Experimental limits to the density of dark matter in the solar system

Øyvind Grøn
Oslo College, Faculty of Engineering, Cort Adelers Gate 30, N-0254 Oslo, Norway
Institute of Physics, University of Oslo, P.O. Box 1048, N-0316 Blindern, Oslo 3, Norway

Harald H. Soleng[1]
CERN, Theory Division, CH-1211 Geneva 23, Switzerland

(November 16, 1994)

Abstract

On the scales of galaxies and beyond there is evidence for unseen dark matter. In this paper we find the experimental limits to the density of dark matter bound in the solar system by studying its effect upon planetary motion.

Subject Headings: Gravitation — Dark Matter (solar system tests)
PACS numbers: 04.20.-q 95.35.+d

CERN-TH.7500/94
November 1994

Typeset using REVTEX

[1]Electronic address: Soleng@surya11.cern.ch

1. INTRODUCTION

According to Newton's inverse square force law, the circular speed around an isolated object of mass M should be

$$v_c = \sqrt{\frac{MG}{r}}. \tag{1}$$

In disk galaxies we do, however, observe that the circular speeds are approximately independent of r at large distances. The standard explanation is that this is due to halos of unseen matter that makes up around 90% of the total mass of the galaxies (Tremaine 1992).The same pattern repeats itself on larger and larger scales, until we reach the cosmic scales where a baryonic density compatible with successful big bang nucleosynthesis is less than 10% of the density predicted by inflation, i.e. the critical density.

The flat rotation curves of galaxies, taken at face value, imply that the effective gravitational force follows a $1/r$ law at large scales. This could either be due to dark matter or to a departure from Newtonian dynamics at small accelerations (Milgrom 1983; Bekenstein 1992) or large scales (Sanders 1990). An effective gravitational acceleration law of the form

$$g = -\frac{\sqrt{GMa_0}}{r} \tag{2}$$

at small accelerations $a \ll a_0$ has been reported (Kent 1987; Milgrom 1988; Begeman, Broeils, & Sanders 1991) to be successful in reproducing the observations of galactic systems.[2].The constant a_0 has been determined by studies of galaxy rotation curves and its value has been found to be $a_0 \approx 10^{-8}$ cm s^{-2}. As noted by Milgrom (1983), this value of $a_0 \approx cH_0$.

With such a $1/r$ force law the circular speed would approach

$$v_c = (GMa_0)^{1/4}. \tag{3}$$

If the luminosity L of a galaxy is proportional to its mass M, then this relation would explain the infrared Tully–Fisher law (Tully & Fisher 1977) which states that circular speeds in galaxies scale as

$$v_c \propto L^{1/4}. \tag{4}$$

The theoretical underpinning for this effective force law is missing. It might be due to a modification of gravity along the lines of Milgrom (1983), but the standard view is that it is caused by dark matter. At this point, it is worth mentioning that a large-distance force law of this type can be reproduced within standard general relativity theory with a very simple, but perhaps unrealistic, matter source (Soleng 1993, 1994a). Our key point is that general relativity is quite capable of explaining the observed gravitational properties of the universe provided we give it the right input. Most likely the dark matter is a mixture of several

---

[2]However, not without debate (Lake 1989; Milgrom 1991).

components, such as weakly interacting particles, black holes, brown dwarfs, neutron stars, as well as energy stored in high-frequency oscillation of Newton's gravitational coupling (Accetta & Steinhardt 1991; Steinhardt & Will, 1994). Whatever the origin of the $1/r$ force law might be, its reported experimental success forces us to take it seriously. Accordingly, we think that it is particularly important to compare the densities of dark matter inferred from large scale dynamics with experimental limits from local tests. If dark matter exists in the form of *microscopic* objects, one would expect that this unknown form of energy penetrates into galaxies and also enter the solar system.

In this paper we focus on a class of dark matter models that have a density profile obeying a power-law. It is well known that a non-zero energy-distribution outside the central mass produces a perihelion precession of the orbits of satellites already at the Newtonian level. Indeed, the gravitational perturbations from Venus, the Earth and Jupiter account for the major part of the observed perihelion precession of Mercury. The general relativistic effect of $43''$ per century represents an additional 8% precession. Since the solar system is characterized by weak gravitational fields and velocities much lower that of light, it is enough to consider the modification to the precession coming from a dark-matter-induced perturbation of g_{tt}. The exact form of g_{rr} is needed only for stronger fields.

2. OUR MODEL

In order to study the gravitational effects of hypothetical dark matter on planetary motion, we need a solution of Einstein's field equations for a static, spherically symmetric metric and a given distribution of dark matter. The line-element for a static, spherically symmetric gravitational field can in general be written as[3]

$$ds^2 = -e^{2\mu}dt^2 + e^{2\lambda}dr^2 + r^2 d\Omega^2. \tag{5}$$

Since we are interested only in the dark-matter-induced perturbation to $g_{tt} = -e^{2\mu}$, we are allowed to assume the symmetry

$$\mu = -\lambda. \tag{6}$$

In principle, this means a restriction of the equation of state of the dark matter fluid, which in this case becomes an imperfect fluid with an anisotropic pressure and radial boost invariance (Soleng 1994b), but it should be noted that for the weak field and low velocities we are considering, the value of the perihelion precession would be the same for a different equation of state.

The dark matter density will be assumed to have a density profile given by $\rho \propto r^{-2+2\beta}$, where β is a dimensionless constant. This distribution covers an interesting part of parameter space ranging from a cosmological constant at $\beta = 1$ with the mass increasing as r^3 to the $1/r^2$ distribution with mass increasing as r, which is needed to explain the flat rotation curves of galaxies at $\beta \to 0$.

---

[3]In the rest of the paper we shall employ geometrized units with $G = c = 1$.

Let us therefore consider a metric with

$$e^{2\mu} = 2\alpha - \frac{2M}{r} - \left(\frac{\ell}{r}\right)^{2\beta}, \tag{7}$$

where α is a dimensionless constant. For this line-element, Einstein's field equations give the following energy-momentum tensor,

$$8\pi T^t{}_t = 8\pi T^r{}_r = \frac{2\alpha - 1 - (2\beta - 1)\left(\frac{\ell}{r}\right)^{2\beta}}{r^2} \tag{8a}$$

and

$$8\pi T^\Omega{}_\Omega = \frac{(1-2\beta)\beta}{r^2}\left(\frac{\ell}{r}\right)^{2\beta}. \tag{8b}$$

[-1.0em]

The metric defined by equations (5)–(7) reduces to the Schwarzschild metric if $\alpha \to 1$ and $\beta \to 0$. Other special cases are $\beta = -1$ and $\alpha = 1/2$, which is the Schwarzschild–de Sitter metric. At least approximately, the metric of the solar system should be asymptotically Minkowskian except if $\beta = -1$, in which case it must be asymptotically (anti)-de Sitter. For all other values of β the metric is asymptotically Minkowskian, provided $\alpha = 1/2$. One should, however, note that if $\beta \approx 0$, then this holds only in a formal sense. Assume, for instance, that $\beta \approx 10^{-12}$. In this case, independent of the value of ℓ as long as it stays within the physically reasonable range of $\ell_{\rm Planck} < \ell < H_0^{-1}$, the term $(\ell/r)^{2\beta}$ is unity to an extremely good approximation. Hence, it is clear that for $|\beta|$ very close to zero, $\alpha = 1$ in order to have the proper behaviour of the metric for large r.

In the Newtonian limit the gravitational potential is given by $\Phi = \frac{1}{2}e^{2\mu}$. Consequently, the gravitational acceleration of a test-particle is given by

$$g = \frac{M}{r^2} + \frac{\beta}{r}\left(\frac{\ell}{r}\right)^{2\beta}. \tag{9}$$

Before looking at the consequences for planetary motion in the case $\beta \neq 0$, it is useful to look at the energy and the gravitational mass density or the density of Tolman's mass (Tolman 1930) that are required to produce such a metric.

For $\alpha = 1/2$, the corresponding energy density from equation (8a) is

$$8\pi\rho = \frac{(1-2\beta)}{r^2}\left(\frac{\ell}{r}\right)^{2\beta}, \tag{10}$$

which means that $\beta < 1/2$. and the gravitational mass density computed from equations (8) is

$$8\pi\rho_{\rm grav} = \frac{2\beta(1-2\beta)}{r^2}\left(\frac{\ell}{r}\right)^{2\beta}, \tag{11}$$

which implies that $\beta \geq 0$ to avoid a negative gravitational mass density.

For $\alpha = 1$ and $|\beta| \ll 1$, the energy density is

$$8\pi\rho = -\frac{2\beta}{r^2}, \tag{12}$$

which means that now $\beta \leq 0$. Also for $\alpha = 1$, the gravitational mass density is given by equation (11). In this case, if the energy density is positive, the gravitational mass density is always negative.

This seems to exclude an attractive $1/r$ force, if the weak energy condition is to be satisfied. This is, however, *not* the case. We have here *assumed* $\lambda = -\mu$ and specified μ, but in the general case of equation (5) the energy density is a function of λ, λ' and r, only. With a different $\lambda(r)$, and a suitable equation of state, it is therefore possible to get an attractive $1/r$ force in general relativity. Because of this loophole, we shall also consider the case $\alpha = 1$ with $\beta \approx 0^+$.

3. PERIHELION PRECESSION

The Lagrange function for a test particle moving in the $\theta = \pi/2$ plane in the geometry specified by equations (5) and (6), is

$$2L = -e^{2\mu}\dot{t}^2 + e^{-2\mu}\dot{r}^2 + r^2\dot{\phi}^2. \tag{13}$$

A dot stands for a derivative with respect to proper time, τ. The momenta $P_x \equiv \partial L/\partial \dot{x}$ are

$$P_t = -e^{2\mu}\dot{t} \tag{14a}$$

$$P_r = e^{-2\mu}\dot{r} \tag{14b}$$

$$P_\phi = r^2\dot{\phi}, \tag{14c}$$

[-1.0em]

where P_t and P_ϕ are constants. The constancy of P_ϕ can be used to rewrite the derivatives with respect to proper time in terms of derivatives with respect to ϕ. Hence, we get

$$\frac{d}{d\tau} = \frac{P_\phi\, d}{r^2 d\phi}. \tag{15}$$

Using the normalization of the momenta

$$g_{\mu\nu}P^\mu P^\nu = -1, \tag{16}$$

the relation (15), the expression (7), and the definition

$$u \equiv 1/r, \tag{17}$$

equation (16) takes the form

$$P_\phi^2(u')^2 + \left[2\alpha - 2Mu - (\ell u)^{2\beta}\right]\left[1 + P_\phi^2 u^2\right] = P_t^2 \tag{18}$$

where the prime stands for a derivation with respect to ϕ.

Differentiation of equation (18), using the fact that P_t and P_ϕ are constants, leads to

$$u'' + 2\alpha u = \frac{M}{P_\phi^2} + 3Mu^2 + \beta\frac{(\ell u)^{2\beta}}{uP_\phi^2} + (\beta+1)(\ell u)^{2\beta}u. \tag{19}$$

The planetary orbits are nearly circular, and we can treat the perihelion precession as a perturbation from the circular solution. Let therefore the circular solution be given by

$$2\alpha u_0 = \frac{M}{P_\phi^2} + 3Mu_0^2 + \beta\frac{(\ell u_0)^{2\beta}}{u_0 P_\phi^2} + (\beta+1)(\ell u_0)^{2\beta}u_0. \tag{20}$$

By defining

$$u = u_0(1+\varepsilon), \tag{21}$$

with $\varepsilon \ll 1$, inserting it in the equation of motion (19), and using equation (20) to eliminate the zeroth-order terms, we get the first-order expression

$$\varepsilon'' = \left\{-2\alpha + 6Mu_0 + (\ell u_0)^{2\beta}\left[1 + 3\beta + 2\beta^2 + (2\beta^2 - \beta)(P_\phi u_0)^{-2}\right]\right\}\varepsilon. \tag{22}$$

Let us first consider the case when $0 < \beta \ll 1$. Then $\alpha = 1$ and $(\ell u)^{2\beta} \approx 1$, and consequently equation (22) reduces to

$$\varepsilon'' + \varepsilon = \left(6Mu_0 - \beta P_\phi^{-2}u_0^{-2}\right)\varepsilon \tag{23}$$

where the right-hand side represents the perihelion precession term. The Einstein term coming from the solar mass $M = M_\odot$ is

$$\Delta\phi_0 = 6\pi M_\odot u_0. \tag{24}$$

In addition, there is a dark matter term. Note that $(P_\phi u_0)^2 = (r_0\dot{\phi})^2 = v^2 = M_\odot u_0 \ll 1$.

$$\Delta\phi_{\text{dark}} = -\pi\frac{\beta}{M_\odot u_0}. \tag{25}$$

Let the observed "anomalous", that is, the non-Newtonian perihelion precession be denoted by $\Delta\phi_{\text{obs}}$, and its uncertainty by $\delta\phi_{\text{obs}}$. Since $\Delta\phi_{\text{obs}} = \Delta\phi_0$ within the uncertainty, the dark matter perihelion precession and the Einstein term are related by

$$|\Delta\phi_{\text{dark}}| \leq |\delta\phi_{\text{obs}}|. \tag{26}$$

For the case $\alpha = 1$ and $0 < \beta \ll 1$, we get

$$\beta \leq \frac{1}{6\pi^2}\,\Delta\phi_{\text{obs}}\,|\delta\phi_{\text{obs}}|. \tag{27}$$

In general, if an experimental limit for β has been established for one object, say Mercury, then the limit for another planet, B, is given by

$$\beta_B = \frac{(\Delta\phi_{\rm obs})_B}{(\Delta\phi_{\rm obs})_M}\frac{|\delta\phi_{\rm obs}|_B}{|\delta\phi_{\rm obs}|_M}\beta_M. \tag{28}$$

A good bound on β is therefore obtained by a high-precision measurement of the orbit of an object with a small perihelion precession. For Mercury $M_\odot u_0 \approx 2.5 \times 10^{-8}$, $\delta\phi_{\rm obs} \approx 1''$ per century, and $\Delta\Phi_{\rm obs} = 43''$ per century, giving

$$\beta_M \lesssim 10^{-16}. \tag{29}$$

The data cited by Weinberg (1972) give $\beta_V = 2\beta_M$, $\beta_E = 0.3\beta_M$, $\beta_I = 0.5\beta_M$ for Venus, Earth, and Icarus, respectively. Hence, with these data, the best limit is found for the Earth's orbit. Using equation (11), this corresponds to a gravitational mass density at 1 A.U. of

$$\rho_{\rm dark} \lesssim 10^{-17} \text{ g/cm}. \tag{30}$$

In terms of the expression in equation (2), we find

$$a_0 = \frac{\beta^2}{M_\odot} \lesssim 5 \times 10^{-18} \text{ cm/s}^2, \tag{31}$$

which is six orders of magnitude below the value for a_0 quoted on galactic scales (Kent 1987; Milgrom 1988; Begeman et al. 1991). If we had used a logarithmic term in the metric, we could have obtained an exact $1/r$ contribution to the effective gravitational acceleration. A metric of this type gives the same result.

Let now $0 < \beta < 1/2$ and $\alpha = 1/2$. If $2\beta^2 - \beta \gg (Mu_0)^2$, which is the case unless $\beta \approx 1/2$, we get

$$\Delta\phi_{\rm dark} = \pi(2\beta^2 - \beta)\frac{(\ell u_0)^{2\beta}}{Mu_0}. \tag{32}$$

Let us for simplicity parametrize the length parameter ℓ by

$$\eta \equiv \ell/M. \tag{33}$$

Then we find

$$\eta \leq \left[\frac{6(Mu_0)^{2-2\beta}}{\beta - 2\beta^2}\frac{|\delta\phi_{\rm obs}|}{\Delta\phi_{\rm obs}}\right]^{1/(2\beta)}. \tag{34}$$

For Mercury, which in this case gives the best bound, the experimental limit varies from $\eta \lesssim 10^{-71}$ or $\ell \lesssim 10^{-66}$ cm at $\beta = 0.1$ to $\eta \lesssim 10^{-9}$ or $\ell \lesssim 10^{-4}$ cm at $\beta = 0.45$. The limit on $\rho_{\rm dark}$ does not depend on β and corresponds to the same restriction on the density as in equation (30).

The case when $\beta \approx 1/2$ and $\alpha = 1/2$ becomes equivalent to a change of the central mass. It is therefore not so interesting as a dark matter model.

Finally, let $-1 \leq \beta < 0$ and $\alpha = 1/2$. Using the η parameter as defined in equation (33), we find

$$\eta \geq \left[\frac{6(Mu_0)^{2-2\beta}}{\beta - 2\beta^2} \frac{|\delta\phi_{\rm obs}|}{\Delta\phi_{\rm obs}} \right]^{1/(2\beta)} . \tag{35}$$

For Mercury the experimental limit is varying from $\eta \gtrsim 10^{16}$ or $\ell \gtrsim 10^{21}$ cm for $\beta = -1$ via $\eta \gtrsim 10^{26}$ or $\ell \gtrsim 10^{31}$ cm at $\beta = -0.45$ to $\eta \gtrsim 10^{87}$ or $\ell \gtrsim 10^{92}$ cm when $\beta = -0.1$. The limit on $\rho_{\rm dark}$ is also in this case given by equation (30).

One should note that either must β be very close to zero, so that the magnitude of ℓ effectively becomes unimportant, or ℓ must have values which seem to be unnaturally small such as $\ell < 10^{-66}$ cm for $\beta = 0.1$, much smaller than the Planck length, or very large such as $\ell > 10^{92}$ cm for $\beta = -0.1$. In any case, the experimental limit to the local density of dark matter is the same.

4. CONCLUDING REMARKS

We have found that solar system experiments put very strict limits on possible corrections to the $1/r^2$ force law in the solar system. An extrapolation of the large distance $\sqrt{MGa_0}/r$ law which has been reported to be successful in explaining the flat galactic rotation curves, does only work if the parameter a_0 is several orders of magnitude smaller at planetary scales. In terms of the dark matter interpretation, this result shows that the density profile which must be of the form $1/r^2$ at galactic scales, must increase less rapidly towards small radii.

Braginsky, Gurevich, and Zybin (1992) have studied the effect of dark matter bound in the galaxy but unbound to the solar system. Such unbound dark matter would produce an anisotropy in the gravitational background of the solar system. The resulting tidal forces induce an additional perihelion precession. They found a limit to the density of unbound dark matter in the solar system of the order of 10^{-24} g/cm^3. Hence, the limit on unbound dark matter is much stronger than the limit on bound dark matter.

REFERENCES

Accetta, F. & Steinhardt, P. J. 1991, Phys. Rev. Lett., 67, 298

Begeman, K. G., Broeils, A. H., & Sanders, R. H. 1991, MNRAS, 249, 523

Bekenstein, J. 1992, in Proceedings of the 6th Marcel Grossmann Meeting on General Relativity, ed. H. Sato & T. Nakamura (Singapore: World Scientific Publ.), 905

Braginsky, V. B., Gurevich, A. V., & Zybin, K. P. 1992, Phys. Lett. A, 171, 275

Kent, S. M. 1987, AJ, 93, 816

Lake, G. 1989, ApJ, 345, L17

Milgrom, M. 1983, ApJ, 270, 365

Milgrom, M. 1988, ApJ, 333, 689

Milgrom, M. 1991, ApJ, 367, 490

Sanders, R. H. 1990, Astron. Astrophys. Rev., 2, 1

Soleng, H. H. 1993, BAAS, 25, 796

Soleng, H. H. 1994a, preprint

Soleng, H. H. 1994b, Gen. Rel. Grav., 26, 149

Steinhardt, P. J. & Will, C. M. 1994, preprint

Tolman, R. C. 1930, Phys. Rev., 35, 875

Tremaine, S. 1992, Physics Today, 45, 28

Tully, R. B. & Fisher, J. R. 1977, A&A, 54, 661

Weinberg, S. W. 1972, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity (New York: John Wiley & Sons)