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An Introduction to T-Duality in String Theory

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Abstract

In these lectures a general introduction to T-duality is given. In the abelian case the approaches of Buscher, and Roček and Verlinde are reviewed. Buscher's prescription for the dilaton transformation is recovered from a careful definition of the gauge integration measure. It is also shown how duality can be understood as a quite simple canonical transformation. Some aspects of non-abelian duality are also discussed, in particular what is known on relation to canonical transformations. Some implications of the existence of duality on the cosmological constant and the definition of distance in String Theory are also suggested.

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1 Introduction

Few words have been used with more different meanings than the word "duality". Even within the restricted framework of string theories, duality originally meant a symmetry between the s and the t-channels in strong interactions (coming from the demands in the S-matrix approach of the sixties of Regge behavior without fixed poles and analiticity, which were shown to imply the existence of an infinite number of resonances) [1]. Somewhat related ideas, also termed "duality", appear in the context of Conformal Field Theory (CFT) as simple consequences of locality and associativity of the operator product expansion (OPE) [2].

Duality symmetry plays an important rôle in Statistical Mechanics (for a review and references to the literature see for instance [3]), in particular in the analysis of the phase diagram of spin systems. It can also be understood as a way to show the equivalence between two apparently different theories. On a lattice system described by a Hamiltonian $H(g_i)$ with coupling constants g_i the duality transformation produces a new Hamiltonian $H^*(g_i^*)$ with coupling constants g_i^* on the dual lattice. In this way one can often relate the strong coupling regime of $H(g)$ with the weak coupling regime of $H^*(g^*)$. An important application was the determination of the exact temperature at which the phase transition of the two-dimensional Ising model takes place [4].

More recently, the word "duality" ("space-time duality") has been introduced in yet another sense. T-duality is a symmetry which relates physical properties corresponding to big spacetime radius with quantities corresponding to small radius. This will be our main theme in this review and from now on we will refer to it as just duality (a general reference is [5]). S-duality is a (conjectural) symmetry relating the strong coupling regime with the weak coupling one, a bold generalization of the original conjecture by Montonen and Olive [6]. Still more interesting (and speculative), there is a "duality of dualities": S-duality for strings corresponds to T-duality for fivebranes and conversely (see [7] for a general review). Another formally very similar property is β-duality, a property of the free energy of strings at finite temperature [8] which relates the high and the low temperature phases. For example, for the 10-dimensional heterotic string

$$F(\beta) = \frac{\pi^2}{\beta^2} F(\pi^2/\beta). \qquad (1.0.\ 1)$$

The physical interpretation of this symmetry is, however, somewhat uncertain due to the presence of the Hagedorn temperature.

In String Theory and Two-Dimensional Conformal Field Theory duality is an important tool to show the equivalence of different geometries and/or topologies and in determining some of the genuinely stringy implications on the structure of the low energy Quantum Field Theory limit. Duality symmetry was first described on the context of toroidal compactifications [9]. For the simplest case of a single compactified dimension of radius R, the entire physics of the interacting theory is left unchanged under the replacement $R \to \alpha'/R$ provided one also transforms the dilaton field $\phi \to \phi - \log(R/\sqrt{\alpha'})$ [10]. This simple case can be generalized to arbitrary toroidal compactifications described by constant metric g_{ij} and antisymmetric tensor b_{ij} [12]. The generalization of duality to this case becomes $(g+b) \to (g+b)^{-1}$ and $\phi \to \phi - \frac{1}{2}\log\det(g+b)$. In fact this transformation is an element of an infinite order discrete symmetry group $O(d,d;Z)$ for d-dimensional toroidal compactifications [13, 14]. The symmetry was later extended to the case of non-flat conformal backgrounds in [16]. In Buscher's construction one starts with a manifold M with metric $g_{ij}, i,j = 0, \ldots d-1$, antisymmetric tensor b_{ij} and dilaton field $\phi(x_i)$. One requires the metric to admit at least one continuous abelian isometry leaving invariant the σ-model action constructed out of (g,b,ϕ). Choosing an adapted coordinate system $(x^0, x^\alpha) = (\theta, x^\alpha), \alpha = 1, \ldots d-1$ where the isometry acts by translations of θ, the change of g, b, ϕ is given by

$$\begin{aligned}
&\tilde{g}_{00} = 1/g_{00}, \qquad \tilde{g}_{0\alpha} = b_{0\alpha}/g_{00}, \\
&\tilde{g}_{\alpha\beta} = g_{\alpha\beta} - (g_{0\alpha}g_{0\beta} - b_{0\alpha}b_{0\beta})/g_{00} \\
&\tilde{b}_{0\alpha} = g_{0\alpha}/g_{00}, \\
&\tilde{b}_{\alpha\beta} = b_{\alpha\beta} - (g_{0\alpha}b_{0\beta} - g_{0\beta}b_{0\alpha})/g_{00}, \\
&\tilde{\phi} = \phi - \frac{1}{2}\log g_{00}.
\end{aligned} \tag{1.0. 2}$$

The final outcome is that for any continuous isometry of the metric which is a symmetry of the action one obtains the equivalence of two apparently very different non-linear σ-models. The transformation (1.0. 2) is referred to in the literature as abelian duality due to the abelian character of the isometry of the original σ-model. If n is the maximal number of commuting isometries, one gets a duality group of the form $O(n,n;Z)$ [18]. Duality symmetries are useful in determining important properties of the low-energy effective action, in particular in questions related to supersymmetry breaking and to the lifting of flat directions from the potential [15]. Although the transformation (1.0. 2) was originally obtained using a method apparently not compatible with general covariance, it is not difficult to modify the construction to eliminate this drawback [19]. A particularly useful interpretation of (1.0. 2) is in terms of the gauging of the isometry symmetry [17]. The duality transformation proceeds in two steps: i) First one gauges the isometry group, thus introducing some auxiliary gauge field variables A. The gauge field is required to be flat and this is implemented by adding a Lagrange multiplier term of the form χdA. It is naively clear that if we first perform the integral ovel χ, this provides a δ-function dA on the measure, implying that $A = dX$ is a pure gauge (we consider a spherical world sheet for simplicity). Fixing $X = 0$ the original model is recovered. ii) The second step consists of integrating first the gauge field A. Since there is no gauge kinetic term, the integration is gaussian, yielding a Lagrangian depending on the original variables and the auxiliary variable χ. After fixing the gauge the dual action follows. In [17] it was further shown that if one starts with a conformal field theory (CFT), conformal invariance is preserved by abelian duality. The proof was based on an analogy between the duality transformation and the GKO construction

[32].

Of more recent history is the notion of non-abelian duality [22, 23, 24, 25], which has no analogue in Statistical Mechanics. The basic idea of [22], inspired in the treatment of abelian duality presented in [17], is to consider a conformal field theory with a non-abelian symmetry group G. In this case the gauge field variables A and the Lagrange multipliers live in the Lie algebra associated to G. The duality transformation proceeds in the two steps described above.

In the abelian case it is also possible to work out the mapping between some operators in the original and dual theories, as well as the global topology of the dual manifold [19]. Thus for G abelian we have a rather thorough understanding of the detailed local and global properties of duality. In the non-abelian case global information can only be extracted for σ-models with chiral currents [25]. For these models it is possible to perform a non-local change of variables in the Lagrange multiplier term such that the Lagrangian keeps its local expression and from it the global properties of the dual model can be worked out. The same construction does not work for general σ-models without chiral isometries.

Some interesting reviews on duality can be found in [5] and [20].

The organization of the lectures is as follows:

1. In section two we review the approaches of Buscher [16] and Roček and Verlinde [17] to abelian duality. We also exhibit the kind of information one can obtain with these formalisms. The approach of De la Ossa and Quevedo [22] to non-abelian duality is explained. Some comments are made concerning the global properties of the dual manifold [25].

2. In section three we show that for non-semisimple isometry groups a mixed gravitational-gauge anomaly may emerge in constructing the non-abelian dual. This explains in particular why the example considered in [24] violates conformal invariance to first order in α'.

3. In section four we study with some detail the transformation of the dilaton needed to preserve conformal invariance (to first order in α') under duality.

4. In section five the problem of the behavior of the cosmological constant under duality is addressed. This study is motivated by the work in [36] where an explicit example in which the cosmological constant changes under a duality transformation is considered.

5. In section six we study the implications that duality has in the definition of a proper distance within String Theory. We consider particular families of correlators, manifestly duality invariant, and discuss the properties a distance based on them would have.

6. In section seven the canonical transformation approach to duality is studied. We shall be following [45]. In the abelian case the explicit generating functional

producing Buscher's formulae is constructed. It is shown that all the information which can be obtained in the formulations above can be derived more easily this way. The general formulation of non-abelian duality as a canonical transformation is so far unknown. We review an example [47] where a non-abelian transformation in the $SU(2)$ principal chiral model is constructed as a canonical transformation of type I, the same type as for abelian duality.

7. Section eight contains a partial list of open problems.

2 Abelian and Non-Abelian Dualities

2.1 Abelian Duality

We start with a summary of Buscher's formulation [16]. Consider a non-linear σ-model defined on a d-dimensional manifold M:

$$S = \frac{1}{4\pi\alpha'}\int d^2\xi[\sqrt{h}h^{\mu\nu}g_{ij}\partial_\mu x^i\partial_\nu x^j + i\epsilon^{\mu\nu}b_{ij}\partial_\mu x^i\partial_\nu x^j + \alpha'\sqrt{h}R^{(2)}\phi(x)], \qquad (2.1.\ 1)$$

where g_{ij} is the target space metric, b_{ij} the torsion and ϕ the dilaton field, coupled to the two dimensional scalar curvature in the world sheet $R^{(2)}$. $h_{\mu\nu}$ is the world sheet metric and α' the inverse of the string tension. Let us assume that the σ-model has an abelian isometry represented by a translation in a coordinate θ in the target space. In the coordinates $\{\theta, x^\alpha\}$, $\alpha = 1,\ldots,d-1$, adapted to the isometry, the metric, torsion and dilaton fields are θ-independent. Then the original theory can be obtained from the following $d+1$-dimensional σ-model:

$$\begin{aligned} S_{d+1} = \frac{1}{4\pi\alpha'}\int d^2\xi[\sqrt{h}h^{\mu\nu}(g_{00}V_\mu V_\nu + 2g_{0\alpha}V_\mu\partial_\nu x^\alpha + g_{\alpha\beta}\partial_\mu x^\alpha\partial_\nu x^\beta) \\ +i\epsilon^{\mu\nu}(2b_{0\alpha}V_\mu\partial_\nu x^\alpha + b_{\alpha\beta}\partial_\mu x^\alpha\partial_\nu x^\beta) + 2i\epsilon^{\mu\nu}\tilde{\theta}\partial_\mu V_\nu + \alpha'\sqrt{h}R^{(2)}\phi(x)], \end{aligned} \qquad (2.1.\ 2)$$

where V is a 1-form defined on M and $\tilde{\theta}$ is an additional variable acting as a Lagrange multiplier. The equation of motion for $\tilde{\theta}$ implies $\epsilon^{\mu\nu}\partial_\mu V_\nu = 0$, which in topologically trivial world sheets forces $V_\mu = \partial_\mu\theta$, leading to the original theory. If instead we integrate over the V_μ-fields:

$$V_\mu = -\frac{1}{g_{00}}(g_{0\alpha}\partial_\mu x^\alpha + i\frac{\epsilon_\mu{}^\nu}{\sqrt{h}}(b_{0\alpha}\partial_\nu x^\alpha + \partial_\nu\tilde{\theta})), \qquad (2.1.\ 3)$$

we obtain the dual action:

$$\begin{aligned} \tilde{S} &= \frac{1}{4\pi\alpha'}\int d^2\xi[\sqrt{h}h^{\mu\nu}(\tilde{g}_{00}\partial_\mu\tilde{\theta}\partial_\nu\tilde{\theta} + 2\tilde{g}_{0\alpha}\partial_\mu\tilde{\theta}\partial_\nu x^\alpha + \tilde{g}_{\alpha\beta}\partial_\mu x^\alpha\partial_\nu x^\beta) \\ &\quad +i\epsilon^{\mu\nu}(2\tilde{b}_{0\alpha}\partial_\mu\tilde{\theta}\partial_\nu x^\alpha + \tilde{b}_{\alpha\beta}\partial_\mu x^\alpha\partial_\nu x^\beta) + \alpha'\sqrt{h}R^{(2)}\phi(x)], \end{aligned} \qquad (2.1.\ 4)$$

where:

$$\begin{aligned}
\tilde{g}_{00} &= \frac{1}{g_{00}} \\
\tilde{g}_{0\alpha} &= \frac{b_{0\alpha}}{g_{00}}, \qquad \tilde{b}_{0\alpha} = \frac{g_{0\alpha}}{g_{00}} \\
\tilde{g}_{\alpha\beta} &= g_{\alpha\beta} - \frac{g_{0\alpha}g_{0\beta} - b_{0\alpha}b_{0\beta}}{g_{00}} \\
\tilde{b}_{\alpha\beta} &= b_{\alpha\beta} - \frac{g_{0\alpha}b_{0\beta} - g_{0\beta}b_{0\alpha}}{g_{00}}.
\end{aligned} \tag{2.1. 5}$$

(2.1. 5) show that duality relates very different geometries. We will see that it may also lead to different topologies. The integration on V_μ produces a factor in the measure $\det g_{00}$ which conveniently regularized yields the shift of the dilaton:

$$\tilde{\phi} = \phi - \frac{1}{2}\log g_{00}. \tag{2.1. 6}$$

The regularization prescription in order to find (2.1. 6) is fixed by requiring conformal invariance of the dual theory. In [16] the following definition was shown to yield the correct dilaton shift satisfying conformal invariance to first order in α':

$$\det A \equiv \frac{\det \Delta_A}{\det \Delta}, \tag{2.1. 7}$$

where $\Delta_A = -\frac{1}{\sqrt{h}}\partial_\mu(\sqrt{h}h^{\mu\nu}A\partial_\nu)$, $\Delta = -\frac{1}{\sqrt{h}}\partial_\mu(\sqrt{h}h^{\mu\nu}\partial_\nu)$. We will further justify this definition for the determinant in section four.

The σ-model defined by $(\tilde{g}, \tilde{b}, \tilde{\phi})$ is independent of the $\tilde{\theta}$ variable, hence the original model can be recovered by performing the duality transformation with respect to $\tilde{\theta}$ shifts.

This formalism has apparently some limitations:

1. It seems that general covariance is broken due to the choice of adapted coordinates needed to perform the duality transformation. This also obscures the issue of the global topology of the dual manifold, which is harder to describe if one works in local coordinates.

2. If the original theory has some isometries not commuting with the one used for duality they generically disappear as local symmetries in the dual model.

3. The original model is recovered from (2.1. 2) only in spherical world sheets. The monodromy of the V variable must be fixed by imposing the absence of modular anomalies. For that we need to know which are the orbits of the Killing vector.

4. When the Killing vector has fixed points, V_μ in (2.1. 3) is singular. In this case it could be much wiser to work with the $d+1$-dimensional action (2.1. 2).

5. What happens to the operator mapping from the above construction?.
6. What are the general properties of the non-abelian generalization?.

All these questions can be addressed with a different way of constructing the dual model. We will follow the work of Roček and Verlinde [17]. The formulation of Roček and Verlinde starts with the same σ-model (2.1. 1) with the abelian isometry represented by $\theta \to \theta + \epsilon$. The key point is to gauge the isometry by introducing some gauge fields A_μ transforming as $\delta A_\mu = -\partial_\mu \epsilon$. With a Lagrange multiplier term the gauge field strength is required to vanish, forcing the constraint that the gauge field is pure gauge. After gauge fixing the original model is then recovered.

Gauging the isometry in (2.1. 1) and adding the Lagrange multipliers term leads to:

$$\begin{aligned} S_{d+1} &= \frac{1}{4\pi\alpha'}\int d^2\xi[\sqrt{h}h^{\mu\nu}(g_{00}(\partial_\mu\theta + A_\mu)(\partial_\nu\theta + A_\nu) + 2g_{0\alpha}(\partial_\mu\theta + A_\mu)\partial_\nu x^\alpha \\ &+ g_{\alpha\beta}\partial_\mu x^\alpha \partial_\nu x^\beta) + i\epsilon^{\mu\nu}(2b_{0\alpha}(\partial_\mu\theta + A_\mu)\partial_\nu x^\alpha + b_{\alpha\beta}\partial_\mu x^\alpha\partial_\nu x^\beta) + 2i\epsilon^{\mu\nu}\tilde{\theta}\partial_\mu A_\nu \\ &+ \alpha'\sqrt{h}R^{(2)}\phi(x)]. \end{aligned} \tag{2.1. 8}$$

The dual theory is obtained integrating the A fields:

$$A_\mu = -\frac{1}{g_{00}}(g_{0\alpha}\partial_\mu x^\alpha + i\frac{\epsilon_\mu{}^\nu}{\sqrt{h}}(b_{0\alpha}\partial_\nu x^\alpha + \partial_\nu\tilde{\theta})), \tag{2.1. 9}$$

and fixing $\theta = 0$.

In [17] it is shown that the original and dual theories can be considered as the vectorial and axial cosets of a given higher dimensional theory with chiral currents in which the abelian symmetry group is gauged. This shows that conformal invariance is preserved by abelian duality to all orders in α' since one can think of the initial and dual theories as two different functional integral representations of the same conformal field theory.

Within this approach the open questions enumerated above can be solved.

The procedure of gauging the isometry can be implemented in arbitrary coordinates [19]. If the original σ-model has a torsion term then Noether's procedure must be followed, as made explicit in [26]. Let us consider the following σ-model:

$$\begin{aligned} S &= \frac{1}{8\pi}\int g_{ij}\partial_\mu x^i\partial^\mu x^j + \frac{i}{8\pi}\int b_{ij}dx^i \wedge dx^j \\ &= \frac{1}{2\pi}\int d^2\xi(g_{ij} + b_{ij})\partial x^i\overline{\partial} x^j, \end{aligned} \tag{2.1. 10}$$

where $\alpha' = 2$. Let k^i be a Killing vector for the metric g:

$$\mathcal{L}_k g_{ij} = \nabla_i k_j + \nabla_j k_i = 0. \tag{2.1. 11}$$

Invariance of S requires also

$$\mathcal{L}_k b = d\omega, \qquad \omega = i_k b - v, \tag{2.1. 12}$$

where $(i_k b)_j \equiv k^i b_{ij}$ and v is a one-form such that $i_k H = -dv$ ($H = db$ locally). The associated conservation law is:

$$\partial \bar{J}_k + \bar{\partial} J_k = 0 \tag{2.1. 13}$$

$$\begin{aligned} J_k &= (k - i_k b + \omega)_i \partial x^i = (k - v)_i \partial x^i \equiv (k - v) \cdot \partial x \\ \bar{J}_k &= (k + i_k b - \omega)_i \bar{\partial} x^i = (k + v)_i \bar{\partial} x^i \equiv (k + v) \cdot \bar{\partial} x. \end{aligned} \tag{2.1. 14}$$

If we wish to gauge the isometry we introduce gauge fields $A, \bar{A}$, with $\delta_\epsilon A = -\partial \epsilon$, $\delta_\epsilon \bar{A} = -\bar{\partial}\epsilon$, and $\delta x^i = \epsilon k^i(x)$ now with ϵ a function on the world sheet. It can be shown [19] that the action:

$$S_{d+1} = \frac{1}{2\pi} \int d^2\xi [(g_{ij} + b_{ij}) \partial x^i \bar{\partial} x^j + (J_k - \partial\chi)\bar{A} + (\bar{J}_k + \bar{\partial}\chi)A + k^2 A\bar{A}], \tag{2.1. 15}$$

is invariant under:

$$\begin{aligned} \delta_\epsilon x^i &= \epsilon k^i(x) \qquad \delta_\epsilon \chi = -\epsilon k \cdot v \\ \delta_\epsilon A &= -\partial\epsilon \qquad \delta_\epsilon \bar{A} = -\bar{\partial}\epsilon. \end{aligned} \tag{2.1. 16}$$

The Lagrange multiplier term forces the gauge field to be flat and at the same time cancels the anomalous variation of the Lagrangian. For a genus g world-sheet Σ_g and compact isometry orbits we may have large gauge transformations. We consider multivalued gauge functions:

$$\oint_\gamma d\epsilon = 2\pi n(\gamma) \qquad n(\gamma) \in Z, \tag{2.1. 17}$$

where γ is a non-trivial homology cycle in Σ_g. Since we are dealing with abelian isometries it suffices to consider only the toroidal case $g = 1$. The variation of S_{d+1} is:

$$\begin{aligned} \delta S_{d+1} &= \frac{1}{2\pi} \int \left(\partial\chi \bar{\partial}\epsilon - \partial\epsilon \bar{\partial}\chi \right) = \frac{i}{4\pi} \int_T d\chi \wedge d\epsilon \\ &= \frac{i}{4\pi} \left(\oint_a d\chi \oint_b d\epsilon - \oint_a d\epsilon \oint_b d\chi \right) \end{aligned} \tag{2.1. 18}$$

where a and b are the two generators of the homology group of the torus T. Since ϵ is multivalued by $2\pi Z$, we learn from (2.1. 18) that χ is multivalued by $4\pi Z$:

$$\oint_\gamma d\chi = 4\pi m(\gamma) \qquad m(\gamma) \in Z. \tag{2.1. 19}$$

For a non-compact isometry $\delta S_{d+1} = 0$ and $d\chi$ may in general have real periods. The original theory is recovered integrating the Lagrange multiplier, which appears in the action in the form of a closed one form. In non trivial world sheets these one forms have exact and harmonic components. The χ-dependence in (2.1. 15) is:

$$S_\chi = -\frac{1}{2\pi} \int (d\chi_0 + \chi_h) \wedge A. \tag{2.1. 20}$$

Integrating by parts in the exact part and using Riemann's bilinear identity we obtain:

$$S_\chi = \frac{1}{2\pi}\int \chi_0 \wedge dA - \frac{1}{2\pi}(\oint_a \chi_h \oint_b A - \oint_a A \oint_b \chi_h). \qquad (2.1.\ 21)$$

Integration on χ_0 yields the constraint $dA = 0$ and integration on the harmonic components leads to:

$$\oint_a A = \oint_b A = 0. \qquad (2.1.\ 22)$$

Both constraints imply that A must be an exact one form. Fixing the gauge the original theory is recovered. By construction S_{d+1} is general covariant, and therefore we have a clear idea of the d-dimensional geometrical interpretation of the model. Locally the dual manifold is equivalent to $(M/S^1) \times S^1$ (for compact isometries), where the quotient means that the gauge is fixed by dividing by the orbits of the isometry group. Generically we expect topology change as a consequence of duality. However the more delicate issue is whether the dual manifold $\tilde{M}$ is indeed a product or a twisted product (non-trivial bundle). It is also useful to notice that in the previous arguments the structure of $\pi_1(M)$ played no rôle. This rises some questions concerning the way the operators in both theories are mapped under duality [19]. The nature of the product relating the gauged original manifold and the Lagrange multipliers space turns out to be dictated by the gauge fixing procedure, in particular by Gribov problems. We use an example to labor this point. This is the $SU(2)$ principal chiral model, which represents a σ-model in S^3. The dual with respect to a fixed point free abelian isometry is locally $S^2 \times S^1$. One knows that this also holds globally when performing the gauge fixing. This reveals that the dual manifold is $S^2 \times S^1$ and not a squeezed S^3 (for details see [19]).

The interest of working with the $d+1$-dimensional theory is that the possible singularities in the dual theory due to the existence of fixed points do not emerge. However if we are interested in the explicit form of the dual σ-model we have to eliminate the gauge field A. Integrating on A in (2.1. 15), the dual model $(\tilde{g}, \tilde{b}, \tilde{\phi})$ reads:

$$\begin{aligned}
\tilde{g}_{00} &= \frac{1}{k^2} \\
\tilde{g}_{0\alpha} &= \frac{v_\alpha}{k^2}, \qquad \tilde{b}_{0\alpha} = \frac{k_\alpha}{k^2} \\
\tilde{g}_{\alpha\beta} &= g_{\alpha\beta} - \frac{k_\alpha k_\beta - v_\alpha v_\beta}{k^2} \\
\tilde{b}_{\alpha\beta} &= b_{\alpha\beta} - \frac{k_\alpha v_\beta - k_\beta v_\alpha}{k^2} \\
\tilde{\phi} &= \phi - \frac{1}{2}\log k^2.
\end{aligned} \qquad (2.1.\ 23)$$

Going to adapted coordinates and fixing the gauge we recover Buscher's formulae since $k^2 = g_{00}$, $v_\alpha = b_{0\alpha}$, $k_\alpha = g_{0\alpha}$. However this choice of coordinate system unables us to obtain global information about the dual manifold.

The explicit operator mapping can be constructed [19]. The duals to vertex operators $V_p = \exp ip\theta$, which are momenta operators in the direction of the isometry, are non-local operators which can be interpreted as winding operators only for flat compact isometries. Thus, the description of duality in toroidal compatifications as the symmetry exchanging momenta and windings is modified. In particular the structure of $\pi_1(M)$ turns out not to be important. The winding operators are associated to compact isometry orbits in the target space manifold and not to homologically non-trivial cycles as is usually interpreted for toroidal compactifications.

The extension to non-abelian isometry groups is easily done in this formalism. The details are worked out in the next section.

2.2 Non-Abelian Duality

The same procedure à la Roček and Verlinde was generalized in [22] to construct the dual with respect to a given no-abelian isometry group G. The gauge fields take values in the Lie algebra associated to the isometry group and they transform under gauge transformations $x^m \rightarrow g^m{}_n x^n$, $m, n = 1, \dots, N$, where $g \in G$, as $A \rightarrow g(A + \partial)g^{-1}$. The isometry is gauged by introducing covariant derivatives[2]:

$$\partial x^m \rightarrow Dx^m = \partial x^m + A^\alpha (T_\alpha)^m{}_n x^n, \tag{2.2. 1}$$

where T_α is a N-dimensional representation for the α generator of the Lie algebra of G. The flatness of the gauge fields is imposed by the term:

$$\int Tr(\chi F), \tag{2.2. 2}$$

with $F = \partial \bar{A} - \bar{\partial} A + [A, \bar{A}]$. The χ-fields take values in the Lie algebra associated to G and transform in the adjoint representation to preserve gauge invariance. Integration on χ fixes $F = 0$ in semisimple groups, then A is pure gauge (in spherical world sheets) and after gauge fixing we recover the original model. As before the dual model is obtained integrating on A and then fixing the gauge. For non-semisimple groups the Lagrange multipliers term must be introduced in a different way since the Cartan-Killing metric is degenerate and the integration on χ does not imply that all the F-components are zero. In this case the χ-fields must be taken in the basis dual to T_α and they transform in the coadjoint representation.

We can write the gauged σ-model action as:

$$\begin{aligned} S_{\text{gauge}} \;\; = \;\; & \frac{1}{2\pi} \int d^2z [Q_{mn} Dx^m \bar{D} x^n + Q_{m\mu} Dx^m \bar{\partial} x^\mu + Q_{\mu n} \partial x^\mu \bar{D} x^n + Q_{\mu\nu} \partial x^\mu \bar{\partial} x^\nu \\ & + Tr(\chi F) + \frac{1}{2} R^{(2)} \phi], \end{aligned} \tag{2.2. 3}$$

[2]Note that this way of gauging a continuous global isometry is only valid for certain σ-models and isometry groups [26].

where $Q = g + b$, latin indices are associated to coordinates adapted to the non-abelian isometry and greek indices to inert coordinates. We can write (2.2. 3) as:

$$S_{\rm gauge} = S[x] + \frac{1}{2\pi}\int d^2z[\bar{A}^\alpha f_{\alpha\beta}A^\beta + \bar{h}_\alpha A^\alpha + h_\alpha \bar{A}^\alpha + \frac{1}{2}R^{(2)}\phi], \tag{2.2. 4}$$

with:

$$\begin{aligned} h_\alpha &= (Q_{mn}\partial x^m + Q_{\mu n}\partial x^\mu)(T_\alpha)^n{}_q x^q - \partial\chi^\alpha T_R \eta_{\alpha\alpha} \\ \bar{h}_\alpha &= (Q_{n\mu}\bar{\partial} x^\mu + Q_{nm}\bar{\partial} x^m)(T_\alpha)^n{}_q x^q + \bar{\partial}\chi^\alpha T_R \eta_{\alpha\alpha} \\ f_{\alpha\beta} &= Q_{mn}(T_\beta)^m{}_r (T_\alpha)^n{}_p x^r x^p + C_{\beta\alpha}{}^\gamma \chi^\gamma T_R \eta_{\gamma\gamma}, \end{aligned} \tag{2.2. 5}$$

where $[T_\alpha, T_\beta] = C_{\alpha\beta}{}^\gamma T_\gamma$ and $Tr(T_\alpha T_\beta) = T_R\eta_{\alpha\beta}$ ($Tr(T^{'}_\alpha T_\beta) = T_R\eta_{\alpha\beta}$ if the group is not semisimple).

Integrating $A, \bar{A}$:

$$\tilde{S} = S[x] + \frac{1}{2\pi}\int d^2z[\bar{h}_\alpha (f^{\alpha\beta})^{-1} h_\beta + \frac{1}{2}R^{(2)}\tilde{\phi}], \tag{2.2. 6}$$

where $\tilde{\phi}$ is given by:

$$\tilde{\phi} = \phi - \frac{1}{2}\log({\rm det} f), \tag{2.2. 7}$$

after regularizing the factor det$f$ coming from the measure as in previous section. In all the examples considered the dual model with this dilaton satifies the conformal invariance conditions to first order in $\alpha^{'}$, but a general proof analogous to that of Buscher in the abelian case is lacking.

The construction above seems to be a straightforward extension of abelian duality. However this is not so. Non-abelian duality is quite different from abelian duality, as it is clearly manifested in the context of Statistical Mechanics. In this context duality transformations are applied to models defined on a lattice L with physical variables taking values on some abelian group G. The duality transformation takes us from the triplet $(L, G, S[g])$, where $S[g]$ is the action depending on some coupling constants labelled collectively by g to a model $(L^*, G^*, S^*[g^*])$ on the dual lattice L^* with variables taking values on the dual group G^* and with some well-defined action $S^*[g^*]$. For abelian groups, G^* is the representation ring, itself a group, and when we apply the duality transformation once again we obtain the original model. As soon as the group is non-abelian the previous construction breaks down because the representation ring of G is not a group [3]. In particular the non-abelian duality transformation cannot be performed again to obtain the model we started with. In the context of String Theory the major problems in stating non-abelian duality as an exact symmetry come when trying to extend it to non-trivial world sheets and when performing the operator mapping (a detailed explanation on this can be found in [19]). With the usual Lagrange multipliers variables is not possible to extract global information. In σ-models with chiral currents a non-local change of variables in the Lagrange multipliers term can be done such that the dual

Lagrangian is local. In these variables the dual theory can be shown to be the product of the coset of the original manifold by the isometry group M/G and the WZW model of group G [25]. This applies in particular to the case of abelian groups, in agreement with the results on abelian duality. The dual variables introduced in [25] are the base of the non-abelian bosonization studied in [27]. As in the abelian case the more delicate issue is to know what kind of a product it is. This can be worked out in the case of WZW models [49]. The partition function at genus one of a WZW model with group G is not a product of any modular invariant partition function for the G/H coset theory and one of the H WZW-model (now H is the gauged isometry group). The Kac-Moody characters of the G_k WZW-model have a well-defined decomposition in terms of products of G/H and H_k characters. This implies that the product is a twisted product. In fact in this case the explicit integration on the A-fields can be made and the result is that the model is self-dual.

For non-abelian isometry groups certain anomalies can arise when performing the non-abelian dual construction [25, 28]. When one analyzes carefully the measure of integration over the gauge fields and its dependence on the world sheet metric, one encounters a mixed gauge and gravitational anomaly [29] when any generator of the isometry group in the adjoint representation has a non-vanishing trace. This can only happen for non-semisimple groups. This mixed anomaly generates a contribution to the trace anomaly which cannot be absorbed in a dilaton shift and imposes a mild anomaly cancellation condition for the consistency of non-abelian duality. We treat this point in the next section.

3 Mixed Anomalies and Effective Actions

Since we are interested in conformal invariance, we introduce an arbitrary metric $h_{\alpha\beta}$ on the world sheet and compute the contribution to the trace anomaly of the auxiliary gauge fields $A^a_\pm$. If for simplicity we work on genus zero surfaces, the most straightforward way to compute the dependence of the effective action on the world sheet metric is to first parametrize $A_\pm$ as:

$$A_+ = L^{-1}\partial_+ L, \;\; A_- = R^{-1}\partial_- R, \tag{3.0. 1}$$

for L, R group elements. We can think of $x^\pm$ as light-cone variables or as complex coordinates, and they depend on the metric being used. In changing variables from $A_\pm$ to (L, R) we encounter jacobians:

$$\mathcal{D}A_+\mathcal{D}A_- = \mathcal{D}L\,\mathcal{D}R\,\det(D_+(A_+)D_-(A_-)) \tag{3.0. 2}$$

with $A_\pm$ given by (3.0. 1) (we take $A_\pm$ as antihermitian matrices). We can write the determinants in (3.0. 2) in terms of a pair of (b,c)-systems (b_{+a}, c^a), $(b_{-a}, \tilde{c}^a)$. $c, \tilde{c}$ are 0-forms transforming in the adjoint representation of the group. For arbitrary groups $b_\pm$ transform in the coadjoint representation. The determinants in (3.0. 2) can be exponentiated in terms of the (b,c)- systems with an action:

$$S[b_\pm, c, \tilde{c}] = \frac{i}{\pi}\int (b_+ D_-(A)c + b_- D_+(A)\tilde{c}), \tag{3.0. 3}$$

which is formally conformal invariant. The variation of S with respect to the metric is given by the energy-momentum tensor $T_{\pm\pm}$. We can ignore momentarily that $A_\pm$ are given by (3.0. 1) and work with arbitrary gauge fields. We can compute the dependence of the effective action for (3.0. 3) on the metric $h_{\alpha\beta}$ and the gauge field using Feynman graphs. Expanding about the flat metric, and using the methods in [29], the first diagrams contributing to the effective action are

$h_{--}(h_{++})$ couples to $T_{++}(T_{--})$, and $A_-(A_+)$ to the ghost currents $j_+(j_-)$ given by:

$$T_{++} = \partial_+ c^a b_{+a}, \;\; T_{--} = \partial_- \tilde{c}^a b_{-a} \tag{3.0. 4}$$

$$j_+^i = b_{+a}(T^i)^a_b c^b, \;\; j_-^i = b_{-a}(T^i)^a_b \tilde{c}^b \tag{3.0. 5}$$

If one keeps track of the $i\epsilon$ prescriptions in the propagators appearing in the graphs, the loop integrals are finite, and we can write their contributions to the effective action as:

$$W^{(2)} = \frac{1}{4\pi} Tr T^a \int d^2p (h_{--}(p)\frac{p_+^2}{p_-}A_-^a(-p) + h_{++}(p)\frac{p_-^2}{p_+}A_+^a(-p)). \tag{3.0. 6}$$

The coefficient of (3.0. 6) and $W^{(2)}$ may also be computed using the OPE:

$$T(z)j_a(w) \sim \frac{TrT_a}{(z-w)^3} + \frac{1}{(z-w)^2}j_a(w) + \frac{1}{(z-w)}\partial j_a(w). \tag{3.0. 7}$$

As it stands, $W^{(2)}$ has a gravitational anomaly, i.e. the energy-momentum tensor is not conserved. However we can still add local counterterms to (3.0. 6) to recover general coordinate invariance. Since to first order in h the two-dimensional scalar curvature has as Fourier transform:

$$R(p) = 2(2p_+p_-h_{+-}(p) - p_+^2 h_{--} - p_-^2 h_{++}), \tag{3.0. 8}$$

if we add the counterterms:

$$\begin{aligned} W_{\text{c.t.}} &= \frac{1}{4\pi} TrT_a \int A_-^a(-p)(h_{++}(p)p_- - 2p_+h_{+-}) \\ &+ \frac{1}{4\pi} TrT_a \int A_+^a(-p)(h_{--}(p)p_+ - 2p_-h_{+-}), \end{aligned} \tag{3.0. 9}$$

we obtain an effective action

$$W^{(2)} = \frac{1}{16\pi} Tr T_a \int R(p) \frac{p_+ A^a_-(-p) + p_- A^a_+(-p)}{p_+ p_-}, \qquad (3.0.\ 10)$$

leading to a conserved energy-momentum tensor, although it contains a trace anomaly which is not proportional to $R(p)$ and therefore it cannot be absorbed in a modification of the dilaton transformation. Varying (3.0. 10) with respect to h_{+-} leads to:

$$\langle T_{+-} \rangle = \frac{\delta W^{(2)}}{\delta h_{+-}} = \frac{1}{4\pi} Tr T_a (p_+ A^a_-(-p) + p_- A^a_+(-p)) \qquad (3.0.\ 11)$$

which in covariant form becomes $\sim Tr T_a \nabla^\alpha A^a_\alpha$.

Similarly we can vary the effective action to this order with respect to gauge transformations to evaluate the corresponding gauge anomaly:

$$\begin{aligned}(D_- \frac{\delta W^{(2)}}{\delta A^a_-} + D_+ \frac{\delta W^{(2)}}{\delta A^a_+}) &\sim p_- \frac{\delta W^{(2)}}{\delta A^a_-(p)} + p_+ \frac{\delta W^{(2)}}{\delta A^a_+(p)} \\ &= p_- \langle j_{a+}(p) \rangle + p_+ \langle j_{a-}(p) \rangle = -\frac{1}{8\pi} Tr T_a R(-p). \qquad (3.0.\ 12)\end{aligned}$$

This is a different way of writing the third order pole in the OPE (3.0. 7). From (3.0. 11) we see that at this order ($W^{(2)}$) the trace anomaly is not proportional to R, and it therefore cannot be absorbed in a contribution to the dilaton or the effective value of c (the central charge of the Virasoro algebra). The contribution in (3.0. 11) spoils the conformal invariance of the dual theory, and further fields should be required to cancel it. However in that case the resulting theory would not agree with the one obtained through a naive duality transformation. Another way to obtain the same conclusion as in (3.0. 11) is to use heat kernel methods. Both methods agree and we conclude that the condition for the duality transformation to respect conformal invariance is that the generators of the duality group in the adjoint representation should have a vanishing trace. The opposite may only happen for non-semisimple groups, as in the example discussed in [24].

4 The Transformation of the Dilaton

It is well known [16] that the transformation (2.1. 5) is not the whole story. Indeed, the dual model is not even conformally invariant in general, unless an appropriate transformation of the dilaton is included, namely

$$\tilde{\phi} = \phi - \frac{1}{2} \log k^2. \qquad (4.0.\ 1)$$

Perhaps the simplest way to realize that something has to change in the dilaton coupling is to insist on the demand that the BRS charge be nilpotent. It is well-known [33, 34] that the BRS charge can be written as:

$$Q = \oint \frac{dz}{2\pi i} c(z)(T^{(x)} + \frac{1}{2} T_{gh}), \qquad (4.0.\ 2)$$

where

$$T^{(x)} \equiv -\frac{1}{2} g_{\mu\nu} \partial x^\mu \partial x^\nu + \frac{1}{2} \partial^2 \phi$$
$$T_{gh} \equiv -2b\partial c - \partial b c \qquad (4.0.\ 3)$$

and $Q^2 = 0$ (in the OPE sense) is equivalent to the consistency conditions of the σ-model β-functions equal to zero [35].

Using the fact that after performing a duality transformation

$$\tilde{T}^{(x)} = T^{(x)} + \frac{1}{2k^2}[(k.\partial x)^2 - ((v-w).\partial x)^2] + \frac{1}{2}\partial^2(\tilde{\phi} - \phi) \qquad (4.0.\ 4)$$

the condition $\tilde{Q}^2 = 0$ necessarily leads to (4.0. 1).

We can trace the need for a transformation of the dilaton to the behavior of the measure under conformal transformations. Under a Weyl rescaling of the 2-d world-sheet metric, $g \to e^\sigma g$, the integration measure over the embeddings behaves (to first order in σ) as:

$$\mathcal{D}_{(e^\sigma g)} x = \mathcal{D}_g x \, e^{\frac{d}{48\pi} S_L(\sigma) + 6\alpha' \int (-\nabla^2 \phi + (\nabla\phi)^2 - \frac{1}{4} R + \frac{1}{48} H^2)\sigma}, \qquad (4.0.\ 5)$$

where $S_L(\sigma)$ is the Liouville action. This means that although they are formally the same, both measures $\mathcal{D}x$ and $\mathcal{D}\tilde{x}$ behave in a very different way under Weyl transformations unless, of course, a compensating transformation of the dilaton is introduced to this purpose.

In the path integral approach the way to obtain the correct dilaton shift yielding to a conformally invariant dual theory can be seen as follows. Let us work with the approach of Roček and Verlinde. In complex coordinates and on spherical world sheets we can parametrize $A = \partial\alpha$, $\bar{A} = \bar{\partial}\beta$ (as we previously did in (3.0. 1)), for some 0-forms α, β in the manifold M. The change of variables from $A, \bar{A}$ to α, β produces a factor in the measure:

$$\mathcal{D}A\mathcal{D}\bar{A} = \mathcal{D}\alpha\mathcal{D}\beta(\det\partial)(\det\bar{\partial}) = \mathcal{D}\alpha\mathcal{D}\beta(\det\Delta). \qquad (4.0.\ 6)$$

Substituting $A, \bar{A}$ as functions of α, β in (2.1. 8) and integrating on α, β, the following determinant emerges:

$$(\det(\partial g_{00} \bar{\partial}))^{-1}. \qquad (4.0.\ 7)$$

In particular, the integration on β produces a delta-function

$$\delta(\bar{\partial}(g_{00}\partial\alpha + (g_{0\alpha} - b_{0\alpha})\partial x^\alpha - \partial\tilde{\theta})), \qquad (4.0.\ 8)$$

which integrated on α yields the factor in the measure (4.0. 7).

What we finally get in the measure is then

$$\frac{\det\Delta}{\det\Delta_{g_{00}}} \qquad (4.0.\ 9)$$

where $\Delta_{g_{00}}$ is given as in (2.1. 7) in complex notation. This formula provides a justification for Buscher's prescription (see also [30]) for the computation of the determinant arising from the naive gaussian integration. As we have just seen some care is needed in order to correctly define the measure of integration over the gauge fields. From (4.0. 9) the dilaton shift (2.1. 6) is obtained in the following way. Writing g_{00} as $g_{00} = 1 + \sigma \approx e^{\sigma}$ we have:

$$\Delta_{g_{00}} = (1+\sigma)\Delta - h^{\mu\nu}\partial_\mu \sigma \partial_\nu. \qquad (4.0.\ 10)$$

Substituting in the infinitesimal variation of Schwinger's formula:

$$\delta \log \det \Delta = Tr \int_\epsilon^\infty dt \delta\Delta e^{-t\Delta} \qquad (4.0.\ 11)$$

we obtain

$$\delta \log \det \Delta_{g_{00}} = -\int d^2\xi \sqrt{h}\Omega \langle \xi | e^{-\epsilon(\Delta + \sigma\Delta - h^{\mu\nu}\partial_\mu\sigma\partial_\nu)} | \xi \rangle, \qquad (4.0.\ 12)$$

where $\delta\Delta_{g_{00}} = -\Omega\Delta_{g_{00}}$ with $\delta h_{\mu\nu} = \Omega\delta_{\mu\nu}$. We can now use the heat kernel expansion [31]:

$$\langle \xi | e^{-\epsilon D} | \xi \rangle = \frac{1}{4\pi\epsilon} + \frac{1}{4\pi}(\frac{1}{6}R^{(2)} - V), \qquad (4.0.\ 13)$$

where

$$D \equiv \Delta - 2ih^{\mu\mu}A_\mu\partial_\nu + (-\frac{i}{\sqrt{h}}\partial_\mu(\sqrt{h}h^{\mu\nu}A_\nu) + h^{\mu\nu}A_\mu A_\nu) + V. \qquad (4.0.\ 14)$$

For $D = \Delta + \sigma\Delta - h^{\mu\nu}\partial_\mu\sigma\partial_\nu$ and after dropping the divergent term $1/4\pi\epsilon$ and the quadratic terms in σ we obtain:

$$\delta \log \det \Delta_{g_{00}} = -\frac{1}{8\pi}\int d^2\xi \sqrt{h} R^{(2)} \log g_{00}. \qquad (4.0.\ 15)$$

Substituting in (2.1. 7):

$$\det g_{00} = \exp(-\frac{1}{8\pi}\int d^2\xi \sqrt{h} R^{(2)} \log g_{00}), \qquad (4.0.\ 16)$$

which implies $\tilde{\phi} = \phi - \frac{1}{2}\log g_{00}$.

5 Duality and the Cosmological Constant

A striking feature of duality is the fact that the cosmological constant, defined as the asymptotic value of the scalar curvature, is not in general invariant under the transformation. This fact was first noticed in [36] for the case of a WZW model with group $\widetilde{SL(2,R)}$ where a discrete subgroup was gauged. This space has negative cosmological constant and under a given duality transformation it is mapped into an asymptotically flat space (into a black string). This implies that the usual definition of the cosmological constant from the low-energy effective action is not satisfactory. Even at large distances, if duality is not broken there is a symmetry between local (momentum) modes and non-local (winding) modes. One is lead to wonder to what extent the cosmological constant is a string observable[3]. The contribution to the cosmological constant of the massless sector might be cancelled by the tower of massive states always present in String Theory (proposals along these lines using the Atkin-Lehner symmetry were advanced by G. Moore [39]).

We study now the behavior of the scalar curvature under duality. If the space-time metric in the σ-model takes the form

$$ds^2 = g_{ij} dx^i dx^j \qquad i,j = 0,1,2,...,d-1, \tag{5.0. 1}$$

where x^0 is adapted to the isometry $\vec{k} = \partial/\partial x^0$, (5.0. 1) can be written as

$$\begin{aligned} ds^2 &= (e^0)^2 + (g_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}) dx^\alpha dx^\beta \\ e^0 &= k dx^0 + \frac{k_\alpha}{k} dx^\alpha \\ k^2 &= k_i k^i = g_{00} \qquad k_\alpha = g_{0\alpha}. \end{aligned} \tag{5.0. 2}$$

Buscher's transformation leads to a dual metric

$$\begin{aligned} d\tilde{s}^2 &= (\tilde{e}^0)^2 + (g_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}) dx^\alpha dx^\beta \\ \tilde{e}^0 &= \frac{1}{k}(d\tilde{x}^0 + v_\alpha dx^\alpha), \end{aligned} \tag{5.0. 3}$$

$\tilde{x}^0$ being the Lagrange multiplier and v is defined as in section 2 by $k^l H_{lij} = -\partial_{[i} v_{j]}$, $H = db$. The dual scalar curvature following from (5.0. 3) is

$$\tilde{R} = R - \frac{4}{k^2} g^{\alpha\beta} \partial_\alpha k \partial_\beta k + \frac{4}{k} \Delta_q^{d-1} k + \frac{1}{k^2} H_{0\alpha\beta} H^{0\alpha\beta} - \frac{k^2}{4} F_{\alpha\beta} F^{\alpha\beta}, \tag{5.0. 4}$$

where Δ_q^{d-1} is the $(d-1)$-dimensional Laplacian for the metric $g^q_{\alpha\beta} = g_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}$, and $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ with $A_\alpha = k_\alpha / k^2$. (5.0. 4) can be rewritten as

$$\tilde{R} = R + 4\Delta \log k + \frac{1}{k^2} H_{0\alpha\beta} H^{0\alpha\beta} - \frac{k^2}{4} F_{\alpha\beta} F^{\alpha\beta}. \tag{5.0. 5}$$

From (5.0. 5) we see that:

- The only way to "flatten" negative curvature is by having torsion in the initial space-time. Otherwise the dual of an asymptotically negatively curved space time is a space of the same type.

[3]Similar remarks would apply to the concept of spacetime singularity in String Theory [37, 38].

- Positive curvature seems easier to flatten.
- In general the asymptotic behaviors of $\tilde{R}$ and R are different, which proves the statement at the beginning of this section.
- In the particular case of constant toroidal compactifications $\tilde{R} = R$, in agreement with the result in [40].

We can also construct the dual torsion

$$\begin{aligned} \tilde{H}_{0\alpha\beta} &= -\frac{1}{2}F_{\alpha\beta} \\ \tilde{H}_{\alpha\beta\rho} &= H_{\alpha\beta\rho} - \frac{3}{k^2}H_{0[\alpha\beta}k_{\rho]} - \frac{3}{2}F_{[\alpha\beta}v_{\rho]}. \end{aligned} \qquad (5.0.\ 6)$$

Since

$$\sqrt{g} = k^2\sqrt{\tilde{g}}, \qquad (5.0.\ 7)$$

and the modulus of k can be expressed in terms of the dilaton transformation properties,

$$\tilde{\phi} = \phi - \log k, \qquad (5.0.\ 8)$$

we obtain

$$\tilde{R} + e^{2(\phi-\tilde{\phi})}\tilde{H}^2_{0\alpha\beta} + \Delta\tilde{\phi} = R + e^{2(\tilde{\phi}-\phi)}H^2_{0\alpha\beta} + \Delta\phi, \qquad (5.0.\ 9)$$

which could be used to show the duality invariance of the string effective action to leading order in α'.

The change of the cosmological constant under duality is not only peculiar to three-dimensions [36] but rather generic. This raises the physical question of whether in the context of String Theory the value of the cosmological constant can be inferred from the asymptotic (long distance) behavior of the Ricci tensor. If duality is not broken, the answer seems to be in the negative, and it makes the issue of what is the correct meaning of the cosmological constant in String Theory yet more misterious.

6 The Physical Definition of Distance

The existence of duality raises the question of the empirical definition of distance. This is, of course, not a well defined question in the absence of a sufficiently developed String Field Theory, but can nevertheless be asked if we assume that the outcome of every possible experiment is some correlation function of the corresponding two dimensional CFT.

Thinking on the simplest situation of closed strings propagating in a spacetime with one coordinate compactified in a circle, it is physically obvious that if we attempt to measure distances through the asymptotic behavior of correlation functions at large separations[4] [41], we would get a completely different answer if we use

[4] At large spatial distances the propagator behaves as $G(\vec{r}, \vec{r}^{\,\prime}; M) \sim e^{-M|\vec{r}-\vec{r}^{\,\prime}|}$ so that a suitable definition of distance is given by $d(\vec{r}, \vec{r}^{\,\prime}) \equiv \frac{1}{M}\log\frac{G(\vec{r},\vec{r}^{\,\prime};M)}{G(\vec{r},\vec{r}^{\,\prime};2M)}$.

pure momentum states (of energy $E_p = n/R$) or pure windings states (of energy $E_w = mR$), which we would most simply reinterpret as momentum states of a torus of radius $1/R$. This would lead us in a natural way to restrict the allowed outcome of our experiments to the interval $d \in (1, \infty)$.

There are some technical complications, stemming from the fact that the Polyakov method only allows to compute on-shell correlators, which means that we cannot probe directly off-shell amplitudes.

In the absence of any clear physical distinction among different classes of states, perhaps the most natural possibility is to define distances out of "unpolarized" correlators, that is, considering the contribution of all states at the same time.

There is still a certain freedom as to how to perform the corresponding Fourier transform in order to define physical quantities in position space. The most sensible thing seems, however, to make use of the fact that momentum and winding states define a lattice [11, 12]. To be specific, sticking for concreteness to the case in which r dimensions (called $\vec{x}$) are compactified in circles of radius R, and denoting by $\vec{y}$ the $(d-r)$-dimensional set of all other coordinates, the above considerations yield:

$$G(\vec{x}, \vec{x}'; \vec{y}-\vec{y}', t-t') \equiv \sum_{\vec{n},\vec{m}} e^{2\pi i(\vec{x}-\vec{x}')(\vec{n}/R+\vec{m}R/2)} \int dp_0 d\mu(\vec{p}) e^{i\vec{p}(\vec{y}-\vec{y}')-ip_0(t-t')} \langle V_{\vec{n},\vec{m}} V_{-\vec{n},-\vec{m}} \rangle f_{\vec{n},\vec{m}}, \tag{6.0. 1}$$

where $V_{\vec{n},\vec{m}}$ represents the vertex operator corresponding to the sector with momentum numbers $\vec{n}$ and winding numbers $\vec{m}$, and we will moreover consider pure solitonic states, without any oscillators $N = \tilde{N} = 0$[5].

The momentum space correlator is then given essentially by the delta function implementing the condition that the vertex operator has conformal dimension 1, that is:

$$p_0^2 - \vec{p}^2 - (\frac{\vec{n}}{R} + \vec{m}\frac{R}{2})^2 = N - 1 + \tilde{N} - 1 = -2. \tag{6.0. 2}$$

A further restriction ($\vec{n}\vec{m} = 0$) comes from invariance under translations in σ ($L_0 = \tilde{L}_0$).

Using the integral representation for the delta functions, the integral over p_0 can be easily performed, and the double sum packed into a Riemann theta-function:

$$G(\vec{x}, \vec{x}'; \vec{y} - \vec{y}', t - t') = \int d\mu(\vec{p}) \int_{-\infty}^{\infty} d\tau \tau^{-1/2} \int_{-\infty}^{\infty} d\lambda e^{-i\frac{(t-t')^2}{4\tau} - i\tau(\vec{p}^2-2) + i\vec{p}(\vec{y}-\vec{y}')} \theta(\vec{z}, \Omega), \tag{6.0. 3}$$

where $\vec{z} \equiv (\vec{x} - \vec{x}')(\frac{R}{2}, \frac{1}{R})$ and

$$\Omega \equiv \frac{1}{\pi} \begin{pmatrix} -\tau R^2/2 & \frac{(\lambda-\tau)}{2} \\ \frac{(\lambda-\tau)}{2} & -\frac{\tau}{R^2} \end{pmatrix}. \tag{6.0. 4}$$

[5]In the unpolarized case we are favouring, the selection function is trivial $f_{\vec{n},\vec{m}} = 1$, but we have written it in the formula in order to allow for more general possibilities.

A different expression can be obtained in terms of a double sum of $(d-r)$-dimensional Pauli-Jordan functions

$$G^{(d)}(\vec{x},\vec{x}^{'};\vec{y}-\vec{y}^{'},t-t^{'}) = \sum_{\vec{m},\vec{n}} G_{PJ}^{(d-r)}(\vec{y}-\vec{y}^{'},t-t^{'};M^2(\vec{n},\vec{m}))e^{2\pi i(\frac{\vec{n}}{R}+\vec{m}\frac{R}{2})(\vec{x}-\vec{x}^{'})}\delta(\vec{n}\vec{m}), \tag{6.0. 5}$$

where the "mass spectrum" is given by

$$M^2(\vec{n},\vec{m}) \equiv (\frac{\vec{n}}{R}+\vec{m}\frac{R}{2})^2 - 2. \tag{6.0. 6}$$

It is plain that any definition of distance based on the preceding ideas lacks any periodicity (which shows only in the particular cases in which pure winding states $f_{\vec{n},\vec{m}} = \delta_{\vec{n},\vec{0}}$ or pure momentum states $f_{\vec{n},\vec{m}} = \delta_{\vec{m},\vec{0}}$ are used).

It is also arguable whether these correlators are indeed the most natural ones to consider from the physical point of view. At extreme (either very high or very low) values of the radius, "pure" states (winding or momentum) are much lighter than all the others, so that it is perhaps more natural to define distances in terms of the lightest states only [42].

One could always consider our suggestion as a concrete implementation of earlier speculations that at very short distances there could be a physical regime at which geometry ceases to be smooth, but distances can nevertheless be defined, and they obey the triangular inequality [43].

7 The Canonical Approach

The procedures to implement duality explained in section 2 look unnecessarily complicated. In the one due to Roček and Verlinde the isometry is gauged, the (non propagating) gauge fields are constrained to be trivial, and the Lagrange multipliers themselves are promoted to the rank of new coordinates once the gaussian integration over the gauge fields is performed. One suspects that all those complicated intermediate steps could be avoided, and that it should be possible to pass directly from the original to the dual theory.

Some suggestions have indeed been made in the literature pointing (at least in the simplified situation where all backgrounds are constant or dependent only on time) towards an understanding of duality as particular instances of canonical transformations [14, 44].

In this section we are going to show that this idea works well when the background admits an abelian isometry [45], laying duality on a simpler setting than before, namely as a (privileged) subgroup of the whole group of (non-anomalous, that is implementable in Quantum Field Theory [46]) canonical transformations on the phase space of the theory.

We will proof that Buscher's transformation formulae can be derived by performing a given canonical transformation on the Hamiltonian of the initial theory. We believe that this is a "minimal" approach in the sense that no extraneous structure has to be introduced, and all standard results in the abelian case (and more) are easily recovered using it. In particular it is possible to perform the duality transformation in arbitrary coordinates not only in the original manifold (which was also possible in Roček and Verlinde's formulation) but also in the dual one. The multivaluedness and periods of the dual variables can be easily worked out from the implementation of the canonical transformation in the path integral. The generalization to arbitrary genus Riemann surfaces is in this approach straightforward. The behavior of currents not commuting with those used to implement duality can also be clarified. In the case of WZW models it becomes rather simple to prove that the full duality group is given by $Aut(G)_L \times Aut(G)_R$, where L, R refer to the left- and right-currents on the model with group G, and $Aut(G)$ are the automorphisms of G, both inner and outer. Due to the chiral conservation of the currents in this case, the canonical transformation leads to a local expression for the dual currents. In the case where the currents are not chirally conserved, then those currents associated to symmetries not commuting with the one used to perform duality become generically non-local in the dual theory and this is why they are not manifest in the dual Lagrangian. All the generators of the full duality group $O(d,d;Z)$ can be described in terms of canonical transformations. This gives the impression that the duality group should be understood in terms of global symplectic diffemorphisms. It would be useful to formulate it in the context of some analogue of the group of disconnected diffeomorphisms, but for the time being such a construction is lacking.

Concerning non-abelian duality, it seems to fall beyond the scope of the Hamiltonian point of view. There is one example [47] in which the non-abelian dual has been constructed out of a canonical transformation but it is still early to say whether the general case can be treated similarly.

7.1 The Abelian Case

We start with a bosonic sigma model written in arbitrary coordinates on a manifold M with Lagrangian

$$L = \frac{1}{2}(g_{ab} + b_{ab})(\phi)\partial_+\phi^a\partial_-\phi^b \tag{7.1. 1}$$

where $x^{\pm} = (\tau \pm \sigma)/2$, $a, b = 1, \ldots, d = \mathrm{dim}M$. The corresponding Hamiltonian is

$$H = \frac{1}{2}(g^{ab}(p_a - b_{ac}\phi'^{\,c})(p_b - b_{bd}\phi'^{\,d}) + g_{ab}\phi'^{\,a}\phi'^{\,b}) \tag{7.1. 2}$$

where $\phi'^{\,a} \equiv d\phi^a/d\sigma$. We assume moreover that there is a Killing vector field k^a, $\mathcal{L}_k g_{ab} = 0$ and $i_k H = -dv$ for some one-form v, where $(i_k H)_{ab} \equiv k^c H_{cab}$ and $H = db$ locally. This guarantees the existence of a particular system of coordinates, "adapted coordinates", which we denote by $x^i \equiv (\theta, x^\alpha)$, such that $\vec{k} = \partial/\partial\theta$. We denote the jacobian matrix by $e^i_a \equiv \partial x^i/\partial\phi^a$.

This defines a point transformation in the original Lagrangian (7.1. 1) which acts on the Hamiltonian as a canonical transformation with generating function

$\Phi = x^i(\phi)p_i$, and yields:

$$p_a = e^i_a p_i$$
$$x^i = x^i(\phi). \tag{7.1. 3}$$

Once in adapted coordinates we can write the sigma model Lagrangian as

$$L = \frac{1}{2}G(\dot{\theta}^2 - \theta'^{\,2}) + (\dot{\theta} + \theta')J_- + (\dot{\theta} - \theta')J_+ + V \tag{7.1. 4}$$

where

$$G = g_{00} = k^2 \qquad V = \frac{1}{2}(g_{\alpha\beta} + b_{\alpha\beta})\partial_+ x^\alpha \partial_- x^\beta$$
$$J_- = \frac{1}{2}(g_{0\alpha} + b_{0\alpha})\partial_- x^\alpha \qquad J_+ = \frac{1}{2}(g_{0\alpha} - b_{0\alpha})\partial_+ x^\alpha. \tag{7.1. 5}$$

In finding the dual with a canonical transformation we can use the Routh function with respect to θ, i.e. we only apply the Legendre transformation to $(\theta, \dot{\theta})$. The canonical momentum is given by

$$p_\theta = G\dot{\theta} + (J_+ + J_-) \tag{7.1. 6}$$

and the Hamiltonian

$$H = p_\theta\dot{\theta} - L = \frac{1}{2}G^{-1}p_\theta^2 - G^{-1}(J_+ + J_-)p_\theta + \frac{1}{2}G\theta'^{\,2} +$$
$$+\frac{1}{2}G^{-1}(J_+ + J_-)^2 + \theta'(J_+ - J_-) - V. \tag{7.1. 7}$$

The Hamilton equations are:

$$\dot{\theta} = \frac{\delta H}{\delta p_\theta} = G^{-1}(p_\theta - J_+ - J_-)$$
$$\dot{p_\theta} = -\frac{\delta H}{\delta\theta} = (G\theta' + J_+ - J_-)' \tag{7.1. 8}$$

and the current components:

$$\mathcal{J}_+ = \frac{1}{2}G\partial_+\theta + J_+ = \frac{1}{2}p_\theta + \frac{1}{2}G\theta' + \frac{J_+ - J_-}{2}$$
$$\mathcal{J}_- = \frac{1}{2}G\partial_-\theta + J_- = \frac{1}{2}p_\theta - \frac{1}{2}G\theta' - \frac{J_+ - J_-}{2}. \tag{7.1. 9}$$

It can easily be seing that the current conservation $\partial_-\mathcal{J}_+ + \partial_+\mathcal{J}_- = 0$ is equivalent to the second Hamilton equation $\dot{p_\theta} = -\delta H/\delta\theta$.

The generator of the canonical transformation we choose is:

$$F = \frac{1}{2}\int_{D,\partial D = S^1} d\tilde{\theta} \wedge d\theta = \frac{1}{2}\oint_{S^1}(\theta'\tilde{\theta} - \theta\tilde{\theta}')d\sigma \tag{7.1. 10}$$

that is,

$$p_\theta = \frac{\delta F}{\delta \theta} = -\tilde{\theta}'$$
$$p_{\tilde{\theta}} = -\frac{\delta F}{\delta \tilde{\theta}} = -\theta'. \qquad (7.1.\ 11)$$

This generating functional does not receive any quantum corrections (as explained in [46]) since it is linear in θ and $\tilde{\theta}$. If θ was not an adapted coordinate to a continuous isometry, the canonical transformation would generically lead to a non-local form of the dual Hamiltonian. Since the Lagrangian and Hamiltonian in our case only depend on the time- and space-derivatives of θ, there are no problems with non-locality. The transformation (7.1. 11) in (7.1. 7) gives:

$$\tilde{H} = \frac{1}{2}G^{-1}\tilde{\theta}'^{\,2} + G^{-1}(J_+ + J_-)\tilde{\theta}' +$$
$$\frac{1}{2}Gp_{\tilde{\theta}}^2 - (J_+ - J_-)p_{\tilde{\theta}} + \frac{1}{2}G^{-1}(J_+ + J_-)^2 - V. \qquad (7.1.\ 12)$$

Since:

$$\dot{\tilde{\theta}} = \frac{\delta \tilde{H}}{\delta p_{\tilde{\theta}}} = Gp_{\tilde{\theta}} - (J_+ - J_-), \qquad (7.1.\ 13)$$

we can perform the inverse Legendre transform:

$$\tilde{L} = \frac{1}{2}G^{-1}(\dot{\tilde{\theta}}^2 - \tilde{\theta}'^{\,2}) + G^{-1}J_+(\dot{\tilde{\theta}} - \tilde{\theta}')$$
$$-G^{-1}J_-(\dot{\tilde{\theta}} + \tilde{\theta}') + V - 2G^{-1}J_+J_-. \qquad (7.1.\ 14)$$

From this expression we can read the dual metric and torsion and check that they are given by Buscher's formulae[6]:

$$\tilde{g}_{00} = 1/g_{00}, \qquad \tilde{g}_{0\alpha} = -b_{0\alpha}/g_{00}, \qquad \tilde{g}_{\alpha\beta} = g_{\alpha\beta} - \frac{g_{0\alpha}g_{0\beta} - b_{0\alpha}b_{0\beta}}{g_{00}}$$
$$\tilde{b}_{0\alpha} = -\frac{g_{0\alpha}}{g_{00}}, \qquad \tilde{b}_{\alpha\beta} = b_{\alpha\beta} - \frac{g_{0\alpha}b_{0\beta} - g_{0\beta}b_{0\alpha}}{g_{00}} \qquad (7.1.\ 15)$$

For the dual theory to be conformal invariant the dilaton must transform as $\Phi' = \Phi - \frac{1}{2}\log g_{00}$ [16] [10]. We have not been able to find any argument justifying this transformation within the canonical transformations approach.

The dual manifold $\tilde{M}$ is automatically expressed in coordinates adapted to the dual Killing vector $\tilde{\vec{k}} = \partial/\partial\tilde{\theta}$. We can now perform another point transformation, with the same jacobian as (7.1. 3) to express the dual manifold in coordinates which are as close as possible to the original ones.

[6] The minus signs in $\tilde{g}_{0\alpha}$ and $\tilde{b}_{0\alpha}$ can be absorbed in a redefinition $\tilde{\theta} \to -\tilde{\theta}$.

The transformations we perform are then: First a point transformation $\phi^a \to \{\theta, x^\alpha\}$, to go to adapted coordinates in the original manifold. Then a canonical transformation $\{\theta, x^\alpha\} \to \{\tilde{\theta}, x^\alpha\}$, which is the true duality transformation. And finally another point transformation $\{\tilde{\theta}, x^\alpha\} \to \tilde{\phi}^a$, with the same jacobian as the first point transformation, to express the dual manifold in general coordinates.

It turns out that the composition of these three transformations can be expressed in geometrical terms using only the Killing vector k^a, $\omega_a \equiv e^0_a$ and the corresponding dual quantities[7].

It is then quite easy to check that the total canonical transformation to be made in (7.1. 1) is just

$$\begin{aligned} k^a p_a &\to \tilde{\omega}_a \tilde{\phi}'^{\,a} \\ \omega_a \phi'^{\,a} &\to \tilde{k}^a \tilde{p}_a, \end{aligned} \tag{7.1. 16}$$

whose generating function is[8]

$$F = \frac{1}{2}\int_D \tilde{\omega} \wedge \omega = \frac{1}{2}\int_D \tilde{\omega}_a d\tilde{\phi}^a \wedge \omega_b d\phi^b. \tag{7.1. 17}$$

One then easily performs the transformations in such a way that the dual metric and torsion can be expressed in geometrical terms as

$$\tilde{g}_{ab} = g_{ab} - \frac{1}{k^2}(k_a k_b - (v_a - \omega_a)(v_b - \omega_b)) \tag{7.1. 18}$$

$$\tilde{g}^{ab} = g^{ab} + \frac{1}{(1+k.v)^2}[(k^2 + (v-\omega)^2)k^a k^b - 2(1+k.v)(k^{(a}(v-\omega)^{b)}] \tag{7.1. 19}$$

and

$$\tilde{b}_{ab} = b_{ab} - \frac{2}{k^2} k_{[a}(v-\omega)_{b]}, \tag{7.1. 20}$$

where

$$\begin{aligned} k_{(a}(v-\omega)_{b)} &= \frac{1}{2}(k_a(v_b - \omega_b) + k_b(v_a - \omega_a)) \\ k_{[a}(v-\omega)_{b]} &= \frac{1}{2}(k_a(v_b - \omega_b) - k_b(v_a - \omega_a)). \end{aligned} \tag{7.1. 21}$$

These formulae are the covariant generalization of (7.1. 15). The canonical approach has been very useful in order to obtain the dual manifold in an arbitrary coordinate system. With the usual approaches it is expressed in adapted coordinates to the dual

[7] Note that we must raise and lower indices with the dual metric, i.e. $\tilde{e}_{ia} = \tilde{g}_{ij}\tilde{e}^j_a, \tilde{e}^{ia} = \tilde{g}^{ab}\tilde{e}^i_b$, which implies $\tilde{\omega}_a = \omega_a$, but $\tilde{\omega}^a = k^a(k^2 + v^2) + \vec{e}^{\,a} \cdot v$ (where $\vec{e}^{\,a} \equiv e^a_\alpha$), $\tilde{k}^a = k^a$ but $\tilde{k}_a = (\omega_a - (\vec{e}_a \cdot v))/k^2$. We have moreover $\tilde{\omega}^{\,2} = k^2 + v^2 + g^{\alpha\beta} v_\beta \omega_\alpha$ and $\tilde{k}^2 = 1/k^2$.

[8] The one-form $\omega \equiv \omega_a d\phi^a$ is dual to the Killing vector $\vec{k}$: $\omega(\vec{k}) = 1$, $\omega(\vec{e}_\alpha) = 0$, but it is of course different from $\underline{k} \equiv k_a/k^2\, d\phi^a$ (the former is an exact form, whereas the latter does not even in general satisfy Frobenius condition $\underline{k} \wedge d\underline{k} = 0$).

isometry. This happens because the dual variables appear as Lagrange multipliers and after an integration by parts only the derivatives of them emerge, being then adapted coordinates automatically.

Some other useful information can be extracted easier in the approach of the canonical transformation.

From the generating functional (7.1. 10) we can learn about the multivaluedness and periods of the dual variables [19]. Since θ is periodic and in the path integral the canonical transformation is implemented by [46]:

$$\psi_k[\tilde{\theta}(\sigma)] = N(k) \int \mathcal{D}\theta(\sigma) e^{iF[\tilde{\theta},\theta(\sigma)]} \phi_k[\theta(\sigma)] \qquad (7.1.\ 22)$$

where $N(k)$ is a normalization factor, $\phi_k(\theta + a) = \phi_k(\theta)$ implies for $\tilde{\theta}$: $\tilde{\theta}(\sigma + 2\pi) - \tilde{\theta}(\sigma) = 4\pi/a$, which means that $\tilde{\theta}$ must live in the dual lattice of θ. Note that (7.1. 22) suffices to construct the dual Hamiltonian. It is a simple exercise to check that acting with (7.1. 12) on the left-hand side of (7.1. 22) and pushing the dual Hamiltonian through the integral we obtain the original Hamiltonian acting on $\phi_k[\theta(\sigma)]$:

$$\tilde{H}\psi_k[\tilde{\theta}(\sigma)] = N(k) \int \mathcal{D}\theta(\sigma) e^{iF[\tilde{\theta},\theta(\sigma)]} H\phi_k[\theta(\sigma)] \qquad (7.1.\ 23)$$

This makes the duality transformation very simple conceptually, and it also implies how it can be applied to arbitrary genus Riemann surfaces, because the state $\phi_k[\theta(\sigma)]$ could be the state obtained by integrating the original theory on an arbitrary Riemann surface with boundary. It is also clear that the arguments generalize straightforwardly when we have several commuting isometries.

One can easily see that under the canonical transformation the Hamilton equations are interchanged:

$$\begin{aligned} \dot{p_\theta} = -\frac{\delta H}{\delta \theta} &= (G\theta' + J_+ - J_-)' \rightarrow \dot{\tilde{\theta}} = Gp_{\tilde{\theta}} - J_+ + J_- \\ \dot{\theta} = \frac{\delta H}{\delta p_\theta} &= G^{-1}(p_\theta - J_+ - J_-) \rightarrow \dot{p_{\tilde{\theta}}} = (G^{-1}(\tilde{\theta}' + J_+ + J_-))', \end{aligned} \qquad (7.1.\ 24)$$

and that the canonical transformed currents conservation law is in this case equivalent to the first Hamilton equation.

In the chiral case $J_- = 0$ (i.e. $g_{0i} = -b_{0i}$) and G is a constant, therefore we can normalize θ to set $G = 1$ and :

$$L = \frac{1}{2}(\dot{\theta}^2 - \theta'^{\,2}) + (\dot{\theta} - \theta')J_+ + V. \qquad (7.1.\ 25)$$

The Hamiltonian is

$$H = \frac{1}{2}p_\theta^2 - J_+ p_\theta + \frac{1}{2}(J_+ + \theta')^2 - V. \qquad (7.1.\ 26)$$

The action is invariant under $\delta\theta = \alpha(x^+)$, a $U(1)_L$ Kac-Moody symmetry. The $U(1)$ Kac-Moody algebra has the automorphism $\mathcal{J}_+ \rightarrow -\mathcal{J}_+$. This is precisely the effect of the canonical transformation. The equation of motion or current conservation is:

$$\partial_-(\partial_+\theta + J_+) = 0. \tag{7.1. 27}$$

$\mathcal{J}_+ = \partial_+\theta + J_+ = p_\theta + \theta'$ transforms under the canonical transformation in $\mathcal{J}_+^{c.t.} = -\tilde{\theta}' - p_{\tilde{\theta}} = -\mathcal{J}_+$.

One can also follow the transformation to the dual model of other continuous symmetries. The simplest case is as usual the WZW-model which is the basic model with chiral currents. Consider for simplicity the level-k $SU(2)$-WZW model with action

$$S[g] = \frac{-k}{2\pi}\int d^2\sigma Tr(g^{-1}\partial_+ g g^{-1}\partial_- g) + \frac{k}{12\pi}\int Tr(g^{-1}dg)^3. \tag{7.1. 28}$$

The left- and right-chiral currents are

$$\mathcal{J}_+ = \frac{k}{2\pi}\partial_+ g g^{-1} \qquad \mathcal{J}_- = -\frac{k}{2\pi}g^{-1}\partial_- g. \tag{7.1. 29}$$

Parametrizing g in terms of Euler angles

$$g = e^{i\alpha\sigma_3/2}e^{i\beta\sigma_2/2}e^{i\gamma\sigma_3/2}, \tag{7.1. 30}$$

$\mathcal{J}_+$ are given by:

$$\begin{aligned}
\mathcal{J}_+^1 &= \frac{k}{2\pi}(-\cos\alpha\sin\beta\partial_+\gamma + \sin\alpha\partial_+\beta)\\
\mathcal{J}_+^2 &= \frac{k}{2\pi}(\sin\alpha\sin\beta\partial_+\gamma + \cos\alpha\partial_+\beta)\\
\mathcal{J}_+^3 &= \frac{k}{2\pi}(\partial_+\alpha + \cos\beta\partial_+\gamma),
\end{aligned} \tag{7.1. 31}$$

and similarly for the right currents. If we perform duality with respect to $\alpha \to \alpha +$ constant, $\mathcal{J}_+^3 \to -\mathcal{J}_+^3, \mathcal{J}_-^3 \to \mathcal{J}_-^3$ since $\mathcal{J}_+^3$ is the current component adapted to the isometry. For these currents it is easy to find the action of the canonical transformation because only the derivatives of α appear. For $\mathcal{J}_+^{1,2}$ there is an explicit dependence on α and it seems that the transform of these currents is very non-local. However due to its chiral nature, one can show that there are similar chirally conserved currents in the dual model. To do this we first combine the currents in terms of root generators:

$$\begin{aligned}
\mathcal{J}_+^{(+)} &= \mathcal{J}_+^1 + i\mathcal{J}_+^2 = e^{-i\alpha}(i\partial_+\beta - \sin\beta\partial_+\gamma) = e^{-i\alpha}j_+^{(+)}\\
\mathcal{J}_+^{(-)} &= \mathcal{J}_+^1 - i\mathcal{J}_+^2 = -e^{i\alpha}(i\partial_+\beta + \sin\beta\partial_+\gamma) = e^{i\alpha}j_+^{(-)}.
\end{aligned} \tag{7.1. 32}$$

From chiral current conservation $\partial_-\mathcal{J}_+^{(\pm)} = 0$ we obtain

$$\partial_- j_+^{(\pm)} = \pm i\partial_-\alpha j_+^{(\pm)}. \tag{7.1. 33}$$

In these equations only $\dot{\alpha}, \alpha'$ appear, and after the canonical transformation we can reconstruct the dual non-abelian currents (in the previous equations the canonical transformation amounts to the replacement $\alpha \to \tilde{\alpha}$) which take the same form as the original ones except that with respect to the transformed $\mathcal{J}_+^3$ the rôles of positive and negative roots get exchanged. One also verifies that $\mathcal{J}_-^a$ are unaffected. This implies therefore that the effect of duality with respect to shifts of α is an automorphism of the current algebra amounting to performing a Weyl transformation on the left currents only while the right ones remain unmodified. This result although known [21] is much easier to derive in the Hamiltonian formalism than in the Lagrangian formalism where one must introduce external sources which carry some ambiguities. The construction for $SU(2)$ can be straightforwardly extended to other groups. This implies that for WZW-models the full duality group is $Aut(G)_L \times Aut(G)_R$, where $Aut(G)$ is the group of automorphisms of the group G, including Weyl transformations and outer automorphisms. For instance if we take $SU(N)$, the transformation $J_+ \to -J_+^T$, i.e. charge conjugation, follows from a canonical transformation of the type discussed. It suffices to take as generating functions for the canonical transformation the sum of the generating functions for each generator in the Cartan subalgebra. It is important to remark that the chiral conservation of the currents is crucial to guarantee the locality of the dual non-abelian currents. If the conserved current with respect to which we dualize is not chirally conserved locality is not obtained. The simplest example to verify this is the principal chiral model for $SU(2)$, which although is not a CFT serves for illustrative purposes. The equations of motion for this model imply the conservation laws:

$$\partial_- \mathcal{J}_+^a + \partial_+ \mathcal{J}_-^a = 0 \qquad (7.1.\ 34)$$

where

$$\mathcal{J}_\pm = \frac{k}{2\pi} \partial_\pm g g^{-1}. \qquad (7.1.\ 35)$$

If we perform duality with respect to the invariance under α translations we know how $\mathcal{J}_\pm^3$ transform, since they are the currents associated to the isometry. With the canonical transformation is possible to see as well which are the other dual conserved currents. Since the dual model is only $U(1)$-invariant one expects the rest of the currents to become non-local [47]. In terms of the root generators introduced in (7.1. 32) the conservation laws

$$\partial_- \mathcal{J}_+^{(\pm)} + \partial_+ \mathcal{J}_-^{(\pm)} = 0 \qquad (7.1.\ 36)$$

are expressed:

$$\partial_- j_+^{(\pm)} + \partial_+ j_-^{(\pm)} \mp i(\partial_- \alpha j_+^{(\pm)} + \partial_+ \alpha j_-^{(\pm)}) = 0. \qquad (7.1.\ 37)$$

Performing the canonical transformation we obtain that the dual conserved currents are given by:

$$\tilde{\mathcal{J}}_\pm^{(+)} = \exp\,(i \int d\sigma(\dot{\tilde{\alpha}} + \cos\beta\gamma'))(i\partial_\pm\beta - \sin\beta\partial_\pm\gamma)$$
$$\tilde{\mathcal{J}}_\pm^{(-)} = -\exp\,(-i \int d\sigma(\dot{\tilde{\alpha}} + \cos\beta\gamma'))(i\partial_\pm\beta + \sin\beta\partial_\pm\gamma) \qquad (7.1.\ 38)$$

which cannot be expressed in a local form.

7.2 The Non-Abelian Case

In view of the simplicity of the canonical approach to abelian duality, one could be tempted to think that the corresponding generalization to the non-abelian case would not be very difficult. Unfortunately this is not the case, the reason being that there are no adapted coordinates to a set of non-commuting isometries, and therefore one is led to a non-local form of the Hamiltonian. In [25] we could carry out the non-abelian duality transformation due to the existence of chiral currents and as a consequence of the Polyakov-Wiegmann property [50] satisfied by WZW-actions. Although in the intermediate steps it was necessary to introduce non-local variables, the final result led to a local action in the new variables as a result of the special properties of WZW-models mentioned. The computations could be carried out exactly until the end to evaluate the form of the effective action in terms of the auxiliary variables needed in the construction of non-abelian duals. We have so far been unable to express these functional integral manipulations in a Hamiltonian setting as in the previous section.

To finish this section we present an example from the literature in which a canonical transformation produces a given non-abelian dual model. This example was presented in [47]. They consider the principal chiral model with group $SU(2)$ and construct a local canonical transformation mapping the model in a theory which turns out to be the non-abelian dual with respect to the left action of the whole group. This example was studied in the context of non-abelian duality in [48, 19].

The initial theory is the principal chiral model defined by the Lagrangian:

$$L = Tr(\partial_\mu g \partial^\mu g^{-1}), \tag{7.2. 1}$$

where $g \in SU(2)$. Parametrizing $g = \phi^0 + i\sigma^j\phi^j$, with ϕ^0, ϕ^j subject to the constraint $(\phi^0)^2 + \phi^2 = 1$ and $\phi^2 \equiv \sum_j (\phi^j)^2$, (7.2. 1) becomes:

$$L = \frac{1}{2}(\delta^{ij} + \frac{\phi^i\phi^j}{1-\phi^2})\partial_\mu\phi^i\partial^\mu\phi^j. \tag{7.2. 2}$$

The generating functional:

$$F[\psi,\phi] = \int_{-\infty}^{+\infty} dx \psi^i (\sqrt{1-\phi^2}\frac{\partial}{\partial x}\phi^i - \phi^i\frac{\partial}{\partial x}\sqrt{1-\phi^2} + \epsilon^{ijk}\phi^j\frac{\partial}{\partial x}\phi^k) \tag{7.2. 3}$$

produces the canonical transformation:

$$\begin{aligned} p_i &= \frac{\delta F[\psi,\phi]}{\delta\psi^i} = (\sqrt{1-\phi^2}\delta^{ij} + \frac{\phi^i\phi^j}{\sqrt{1-\phi^2}} - \epsilon^{ijk}\phi^k)\frac{\partial}{\partial x}\phi^j \\ \tilde{p}_i &= -\frac{\delta F[\psi,\phi]}{\delta\phi^i} = (\sqrt{1-\phi^2}\delta^{ij} + \frac{\phi^i\phi^j}{\sqrt{1-\phi^2}} + \epsilon^{ijk}\phi^k)\frac{\partial}{\partial x}\psi^j + \\ & (\frac{2}{\sqrt{1-\phi^2}}(\phi^i\psi^j - \psi^i\phi^j) - 2\epsilon^{ijk}\psi^k)\frac{\partial}{\partial x}\phi^j, \end{aligned} \tag{7.2. 4}$$

which transforms (7.2. 2) into:

$$\tilde{L} = \frac{1}{1+4\psi^2}[\frac{1}{2}(\delta^{ij} + 4\psi^i\psi^j)\partial_\mu\psi^i\partial^\mu\psi^j - \epsilon^{\mu\nu}\epsilon^{ijk}\psi^i\partial_\mu\psi^j\partial_\nu\psi^k]. \qquad (7.2.\ 5)$$

This is the non-abelian dual of (7.2. 1) with respect to the left action of the whole group [48, 19]. The generating functional (7.2. 3) can be written as:

$$F[\psi, \phi] = \int_{-\infty}^{+\infty} dx \psi^i J_i^1[\phi] \qquad (7.2.\ 6)$$

where $J_i^1[\phi]$ are the spatial components of the conserved currents of the initial theory. (7.2. 6) is linear in the dual variables but not in the initial ones, so it will receive quantum corrections when implemented in the path integral. From (7.2. 4) it is not obvious that the dual model will not depend on the original variables ϕ^i. However this is so. Whether this way of constructing the generating functional of non-abelian duality is general or only works in this particular example is still an open question.

8 Conclusions and Open Problems

In these lectures a general exposition of abelian and non-abelian duality has been given. The usual approaches in the literature to both kinds of dualities have been reviewed. The goals of these approaches have been also exhibited, and some of them derived explicitly, as the formulation of abelian duality in an arbitrary coordinate system. The canonical transformations approach to abelian duality presented in [45] has been studied in detail, focussing especially in the problems that could not be solved in the usual approaches of Buscher or Roček and Verlinde or were difficult to study. The non-abelian case has been also considered, although the general construction as a canonical transformation is not yet understood. As was mentioned in the lectures the example given by Curtright and Zachos in [47] opens the possibility for non-abelian duality to be formulated in this way, in spite of the difficulties already mentioned concerning the impossibility of finding an adapted coordinate system to the whole set on non-commuting isometries.

The relation between duality and external automorphisms [51, 52, 53] is also much in need of further clarification.

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