{W}_3$, we find the following relations in the 'matter' sector. We write down the relations between the heighest weight fields T and W_3 before (V) and after (W) the transformation with γ_α with the (conventional) normalizations as in eq.(29):

$$
\begin{aligned}
T_M(W) &= T_M(V) + \{2\, b_1\, \partial c_1 + 3\, b_2\, \partial c_2 + \partial b_1\, c_1 + 2\, \partial b_2\, c_2\}/2 \\
&\qquad +5\, b_1\, \partial b_1\, c_2\, \partial c_2 / 48\, \kappa_M \\
W_{3,M}(W) &= W_{3,M}(V) - \{-\partial b_2\, c_1/2 - 3\, b_2\, \partial c_1/2 \\
&\qquad 5\, b_1\, \partial^3 c_2/12 + 5\, \partial b_1\, \partial^2 c_2/8 + 3\, \partial^2 b_1\, \partial c_2/8 + \partial^3 b_1\, c_2/12 \\
&\qquad -T_G(V)\, b_1\, \partial c_2/6\, \kappa_G - T_G(V)\, \partial b_1\, c_2/16\, \kappa_G \\
&\qquad -T_M(V)\, b_1\, \partial c_2/2\, \kappa_M - 13\, T_M(V)\, \partial b_1\, c_2/48\, \kappa_M \\
&\qquad -5\, \partial T_G(V)\, b_1\, c_2/48\, \kappa_G - 11\, \partial T_M(V)\, b_1\, c_2/48\, \kappa_M\} \\
&\qquad +\{-24\, b_1\, b_2\, c_2\, \partial^2 c_2 - 8\, b_1\, \partial b_2\, c_2\, \partial c_2 - 24\, \partial b_1\, b_2\, c_2\, \partial c_2 \\
&\qquad +8\, b_1\, \partial b_1\, c_1\, \partial c_2 - b_1\, \partial b_1\, \partial c_1\, c_2 + 3\, b_1\, \partial^2 b_1\, c_1\, c_2\}/48\, \kappa_M
\end{aligned} \tag{62}
$$

For the 'Liouville' sector, the same relations obtain, mutatis mutandis. We do not write down, the corresponding expansions for γ_α. We now construct the gauge fixed action explicitly. The Polyakov-Wiegmann formula is used repeatedly to extract the ghost dependence from the WZW functionals. It turns out that all ghost contributions vanish, for a variety of reasons: partial integration, highest weight properties, Grassman algebra, and $\kappa_M + \kappa_G = 0$. The total currents $V_M + V_G$ in eq.(58) can be obtained by inverting the relations eqs.(62) above. Due to the presence of the four-ghost terms, this is actually simpler for the sum than for V_M and V_G separately, by virtue of the relation $\kappa_M + \kappa_G = 0$ which causes the four-ghost terms to cancel. The result is that $V_M + V_G = W_M + W_G + W_{gh}$, with the ghost contributions given by eq.(32). Thus we fulfilled our promise in section III. We emphasize that the method used here was completely constructive. Finally, the terms of eq.(58) involving antifields are immaterial for the gauge fixed action (they determine the final constraint algebra), and need not be discussed here.

Having demonstrated the method, let us now comment on the general situation. First of all, the gauge fixed action that is implied by the eq.(58) has all the suitable variables and symmetries. The dependence on the ghost fields, as emphasized, is only given implicitly through the shifted currents of eq.(59), making the ghost Lagrangian paticularly untransparant. The strategy applied above for $\mathcal{W}_3$ may be developed for the general case also, but a couple of possible obstructions to this straightforward line should be mentioned. First, whereas in section II A the finiteness of the iteration in eq.(6) was guaranteed, for eq.(62) we do not have such a proof, although we do believe that there is no problem in this respect. In particular, for reductions of Lie algebras (not superalgebras) all ghosts are fermionic, and the finiteness of the expansions of γ_α and W_α follows from dimensional arguments. Perhaps more serious is the fact that, in the general case, we have no reason to expect that the WZW functionals with argument $e^{\gamma_\alpha} w_\alpha$ will always simplify as for $\mathcal{W}_3$ above. In general, this could entail a non-standard ghost Lagrangian, and quite possibly a further transformation may be needed, of variables from the set $\{V_\alpha\ ,\ \text{ghosts}\}$ to $\{W_\alpha(V_\alpha, \text{ghosts})\,,\ \text{ghosts'}(V_\alpha, \text{ghosts})\}$that mixes the ghosts with the matter and gravity currents. This transformation should be such that in the end the redefined ghost fields decouple from the WZW models. Also, the inversion of the relations expressing the W curents in terms of the V currents may be considerably more involved in general. We leave the treatment of these complications to the future.

V. CONCLUSIONS

To recapitulate, we realized any $\mathcal{W}$ symmetry that is obtained from a Drinfeld-Sokolov reduction, for non-critical values of the central extension, in a generic way by coupling a WZW model representing the matter fields to a WZW model representing the (generalized) gravitational degrees of freedom. The constrained classical models give rise to two separate $\mathcal{W}$ algebra realizations, and the constaints entail the presence of ghosts. A condition for $\mathcal{W}$-invariance of the full theory is always the vanishing of the sum of the central charges. We showed (using the field-antifield formalism) how to derive the BRST charge, always on the classical level. We showed explicitly the workability of our scheme by applying it to $\mathcal{W}_3$. Since the central extensions are already present at the classical level, the eventual transition to the quantum level was shown to be rather trivial, involving (in that case) only the renormalization of a single coefficient, without additional terms. This suggests that our method may be used to advantage in all these cases where the transition to the quantum algebra's seems impossible or problematic. We hope to come back to these questions in the future.

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REFERENCES

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[1] A.B. Zamolodchikov, Teor. Mat. Fiz. **65** (1985) 1205.

[2] T. Tjin, *Finite and Infinite W Algebras and their Applications*, PhD thesis, Amsterdam 1993;
Phys. Lett. **B292** (1992) 60; J. de Boer and T. Tjin, Comm. Math. Phys. **158** (1993) 485.

[3] P. Bouwknegt, J. McCarthy and K. Pilch, Nucl. Phys. **B352** (1991) 139; Lett. Math. Phys. **29** (1993) 91.

[4] M. Bershadsky, W. Lerche, D. Nemeschansky and N.P. Warner, Phys. Lett. **B292** (1992) 35;
E. Bergshoeff, A. Sevrin and X. Shen, Phys. Lett. **B296** (1992) 95.

[5] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. **B352** (1991) 59; K. Li, Nucl. Phys. **B354** (1991) 711, Nucl. Phys. **B354** (1991) 725.

[6] T. Eguchi, Y. Yamada and S.-K. Yang, Phys. Lett. **B305** (1993) 235.

[7] W. Lerche and A. Sevrin, CERN-TH.7210/94, hep-th/9403183, Nucl. Phys B, in press.

[8] B. Gato-Rivera and A. M. Semikhatov, Phys. Lett. **B293** (1992).

[9] M. Bershadsky, W. Lerche, D. Nemeschansky and N. P. Warner, Nucl. Phys. **B401** (1993) 304.

[10] A. Boresch, K. Landsteiner, W. Lerche and A. Sevrin, preprint, CERN-TH.7379/94, hep-th/9408033.

[11] A. Sevrin, K. Thielemans and W. Troost, Nucl. Phys. **B407** (1993) 459
A. Sevrin and W. Troost, Phys. Lett. **B315** (1993) 145.

[12] R. Siebelink, *Regularisation of Two-Dimensional Induced Models and Non-Critical Strings*, PhD thesis, Leuven 1994.

[13] A. Deckmyn, *Extended Conformal Algebras and their Realizations*, PhD thesis, Leuven 1994.

[14] F. A. Bais, T. Tjin and P. van Driel, Nucl. Phys. **B357** (1991) 632; L. Fehér, L. O'Raifeartaigh, P. Ruelle, I. Tsutsui and A. Wipf, Phys. Rep. **222** (1992) 1.

[15] M. Bershadsky and H. Ooguri, Phys. Lett. **B229** (1989) 374.

[16] J. de Boer and J. Goeree, Nucl. Phys. **B401** (1993) 348.

[17] V.G. Knizhnik, A.M. Polyakov and A.B. Zamolodchikov, Mod. Phys. Lett. **A3** (1988) 83.

[18] H. Ooguri, K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Comm. Math. Phys. **145** (1992) 515.

[19] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Nucl. Phys. **B371** (1992) 315.

[20] A.M. Polyakov and P.B. Wiegmann, Phys. Lett. **B141** (1983) 121; **B141** (1984) 223.

[21] W. Troost and A. Van Proeyen, *An introduction to Batalin-Vilkovisky Lagrangian quantization*, Leuven Notes in Math. Theor. Phys., in preparation.