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SCHEME INDEPENDENCE AND THE EXACT RENORMALIZATION GROUP *

Richard D. Ball,[a †] Peter E. Haagensen,[b] José I. Latorre,[c] and Enrique Moreno[c]

Theory Division, CERN,
CH-1211 Genève 23, Switzerland,[a]

Physics Department, McGill University
3600 University St., Montréal H3A 2T8, Canada, [b]

Departament d'Estructura i Constituents de la Matèria,
Facultat de Física, Universitat de Barcelona,
Diagonal 647, 08028 Barcelona, Spain.[c]

Abstract

We compute critical exponents in a Z_2 symmetric scalar field theory in three dimensions, using Wilson's exact renormalization group equations expanded in powers of derivatives. A nontrivial relation between these exponents is confirmed explicitly at the first two orders in the derivative expansion. At leading order all our results are cutoff independent, while at next-to-leading order they are not, and the determination of critical exponents becomes ambiguous. We discuss the possible ways in which this scheme ambiguity might be resolved.

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[a] E-mail: rball@surya11.cern.ch

[b] E-mail: haagense@cinelli.physics.mcgill.ca

[c] E-mail: latorre@ecm.ub.es, moreno@ecm.ub.es

The exact renormalization group [1-4] is in many instances a powerful analytical technique for the study of nonperturbative renormalization group flows. In practice one must always truncate an infinite tower of equations, and in the past surprisingly good results for critical exponents and renormalization group flows of scalar field theories in $2 \leq d \leq 4$ have been obtained [5-10] by suppressing all the equations except that for the effective potential. There have been several attempts [11-14] to improve the accuracy of these calculations by also keeping terms in the effective action containing two derivatives of the fields, and thus allowing for wave function renormalization. In this letter we will consider the important issue of scheme dependence within this context.

A Euclidean field theory may in general be regularized by employing a certain cutoff function $K_\Lambda(p)$, which suppresses modes with momenta higher than some scale Λ. The exact renormalization group equations are then constructed so as to ensure that as Λ is reduced, and thus more and more modes are integrated out, the vertex functions of the theory are varied in such a way that the Green's functions remain unchanged. This provides a nonperturbative definition of the theory. Now in principle physical observables should be determined unambiguously for a broad class of cutoff functions $K_\Lambda(p)$. This is what we mean by scheme independence. Indeed in certain truncation schemes this independence can be proven; for instance, it can be shown that S-matrix elements are scheme independent to any given order in perturbation theory[15], even though beta-functions and anomalous dimensions are in general scheme dependent beyond leading order. However in other truncations scheme independence cannot be taken for granted. In the derivative expansion, in particular, this problem has not so far been considered: a particular choice of cutoff function is made (e.g., sharp cutoffs [5-9], power-like cutoffs [13], or exponential cutoffs [12,14]), giving perhaps the impression that the results obtained are in some sense unique.

Here we will compute various critical exponents for a Z_2-symmetric scalar field theory in $d=3$ dimensions using the simple and elegant exact renormalization group equations due to Wilson [1] and Polchinski [4]: previous authors [5-9,13,14] have based their computations on the Wegner-Houghton [2] and Weinberg [3] equations. We work to all orders in the fields, but expand the (Wilson) effective action in powers of derivatives. We will show that all the scheme dependence can be absorbed into $2n$ parameters at n-th order in the expansion. At leading order, $n = 0$, the critical exponents are then scheme independent: they are independent of the particular choice of cutoff function. However at next-to-leading order the results become scheme dependent. This makes it particularly difficult to compute

unambiguously the critical exponent η (corresponding to wave function renormalization). We will attempt to control this dependence by employing a minimal sensitivity criterion.

We also discuss the 'scaling' relation between the magnetic deformation exponent λ_H and η, showing explicitly that this is satisfied exactly (in any scheme) at the first two orders in the derivative expansion.

1. The Exact Renormalization Group

Exact renormalization group equations are normally obtained in two steps: incomplete integration of modes, followed by rescaling. The first step is most easily described in the path integral approach developed by Polchinski [4]. Consider the following action for a scalar field theory

$$S[\phi;\Lambda] \equiv \tfrac{1}{2}\int_p \phi_p \phi_{-p} P_\Lambda^{-1}(p^2) + S_{\rm int}[\phi;\Lambda], \tag{1.1}$$

where $\int_p \equiv \int \frac{d^d p}{(2\pi)^d}$, $P_\Lambda(p^2)$ is an analytic function [15] with a single pole on the negative real axis, which falls sufficiently fast as $p^2/\Lambda^2 \to \infty$ so that all modes with Euclidean momenta $p^2 \gg \Lambda^2$ are suppressed, and $S_{\rm int}$ contains all higher order interactions. We will write generically,

$$P_\Lambda(p^2) = \frac{K(p^2/\Lambda^2)}{p^2}\ , \tag{1.2}$$

where $K(p^2/\Lambda^2)$ is the cutoff function responsible for damping high momentum modes.[1] The remaining part of the action, $S_{\rm int}[\phi;\Lambda]$, is then evolved in such a way that as Λ is reduced from some starting value Λ_0 the amputated Green's functions of the theory remain unaltered; this can be achieved provided $S_{\rm int}$ satisfies the following renormalization group equation [4]:

$$\Lambda\frac{d}{d\Lambda}S_{\rm int} = \tfrac{1}{2}\int_p \Lambda\frac{\partial P_\Lambda(p^2)}{\partial\Lambda}\left(\frac{\delta S_{\rm int}}{\delta\phi_p}\frac{\delta S_{\rm int}}{\delta\phi_{-p}} - \frac{\delta^2 S_{\rm int}}{\delta\phi_p\delta\phi_{-p}}\right)\ . \tag{1.3}$$

(Contact terms have been suppressed here; see ref.[15] for more details.) The exact renormalization group thus makes use of two momentum scales, Λ_0, which is held fixed, and Λ which controls the scale down to which high momentum modes have been (loosely speaking) integrated out. What is of relevance for the flow is their ratio or, equivalently, $t \equiv \ln\frac{\Lambda_0}{\Lambda}$;

[1] More properly we should also include a mass scale $m \ll \Lambda_0$ (the position of the zero in P_Λ); while important in the extreme infrared ($\Lambda \ll m$), it may be ignored in the vicinity of the Wilson fixed point, since there m/Λ will be very small.

renormalization group flows then go from the far ultraviolet ($t = 0, \Lambda = \Lambda_0$) to the far infrared ($t \to \infty, \Lambda \to 0$).

The above renormalization group equation may be further simplified by rewriting it in terms of the complete effective action $S[\phi;\Lambda]$: suppressing the contact term

$$\Lambda \frac{d}{d\Lambda} S = \tfrac{1}{2} \int_p \Lambda \frac{d}{d\Lambda} P_\Lambda(p^2) \left(\frac{\delta S}{\delta \phi_p} \frac{\delta S}{\delta \phi_{-p}} - \frac{\delta^2 S}{\delta \phi_p \delta \phi_{-p}} - 2 P_\Lambda^{-1}(p^2) \phi_p \frac{\delta S}{\delta \phi_p} \right). \qquad (1.4)$$

As we shall see, this equation is rather easier to use than the other exact renormalization group equations [2,3] employed in [5-9,13,14], due to the relative simplicity of its non-linear term.

We now consider the rescaling step. Firstly, along the renormalization group flow, $S_{\rm int}$ will develop quadratic terms of the form $\frac{1}{2} z_0(t) p^2 \phi_p \phi_{-p}$ which change the residue of the pole in the propagator of the ϕ_p field. By defining a new rescaled field

$$\varphi_p \equiv \sqrt{Z(t)}\ \phi_p \equiv \sqrt{1 + z_0(t)}\ \phi_p \qquad (1.5)$$

we can choose the renormalization factor $Z(t)$ such as to redefine the kinetic term and fix the residue of the pole

$$\frac{\partial}{\partial p^2} \left. \frac{\delta^2 S}{\delta \varphi_p \varphi_{-p}} \right|_{\substack{p^2=0 \\ \varphi=0}} = \delta(0). \qquad (1.6)$$

The anomalous dimension of φ_p is $\eta(t) \equiv \dot{Z}(t)/Z(t)$, the dot denoting a partial derivative with respect to t. At fixed points of the flow this quantity becomes equal to the critical exponent η_*.

Finally, it is important that we write the couplings in $S_{\rm int}$ in a manifestly dimensionless way, since it is dimensionless couplings which parameterize the theory and whose flow we should study. Thus, we write

$$S_{\rm int} = \sum_{n={\rm even}} \Lambda^{d+n(1-\frac{d}{2})} \int_{p_1 \ldots p_n} s_n(\hat{p}_1, \ldots, \hat{p}_n; t)\ \varphi_{p_1} \cdots \varphi_{p_n}\ \delta(p_1 + \ldots + p_n)\ , \qquad (1.7)$$

where $\hat{p}_i \equiv p_i/\Lambda$ and the couplings (or 'vertex functions') $s_n(\hat{p}_1, \ldots, \hat{p}_n; t)$ are dimensionless. This completes the rescaling step, as now we can rewrite (1.3) as a set of equations for the flow of the couplings s_n:

$$\begin{aligned} \dot{S} =& \tfrac{1}{2} \int_p K'(\tfrac{p^2}{\Lambda^2}) \left(\frac{\delta S}{\delta \varphi_p} \frac{\delta S}{\delta \varphi_{-p}} - \frac{\delta^2 S}{\delta \varphi_p \delta \varphi_{-p}} \right) + d\ S \\ &+ \int_p \left(1 - \tfrac{d}{2} - \tfrac{\eta(t)}{2} - 2p^2 \frac{K'(\frac{p^2}{\Lambda^2})}{K(\frac{p^2}{\Lambda^2})} \right) \varphi_p\ \frac{\delta S}{\delta \varphi_p} - \int_p \varphi_p\ p \cdot \frac{\partial'}{\partial p} \frac{\delta S}{\delta \varphi_p}\ , \end{aligned} \qquad (1.8)$$

where in $\dot{S}$ the partial derivative with respect to t now acts only on the dimensionless couplings s_n in (1.7), and the prime in the momentum derivative means that it does not act on the momentum conserving delta functions of (1.7).

2. Derivative Expansion and Scheme Dependence

The overwhelming complexity of the full exact renormalization group equations means that in practice some sort of truncation must be employed. Since the action may be naturally expanded in powers of the fields and their derivatives, it is natural to consider truncations of either or both of these expansions. Truncations in the number of powers of the fields, as employed in [6-8] seem to be poorly convergent, with many spurious solutions (see [16] for a detailed analysis), although the convergence seems to improve if the expansion is made about the minimum of the potential [9,14]. A better procedure [5,11-13] seems to be to work to all orders in the number of fields, and expand the interaction only in powers of derivatives. In this way it is hoped that the essential features of the long-wavelength physics will be retained.

Truncating at second order in derivatives, we thus write

$$S_{\rm int} = \int d^4x \left(v(\varphi,t) + \tfrac{1}{2} z(\varphi,t) \big(\partial_\mu \varphi\big)^2 + \cdots \right) \tag{2.1}$$

or, alternatively,

$$s_n(p_1,\ldots,p_n;t) = v_n(t) + \frac{1}{n(n-1)} \frac{p_1^2 + \ldots + p_n^2}{\Lambda^2}\ z_{n-2}(t) + \cdots\ , \tag{2.2}$$

with $v(\varphi,t) = \sum v_n(t)\varphi^n$, $z(\varphi,t) = \sum z_n(t)\varphi^n$, and where now $z_0(t) = 0$ due to the normalization condition (1.6). The exact renormalization group equation (1.8) can now be projected onto a series of equations, order by order in p^2, in which φ is taken to be effectively a constant (and denoted by x). This procedure is somewhat lengthy but unambiguous, and results in an infinite set of coupled partial differential equations:[2]

$$\begin{aligned}
\dot{v} &= I_0 v'' + 2I_1 z - K_0 v'^2 + (1 - \tfrac{d}{2} - \tfrac{\eta}{2}) x v' + d v\ , \\
\dot{z} &= I_0 z'' + K_1 v''^2 - 4K_0 z v'' - 2K_0 z' v' + (1 - \tfrac{d}{2} - \tfrac{\eta}{2}) x z' - \eta z - \tfrac{\eta}{2}\ , \\
&\ \ \vdots
\end{aligned} \tag{2.3}$$

where dots and primes denote partial derivatives with respect to t and x respectively. The number of equations grows rather fast with increasing powers of p^2; there are three more equations at the next order. So far, there seems to have been no attempt to go to order

[2] The zeroth order equation (with $z = \eta = 0$) seems to have been first derived in [17]; different versions of the first order equations may be found in [11,12].

p^4; here we will just study the two equations (2.3). The numbers K_n, I_n, $n = 0, 1, \ldots$ are remnants of the cutoff function $K(z)$ and thus parametrize the scheme dependence of the equations:

$$K_n \equiv (-)^{n+1} K^{(n+1)}(0), \qquad I_n \equiv -\int_p (p^2)^n K'(p^2) = \Omega_d \int_0^\infty dz z^n K'(z) \ , \tag{2.4}$$

where $K^{(n)}$ is the n-th derivative of K, and $\Omega_d = 2/\big(\Gamma(\frac{d}{2})(4\pi)^d\big)$. At zeroth order in p^2, all the cutoff dependence is parametrized in terms of K_0 and I_0, while at first order K_1 and I_1 are also needed; at n-th order we need K_n and I_n. Thus the further we go in the derivative expansion, the more information about the cutoff function is needed; short-distance properties of it only gain relevance as the derivative expansion is carried to higher orders. Incidentally, we note that in a calculation of perturbative renormalization group functions in this setting, these same remnants of the cutoff function are present and determine the scheme dependence of some of the results[18].

We have just shown that the cutoff dependence of eq. (2.3) is given in terms of at most $2n + 2$ arbitrary parameters at order p^{2n}. In fact we can exploit global rescalings of the variable x and of the functions v and z to further simplify the first two equations to

$$\begin{aligned} \dot{v} &= v'' + 2Az - v'^2 + (1 - \tfrac{d}{2} - \tfrac{\eta}{2}) x v' + dv \ , \\ \dot{z} &= z'' + Bv''^2 - 4zv'' - 2z'v' + (1 - \tfrac{d}{2} - \tfrac{\eta}{2}) x z' - \eta z - \tfrac{\eta}{2} \ , \end{aligned} \tag{2.5}$$

where

$$A \equiv \frac{I_1 K_0}{I_0}, \qquad B \equiv \frac{K_1}{K_0^2} \ . \tag{2.6}$$

It follows that at zeroth order ($z = 0$) there is no cutoff dependence at all, while all cutoff dependence at order p^2 is reduced to a two-parameter family (A, B). Furthermore, reasonable cutoff functions (for example, exponentials of polynomials[15]) will decrease smoothly and monotonically, so all moments I_n will be positive and of similar magnitude; the derivatives at the origin, K_n, will also be loosely of the same order. In other words, A and B will both be positive and loosely of order unity. For example, with an exponential cutoff [12,14] $K(z) = e^{-z}$, $A = d/2$ and $B = 1$. On the other hand a sharp cutoff $K(z) = \theta(1 - z)$, corresponds to a singular point in the A-B parameter space, since $A = 0$ while B is undetermined, and can thus only be made well-defined through some limiting procedure (as discussed in [8]). The power-like cut-off $K(z) = (1 + z^2)^{-1}$ employed in [13] is also singular since $A = 0$ and B is infinite. In these cases it becomes necessary to use other more complicated exact renormalization group equations [2-3].

3. Leading Order Numerical Results

The truncation of the derivative expansion of the exact renormalization group equation at zeroth order is obtained trivially from eq. (2.5) by writing $z(x,t) = 0$ and $\eta = 0$. Setting η to zero is justifed only by the a priori suppression of all momentum dependent terms in the interaction; it turns out to be a good approximation at zeroth order because the actual value of η_* is very small. Writing $f \equiv v'$ (to eliminate the arbitrary field independent part of v) the leading order exact renormalization group equation is then

$$\dot{f} = f'' - 2ff' + (1 + \tfrac{d}{2})f + (1 - \tfrac{d}{2})xf' \ . \tag{3.1}$$

This is considerably simpler than equivalent truncations of other exact renormalization group equations[2,3], since all non-linearity is reduced to a quadratic term which is easily controlled numerically. Furthermore, eq. (3.1) is manifestly scheme independent (that is, independent of the particular choice of cutoff function). This is of great importance since the results we will display shortly are the leading orders of the expansion.

We begin by solving the fixed point equation

$$0 = {f_*}'' - 2f_*{f_*}' + (1 + \tfrac{d}{2})f_* + (1 - \tfrac{d}{2})x{f_*}' \ , \tag{3.2}$$

with initial conditions $f_*(0) = 0$ (by the Z_2 symmetry) and ${f_*}'(0) = \gamma$. We now concentrate for definiteness on dimension $d = 3$ where a comparison with other results is readily available. The nontrivial solution at the ultraviolet fixed point only exists for one value of the initial condition $\gamma = \gamma_*$. Numerically, this is found by tuning γ so as to let f_* run as far as possible; in this way we determine $\gamma_* = -.228601293102\ldots$ (where all figures are significant: the behaviour of $f_*(x)$ at large x is extremely sensitive to the precise value of γ_*, always diverging at some finite value of x for $\gamma \neq \gamma_*$).

It is in fact trivial to determine the asymptotic behavior of the solution $f_*(x)$ to (3.2) at large x: $f_*(x) \sim x$, and thus $v_*(x) \sim x^2$. This is in marked contrast to the x^5 asymptotic behavior of solutions to the equations considered in [5,13], resulting from their rescaling terms. This difference stems from the different structure of the non-linearities in the equations; we find it intriguing since it is ultimately responsible for the differences in the numerical results.

Departures from the fixed point solution are governed by operators whose critical exponents are universal. In our case, the eigenvalue equation to be solved is given by the linearization of eq. (3.1): writing $f(x,t) = f_*(x) + \sum_n a_n g_n(x) e^{\lambda_n t}$, with a_n small,

$$g_n'' - 2(f_* g_n' + f_*{}' g_n) + \tfrac{5}{2} g_n - \tfrac{x}{2} g_n' = \lambda_n g_n \ , \tag{3.3}$$

where g_n is the n-th eigenfunction with eigenvalue λ_n. The critical exponent of the only relevant, Z_2-symmetric, operator at the infrared fixed point is known as $\nu \equiv 1/\lambda_1$, while the first irrelevant one is known as $\omega \equiv -\lambda_2$. We investigate eq. (3.3) numerically and find the results given in Table III below (LO). Further eigenvalues for irrelevant operators may be determined similarly, but become increasingly meaningless as they depend more and more on short-distance dynamics. Curiously our leading order result $\nu = 0.649$ is rather closer to the values obtained using more traditional methods [19] than that obtained using the leading order approximation to other exact renormalization group equations (i.e. $\nu = 0.690$ for a Wegner-Houghton equation [5-9], $\nu = 0.660$ for an equation for one particle irreducible vertices with a power cut-off [13]). However before we can attatch any significance to this observation we need to determine the size of the error induced by the truncation of the derivative expansion, and this we can only do by calculating the next order correction.

4. Next-to-Leading Order

The main drawback of the simple zeroth order approximation described above is the absence of wave function renormalization: η is set to zero. Equation (3.1) thus includes only the physics of zero modes, whereas wave function renormalization is precisely related to the kinetic term which carries derivatives. This leads us to consider the next-to-leading corrections, of order p^2 in the derivative expansion.

To study numerically the order p^2 approximation of the exact renormalization group equation eqns. (2.5) we first consider the fixed point differential equations $\dot{f}(t,x) = 0$, $\dot{z}(t,x) = 0$:

$$\begin{aligned} 0 &= f_*'' + 2A z_*' - 2 f_* f_*' + \Delta^+ f_* + \Delta^- x f_*' \ , \\ 0 &= z_*'' + B f_*'^2 - 4 z f_*' - 2 z' f_* + \Delta^- x z_*' - \eta z_* - \tfrac{\eta}{2} \ , \end{aligned} \tag{4.1}$$

where $\Delta^\pm \equiv 1 \pm d/2 - \eta/2$. This pair of non-linear coupled ordinary differential equations, together with appropriate initial conditions, determine the shape of the Wilson fixed point action at order p^2. Three of the initial conditions are already fixed: two by invoking Z_2

$\Delta\eta \times 10^3$	$B = 0.25$	0.5	0.75
$A = 0.5$	-0.76	- 2.8	-6.1
0.7	-0.28	- 0.9	-1.9
0.8	-0.03	0.0	0.2
0.9	0.21	1.0	2.5
1.1	0.71	3.0	7.1

Table I: The difference $\Delta\eta \equiv \eta_2 - \eta_1$ for a range of values of the scheme parameters A and B.

symmetry, $f_*(0) = 0$ and $z'_*(0) = 0$, and the third by the normalization condition eq. (1.6), $z_*(0) = 0$. For any fixed values of the parameters A and B, a well-behaved solution of eqs. (2.5) (*i.e.*, a solution defined for all finite x) only exists for the unique value $\eta = \eta_*$ of the anomalous dimension and a unique value γ_* of the initial condition $f'_*(0) = \gamma$. For any other choice of η and $f'_*(0)$ the solutions $f_*(x)$ and $z_*(x)$ end at singularities at finite x. By contrast, the correct solutions behave asymptotically, as $x \to \infty$, as

$$
\begin{aligned}
f_*(x) &\sim (1 - \eta_*/2)\, x + k x^{-\Delta^+/\Delta^-} - 2k^2 \frac{\Delta^-}{\Delta^+(2-\eta_*)}\, x^{-2\Delta^+/\Delta^- - 1} + O(x^{-3\Delta^+/\Delta^- - 2}), \\
z_*(x) &\sim \frac{B(1 - \frac{\eta_*}{2})^2 - \frac{\eta_*}{2}}{4 - \eta_*} + k \frac{\Delta^-}{\Delta^+} \frac{B(\eta_* - 2) - \eta_*}{4 - \eta_*}\, x^{-\Delta^+/\Delta^- - 1} + O(x^{-2\Delta^+/\Delta^- - 2}),
\end{aligned}
\tag{4.2}
$$

where k is a constant.

Equations (4.1) are nonlinear and stiff. Furthermore, finding the correct solution involves a double fine tuning in η and $f'_*(0)$. Thus it becomes extremely difficult to simply integrate up the equations as we did in the zeroth order case, and it is much easier to proceed recursively. So first we set $z_*(x)$ and η to zero and find the unique value γ_0 of $f'(0)$ for which the first equation (4.1) has a non-singular solution, $f_0(x)$, just as we did for the zeroth order approximation. Now we use this solution in the second equation (4.1) and solve for $z_1(x)$ by fine-tuning η to the appropriate value η_1. With the values of η_1 and $z_1(x)$ we return to the first equation and obtain a new solution $f_1(x)$, with $f'_1(0) = \gamma_1$, and so on. In this way we obtain a sequence of functions $f_0, z_1, f_1, z_2, f_2, \ldots$ and constants $\gamma_0, \eta_1, \gamma_1, \eta_2, \gamma_2, \ldots$ which (at least for a reasonable range of A and B) converge rather rapidly to the exact solution $f_*(x), z_*(x), \gamma_*, \eta_*$. Solutions $f_*(x)$ and $z_*(x)$ are shown in fig. 1.

	η	ν	ω
B = 0.25	0.01898	0.63706	0.7016
0.5	0.03762	0.62522	0.7733
0.75	0.05595	0.61595	0.8501

Table II: Critical exponents at next-to-leading order for $A = 0.8$ and a range of values of B.

Again for $d = 3$, a numerical analysis of this sequence of solutions shows that the rate of convergence of the iterative procedure is in effect controlled by A. Indeed for any fixed B, the convergence is fastest (in the sense that corrections to η in sucessive iterations are minimized) when $A = 0.8$. Moreover, the function $\eta(A)$ has an inflexion point at this particular value of A. This is demonstrated in Table I where we show the difference $\Delta\eta \equiv \eta_2 - \eta_1$ for different values of the parameters A and B. We will use this criterion to fix, from now on, the value of A to 0.8.

Within the accuracy of our computations, we could not find a similar minimal sensitivity criterion for the parameter B either in the fixed point solutions or in the critical exponents. The values of η_* are given in Table II. The dependence of the anomalous dimension η_* in the parameter B is almost linear, as can be inferred by the following argument. By varying the initial conditions of f_* and z_* at $x = 0$ in the first equation (2.5) and assuming a (fast) convergence of the iterative process, we can relate the second derivative of z_1 at the origin to the parameter γ_0. Using this relation and the projection onto $x = 0$ of the second equation (2.5) one can deduce the approximate relation:

$$\eta_* \approx 2B \frac{\gamma^{*2}}{1 - \frac{\gamma^*}{A}} \ , \tag{4.3}$$

in good qualitative agreement with our numerical results. Although no serious results can be obtained from this estimate of η_*, it is interesting that it has no stationary points in either A or B. The optimal value of $A = 0.8$ is, thus, due to higher order iterations. However, to the precision we work with, the dependence of η_* on B remains almost linear even when subsequent iterations are considered.

Once a fixed point solution is found we can study the critical exponents associated with this solution, by again linearizing equations (2.5) around the fixed point. This gives rise to the eigenvalue problem

$$\begin{aligned} &g_n''(x) - 2g_n(x)f_*{}'(x) - 2f_*(x)g_n'(x) + \Delta^+ g_n(x) + \Delta^- x g_n'(x) + 2Ah_n'(x) = \lambda_n g_n(x), \\ &h_n''(x) - 2f_*(x)h_n'(x) - 2g_n(x)z_*{}'(x) - 4f_*{}'(x)h_n(x) - 4g_n'(x)z_*(x) \\ &\qquad\qquad + \Delta^- x h_n'(x) - \eta_* h_n(x) + 2Bf_*{}'(x)g_n'(x) = \lambda_n h_n(x), \end{aligned} \tag{4.4}$$

where $f_*(x)$ and $z_*(x)$ are the fixed point solutions, η_* the corresponding anomalous dimension, and $(g_n(x), h_n(x))$ is the eigenvector (scaling operator) corresponding to the critical exponent λ_n.

For Z_2-symmetric perturbations, we now use as initial conditions $g_n(0) = 0, h_n'(0) = 0$, which are fixed by the Z_2-symmetry, and $g_n'(0) = 1$ as a normalization. Solutions are again found by an iterative procedure, where the two parameters to be tuned are now λ_n and $h_n(0) = h_0$.[3] When $d = 3$, we find as expected that for $A = 0.8$ and a reasonable range of values of B all the eigenvalues were very close to those found at zeroth order. Results after four iterations, for various values of B, are given in Table II.

5. The Magnetic Deformation Exponent

Thus far we have only considered Z_2-symmetric perturbations about the fixed point. It is also interesting to consider Z_2-antisymmetric deviations from the fixed point, since these yield another critical exponent λ_H. In the analogous Ising system, such perturbations measure the response of the system to a weak external magnetic field.

To find these antisymmetric eigenvectors to eq. (4.4), we use as initial conditions $g_n(0) = 1, g_n'(0) = 0, h_n(0) = h_0$ small and $h_n'(0) = 0$. This does not give an independent critical exponent for the theory, but it does allow us to verify for $d = 3$ the 'scaling relation' [19]

$$\lambda_H = 1 + \frac{d}{2} - \frac{\eta_*}{2} \ , \tag{5.1}$$

[3] Although h_0 is very small, it is nonzero due to the fact that we have implicitly relaxed the normalization condition eq. (1.6) by setting $\eta(t) = \eta_*$ away from the fixed point.

	η	ν	ω
LO	0	0.649	0.66
NLO	0.019 - 0.056	0.616 - 0.637	0.70 - 0.85
NLOS	0.035	0.627	0.76
Ref.[19]	0.030 - 0.040	0.630 - 0.631	0.75 - 0.85

Table III: Critical exponents at leading order (LO), next-to-leading order for B in the range .25 - .75 (NLO) , next-to-leading order with the scheme dependence fixed to give η correctly (NLOS), and finally the average value obtained from other methods [19].

giving us a nontrivial consistency check on our results for η_*. At order p^0 and at order p^2, with $A = 0.8$ and several values of B, we have calculated λ_H, and have confirmed numerically that the relation (5.1) is always satisfied.

Indeed it is possible to prove (5.1) rigorously for any d, any scheme, and at leading or next to leading order in the derivative expansion: it is straightforward to verify that

$$(g_n(x), h_n(x)) = (f'_*(x) - (1 - \tfrac{\eta}{2}), z'_*(x)) \tag{5.2}$$

is an exact eigenvector of eq. (4.4), with eigenvalue λ_H given by eq. (5.1).

Incidentally,

$$(g_n(x), h_n(x)) = (f'_*(x), z'_*(x)) \tag{5.3}$$

is also an exact eigenvector, with eigenvalue $\lambda_I = -1 + \frac{d}{2} + \frac{\eta}{2}$. At $d = 3$, there are thus three relevant eigenvalues: $\lambda_T \equiv 1/\nu$, λ_H and λ_I. Just as λ_H and λ_T correspond to perturbations proportional to the operators φ and φ^2, respectively, λ_I corresponds to the operator φ^3, which however is not independent from φ, due to the equations of motion.

6. Conclusions

Like perturbation theory, the derivative expansion to any finite order represents a truncation of the renormalization group. We find it interesting that our results for anomalous dimensions are scheme independent (or, more precisely, cutoff independent) to first nontrivial order and scheme dependent beyond that.

Our results for the leading critical exponents of a scalar field theory with Z_2 symmetry in three dimensions at the Wilson fixed point are summarized in Table III, together with values obtained from different methods [19](resummed perturbation series, ϵ-expansions, and lattice simulations). At leading order our (scheme independent) equation seems to do remarkably well; essentially this is because the true value of η is very small. At next-to-leading order the results improve in the right direction (and are consistent with those in [13,14]), but scheme dependence makes it impossible to give a precise estimate of the errors at this order. Our attempts to fix the scheme by some sort of 'minimal sensitivity' criterion were only partially successful, because η depends approximately linearly on the scheme parameter B. A next-to-next-to-leading order computation might give results with a weaker scheme dependence, allowing perhaps for a determination of η and thus a sensible estimation of the errors incurred in other quantities also. However, at present scheme dependence severely limits the accuracy of the technique, in particular for quantities which depend sensitively on η. The scheme ambiguity of our results may however be significantly reduced if we simply choose a scheme which gives the 'known' value for η: our results for ν and ω are then both competitive and consistent with the 'known' results for these exponents.

We also discussed two other relevant eigenvalues, corresponding to asymmetric perturbations about the fixed point. We showed that the scaling relations relating these relevant eigenvalues to the other critical exponents are satisfied exactly by our truncated set of equations, and thus give a useful consistency check on the numerical calculations.

It should be possible in the future to use these truncated exact renormalization group equations to compute other quantities of physical interest (and in particular Green's functions), in a wider range of models. In particular it would be possible [20] to include (chiral) fermions, and thus study (for example) the electroweak sector of the standard model (with gauge couplings suppressed).

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References

[1] K. Wilson and J. Kogut, *Phys. Rep.* .**12** (1974) 75;
K. Wilson, *Rev. Mod. Phys.* **47** (1975) 773.

[2] F.J. Wegner and A. Houghton, *Phys. Rev.* **A8** (1972) 401.

[3] S. Weinberg, in "Understanding the Fundamental Constituents of Matter", Erice 1976, ed. A. Zichichi (Plenum, 1978).

[4] J. Polchinski, *Nucl. Phys.* **B231** (1984) 269.

[5] A. Hasenfratz and P. Hasenfratz, *Nucl. Phys.* **B270** (1986) 687.

[6] A. Margaritis, G. Odor and A. Patkos, *Zeit. Phys.* **C39** (1988) 109.

[7] P.E. Haagensen, Yu. Kubyshin, J.I. Latorre and E. Moreno, *Phys. Lett.* **323B** (1994) 330.

[8] T.R. Morris, *Int. Jour. Mod. Phys.* **A9** (1994) 2411.

[9] M. Alford, *Phys. Lett.* **336B** (1994) 237.

[10] A.E. Filippov and S.A. Breus, *Phys. Lett.* **A158** (1991) 300.

[11] G.R. Golner, *Phys. Rev.* **B33** (1986) 7863.

[12] A.E. Filippov and A.V. Radievsky, *Phys. Lett.* **A169** (1992) 195.

[13] T.R. Morris, *Phys. Lett.* **329B** (1994) 241.

[14] N. Tetradis and C. Wetterich, *Critical Exponents from the Effective Average Action* DESY-93-094.

[15] R.D. Ball and R.S. Thorne, *Renormalizability of Effective Scalar Field Theory*, CERN-TH.7067/93, *Ann. Phys.* (in press).

[16] T.R. Morris, *Phys. Lett.* **334B** (1994) 355.

[17] V.I. Tokar, *Phys. Lett.* **A104** (1984) 135.

[18] J. Hughes and J. Liu, *Nucl. Phys.* **B307** (1988) 183.

[19] C. Itzykson and J.M. Drouffe, *Statistical Field Theory, Vol. 1* (Cambridge University Press, 1989);
J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, (Oxford University Press, 1990).

[20] M. Maggiore, *Zeit. Phys.* **C41** (1989) 687.

Figure Captions

Fig. 1. The fixed point solution at next to leading order for $A = 0.8$ and $B = 0.25, 0.5, 0.75$: a) $f_*(x)$ b) $z_*(x)$. The (scheme independent) leading order solution is also shown (dashed).

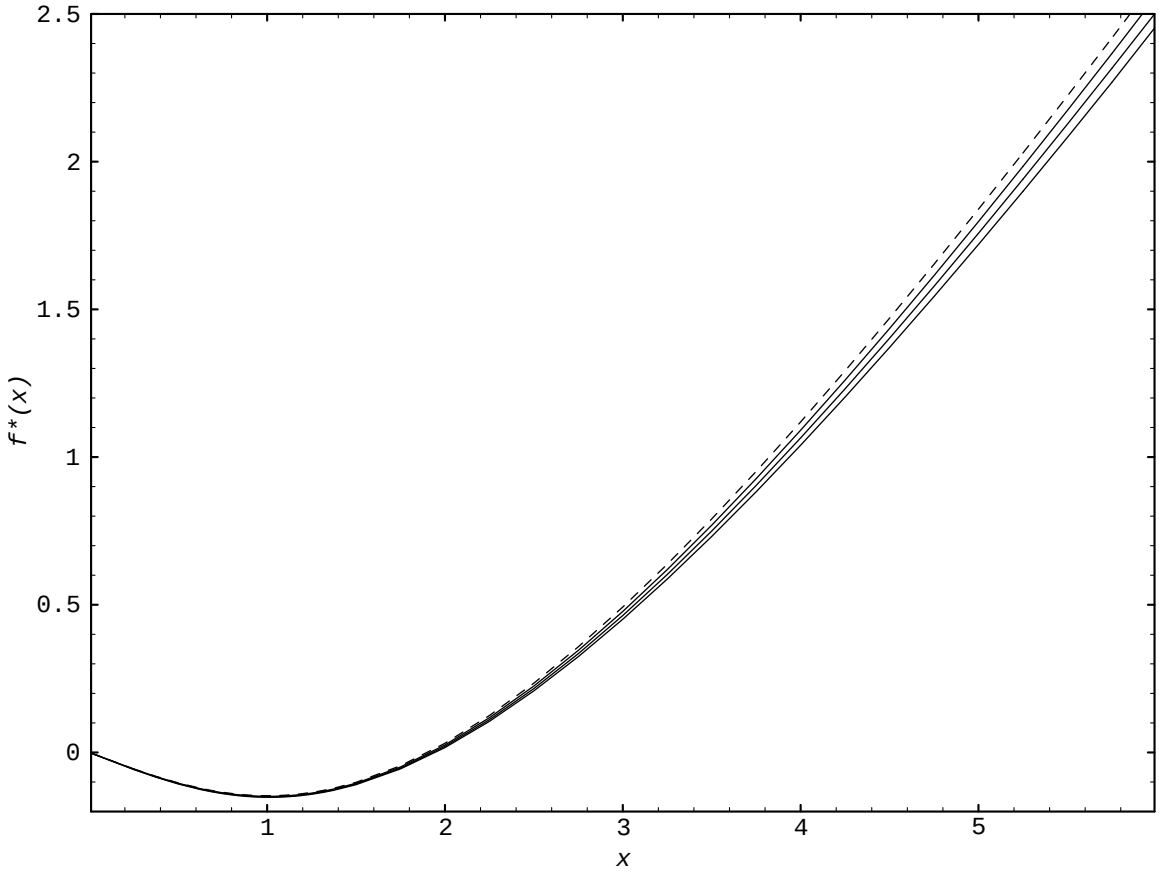

Fig.1.a

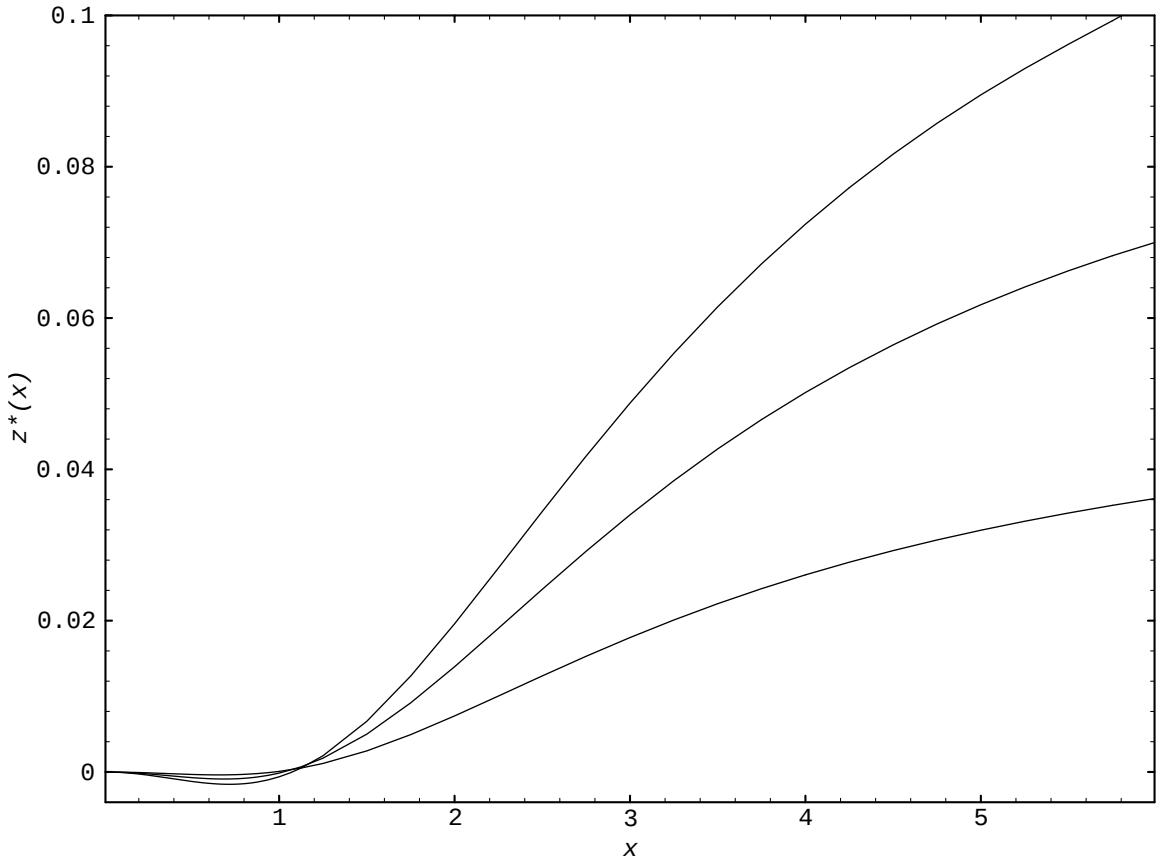

Fig.1.b