Heavy Quarkonium Effective Theory

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Abstract

We formulate a QCD-based effective theory approach to heavy quarkonia-like systems as $\bar{c}c$ and $\bar{b}b$ resonances and B_c states. We apply the method to inclusive decays, working out a few examples in detail.

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1 Introduction

One of the most important recent developments in the field of heavy-quark physics has been the formulation of the static limit for a heavy quark as an effective theory [1]. The so-called Heavy Quark Effective Theory (HQET) has put the description of heavy-hadron physics on a QCD-related and model-independent basis. Aside from this theoretical progress there have been numerous phenomenological applications of this idea, in which the model dependence has completely disappeared or is substantially decreased [2].

Most of these applications deal with systems involving a single heavy quark. In HQET both the particle and the antiparticle numbers are separately conserved, and the applications considered so far deal mainly with the one-(anti)particle sector of HQET. Thus the one-particle sector of HQET is already explored in some detail, including leading and next-to-leading QCD corrections.

However, there have also been first attempts to deal with the two-(anti)particle sector and the particle-antiparticle sector [3, 4]. Here it turns out that complex anomalous dimensions are obtained by a naïve analysis. In HQET the anomalous dimensions in general depend on the velocities of the heavy quarks and the imaginary parts of the anomalous dimensions in the two-particle sector diverge as $1/\sqrt{(vv')^2 - 1}$ for $v \to v'$, if v and v' are the velocities of the two heavy quarks. Subsequent investigations have shown that this singularity is related to the long-range part of the quark-antiquark potential and that one may remove this divergence by a suitable definition of the multiparticle states of HQET. In a basis consisting of the redefined states one then has real anomalous dimensions.

The physical origin of these phases is the same as that of the well-known Coulomb phases. The one-gluon (one-photon) exchange potential decreases too slowly, leading to the well-known singularities of the form $1/|\vec{v}|$, where $\vec{v}$ is the relative velocity of the outgoing coloured (charged) particles. From its physical interpretation as well as from the fact that the divergent phase is a property of the states, it is clear that these divergent contributions are a long-distance effect belonging to the infrared dynamics of the state with two heavy quarks.

Heavy quarkonium states have to show up as bound states in the particle-antiparticle sector of HQET. In such a state, however, the two velocities of the heavy quarks differ by an amount of only Λ_{QCD}/m (m is the heavy-quark mass) and hence we would rather like to switch to a description in which the two heavy-quark velocities become equal. This limit, however, cannot be taken in a naïve way, since the static limit of HQET does not reproduce the above phases, which are related to the potential between the two heavy quarks and hence to the binding mechanism of the quarkonium state.

It turns out that this limit may be performed if the evolution of the states is determined not by the static Lagrangian, but rather by a Lagrangian consisting of the static HQET part and the first subleading spin symmetric contribution to the Lagrangian, i.e. the kinetic energy operator. In other words, the presence of the divergent phase forces us to go beyond the static limit, if we want to consider heavy-quarkonium states, in which the velocities of the heavy quarks differ only by an amount of order Λ_{QCD}/m. This in turn introduces a mass dependence into the lowest-order dynamics such that the inverse "Bohr radius" $\widetilde{\Lambda}$, which sets a small non-perturbative scale, is in general not independent of m. For this reason we lose heavy-flavour symmetry, but spin symmetry is still present, since the kinetic energy operator is still spin symmetric.

In the present paper we set up an effective theory approach to processes involving heavy quarkonia, which is based on the heavy-mass limit of QCD. We focus on inclusive annihilation-type decays for which we perform a systematic expansion in powers of $\tilde{\Lambda}/m$, up to logarithms, which may be accessed via the renormalization group.

The approach proposed here has many features common to the one of Bodwin, Braaten and Lepage (BBL) [5]. Their method is based on non-relativistic QCD (NRQCD), where the lowest-order dynamics is determined by the Lagrangian of the Schrödinger equation for the heavy quarks, which is basically that obtained by including in addition to the static HQET part also the kinetic energy operator. Based on this they perform an expansion in terms of v/c, where v is the typical relative velocity of the two heavy quarks. We shall compare the two approaches between them as well as with previous approaches to inclusive quarkonium decays (for reviews see e.g. [6, 7]) as we go along.

Our general strategy is as follows. We shall first discuss (section 2) the heavy-mass limit for quarkonia-like systems and show that the two heavy-particle velocities may be chosen to become equal, once the subleading kinetic energy operator is added to the lowest-order dynamics. This however implies that there will be no mass-independent static limit for the quarkonia, in other words, the "binding energy" $\tilde{\Lambda}$ will depend on the heavy mass in a non-trivial way, but it is still a small scale $\tilde{\Lambda} \ll m$, so that an expansion in powers of $\tilde{\Lambda}/m$ is useful.

The first step to access the inclusive annihilation decays is to separate long and short distances, where the distance scale is set by the Compton wavelength of the heavy quark. Technically speaking, the first step is to set up an operator product expansion for the inclusive decay rate of a heavy quarkonium into light degrees of freedom (section 3). In section 4 we write down the $\tilde{\Lambda}/m$ expansion for the rate of inclusive heavy quarkonia annihilation up to and including $\tilde{\Lambda}^2/m^2$; up to this order the inclusive decay rate involves matrix elements of dimension-six and dimension-eight operators.

The coefficients of this expansion may be calculated in perturbation theory, and some

simple examples are considered in section 5. These perturbatively calculated coefficients are the ones at the matching scale, i.e. at the scale of the heavy quark mass. In general the coefficients are scale-dependent, since the operators need to be renormalized. In section 6 we consider the renormalization group flow of the coefficients. Given their values at the matching scale one may then use the renormalization group to scale down to some small scale μ, at which one then tries to estimate the matrix elements of the corresponding operators. These matrix elements are non-perturbative quantities, but heavy quark symmetries restrict the number of independent parameters. This is studied in section 7. Finally, we apply this machinery to some simple examples in section 8. Section 9 contains a further discussion of previous approaches [6, 7] as well as our conclusions. Some technical details are left for the appendices.

2   Heavy-Mass Limit for Quarkonia

We shall first set up the heavy-mass limit appropriate for heavy-quarkonia states. We shall start from the Lagrangian and the fields of QCD. We denote the heavy-quark field of full QCD by Q and define

$$Q_v^{(+)}(x) = \exp(im\ vx)Q(x) = h_v^{(+)}(x) + H_v^{(+)}(x) \ , \tag{1}$$

where v is a velocity ($v^2 = 1$), which is later identified with the velocity of the heavy hadron. Extracting this phase factor from the full QCD field Q removes the dominant part mv of the heavy-quark momentum, since this phase redefinition corresponds to a splitting of the heavy-quark momentum according to $p = mv + k$, where the residual momentum k is small, of the order of Λ_{QCD}. Furthermore, $h_v^{(+)}$ ($H_v^{(+)}$) is the large (small) component field, corresponding to the projections

$$h_v^{(+)} = P_+ Q_v^{(+)}, \quad H_v^{(+)} = P_- Q_v^{(+)} \text{ with } P_\pm = \frac{1}{2}(1 \pm \slashed{v}) \ . \tag{2}$$

The small component field $H_v^{(+)}$ is related to the large scale m; integrating out $H_v^{(+)}$ from the generating functional of QCD Green's functions corresponds to the replacement

$$H_v^{(+)} = P_- \left(\frac{1}{2m + ivD} \right) i\slashed{D} h_v^{(+)} \tag{3}$$

and this yields a non-local Lagrangian of the form [8]

$$\mathcal{L} = \bar{h}_v^{(+)}(ivD)h_v^{(+)} + \bar{h}_v^{(+)} i\slashed{D} P_- \left(\frac{1}{2m + ivD} \right) i\slashed{D} h_v^{(+)} \ , \tag{4}$$

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which still contains all orders in $1/m$. However, the non-locality appearing in the second term of (4) may be expanded into an infinite series of local terms, which come with increasing powers of $1/m$. Hence one may in this way establish the desired heavy-mass expansion. The first few terms of the expansion for the Lagrangian are

$$\mathcal{L} = \bar{h}_v^{(+)}(ivD)h_v^{(+)} + \left(\frac{1}{2m}\right)\bar{h}_v^{(+)}i\slashed{D}P_-i\slashed{D}h_v^{(+)}$$
$$+ \left(\frac{1}{2m}\right)^2 \bar{h}_v^{(+)}i\slashed{D}P_-(-ivD)i\slashed{D}h_v^{(+)} + \cdots \ , \tag{5}$$

while the field is given by

$$Q_v^{(+)} = \left(1 + \frac{1}{2m}P_-i\slashed{D} + \frac{1}{4m^2}(-ivD)P_-i\slashed{D} + \mathcal{O}(1/m^3)\right)h^{(+)}(x) \ . \tag{6}$$

The non-local expression (4) is still equivalent to full QCD; in particular it is independent of the still arbitrary velocity vector v. In fact, the Lagrangian (4) is invariant under an infinitesimal shift of the velocity

$$v \rightarrow v + \delta v \qquad v \cdot \delta v = 0$$
$$h_v^{(+)} \rightarrow h_v^{(+)} + \frac{\slashed{\delta v}}{2}\left(1 + P_-\frac{1}{2m+ivD}i\slashed{D}\right)h_v^{(+)}$$
$$iD \rightarrow -m\,\delta v \ . \tag{7}$$

This invariance is the so-called reparametrization invariance [9], which will play some role in what follows.

The Lagrangian (4) and its expansion in powers of $1/m$ (5) is the Lagrangian for a heavy quark. In the infinite mass limit, the quarks and the antiquarks are separated by an infinitely large mass gap, and hence in order to describe heavy quark-antiquark systems we have to introduce the antiquark field as a separate field, the Lagrangian of which is obtained from (4) or (5) by the replacement $v \rightarrow -v$.

Heavy quarkonia should appear in the particle-antiparticle sector of HQET as bound states. The starting point of our considerations is the Lagrangian of a heavy-quark and a heavy-antiquark field. In the static limit we obtain from (5)

$$\mathcal{L} = \bar{h}_v^{(+)}(ivD)h_v^{(+)} - \bar{h}_w^{(-)}(iwD)h_w^{(-)} \ , \tag{8}$$

where the static quark field $h_v^{(+)}$ ($h_w^{(-)}$) moves with velocity v (w). From (8) we obtain the equations of motion

$$(ivD)h_v^{(+)} = 0 \qquad (iwD)h_w^{(-)} = 0 \ . \tag{9}$$

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However, (8) or (9) cannot be used to describe heavy quarkonia states. In order to discuss this we shall consider the matrix elements

$$\langle A|\bar{Q}(x)\Gamma Q(x)|0\rangle \tag{10}$$

where A is a state containing a heavy quark and a heavy antiquark moving with velocities v and w respectively. In the static limit this matrix element becomes

$$\widetilde{\langle A|}\bar{h}_v^{(+)}(x)\Gamma h_w^{(-)}(x)|0\rangle \tag{11}$$

where the tilde denotes the static limit of the state.

Matrix elements of this kind have been considered already in [3, 4], where the short-distance corrections have been calculated. It has been observed that for the naïve definition of the states, the anomalous dimension of the current $\bar{h}_v^{(+)}(x)\Gamma h_w^{(-)}(x)$ acquires an imaginary part of the general structure

$$\text{Im } \gamma = f(\alpha_s)\frac{1}{\sqrt{(vw)^2 - 1}} \tag{12}$$

as $v \to w$, where the function f is known as power series in α_s up to two loops [4]

$$f(\alpha_s) = \frac{4}{3}\alpha_s \left(1 + \frac{\alpha_s}{4\pi}\left[\frac{31}{3} - \frac{10}{9}n_f\right] + \cdots\right) . \tag{13}$$

The real part of the anomalous dimension vanishes in the limit $v \to w$ to all orders, since the current $\bar{h}_v^{(+)}\gamma_\mu h_v^{(+)}$ is a generator of heavy-flavour symmetry. On the other hand, the anomalous dimensions of $\bar{h}_v^{(+)}\gamma_\mu h_w^{(+)}$ and $\bar{h}_v^{(+)}\gamma_\mu h_w^{(-)}$ are related by analytic continuation and hence the real part is identical.

Thus the only problem is in fact the imaginary part given in (12), which diverges in the limit $v \to w$, and this is the major obstacle in taking the naïve limit $v \to w$ in (8). Keeping only this imaginary part, the solution of the renormalization group equation yields a phase factor

$$\exp(i\phi(vw)) = \exp\left\{i\frac{vw}{\sqrt{(vw)^2 - 1}}\int\limits_{\alpha_s(m)}^{\alpha_s(\mu)}d\alpha\frac{f(\alpha)}{\beta(\alpha)}\right\} \quad ; \quad \beta(\alpha(\mu)) = \mu\frac{\partial}{\partial\mu}\alpha(\mu) \tag{14}$$

for matrix elements such as (11), which is not well defined in the limit $v \to w$.

It has been pointed out in [4] that these phases are related to the Coulombic part of the one-gluon exchange and that the phase may be removed by a suitable definition

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of the multiparticle states of HQET. After the redefinition of the states, the anomalous dimensions are real and well behaved in the limit $v \to w$. Furthermore, the phase in (14) is related to the "long-range" part of the one-gluon exchange potential and indicates the possibility of having bound states in some of the quark-(anti)quark channels.

Consequently, the phase appearing in (14) is an infrared contribution which should be contained in the dynamics of the effective theory. In other words, if we want to describe heavy quarkonia, the phase factor is related to the binding mechanism, which is an infrared effect. In a state such as a quarkonium, the two velocities differ only by a small amount of order $1/m$ which is a hint that we need to go beyond the static limit to describe quarkonia states.

In order to see how higher-order terms in the Lagrangian cure the problem, we use the reparametrization invariance (7). The expressions

$$\mathcal{V} = v + \frac{\overleftarrow{iD}}{m} \text{ and } \mathcal{W} = w + \frac{iD}{m}$$

$$\tilde{h}_v^{(+)} = \left(1 + \frac{i\not{D}}{2m}\right) h_v^{(+)} \text{ and } \tilde{h}_w^{(-)} = \left(1 + \frac{i\not{D}}{2m}\right) h_w^{(-)} \tag{15}$$

are invariant under the reparametrizations $v + \delta v$ and $w + \delta w$, where this is true for the second line only to order $1/m$. This observation has been used in [10] to obtain, for heavy-heavy currents the renormalization of the subleading terms from the leading ones. Using reparametrization invariance, we have

$$\exp(i\phi(vw))\bar{h}_v^{(+)}\Gamma h_{-w}^{(-)} \quad \overset{\text{RPI}}{\longrightarrow} \quad \tilde{\bar{h}}_v^{(+)} \exp(i\phi(\mathcal{V}\mathcal{W}))\Gamma \tilde{h}_{-w}^{(-)} \ . \tag{16}$$

If we now consider the limit $v \to w$ we have also $\mathcal{V} \to \mathcal{W}$, which yields in the phase factor the formal expression

$$\exp(i\phi(\mathcal{V}\mathcal{W})) \to \exp(i\phi(\mathcal{V}^2)) = \exp\left\{i\frac{1}{\sqrt{\mathcal{V}^2 - 1}} \int\limits_{\alpha_s(m)}^{\alpha_s(\mu)} d\alpha \frac{f(\alpha)}{\beta\alpha}\right\} \ , \tag{17}$$

where we have only kept the singular term in the last step.

Hence the divergent phase factor may be shifted from the velocities into the residual momenta, if the equation of motion of the heavy-quark field is not the one given by (9), but rather by

$$\frac{m}{2}\left(\mathcal{V}^2 - 1\right) h_v^{(+)} = 0 \quad \text{and} \quad \frac{m}{2}\left(\mathcal{W}^2 - 1\right) h_{-w}^{(-)} = 0 \ . \tag{18}$$

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In other words, the Lagrangian leading to these equations of motion is the combination

$$\mathcal{L}_0 = \frac{m}{2}\bar{h}_v^{(+)}(\mathcal{V}^2 - 1)h_v^{(+)} + \frac{m}{2}\bar{h}_{-w}^{(-)}(\mathcal{W}^2 - 1)h_{-w}^{(-)}$$

$$= \bar{h}_v^{(+)}(ivD)h_v^{(+)} + \bar{h}_v^{(+)}\frac{(i\not{D})^2}{2m}h_v^{(+)} - \bar{h}_w^{(-)}(iwD)h_w^{(-)} + \bar{h}_w^{(-)}\frac{(i\not{D})^2}{2m}h_w^{(-)} \ , \qquad (19)$$

where we have replaced $w \to -w$ in the last step. This expression for the Lagrangian is reparametrization-invariant; it is also the same invariance which ensures that the static and the kinetic terms of the Lagrangian renormalize in the same way. Now one may perform the limit $v \to w$ without encountering a problem in the short-distance contributions as in (12). Of course, the phase (14) has not disappeared: it will show up as a singularity in the residual relative momentum, which will be generated by the infrared dynamics as given in (19). This also means that it is a long-distance effect and may be absorbed into the states as discussed in [4].

However, if it is (19) that determines the leading term of our expansion, we will not be able to perform a strict infinite-mass limit, since now $\mathcal{L}_0$ depends explicitly on the heavy mass. In the case of heavy quarkonium, non-perturbative effects will generate binding of the two heavy objects that will introduce a small scale $\tilde{\Lambda}$, which now in general depends on the heavy mass.

The bottomonium and the charmonium are far from being Coulombic systems and there is no obvious reason why in the heavy-mass limit a heavy quarkonium should become Coulombic; still the case of a Coulombic system is instructive. Neglecting any running of α, the size of such a Coulombic system is $R_{Bohr} = 1/(\alpha m)$, which is large compared to the Compton wavelength $\lambda_Q = 1/m$ and hence disparate scales appear allowing for an effective field theory treatment. However, the small scale $1/R_{Bohr}$ depends on the mass such that it does not approach a finite limit as $m \to \infty$, even for running α_s.

For the heavy quarkonia we shall not assume any type of potential or binding model, but rather leave the matrix elements of the operators as non-perturbative entities, which may not be determined within the effective theory. The only thing that has to be kept in mind is that these matrix elements depend on the mass, as can be seen from the Coulombic example. The Coulombic system is, however, an extreme case, since all scales are set there by the mass of the constituents; we shall argue below that the mass dependence might, in reality, be much weaker.

Based on the new equations of motion we get for tree-level matrix elements the relations

$$(ivD)h_v^{(+)} = \frac{(iD)^2}{2m}h_v^{(+)} = \mathcal{O}(1/m) \quad , \quad (ivD)h_v^{(-)} = -\frac{(iD)^2}{2m}h_v^{(-)} = \mathcal{O}(1/m) \ . \qquad (20)$$

As we shall see, this means that the static equations of motion hold up to terms one order higher in the heavy-mass expansion. This is very similar to what happens in the v/c expansion of [5], where the right-hand side of (20) is suppressed by an additional power in v/c.

The magnetic-moment term of order $1/m$ appearing in the Lagrangian and all higher-order terms are treated as perturbations and lead to time-ordered products; these terms may be identified with the corrections to the states. The expansion of the fields $Q_v^{(\pm)}$ leads to corrections, which are in general local operators. However, as in any effective theory we are free to perform field redefinitions [11], and hence the expansions (5) and (6) are not uniquely defined; one may always move terms appearing in the expansion of the field into the Lagrangian; these contributions appear in the Lagrangian as terms that would vanish by a naïve use of the equations of motion (9). However, inserted into time-ordered products they yield local terms according to

$$\langle\psi|T\{\bar{h}^{(+)}\Gamma(ivD)h^{(+)}\bar{h}^{(+)}\Gamma'h^{(+)}\}|\psi\rangle = i\delta^4(x)\langle\psi|\bar{h}^{(+)}\Gamma P_+\Gamma'h^{(+)}|\psi\rangle + \mathcal{O}(1/m) \ , \qquad (21)$$

where the higher-order terms appear, since the equation of motion is not quite the static one. However, this ambiguity of shifting contributions between the Lagrangian and the fields appears only in terms of order $1/m^2$ or higher and thus does not affect our arguments concerning the heavy-mass limit based on reparametrization invariance.

Hence only the combination of (5) and (6) has physical significance and for the present application it is convenient to use expansions somewhat different from (5) and (6). By a linear field redefinition we arrive at equivalent expansions, which are up to $1/m^2$:

$$\mathcal{L} = \mathcal{L}_{\text{static}} + \mathcal{L}_I \qquad (22)$$

where

$$\mathcal{L}_{\text{static}} = \bar{h}^{(+)} ivD\, h^{(-)} - \bar{h}^{(-)} ivD\, h^{(-)}$$

$$\mathcal{L}_I = \left(\frac{1}{2m}\right) L_1 + \left(\frac{1}{2m}\right)^2 L_2$$

$$= \left(\frac{1}{2m}\right)(K_1 + G_1) + \left(\frac{1}{2m}\right)^2 (K_2 + G_2) + \mathcal{O}(1/m^3) \ . \qquad (23)$$

Here we have defined

$$\begin{aligned}
K_1 &= K_1^{(+)} + K_1^{(-)} & K_1^{(\pm)} &= \bar{h}^{(\pm)}[(iD)^2 - (ivD)^2]h^{(\pm)} \\
G_1 &= G_1^{(+)} + G_1^{(-)} & G_1^{(\pm)} &= (-i)\bar{h}^{(\pm)}\sigma_{\mu\nu}(iD^\mu)(iD^\nu)h^{(\pm)} \\
K_2 &= K_2^{(+)} + K_2^{(-)} & K_2^{(\pm)} &= \bar{h}^{(\pm)}[(iD_\mu),[(-ivD),(iD^\mu)]]h^{(\pm)} \\
G_2 &= G_2^{(+)} + G_2^{(-)} & G_2^{(\pm)} &= (-i)\bar{h}^{(\pm)}\sigma_{\mu\nu}\{(iD^\mu),[(-ivD),(iD^\nu)]\}h^{(\pm)} \ .
\end{aligned} \qquad (24)$$

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The corresponding expansion of the field $\bar{Q}_v^{(+)}$ reads

$$Q_v^{(+)}(x) = \left(1 + \frac{1}{2m}P_- i\!\!\not{D} - \frac{1}{8m^2}(ivD)P_- i\!\!\not{D}\right.$$
$$\left. - \frac{1}{8m^2}\left((iD)^2 - (ivD)^2 - i\sigma_{\mu\nu}iD^\mu iD^\nu\right) + \mathcal{O}(1/m^3)\right) h^{(+)}(x) . \quad (25)$$

In fact this is the form that has been obtained from QCD by a sequence of Foldy-Wouthuysen transformations [12].

The advantage of this form is that now the Lagrangian no longer contains terms that would vanish by a naïve use of the equation of motion; all these terms have been shifted into the expansion of the field $\bar{Q}_v^{(+)}$. Again the corresponding expressions for the antiparticle fields are obtained by the replacement $v \to -v$.

In this way one may obtain all terms which explicitly contain a heavy-quark field. However, there are more contributions appearing in order $1/m^2$ which are due to closed loops of heavy quarks in the full theory. These contributions may be expressed as local higher-dimensional operators involving only gluon fields. To order $1/m^2$ there are only two independent operators, leading at tree level to a contribution $\mathcal{L}_{glue}$ to the Lagrangian ($g_s^2 = 4\pi\alpha_s$):

$$\left(\frac{1}{2m}\right)^2 \mathcal{L}_{glue} = \frac{\alpha_s}{30\pi m^2}\,\mathrm{Tr}\{[iD_\mu, G^{\mu\nu}][iD^\lambda, G_{\lambda\nu}]\} + \frac{i\alpha_s g_s}{360\pi m^2}\,\mathrm{Tr}\{G^{\mu\nu}[G_{\nu\rho}, G^\rho{}_\mu]\} \quad (26)$$

where the gluon field strength is defined as

$$[iD_\mu, iD_\nu] = ig_s G_{\mu\nu} . \quad (27)$$

We may use the equations of motion of the gluon field to rewrite two of the $1/m^2$ operators as

$$\bar{h}^{(+)}[(iD_\mu), [(-ivD), (iD^\mu)]]h^{(+)} = 4\pi\alpha_s \bar{h}^{(+)}\gamma_\mu T^a h^{(+)} \sum_q \bar{q}\gamma^\mu T^a q$$
$$\mathrm{Tr}\{[iD_\mu, G^{\mu\nu}][iD^\lambda, G_{\lambda\nu}]\} = -2\sum_{qq'}(\bar{q}\gamma^\mu T^a q)(\bar{q}'\gamma_\mu T^a q') . \quad (28)$$

In this form they have a simple interpretation: The first one is the interaction of the heavy quark with the (virtual) light quarks in the quarkonium, and the second one is the interaction among these (virtual) light quarks, which is introduced by heavy-quark loops.

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In order to calculate short-distance QCD corrections within this effective theory, one has to start from the Feynman rules as derived from $\mathcal{L}_0$. The propagator $H(k)$ obtained in this way is

$$H(k) = P_+ \frac{i}{vk + \frac{1}{2m}k^2 + i\epsilon} \tag{29}$$

and hence contains all orders of $1/m$. However, the reason why we had to include these higher-order terms was that this removes the divergent phase occurring in the limit of small relative velocity. This phase is a long-distance effect and may be absorbed into the states, which thus have to evolve according to the dynamics dictated by $\mathcal{L}_0$ given in (19). Once the phase is removed, we may expand the remaining expression for the short-distance contribution in powers of $1/m^n$. Hence, as far as practical calculations are concerned, we may simply use the static propagator

$$H_{stat}(k) = P_+ \frac{i}{vk + i\epsilon} \tag{30}$$

as in usual HQET, which is the leading term of (29). If we chose the velocities of the heavy quarks to be equal, then ill-defined imaginary parts such as

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \delta(vk)\delta(v'k) \to \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} (\delta(vk))^2 \text{ as } v \to v' \tag{31}$$

will show up at the one-loop level, which are contributions to the divergent phase. However, these are absorbed into the states and the real parts may be calculated simply in the static limit, even for equal velocities of the heavy quarks.

In this way we may exploit the full machinery of the static limit, i.e. of HQET. However, although the short-distance contributions are calculable in the static limit, the matrix elements of operators composed of static fields will not be flavour-independent in the case of a quarkonium, since the states contain a non-trivial mass dependence.

This concludes the set-up for the heavy-mass expansion for heavy quarkonia. In the following section we shall apply these ideas to inclusive annihilation decays of heavy quarkonia.

3   Separation Between Long and Short Distances

In this section we shall discuss the general set-up for a QCD-based calculation of the inclusive annihilation of a heavy-quarkonium state. The aim is to establish a separation

between long- and short-distance physics; the latter may be calculated in perturbation theory, while the long-distance part is non-perturbative and is parametrized in terms of hadronic matrix elements.

The starting point is the inclusive transition rate Γ of a quarkonium state $|\psi\rangle$, which is given by the optical theorem in terms of the forward matrix element of the transition operator T

$$\Gamma = 2 \text{ Im } \langle\psi|T|\psi\rangle \ , \tag{32}$$

which is itself related to the discontinuities across the cuts of the two-point function of two fields Ψ interpolating the state $|\psi\rangle$

$$G(p) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \langle 0|T\{\Psi(x)\Psi(0)\}|0\rangle \ . \tag{33}$$

For the case at hand, (33) has a cut along the real axis of the complex p^2 plane starting at m_{light}^2, where the mass m_{light} corresponds to the lightest hadronic state the heavy quarkonium may decay into.

On the other hand, the transition rate Γ is related to the discontinuity across the cut at the mass of the heavy quarkonium state, which is much larger than m_{light}. If we consider only decays into light hadrons we are thus far away from the resonance region, for which p^2 is of order m_{light}^2. Using local duality we may calculate the annihilation of the two heavy quarks into light hadrons in perturbation theory, since the scale for this process is set by the heavy-quark mass. In other words, the short-distance piece of the transition rate Γ has an expansion in inverse powers of the heavy-quark mass, up to logarithms induced by renormalization, and the coefficients of this expansion may be calculated in perturbation theory.

The annihilation part of the transition operator T in terms of the heavy-quark fields is in general given by

$$T = \int d^4X \, d^4\rho \, d^4\xi \sum_{C=1,8} \sum_{j,k} K_{ij}^{(C)}(X,\rho,\xi) \, \bar{Q}(X+\rho)\Gamma_j CQ(X-\rho)\bar{Q}(-\xi)\bar{\Gamma}_k CQ(\xi) \ , \tag{34}$$

where C is a matrix in colour space; $C = 1$ ($C = 8$) corresponds to the colour combination $C \otimes C = 1 \otimes 1$ ($C \otimes C = T^a \otimes T^a$) and the sum over j and k runs over the sixteen Dirac matrices. The kernel K_{jk} depends on the c.m.s.-coordinate X of the heavy-quark pair as well as on the two relative coordinates ξ and ρ.

To set up an expansion in inverse powers of the heavy-quark mass we perform a phase redefinition of the heavy-quark fields

$$Q(x) = \exp(-im(vx))Q_v^{(+)}(x), \quad \bar{Q}(x) = \exp(-im(vx))\bar{Q}_v^{(-)}(x) \tag{35}$$

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where the superscript $(+)$ $((-))$ refers to the annihilation part of the fields for quarks (antiquarks). Here v is a velocity vector, which we chose to be the same for both the quark and the antiquark. Hence we shall work in the limit we have discussed in the last section and finally identify the velocity with the velocity of the heavy quarkonium.

This phase redefinition removes the dominant piece of the space-time dependence of the heavy-quark field operator; the remaining dependence is only due to the residual momentum of the heavy quark in the heavy hadron.

Inserting these redefined fields the transition operator takes the form

$$T = \int d^4X \, d^4\rho \, d^4\xi \sum_{C=1,8} \sum_{j,k} K_{ij}^{(C)}(X, \rho, \xi) \exp(i2mXv)$$
$$\bar{Q}_v^{(+)}(X+\rho)\Gamma_j C Q_v^{(-)}(X-\rho)\bar{Q}_v^{(-)}(-\xi)\bar{\Gamma}_k C Q_v^{(+)}(\xi) \; . \tag{36}$$

Introducing the Fourier-transform of the kernel as

$$K_{ij}^{(C)}(X, \rho, \xi) = \int \frac{d^4\tilde{P}}{(2\pi)^4} \frac{d^4\pi}{(2\pi)^4} \frac{d^4\eta}{(2\pi)^4} e^{-i\tilde{P}X} e^{-i\pi\rho} e^{i\eta\xi} \widetilde{\mathcal{K}}_{jk}^{(C)}(\tilde{P}, \pi, \eta) \; , \tag{37}$$

we obtain

$$T = \int d^4X \, d^4\rho \, d^4\xi \sum_{C=1,8} \sum_{j,k} \int \frac{d^4\tilde{P}}{(2\pi)^4} \frac{d^4\pi}{(2\pi)^4} \frac{d^4\eta}{(2\pi)^4} \widetilde{\mathcal{K}}_{jk}^{(C)}(\tilde{P}, \pi, \eta) \, e^{-iX(\tilde{P}-2mv)} e^{-i\pi\rho} e^{i\eta\xi}$$
$$\bar{Q}_v^{(+)}(X+\rho)\Gamma_j C Q_v^{(-)}(X-\rho)\bar{Q}_v^{(-)}(-\xi)\bar{\Gamma}_k C Q_v^{(+)}(\xi) \; . \tag{38}$$

The dominant momentum dependence is $2mv$ and we can remove this large piece by redefining the c.m.s. momentum; hence we introduce the residual c.m.s. momentum

$$P = \tilde{P} - 2mv \tag{39}$$

and obtain

$$T = \int d^4X \, d^4\rho \, d^4\xi \sum_{C=1,8} \sum_{j,k} \int \frac{d^4P}{(2\pi)^4} \frac{d^4\pi}{(2\pi)^4} \frac{d^4\eta}{(2\pi)^4} \mathcal{K}_{jk}^{(C)}(m; P, \pi, \eta) \, e^{-iXP} e^{-i\pi\rho} e^{i\eta\xi}$$
$$\bar{Q}_v^{(+)}(X+\rho)\Gamma_j C Q_v^{(-)}(X-\rho)\bar{Q}_v^{(-)}(-\xi)\bar{\Gamma}_k C Q_v^{(+)}(\xi) \; , \tag{40}$$

where the kernel $\mathcal{K}_{jk}^{(C)}$ now depends only on small, i.e. residual, momenta and on the heavy mass m.

Furthermore, the kernel has an expansion in inverse powers of the heavy-quark mass m, up to logarithms of the form $\ln(m/\mu)$, where μ is a factorization scale[2]. The transition rate Γ is independent of the factorization scale, and thus the scale dependence of the expansion of the kernel has to be compensated by the corresponding dependence of the matrix elements, which appear in the expansion. As usual, this scale dependence is governed by the renormalization group, and hence one may recover the logarithmic dependence of the transition rate on the heavy-quark mass m.

The expansion of the kernel thus takes the general form $(\lambda = \mu/m)$

$$
\begin{aligned}
\mathcal{K}_{jk}^{(C)}(m; P, \pi, \eta) = {} & R_{jk}^{(C)}(\lambda) + \frac{1}{2m}\left(P_\mu S_{ij}^{(C)\mu}(\lambda) + \pi_\mu T_{ij}^{(C)\mu}(\lambda) + \eta_\mu U_{ij}^{(C)\mu}(\lambda)\right) \\
& + \left(\frac{1}{2m}\right)^2 \left(P_\mu P_\nu S_{ij}^{(C)(1)\mu\nu}(\lambda) + P_\mu \pi_\nu S_{ij}^{(C)(2)\mu\nu}(\lambda) + P_\mu \eta_\nu S_{ij}^{(C)(3)\mu\nu}(\lambda)\right. \\
& \qquad\qquad + \pi_\mu P_\nu T_{ij}^{(C)(1)\mu\nu}(\lambda) + \pi_\mu \pi_\nu T_{ij}^{(C)(2)\mu\nu}(\lambda) + \pi_\mu \eta_\nu T_{ij}^{(C)(3)\mu\nu}(\lambda) \\
& \qquad\qquad \left. + \eta_\mu P_\nu U_{ij}^{(C)(1)\mu\nu}(\lambda) + \eta_\mu \pi_\nu U_{ij}^{(C)(2)\mu\nu}(\lambda) + \eta_\mu \eta_\nu U_{ij}^{(C)(3)\mu\nu}(\lambda)\right) \\
& + \mathcal{O}(1/m^3) \, .
\end{aligned}
\tag{41}
$$

The powers of the momenta will simply yield matrix elements of operators involving derivatives. For later use it is convenient to introduce those that correspond to the residual-c.m.s.-momentum (RCM)

$$
i\partial_\mu\left(\bar{Q}_v^{(+)}\Gamma Q_v^{(-)}\right) = \left(\bar{Q}_v^{(+)}\Gamma(iD_\mu)Q_v^{(-)}\right) + \left(\bar{Q}_v^{(+)}(i\overleftarrow{D}_\mu)\Gamma Q_v^{(-)}\right)
\tag{42}
$$

and to the residual-relative-momentum (RRM)

$$
\left(\bar{Q}_v^{(+)}(i\overrightarrow{D}_\mu)\Gamma Q_v^{(-)}\right) = \left(\bar{Q}_v^{(+)}\Gamma(iD_\mu)Q_v^{(-)}\right) - \left(\bar{Q}_v^{(+)}(i\overleftarrow{D}_\mu)\Gamma Q_v^{(-)}\right)
\tag{43}
$$

where $D = \partial - ig\mathcal{A}$ is the QCD covariant derivative for fields in the fundamental representation. For the other bilinear $\bar{Q}_v^{(-)} \cdots Q_v^{(+)}$ we define the derivatives in the same way.

---

[2]We restrict our attention here to the purely perturbative aspects of the operator expansion and ignore possible problems induced by renormalon poles. These have been discussed recently in the context of the heavy mass expansion in [13].

Inserting (41) into the expression for the T operator and using the definitions (42) and (43) one finds for the transition rate Γ up to and including order $1/m^2$:

$$
\begin{aligned}
\langle\psi|T|\psi\rangle = \sum_{C=1,8}\sum_{j,k}\Big\{ &R_{jk}^{(C)}(\mu/m)\langle\psi|[\bar{Q}_v^{(+)}\Gamma_j C Q_v^{(-)}][\bar{Q}_v^{(-)}\bar{\Gamma}_k C Q_v^{(+)}]|\psi\rangle_\mu \\
+ \frac{1}{2m}\Big[&S_{ij}^{(C)\mu}(\lambda)\langle\psi|[i\partial_\mu(\bar{Q}_v^{(+)}\Gamma_j C Q_v^{(-)})][\bar{Q}_v^{(-)}\bar{\Gamma}_k C Q_v^{(+)}]|\psi\rangle_\mu \\
+ &T_{ij}^{(C)\mu}(\lambda)\langle\psi|[(\bar{Q}_v^{(+)}\Gamma_j(i\overleftrightarrow{D}_\mu)C Q_v^{(-)})][\bar{Q}_v^{(-)}\bar{\Gamma}_k C Q_v^{(+)}]|\psi\rangle_\mu \\
+ &U_{ij}^{(C)\mu}(\lambda)\langle\psi|[(\bar{Q}_v^{(+)}\Gamma_j C Q_v^{(-)})][\bar{Q}_v^{(-)}\bar{\Gamma}_k C(i\overleftrightarrow{D}_\mu)Q_v^{(+)}]|\psi\rangle_\mu\Big] \\
+ \left(\frac{1}{2m}\right)^2\Big[&S_{ij}^{(C)(1)\mu\nu}(\lambda)\langle\psi|[i\partial_\mu(\bar{Q}_v^{(+)}\Gamma_j C Q_v^{(-)})][i\partial_\mu(\bar{Q}_v^{(-)}\bar{\Gamma}_k C Q_v^{(+)})]|\psi\rangle_\mu \\
+ &S_{ij}^{(C)(2)\mu\nu}(\lambda)\langle\psi|[i\partial_\mu(\bar{Q}_v^{(+)}\Gamma_j(i\overleftrightarrow{D}_\nu)C Q_v^{(-)})][\bar{Q}_v^{(-)}\bar{\Gamma}_k C Q_v^{(+)}]|\psi\rangle_\mu \\
+ &S_{ij}^{(C)(3)\mu\nu}(\lambda)\langle\psi|[i\partial_\mu(\bar{Q}_v^{(+)}\Gamma_j C Q_v^{(-)})][\bar{Q}_v^{(-)}\bar{\Gamma}_k C(i\overleftrightarrow{D}_\nu)Q_v^{(+)}]|\psi\rangle_\mu \\
+ &T_{ij}^{(C)(1)\mu\nu}(\lambda)\langle\psi|[\bar{Q}_v^{(+)}\Gamma_j(i\overleftrightarrow{D}_\mu)C Q_v^{(-)})][i\partial_\nu(\bar{Q}_v^{(-)}\bar{\Gamma}_k C Q_v^{(+)})]|\psi\rangle_\mu \\
+ &T_{ij}^{(C)(2)\mu\nu}(\lambda)\langle\psi|[\bar{Q}_v^{(+)}\Gamma_j(i\overleftrightarrow{D}_\mu)(i\overleftrightarrow{D}_\nu)C Q_v^{(-)}][\bar{Q}_v^{(-)}\bar{\Gamma}_k C Q_v^{(+)}]|\psi\rangle_\mu \\
+ &T_{ij}^{(C)(3)\mu\nu}(\lambda)\langle\psi|[\bar{Q}_v^{(+)}\Gamma_j(i\overleftrightarrow{D}_\mu)C Q_v^{(-)}][\bar{Q}_v^{(-)}\bar{\Gamma}_k C(i\overleftrightarrow{D}_\nu)Q_v^{(+)}]|\psi\rangle_\mu \\
+ &U_{ij}^{(C)(1)\mu\nu}(\lambda)\langle\psi|[i\partial_\nu(\bar{Q}_v^{(+)}\Gamma_j C Q_v^{(-)})][\bar{Q}_v^{(-)}\bar{\Gamma}_k C(i\overleftrightarrow{D}_\mu)Q_v^{(+)}]|\psi\rangle_\mu \\
+ &U_{ij}^{(C)(2)\mu\nu}(\lambda)\langle\psi|[\bar{Q}_v^{(+)}\Gamma_j(i\overleftrightarrow{D}_\nu)C Q_v^{(-)}][\bar{Q}_v^{(-)}\bar{\Gamma}_k C(i\overleftrightarrow{D}_\mu)Q_v^{(+)}]|\psi\rangle_\mu \\
+ &U_{ij}^{(C)(3)\mu\nu}(\lambda)\langle\psi|[\bar{Q}_v^{(+)}\Gamma_j C Q_v^{(-)}][\bar{Q}_v^{(-)}(i\overleftrightarrow{D}_\mu)(i\overleftrightarrow{D}_\nu)Q_v^{(+)}]|\psi\rangle_\mu\Big] \\
+ &\mathcal{O}(1/m^3)\Big\}
\end{aligned}
$$

$$(44)$$

where the subscript μ at the matrix elements indicates their renormalization point and all field operators have to be taken at $x = 0$. In what follows we shall not display the dependence on x any more, if $x = 0$. Furthermore, we have replaced the ordinary derivatives appearing in the Taylor expansion of the matrix elements by covariant ones in order to ensure gauge invariance; this could be implemented from the very beginning by defining the T operator including appropriate Wilson-line operators.

The lengthy expression (44) is the most general short-distance expansion for the matrix elements of the T operator for the case of heavy quarkonia decay up to $1/m^2$, and this is the first ingredient for a $1/m$ expansion of the transition rate Γ. In a second step, one has to expand the matrix elements appearing in (44) in powers of $1/m$.

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4 $1/m$ Expansion for Inclusive Quarkonia Annihilation

In this section we write down the complete heavy-mass expansion for the matrix elements appearing in (44). The result we are aiming at is an expansion of the transition rate Γ of the form

$$\Gamma = \Gamma_0 + \left(\frac{1}{2m}\right)\Gamma_1 + \left(\frac{1}{2m}\right)^2 \Gamma_2 + \cdots \tag{45}$$

where we shall explicitly construct the terms up to and including $1/m^2$. In this expansion, the ratio of two successive Γ_j is set by a small scale

$$\frac{\Gamma_{j+1}}{\Gamma_j} = \widetilde{\Lambda} \tag{46}$$

where $\widetilde{\Lambda}$ is small compared to the heavy-quark mass. However, there is some difference between the heavy quarkonia and the heavy-light systems. In the latter the parameter corresponding to $\widetilde{\Lambda}$ is $\bar{\Lambda} = M_{hadron} - m$, which becomes independent of the heavy quark mass in the heavy-mass limit.

As we have discussed above, in heavy quarkonia systems this is not true, since in order to perform the limit $v \to w$ we had to include subleading terms of the $1/m$ expansion into $\mathcal{L}_0$, which determines the dynamics of the states. In this way a non-trivial dependence on m is introduced into the scale $\widetilde{\Lambda}$ induced by the binding of the two heavy quarks. Nevertheless, this is a small scale and the expansion will be useful.

Since this means that we are not able to perform a static limit for a heavy quarkonium, we shall no longer have heavy-flavour symmetry. However, if we define the scale $\widetilde{\Lambda}$ in a similar way as $\bar{\Lambda}$ for heavy light systems

$$\widetilde{\Lambda} = \frac{iv\partial\langle 0|\bar{h}_v^{(-)}\Gamma h_v^{(+)}|\psi(v)\rangle}{\langle 0|\bar{h}_v^{(-)}\Gamma h_v^{(+)}|\psi(v)\rangle} \tag{47}$$

we may get some idea of the flavour dependence by looking at the level spacings of the (n^3S_1) quarkonia of the c and the b quarks. Here it is a well-known fact that the level spacings are almost the same in the $(\bar{c}c)$ and $(\bar{b}b)$ system, which we take as a hint that $\widetilde{\Lambda}$ is only weakly dependent on the heavy flavour. We shall return to this point when we discuss heavy-quark symmetries in section 7.

The way we shall set up the expansion (45) is to use the expansions (23) and (25). The fields $Q_v^{(\pm)}$ are expanded as in (25) while the chromomagnetic term G_1 and all terms

of order $1/m^2$ of the Lagrangian are treated as perturbations, which lead to time-ordered products. Up to and including $1/m^2$ only the double insertion of G_1 and the single insertion of the order $1/m^2$ terms of the Lagrangian will play a role.

The leading-order term Γ_0 in the heavy-mass expansion is given by forward-matrix-elements of dimension-six operators involving the static fields $h^{(\pm)}$. In general, there are four dimension-six operators, two with colour structure $1 \otimes 1$

$$
\begin{aligned}
A_1^{(1)} &= \left(\bar{h}^{(+)} \gamma_5 h^{(-)} \right) \left(\bar{h}^{(-)} \gamma_5 h^{(+)} \right) \\
A_2^{(1)} &= \left(\bar{h}^{(+)} \gamma_\mu h^{(-)} \right) \left(\bar{h}^{(-)} \gamma^\mu h^{(+)} \right)
\end{aligned}
\tag{48}
$$

and the corresponding two operators $A_1^{(8)}$, $A_2^{(8)}$ with colour structure $T^a \otimes T^a$. In fact, due to the projections $P_\pm = (1 \pm \slashed{v})/2$ implicit in the static operators these are the only spinor structures that can emerge (see appendix for details).

Thus the leading term Γ_0 is always given as a linear combination of the four dimension-six operators $A_i^{(C)}$ (the factor $1/2$ arises from (32))

$$
\frac{1}{2} \Gamma_0 = \sum_{i=1}^{2} \sum_{C=1,8} \mathcal{C}(A_i^{(C)}) \langle \psi | A_i^{(C)} | \psi \rangle \ ,
\tag{49}
$$

where the coefficients $\mathcal{C}(A_i^{(C)})$ depend on the specific process, i.e. the kernel K_{ij} appearing in (44). We shall give a few examples of this "matching procedure" below.

In higher orders of the heavy-mass expansion matrix elements of higher-dimensional operators appear from the expansion of the $Q_v^{(\pm)}$, which have increasing powers of covariant derivatives acting on the static fields and it is convenient to use the RCM and RRM derivatives introduced in (42) and (43).

All dimension-seven operators either vanish due to symmetries or they are proportional to (ivD). The latter may be rewritten using the equations of motion; in the static case their tree-level matrix element vanishes, while one obtains a dimension-eight contribution, if we include the higher-order terms of the equation of motion (9). Whatever choice one prefers, there are no tree-level matrix elements of local dimension-seven operators. The only contribution which is allowed at order $1/m$ is a single insertion of the first-order chromomagnetic-moment operator

$$
\langle \psi | T \left\{ [G_1^{(+)}(x) + G_1^{(-)}(x) [A_i^{(C)} \right\} | \psi \rangle
\tag{50}
$$

for which we shall show below that it vanishes due to spin symmetry. Thus for the inclusive heavy quarkonia decays there is no contribution of order $\widetilde{\Lambda}/m$.

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The second-order term in the $1/m$ expansion has contributions from both the Lagrangian and the local operators. The local dimension-eight operators may be classified according to the derivatives defined above. The first set of operators consists of the contributions with two RCM derivatives (RCM × RCM). There are in general six independent contributions of this type, three of which are colour $1 \otimes 1$ and the other four are the corresponding $T^a \otimes T^a$ operators. The colour $1 \otimes 1$ operators are

$$
\begin{aligned}
B_1^{(1)} &= \left[i\partial_\mu \left(\bar{h}^{(+)}\gamma_5 h^{(-)} \right) \right] \left[i\partial^\mu \left(\bar{h}^{(-)}\gamma_5 h^{(+)} \right) \right] \\
B_2^{(1)} &= \left[i\partial_\mu \left(\bar{h}^{(+)}\gamma^\mu h^{(-)} \right) \right] \left[i\partial^\nu \left(\bar{h}^{(-)}\gamma_\nu h^{(+)} \right) \right] \\
B_3^{(1)} &= \left[i\partial_\mu \left(\bar{h}^{(+)}\gamma^\nu h^{(-)} \right) \right] \left[i\partial^\mu \left(\bar{h}^{(-)}\gamma_\nu h^{(+)} \right) \right] \ .
\end{aligned}
\tag{51}
$$

Note that one may flip the RCM derivatives from one heavy-quark bilinear to the other, thereby picking up a total derivative, which will not contribute to the forward matrix elements we shall consider. This is the reason why no terms appear in which both derivatives act on the same heavy-quark bilinear; these may be rewritten into the B operators.

In the same way one obtains six colour $1 \otimes 1$ RCM × RRM operators

$$
\begin{aligned}
C_1^{(1)} &= \left\{ \left(\bar{h}^{(+)}\gamma_5 (i\overleftrightarrow{D}_\mu) h^{(-)} \right) \left[i\partial^\mu \left(\bar{h}^{(-)}\gamma_5 h^{(+)} \right) \right] + \text{h.c.} \right\} \\
C_2^{(1)} &= \left\{ \left(\bar{h}^{(+)}(i\overleftrightarrow{D\!\!\!/}) h^{(-)} \right) \left[i\partial^\mu \left(\bar{h}^{(-)}\gamma_\mu h^{(+)} \right) \right] + \text{h.c.} \right\} \\
C_3^{(1)} &= \left\{ \left(\bar{h}^{(+)}\gamma_\nu (i\overleftrightarrow{D}_\mu) h^{(-)} \right) \left[i\partial^\mu \left(\bar{h}^{(-)}\gamma_\nu h^{(+)} \right) \right] + \text{h.c.} \right\} \\
C_4^{(1)} &= \left\{ \left(\bar{h}^{(+)}\gamma_\nu (i\overleftrightarrow{D}_\mu) h^{(-)} \right) \left[i\partial^\nu \left(\bar{h}^{(-)}\gamma_\mu h^{(+)} \right) \right] + \text{h.c.} \right\} \\
C_5^{(1)} &= \left\{ i\varepsilon^{\alpha\beta\kappa\mu} v_\alpha \left(\bar{h}^{(+)}\gamma_\beta (i\overrightarrow{D}_\kappa) h^{(-)} \right) \left[i\partial_\mu \left(\bar{h}^{(-)}\gamma_5 h^{(+)} \right) \right] + \text{h.c.} \right\} \\
C_6^{(1)} &= \left\{ i\varepsilon^{\alpha\beta\kappa\mu} v_\alpha \left(\bar{h}^{(+)}\gamma_5 (i\overrightarrow{D}_\kappa) h^{(-)} \right) \left[i\partial_\mu \left(\bar{h}^{(-)}\gamma_\beta h^{(+)} \right) \right] + \text{h.c.} \right\}
\end{aligned}
\tag{52}
$$

and the corresponding $T^a \otimes T^a$ operators.

Finally, there are 22 RRM × RRM operators, half of which are $1 \otimes 1$ and the other half are the corresponding colour $T^a \otimes T^a$. These operators fall into two categories, namely (for the colour singlets)

$$
\begin{aligned}
D_1^{(1)} &= \left(\bar{h}^{(+)}\gamma_5 (i\overleftrightarrow{D}_\mu) h^{(-)} \right) \left(\bar{h}^{(-)}\gamma_5 (i\overleftrightarrow{D}^\mu) h^{(+)} \right) \\
D_2^{(1)} &= \left(\bar{h}^{(+)}(i\overleftrightarrow{D\!\!\!/}) h^{(-)} \right) \left(\bar{h}^{(-)}(i\overleftrightarrow{D\!\!\!/}) h^{(+)} \right)
\end{aligned}
$$

$$D_3^{(1)} = \left(\bar{h}^{(+)}\gamma_\mu(i\overrightarrow{D}_\nu)h^{(-)}\right)\left(\bar{h}^{(-)}\gamma^\mu(i\overleftarrow{D}^\nu)h^{(+)}\right)$$

$$D_4^{(1)} = \left(\bar{h}^{(+)}\gamma_\nu(i\overrightarrow{D}_\mu)h^{(-)}\right)\left(\bar{h}^{(-)}\gamma^\mu(i\overleftarrow{D}^\nu)h^{(+)}\right)$$

$$D_5^{(1)} = \left\{i\varepsilon^{\alpha\beta\kappa\mu}v_\alpha\left(\bar{h}^{(+)}\gamma_\beta(i\overrightarrow{D}_\kappa)h^{(-)}\right)\left(\bar{h}^{(-)}\gamma_5(i\overrightarrow{D}_\mu)h^{(+)}\right) + \text{h.c.}\right\} \qquad (53)$$

and

$$E_1^{(1)} = \left\{\left(\bar{h}^{(+)}\gamma_5(i\overrightarrow{D})^2h^{(-)}\right)\left(\bar{h}^{(-)}\gamma_5 h^{(+)}\right) + \text{h.c.}\right\}$$

$$E_2^{(1)} = \left\{\left(\bar{h}^{(+)}(i\overrightarrow{\slashed{D}})(i\overrightarrow{D}_\mu)h^{(-)}\right)\left(\bar{h}^{(-)}\gamma^\mu h^{(+)}\right) + \text{h.c.}\right\}$$

$$E_3^{(1)} = \left\{\left(\bar{h}^{(+)}(i\overrightarrow{D}_\mu)(i\overrightarrow{\slashed{D}})h^{(-)}\right)\left(\bar{h}^{(-)}\gamma^\mu h^{(+)}\right) + \text{h.c.}\right\}$$

$$E_4^{(1)} = \left\{\left(\bar{h}^{(+)}(i\overrightarrow{D})^2\gamma_\mu h^{(-)}\right)\left(\bar{h}^{(-)}\gamma^\mu h^{(+)}\right) + \text{h.c.}\right\}$$

$$E_5^{(1)} = \left\{i\varepsilon^{\alpha\beta\kappa\mu}v_\alpha\left(\bar{h}^{(+)}\gamma_\beta(i\overrightarrow{D}_\kappa)(i\overrightarrow{D}_\mu)h^{(-)}\right)\left(\bar{h}^{(-)}\gamma_5 h^{(+)}\right) + \text{h.c.}\right\}$$

$$E_6^{(1)} = \left\{i\varepsilon^{\alpha\beta\kappa\mu}v_\alpha\left(\bar{h}^{(+)}\gamma_5(i\overrightarrow{D}_\kappa)(i\overrightarrow{D}_\mu)h^{(-)}\right)\left(\bar{h}^{(-)}\gamma_\beta h^{(+)}\right) + \text{h.c.}\right\} . \qquad (54)$$

Thus there are in general 40 dimension-eight operators. Note that we have already dropped the operators which vanish because of the equations of motion for the heavy quark.

The general expression for Γ_2 is thus given by a linear combination of the 40 local dimension-eight operators and the non-local contributions from the time-ordered products with the Lagrangian (recall the factor $1/(4m^2)$ in the definitions (45), (23), and (26) and the factor $1/2$ in (32))

$$\frac{1}{2}\Gamma_2 = \sum_{i=1}^{3}\sum_{C=1,8}\mathcal{C}(B_i^{(C)})\langle\psi|B_i^{(C)}|\psi\rangle + \sum_{i=1}^{6}\sum_{C=1,8}\mathcal{C}(C_i^{(C)})\langle\psi|C_i^{(C)}|\psi\rangle$$

$$+ \sum_{i=1}^{5}\sum_{C=1,8}\mathcal{C}(D_i^{(C)})\langle\psi|D_i^{(C)}|\psi\rangle + \sum_{i=1}^{6}\sum_{C=1,8}\mathcal{C}(E_i^{(C)})\langle\psi|E_i^{(C)}|\psi\rangle$$

$$+ \sum_{i=1}^{2}\sum_{C=1,8}\mathcal{C}(A_i^{(C)})(-i)\int d^4x\,\langle\psi|T\left\{L_2(x)A_i^{(C)}(0)\right\}|\psi\rangle$$

$$+ \frac{(-i)^2}{2}\sum_{i=1}^{2}\sum_{C=1,8}\mathcal{C}(A_i^{(C)})\int d^4x\,d^4y\,\langle\psi|T\left\{G_1(x)G_1(y)A_i^{(C)}(0)\right\}|\psi\rangle$$
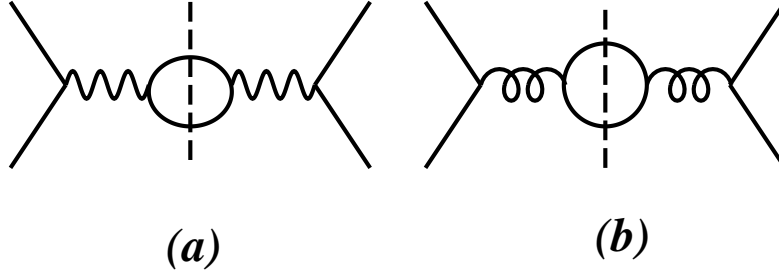
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Figure 1: Fermion loop diagrams. Diagram (a) contributes to the electromagnetic kernel for the quarkonia decays into e^+e^-, diagram (b) contributes to the strong interaction kernel for the quarkonia decays into light hadrons. The dashed line means that the intermediate states are on-shell.

$$+ \sum_{i=1}^{2} \sum_{C=1,8} \mathcal{C}(A_i^{(C)})(-i) \int d^4x \, \langle \psi | T \left\{ \mathcal{L}_{glue}(x) A_i^{(C)}(0) \right\} | \psi \rangle \tag{55}$$

where the coefficients in the time-ordered product terms are again given by the lowest-order coefficients $\mathcal{C}(A_i^{(C)})$. This expression is still quite general and simplifies once the heavy-quark symmetry and the fact that we have to take forward matrix elements is taken into account.

5 The Operator Coefficients to Leading Non-Trivial Order

In this section we shall calculate two simple kernels in order to show how our method is applied, namely the fermion loop diagrams depicted in fig. 1.

These diagrams yield contributions of order α (fig. 1a) and $\alpha_s(m)$ (fig. 1b). Calculating the electromagnetic contribution (fig. 1a) one obtains for the kernel

$$\mathcal{K}_{\mu\nu}(P, \pi, \eta) = -\frac{4\pi}{3P^2} \alpha^2 c_Q^2 g_{\mu\nu} \ , \tag{56}$$

where c_Q is the charge of the heavy quark in units of the electron charge. Furthermore, $P = p + \bar{p}$ is the sum of the momenta of the quark p and the antiquark $\bar{p}$. Hence only the Dirac matrix combination $\gamma_\mu \otimes \gamma^\mu$ contributes in the sum over the Dirac matrices in (44).

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The kernel is expanded using

$$p = v\sqrt{m^2 - p_\perp^2} + p_\perp = v\left(m - \frac{1}{2m}p_\perp^2 + \cdots\right) + p_\perp$$

and the result is matched to the operators by identifying the momenta with the derivatives appearing in the operators (see appendix for more details). The only non-vanishing coefficients are

$$\mathcal{C}^{ee}(A_2^{(1)}, m) = \mathcal{C}^{ee}(B_3^{(1)}, m) = -\frac{\pi}{3m^2}\alpha^2 c_Q^2$$
$$\mathcal{C}^{ee}(E_2^{(1)}, m) = \mathcal{C}^{ee}(E_3^{(1)}, m) = -\frac{\pi}{12m^2}\alpha^2 c_Q^2$$
$$\mathcal{C}^{ee}(E_4^{(1)}, m) = -\frac{\pi}{6m^2}\alpha^2 c_Q^2. \tag{57}$$

From this one may obtain the coefficients for the quark-loop diagram of fig. 1b by the replacements $c_Q^2 \alpha^2 \to n_f \alpha_s^2(m)/2$ in the coefficients and $1 \otimes 1 \to T^a \otimes T^a$ in the operators, where n_f is the number of light flavours which are allowed in the quark loop. In this way we obtain

$$\mathcal{C}^{qq}(A_2^{(8)}, m) = \mathcal{C}^{qq}(B_3^{(8)}, m) = -\frac{\pi}{6m^2}\alpha_s^2(m)n_f$$
$$\mathcal{C}^{qq}(E_2^{(8)}, m) = \mathcal{C}^{qq}(E_3^{(8)}, m) = -\frac{\pi}{24m^2}\alpha_s^2(m)n_f$$
$$\mathcal{C}^{qq}(E_4^{(8)}, m) = -\frac{\pi}{12m^2}\alpha_s^2(m)n_f . \tag{58}$$

The matching calculation yields these coefficients at the scale $\mu = m$ and hence it is α_s taken at this scale that enters the expression. We have indicated the μ dependence of the coefficients by an additional argument for these functions. A change of this scale is governed by the renormalization group of the effective theory, which is discussed in the next section.

6   QCD Evolution of the Coefficients C

In general QCD corrections render the operators and the coefficients scale-dependent in such a way that the transition rate Γ is scale-independent. Schematically this may be written as

$$\Gamma = \sum_i \mathcal{C}_i(\mu)\langle\mathcal{O}_i\rangle|_\mu \tag{59}$$

where $\langle\rangle|_\mu$ means that the matrix element is normalized at scale μ. The scale dependence of the matrix elements and the coefficients $\mathcal{C}_i$ is governed by the anomalous-dimension matrix γ

$$\mu\frac{\partial}{\partial\mu}\langle\mathcal{O}_i\rangle|_\mu = -\gamma_{ij}\langle\mathcal{O}_j\rangle|_\mu , \qquad (60)$$

which is obtained in the standard fashion from the ultraviolet divergences of the matrix elements. In order to have the transition rate scale-independent, the coefficients $\mathcal{C}_i(\mu)$ have to obey the renormalization group equation

$$\mu\frac{d}{d\mu}\mathcal{C}_j = \mathcal{C}_i\,\gamma_{ij} , \qquad (61)$$

where the initial condition at the scale $\mu = m$ is given from the matching calculation, i.e. the calculation of the hard kernels as performed in the last section.

The four dimension-six operators $A_i^{(C)}$ have a vanishing anomalous-dimension matrix and hence there is no mixing between these four operators. Keeping the velocity of the quark operator different from the one for the antiquark we find that the anomalous-dimension matrix – after the redefinition of the states – vanishes as

$$\gamma_{ij}^{(6)} \propto \omega r(\omega) - 1 , \qquad (62)$$

where ω is the product of the two velocities and

$$r(x) = \frac{1}{\sqrt{x^2-1}}\ln(x+\sqrt{x^2-1}) . \qquad (63)$$

Hence the coefficient functions $\mathcal{C}(A_i^{(C)})$ are scale-independent.

In a similar way we find that there is no mixing between the local dimension-eight operators. In order to access the renormalization of the non-local time-ordered product terms, we have to first study the renormalization of the terms in the Lagrangian. The renormalization of the first-order Lagrangian is [14]

$$K_1\bigg|_\mu = C_0 C_1\, K_1\bigg|_m \qquad C_1 = 1$$
$$G_1\bigg|_\mu = C_0 C_2\, G_1\bigg|_m \qquad C_2 = \eta^{-9/(33-2n_f)} \qquad (64)$$

where

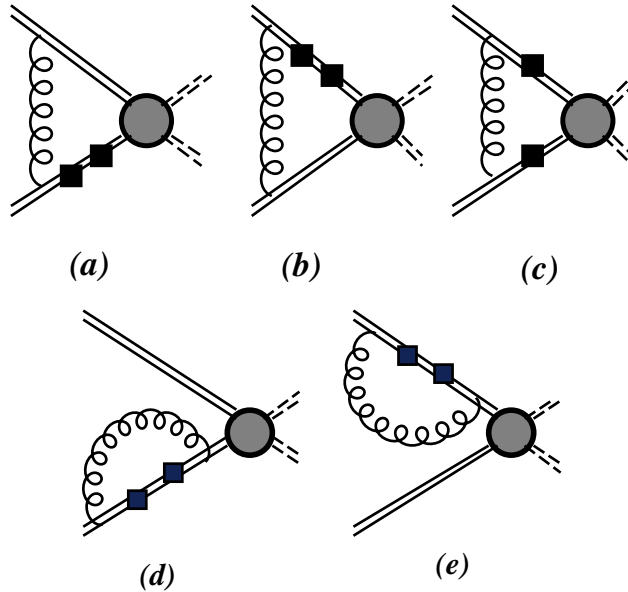$$\eta = \frac{\alpha_s(\mu)}{\alpha_s(m)} \qquad (65)$$

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Figure 2: Feynman diagrams leading to mixing between the non-local and the local operators. The square boxes correspond to insertions of the first-order kinetic energy operator K_1 and the shaded blob corresponds to the dimension-six operators $A_i^{(C)}$. The diagrams (a) to (e) have to be calculated for all permutations of the external lines.

and C_0 is the wave function renormalization constant of the static field, which reads in Feynman gauge

$$C_0 = \eta^{8/(33-2n_f)} \ . \tag{66}$$

The subscripts μ and m label at which scale the matrix elements, in which the operators are inserted, have to be normalized. We note that $C_1 = 1$ ensures that both terms of the Lagrangian (19) renormalize in the same way; in other words, the wave-function renormalization is the same in both the static and the effective theory based on (19).

Of the second-order Lagrangian we need only the contributions $K_2^{(\pm)}$ and $\mathcal{L}_{glue}$, since there will be no contribution from a single insertion of $G_2^{(\pm)}$ due to spin symmetry. Furthermore, we shall not include the renormalization of the second-order Lagrangian, since its contribution is hard to estimate, no matter at which scale we consider the matrix elements.

One important effect of the renormalization in the present application is the mixing of the time-ordered products of the Lagrangian into the local operators, since non-local

operators may require a local counterterm. The only contribution that leads to an ultraviolet divergence is a double insertion of the first-order kinetic energy operator K_1. We define the non-local operators

$$T_i^{(C)} = \frac{(-i)^2}{2} \int d^4x \, d^4y \, T\left\{ K_1(x) K_1(y) A_i^{(C)}(0) \right\} \,, \tag{67}$$

and the one-loop diagram types leading to ultraviolet-divergent contributions are depicted in fig. 2. Diagrams (a) to (c) are one-particle irreducible pieces, while (d) and (e) may be interpreted as the mixing of the time-ordered product $T\{K_1(x)K_1(0)\}$ into local terms of the Lagrangian.

We define two sets of seven operators, the first corresponds to the spin singlet, the second to the spin triplet coupling

$$(\mathcal{O}_i^{[1]}) = (T_1^{(1)}, T_1^{(8)}, D_1^{(1)}, E_1^{(1)}, B_1^{(8)}, D_1^{(8)}, E_1^{(8)})$$
$$(\mathcal{O}_i^{[3]}) = (T_2^{(1)}, T_2^{(8)}, D_3^{(1)}, E_4^{(1)}, B_3^{(8)}, D_3^{(8)}, E_4^{(8)}) \,. \tag{68}$$

These two sets have the same anomalous-dimension matrix and we find

$$\gamma_{ij} = -\frac{g^2}{6\pi^2} \begin{bmatrix} 0 & 0 & 0 & -2 & 0 & 4 & 0 \\ 0 & 0 & 8/9 & 0 & -6 & 5/3 & -7/6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \,. \tag{69}$$

Using the anomalous-dimension matrix, one may calculate the scale dependence of the coefficient functions

$$(\mathcal{C}_i^{[1]}) = (\mathcal{C}(T_1^{(1)}), \mathcal{C}(T_1^{(8)}), \mathcal{C}(D_1^{(1)}), \mathcal{C}(E_1^{(1)}), \mathcal{C}(B_1^{(8)}), \mathcal{C}(D_1^{(8)}), \mathcal{C}(E_1^{(8)}))$$
$$(\mathcal{C}_i^{[3]}) = (\mathcal{C}(T_2^{(1)}), \mathcal{C}(T_2^{(8)}), \mathcal{C}(D_3^{(1)}), \mathcal{C}(E_4^{(1)}), \mathcal{C}(B_3^{(8)}), \mathcal{C}(D_3^{(8)}), \mathcal{C}(E_4^{(8)})) \,, \tag{70}$$

by solving the renormalization group equation (61) with the initial conditions given at $\mu_0 = m$. The solutions may be expressed in terms of the value of the coefficients at the scale m and one obtains

$$\mathcal{C}(T_1^{(1)}, \mu) = \mathcal{C}(T_1^{(1)}, m) = \mathcal{C}(A_1^{(1)}, m)$$
$$\mathcal{C}(T_1^{(8)}, \mu) = \mathcal{C}(T_1^{(8)}, m) = \mathcal{C}(A_1^{(8)}, m)$$

$$\mathcal{C}(D_1^{(1)}, \mu) = \mathcal{C}(D_1^{(1)}, m) + \frac{32}{9} \frac{1}{33 - 2n_f} \mathcal{C}(T_1^{(8)}, m) \ln \eta$$

$$\mathcal{C}(E_1^{(1)}, \mu) = \mathcal{C}(E_1^{(1)}, m) - \frac{8}{33 - 2n_f} \mathcal{C}(T_1^{(1)}, m) \ln \eta$$

$$\mathcal{C}(B_1^{(8)}, \mu) = \mathcal{C}(B_1^{(8)}, m) - \frac{24}{33 - 2n_f} \mathcal{C}(T_1^{(8)}, m) \ln \eta$$

$$\mathcal{C}(D_1^{(8)}, \mu) = \mathcal{C}(D_1^{(8)}, m) + \frac{16}{33 - 2n_f} \mathcal{C}(T_1^{(1)}, m) \ln \eta$$

$$+ \frac{20}{3} \frac{1}{33 - 2n_f} \mathcal{C}(T_1^{(8)}, m) \ln \eta$$

$$\mathcal{C}(E_1^{(8)}, \mu) = \mathcal{C}(E_1^{(8)}, m) - \frac{14}{3} \frac{1}{33 - 2n_f} \mathcal{C}(T_1^{(8)}, m) \ln \eta \tag{71}$$

and the same solution for the second set of coefficients.

The non-trivial evolution of all the coefficients of the local operators is driven by the coefficient of the time-ordered product corresponding to a double insertion of K_1; in turn, this coefficient is simply the one of the dimension-six operator with the corresponding spin structure.

The scale μ_0 is the matching scale, which we chose to be $\mu_0 = m$. On the other hand, we want to study the matrix elements of the operators at some lower scale, say the scale of the binding energy of the quarkonium, where we expect to have a reasonable approximation using a wave-function model. The evolution equation (61) allows us to change the renormalization scale, thereby inducing mixing of the non-local operators into the local ones.

7 Forward Matrix Elements and Heavy Quark Symmetry

Once we have scaled down to the small scale of the order of the binding energy of the quarkonium, we have to evaluate the forward matrix elements appearing in (55). This requires in general non-perturbative input, which has to be supplied by other methods such as lattice gauge theory or by model estimates. However, we may use heavy-quark symmetry to count the number of independent non-perturbative parameters.

The Lagrangian $\mathcal{L}_0$ given in (19) still has the heavy-quark spin symmetry. As a consequence, the heavy quarkonia systems fall in general into spin symmetry quartets:

For a given orbital angular momentum ℓ and radial excitation quantum number n, the four states (in the spectroscopic notation $^{2S+1}\ell_J$)

$$[n^1\ell_\ell \quad n^3\ell_{\ell-1} \quad n^3\ell_\ell \quad n^3\ell_{\ell+1}] \tag{72}$$

form such a spin symmetry quartet. An exception are the S waves ($\ell = 0$), for which the three polarization directions of the n^3S_1 and the n^1S_0 form the spin symmetry quartet.

In order to exploit the consequences of the spin symmetry for the inclusive hadronic decays we shall use the trace formalism. We denote with $|Y_\ell\rangle$ the spin symmetry quartet consisting of the spin singlet and the spin triplet for a given orbital angular momentum ℓ. The coupling of the heavy-quark spins may be represented by the matrices

$$H_Y(v) = \begin{cases} P_+\gamma_5 \text{ for the spin singlet} \\ P_+\slashed{\epsilon} \text{ for the spin triplet} \end{cases}, \tag{73}$$

where $P_+ = (1+\slashed{v})/2$ is the projection of the "large components". Note that the matrices H_Y are independent of ℓ.

For the dimension-six operators this implies

$$\langle Y_\ell| \left(\bar{h}^{(+)} C\,\Gamma\,h^{(-)}\right) \left(\bar{h}^{(-)} C\,\Gamma'\,h^{(+)}\right) |Y_\ell\rangle = a_\ell^{(C)}\,\text{Tr}\left\{\overline{H}_Y\Gamma\right\}\,\text{Tr}\left\{\Gamma' H_Y\right\}\,, \tag{74}$$

which means that there are only two independent parameters for a given ℓ describing the dimension-six matrix elements, namely $a_\ell^{(1)}$ and $a_\ell^{(8)}$. In the study of exclusive non-leptonic decays of heavy-light mesons, matrix elements of four-quark operators are usually estimated in vacuum insertion. Applying this approximation to (74) suggests that $a_0^{(1)}$ is the dominant coefficient.

The generic dimension-eight colour $1 \otimes 1$ operators may be written in terms of these representations as

$$\langle Y_\ell|[i\partial^\mu \left(\bar{h}^{(+)}\Gamma h^{(-)}\right)][i\partial^\nu \left(\bar{h}^{(-)}\Gamma' h^{(+)}\right)]|Y_\ell\rangle = b_\ell^{\mu\nu}\,\text{Tr}\left\{\overline{H}_Y\Gamma\right\}\,\text{Tr}\left\{\Gamma' H_Y\right\}$$

$$\left[\langle Y_\ell| \left(\bar{h}^{(+)}(i\overrightarrow{D}^\mu)\Gamma h^{(-)}\right)[i\partial^\nu \left(\bar{h}^{(-)}\Gamma' h^{(+)}\right)]|Y_\ell\rangle +\ \text{h.c.}\right] = c_\ell^{\mu\nu}\,\text{Tr}\left\{\overline{H}_Y\Gamma\right\}\,\text{Tr}\left\{\Gamma' H_Y\right\}$$

$$\langle Y_\ell| \left(\bar{h}^{(+)}(i\overrightarrow{D}^\mu)\Gamma h^{(-)}\right) \left(\bar{h}^{(-)}\Gamma'(i\overrightarrow{D}^\nu)h^{(+)}\right)]|Y_\ell\rangle = d_\ell^{\mu\nu}\,\text{Tr}\left\{\overline{H}_Y\Gamma\right\}\,\text{Tr}\left\{\Gamma' H_Y\right\}$$

$$\left[\langle Y_\ell| \left(\bar{h}^{(+)}(i\overrightarrow{D}^\mu)(i\overrightarrow{D}^\nu)\Gamma h^{(-)}\right) \left(\bar{h}^{(-)}\Gamma' h^{(+)}\right)]|Y_\ell\rangle +\ \text{h.c.}\right] = e_\ell^{\mu\nu}\,\text{Tr}\left\{\overline{H}_Y\Gamma\right\}\,\text{Tr}\left\{\Gamma' H_Y\right\}\,. \tag{75}$$

The tensors $b_\ell^{\mu\nu}$, $c_\ell^{\mu\nu}$, $d_\ell^{\mu\nu}$ and $e_\ell^{\mu\nu}$ have to be constructed from the velocity vector v and the metric tensor. Furthermore, if one contracts one of the indices with v, one may use the

equation of motion of the fields $h_v^{(\pm)}$; in the static limit this simply vanishes, while with an equation of motion as (18) one obtains a term one order higher in the $1/m$ expansion, which may be dropped, since we work only to order $1/m^2$. Consequently, the tensors are given by

$$
\begin{aligned}
b_\ell^{\mu\nu} &= b_\ell(g^{\mu\nu} - v^\mu v^\nu) &&, & c_\ell^{\mu\nu} &= c_\ell(g^{\mu\nu} - v^\mu v^\nu) \,, \\
d_\ell^{\mu\nu} &= d_\ell(g^{\mu\nu} - v^\mu v^\nu) &&, & e_\ell^{\mu\nu} &= e_\ell(g^{\mu\nu} - v^\mu v^\nu) \,.
\end{aligned} \tag{76}
$$

Spin symmetry thus places a very strong restriction on the local matrix elements, since for a given orbital angular momentum ℓ there are only eight non-perturbative parameters $b_\ell^{(1)}, \cdots e_\ell^{(1)}$ and the corresponding octet partners $b_\ell^{(8)}, \cdots e_\ell^{(8)}$, which describe all the forward matrix elements of the local dimension-eight operators given above.

From an estimate based on vacuum insertion one would guess that $b_0^{(1)}$, $d_1^{(1)}$ and $e_0^{(1)}$ dominate all other coefficients; in particular all the $c_\ell^{(C)}$ are expected to be small, if vacuum insertion makes any sense.

In a very similar way one may analyse the spin symmetry structure of the non-local terms. The kinetic energy terms are spin symmetric; furthermore there is no contribution from insertions of K_1, since this is already contained in the dynamics of the states; however, we note that this does not mean that K_1 does not show up at all: A double insertion of K_1 mixes under renormalization into local dimension-eight operators, but this is a short-distance effect which was calculated in the last section.

For an insertion of the second-order kinetic energy term K_2 we obtain from the trace formalism

$$
(-i) \int d^4x \, \langle Y_\ell | T\{ K_2(x) \left(\bar{h}^{(+)} \Gamma h^{(-)} \right) \left(\bar{h}^{(-)} \Gamma' h^{(+)} \right) | Y_\ell \rangle = k_2^{(1)} \, \mathrm{Tr} \left\{ \overline{H}_Y \Gamma \right\} \, \mathrm{Tr} \left\{ \Gamma' H_Y \right\} \tag{77}
$$

and a corresponding expression for the colour $T^a \otimes T^a$ contribution.

Furthermore, spin symmetry implies that a single insertion of a chromomagnetic moment operator vanishes. We have $(j = 1, 2)$:

$$
\begin{aligned}
(-i) \int d^4x \, &\langle Y_\ell | T\{ G_j(x) \left(\bar{h}^{(+)} \Gamma h^{(-)} \right) \left(\bar{h}^{(-)} \Gamma' h^{(+)} \right) | Y_\ell \rangle = \\
g_j^{\mu\nu} \Big[\, &\mathrm{Tr} \left\{ \overline{H}_Y \sigma_{\mu\nu} P_+ \Gamma \right\} \, \mathrm{Tr} \left\{ \Gamma' H_Y \right\} + \, \mathrm{Tr} \left\{ \overline{H}_Y \Gamma P_- \sigma_{\mu\nu} \right\} \, \mathrm{Tr} \left\{ \Gamma' H_Y \right\} \\
&+ \mathrm{Tr} \left\{ \overline{H}_Y \Gamma \right\} \, \mathrm{Tr} \left\{ \Gamma' P_+ \sigma_{\mu\nu} H_Y \right\} + \, \mathrm{Tr} \left\{ \overline{H}_Y \Gamma \right\} \, \mathrm{Tr} \left\{ \sigma_{\mu\nu} P_- \Gamma' H_Y \right\} \Big] \,.
\end{aligned} \tag{78}
$$

Thus the tensor $g_j^{\mu\nu}$ has to be antisymmetric, but it may only be built from the velocity vector and the metric tensor; from these only symmetric combinations are possible, and hence these contributions have to vanish.

Finally, there is also a double insertion of the chromomagnetic moment operator G_1. Analysing the spin structure of this term gives

$$\frac{(-i)^2}{2} \int d^4x \, d^4y \, \langle Y_\ell | T\{G_1(x)G_1(y) \left(\bar{h}^{(+)}\Gamma h^{(-)}\right) \left(\bar{h}^{(-)}\Gamma' h^{(+)}\right) |Y_\ell\rangle = S^{\mu\nu\alpha\beta}$$

$$\left[\text{Tr} \left\{ \overline{H}_Y \sigma_{\mu\nu} P_+ \sigma_{\alpha\beta} P_+ \Gamma \right\} \text{Tr} \left\{ \Gamma' H_Y \right\} + \text{Tr} \left\{ \overline{H}_Y P_+ \sigma_{\mu\nu} \Gamma P_- \sigma_{\alpha\beta} \right\} \text{Tr} \left\{ \Gamma' H_Y \right\} + \cdots \right] \quad (79)$$

where the ellipses stand for all other possible insertions of the two sigma-matrices. The tensor S now has to be antisymmetric and it may only be built from the metric tensor $g_{\mu\nu}$, since we have $P_\pm \sigma_{\mu\nu} v^\mu P_\pm = 0$ as well as $P_\pm \sigma_{\mu\nu} v^\mu P_\mp = 0$. Hence we have only a single parameter for this contribution

$$\frac{(-i)^2}{2} \int d^4x \, d^4y \, \langle Y_\ell | T\{G_1(x)G_1(y) \left(\bar{h}^{(+)}\Gamma h^{(-)}\right) \left(\bar{h}^{(-)}\Gamma' h^{(+)}\right) |Y_\ell\rangle = G^{(1)}$$

$$\left[\text{Tr} \left\{ \overline{H}_Y \sigma_{\mu\nu} P_+ \sigma^{\mu\nu} P_+ \Gamma \right\} \text{Tr} \left\{ \Gamma' H_Y \right\} + \text{Tr} \left\{ \overline{H}_Y P_+ \sigma_{\mu\nu} \Gamma P_- \sigma^{\mu\nu} \right\} \text{Tr} \left\{ \Gamma' H_Y \right\} + \cdots \right] \quad (80)$$

and a corresponding relation for the colour combination $T^a \otimes T^a$.

The constraints from spin symmetry thus allow us to drop the operators $C_5^{(C)}$, $C_6^{(C)}$, $D_5^{(C)}$, $E_5^{(C)}$ and $E_6^{(C)}$ in a calculation for the total rate. Of course, the contribution of $\mathcal{L}_{glue}$, (26), is spin-symmetric, since it involves only the light degrees of freedom. Furthermore, the only contribution of chromomagnetic-moment operators is the double insertion of G_1, which in turn is the only spin-symmetry-violating contribution that appears.

At the end of this section we want to discuss the flavour symmetry. Let us recall that for heavy-light systems both the spin and the flavour symmetries hold. As already mentioned several times, the flavour symmetry is broken in the present case of heavy quarkonia, because the states acquired a mass dependence. The interesting question arises about the amount of flavour-symmetry violation. As a simple example, we study the simpler problem of flavour-symmetry breaking for mass splittings. In the heavy-mass limit the four states in (72) should be degenerate; splitting between the members of the spin-symmetry quartet is induced by spin-orbit and spin-spin interactions. We may analyse this by looking at the mass of a heavy quarkonium in terms of the Lagrangian

$$M(n^{2S+1}\ell_J) = 2m + \tilde{\Lambda}(n, \ell) + \frac{1}{2m}\langle\psi|[G_1^{(+)} + G_1^{(-)}]|\psi\rangle$$

$$+ \left(\frac{1}{2m}\right)^2 \langle\psi|[K_2^{(+)} + K_2^{(-)} + G_2^{(+)} + G_2^{(-)}]|\psi\rangle$$

$$+ (-i)\left(\frac{1}{2m}\right)^2 \int d^4x \, \langle\psi|T\{[G_1^{(+)}(x) + G_1^{(-)}(x)][G_1^{(+)} + G_1^{(-)}]\}|\psi\rangle$$

$$+ \mathcal{O}(1/m^3) \, . \quad (81)$$

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Note that no insertions of $K_1^{(\pm)}$ appear, since this term is already included in $\mathcal{L}_0$.

The first contribution $\widetilde{\Lambda}(n, \ell)$ is the "binding energy", as one would obtain from the solution of the Schrödinger-type of equations corresponding to $\mathcal{L}_0$. This parameter can only depend on the quantum numbers n and ℓ, since spin symmetry is still unbroken.

Breaking of spin symmetry occurs first at order $1/m$; it is given by the expectation value of the first-order chromomagnetic moment operator. In a non-relativistic language, this would correspond to a coupling of the form $(\vec{s}^{(+)} + \vec{s}^{(-)}) \cdot \vec{B}$, where $\vec{B}$ is the chromomagnetic field, which is created by the orbital motion. Hence we expect $\vec{B} \propto \vec{L}$, where $\vec{L}$ is the orbital angular momentum. In other words, the first-order term is a spin-orbit coupling term.

The spin symmetric local terms of order $1/m^2$ are corrections to $\widetilde{\Lambda}(n, \ell)$, while the second-order chromomagnetic operator yields a correction to the spin-orbit term. The time-ordered product of the two first-order chromomagnetic moment operators will give (aside from a correction term to $\widetilde{\Lambda}(n, \ell)$) a spin-spin coupling term, which in a non-relativistic form is $\vec{s}^{(+)} \cdot \vec{s}^{(-)}$ for $\ell = 0$. Hence we arrive at a mass formula of the form (for $\ell = 0$):

$$
M(n^{2S+1}\ell_J) = 2m + \widetilde{\Lambda}(n, \ell) + \frac{\omega(n, \ell)}{2m} \frac{1}{2} \left[J(J+1) - \ell(\ell+1) - S(S+1) \right]
$$
$$
+ \frac{\tau(n, \ell)}{4m^2} \left[\frac{1}{2} S(S+1) - \frac{3}{4} \right] , \tag{82}
$$

where $\omega(n, \ell)$ and $\tau(n, \ell)$ are given in terms of matrix elements involving the Lagrangian.

Spin-dependent relativistic corrections can be systematically studied in the Wegner–Wilson loop approach. Their general structure up to order $1/m^2$ (under the assumption of a fixed background field $A_\mu(x)$, cf. ref. [15]) has been obtained by Eichten and Feinberg (EF) [16]

$$
V(r) = V_0(r) + \frac{V_0(r)' + 2V_1(r)' + 2V_2(r)'}{2m^2 r} \vec{L} \cdot \vec{S} + \frac{V_4(r)}{3m^2} \vec{s}_1 \cdot \vec{s}_2
$$
$$
+ \frac{V_3(r)}{m^2} \left(\frac{\vec{r} \cdot \vec{s}_1 \, \vec{r} \cdot \vec{s}_2}{r^2} - \frac{1}{3} \vec{s}_1 \cdot \vec{s}_2 \right)
$$
$$
+ \text{ spin-independent corrections} , \tag{83}
$$

where $V_0(r)$ is the spin-independent potential and $V_i(r)$ are related to expectation values of the colour-electric and magnetic fields, e.g. ($\vec{r} = \vec{r}_1 - \vec{r}_2$ and the limit $T \to \infty$ is understood):

$$
\left(\frac{r_i r_j}{r^2} - \frac{\delta_{ij}}{3} \right) V_3(r) + \frac{\delta_{ij}}{3} V_4(r) = \frac{g^2}{T} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} \langle B_i(\vec{r}_1, t) \, B_j(\vec{r}_2, t') \rangle \, \mathrm{d}t \, \mathrm{d}t' \, / \, \langle 1 \rangle . \tag{84}
$$

While (83) is exact for the spin-dependent relativistic corrections (for equal masses), the question of the spin-independent relativistic corrections appears not to be yet completely understood at the present time (for a review see e.g. [15]).

The potentials $V_i(r)$ are usually assumed to be flavour-independent. This assumption is not only in agreement with observed mass splittings of charmonium and bottomonium, but also confirmed by lattice-QCD studies (for a recent study see e.g. ref. [17]). The flavour dependence of mass splittings in the EF formulation (83) enters through the explicit $1/m^2$ factors, but also through the mass dependence of the expectation values

$$\langle r^q V_i^{(p)}(r) \rangle_{nl} = \int_0^\infty \mathrm{d}r \; r^2 \, R_{nl}^2(r) \, r^q \, V_i^{(p)}(r) \; , \qquad (85)$$

where $R_{nl}(r)$ are the radial wave functions (normnalized to $\int_0^\infty \mathrm{d}r \; r^2 \, R_{nl}^2(r) = 1$). Our result (81,82) has a structure similar to that of (83). For example, we may identify

$$\tau(n,l) = \langle \frac{3}{4} V_4(r) \rangle_{nl} \; . \qquad (86)$$

On the other hand, it is straightforward to include one-loop corrections to our result (81), while the EF representations seem to be non-renormalizable objects, and hence any attempt to calculate them in one-loop order should end up with a divergent result [15].

Although the functions $\omega(n,\ell)$ and $\tau(n,\ell)$ are matrix elements of flavour-independent operators, they in fact depend on the flavour through the states. We may obtain some idea on the flavour dependence of $\omega(n,\ell)$ and $\tau(n,\ell)$ by comparing the spectra of bottomonium and charmonium. For example, the spin-orbit term (proportional to $\vec{L}\cdot\vec{S}$ in (83)) and the tensor interaction (last term in (83)) do not contribute to either of the mass differences $m(2^3S_1) - m(1^3S_1)$ or $\bar{m}(1^3P_J) - m(1^3S_1)$, where $\bar{m}(1^3P_J)$ denotes the centre of gravity of the three ^3P_J ($J = 0, 1, 2$) states. Experimentally these mass differences are $563\,\mathrm{MeV}$ ($589\,\mathrm{MeV}$) and $440\,\mathrm{MeV}$ ($428\,\mathrm{MeV}$), respectively, for bottomonium (charmonium). Hence, certainly $\widetilde{\Lambda}(n,l)$ and $\tau(n,l)$ are basically flavour-independent.

Thus the parameters entering (82) seem to be only weakly dependent on the heavy quark flavour for the observable states of heavy quarkonia, although heavy-flavour symmetry is not present any more once we include subleading terms into $\mathcal{L}_0$. However, it is generally believed that in the heavy-mass limit a quarkonium should behave as an almost Coulombic system. This would imply that $\widetilde{\Lambda}(n) \sim m$, $\omega(n) \sim m^2$ and $\tau(n) \sim m^3$, which is not compatible with the data from charmonium and bottomonium. (For (83), the vector-Coulomb case means spin-independent terms are proportional to $\alpha_s^2 m$, while all spin-dependent terms are $\propto \alpha_s^4 m$.) This could mean that either these systems are not close enough to the heavy-mass limit to become Coulombic, or that the heavy-mass limit

is not the Coulombic one. Let us recall that the heavy-light systems with a b quark and to a lesser extent the ones with a c quark already behave as one would expect in the heavy-mass limit. This clearly indicates that the flavour dependence of the matrix elements parametrizing the long-distance effects is smaller than suggested by the Coulombic limit.

8   Some Phenomenology

In this section we shall consider a few applications of the formalism outlined above. We shall not give a full discussion of the phenomenological applications of the heavy-mass expansion for heavy quarkonia, but rather study a few simple examples in some detail, based on the matching calculation performed in section 5.

Compared to the conventional approach, and also compared to the v/c expansion advocated by BBL [5], we have a much larger list of parameters, although spin symmetry reduces this number to some extent. In a given order of the $1/m$ expansion they are all of the same dimension and there is no a priori reason why some of them should be less important than some others; at least the $1/m$ expansion does not give any hint.

The most simple example (although maybe academic) is the semi-inclusive decays of a heavy quarkonium into e^+e^- and light hadrons. In leading order of the expansion we have that $\eta_Q \to e^+e^-$ light hadrons vanishes, while we have for the ψ_Q

$$\frac{1}{2}\,\Gamma_0(\psi_Q \to e^+e^- \text{ light hadrons}) = \mathcal{C}^{ee}(A_2^{(1)},\mu)\langle\psi|A_2^{(1)}|\psi\rangle = \frac{4\pi}{3m^2}\alpha^2 c_Q^2 a_0^{(1)} \qquad (87)$$

with

$$\mathcal{C}^{ee}(A_2^{(1)},\mu) = -\frac{\pi}{3m^2}\alpha^2 c_Q^2 \ . \qquad (88)$$

There are no corrections of order $1/m$, since there are no local dimension-seven operators having non-vanishing matrix elements, and also the only non-local term, which is an insertion of a first-order chromomagnetic moment operator $G_1^{(\pm)}$, vanishes due to spin symmetry.

The first subleading contribution appears at order $1/m^2$. Using the matching calculation of section 5 and taking into account the running considered in section 6, we obtain

$$\begin{aligned}
\frac{1}{2}\,\Gamma_2 = {}& \mathcal{C}^{ee}(B_3^{(1)},\mu)\langle\psi|B_3^{(1)}|\psi\rangle_\mu + \mathcal{C}^{ee}(D_3^{(8)},\mu)\langle\psi|D_3^{(8)}|\psi\rangle_\mu \\
& + \mathcal{C}^{ee}(E_2^{(1)},\mu)\langle\psi|E_2^{(1)}|\psi\rangle_\mu + \mathcal{C}^{ee}(E_3^{(1)},\mu)\langle\psi|E_3^{(1)}|\psi\rangle_\mu \\
& + \mathcal{C}^{ee}(E_4^{(1)},\mu)\langle\psi|E_4^{(1)}|\psi\rangle_\mu + \mathcal{C}^{ee}(A_2^{(1)},\mu)(k_2^{(1)} + G^{(1)} + F^{(1)})
\end{aligned} \qquad (89)$$

where μ is the renormalization point $\mu < m$, and $F^{(1)}$ is the contribution of the single insertion of the purely gluonic piece $\mathcal{L}_{glue}$ of the $1/m^2$ Lagrangian.

The coefficients are given by

$$\mathcal{C}^{ee}(B_3^{(1)}, \mu) = -\frac{\pi}{3m^2}\alpha^2 c_Q^2$$

$$\mathcal{C}^{ee}(D_3^{(8)}) = -\frac{\pi}{3m^2}\alpha^2 c_Q^2 \frac{16}{33 - 2n_f}\ln\eta$$

$$\mathcal{C}^{ee}(E_2^{(1)}, \mu) = \mathcal{C}^{ee}(E_3^{(1)}, \mu) = -\frac{\pi}{12m^2}\alpha^2 c_Q^2$$

$$\mathcal{C}^{ee}(E_4^{(1)}, \mu) = -\frac{\pi}{3m^2}\alpha^2 c_Q^2 \left[\frac{1}{2} - \frac{8}{33 - 2n_f}\ln\eta\right] \ . \tag{90}$$

Note that we have an additional factor $1/(2m)^2$ in front of the second-order contribution according to our definition (45).

This expression looks relatively complicated, but it simplifies somewhat due to spin symmetry. The matrix elements of the local operators are given in terms of the three parameters $b_0^{(1)}$, $e_0^{(1)}$ and $d_0^{(8)}$, while the non-local terms introduce another three parameters: $k_2^{(1)}$, $G^{(1)}$, and $F^{(1)}$. At order $1/m^2$ there are thus six parameters describing the decay rate, which may then be expressed as

$$\frac{1}{2}\Gamma_2 = \frac{\pi}{3m^2}\alpha^2 c_Q^2 \left[12b_0^{(1)} + 8e_0^{(1)}\left(1 - \frac{12}{33 - 2n_f}\ln\eta\right)\right.$$
$$\left. + 12d_0^{(8)}\frac{16}{33 - 2n_f}\ln\eta - k_2^{(1)} - G^{(1)} - F^{(1)}\right] \ . \tag{91}$$

A further simplification can only be achieved with additional theoretical prejudices. A popular assumption (although ad hoc) for non-leptonic inclusive decays of heavy-light mesons is vacuum insertion. Note, however, that this is not a scale-invariant statement. Usually vacuum insertion is applied at some small scale, where the matrix elements are estimated by e.g. a wave-function model. There the local contributions are related to the wave function and its derivatives at the origin $\vec{x} = \vec{0}$, and the non-local contributions correspond to corrections to the wave function. The application of vacuum insertion to the present case of heavy-quarkonium decays is quite a strong assumption, because it removes already all the operators with a RCM derivative. This follows from the fact that, with this assumption, the c.m.s. frame of the hard annihilation process has to be the same as the one of the heavy quarkonium. Furthermore, this assumption leads to vanishing matrix elements for all colour $T^a \otimes T^a$ operators at the small scale μ. And

finally, in such a picture the contributions of the purely gluonic piece (26) vanish (e.g. the term $F^{(1)}$ in (91)).

If we nonetheless use vacuum insertion at a small scale μ we find

$$\Gamma = \frac{\alpha^2 c_Q^2}{m^2} \left[\frac{8\pi}{3} a_0^{(1)} + \frac{4\pi}{3m^2} e_0^{(1)} \left(1 - \frac{12}{33 - 2n_f} \ln \eta \right) - \frac{\pi}{6m^2} \left(k_2^{(1)} + G^{(1)} \right) \right] . \tag{92}$$

The first term is the one familiar from non-relativistic potential models: Identifying

$$a_0^{(1)} = \frac{3}{2} |\Psi(0)|^2 = \frac{3}{8\pi} |R(0)|^2 \tag{93}$$

we recover the well-known Royen–Weisskopf formula ($M = M(^3S_1) \approx 2m$):

$$\Gamma(^3S_1 \to \ell\bar{\ell}) = \frac{4\alpha^2 c_Q^2}{M^2} |R(0)|^2 , \tag{94}$$

which holds for the exclusive decay. However, it is this expression we obtain to leading order also for the inclusive decay; this indicates that the exclusive mode $\psi \to e^+ e^-$ will saturate a large portion of the inclusive decay $\psi \to e^+ e^- +$ light hadrons.

The non-logarithmic part of the $e_0^{(1)}$ $1/m^2$ correction in (92) has also been discussed by BBL [5] and by Keung and Muzinich (KM) [18]. It is in fact the only correction that occurs in either approach. BBL propose to identify (the analogon to) $e_0^{(1)}$ with the limit as $r \to 0$ of $-\vec{\Delta}^2 R(r)$ with appropriate regularization. KM calculate kinematical corrections to the leading-order amplitude. They denote by ϵ the binding energy and by $\vec{p}$ the relative three-momentum of the heavy quark and antiquark. Then they evaluate the amplitude at $\vec{p}^2 = m\epsilon$ rather than at $\vec{p}^2 = 0$, as is the usual wave-function prescription for S-wave decays. Their corrections proportional to ϵ/m are thus $\vec{p}^2/m^2$ corrections and can be identified with those arising from the E-operators in our approach.

Let us emphasize that the appearance of the logarithm in (92) indicates the breakdown of the naïve potential-model calculations also for S-wave decays as was conjectured a long time ago [19]. Expanding

$$\frac{6}{33 - 2n_f} \ln \eta \approx \frac{\alpha_s(m)}{\pi} \ln \frac{m}{\mu} , \tag{95}$$

we obtain a contribution to the decay width proportional to $|R''(0)|^2 \ln(m/\mu)$. This manifestly violates the potential-model ansatz that the infrared dynamics is given by the meson's non-relativistic wave function (and its derivatives), while the short-distance part is free of infrared singularities.

As a further example, let us study the hadronic decay of spin-triplet P-wave quarkonia $n^3P_J = \chi_{QJ}(nP)$. In the wave-function approach, the leading-order (in $\alpha_s(m)$) decay is into two gluons for the $J = 0, 2$ states, $\Gamma(^3P_{0,2} \to gg) = O(\alpha_s^2)$, while the $J = 1$ state can first decay at $O(\alpha_s^3)$ into either three gluons or a quark-antiquark-pair plus a gluon. Moreover, it was found [19, 20, 21] that the quark-antiquark-gluon cuts are singular in the limit of zero binding energy ϵ

$$\Gamma(^3P_J \to q\bar{q}g) = \frac{n_f}{3} \frac{128}{3\pi} \frac{\alpha_s(m)^3}{M^4} \ln \frac{m}{\epsilon} \ |R'_{1P}(0)|^2 \ . \tag{96}$$

The magnitude of the logarithm is usually estimated by identifying $1/\epsilon$ with the average radius of the ^3P states. Again, the presence of an infrared sensitive logarithm signals the breakdown of the usual factorization assumption that is behind the wave-function approach. It has been argued that the decay into $q\bar{q}g$ has to be considered as being of the same perturbative order $O(\alpha_s^2)$ as the two-gluon decay since the expresion $\alpha_s \ln(m/\epsilon)R'(0)^2$ has to be considered as a new nonperturbative parameter besides $R'(0)^2$ describing P-wave decays [22, 7].

The present formalism reproduces this result by generating the large logarithm through the renormalization group running. In fact, formally the decay into a pair of quarks is the dominant one, since it is of the same order in α_s as the two-gluon decay but logarithmically enhanced, see (102) below. To see how this comes about consider the $q\bar{q}$ decays of the spin-triplet P-wave states in more detail. At the matching scale m only colour-octet contributions are present (section 5) and hence the leading-order result is

$$\frac{1}{2} \Gamma_0(3\chi \to q\bar{q} \to \text{ light hadrons}) = -\frac{\pi}{6m^2}\alpha_s^2(m)n_f\langle\psi|A_2^{(8)}|\psi\rangle = \frac{2\pi}{3m^2}\alpha_s^2(m)n_f\, a_0^{(8)} \ . \tag{97}$$

For the second-order contributions we find

$$\frac{1}{2} \Gamma_2 = +\mathcal{C}^{qq}(B_3^{(8)}, \mu)\langle\psi|B_3^{(8)}|\psi\rangle_\mu + \mathcal{C}^{qq}(D_3^{(8)}, \mu)\langle\psi|D_3^{(8)}|\psi\rangle_\mu + \mathcal{C}^{qq}(D_3^{(1)}, \mu)\langle\psi|D_3^{(1)}|\psi\rangle_\mu$$
$$+\mathcal{C}^{qq}(E_2^{(8)}, \mu)\langle\psi|E_2^{(8)}|\psi\rangle_\mu + \mathcal{C}^{qq}(E_3^{(8)}, \mu)\langle\psi|E_3^{(8)}|\psi\rangle_\mu$$
$$+\mathcal{C}^{qq}(E_4^{(8)}, \mu)\langle\psi|E_4^{(8)}|\psi\rangle_\mu + \mathcal{C}^{qq}(A_2^{(8)}, \mu)(k_2^{(8)} + G^{(8)} + F^{(8)}) \ . \tag{98}$$

The coefficients of the local operators are (at a scale $\mu < m$):

$$\mathcal{C}^{qq}(B_3^{(8)}, \mu) = -\frac{\pi}{6m^2}\alpha_s^2(m)n_f\left(1 - \frac{24}{33 - 2n_f}\ln\eta\right)$$

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$$\mathcal{C}^{qq}(D_3^{(8)}) = -\frac{10\pi}{9m^2}\alpha_s^2(m)n_f\frac{1}{33-2n_f}\ln\eta$$

$$\mathcal{C}^{qq}(D_3^{(1)}) = -\frac{16\pi}{27m^2}\alpha_s^2(m)n_f\frac{1}{33-2n_f}\ln\eta$$

$$\mathcal{C}^{qq}(E_2^{(8)},\mu) = \mathcal{C}^{qq}(E_3^{(8)},\mu) = -\frac{\pi}{24m^2}\alpha_s^2(m)$$

$$\mathcal{C}^{qq}(E_4^{(8)},\mu) = -\frac{\pi}{12m^2}\alpha_s^2(m)2\left(1-\frac{28}{3}\frac{1}{33-2n_f}\ln\eta\right) . \tag{99}$$

It is interesting to note that the renormalization group flow induces operators ($D_3^{(8)}$ and $D_3^{(1)}$) which have not been present at the matching scale. We may express the local contributions to the second-order contribution in terms of the parameters $b_1^{(8)}$, $d_1^{(8)}$, $d_1^{(1)}$, and $e_1^{(8)}$:

$$\frac{1}{2}\Gamma_2 = \frac{\pi}{6m^2}\alpha_s^2(m)n_f\left[12b_1^{(8)}\left(1-\frac{24}{33-2n_f}\ln\eta\right)+8e_1^{(8)}\left(1-\frac{7}{33-2n_f}\ln\eta\right)\right.$$

$$\left.+\frac{128}{3}d_1^{(1)}\frac{1}{33-2n_f}\ln\eta+80d_1^{(8)}\frac{1}{33-2n_f}\ln\eta-k_2^{(8)}-G^{(8)}-F^{(8)}\right].\tag{100}$$

We note that not only the leading contribution (97), but also the first subleading one (100) is the same for all three states $^3P_0$, $^3P_1$, and $^3P_2$. Thus, if the channel involving a $q\bar{q}$-pair really dominates the decays of these states into light hadrons due to the logarithmic enhancement, we have

$$\Gamma(^3P_2 \to \text{ light hadrons}) = \Gamma(^3P_1 \to \text{ light hadrons}) = \Gamma(^3P_0 \to \text{ light hadrons}) . \tag{101}$$

The second-order contribution (100) is given in terms of seven parameters, but this number reduces once vacuum insertion is assumed. Then only one of them is non-zero, namely the parameter $d_1^{(1)}$, and the rate takes the simple form

$$\Gamma(^3\chi_Q \to q\bar{q}(g) \to \text{ light hadrons }) = \frac{32\pi}{9m^4}\alpha_s^2(m)n_fd_1^{(1)}\frac{1}{33-2n_f}\ln\eta . \tag{102}$$

Furthermore, in a wave-function model the parameter $d_1^{(1)}$ is given in terms of the derivative of the wave function at the origin

$$d_1^{(1)}(n) = \frac{3}{2\pi}\left|R'_{nP}(0)\right|^2 , \tag{103}$$

and if one expands (102) again in powers of $\alpha_s(m)$, see (95), we reproduce the result (96).

9 Comparison with previous approaches and conclusion

Traditionally [6] inclusive hadronic (and electromagnetic) decays of heavy quarkonia are calculated in the framework of the Bethe–Salpeter description of the bound state. Assuming an instantaneous potential and working in the extreme non-relativistic limit, the decay width of a heavy quark-antiquark bound state into light hadrons (l.h.) is written as

$$\Gamma(n\,^{2S+1}L_J \to \text{l.h.}) = G(n)\,\hat{\Gamma}(Q\bar{Q}(^{2S+1}L_J) \to \text{partons})\,. \tag{104}$$

This ansatz of separation into a short-distance part $\hat{\Gamma}$ describing the decay of a free (unbound) $Q\bar{Q}$-pair to decay into partons ($q\bar{q}$-pairs and gluons) and a long-distance one $G(n)$ representing the non-perturbative bound-state formation, is motivated by the observation that the problem involves two widely separated scales, the $Q\bar{Q}$ radius of the order of $1\,\text{fm}$ and the heavy-quark Compton wavelength $\lambda_Q \sim 1/m \ll r_{Q\bar{Q}}$. The non-perturbative part $G(n)$ is expressed in terms of the non-relativistic (Schrödinger) wave function at zero relative coordinate $R_{nS}(0)$ (for S-wave decays, and the derivative $R'_{nP}(0)$ for P-wave decays), which is calculated in a potential model or extracted from data. The factor $G(n)$ thus depends explicitly on the binding energy ϵ. The $Q\bar{Q}$ decay, on the other hand, is governed by a scale of the order of $\mu \sim m$, and can hence be expanded in a series of $\alpha_s(m)$. The decay rate is called factorizable into a long- and a short-distance contribution [19] if the short-distance factor $\hat{\Gamma}$ is calculable without encountering infrared divergencies.

Let us elaborate on the assumptions of the conventional approach in some more detail. Consider the transition amplitude $\mathcal{M}(J^P, P; k_i)$ of a J^P quark-antiquark bound state of mass M (and four-momentum $P = (M, \vec{0})$) into partons (light $q\bar{q}$-pairs and gluons) of four-momenta k_i in terms of which the width is given by $\Gamma \sim |\mathcal{M}|^2\,\text{dPS}$. (For illustration we suppress colour and polarization indices.) Introducing the Bethe–Salpeter bound-state wave function $\Phi(J^P, P; q)$, dependent on the relative quark and antiquark momenta q ($p_{Q,\bar{Q}} = \frac{1}{2}P \pm q$), the transition amplitude is written in this picture as

$$\mathcal{M} = \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \text{Tr}\,\Phi(P, q)\mathcal{O}(P, q, k_i) \tag{105}$$

such that $\mathcal{O}(P, q, k_i)$ is the amplitude of a free (unbound) $Q\bar{Q}$-pair to decay into light hadrons. Upon a non-relativistic reduction

$$\Phi(P, q) = 2\pi\delta(q^0) \sum_{m, S_z} \psi_{nlm}(\vec{q})\, \langle lm S S_z\,|J J_z\rangle\, P_{S S_z}(P, q)\,, \tag{106}$$

the amplitude becomes

$$\mathcal{M} = \sum_{m,S_z} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \psi_{nlm}(\vec{q}\,) \underbrace{\langle lmSS_z | JJ_z \rangle \operatorname{Tr} P_{SS_z}(P,q)\mathcal{O}(P,q,k_i)}_{A(\vec{q}^{\,2},\epsilon)} \tag{107}$$

where $p_i^2 = M^2/4 - \vec{q}^{\,2} = m^2 + \epsilon m - \vec{q}^{\,2}$, ϵ is the binding energy, and P_{SS_z} is a projection operator. The Schrödinger wave function in coordinate space extends over distances of the order of the Bohr radius. Correspondingly, the Fourier transform $\psi_{nlm}(\vec{q}\,)$ is non-zero only for $|\vec{q}\,|/m \ll 1$, and hence $|\vec{q}\,|/m$ becomes a reasonable expansion parameter. For S-wave decays, the leading term of this expansion yields the usual expression for the decay amplitude proportional to the non-relativistic wave function calculated at the origin in the relative coordinate space:

$$\mathcal{M} \approx \frac{1}{\sqrt{4\pi}} R_{nS}(0) A(\vec{q}^{\,2} = 0, \epsilon) \ . \tag{108}$$

Here $R_{nl}(0) = 4\pi\psi_{nl}(0)$ ($l = S, P, D$, etc.) is the radial wave function at zero distance $r = 0$. For P-wave decays, the terms linear in $\vec{q}$ in the expansion of $\mathcal{O}$, as well as terms linear in $\vec{q}$ coming from the small components of the relativistic wave function, must be retained yielding a final expression for the amplitude $\mathcal{M}$ proportional to the derivative of the wave function for the $l = P$ state at zero r, $R'_{nP}(0)$. Factorization is now said to hold if the limit $\epsilon \to 0$ exists for the amplitude A.

Within this potential-model approach, next-to-leading-order (NLO) perturbative QCD corrections have been calculated. Both the NLO corrections to $^1S_0$ (i.e. η_c) [23] and $^3S_1$ (i.e. J/ψ) [24] decays indeed obey the factorized form (104). However, there are three observations from which it becomes obvious that this picture is too simple:

- Infrared sensitive logarithms $\sim \ln(m/\epsilon)$ appear in the calculation of P-wave decays, to be precise in the NLO corrections to $^3P_{0,2}$ decays [20] and already in the leading-order expressions of $^1P_1$ and $^3P_1$ decays [19]. That is, without keeping the binding energy non-zero, the perturbative part of the calculation would diverge. The factorization (104) thus breaks down.

- The description of S-wave decays based on the strict non-relativistic limit is not in agreement with data. For example, a reasonable α_s determination from the ratio $\Gamma(^3S_1 \to \text{l.h.})/\Gamma(^3S_1 \to \ell\bar{\ell})$ is possible only once a (rather large) adhoc relativistic correction factor is applied. Other failures are the photon spectrum of $^3S_1$ decays and photo- and hadroproduction of J/ψ [7]. In fact, logarithmic infrared divergences,

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predicted to arise also in relativistic corrections to S-wave decays a long time ago [19], show up, cf. (92,95). Within the wave-function approach one can distinguish two sources of relativistic corrections, namely corrections to (i) the amplitude $\mathcal{O}$ and the (ii) the wave function ψ, cf. (107). Relativistic corrections to the wave function are very difficult to access since at present no full analysis of the spin-independent relativistic corrections to the potential exists, cf. (83). Kinematical corrections of type (i) have been discussed by Keung and Muzinich some time ago [18] for S-wave decays and more recently applied also to J/ψ photo- [25] and hadroproduction [7]. The prescription is to evaluate the amplitude $A(\vec{q}^{\,2}, \epsilon)$, (107), at $\vec{q}^{\,2} = m\epsilon$ rather than at $\vec{q}^{\,2} = 0$. In this way certain binding-energy corrections $\propto \epsilon/m = \vec{q}^{\,2}/m^2$ are kept.

- Sizeable, non-perturbative corrections have been predicted [26] for annihilation decays of heavy quarkonia, at least for the charmonium system. Such higher-twist corrections could arise from non-zero condensates (in particular, the gluon condensate) and/or colour-octet intermediate states, e.g. via a non-perturbative colour-E1 transition of the J/ψ into a (coloured) χ_{cJ} state ${}^3S_1 \to {}^3P_J^{(8)} + g$, followed by a "hard" ($\mu \sim m$) decay ${}^3P_J^{(8)} \to gg$.

In this paper we aim at a systematic, QCD-based treatment of inclusive annihilation decays of heavy quarkonia. In our approach, the factorization of long- and short-distance contributions is well defined. Furthermore, "genuine" relativistic corrections, i.e. the ones proportional to the relative velocity of the heavy quarks in the conventional language, and what is usually called non-perturbative corrections come out to have the same origin, namely the higher-order terms of the $1/m$ expansion. In fact, since we may shift certain contributions from the operators into the states in this expansion, there is no unique distinction between the two kinds of corrections.

To achieve such a systematic description of quarkonia decays we formulate an effective field theory for heavy quarkonia, which follows rigorously from the QCD Lagrangian. As a first step we write down an operator product expansion for the inclusive annihilation decays of heavy quarkonia in order to separate long- and short-distance contributions. Herein the distance scale is set by the Compton wavelength of the heavy quark. This provides us with a systematic $1/m$ expansion of the short-distance contributions with mass-independent operators. As the next step we expand the heavy-quark fields $Q_v^{(\pm)}(x)$ and the Lagrangian. In the wave-function picture, the expansions of the states and of the kernel correspond to corrections to the amplitude A in (107), while time-ordered insertion of the higher-order Lagrangian correspond to corrections of the wave function.

In the set-up of an effective theory approach to heavy quarkonia we now observe a crucial difference compared to heavy-light systems: The static limit does not exist for a heavy quarkonium state. Divergent phases appear in the quark-antiquark sector of HQET once the velocities of the two heavy quarks differ by an amount of only Λ_{QCD}/m. These divergent phases can be and have to be absorbed into the heavy-quark states. Using reparametrization invariance, we have shown that it is necessary and sufficient to include the $1/m$ kinetic term into the leading-order Lagrangian in order to consider two heavy quarks moving with the same velocity. The dynamics defined by this static-plus-$1/m$-kinetic-energy Lagrangian already contains the divergent phases (and the binding of the two heavy quarks) as an infrared effect, and this is the physical reason why the limit $v \to v'$ exists. Since the redefined states of a heavy quarkonium (seemingly) do not have a static limit, it is, in contrast to heavy-light systems, not possible to describe the full mass dependence of the relevant matrix elements in a $1/m$ expansion. Although the short-distance part may still be written as a $1/m$ expansion with mass-independent operators (once divergent imaginary parts have been shifted from the operators into the states by a suitable redefinition), the matrix elements of these operators with the redefined states become mass-dependent due to the mass dependence of the states.

As a result of the mass dependence of the states, heavy-flavour symmetry does not hold anymore. However, judging from the spectra and also from the widths of charmonium compared to bottomonium, the flavour dependence of the matrix elements does not seem to be very strong; at least it appears to be much weaker than one would expect for a Coulombic system.

On the other hand, spin symmetry is still preserved since the extra (kinetic) $1/m$ term in the leading-order Lagrangian is spin-symmetric. Hence the heavy quarkonia states described by our leading-order Lagrangian have to fall into degenerate spin symmetry quartets. Spin symmetry, furthermore, allows a restriction of the number of independent parameters describing matrix elements involving heavy quarkonia states.

We have applied our approach to inclusive heavy-quarkonia decays into light quarks and leptons. The effective theory machinery allows us to calculate the logarithmic dependence on the heavy mass m by studying the renormalization of the operators mediating the decay. The matching of QCD to the effective theory is performed at the large scale m, where the coefficients of the operator are determined by comparing the QCD result with the effective theory. This determines the initial conditions of the operator coefficients. The renormalization group of the effective theory then allows the coefficients to be run down to some smaller scale μ. We now see how the factorization assumption of (104) is generalized in a proper QCD treatment: A given inclusive annihilation decay is, in general, the sum of various contributions, each of which is the product of a non-perturbative

contribution, parametrized as the matrix element of an operator renormalized at a scale μ, and a Wilson coefficient evaluated at the same scale.

In addition to matrix elements of local operators also non-local contributions appear. From the conventional point of view these correspond to corrections to the wave functions of the states. Among these non-local terms one may identify spin-symmetry breaking corrections, which in the wave-function language generate differences between, for example, the $^1S_0$ and the $^3S_1$ wave functions.

The rate to leading order is given in terms of forward matrix elements of dimension-six operators, which at the one-loop level are scale-invariant, and hence no logarithmic enhancement through terms as $\ln(m/\mu)$ is present. Due to symmetries, the leading-order contribution is in general given in terms of only two parameters ($a_\ell^{(1)}$ and $a_\ell^{(8)}$) for a given orbital angular momentum ℓ (and a given n) of the heavy quarkonium state.

The first subleading contribution is suppressed by two powers of the heavy mass and involves forward matrix elements of dimension-eight operators as well as non-local contributions corresponding to corrections to the states. The number of dimension-eight operators is quite large, although it is somewhat reduced by spin symmetry. In total, the number of independent parameters is in general still large, namely 14 ($b_\ell^{(C)}$, $c_\ell^{(C)}$, $d_\ell^{(C)}$, $e_\ell^{(C)}$, $k_2^{(C)}$, $G_\ell^{(C)}$ and $F_\ell^{(C)}$) for each heavy quarkonium angular momentum ℓ.

Intuitively one may expect that some of the parameters are smaller than others. If one assumes vacuum insertion, then all matrix elements may be interpreted as wave functions and its derivatives taken at the origin. In this way one is led to assume that all the colour combinations $T^a \otimes T^a$ are suppressed compared to the $1 \otimes 1$ operators. Furthermore, the wave functions for P-wave states vanish at the origin, and this suggests that the matrix elements of the operators $D_n^{(1)}$ dominate for the P-wave states. Let us emphasize, however, that our analysis is independent of these assumptions. Although our approach has many common features with the expansion advocated by BBL [5] it is more general since we give the expansion before additional, less rigorous assumptions have been applied:

1. Within a set of operators of a given dimension we do not neglect those that are suppressed by powers of the relative $Q\bar{Q}$ velocity v/c in the wave-function language. For instance, we do not neglect an operator involving a gluon field strength $[iD, iD]$ in comparison to $(iD)^2$, although the two operators correspond to different powers of v/c in the language of [5]. Such a procedure is adequate for Coulombic systems but may be a bad approximation for charmonium and bottomonium, which certainly do not behave Coulomb-like. Yet, up to order $(\tilde{\Lambda}/m)^2$ no difference appears between the two approaches, since the matrix elements of antisymmetric products of covariant derivatives vanish due to spin symmetry.

2. We do not restrict the heavy quarkonium to the leading Fock state. Corresponding arguments are again based on counting of powers of v/c: Using perturbation theory, each additional gluon associated with the (assumed) dominant $Q\bar{Q}$ pair is ascribed an extra power of v/c [5] via the identification $v \sim \alpha_s(m\,v)$, valid for a colour-Coulomb potential. Although this estimate may be underlined by the multipole expansion, we do not know of any rigorous derivation.

3. The non-local contributions from the time-ordered products with the Lagrangian in Γ_2 (55) allows us to rigorously define the spin-symmetry breaking terms that appear at order $1/m^2$. Thus in principle we can calculate the coefficients that appear in relations as $R_\psi = R_{\eta_c}[1 + O(v^2/c^2)]$, although in practise additional unknown parameters enter, namely $G_l^{(c)}$ for each l.

If we apply these additional assumptions, a large number of our non-perturbative parameters may be dropped, and our approach yields the same result up to order $(\tilde{\Lambda}/m)^2$ as the one of ref. [5].

In conclusion, the approach presented here is based on QCD and hence provides a model-independent basis for the description of heavy-quarkonium physics. It allows us to separate long and short distances, where the short-distance contribution may be evaluated perturbatively. The long-distance part is parametrized in terms of matrix elements, which involve up to order $1/m^2$ operators of dimension six and dimension eight. These matrix elements are beyond the effective theory approach and have to be taken either from data, estimated via non-perturbative methods such as sum rules, or they may eventually be calculated from lattice QCD.

APPENDIX

A Spinorology

Similar to the case of only upper components [27], we also have only four independent matrices once we project on upper and lower components. The projectors are

$$P_+ = \frac{1}{2}(1 + \not{v}) \quad P_+ = \frac{1}{2}(1 - \not{v}) \tag{109}$$

and we have the mapping of the sixteen Dirac matrices

$$
\begin{aligned}
1 &\longrightarrow P_+P_- = 0 & \gamma_\mu &\longrightarrow P_+\gamma_\mu P_- \\
\gamma_\mu\gamma_5 &\longrightarrow P_+\gamma_\mu\gamma_5 P_- = P_+\gamma_5 P_- v_\mu & \gamma_5 &\longrightarrow P_+\gamma_5 P_- \\
(-i)\sigma_{\mu\nu} &\longrightarrow P_+(-i)\sigma_{\mu\nu}P_- = P_+(v_\mu\gamma_\nu - v_\nu\gamma_\mu)P_- \; .
\end{aligned}
\tag{110}
$$

We chose the four matrices to be $P_+\gamma_5 P_-$ (corresponding to the unit matrix) and $P_+\gamma_\mu P_-$ (corresponding to the three Pauli matrices). Note that $v^\mu P_+\gamma_\mu P_- = 0$, so these are indeed only three matrices. Hence, stricktly speaking, all Lorentz indices in (48) and (51–54) should be written as perpendicular indices only defined by

$$a_\mu^\perp = a_\rho \left(g^{\rho\mu} - v^\rho v^\mu\right) \; . \tag{111}$$

Note that

$$g^{\mu\nu} \, g_{\mu\nu}^\perp = 3 \; . \tag{112}$$

One may obtain the projections for any Dirac matrix Γ in terms of these four matrices by the trace formula

$$P_+\Gamma P_- = \frac{1}{2} \, \mathrm{Tr} \, \{\Gamma P_+\gamma_5 P_-\} \, P_+\gamma_5 P_- + \frac{1}{2} \, \mathrm{Tr} \, \{\Gamma P_+\gamma^\mu P_-\} \, P_+\gamma_\mu^\perp P_- \; . \tag{113}$$

The projections with P_- and P_+ interchanged are obtained by replacing $v \to -v$ in the above equations.

B Matching calculation to $O(1/m^2)$

In this appendix we collect a few useful relations that appear frequently in the matching calculation.

B.1 Expansions of bilinears

Using (25), (42), and (43) we obtain, for the bilinears up to and including order $1/m^2$:

$$\bar{Q}^{(+)}\Gamma Q^{(-)} = \bar{h}^{(+)}\Gamma h^{(-)}$$
$$+ \frac{1}{4m} \left(i\, \partial_\mu^\perp \left(\bar{h}^{(+)} [\Gamma, \gamma^\mu] h^{(-)} \right) + \bar{h}^{(+)} \left\{ \Gamma, i\, \overleftrightarrow{\slashed{D}^\perp} \right\} h^{(-)} \right)$$
$$+ \frac{1}{16m^2} \left(i\, \partial_\mu^\perp\, i\, \partial_\nu^\perp \left(\bar{h}^{(+)}\gamma^\mu\Gamma\gamma^\nu h^{(-)} \right) \right.$$
$$+ i\, \partial_\mu^\perp \left(\bar{h}^{(+)}\gamma^\mu\Gamma i\, \overrightarrow{\slashed{D}^\perp}\, h^{(-)} \right) - i\, \partial_\mu^\perp \left(\bar{h}^{(+)} i\, \overleftarrow{\slashed{D}^\perp}\, \Gamma\gamma^\mu h^{(-)} \right)$$
$$\left. -\bar{h}^{(+)} i\, \overleftarrow{\slashed{D}^\perp}\, \Gamma i\, \overrightarrow{\slashed{D}^\perp}\, h^{(-)} \right) + O\left(\frac{1}{m^3} \right) . \tag{114}$$

With the help of (113) and the equations of motion $i\, v{\cdot}\partial^\perp \left(\bar{h}^{(+)}\Gamma h^{(-)} \right) = 0 = \bar{h}^{(+)}\, v{\cdot}i\, \overleftrightarrow{\slashed{D}^\perp}$ $\Gamma h^{(-)}$ we find the following expansions of the bilinears up to and including order $1/m^2$:

$$\bar{Q}^{(+)} Q^{(-)} = \frac{1}{2\,m}\, A_6$$

$$\bar{Q}^{(+)} \gamma_5\, Q^{(-)} = A_1 + \frac{1}{16\,m^2}\, (-A_9 + 2\,A_{12} + A_{14} - A_{17})$$

$$\bar{Q}^{(+)} \gamma^\alpha\, Q^{(-)} = A_2^\alpha + \frac{1}{2\,m}\, v^\alpha\, A_3$$
$$+ \frac{1}{16\,m^2}\, (-A_{10}^\alpha + 2\,A_{11}^\alpha - 2\,A_{13}^\alpha + A_{15}^\alpha - A_{16}^\alpha + A_{18}^\alpha)$$

$$\bar{Q}^{(+)} \gamma_5\, \gamma^\alpha\, Q^{(-)} = -v^\alpha\, A_1 + \frac{1}{2\,m}\, (A_4^\alpha + A_7^\alpha)$$
$$+ \frac{1}{16\,m^2}\, v^\alpha\, (-A_9 + 2\,A_{12} + A_{14} - A_{17})$$

$$\bar{Q}^{(+)} (-i)\, \sigma^{\alpha\beta}\, Q^{(-)} = v^\alpha\, A_2^\beta - v^\beta\, A_2^\alpha$$
$$+ \frac{1}{2\,m}\, \left(A_5^{\beta\alpha} - A_5^{\alpha\beta} - A_8^{\alpha\beta} \right)$$
$$+ \frac{1}{16\,m^2}\, \left(v^\alpha\, \left(A_{10}^\beta - 2\,A_{11}^\beta + 2\,A_{13}^\beta - A_{15}^\beta + A_{16}^\beta - A_{18}^\beta \right) - (\alpha \leftrightarrow \beta) \right) . \tag{115}$$

Here we have introduced

$$A_1 = \bar{h}^{(+)}\, \gamma_5\, h^{(-)}$$

$$A_2^\alpha = \bar{h}^{(+)} \gamma^{\perp\alpha} h^{(-)}$$

$$A_3 = i\,\partial^{\perp\alpha} \left(\bar{h}^{(+)} \gamma_\alpha^\perp h^{(-)} \right)$$

$$A_4^\alpha = i\,\partial^{\perp\alpha} \left(\bar{h}^{(+)} \gamma_5 h^{(-)} \right)$$

$$A_5^{\alpha\beta} = i\,\partial^{\perp\alpha} \left(\bar{h}^{(+)} \gamma^{\perp\beta} h^{(-)} \right)$$

$$A_6 = \bar{h}^{(+)} i\,\overleftrightarrow{\slashed{D}^\perp}\, h^{(-)}$$

$$A_7^\alpha = i\,\epsilon^{\alpha\mu\nu\rho} v_\mu\, \bar{h}^{(+)} \gamma_\nu^\perp\, i\,\overleftrightarrow{D_\rho^\perp}\, h^{(-)}$$

$$A_8^{\alpha\beta} = i\,\epsilon^{\alpha\beta\mu\rho} v_\mu\, \bar{h}^{(+)} \gamma_5\, i\,\overleftrightarrow{D_\rho^\perp}\, h^{(-)}$$

$$A_9 = i\,\partial^{\perp\rho}\, i\,\partial_\rho^\perp\, \bar{h}^{(+)} \gamma_5 h^{(-)}$$

$$A_{10}^\alpha = i\,\partial^{\perp\rho}\, i\,\partial_\rho^\perp\, \bar{h}^{(+)} \gamma^{\perp\alpha} h^{(-)}$$

$$A_{11}^\alpha = i\,\partial^{\perp\alpha}\, i\,\partial^{\perp\rho}\, \bar{h}^{(+)} \gamma_\rho^\perp h^{(-)}$$

$$A_{12} = i\,\epsilon^{\alpha\mu\nu\rho} v_\alpha\, i\,\partial_\mu^\perp\, \bar{h}^{(+)} \gamma_\nu^\perp\, i\,\overleftrightarrow{D_\rho^\perp}\, h^{(-)}$$

$$A_{13}^\alpha = i\,\epsilon^{\alpha\mu\nu\rho} v_\mu\, i\,\partial_\nu^\perp\, \bar{h}^{(+)} \gamma_5\, i\,\overleftrightarrow{D_\rho^\perp}\, h^{(-)}$$

$$A_{14} = \bar{h}^{(+)} i\,\overleftrightarrow{D^{\perp\rho}}\, \gamma_5\, i\,\overleftrightarrow{D_\rho^\perp}\, h^{(-)}$$

$$A_{15}^\alpha = \bar{h}^{(+)} i\,\overleftrightarrow{D^{\perp\rho}}\, \gamma^{\perp\alpha}\, i\,\overleftrightarrow{D_\rho^\perp}\, h^{(-)}$$

$$A_{16}^\alpha = \bar{h}^{(+)} i\,\overleftrightarrow{D^{\perp\alpha}}\, i\,\overleftrightarrow{\slashed{D}^\perp}\, h^{(-)} + \bar{h}^{(+)} i\,\overleftrightarrow{\slashed{D}^\perp}\, i\,\overleftrightarrow{D^{\perp\alpha}}\, h^{(-)}$$

$$A_{17} = i\,\epsilon^{\alpha\rho\mu\sigma} v_\alpha\, \bar{h}^{(+)} i\,\overleftrightarrow{D_\rho^\perp}\, \gamma_\mu^\perp\, i\,\overleftrightarrow{D_\sigma^\perp}\, h^{(-)}$$

$$A_{18}^\alpha = i\,\epsilon^{\alpha\mu\rho\sigma} v_\mu\, \bar{h}^{(+)} i\,\overleftrightarrow{D_\rho^\perp}\, \gamma_5\, i\,\overleftrightarrow{D_\sigma^\perp}\, h^{(-)} \ . \tag{116}$$

B.2 Expansions of quatrilinears

With the help of the bilinears (116), we obtain $1/m$ expansions of quatrilinears $A_i \otimes \{g, \epsilon\} \otimes A_j^*$. Owing to the fact that quatrilinears have to be parity-even, and that total derivatives may be neglected, we find that there are two dimension-six operators, no dimension-seven operator, and 17 dimension-eight operators. Three more dimension-eight operators $(C_3 + C_4,\ D_3 + D_4,\ E_2 - E_3)$ appear only if also the kernel is expanded. Including colour, there are twice as many operators

$$O_i^{(8)} = T^a \otimes T^a\, O_i^{(1)} \ . \tag{117}$$

The dimension-six operators are

$$-A_1^{(1)} = A_1 A_1^*$$
$$A_2^{(1)} = A_2^\alpha A_{2\alpha}^* \ . \tag{118}$$

The dimension-eight operators that can be constructed from (116) are:

$$B_1^{(1)} = A_4^\alpha A_{4\alpha}^* = A_9 A_1^*$$
$$-B_2^{(1)} = A_3 A_3^* = A_{11}^\alpha A_{2\alpha}^*$$
$$-B_3^{(1)} = A_5^{\alpha\beta} A_{5\alpha\beta}^* = A_{10}^\alpha A_{2\alpha}^* \tag{119}$$

$$C_1^{(1)} = \frac{1}{2} \, i \, \epsilon^{\alpha\beta\mu\nu} \, v_\mu \, A_{8\alpha\beta} \, A_{4\nu}^*$$
$$-C_2^{(1)} = A_6 A_3^* + A_3 A_6^*$$
$$C_3^{(1)} - C_4^{(1)} = i \, \epsilon^{\alpha\beta\mu\nu} \, v_\mu \, A_{5\alpha\beta} \, A_{7\nu}^*$$
$$C_5^{(1)} = A_{12} A_1^* + A_1 A_{12}^* = - \left(A_7^\alpha A_{4\alpha}^* + A_4^\alpha A_{7\alpha}^* \right)$$
$$-C_6^{(1)} = A_{13}^\alpha A_{2\alpha}^* + A_2^\alpha A_{13\alpha}^* = A_8^{\alpha\beta} A_{5\alpha\beta}^* \tag{120}$$

$$-D_1^{(1)} = \frac{1}{2} A_8^{\alpha\beta} A_{8\alpha\beta}^*$$
$$-D_2^{(1)} = A_6 A_6^*$$
$$D_3^{(1)} - D_4^{(1)} = A_7^\alpha A_{7\alpha}^*$$
$$D_5^{(1)} = -\frac{1}{2} \, i \, \epsilon^{\alpha\beta\mu\nu} \, v_\mu \, A_{8\alpha\beta} \, A_{7\nu}^* + \text{h.c.} \tag{121}$$

$$-E_1^{(1)} = A_{14} A_1^* + A_1 A_{14}^*$$
$$E_2^{(1)} + E_3^{(1)} = A_{16}^\alpha A_{2\alpha}^* + A_2^\alpha A_{16\alpha}^*$$
$$E_4^{(1)} = A_{15}^\alpha A_{2\alpha}^* + A_2^\alpha A_{15\alpha}^*$$
$$E_5^{(1)} = A_{17} A_1^* + A_1 A_{17}^*$$
$$-E_6^{(1)} = A_{18}^\alpha A_{2\alpha}^* + A_2^\alpha A_{18\alpha}^* \ . \tag{122}$$

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B.3 Two-component representation

In the rest frame of the quarkonium we have $v^\mu = (1, \vec{0})$. Using the Dirac representation of the gamma-matrices let us introduce the two-component spinors ψ and χ via

$$h^{(+)} = \begin{pmatrix} \psi \\ 0 \end{pmatrix} \quad , \quad h^{(-)} = \begin{pmatrix} 0 \\ \chi \end{pmatrix} . \tag{123}$$

Then we can express the bilinears as follows

$$A_1 = \psi^\dagger \chi$$

$$\vec{A}_2 = \psi^\dagger \vec{\sigma} \chi \quad (A_2^0 = 0)$$

$$A_3 = -i \, \vec{\partial} \cdot \left(\psi^\dagger \vec{\sigma} \chi \right)$$

$$\vec{A}_4 = i \, \vec{\partial} \left(\psi^\dagger \chi \right) \quad (A_4^0 = 0)$$

$$A_5^{ij} = i \, \partial^i \left(\psi^\dagger \sigma^j \chi \right)$$

$$A_6 = -\psi^\dagger i \, \overset{\leftrightarrow}{\vec{D}} \cdot \vec{\sigma} \chi$$

$$A_7^i = -i \, \psi^\dagger \left(\vec{\sigma} \times i \, \overset{\leftrightarrow}{\vec{D}} \right)_i \chi$$

$$A_8^{ij} = i \, \epsilon^{ijk} \psi^\dagger i \, \overset{\leftrightarrow}{D_k} \chi$$

$$A_9 = - \left(i \, \vec{\partial} \right)^2 \psi^\dagger \chi$$

$$\vec{A}_{10} = - \left(i \, \vec{\partial} \right)^2 \psi^\dagger \vec{\sigma} \chi$$

$$\vec{A}_{11} = -i \, \vec{\partial} \left(i \, \vec{\partial} \cdot \left(\psi^\dagger \vec{\sigma} \chi \right) \right)$$

$$A_{12} = i \, \epsilon^{ijk} i \, \partial_i \psi^\dagger \sigma_j \, i \, \overset{\leftrightarrow}{D_k} \chi$$

$$A_{13}^i = -i \, \epsilon^{ijk} i \, \partial_j \psi^\dagger i \, \overset{\leftrightarrow}{D_k} \chi$$

$$A_{14} = -\psi^\dagger \left(i \, \overset{\leftrightarrow}{\vec{D}} \right)^2 \chi$$

$$\vec{A}_{15} = -\psi^\dagger \vec{\sigma} \left(i \, \overset{\leftrightarrow}{\vec{D}} \right)^2 \chi$$

$$\vec{A}_{16} = -\psi^\dagger \left[i \, \overset{\leftrightarrow}{\vec{D}} \left(i \, \overset{\leftrightarrow}{\vec{D}} \cdot \vec{\sigma} \right) + \left(i \, \overset{\leftrightarrow}{\vec{D}} \cdot \vec{\sigma} \right) i \, \overset{\leftrightarrow}{\vec{D}} \right] \chi$$

$$A_{17} = i \, \epsilon^{ijk} \psi^\dagger i \, \overset{\leftrightarrow}{D_i} \sigma_j \, i \, \overset{\leftrightarrow}{D_k} \chi$$

$$A_{18}^k = -i\,\epsilon^{ijk}\,\psi^\dagger\,i\,\overleftrightarrow{D_i}\,i\,\overleftrightarrow{D_k}\,\chi\,. \tag{124}$$

The complex-conjugate expressions are obtained from (124) by the replacements $\psi^\dagger \to \chi^\dagger$, $\chi \to \psi$, $i \to -i$ (using $\sigma_i^\dagger = \sigma_i$). The quatrilinears are obtained using $\epsilon_{ijk}\,\epsilon_{ijl} = 2\,\delta_{kl}$. The connection to the operators of ref. [5] is then straightforward ($c = 1$ or 8)

$$A_1^{(c)} = -\mathcal{O}_c(^1S_0)$$
$$A_2^{(c)} = -\mathcal{O}_c(^3S_1)$$
$$\frac{1}{4}\,D_1^{(c)} = \mathcal{O}_c(^1P_1)$$
$$\frac{1}{4}\,D_2^{(c)} = 3\,\mathcal{O}_c(^3P_0)$$
$$\frac{1}{4}\,D_3^{(c)} = \mathcal{O}_c(^3P_0) + \mathcal{O}_c(^3P_1) + \mathcal{O}_c(^3P_2)$$
$$\frac{1}{4}\,D_4^{(c)} = \mathcal{O}_c(^3P_0) - \mathcal{O}_c(^3P_1) + \mathcal{O}_c(^3P_2)$$
$$\frac{1}{4}\,E_1^{(c)} = 2\,\mathcal{P}_c(^1S_0)$$
$$\frac{1}{4}\,E_4^{(c)} = 2\,\mathcal{P}_c(^3S_1)$$
$$\frac{1}{4}\left[E_2^{(c)} + E_3^{(c)}\right] = 4\,\mathcal{P}_c(^3S_1,{}^3D_1) + \frac{4}{3}\,\mathcal{P}_c(^3S_1)\,. \tag{125}$$

B.4   Matching calculation

The short-distance coefficients appearing in (49) and (55) at the scale $\mu = m$ are obtained from the annihilation part of the scattering amplitude $\mathcal{M}$ for $Q\bar{Q} \to Q\bar{Q}$ computed in full QCD. The amplitude, calculated in QCD perturbation theory for on-shell quarks and antiquarks, is Taylor-expanded in the "small" components of the heavy quark momenta. The small component of a momentum is defined as the component perpendicular to the direction of the quarkonium momentum Mv. Denoting the momenta of the initial $Q\bar{Q}$ pair by p and $\bar{p}$ for Q and $\bar{Q}$, respectively, and the ones of the final $Q\bar{Q}$-pair by p' and $\bar{p}'$ for Q and $\bar{Q}$, respectively we have:

$$p = v\sqrt{m^2 - p_\perp^2} + p_\perp$$
$$= mv\left[1 - \frac{1}{2m^2}\,(P+\pi)^2 + \ldots\right] + P + \pi$$

$$\bar{p} = mv \left[1 - \frac{1}{2m^2} \left(P - \pi \right)^2 + \ldots \right] + P - \pi$$

$$p' = mv \left[1 - \frac{1}{2m^2} \left(P + \eta \right)^2 + \ldots \right] + P + \eta$$

$$\bar{p}' = mv \left[1 - \frac{1}{2m^2} \left(P - \eta \right)^2 + \ldots \right] + P - \eta \ , \tag{126}$$

where $\pi^2 = \eta^2$. Note that both the kernel and the spinors have to expanded to the desired order. The latter expansion follows from (25). For example, up to order $1/m^2$ we have

$$Q^{(+)} = \left(1 + \frac{P^2 + \pi^2}{8m^2} \right) \left(1 + \frac{\slashed{P} + \slashed{\pi}}{2m} \right) P_+ \ . \tag{127}$$

The result of this procedure is then matched to the operators in (49) and (55) by identifying the momenta with the derivatives appearing in the operators

$$\bar{h}^{(+)} \, \Gamma \, i \, \overleftrightarrow{D}_\mu \, h^{(-)} \rightarrow \bar{h}^{(+)} \, \Gamma \, \eta_\mu \, h^{(-)}$$

$$\bar{h}^{(-)} \, \Gamma \, i \, \overleftrightarrow{D}_\mu \, h^{(+)} \rightarrow \bar{h}^{(-)} \, \Gamma \, \pi_\mu \, h^{(+)}$$

$$i \partial_\mu \left(\bar{h}^{(\pm)} \, \Gamma \, h^{(\mp)} \right) \rightarrow \bar{h}^{(\pm)} \, \Gamma \, P_\mu \, h^{(\mp)} \ . \tag{128}$$

Then the coefficients $\mathcal{C}(O_i^{(c)})$ of (55) can be read off from the dimension-eight contribution to the amplitude

$$\mathcal{M}_{d=8} = \left(\frac{1}{2m} \right)^2 \sum_{c=1,8} \left[\sum_{i=1}^{3} \mathcal{C}(B_i^{(c)}, m) \, B_i^{(c)} + \sum_{i=1}^{6} \mathcal{C}(C_i^{(c)}, m) \, C_i^{(c)} \right.$$

$$\left. + \sum_{i=1}^{5} \mathcal{C}(D_i^{(c)}, m) \, D_i^{(c)} + \sum_{i=1}^{6} \mathcal{C}(E_i^{(c)}, m) \, E_i^{(c)} \right]$$

$$\equiv \frac{1}{4m^2} \sum_{c=1,8} X_c \left\{ \mathcal{C}(B_1^{(c)}, m) \, P^2 \, \gamma_5 \otimes \gamma_5 \right.$$

$$+ \mathcal{C}(B_2^{(c)}, m) \, \slashed{P} \otimes \slashed{P} + \mathcal{C}(B_3^{(c)}, m) \, P^2 \, \gamma^\alpha \otimes \gamma_\alpha$$

$$+ \mathcal{C}(C_1^{(c)}, m) \, (\pi + \eta) \cdot P \, \gamma_5 \otimes \gamma_5 + \mathcal{C}(C_2^{(c)}, m) \, [\slashed{\pi} \otimes \slashed{P} + \slashed{P} \otimes \slashed{\eta}]$$

$$+ \mathcal{C}(C_3^{(c)}, m) \, (\pi + \eta) \cdot P \, \gamma^\alpha \otimes \gamma_\alpha + \mathcal{C}(C_4^{(c)}, m) \, [\slashed{P} \otimes \slashed{\pi} + \slashed{\eta} \otimes \slashed{P}]$$

$$+ \mathcal{C}(C_5^{(c)}, m) \, (-i) \, \epsilon^{\alpha\beta\gamma\delta} \, v_\alpha \, P_\beta \, [\pi_\gamma \, \gamma_\delta \otimes \gamma_5 + \eta_\gamma \, \gamma_5 \otimes \gamma_\delta]$$

$$+ \mathcal{C}(C_6^{(c)}, m) \, (-i) \, \epsilon^{\alpha\beta\gamma\delta} \, v_\alpha \, P_\beta \, [\pi_\gamma \, \gamma_5 \otimes \gamma_\delta + \eta_\gamma \, \gamma_\delta \otimes \gamma_5]$$

$$+ \mathcal{C}(D_1^{(c)}, m) \, \pi \cdot \eta \, \gamma_5 \otimes \gamma_5 + \mathcal{C}(D_2^{(c)}, m) \, \slashed{\pi} \otimes \slashed{\eta}$$

$$+\mathcal{C}(D_3^{(c)},m)\,\pi\cdot\eta\,\gamma^\alpha\otimes\gamma_\alpha+\mathcal{C}(D_4^{(c)},m)\,\slashed{\eta}\otimes\slashed{\pi}$$
$$+\mathcal{C}(D_5^{(c)},m)\,(-i)\,\epsilon^{\alpha\beta\gamma\delta}\,v_\alpha\,\pi_\beta\,\eta_\gamma\,[\gamma_5\otimes\gamma_\delta+\gamma_\delta\otimes\gamma_5]$$
$$+\mathcal{C}(E_1^{(c)},m)\,\left[\pi^2+\eta^2\right]\gamma_5\otimes\gamma_5+\left\{\mathcal{C}(E_2^{(c)},m)+\mathcal{C}(E_3^{(c)},m)\right\}[\slashed{\pi}\otimes\slashed{\pi}+\slashed{\eta}\otimes\slashed{\eta}]$$
$$+\mathcal{C}(E_4^{(c)},m)\,\left[\pi^2+\eta^2\right]\gamma^\alpha\otimes\gamma_\alpha\Big\}\;, \tag{129}$$

where we suppressed the spinors and abbreviated the colour structure

$$X_1=1\otimes 1\quad,\quad X_8=T^a\otimes T^a\;. \tag{130}$$

Note that E_5 and E_6 give only field-strength terms, i.e. vanish for commuting Dirac matrices. The dimension-six part is simply

$$\begin{aligned}\mathcal{M}_{d=6}&=\frac{1}{m^2}\sum_{c=1,8}\sum_{i=1}^{2}\mathcal{C}(A_i^{(c)},m)\,A_i^{(c)}\\&\equiv\frac{1}{m^2}\sum_{c=1,8}X_c\left\{\mathcal{C}(A_1^{(c)},m)\,\gamma_5\otimes\gamma_5+\mathcal{C}(A_2^{(c)},m)\,\gamma^\alpha\otimes\gamma_\alpha\right\}\;.\end{aligned} \tag{131}$$

C   Operators in terms of quarkonia quantum numbers

In order to calculate the hadronic matrix elements using vacuum insertion, it is convenient to re-express the operators B, C and D in terms of other operators in which for both of the heavy-quark bilinears the "orbital angular momentum" corresponding to the derivative is coupled to the total spin of the quarks to some total angular momentum J. If ∇_μ is either the RRM or the RCM derivative ($i\overleftrightarrow{D}_\mu$ or $i\partial_\mu$, respectively), these couplings are

$$\begin{aligned}{}^1P_1\otimes{}^1P_1&\cong\left(\bar{h}^{(-)}\gamma_5\nabla_\mu h^{(+)}\right)\left(\bar{h}^{(-)}\gamma_5\nabla^\mu h^{(+)}\right)\\{}^3P_0\otimes{}^3P_0&\cong\frac{1}{3}\left(\bar{h}^{(-)}\gamma^\mu\nabla_\mu h^{(+)}\right)\left(\bar{h}^{(-)}\gamma^\nu\nabla_\nu h^{(+)}\right)\\{}^3P_1\otimes{}^3P_1&\cong\frac{1}{2}\,i\varepsilon_{\alpha\beta\mu\nu}v^\beta\left(\bar{h}^{(-)}\gamma^\mu\nabla^\nu h^{(+)}\right)i\varepsilon^{\alpha\beta'\mu'\nu'}v_{\beta'}\left(\bar{h}^{(-)}\gamma_{\mu'}\nabla_{\nu'}h^{(+)}\right)\\{}^3P_2\otimes{}^3P_2&\cong\left(\bar{h}^{(-)}\gamma^{[\mu}\nabla^{\nu]}h^{(+)}\right)\left(\bar{h}^{(-)}\gamma_{[\mu}\nabla_{\nu]}h^{(+)}\right)\end{aligned} \tag{132}$$

where the bracket denotes the symmetric traceless combination of a tensor (cf. (111,112))

$$X_\perp^{[\mu\nu]}=\frac{1}{2}\left(X_\perp^{\mu\nu}+X_\perp^{\nu\mu}\right)-\frac{1}{3}\,g_\perp^{\mu\nu}\,X_{\perp\mu}^\mu\;. \tag{133}$$

The first operator of the three sets B, C and D is already the $^1P_1$ combination; the other three may be rewritten as linear combinations

$$\begin{pmatrix} ^3P_0 \otimes {}^3P_0 \\ ^3P_1 \otimes {}^3P_1 \\ ^3P_2 \otimes {}^3P_2 \end{pmatrix} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & -1/2 \\ -1/3 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} \nabla_\mu \gamma^\mu \otimes \nabla^\nu \gamma_\nu \\ \nabla_\mu \gamma_\nu \otimes \nabla^\mu \gamma^\nu \\ \nabla_\mu \gamma_\nu \otimes \nabla^\nu \gamma^\mu \end{pmatrix} . \tag{134}$$

By inverting (134) we have

$$\begin{pmatrix} \nabla_\mu \gamma^\mu \otimes \nabla^\nu \gamma_\nu \\ \nabla_\mu \gamma_\nu \otimes \nabla^\mu \gamma^\nu \\ \nabla_\mu \gamma_\nu \otimes \nabla^\nu \gamma^\mu \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} ^3P_0 \otimes {}^3P_0 \\ ^3P_1 \otimes {}^3P_1 \\ ^3P_2 \otimes {}^3P_2 \end{pmatrix} . \tag{135}$$

Explicitly, for the D-operators we have, for example,

$$D^{(c)}(^1P_1) = D_1^{(c)}$$
$$D^{(c)}(^3P_0) = \frac{1}{3} D_2^{(c)}$$
$$D^{(c)}(^3P_1) = \frac{1}{2} \left\{ D_3^{(c)} - D_4^{(c)} \right\}$$
$$D^{(c)}(^3P_2) = \frac{1}{2} \left\{ D_3^{(c)} + D_4^{(c)} \right\} - \frac{1}{3} D_2^{(c)} . \tag{136}$$

Similarly we can define operators with appropriate S-wave quantum numbers

$$E^{(c)}(^1S_0) = \frac{1}{2} E_1^{(c)}$$
$$E^{(c)}(^3S_1) = \frac{1}{2} E_4^{(c)}$$
$$E^{(c)}(^3S_1, {}^3D_1) = \frac{1}{4} \left(E_2^{(c)} + E_3^{(c)} \right) - \frac{1}{6} E_4^{(c)} . \tag{137}$$

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