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The exact mass-gap of the supersymmetric $\mathbb{CP}^{n-1}$ sigma model

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ABSTRACT

A formula for the mass-gap of the supersymmetric $\mathbb{CP}^{n-1}$ sigma model ($n > 1$) in two dimensions is derived: $m/\Lambda_{\overline{\mathrm{MS}}} = \sin(\pi\Delta)/(\pi\Delta)$ where $\Delta = 1/n$ and m is the mass of the fundamental particle multiplet. This result is obtained by comparing two expressions for the free-energy density in the presence of a coupling to a conserved charge; one expression is computed from the exact S-matrix of Köberle and Kurak via the thermodynamic Bethe ansatz and the other is computed using conventional perturbation theory. These calculations provide a stringent test of the S-matrix, showing that it correctly reproduces the universal part of the beta-function and resolving the problem of CDD ambiguities.

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1. Introduction

This paper is the second of a pair (see [1]) concerned with exact results for supersymmetric sigma-models in two dimensions. The general strategy, which has already been successfully applied to non-supersymmetric theories in [2,3,4,5,6,7,8], is to calculate the free energy of a theory in the presence of a background field by using the Thermodynamic Bethe Ansatz (TBA) equations derived from some postulated exact S-matrix. By comparing this result with a calculation of the same quantity in standard perturbation theory, the validity of the S-matrix can be checked and the exact mass-gap for the model can be extracted. In [1] the supersymmetric $O(N)$ sigma model ($N > 4$) was considered and it was shown how the novel difficulties associated with a supersymmetric theory—in particular the problem of diagonalizing the resulting TBA system—could be overcome for this simplest family of examples. In this sequel we consider the supersymmetric $\mathbb{CP}^{n-1}$ sigma-models. These theories differ from the $O(N)$ models in having $N = 2$ supersymmetry and also in a number of other important respects. For our purposes, the most important difference is that the TBA equations we shall have to solve involve anti-particles as well as particles. We shall see that the methods described above can, nevertheless, be applied successfully in these cases too.

The bosonic $\mathbb{CP}^{n-1}$ models ($n > 1$) first attracted attention as "toy" field theories with instanton solutions [9] generalizing the bosonic $O(3)$ model (the $O(3)$ and $\mathbb{CP}^1$ models coincide but there are no instanton solutions in the $O(N)$ models for $N > 3$). The analogy with QCD is reinforced by the fact that the bosonic $\mathbb{CP}^{n-1}$ theories are asymptotically free, they generate their mass dynamically, and they are confining. Unfortunately, the classical integrability of the bosonic $\mathbb{CP}^{n-1}$ models appears to be vitiated at the quantum level by anomalies in the conserved currents. For the supersymmetric $\mathbb{CP}^{n-1}$ models [10,11,12] on the other hand, integrability is maintained at the quantum level due to a cancelling of the anomalies between the bosonic and fermionic degrees of freedom [13] and exact S-matrices have been proposed by Köberle and Kurak [14]. The structure of the quantum supersymmetric $\mathbb{CP}^{n-1}$ models is simpler than that of their bosonic counterparts in that they do not display confinement [10,11]. Even so they are highly non-trivial, asymptotically-free quantum field theories with dynamically generated mass, and they have been used—along with their bosonic counterparts—to gain some profound insights into non-perturbative effects in field theories, such as the relationship between instantons and the $1/n$ expansion (see *eg.* [9,10,11,15,16] and references therein.) More recently they have also attracted attention as massive integrable perturbations of $N = 2$ superconformal field theories and as useful examples to which new non-perturbative techniques can be applied [17,18,19].

2. The model and its S-matrix

The supersymmetric $\mathbb{CP}^{n-1}$ model is defined by a lagrangian density

$$\mathcal{L} = \frac{1}{2g}\left\{|(\partial_\mu - A_\mu)z_a|^2 + i\bar{\psi}_a\gamma^\mu(\partial_\mu - A_\mu)\psi_a \right.$$
$$\left. +\frac{1}{4}\left[\left(\bar{\psi}_a\psi_a\right)^2 + \left(\bar{\psi}_a\gamma_5\psi_a\right)^2 - \left(\bar{\psi}_a\gamma_\mu\psi_a\right)^2\right]\right\}, \tag{2.1}$$
$$\text{where} \qquad A_\mu = \tfrac{1}{2}\left(z_a^*\partial_\mu z_a - z_a\partial_\mu z_a^*\right).$$

We work throughout in Minkowski space and our conventions agree with those of [20]. The fields z_a and ψ_a are an n-component complex scalar field and an n-component complex Dirac fermion, respectively, which satisfy the constraints $z_a^* z_a = 1$ and $z_a^*\psi_a = 0$. The theory is clearly invariant under global SU(n) transformations on these fields. There is also a local U(1) invariance under which the quantity A_μ above transforms as a gauge field; this means that the theory can be interpreted as a sigma-model whose target manifold is the complex projective space $\mathbb{CP}^{n-1}$.

Applying well-known general results (see *eg.* [20] and [21]), the two-loop beta-function for this model and the corresponding behaviour of the running coupling constant can be written

$$\beta(g) = -\beta_1 g^2 - \beta_2 g^3 + \mathcal{O}(g^4),$$
$$\text{so} \quad \frac{1}{g(\mu/\Lambda)} = \beta_1 \ln\frac{\mu}{\Lambda} + \frac{\beta_2}{\beta_1}\ln\ln\frac{\mu}{\Lambda} + \mathcal{O}\left(\frac{\ln\ln(\mu/\Lambda)}{\ln(\mu/\Lambda)}\right), \tag{2.2}$$
$$\text{where} \quad \beta_1 = n/\pi, \quad \beta_2 = 0.$$

From these expressions we see that the theory is asymptotically free, with dynamical mass generation.

It is natural to expect the spectrum of the theory to contain supersymmetric multiplets of particles, some of which are "fundamental", in the sense that they carry the quantum numbers of the fields in the classical lagrangian, and others which can be regarded as bound states. We denote the quantum states corresponding to the fundamental particles by $|a, i, \theta\rangle$, where $i = 0, 1$ denotes a boson or fermion respectively, a is the SU(n) vector index of the n dimensional representation and θ is the rapidity of the particle, so its velocity is $v = \tanh(\theta)$. The spectrum also includes a set of fundamental anti-particles $|\bar{a}, i, \theta\rangle$ transforming in the $\bar{n}$ representation of SU(n). It turns out that these fundamental anti-particles can be formed as bound-states of the fundamental particles (or vice-versa; an example of "nuclear democracy").

The integrability of the model [13] implies that the S-matrix factorizes and that all S-matrix elements can be deduced from the two-body one [22]. On the basis of this, Köberle and Kurak [14] (see also [17,23,24,25]) proposed an S-matrix to describe the scattering of the fundamental particle multiplet in accordance with the SU(n) symmetry of the model and all the usual axioms of S-matrix theory. Their proposal can be written in an illuminating way in which the supersymmetric and SU(n) degrees of freedom are factored. This was made explicit in [17] following the discussion for a general supersymmetric theory in [26].

In detail, the two-body S-matrix elements for the fundamental particles can be written

$$\langle c,k,\theta_2;d,l,\theta_1,\mathrm{out}|a,i,\theta_1;b,j,\theta_2,\mathrm{in}\rangle = S_{N=2}(\theta_1-\theta_2)_{ij}^{kl}S_{\mathrm{CGN}}(\theta_1-\theta_2)_{ab}^{cd}. \tag{2.3}$$

The SU(n) part of the S-matrix is the factorizable S-matrix of the fundamental vector particle of the SU(n) chiral Gross-Neveu model [27]:

$$S_{\mathrm{CGN}}(\theta)_{ab}^{cd} = Y_1(\theta)\left[\delta^{ad}\delta^{bc} - \frac{2\pi i\Delta}{\theta}\delta^{ac}\delta^{bd}\right], \tag{2.4}$$

with $\Delta = 1/n$ and with the unitarizing/crossing scalar factor

$$Y_1(\theta) = \frac{\Gamma(1+i\theta/2\pi)\Gamma(-\Delta-i\theta/2\pi)}{\Gamma(-i\theta/2\pi)\Gamma(1-\Delta+i\theta/2\pi)}. \tag{2.5}$$

The supersymmetric part of the S-matrix is the $N=2$ supersymmetric $\mathbb{Z}_n$ minimal S-matrix [17] which has the form

$$\begin{aligned} S_{N=2}(\theta) &= Y_2(\theta) \\ &\times \begin{pmatrix} \sinh(\theta/2+i\pi\Delta) & 0 & 0 & 0 \\ 0 & i\sin(\pi\Delta) & \sinh(\theta/2) & 0 \\ 0 & \sinh(\theta/2) & i\sin(\pi\Delta) & 0 \\ 0 & 0 & 0 & -\sinh(\theta/2-i\pi\Delta) \end{pmatrix}. \end{aligned} \tag{2.6}$$

where the rows and columns are labelled in the order $(0,0),(0,1),(1,0),(1,1)$ and where

$$\begin{aligned} Y_2(\theta) &= \frac{1}{\sinh(\theta/2+i\pi\Delta)} \\ &\times \prod_{j=1}^{\infty}\frac{\Gamma^2(i\theta/2\pi+j)\Gamma(-i\theta/2\pi+j+\Delta)\Gamma(-i\theta/2\pi+j-\Delta)}{\Gamma^2(-i\theta/2\pi+j)\Gamma(i\theta/2\pi+j+\Delta)\Gamma(i\theta/2\pi+j-\Delta)}. \end{aligned} \tag{2.7}$$

The S-matrix elements of the fundamental anti-particles can be found by crossing the elements (2.3).

It is important that in (2.3) we have chosen an ordering in the final state where the particle of rapidity θ_2 is to the left of the particle of rapidity θ_1; it is only this "modified" S-matrix (using the nomenclature of [26]) which exhibits the factorization between the supersymmetric and bosonic degrees of freedom manifested in (2.3). This is opposite to the choice made in the original treatment of [14], which means that some care is required in comparing the signs for amplitudes involving two fermions. It is also worth pointing out that although it is this same "modified" S-matrix which appears in [17], we are including the simple pole in the SU(n) factor rather than in the supersymmetric factor; the net result is easily seen to agree with [17].

The expression (2.3) is "minimal" in the sense that it has the minimum number of poles and zeros on the physical strip (the region $0 \le \mathrm{Im}(\theta) \le \pi$) consistent with the requirements of symmetry, existence of a bound-state and the axioms of S-matrix theory. But this still leaves open the possibility of adding CDD factors to the S-matrix; these spoil none of the axioms, they introduce no new poles on the physical strip and they passively respect the bootstrap equations. For our model the CDD ambiguities correspond to multiplying the S-matrix of the fundamental particles (2.3) by factors of the form

$$\frac{\sinh\left(\frac{\theta}{2}-\frac{i\pi}{2n}(2-\alpha)\right)\sinh\left(\frac{\theta}{2}-\frac{i\pi}{2n}\alpha\right)}{\sinh\left(\frac{\theta}{2}+\frac{i\pi}{2n}(2-\alpha)\right)\sinh\left(\frac{\theta}{2}+\frac{i\pi}{2n}\alpha\right)}. \tag{2.8}$$

where $0<\alpha<2$. One of the conclusions of this paper will be that the minimal form (2.3) is the true S-matrix of the theory, so that all CDD factors are ruled out. As in previous work [1-8] this will follow from the consistency of our calculation using the minimal S-matrix with a calculation in perturbation theory.

Although it will not concern us directly here, we point out that the complete spectrum of the model can be determined using the bootstrap procedure. The S-matrix (2.3) for the fundamental particles has a simple pole on the physical strip at $\theta = 2\pi i\Delta$ which corresponds to a bound state transforming in the antisymmetric tensor representation of SU(n). Continuing the bootstrap in this way one finds a spectrum of bound-states which is identical to the SU(n) chiral Gross-Neveu model, namely $m_r = m\sin(\pi r\Delta)/\sin(\pi\Delta)$, $1 \le r < n$, where the r^{th} bound-state transforms in the r^{th} fundamental representation of SU(n) (each particle carries in addition supersymmetric quantum numbers). The fundamental anti-particles correspond to the $n{-}1^{\mathrm{th}}$ bound-state. We shall only require the S-matrix elements of the fundamental particles and the fundamental anti-particles for our calculation.

3. Coupling to a conserved charge

As summarized in [1], we wish to couple the model to a chemical potential h via a conserved charge Q and to calculate the resulting free energy as a function of h. In other words, we seek the response $\delta f(h) = f(h) - f(0)$ of the ground state energy density for the system with Hamiltonian modified from H to $H - hQ$. The TBA equations allow one to find an expansion for $\delta f(h)$ of the form $h^2 F_1(h/m)$ valid when $h \gg m$ and in this regime one can calculate the same quantity in perturbation theory to obtain an expression $h^2 F_2(h/\Lambda)$, where Λ is the usual dimension-full parameter of perturbation theory. Setting $F_1(h/m) = F_2(h/\Lambda)$ and comparing the first few leading order terms gives a powerful test of the S-matrix and allows one to extract the mass gap m/Λ.

The analysis of the TBA system is only tractable if we can choose Q so that the new ground-state consists of a restricted number of particle types. In [1] we found that the presence of supersymmetry complicated matters because the ground state formed a supersymmetric doublet with non-trivial scattering, unlike the examples in [3–5] where one could select Q so as to give a single particle type. Despite this complication, the problem considered in [1] could still be solved. In the present case we might be tempted to make the same choice as for the SU(n) principle chiral model [4] and the chiral Gross-Neveu model [6]:

$$Q = \operatorname{diag}\left(1, -\frac{1}{n-1}, \ldots, -\frac{1}{n-1}\right), \tag{3.1}$$

for which the fundamental multiplet $|1, j, \theta\rangle$ has the largest charge/mass ratio. As we shall see in the next section, however, this choice is inconvenient because, even if the TBA system proved tractable, it would require a three-loop perturbation theory computation to extract the mass-gap. To avoid this we are motivated to consider the alternative choice

$$Q = \operatorname{diag}(1, -1, 0, \ldots, 0). \tag{3.2}$$

With this coupling the situation is still more complicated than in [1] because there are now two fundamental doublets with the largest charge/mass ratio, namely $|1, j, \theta\rangle$ and $|\bar{2}, j, \theta\rangle$, a feature which clearly arises because of the presence of distinct antiparticles. Nevertheless we shall see that the resulting TBA system can be analyzed successfully and that the mass gap can again be extracted by comparison with a perturbation theory calculation to just one loop.

It is worth re-iterating the point already emphasized in [1] that it is an assumption that only $|1, j, \theta\rangle$ and $|\bar{2}, j, \theta\rangle$ appear in the ground state and that in particular no bound states contribute. As in most previous work of this type, we rely on the consistency of our final results to vindicate this assumption. An important property of the states which we are assuming appear, is that their scattering is purely elastic in the space of SU(n) quantum numbers. Of course the scattering is still non-diagonal in the supersymmetric subspace. The SU(n) part of the S-matrix elements of 1 with 1 and $\bar{2}$ with $\bar{2}$ coincide:

$$S_{\rm CGN}(\theta)_{11}^{11} = S_{\rm CGN}(\theta)_{\bar{2}\bar{2}}^{\bar{2}\bar{2}} = \frac{\Gamma(1 + i\theta/2\pi)\Gamma(1 - \Delta - i\theta/2\pi)}{\Gamma(1 - i\theta/2\pi)\Gamma(1 - \Delta + i\theta/2\pi)}. \tag{3.3}$$

The SU(n) part of the S-matrix for 1 interacting with $\bar{2}$ can be found by crossing:

$$S_{\rm CGN}(\theta)_{1\bar{2}}^{\bar{2}1} = S_{\rm CGN}(\theta)_{\bar{2}1}^{1\bar{2}} = \frac{\Gamma(\frac{1}{2} - i\theta/2\pi)\Gamma(\frac{1}{2} - \Delta + i\theta/2\pi)}{\Gamma(\frac{1}{2} + i\theta/2\pi)\Gamma(\frac{1}{2} - \Delta - i\theta/2\pi)}. \tag{3.4}$$

A crucial point is that there is no reflection amplitude for the interaction of 1 and $\bar{2}$: the scattering is purely elastic.

4. Free-energy from perturbation theory

To couple the theory to the charge (3.2) by changing the Hamiltonian $H \to H - hQ$ we can make the replacement $\partial_0 \to \partial_0 + ihQ$ in the Lagrangian (2.1). It will turn out to be sufficient to calculate the ground-state energy density of this theory to one loop. We must therefore expand the Lagrangian to quadratic order in an independent set of fields and we can drop all terms which are independent of h to this order because we are interested only in the response of the free-energy density, $\delta f(h) = f(h) - f(0)$. By exploiting the local U(1) invariance of the action, we can take z_1 to be real and we can solve the bosonic constraint by writing $z_1 = \sqrt{(1-|\pi|^2)(\frac{1}{2}+\phi)}$ and $z_2 = e^{i\theta}\sqrt{(1-|\pi|^2)(\frac{1}{2}-\phi)}$ where $\pi = (z_3, \ldots, z_n)$ and θ, ϕ are real. The fermionic degrees of freedom and the variable θ decouple to this order, and we are left with the expression

$$\mathcal{L}_{1-\mathrm{loop}} = \frac{1}{2g}\left\{(\partial_\mu\phi)^2 + |\partial_\mu\pi|^2 + h^2 - 4h^2\phi^2 - h^2|\pi|^2\right\}. \qquad (4.1)$$

Using dimensional regularization with the $\overline{\mathrm{MS}}$-scheme gives the one-loop free energy

$$\delta f(h) = -\frac{h^2}{2g} - \frac{h^2}{\pi}\ln 2 + \frac{nh^2}{4\pi}(1 - \ln(h^2/\mu^2)). \qquad (4.2)$$

We now substitute for the running coupling. The result, expressed in terms of the one-loop and two-loop beta functions coefficients, is:

$$\delta f(h) = \\ -h^2\frac{\beta_1}{2}\left[\ln\frac{h}{\Lambda_{\overline{\mathrm{MS}}}} - \frac{1}{2} + \frac{2\ln 2}{\pi\beta_1} + \frac{\beta_2}{\beta_1^2}\ln\ln\frac{h}{\Lambda_{\overline{\mathrm{MS}}}} + \mathcal{O}\left(\frac{\ln\ln(h/\Lambda_{\overline{\mathrm{MS}}})}{\ln(h/\Lambda_{\overline{\mathrm{MS}}})}\right)\right]. \qquad (4.3)$$

Substituting the specific values for these coefficients given in (2.2) we obtain

$$\delta f(h) = -\frac{h^2 n}{2\pi}\left[\ln\frac{h}{\Lambda_{\overline{\mathrm{MS}}}} - \frac{1}{2} + \frac{2\ln 2}{n} + \mathcal{O}\left(\frac{\ln\ln(h/\Lambda_{\overline{\mathrm{MS}}})}{\ln(h/\Lambda_{\overline{\mathrm{MS}}})}\right)\right]. \qquad (4.4)$$

Notice that if we chose the charge Q to be (3.1) then there would be no "tree-level" $\mathcal{O}(1/g)$ term. In this case it would be necessary to do a three-loop computation in order to extract the mass-gap.

5. Free-energy from the S-matrix

In this section we write down the TBA equations for the model and solve them in the limit $h \gg m$. We must follow the hypothesis made earlier that only the multiplets $|1, j, \theta\rangle$

and $|\bar{2}, j, \theta\rangle$ contribute to the ground-state. Since the scattering of these multiplets is purely elastic, it will not be necessary to perform a diagonalization in the space of SU(n) quantum numbers (although this diagonalization can be done [28]). The remaining difficulty is that the S-matrix for these favoured states is still non-diagonal in the supersymmetric subspace. But this problem can also be solved since it has been shown by Fendley and Intriligator [17] that it is equivalent to diagonalizing the transfer matrix of the six vertex model at the free fermion point.

In [17] a system of equations is derived which links the density of single particle states in rapidity space $\varrho_a(\theta)$ to the density of occupied single particle states $\sigma_a(\theta)$, where $a = 1, \bar{2}$. The equations also involve densities $P_l^+(\theta)$ and $P_l^-(\theta)$, $l = 0, \bar{0}$ corresponding to two "supersymmetric magnons", reflecting the fact that there are two supersymmetries:

$$\begin{aligned} \varrho_a(\theta) &= \frac{m}{2\pi}\cosh\theta + \phi_{ab} * \sigma_b(\theta) + \phi_{al} * P_l^+(\theta), \\ P_l^+(\theta) + P_l^-(\theta) &= \phi_{al} * \sigma_a(\theta), \end{aligned} \tag{5.1}$$

where $f * g(\theta) = \int_{-\infty}^{\infty} d\theta' f(\theta - \theta')g(\theta')$. The kernels appearing here are

$$\begin{aligned} \phi_{ab}(\theta) &= \frac{1}{2\pi i}\frac{d}{d\theta}\ln S_{\rm CGN}(\theta)_{ab}^{ba}, \\ \phi_{10}(\theta) = \phi_{\bar{2}\bar{0}}(\theta) &= \frac{\sin(\pi\Delta)}{\cosh\theta - \cos(\pi\Delta)}, \\ \phi_{\bar{2}0}(\theta) = \phi_{1\bar{0}}(\theta) &= \frac{\sin(\pi\Delta)}{\cosh\theta + \cos(\pi\Delta)}, \end{aligned} \tag{5.2}$$

where as before $a, b = 1, \bar{2}$ and the elements $S_{\rm CGN}(\theta)_{ab}^{ba}$ are written down in (3.3) and (3.4). Notice that the kernel which multiplies the densities of the particles $\sigma_a(\theta)$ involves a contribution only from the chiral Gross-Neveu part of the S-matrix—so if we removed the magnon terms the equations would be identical to those for the chiral Gross-Neveu model. Notice also that the super-magnons do not interact amongst themselves.

To find the TBA equations [29] we define the "excitation energies" of the particles $\epsilon_a(\theta)$ and the magnons $\xi_l(\theta)$ at finite temperature T by

$$\frac{\sigma_a(\theta)}{\varrho_a(\theta)} = \frac{1}{e^{\epsilon_a(\theta)/T} + 1}, \qquad \frac{P_l^-(\theta)}{P_l^+(\theta)} = e^{\xi_l(\theta)/T}. \tag{5.3}$$

We shall require the TBA equations at zero temperature with the field h coupling via Q acting as a chemical potential. At $T = 0$, $\epsilon_a(\pm\theta_{\rm F}^a) = 0$, where $\theta_{\rm F}^a$ are Fermi rapidities and $\epsilon_a(\theta)$ is negative precisely when $-\theta_{\rm F}^a < \theta < \theta_{\rm F}^a$. It is convenient to introduce the following notation

$$f^{\pm}(\theta) = \begin{cases} f(\theta) & f(\theta) \gtrless 0 \\ 0 & \text{otherwise.} \end{cases} \tag{5.4}$$

The free-energy per unit volume at $T = 0$ is

$$\delta f(h) = f(h) - f(0) = \frac{m}{2\pi}\int_{-\infty}^{\infty} d\theta \left[\epsilon_1^-(\theta) + \epsilon_{\bar{2}}^-(\theta)\right]\cosh\theta, \tag{5.5}$$

where $\epsilon_a(\theta)$, $a=1,\bar{2}$, are the solutions of the $T=0$ TBA equations:

$$\begin{aligned}\epsilon_a(\theta)-\phi_{ab}*\epsilon_b^-(\theta)-\phi_{al}*\xi_l^-(\theta)&=m\cosh\theta-h,\\ \xi_l(\theta)-\phi_{al}*\epsilon_a^-(\theta)&=0.\end{aligned}\tag{5.6}$$

To simplify (5.6) it is important to notice that $\phi_{al}(\theta)$ is a positive kernel, which implies that the magnon variables are given by $\xi_l^+(\theta)=0$ and $\xi_l^-(t)=\phi_{al}*\epsilon_a^-(\theta)$. Furthermore, the solution does not distinguish between the values of the favoured SU(n) quantum numbers and so we have $\epsilon_1(\theta)=\epsilon_{\bar{2}}(\theta)\equiv\epsilon(\theta)$. The four equations in (5.6) then reduce to a single equation for $\epsilon(\theta)$:

$$\epsilon^+(\theta)+R*\epsilon^-(\theta)=m\cosh\theta-h,\tag{5.7}$$

where the kernel is

$$R(\theta)=\delta(\theta)-\phi_{11}(\theta)-\phi_{1\bar{2}}(\theta)-[\phi_{10}+\phi_{1\bar{0}}]*[\phi_{10}+\phi_{1\bar{0}}](\theta).\tag{5.8}$$

and the expression (5.5) for the free-energy density becomes

$$\delta f(h)=\frac{m}{\pi}\int_{-\infty}^{\infty}d\theta\,\epsilon^-(\theta)\cosh\theta.\tag{5.9}$$

The Fourier transform of the kernel in (5.7) is

$$R(\theta)=\int_0^\infty\frac{d\omega}{\pi}\cos(\omega\theta)\frac{\cosh((1-2\Delta)\pi\omega/2)\sinh(\pi\Delta\omega)}{\cosh^2(\pi\omega/2)}e^{\pi\omega/2},\tag{5.10}$$

where $\Delta=1/n$. As in [1] we see that this Fourier transform vanishes at the origin so that the solution resembles those for the bosonic models discussed in [3-4]. We now seek an expression for the Fourier transform of the kernel $1/(G_+(\omega)G_-(\omega))$ where $G_\pm(\omega)$ are analytic in the upper (lower) half planes and $G_-(\omega)=G_+(-\omega)$. The unique solution is

$$\begin{aligned}G_+(\omega)=&\frac{\Gamma(\frac{1}{2}-i(1-2\Delta)\omega/2)\Gamma(1-i\Delta\omega)}{\Gamma^2(\frac{1}{2}-i\omega/2)}e^{-\frac{1}{2}\ln(-i\Delta\omega)}\\ &\times e^{-i\omega(\frac{1}{2}-\Delta)(1-\ln(-i\omega(\frac{1}{2}-\Delta)))-i\omega\Delta(1-\ln(-i\omega\Delta))+i\omega(1-\ln(-i\frac{1}{2}\omega))}.\end{aligned}\tag{5.11}$$

From [4] we know that if $G_+(i\xi)$ has an expansion for small ξ like

$$G_+(i\xi)=\frac{k}{\sqrt{\xi}}e^{-a\xi\ln\xi}\left(1-b\xi+\mathcal{O}(\xi^2)\right),\tag{5.12}$$

then the free-energy density for $h\gg m$ takes the form

$$\begin{aligned}\delta f(h)=-\frac{h^2k^2}{2}\Bigg[&\ln\frac{h}{m}+\ln\left(\frac{\sqrt{2\pi}ke^{-b}}{G_+(i)}\right)-1+a(\gamma_{\rm E}-1+\ln 8)\\ &+(a+\tfrac{1}{2})\ln\ln\frac{h}{m}+\mathcal{O}\left(\frac{\ln\ln(h/m)}{\ln(h/m)}\right)\Bigg].\end{aligned}\tag{5.13}$$

Our kernel does indeed have an expansion of the form (5.12) with

$$k=\frac{1}{\sqrt{\pi\Delta}},\quad a=-\frac{1}{2},\quad\frac{\sqrt{2\pi}ke^{-b}}{G_+(i)}=\frac{\sin(\pi\Delta)}{\pi\Delta}e^{\gamma_{\rm E}/2+\left(\frac{3}{2}+2\Delta\right)\ln 2}.\tag{5.14}$$

The resulting expression for the free energy is

$$\delta f(h)=-\frac{h^2}{2\pi\Delta}\left[\ln\frac{h}{m}+\ln\left(\frac{\sin(\pi\Delta)}{\pi\Delta}2^{2\Delta}\right)-\frac{1}{2}+\mathcal{O}\left(\frac{\ln\ln(h/m)}{\ln(h/m)}\right)\right].\tag{5.15}$$

6. Comparison and conclusions

Comparing (5.15) with (4.4) we see that the result from the TBA calculation correctly reproduces the universal coefficients of the beta-function and we extract the following value for the mass-gap for the supersymmetric $\mathbb{C}\mathrm{P}^{n-1}$ model

$$\frac{m}{\Lambda_{\overline{\mathrm{MS}}}} = \frac{\sin(\pi\Delta)}{\pi\Delta}, \qquad \Delta = \frac{1}{n}, \qquad n > 1. \tag{6.1}$$

It is instructive to consider what would have happened if we had chosen the other charge (3.1), rather than (3.2). In that case only the multiplet $|1, j, \theta\rangle$ would have appeared in the ground-state and the resulting TBA equation would have had the same form as (5.7), but with a different kernel:

$$R(\theta) = \delta(\theta) - \phi_{11}(\theta) - \phi_{10} * \phi_{10}(\theta) - \phi_{1\bar{0}} * \phi_{1\bar{0}}(\theta). \tag{6.2}$$

The Fourier transform of this kernel does not vanish at the origin, and so the situation is analogous to the Gross-Neveu models [6,5]. In these cases, the expansion of the free-energy density for $h \gg m$ has a different form which has a well-defined limit as $h \to \infty$, unlike (5.15). This matches precisely the fact that with this different choice of charge the perturbative expansion of the free-energy density is also markedly different since there is no "tree-level" contribution and it would require a three-loop calculation to extract the mass-gap.

In addition to our result for the mass gap, an important conclusion of our paper is that we have resolved the problem of CDD ambiguities in the ansatz of Köberle and Kurak. Any additional CDD factors of the form (2.8) would alter the kernel $R(\theta)$ of the TBA equation and the thermodynamics of the system depends so sensitively on this that we can argue with some confidence that the solution for the free-energy density would not match that found in perturbation theory.

It is interesting to compare our result for the mass-gap with the spectrum for the super-$\mathbb{C}\mathrm{P}^{n-1}$ model proposed in [18] on the basis of very different methods. It is clear that the the n-dependence of these expressions is in complete agreement (setting $r = 1$ in the formula given in the discussion following equation (6) in the first reference in [18] should give the mass of the fundamental multiplet) but the connection between the overall scale factors is less clear. It would be interesting to investigate in detail how the results of [18] could be expressed in terms of the more conventional scale parameter used in this paper.

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