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Strings in Spacetime Cotangent Bundle and T-duality

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Abstract

A simple geometric description of T-duality is given by identifying the cotangent bundles of the original and the dual manifold. Strings propagate naturally in the cotangent bundle and the original and the dual string phase spaces are obtained by different projections. Buscher's transformation follows readily and it is literally projective. As an application of the formalism, we prove that the duality is a symplectomorphism of the string phase spaces.

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1 Introduction

Target space duality keeps attracting the attention of string theorists (see e.g. [1, 2, 3, 4]) mainly because it deepens our understanding of the geometry of spacetime from the string point of view and it is an important tool for disentangling the full symmetry structure of the string theory.

Duality is usually derived in the σ-model context in appropriate coordinates, respecting the isometry of the action. The symmetry is gauged and the gauge fields are constrained to be trivial by introducing a Lagrange multiplier. The latter is understood as a new (dual) coordinate and the gauge fields are integrated out to end up with the dual σ-model. The dual metric $\tilde{G}$ and the skew-symmetric tensor $\tilde{B}$ are then given by Buscher's formula.

This approach is not very transparent from the geometrical point of view, however. An important conceptual simplification was undertaken in [5] where the duality was described in a global geometric setting and, following the suggestion in [6], it was interpreted as just a canonical transformation of the phase space of the theory [7]. Attempting to further clarify the concept, we give a very simple global geometric description of duality for a large class of systems including the σ-models. We show that the dual target, its cotangent bundle, and the elements of the original and the dual phase spaces can be naturally embedded into the cotangent bundle of the original target. The relation between the original and the dual quantities is now simply given by different projections of invariant objects living in the cotangent bundle. Buscher's formula follows extremely straightforwardly and it is literally projective. As an application of the developed formalism we present a simple proof that duality acts as a symplectomorphism of the original and the dual phase spaces, understood as appropriate submanifolds of the cotangent bundles of the loop spaces of the targets. The whole picture seems quite natural. We may even guess that the cotangent bundle will play an important role in developing a natural framework for the description of string symmetries.

The plan of the paper is as follows. In section 2 we identify the cotangent bundles of the original and the dual manifolds and compare their canonical symplectic structures. In the next part we lift the dynamical characteristics of strings into the cotangent bundle and by different projections we obtain Buscher's formula. In section 4 we prove that the duality is a symplectomorphism and in section 5 we consider σ-models only, recovering the standard results in a very compact way.

2 Dual symplectic structures on the cotangent bundle

Consider a manifold M with a global vector field v and a global closed 1-form ω such that $\omega(v)=1$.[2] Let T^*M be the cotangent bundle of M. The fields v and ω are naturally extended to the whole T^*M. On T^*M there is the canonical symplectic form Ω. We define the dual fields $\tilde{v}$ and $\tilde{\omega}$ on T^*M in this way:

$$\tilde{\omega}=\Omega(.,v) \qquad \text{and} \qquad \omega=\Omega(\tilde{v},.) \tag{1}$$

(so that $d\tilde{\omega}=0$ and $\tilde{\omega}(\tilde{v})=1$).

There exists another symplectic form $\tilde{\Omega}$ on T^*M such that

$$\tilde{\Omega}+\tilde{\omega}\wedge\omega=\Omega+\omega\wedge\tilde{\omega}. \tag{2}$$

Then obviously

$$\omega=\tilde{\Omega}(.,\tilde{v}) \qquad \text{and} \qquad \tilde{\omega}=\tilde{\Omega}(v,.) \tag{3}$$

These relations are manifestly dual to (1). This suggests that T^*M can be interpreted as the cotangent bundle of some dual manifold $\tilde{M}$ whose canonical symplectic form is $\tilde{\Omega}$. We obtain $\tilde{M}$ by the action of the vector field $\tilde{v}$ on a hypersurface $M_0 \subset M$ such that $\omega|_{M_0}=0$. Then clearly $M_0=M\cap\tilde{M}$ and $\tilde{\omega}|_{M_0}=0$. The dual projection $\tilde{\pi}$ maps $(P\in M,\alpha\in T^*_PM)\in T^*M$ into $(P_0\in M_0,\alpha(v)\omega)\in\tilde{M}$, where P_0 lies on the same integral curve of v as P does. Every point $(P,\alpha)\in T^*M$ can be understood as a 1-form on $\tilde{M}$ at the point $\tilde{\pi}(P,\alpha)=(P_0,\alpha(v)\omega)$. It associates , to any vector t_0 annihilated by $\tilde{\omega}$, the number $\alpha(t_0)$, and to $\tilde{v}$ the number $\int_{P_0}^P\omega$. If the integral curves of v are closed then the function $f(P)=\int_{P_0}^P\omega$ is multivalued. In this case we have to identify T^*M with $T^*\tilde{M}$ factorized by an appropriate discrete group.

There remains to show that Eq. (2) holds. Because $\tilde{\Omega}=d\tilde{\theta}$ (and $\Omega=d\theta$) where θ and $\tilde{\theta}$ are canonically defined, it is sufficient to compare $\tilde{\theta}$ and θ. Every vector $t\in T_X(T^*M)$ can be uniquely written as $t=t_0+\omega(t)v+\tilde{\omega}(t)\tilde{v}$ where $\omega(t_0)=\tilde{\omega}(t_0)=0$. Then by definition $\theta(t_0)=\tilde{\theta}(t_0)$, $\theta(\tilde{v})=0$, $\tilde{\theta}(v)=0$, $\theta(v)=\int^X\tilde{\omega}$ and $\tilde{\theta}(\tilde{v})=\int^X\omega$. Hence

$$\tilde{\theta}(t)=\theta(t)-\omega(t)\int^X\tilde{\omega}+\tilde{\omega}(t)\int^X\omega$$

and Eq. (2) follows. As the formulae suggest, starting with the dual fibration and repeating the procedure will bring us back to the original one.

We shall argue that the framework described above is very well suited for the description of the (Abelian) target space duality in string theory.

[2]The form ω is locally the differential of a coordinate along v.

3 Strings embedded into T^*M and Buscher's formula

Suppose M is a spacetime in which string propagation is governed by an action S, invariant with respect to the vector field v. There is no need to assume that S is a σ-model action; we only suppose that S is local, reparametrization-invariant and depending on the first derivatives of the embedding of the string worldsheet into M.

Because v is a symmetry, by Noether theorem there is a closed 1-form α (the density of the v-component of the momentum of the string) on any surface extremizing S. Due to this fact, any on-shell string can be naturally lifted to T^*M along $\tilde{v}$ so that the form $\tilde{\omega}$ restricted to the lifted surface gives precisely α.[3]

Then we project the lifted surface to $\tilde{M}$. We shall show that there exists a $\tilde{v}$-invariant action $\tilde{S}$ whose extremal surfaces are just the projections to $\tilde{M}$ of the lifted surfaces. The action $\tilde{S}$ will obey the duality property, namely the lift of the dual extremal surface along v, such that the dual Noether form $\tilde{\alpha}$ is restricted ω, coincides with the original lift.

All that picture should be refined, however, in the case of closed strings. Then the form α need not be exact and the lift of the non-contractible loop on the worldsheet may give an open curve in T^*M. Therefore, we have to identify the points of T^*M along $\tilde{v}$ in such a way that

$$n\oint_{orbit\ of\ \tilde{v}} \tilde{\omega} = \oint_{loop} \alpha = p,$$

where p is the v-component of the total momentum of the string and n is an integer. As a consequence, the momentum of the string is an integer multiple of some minimal momentum and the dual string winds n times around the orbit of $\tilde{v}$. The picture holds in the dual version, of course.

In order to find the dual action, we have to identify a geometric object on M which encodes the original action, can be naturally lifted to T^*M, and then projected to the dual manifold $\tilde{M}$. Because the action is reparametrization-invariant and depends at most on the first derivatives, the Lagrangian L is a function of decomposable bivectors b at any point P of M such that

$$L(\lambda b) = \lambda L(b), \qquad \lambda > 0. \tag{4}$$

This object is not convenient for lifting to T^*M. Instead, we define a bilinear mapping $\mathbf{p}$

$$\mathbf{p}_b(t,u) \equiv \frac{d}{d\epsilon} L(b + \epsilon t \wedge u)|_{\epsilon=0}, \tag{5}$$

[3] This lift is defined uniquely up to a uniform shift by $\tilde{v}$. We shall discuss the ambiguity in the next section.

where $t, u \in T_P M$ and $t \wedge b = 0$ (the last condition means that the vector t lies in the plane of b and in fact ensures that the argument of L in (5) is decomposable). Note that $\mathbf{p}_{\lambda b}(t, u) = \mathbf{p}_b(t, u)$ or in other words the mapping $\mathbf{p}$ depends only on the plane in which b lies. Physically speaking, $\mathbf{p}_b(t, u)$ is the density of the u-component of the momentum.[4]

We see that the mapping $\mathbf{p}_b(t, u)$ contains the essence of dynamical properties of string. Obviously, $\mathbf{p}_b(b) = L(b)$ and $\mathbf{p}_b(t, v) = \alpha(t)$ where α is the mentioned Noether 1-form.

A bivector $b = v_1 \wedge v_2$ at a point $P \in M$ can be naturally lifted to a decomposable bivector $b^* \in \Lambda^2 T_P(T^*M) \simeq \Lambda^2(T_P M + T_P^* M)$:

$$b^* = (v_1 + \mathbf{p}_b(v_1, .)) \wedge (v_2 + \mathbf{p}_b(v_2, .)). \tag{6}$$

We observe the simple formula

$$2L(b) = \Omega(b^*). \tag{7}$$

The dual Lagrangian $\tilde{L}$ is defined in such a way that lifting $\tilde{b} = \tilde{\pi}(b^*)$ by $\tilde{L}$ gives b^*. Then obviously[5]

$$2\tilde{L}(\tilde{b}) = \tilde{\Omega}(b^*). \tag{8}$$

From (2), (7) and (8) Buscher's duality transformation follows:

$$\tilde{L}(\tilde{b}) = L(b) - (\tilde{\omega} \wedge \omega)(b^*) = L(b) - (\alpha \wedge \omega)(b). \tag{9}$$

By construction, the dual Lagrangian is $\tilde{v}$-invariant. In a way, it may be interpreted as the Routh function, because ω is the differential of the coordinate along the symmetry field and α is the corresponding momentum. Later we shall write the formula in the familiar σ-model context. The formula can be derived also relaxing the condition of reparametrization invariance. However, its derivation is slightly less straightforward.

We should demonstrate that we obtain, by minimizing the dual action, the surfaces in $\tilde{M}$ projected from the lifted extremal surfaces of the original action. First notice that for lifting a surface $F \subset M$ we only need $d\alpha = 0$. Upon gauging the symmetry v the variation of the action $S(F + \epsilon v) - S(F)$ is equal to $\int_F d\epsilon \wedge \alpha$. It means that $d\alpha = 0$ iff F is extremal with respect to arbitrary (non-uniform) variations in the direction of v. By construction, the same thing

[4]We may say that $\mathbf{p}_b(t, u)$ defines for any embedding of string into M a 1-form on the worldsheet with values in 1-form on the target. In arbitrary coordinates ζ^α on the worldsheet and X^μ on the target it can be written as $\mathbf{p}_{\alpha\mu} = \partial(\mathcal{L}\epsilon_{\alpha\beta})/\partial(\partial_\beta X^\mu)$, where $S = \int \mathcal{L} d\zeta^0 d\zeta^1$.

[5]Speaking more exactly, the value of Ω is the same for any bivector on T^*M obtained by acting by the vector fields $v, \tilde{v}$ on b^*, because the symplectic form Ω is $v, \tilde{v}$-invariant. This means that we can transport b^* into a point of the dual manifold $\tilde{M}$ embedded in T^*M and write the formula (8) there.

is valid for dual objects, i.e. the liftable surfaces are exactly those extremizing S and $\tilde{S}$ with respect to the v- and $\tilde{v}$-direction respectively. Therefore we shall restrict our attention to these surfaces only. Now from (9) one easily observes

$$\tilde{S}(\tilde{F}) = S(F) - \int_{F^*} \tilde{\omega} \wedge \omega, \tag{10}$$

where F^* is the common lift of $F \subset M$ and $\tilde{F} \subset \tilde{M}$. The difference between the actions is an integral of a closed form so it does not feel any variation.

4 Duality as a symplectomorphism of string phase space

By the phase space of a closed string theory we understand the space of all classical solutions having the topology of a cylinder. On this space there is a natural symplectic form Ω_{Ph} coming from the action. However, we had to quantize the momentum for the duality to make sense. If we fix the momentum then we obtain a hypersurface in the phase space. This is not a symplectic space because Ω_{Ph} is not invertible on it. To obtain a symplectic structure one has to factorize this hypersurface: one identifies each string F with all the strings obtained by tranlating F by the vector field v. This is (the simplest case of) the Marsden-Weinstein reduction [8]: if there is a function p on a symplectic space generating a vector field w then one sets $p = \text{const.}$ to obtain a hypersurface and then factorizes by w; the result is a symplectic space. This factorisation is perfectly suited for the duality because the dual string $\tilde{F}$ is only defined up to a shift by $\tilde{v}$ and does not depend on shifting F by v. So, duality is a one-to-one mapping of the reductions. There is a pretty physical reason for the factorization: if a string has an exact value of the momentum then its state does not change if we shift it by v.

Now we prove the following proposition: the duality is a symplectomorphism of the (reduced) phase spaces.

Proof: Let LM be the loop space of the target M. As usual, we obtain the phase space from the cotangent bundle T^*LM, on which there is the canonical symplectic form $\Omega_{Ph} = d\theta_{Ph}$. Namely, we identify some submanifold in T^*LM and then factorize it appropriately.[6] The construction goes as follows: if we have a string worldsheet F and a loop l on it, then we define a corresponding element $l_F \in T^*_l LM$. To describe how l_F acts on a vector $u \in T_l LM$, first realize that u can be thought of as a family of vectors $u(X) \in T_X M$ where X

[6] We proceed conceptually as in the case of a relativistic particle in a background; in the σ-model case the submanifold is defined by the Virasoro constraints.

runs along l. Then

$$l_F(u) \equiv \oint_l \mathbf{p}_b(.,u(X)). \tag{11}$$

If we take all l_F's for all possible F's we obtain the mentioned submanifold of T^*LM. Now we identify all l_F's coming from the same extremal F and obtain the phase space.

In this framework we can easily prove the proposition. Let H be a surface in the original phase space, i.e. a 2-parametric family of on-shell strings, and let on each $F \in H$ be a loop $l(F)$. Then by (11)

$$\int_H \Omega_{Ph} = \oint_{\partial H} \theta_{Ph} = \oint_{\bigcup_{F\in\partial H} l(F)} \mathbf{p}_{b(F)}(.,.)$$

The last integral is over a closed surface in M. We will prove that if $\tilde{H}$ is a corresponding family in $\tilde{M}$ (defined up to an independent shift of each $\tilde{F}$ by $\tilde{v}$) then

$$\oint_{\bigcup_{F\in\partial H} l(F)} \mathbf{p} = \oint_{\bigcup_{\tilde{F}\in\partial\tilde{H}} l(\tilde{F})} \tilde{\mathbf{p}}.$$

We will compare the two expressions using the common lifted family H^*. One immediately checks that if t_* and u_* are vectors at a point of a lifted surface F^*, t_* tangent to F^* and u_* arbitrary, then

$$\tilde{\mathbf{p}}(\tilde{t},\tilde{u}) - \mathbf{p}(t,u) = (\omega \wedge \tilde{\omega})(t_* \wedge u_*)$$

so that

$$\oint_{\bigcup_{\tilde{F}\in\partial\tilde{H}} l(\tilde{F})} \tilde{\mathbf{p}} - \oint_{\bigcup_{F\in\partial H} l(F)} \mathbf{p} = \oint_{\bigcup_{F^*\in\partial H^*} l(F^*)} \omega \wedge \tilde{\omega} = 0$$

because $\omega \wedge \tilde{\omega}$ is closed and the closed surface over which we integrate is a boundary.

5 σ-model and projective transformations

In what follows we shall study the duality in the familiar context of the non-linear σ-model. In this case there is a metric G and a 2-form B on the manifold. The action S is minus the area of the surface plus the integral of B over the surface. We assume the surface to be time-like everywhere, i.e. there are two real light-like tangent vectors k, l at any point of the surface; we always choose both of them lying on the same light cone (future or past). Then the area of $k \wedge l$ is simply $-k \cdot l$, i.e.

$$L(k \wedge l) = G(k,l) + B(k,l) \equiv E(k,l). \tag{12}$$

It means that

$$\mathbf{p}_{k\wedge l}(k,.) = E(k,.) \qquad \text{and} \qquad \mathbf{p}_{k\wedge l}(l,.) = -E(.,l). \tag{13}$$

The duality transformation follows directly from Buscher's formula (9), but there is a simpler geometric way of deriving it in the σ-model context. Using Eq. (13) and the decomposition $T_P(T^*M) \simeq T_PM + T_P^*M \simeq T_P\tilde{M} + T_P^*\tilde{M}$ we interpret $k^* = k + \mathbf{p}_{k\wedge l}(k,.) = k + E(k,.)$ as $\tilde{k} + \tilde{E}(\tilde{k},.)$, and accordingly for l^*, thus obtaining the dual bilinear form $\tilde{E} = \tilde{G} + \tilde{B}$. That is, we interpret the graph of E, $\mathbf{E} = \{t + E(t,.)|t \in T_PM\}$, from the dual point of view as the graph of $\tilde{E}$.

The projective formula for $\tilde{E}$ is self-evident now. One simply exchanges T_PM and $T_P\tilde{M}$ by the linear transformation R of $T_P(T^*M)$, $v \leftrightarrow \tilde{v}$ and $w \mapsto w$, if $\omega(w) = \tilde{\omega}(w) = 0$. Then $\tilde{t} + \tilde{E}(\tilde{t},.) = R(t + E(t,.))$.

Our formalism can be easily extended to the case of d commuting symmetries. Then R is an element of a group $O(d,d;\mathbf{Z})$ preserving the natural metric on $T_P(T^*M)$, $(t+\beta)^2 = \beta(t)$.

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