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REMARKS ON NON-ABELIAN DUALITY

S. Elitzur[1], **A. Giveon**[2], **E. Rabinovici**[3]
Racah Institute of Physics, The Hebrew University
Jerusalem, 91904, Israel

A. Schwimmer[4]
Weizmann Institute, Rehovot, Israel
and SISSA and INFN Trieste, Italy

G. Veneziano[5]
Theoretical Physics Division, CERN CH - 1211 Geneva 23
and Observatoire de Paris and ENS, Paris, France

ABSTRACT

A class of two-dimensional globally scale-invariant, but not conformally invariant, theories is obtained. These systems are identified in the process of discussing global and local scaling properties of models related by duality transformations, based on non-semisimple isometry groups. The construction of the dual partner of a given model is followed through; non-local as well as local versions of the former are discussed.

[1] e-mail address: elitzur@vms.huij.ac.il
[2] e-mail address: giveon@vms.huji.ac.il
[3] e-mail address: eliezer@vms.huji.ac.il
[4] e-mail address: ftschwim@weizmann.weizmann.ac.il
[5] e-mail address: VENEZIA@NXTH04.CERN.CH

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1 Introduction and Discussion

The large amount of constraints imposed by conformal invariance in two dimensions has enabled the exact solution of many such systems. The question has arisen time and again whether the same symmetry and results could be obtained by requiring only the global scale symmetry. In this paper we deal with various aspects of this issue in two-dimensional field theories, aspects that have emerged rather unexpectedly in the process of mapping one theory into another by discrete transformations of target-space-duality type.

Duality relates two different geometries by establishing an isomorphism between the sets of harmonic maps from S^2 into the two manifolds. A standard procedure exists for discovering the dual partner of a given manifold, when the latter possesses a continuous group of isometries [1].

In ref. [2], a flat manifold was discovered for which the dual partner resulting from this procedure, when applied with respect to a certain group of isometries, is a manifold whose corresponding two-dimensional sigma model is not conformal. An explicit calculation of the β-functions corresponding to this model shows not only that they do not vanish, but also that they cannot be cancelled by an appropriate dilaton term. This shows that duality, in the above sense of correspondence between classical sigma model solutions, is not sufficient for the equivalence of the two quantum models on a curved worldsheet background.

It was pointed out in ref. [3] that the group of isometries used in this example is non-semisimple, and contains traceful structure constants. It was also suggested that these features are related to an anomaly. The anomaly was identified in ref. [4] as a mixed gravitational-gauge or conformal-gauge anomaly. Its origin is in the possible dependence of the Jacobians – related to the passage from the original coordinates to the dual ones – on the worldsheet metric and the traceful isometry generators.

In this paper we show that the Jacobian appears in the form

$$\det(MN)/(\det M \det N),$$

and therefore, to have an overall anomaly it is crucial to have a multiplicative anomaly [5], *i.e.*, $\det(MN) \neq \det M \det N$. It turns out that, while the dependence survives for the mixed gauge-conformal anomaly, for the pure conformal anomaly it cancels. As a result, in the dual sigma-model action the duality process generates an additional non-local term proportional to the trace of the generator and to $\frac{1}{\Box}R^{(2)}$, where $R^{(2)}$ is the worldsheet curvature. (This term is essentially the additional term which duality induces from the anomaly term of [4].)

We use these results to study the worldsheet-metric dependence of models emerging from such an anomalous duality process. Let A be the original model with the invariance group of isometries, and B' the sigma model resulting from the duality procedure (the "prime" on B will be explained later). We first stress that, since the only difference between B' and A is the calculable and well-controlled non-local anomalous term, the extra dependence of B' on the conformal background should cancel exactly the background dependence of this term. In particular, if A is conformal, the ordinary geometric calculation of the β-function of B' should be cancelled by the variation of the anomaly term with respect to the background conformal factor. We check this in detail for a few examples, including that studied in [2].

On a flat worldsheet background, the sigma models A and B' are already equivalent. In particular, their spectra are identical. This is confirmed by the result that the anomaly is proportional to $R^{(2)}$. In the case in which A is conformal, the spectrum of B' must be massless as well. Next we recall some general features of massless 2-d sigma models.

The simplest situation is when the model is truly conformal, *i.e.*, when there exists a local, traceless energy-momentum tensor $T_{\alpha\beta}$. The β-function equations give a vanishing result. As a consequence, in this case the model can be coupled to a two-dimensional background metric in a Weyl-invariant manner at the quantum level. It could happen that such a local traceless tensor does not exist. Since the spectrum is massless, there should exist a local, conserved dilation current D_α. From the relation between $T_{\alpha\beta}$ and D_α it follows [6] that in this case the trace of the energy-momentum tensor must be a total derivative. The β-function equations no longer give zero; however, the deviation from zero can be absorbed in a "wave function renormalization", *i.e.*, a reparametrization of the fields, and in a gauge transformation of the antisymmetric tensor which enters the sigma model [7, 6]. Explicitly, the β-function with respect to the target-space metric $G_{\mu\nu}$ has the form:

$$\beta_{G_{\mu\nu}} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \tag{1.1}$$

where ξ_μ defines the reparametrization. For a general ξ_μ one cannot define a local Weyl-invariant coupling to the background metric. If, however, ξ_μ has the special form:

$$\xi_\mu = \partial_\mu \Phi, \tag{1.2}$$

where Φ is the "dilaton" field, a local coupling can be defined involving the curvature, and an "improved" traceless energy-momentum tensor exists.

Our case of a sigma model B', resulting by an anomalous duality process from an originally conformal theory, is of the type of a massless *nonconformal* theory. In fact, the representation of its β-function as resulting from the variation of the anomalous term with respect to the conformal worldsheet factor, makes it explicit that these β-functions are total derivatives representable in the form of eq. (1.1).

We will study two such examples. One will turn out to be improvable by a dilaton term. The other example, that of [2], develops a β-function which is a total derivative, but not improvable by a local background-dependent term. We have thus found a genuine example of a two-dimensional, scale-invariant, interacting non-conformal model. In ref. [6] it was proved that for a unitary theory in $d = 2$ with a discrete operator spectrum, an improved energy-momentum tensor always exists. Our example is, necessarily, a non-compact sigma model. Similar situations arise in higher dimensions at the tree level for gauge theories based on n-forms [8].

Note, however, that this apparently non-locally improvable B' model is actually improvable in a more general sense. One can "improve" it by adding to its action the non-local background-dependent anomaly term, and then re-express it in a local form by passing to the language of the A-model which, in the presence of the anomaly term, is completely equivalent to it.

The paper is organized as follows: in section 2, we discuss in detail the procedure of non-Abelian duality [9] (for an isometry group acting with no fixed points [3]), in particular the appearance of the anomaly. In section 3, we show how the original A-model can be recovered. In section 4, we discuss the B-model related to it, *i.e*, the B' sigma model together with the (non-local) improvement. In Section 5, we discuss an example where the reparametrization needed is of the form (1.2) and, therefore, models A and B are equivalent after a dilaton-type correction to B' is made. In section 6, we discuss the model proposed in [2]: model A is conformal but model B' is just dilation invariant. In section 7, we discuss how the anomaly can be understood in terms of Ward identities in a flat worldsheet background. In section 8, we discuss some alternative local representations of the mixed anomaly term, in a higher dimensional sigma model. A general form of the anomaly for a non-compact group and a general proof of the dilaton corrections [10] are discussed in two appendices.

2 The general case (without isotropy)

As in ref. [3], we can consider a target space with coordinates g that transform as $g \to ug$ for u in some group G, and further coordinates x^i that are inert. A general action can be written in the form

$$\begin{aligned} S[g,x] = \frac{1}{2\pi}\int d^2z\,\Big(E_{ab}(x)(g^{-1}\partial g)^a(g^{-1}\bar\partial g)^b \;&+\; F^R_{aj}(x)(g^{-1}\partial g)^a\bar\partial x^j \\ +\; F^L_{ib}(x)\partial x^i(g^{-1}\bar\partial g)^b \;&+\; F_{ij}(x)\partial x^i\bar\partial x^j \\ &-\; \Phi(x)\partial\bar\partial\sigma\Big)\,, \end{aligned} \tag{2.1}$$

where

$$(g^{-1}\partial g)^a \equiv tr(\tilde T^a g^{-1}\partial g) \quad \Leftrightarrow \quad g^{-1}\partial g = (g^{-1}\partial g)^a T_a\ , \quad etc. \tag{2.2}$$

The generators T_a, $a = 1, ..., dim(G)$, obey

$$[T_a, T_b] = f^c_{ab}T_c, \tag{2.3}$$

and the "dual generators" $\tilde T^a$ are defined by the condition

$$tr(T_a\tilde T^b) = \delta^b_a. \tag{2.4}$$

In eq. (2.1) the background matrices $E_{ab}, F^L_{ib}, F^R_{aj}, F_{ij}$ and the dilaton Φ depend only on the coordinates x^i. Here $z = (\zeta_1 + i\zeta_2)/\sqrt2$, $\bar z = (\zeta_1 - i\zeta_2)/\sqrt2$ are complex worldsheet coordinates, $\partial \equiv \partial/\partial z = (\partial_1 - i\partial_2)/\sqrt2$, $\bar\partial \equiv \partial/\partial\bar z = (\partial_1 + i\partial_2)/\sqrt2$, and $\sigma(z,\bar z)$ is the worldsheet conformal factor, *i.e.*

$$\partial\bar\partial\sigma = \frac14\sqrt h R^{(2)}, \qquad h_{z\bar z} = e^{-2\sigma}, \tag{2.5}$$

where h is the worldsheet metric (in the conformal gauge) and $R^{(2)}$ is the worldsheet curvature.

The quantum field theory is defined by the functional integral

$$Z_O = \int D_Lg\ Dx\ e^{-S[g,x]} \tag{2.6}$$

(for future use, we label this partition function as Z_O). Here D_Lg is the left invariant measure (which is required for consistency with the isometry that acts from the left, $g \to ug$) [1]

$$D_Lg \equiv \prod_{a,z,\bar z}(g^{-1}dg)^a, \tag{2.7}$$

[1] For semi-simple groups there is no difference between left and right invariant measures; however, for general groups they may differ.

and Dx is the rest of the sigma-model measure ($Dx \equiv \prod_{z,\bar z}\sqrt{\det(F_{ij})\det(E_{ab})}dx$, if $F^L = F^R = 0$).

We now rewrite the theory (2.6) by inserting the identity $I = \int D_L\bar g\ \delta_L(g,\bar g)$. The (left-invariant) delta-function sets $\bar g = g$ and allows, in particular, the replacement of $g^{-1}\bar\partial g$ by $\bar g^{-1}\bar\partial\bar g$:

$$Z_O = \int D_Lg\ D_L\bar g\ \delta_L(g,\bar g)\ Dx\ e^{-S[g,\bar g,x]}, \tag{2.8}$$

where

$$\begin{aligned} S[g,\bar g,x] &= \frac{1}{2\pi}\int d^2z\,\Big(E_{ab}(x)(g^{-1}\partial g)^a(\bar g^{-1}\bar\partial\bar g)^b \ +\ F^R_{aj}(x)(g^{-1}\partial g)^a\bar\partial x^j \\ &+\ F^L_{ib}(x)\partial x^i(\bar g^{-1}\bar\partial\bar g)^b \ +\ F_{ij}(x)\partial x^i\bar\partial x^j \\ &-\ \frac{1}{4}\Phi(x)\sqrt{h}R^{(2)}\Big)\,. \end{aligned} \tag{2.9}$$

At this point we define a vector field $A, \bar A$ in terms of the group variables $g, \bar g \in G$:

$$A = g^{-1}\partial g, \qquad \bar A = \bar g^{-1}\bar\partial\bar g. \tag{2.10}$$

Changing the integration variables from $g, \bar g$ to $A, \bar A$ one gets

$$\begin{aligned} Z_O \to Z &= \int DA\ D\bar A\ \det(DA/D_Lg)^{-1}\ \det(D\bar A/D_L\bar g)^{-1} \\ &\times\ \delta(F)\ \det(DF/D_Lg)|_{\bar g=g}\ e^{-S[A,\bar A,x]}, \end{aligned} \tag{2.11}$$

where $S[A,\bar A,x]$ is given by inserting eq. (2.10) into $S[g,\bar g,x]$ (2.9). In eq. (2.11) F is the field strength,

$$\begin{aligned} F(A,\bar A) &= \partial\bar A - \bar\partial A + [A,\bar A] \ =\ g^{-1}\partial(f^{-1}\bar\partial f)g \ =\ \bar g^{-1}\bar\partial(\partial f f^{-1})\bar g, \\ f &\equiv \bar g g^{-1}, \end{aligned} \tag{2.12}$$

and we have used the equality[2]

$$\delta_L(g,\bar g) = \delta(F)\det N, \qquad N = (DF/D_Lg)|_{\bar g=g}. \tag{2.13}$$

Calculating N (by using eqs. (2.7) and (2.12)) we obtain formally

$$N = M(A)\bar M(\bar A), \tag{2.14}$$

where

$$M(A) = \partial + [A,\cdot], \qquad \bar M(\bar A) = \bar\partial + [\bar A,\cdot]. \tag{2.15}$$

[2]The equality (2.13) is only correct locally: $\delta(F)$ constrains the curvature to vanish, but does not force the connection A to be trivial on a non-zero genus worldsheet. Therefore, to correct eq. (2.13), we need to sum $\delta_L(g,\bar g)$ over all flat connections. We will not address such global issues [11, 3, 12] here. Moreover, as $F = 0$, one could involve in the process of duality any function of F which would vanish for $F = 0$. Such terms lead, in general, to very complicated non-local theories, all equivalent. This structure can, however, be removed by applying a non-local field redefinition.

Next we replace the $F = 0$ delta-function with the constraint imposed by a Lagrange multiplier term in the action, namely,

$$\delta(F) = \int D\lambda \; e^{-\frac{1}{2\pi}\int d^2z \, tr\lambda F}. \tag{2.16}$$

Here

$$F \equiv F^a T_a, \quad \lambda \equiv \lambda_a \tilde{T}^a \;\Rightarrow\; tr\lambda F = \lambda_a F^a. \tag{2.17}$$

Finally, we should deal with the Jacobian of the change of variables from $g, \bar{g}$ to $A, \bar{A}$:

$$\det(DA/D_L g)\det(D\bar{A}/D_L\bar{g}) = \det M(A)\det \bar{M}(\bar{A}), \tag{2.18}$$

where $M(A), \bar{M}(\bar{A})$ are given in (2.15). The precise manner of calculating these determinants is given in [4].

Altogether, the partition function takes the form:

$$\begin{aligned} Z &= \int D\lambda \; Dx \; DA \; D\bar{A} \; e^{J(A,\bar{A})} \; e^{-S[A,\bar{A},\lambda,x]}, \\ e^{J(A,\bar{A})} &= \frac{\det(M(A)\bar{M}(\bar{A}))}{\det M(A)\det \bar{M}(\bar{A})}, \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} S[A,\bar{A},\lambda,x] &= \frac{1}{2\pi}\int d^2z \Big(E_{ab}(x)A^a\bar{A}^b \;+\; F^R_{aj}(x)A^a\bar{\partial}x^j \\ &+ \; F^L_{ib}(x)\partial x^i\bar{A}^b \;+\; F_{ij}(x)\partial x^i\bar{\partial}x^j \;-\; \frac{1}{4}\Phi(x)\sqrt{h}R^{(2)} \\ &+ \; \lambda_a(\partial\bar{A}^a - \bar{\partial}A^a + f^a_{bc}A^b\bar{A}^c)\Big) \; . \end{aligned} \tag{2.20}$$

In the following, we discuss the evaluation of the determinants term in Z (2.19): $\exp J(A,\bar{A})$. Naively the determinants would cancel; however, there is a possibility of a multiplicative anomaly [5]. It is convenient to use a covariant notation and to replace the determinants by actions involving bosonic ghosts $\beta_\alpha, \gamma, \gamma'$ (here $\alpha = 1,2$ is a worldsheet index) and fermionic ghosts b, c:

$$\begin{aligned} e^{J(A,\bar{A})} &= \int D\beta \; D\gamma \; D\gamma' \; Db \; Dc \\ &\times \; \exp\int d^2\zeta \; tr[\sqrt{h}h^{\alpha\delta}(\beta_\alpha D_\delta\gamma + bD_\alpha D_\delta c) + \beta_\alpha\tilde{D}^\alpha\gamma'], \end{aligned} \tag{2.21}$$

where

$$\beta_\alpha \equiv \beta_{\alpha a}\tilde{T}^a, \qquad \gamma \equiv \gamma^a T_a, \qquad \gamma' \equiv \gamma'^a T_a, \qquad b \equiv b_a\tilde{T}^a, \qquad c \equiv c^a T_a, \tag{2.22}$$

$$D_\alpha \equiv \partial_\alpha + [g^{-1}\partial_\alpha g, \cdot], \qquad \tilde{D}^\alpha \equiv \epsilon^{\alpha\beta}D_\beta. \tag{2.23}$$

In deriving (2.21) we used the fact that

$$(D^\alpha + \tilde{D}^\alpha)(D_\alpha - \tilde{D}_\alpha) = 2D^\alpha D_\alpha \tag{2.24}$$

if the field entering D_α is a pure gauge.

The ghost actions are formally conformal invariant as well as gauge invariant under $g \to gu$, with all ghost fields transforming in the adjoint representation: $c \to u^{-1}cu$, $b \to u^{-1}bu$. A new type of anomaly may appear coupling the background worldsheet metric to the gauge field [3, 4]. This anomaly can be studied by considering for the free part of (2.21) the correlator between the energy-momentum tensor and the vector current V_a^α coupled to the gauge field (note that by giving a negative intrinsic parity to γ', its gauge coupling can also be considered vectorlike):

$$\frac{\delta^2 J}{\sqrt{h}\delta h_{\alpha\beta}(\zeta)\sqrt{h}\delta A^a_\gamma(\xi)} = \langle T^{\alpha\beta}(\zeta)V_a^\gamma(\xi)\rangle \equiv S_a^{\alpha\beta\gamma}(\zeta-\xi), \tag{2.25}$$

$$T^{\alpha\beta} = \frac{1}{2}\sqrt{h}\ tr\Big(\beta^\alpha D^\beta\gamma - \frac{1}{2}h^{\alpha\beta}\beta^\delta D_\delta\gamma\Big) + (\alpha\leftrightarrow\beta),$$
$$V_a^\alpha \equiv \frac{1}{\sqrt{h}}\frac{\delta J}{\delta A^a_\alpha} = f^e_{ad}(\beta^\alpha_e\gamma^d + \epsilon^{\delta\alpha}\beta_{\delta e}\gamma'^d + 2b_e\nabla^\alpha c^d). \tag{2.26}$$

Note that $T^{\alpha\beta}$ in (2.26) does not involve γ'. This energy-momentum tensor is invariant under the group G. Taking the expectation value of the commutator of $T^{\alpha\beta}V_a^\gamma$ with an isometry generator Q_b, and using $[Q_b, T^{\alpha\beta}] = 0$, and $Q_b|0\rangle = 0$ which implies $\langle[Q_b, T^{\alpha\beta}V_a^\gamma]\rangle = 0$, one obtains

$$\langle T^{\alpha\beta}[Q_b, V_a^\gamma]\rangle = 0. \tag{2.27}$$

Therefore, the correlator (2.25) can be non-zero only for a group index which does not appear on the right-hand side of a commutator $[Q_b, V_a]$. Note that all the generators which do appear on this right-hand side necessarily have zero trace (because the trace of a commutator is zero). The necessary and sufficient condition to have such "quasi-Abelian" directions is for the group to be non-semisimple [13].

The kinematical constraints on the correlator (2.25) on a flat background are very simple to analyse: the correlator depends on $\zeta - \xi$ and, therefore, its Fourier transform $\tilde{S}$ depends on a single momentum q^α, and its general kinematical decomposition has the form

$$\tilde{S}_a^{\alpha\beta\gamma} = A_a\eta^{\alpha\beta}q^\gamma + B_a(\eta^{\alpha\gamma}q^\beta + \eta^{\beta\gamma}q^\alpha) + C_a q^\alpha q^\beta q^\gamma, \tag{2.28}$$

where A_a, B_a, C_a are invariant amplitudes depending on q^2 only. Using eq. (2.28) it is easy to show that at least two out of the three Ward identities, following from the conservation and tracelessness of $T^{\alpha\beta}$ and the conservation of V_a^α, should be violated. In particular, if the conservation of $T^{\alpha\beta}$ is imposed, $\tilde{S}_a^{\alpha\beta\gamma}$ has a unique form in any dimension d:

$$\tilde{S}_a^{\alpha\beta\gamma} = K_a(\eta^{\alpha\beta}q^2 - q^\alpha q^\beta)(q^2)^{(d-4)/2}q^\gamma, \tag{2.29}$$

where K_a are dimensionless constants. The expression (2.29) violates the conservation of the vector current. It follows that, if dimensional regularization can be used, the correlator will vanish identically. Therefore, the only possibility to obtain an anomalous correlator is to have in the action a γ_5 or $\epsilon^{\alpha\beta}$. We conclude, therefore, that the b, c action corresponding to $\det(M(A)\bar{M}(\bar{A}))$ does not have an anomaly [3]. In order to calculate the coefficient K we isolate

[3]If we studied $\det(M(A)\bar{M}(\bar{A}))$ in a general A-background, the anomaly could appear only in the unnatural parity part; for this type of anomaly the current conservation can be imposed and, therefore, the anomaly vanishes for a pure gauge configuration, leading again to the same conclusion.

the convergent C amplitude in the Feynman diagram, the other amplitudes being determined by the choice of which Ward identity is preserved. We obtain: $K_a = 1/4\pi\ (tr\ T_a)$.

On a general background, covariantizing (2.29) and using eq. (2.25), the simultaneous variation of J with respect to the background metric and gauge field is

$$\begin{aligned} \delta J &= \frac{1}{4\pi}(tr\ T_a)\int d^2\zeta d^2\xi\ S_a^{\alpha\beta\gamma}(\zeta-\xi)\sqrt{h}\delta h_{\alpha\beta}(\zeta)\sqrt{h}\delta A_\gamma^a(\xi) \\ &= \frac{1}{4\pi}(tr\ T_a)\int d^2\zeta d^2\xi\sqrt{h(\zeta)}\sqrt{h(\xi)} \\ &\times\ \Big(\frac{h^{\alpha\beta}}{\sqrt{h}}\delta^2(\zeta-\xi)-\frac{\nabla^\alpha\nabla^\beta}{\Box}\Big)\delta\nabla^\gamma A_\gamma^a(\xi)\delta h_{\alpha\beta}(\zeta). \end{aligned} \tag{2.30}$$

In the conformal gauge, $h_{\alpha\beta} = \exp(-2\sigma)\eta_{\alpha\beta}$ (2.5), one finds

$$J = -\frac{1}{2\pi}(trT_a)\int d^2\zeta\sqrt{h}h^{\alpha\beta}\ \nabla_\alpha A_\beta^a\ \sigma. \tag{2.31}$$

We remark that, potentially, there could be an anomalous contribution involving just the metric (the standard Polyakov anomaly, $\sqrt{h}R^{(2)}\frac{1}{\Box}R^{(2)}$). The explicit expression (2.21) shows that we have a bosonic β-γ system, with $c = 2$, and an anticommuting complex scalar (b-c), with $c = -2$, coupled to the background worldsheet metric. Therefore, the anomalies cancel and there is no multiplicative anomaly depending just on the metric. The central charge, c_t, of the system described by eq. (2.19) is equal, by construction, to the system described by eq. (2.1). For semisimple groups, one can show that the system in eq. (2.20) already has a central charge c_t, leaving for the rest of the system, e^J in (2.19), only a zero central charge. This result was obtained above by assuming that (b, c) was a scalar pair. With such a scalar pair, one can show that the numerator in e^J (2.19) can be regularized in such a manner that it leads to no mixed anomaly.

From eq. (2.31), we see that for a non-trivial worldsheet metric and gauge fields, an anomalous (non-local) piece appears in the effective action:

$$\begin{aligned} S_{nonlocal} &= \frac{1}{8\pi}(tr\ T_a)\int d^2z\Big(\frac{1}{\partial}A^a+\frac{1}{\bar\partial}\bar A^a\Big)\sqrt{h}R^{(2)} \\ &= \frac{1}{2\pi}\int d^2z\ ln(\det M(g)\det M(\bar g))\partial\bar\partial\sigma, \end{aligned} \tag{2.32}$$

(here we are back to complex worldsheet coordinates) where ${M_a}^b(g)$ is the adjoint representation matrix, *i.e.*

$$gT_ag^{-1} = {M_a}^b(g)T_b\ \Rightarrow\ {M_a}^b(g) = tr(gT_ag^{-1}\tilde T^b). \tag{2.33}$$

The proof of the second equality in (2.32) is given in Appendix A. We should note here that $\det M$ is *not* a constant if T_a is not traceless for some a:

$$tr\ T_a \neq 0\ \Leftrightarrow\ \det\ M \neq 1. \tag{2.34}$$

Explicitly, if we parametrize

$$g(\theta) = e^{\theta^a T_a}, \tag{2.35}$$

then

$$ln \ \det \ g = tr \ ln \ g = tr \ \theta^a T_a. \tag{2.36}$$

(Obviously, the matrix g is equivalent to $M(g)$ if the generators T_a are in the adjoint representation.)

To summarize, the theory is now defined by the functional integral

$$Z \ = \ \int DA \ D\bar{A} \ D\lambda \ Dx \ e^{-(S[A,\bar{A},\lambda,x]+\frac{1}{2\pi}\int d^2z \, ln(\det M(g) \det M(\bar{g}))\partial\bar{\partial}\sigma)}, \tag{2.37}$$

where $S[A, \bar{A}, \lambda, x]$ is given in (2.20).

Starting with the theory (2.37), we now consider two avenues which we call the "A-model" and the "B-model". One gets the A-model by integrating out the Lagrange multiplier. On the other hand, one gets the B-model by integrating out the gauge field[4].

3 The A-model

By construction, integrating out λ in (2.37) constrains the gauge field to be pure gauge ($g = \bar{g}$), and we get

$$\begin{aligned} Z_A \ &= \ \int DA \ D\bar{A} \ \delta(F) \ Dx \ e^{-(S[A,\bar{A},\lambda,x]+S_{nonlocal}[A,\bar{A},\sigma])} \\ &= \ \int D_L g \ D_L\bar{g} \ \delta_L(g,\bar{g}) \ Dx \ e^{-S[g,\bar{g},x]} \\ &= \ \int D_L g \ Dx \ e^{-S[g,x]}. \end{aligned} \tag{3.1}$$

Here $S[g, x]$ is given by (2.1) and $S_{nonlocal}[A, \bar{A}, \sigma]$ is given in (2.32). Therefore, we get that the A-model is equivalent to the original model

$$Z_A = Z_O = \int D_L g Dx e^{-S[g,x]} \tag{3.2}$$

(up to global issues [3, 12] that we do not address here).

[4]An alternative way [11] to get (2.37) from (2.6) is to gauge the G-symmetry of the action $S[g, x]$ with non-dynamical gauge fields (*i.e.*, without a F^2 term) by minimal coupling:

$$g^{-1}\partial g \to g^{-1}(\partial + A)g \ , \quad g^{-1}\bar{\partial} g \to g^{-1}(\bar{\partial} + \bar{A})g,$$

(here $A \equiv A^a T_a$ transforms as $A \to u(\partial + A)u^{-1}$, *etc.*), and add the Lagrange multiplier term, $tr\lambda F$, that constrains the gauge field to be (locally) pure gauge. Then, after choosing a unitary gauge, $g = 1$, and adding the non-local term to cancel the trace anomaly [3, 4], one recovers (2.37). We chose not to work with such a gauge theory, in order not to deal with the problem of a gauge-invariant measure for the gauge field in non-semisimple groups.

4 The B-model

To integrate out the gauge field $A, \bar{A}$ in (2.37) it is convenient to re-express the non-local part of the action in terms of $A, \bar{A}$ and the conformal factor σ. The non-local part takes the form

$$\begin{aligned} S_{nonlocal}[A,\bar{A},\sigma] &= \frac{1}{2\pi}\int d^2z\, ln(\det M(g)\det M(\bar{g}))\partial\bar{\partial}\sigma \\ &= \frac{1}{8\pi}(tr\ T_a)\int d^2z\Big(\frac{1}{\partial}A^a + \frac{1}{\bar{\partial}}\bar{A}^a\Big)\sqrt{h}R^{(2)} \\ &= -\frac{trT_a}{2\pi}\int d^2z(A^a\bar{\partial}\sigma + \bar{A}^a\partial\sigma). \end{aligned} \tag{4.1}$$

The action $S[A,\bar{A},\lambda,x] + S_{nonlocal}[A,\bar{A},\sigma]$ is bilinear in $A, \bar{A}$ (it is actually linear in A and in $\bar{A}$, separately). Therefore, integrating out $A, \bar{A}$ is simple and leads to

$$\begin{aligned} Z_B &= \int D\lambda\ Dx\ DA\ D\bar{A}\ e^{-(S[A,\bar{A},\lambda,x]+S_{nonlocal}[A,\bar{A},\sigma])} \\ &= \int D\lambda\ Dx\ \det N(x,\lambda)\ e^{-S_B[\lambda,x,\sigma]}, \end{aligned} \tag{4.2}$$

where the *dual* action S_B is

$$\begin{aligned} S_B[\lambda,x,\sigma] &= \frac{1}{2\pi}\int d^2z\,\Big(F_{ij}\partial x^i\bar{\partial}x^j - (\Phi - ln\det N)\partial\bar{\partial}\sigma \\ &+ (\partial\lambda_a - \partial x^i F^L_{ia} + trT_a\partial\sigma)N^{ab}(\bar{\partial}\lambda_b + F^R_{bj}\bar{\partial}x^j - trT_b\bar{\partial}\sigma)\Big), \end{aligned} \tag{4.3}$$

and

$$N^{ab}(x,\lambda) = [(E(x) + \lambda_c f^c)^{-1}]^{ab}. \tag{4.4}$$

Here the shift of the dilaton comes from a Jacobian factor that arises from integrating over the gauge field [10] (see Appendix B), and the matrices f^c have the structure constants as components $(f^c)_{ab} = f^c_{ab}$.

Equation (4.3) is our key result: generally it is non-local, since σ appears explicitly in a form which does not allow its replacement by the curvature, without the use of the inverse Laplacian. Since by construction (4.3) is the form equivalent to model A, after the quantum corrections are taken into account we can read off the non-local corrections to the energy-momentum tensor needed to make the two models equivalent. The variation with respect to σ gives the trace of the energy-momentum. Therefore, the explicit σ-dependent terms in (4.3) give the difference at one-loop level between the traces of the energy-momentum tensors for models A and B' (recall that B' is the sigma-model part of the B model, without the (non-local) dilaton-type correction). From the explicit expressions we see that this difference will be *always* a total derivative. This means, as discussed in the introduction, that if model A was conformally invariant, then model B' will have at least a local, conserved dilation current. Moreover, if model A is conformal invariant, then (4.3) is by construction σ-independent. Therefore, since the total variation with respect to σ is obtained by the β-function equations in addition to the explicit σ dependence, we can directly obtain the β-functions of model B' by taking (with negative sign) the σ-variations in (4.3).

We discuss now two typical examples.

5 Example 1

In this section we present the first example. We discuss a two-dimensional group G_{JP} with generators T_a, $a = 1, 2$, that we write in the adjoint representation as

$$T_2 = P = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad T_1 = J = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.1}$$

They obey the algebra

$$[J, P] = P \;\Rightarrow\; f^P_{JP} = -f^P_{PJ} = 1, \tag{5.2}$$

and all other commutators and structure constants are 0. The dual generators $\tilde{T}^a$ are given by

$$\tilde{T}^2 = P^t = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \tilde{T}^1 = J^t = J. \tag{5.3}$$

It is easy to check that (2.4) is satisfied.

We parametrize elements $u(\alpha, \beta) \in G_{JP}$ by

$$u(\alpha, \beta) = e^{\alpha P} e^{\beta J} = \begin{pmatrix} 1 & 0 \\ \alpha & e^\beta \end{pmatrix}. \tag{5.4}$$

It is easy to check that $u(\alpha, \beta)$ obey the group product

$$u(\alpha, \beta) u(\alpha', \beta') = u(\alpha + e^\beta \alpha', \beta + \beta'), \tag{5.5}$$

and the inverse is

$$u^{-1}(\alpha, \beta) = u(-\alpha e^{-\beta}, -\beta) = \begin{pmatrix} 1 & 0 \\ -\alpha e^{-\beta} & e^{-\beta} \end{pmatrix}. \tag{5.6}$$

To write G_{JP} invariant actions it is convenient to parametrize $g \in G_{JP}$ by

$$g(\phi, \chi) = \begin{pmatrix} 1 & 0 \\ \phi & \chi^{-1} \end{pmatrix}. \tag{5.7}$$

Then the isometry acts as

$$g(\phi, \chi) \to g'(\phi', \chi') = u(\alpha, \beta) g(\phi, \chi) = \begin{pmatrix} 1 & 0 \\ \phi' & (\chi')^{-1} \end{pmatrix}, \tag{5.8}$$

where

$$\phi' = e^\beta \phi + \alpha, \qquad \chi' = e^{-\beta} \chi. \tag{5.9}$$

The G_{JP} invariant elements in the action are constructed out of

$$\begin{aligned} (g^{-1}\partial_\mu g)^P &\equiv tr(P^t g^{-1} \partial_\mu g) = \chi \partial_\mu \phi, \\ (g^{-1}\partial_\mu g)^J &\equiv tr(J^t g^{-1} \partial_\mu g) = -\chi^{-1} \partial_\mu \chi. \end{aligned} \tag{5.10}$$

For example, we will choose the action

$$\begin{aligned} S[\phi, \chi] &= \frac{1}{2\pi} \int d^2 z \left((g^{-1}\partial g)^P (g^{-1}\bar\partial g)^J + (g^{-1}\partial g)^J (g^{-1}\bar\partial g)^P \right) \\ &= -\frac{1}{2\pi} \int d^2 z \, (\partial\phi \bar\partial\chi + \partial\chi \bar\partial\phi). \end{aligned} \tag{5.11}$$

This action is of the form (2.1) with $E_{PJ} = E_{JP} = 1$, and all other backgrounds are 0. The background is a flat Minkowski space in two dimensions and, therefore, the A-model is conformal.

To find the B-model we need the background matrix N (4.4):

$$N^{ab}(x,\lambda) = [(E(x) + \lambda_c f^c)^{-1}]^{ab} = \frac{1}{1-\lambda_P^2}\begin{pmatrix} 0 & 1+\lambda_P \\ 1-\lambda_P & 0 \end{pmatrix}, \tag{5.12}$$

Using

$$\det N = \frac{1}{\lambda_P^2 - 1}, \qquad trT_a = (tr\ J)\delta_a^J = \delta_a^J, \tag{5.13}$$

one finds

$$\begin{aligned} S_B[\lambda] &= \frac{1}{2\pi}\int d^2z\,\Big((\partial\lambda_a + \delta_a^J\partial\sigma)N^{ab}(\bar\partial\lambda_b - \delta_b^J\bar\partial\sigma) + ln\ \det N\ \partial\bar\partial\sigma\Big) \\ &= \frac{1}{2\pi}\int d^2z\,\Big(\frac{1}{1-\lambda_P^2}(\partial\lambda_P\bar\partial\lambda_J + \partial\lambda_J\bar\partial\lambda_P) + total\ derivative\Big), \end{aligned} \tag{5.14}$$

and

$$Z_B = \int \frac{D\lambda_P D\lambda_J}{\lambda_P^2 - 1}\ e^{-S_B[\lambda]}. \tag{5.15}$$

The background in S_B is a 2-d flat Minkowski space up to a conformal factor:

$$ds^2 = e^{\rho(\lambda_P)}(\partial\lambda_P\bar\partial\lambda_J + \partial\lambda_J\bar\partial\lambda_P), \qquad \rho(\lambda_P) = -ln(1-\lambda_P^2), \tag{5.16}$$

and the curvature is

$$R = -4e^{-\rho(\lambda_P)}\frac{\partial}{\partial_{\lambda_J}}\frac{\partial}{\partial_{\lambda_P}}\rho(\lambda_J) = 0. \tag{5.17}$$

Therefore, the B-model is flat. Changing coordinates to

$$d\phi = \frac{1}{\lambda_P^2 - 1}d\lambda_P, \qquad d\chi = d\lambda_J, \tag{5.18}$$

we get

$$Z_B = \int D\phi\ D\chi\ e^{-S[\phi,\chi]}, \qquad S[\phi,\chi] = -\frac{1}{2\pi}\int d^2z\,(\partial\phi\bar\partial\chi + \partial\chi\bar\partial\phi). \tag{5.19}$$

Therefore, the B-model (5.15) is equivalent to the A-model (5.11).

The fact that the B-model could be written in a local form, although the anomaly was present, derives from the two-dimensional target-space nature. In two dimensions any ξ appearing in eq. (1.1) is of the type (1.2). Therefore, the energy-momentum tensor is improvable by a dilaton term. Indeed, in the example above, the term induced by the anomaly is of a dilaton type (up to a total derivative), and moreover, it exactly cancels the usual dilaton term arising by duality.

6 Example 2

In this section we present the four-dimensional Bianchi V cosmological example considered in ref. [2]. We discuss a three-dimensional group G_V with generators T_a, $a = 1, 2, 3$, that we write in the adjoint representation as

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad T_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{6.1}$$

They obey the algebra

$$\begin{gathered} [T_1, T_2] = T_2, \qquad [T_1, T_3] = T_3, \qquad [T_2, T_3] = 0 \;\Rightarrow \\ f_{12}^2 = f_{13}^3 = 1 = -f_{21}^2 = -f_{31}^3, \qquad f_{ab}^c = 0 \quad otherwise. \end{gathered} \tag{6.2}$$

The dual generators $\tilde{T}^a$ are given by

$$\tilde{T}^1 = \frac{1}{2} T_1, \qquad \tilde{T}^2 = (T_2)^t, \qquad \tilde{T}^3 = (T_3)^t. \tag{6.3}$$

It is easy to check that (2.4) is satisfied.

We parametrize elements $u(\alpha, \vec{\beta}) \in G_V$, $\vec{\beta} = (\beta_1, \beta_2)$ by

$$u(\alpha, \vec{\beta}) = e^{\beta_1 T_2 + \beta_2 T_3} e^{\alpha T_1} = \begin{pmatrix} 1 & 0 & 0 \\ \beta_1 & e^\alpha & 0 \\ \beta_2 & 0 & e^\alpha \end{pmatrix}. \tag{6.4}$$

It is easy to check that $u(\alpha, \vec{\beta})$ obey the group product

$$u(\alpha, \vec{\beta}) u(\alpha', \vec{\beta}') = u(\alpha + \alpha', \vec{\beta} + e^\alpha \vec{\beta}'), \tag{6.5}$$

and the inverse is

$$u^{-1}(\alpha, \vec{\beta}) = u(-\alpha, -\vec{\beta} e^{-\alpha}) = \begin{pmatrix} 1 & 0 & 0 \\ -\beta_1 e^{-\alpha} & e^{-\alpha} & 0 \\ -\beta_2 e^{-\alpha} & 0 & e^{-\alpha} \end{pmatrix}. \tag{6.6}$$

Let $g(\phi, \vec{\chi}) \in G_V$, $\vec{\chi} = (\chi_1, \chi_2)$ be

$$g(\phi, \vec{\chi}) = \begin{pmatrix} 1 & 0 & 0 \\ \chi_1 & e^\phi & 0 \\ \chi_2 & 0 & e^\phi \end{pmatrix}. \tag{6.7}$$

Then the isometry u acts on g by a left multiplication

$$g(\phi, \vec{\chi}) \to g'(\phi', \vec{\chi}') = u(\alpha, \vec{\beta}) g(\phi, \vec{\chi}) = g(\phi + \alpha, e^\alpha \vec{\chi} + \vec{\beta}). \tag{6.8}$$

The G_V invariant elements in the action are constructed out of

$$\begin{aligned} (g^{-1}\partial_\mu g)^1 &\equiv tr(\tilde{T}^1 g^{-1} \partial_\mu g) = \partial_\mu \phi, \\ (g^{-1}\partial_\mu g)^2 &\equiv tr(T^{2t} g^{-1} \partial_\mu g) = e^{-\phi} \partial_\mu \chi_1, \\ (g^{-1}\partial_\mu g)^3 &\equiv tr(T^{3t} g^{-1} \partial_\mu g) = e^{-\phi} \partial_\mu \chi_2. \end{aligned} \tag{6.9}$$

To write the four-dimensional Bianchi V model we choose to insert in eq. (2.1):

$$x^i \equiv \{t\}, \qquad E_{ab}(t) = a(t)^2\delta_{ab}, \qquad F(t) = -1, \qquad F^L = F^R = \Phi = 0. \tag{6.10}$$

The action takes the form

$$\begin{aligned} S[\phi, \vec{\chi}, t] &= \frac{1}{2\pi}\int d^2z \left(a(t)^2\delta_{ab}(g^{-1}\partial g)^a(g^{-1}\bar{\partial}g)^b - \partial t\bar{\partial}t\right) \\ &= \frac{1}{2\pi}\int d^2z \left(- \partial t\bar{\partial}t + a(t)^2[\partial\phi\bar{\partial}\phi + e^{-2\phi}(\partial\chi_1\bar{\partial}\chi_1 + \partial\chi_2\bar{\partial}\chi_2)]\right). \end{aligned} \tag{6.11}$$

Conformal invariance requires

$$a(t) = t. \tag{6.12}$$

In this case the metric is flat (it is the interior of the light-cone in Minkowski space). In the following we consider the conformal model (6.11),(6.12).

To find the B-model we need the background matrix N (4.4):

$$N^{ab}(t,\lambda) = [(E(t) + \lambda_c f^c)^{-1}]^{ab} = \frac{1}{t^2(t^4 + \lambda_2^2 + \lambda_3^2)}\begin{pmatrix} t^4 & -\lambda_2 t^2 & -\lambda_3 t^2 \\ \lambda_2 t^2 & t^4 + \lambda_3^2 & -\lambda_2\lambda_3 \\ \lambda_3 t^2 & -\lambda_2\lambda_3 & t^4 + \lambda_2^2 \end{pmatrix}, \tag{6.13}$$

Using

$$\det N = \frac{1}{t^2(t^4 + \lambda_2^2 + \lambda_3^2)}, \qquad trT_a = (tr\ T_1)\delta_a^1 = 2\delta_a^1, \tag{6.14}$$

one finds

$$\begin{aligned} S_B[\lambda, t] &= \frac{1}{2\pi}\int d^2z \Big((\partial\lambda_a + 2\delta_a^1\partial\sigma)N^{ab}(\bar{\partial}\lambda_b - 2\delta_b^1\bar{\partial}\sigma) \\ &- \partial t\bar{\partial}t + ln\ \det N\ \partial\bar{\partial}\sigma\Big) \\ &= S_0[\lambda, t] + S_1[\lambda, t] + S_2[\lambda, t], \end{aligned} \tag{6.15}$$

$$S_0[\lambda, t] = \frac{1}{2\pi}\int d^2z \left(\partial\lambda_a N^{ab}(t,\lambda)\bar{\partial}\lambda_b + ln\ \det N\partial\bar{\partial}\sigma\right) \tag{6.16}$$

$$S_1[\lambda, t] = \frac{2}{2\pi}\int d^2z\ \sigma\Big(\bar{\partial}(N^{a1}(t,\lambda)\partial\lambda_a) - \partial(N^{1b}(t,\lambda)\bar{\partial}\lambda_b)\Big). \tag{6.17}$$

$$S_2[\lambda, t] = -\frac{2^2}{2\pi}\int d^2z\ N^{11}(t,\lambda)\partial\sigma\bar{\partial}\sigma. \tag{6.18}$$

The A-model is conformal and, therefore, one expects the B-model to be conformal, *i.e.*,

$$\pi\frac{\delta S_B}{\delta\sigma} = T_{z\bar{z}} = 0 = \pi\frac{\delta(S_0 + S_1 + S_2)}{\delta\sigma} = T^0_{z\bar{z}} + T^1_{z\bar{z}} + T^2_{z\bar{z}}. \tag{6.19}$$

Variation of S_0 with respect to the conformal factor σ gives

$$\pi\frac{\delta S_0}{\delta\sigma} = T^0_{z\bar{z}} = \frac{1}{2}[\beta_{G_{IJ}}(X) + \beta_{B_{IJ}}(X)]\partial X^I\bar{\partial}X^J + \frac{1}{2}\beta_\Phi(X)\partial\bar{\partial}\sigma, \tag{6.20}$$

where

$$X^I \equiv \{\lambda^a, t\}, \qquad \Phi = -ln\ \det N,$$
$$G_{ab} = \frac{1}{2}(N+N^t)^{ab}, \quad G_{tt} = -1, \qquad otherwise\ \ G_{IJ} = 0,$$
$$B_{ab} = \frac{1}{2}(N-N^t)^{ab}, \qquad otherwise\ \ B_{IJ} = 0, \tag{6.21}$$

and to one-loop order the beta-functions are [14]

$$\begin{aligned}
\beta_{G_{IJ}} &= R_{IJ} - \frac{1}{4}H^2_{IJ} - \nabla_I\nabla_J\Phi, \\
\beta_{B_{IJ}} &= -\frac{1}{2}\nabla^K H_{IJK} - \frac{1}{2}\nabla^K\Phi H_{IJK}, \\
\beta_\Phi &= R - \frac{1}{12}H^2 - 2\nabla^2\Phi - (\nabla\Phi)^2 - \frac{2(c-4)}{3},
\end{aligned} \tag{6.22}$$

where

$$H_{IJK} = \nabla_{[I}B_{JK]}, \qquad H^2_{IJ} = H_{IKL}H_J^{KL}. \tag{6.23}$$

Variation of S_1 and S_2 with respect to σ gives

$$\begin{aligned}
\pi\frac{\delta S_1}{\delta\sigma} &= T^1_{z\bar z} = \bar\partial(N^{a1}(t,\lambda)\partial\lambda_a) - \partial(N^{1b}(t,\lambda)\bar\partial\lambda_b) + \mathcal{O}(\sigma), \\
\pi\frac{\delta S_2}{\delta\sigma} &= T^2_{z\bar z} = 0 + \mathcal{O}(\sigma).
\end{aligned} \tag{6.24}$$

Here the leading order comes from naive variation with respect to σ; higher order corrections in σ require more careful regularization of (6.17),(6.18), as done in deriving $T^0_{z\bar z}$ (6.20). We should note that the leading order contribution to $T^1_{z\bar z} + T^2_{z\bar z}$ is a total derivative. As will be shown below, using the equation of motion

$$\partial(N^{ab}\bar\partial\lambda_b) + \bar\partial(N^{ba}\partial\lambda_b) - \frac{\delta N^{bc}}{\delta\lambda_a}\partial\lambda_b\bar\partial\lambda_c = 0 + \mathcal{O}(\sigma), \tag{6.25}$$

one finds that

$$T^0_{z\bar z} + T^1_{z\bar z} + T^2_{z\bar z} = 0 + \mathcal{O}(\sigma). \tag{6.26}$$

In the following we present the detailed calculations. The σ-model described by the action S_0 (6.16) has non-zero β-functions [2]. Explicitly, we have the metric

$$ds^2 = -dt^2 + \frac{t^2}{4x(t^4+x)}dx^2 + \frac{x}{t^2}dy^2 + \frac{t^2}{t^4+x}dz^2, \tag{6.27}$$

the antisymmetric tensor

$$B = \frac{1}{2(t^4+x)}dx\wedge dz, \tag{6.28}$$

and the dilaton

$$\Phi = ln\ t^2 + ln(t^4+x). \tag{6.29}$$

These are expressed in terms of the variables $x^\mu = \{t, x, y, z\}$ defined by:

$$\lambda_1 = z, \qquad \lambda_2 = \sqrt{x}\cos y, \qquad \lambda_3 = \sqrt{x}\sin y. \tag{6.30}$$

In our case, we have for the Ricci tensor the following non-vanishing components:

$$\begin{aligned}
R_{tt} &= \frac{-6t^8 + 16t^4x - 2x^2}{t^2(t^4+x)^2}, \qquad R_{tx} = \frac{-t^4 + x}{t(t^4+x)^2}, \\
R_{xx} &= \frac{2t^8 - 7t^4x - x^2}{2x(t^4+x)^3}, \qquad R_{yy} = \frac{4x}{t^4+x}, \\
R_{zz} &= \frac{6t^8 - 10t^4x}{(t^4+x)^3}.
\end{aligned} \tag{6.31}$$

For $H^2_{\mu\nu}$ we get the non-vanishing components

$$(H^2)_{tt} = \frac{32t^2x}{(t^4+x)^2}, \qquad (H^2)_{xx} = \frac{-8t^4}{(t^4+x)^3}, \qquad (H^2)_{zz} = \frac{-32t^4x}{(t^4+x)^3}. \tag{6.32}$$

Including the $\nabla_\mu\nabla_\nu\Phi$ contributions,

$$\begin{aligned}
\nabla_t\nabla_t\Phi &= \frac{2(-3t^8 + 4t^4x - x^2)}{t^2(t^4+x)^2}, \qquad \nabla_t\nabla_x\Phi = \frac{-(3t^4+x)}{t(t^4+x)^2}, \\
\nabla_x\nabla_x\Phi &= \frac{4t^8 - t^4x - x^2}{2x(t^4+x)^3}, \\
\nabla_y\nabla_y\Phi &= \frac{4x(2t^4+x)}{t^4(t^4+x)}, \qquad \nabla_z\nabla_z\Phi = \frac{2(3t^8 - 3t^4x - 2x^2)}{(t^4+x)^3},
\end{aligned} \tag{6.33}$$

we have for $\beta_{G_{\mu\nu}}$ in (6.22):

$$\begin{aligned}
\beta_{G_{tx}} &= \frac{2}{t(t^4+x)}, \qquad \beta_{G_{xx}} = \frac{-t^4}{x(t^4+x)^2}, \qquad \beta_{G_{yy}} = \frac{-4x}{t^4}, \\
\beta_{G_{zz}} &= \frac{4x}{(t^4+x)^2},
\end{aligned} \tag{6.34}$$

and for $\beta_{B_{\mu\nu}}$ in (6.22):

$$\begin{aligned}
\beta_{B_{\mu\nu}} &= -\frac{1}{2}\nabla^\rho H_{\mu\nu\rho} - \frac{1}{2}\nabla^\rho\Phi H_{\mu\nu\rho} \\
\beta_{B_{tz}} &= \frac{-4t^5}{(t^4+x)^2} - \frac{4tx}{(t^4+x)^2} = \frac{-4t}{t^4+x}, \\
\beta_{B_{xz}} &= \frac{4t^6}{(t^4+x)^3} - \frac{2t^2(3t^4+x)}{(t^4+x)^3} = \frac{-2t^2}{(t^4+x)^2}.
\end{aligned} \tag{6.35}$$

All other components of β_G and β_B are zero. The dilaton β-function in (6.22) turns out to be a constant:

$$\begin{aligned}
\beta_\Phi &= R - \frac{1}{12}H^2 - [2\nabla^2\Phi - (\nabla\Phi)^2] - \frac{2(c-4)}{3} \\
&= \frac{4t^4(5t^4 - 9x)}{t^2(t^4+x)^2} - \frac{8t^2x}{(t^4+x)^2} - \frac{4t^2(5t^4-7x)}{(t^4+x)^2} - \frac{2(c-4)}{3} \\
&= \frac{2(4-c)}{3},
\end{aligned} \tag{6.36}$$

and therefore, β_Φ vanishes if the central charge is $c=4$, as for the A-model [5].

Although we have found non-vanishing contributions to the β-functions, the additional non-local term (6.17), resulting from the anomaly (2.32), has an extra explicit σ dependence which should be taken into account. The variation of this term with respect to σ exactly cancels the above β-functions, recovering the conformal symmetry of the original model. Indeed, from (6.17) one finds

$$\begin{aligned}
\delta_\sigma S_1 &= \frac{1}{\pi}\int d^2z\ \delta\sigma\Big[\bar\partial\Big(\frac{1}{t^4+x}(t^2\partial z+\frac{1}{2}\partial x)\Big)-\partial\Big(\frac{1}{t^4+x}(t^2\bar\partial z-\frac{1}{2}\bar\partial x)\Big)\Big]\\
&= \frac{1}{\pi}\int d^2z\ \delta\sigma\Big[\frac{2t(-t^4+x)}{(t^4+x)^2}(\bar\partial t\partial z-\partial t\bar\partial z)-\frac{t^2}{(t^4+x)^2}(\bar\partial x\partial z-\partial x\bar\partial z)\\
&- \frac{1}{(t^4+x)^2}[2t^3(\bar\partial t\partial x+\partial t\bar\partial x)+\bar\partial x\partial x]+\frac{1}{t^4+x}\partial\bar\partial x\Big].
\end{aligned} \tag{6.37}$$

Substituting the equation of motion,

$$\begin{aligned}
\partial\bar\partial x &= \frac{t^4+2x}{2x(t^4+x)}\partial x\bar\partial x\ +\ \frac{t^4-x}{t(t^4+x)}(\partial t\bar\partial x+\bar\partial t\partial x)\\
&+ \frac{2x(t^4+x)}{t^4}\partial y\bar\partial y\ -\ \frac{2x}{t^4+x}\partial z\bar\partial z\ +\ \frac{4tx}{t^4+x}(\partial t\bar\partial z-\bar\partial t\partial z),
\end{aligned} \tag{6.38}$$

into eq. (6.37), one finds

$$\delta_\sigma S_1=-\frac{1}{2\pi}\int d^2z\ [(\beta_{G_{\mu\nu}}+\beta_{B_{\mu\nu}})\partial x^\mu\bar\partial x^\nu]\delta\sigma, \tag{6.39}$$

with the same β_G and β_B of eqs. (6.34), (6.35). Therefore, the total action S_0+S_1 is σ independent, as it should be.

Finally, let us emphasize that we have checked cancellation only to leading order in σ, namely, for $\sigma=0$. However, we should note that since the $\mathcal{O}(\sigma)$ corrections in (6.26) must vanish, this can be used to find the $\mathcal{O}(\sigma)$ contribution to the variation of the non-local action with respect to the conformal factor: $\delta(S_1+S_2)/\delta\sigma$.

To appreciate the possible effect of the mixed anomaly term on the actual value of the Virasoro central charge, consider the extremely simple system of a single scalar with a linear dilaton term:

$$L_A=\partial x\bar\partial x+Q\sqrt{h}R^{(2)}x. \tag{6.40}$$

This is a conformal system with central charge $c=1+3Q^2$. Applying the above procedure we get in the new variables A and λ:

$$L=A\bar A+\lambda F+Q(\partial\bar A+\bar\partial A)\frac{1}{\Box}\sqrt{h}R^{(2)}, \tag{6.41}$$

[5] Note that here β_G and β_B do not vanish and, therefore, the fact that β_Φ is a constant (actually zero) is not obvious. Even if the dilaton beta-function would not vanish, there could be $\mathcal{O}(\sigma)$ contributions in (6.24) that would cancel $\beta_\Phi\partial\bar\partial\sigma$ in $T_{z\bar z}$. In our case we conclude that such $\mathcal{O}(\sigma)$ corrections must vanish.

where now the non-local term does not result from a Jacobian anomaly but rather from the explicit Q term in the original action. Integrating out A will now give:

$$L_B = \partial\lambda\bar{\partial}\lambda + \frac{1}{2}Q^2\sqrt{h}R^{(2)}\frac{1}{\Box}\sqrt{h}R^{(2)}. \tag{6.42}$$

The scalar λ is now a normal massless scalar field with $c = 1$, and the difference between the original central charge and the final one is compensated by the explicit non-local Q^2 term in L_B. Note that this is an order σ^2 term (exact, in this example) as expected.

7 The relation between local energy-momentum tensors in models A and B

In this work we considered the two-dimensional quantum field theory (2.19). For the purpose of the discussion in this section, let us neglect the determinants term $\exp J(A, \bar{A})$ in (2.19), and consider the theory

$$\int D\lambda\ Dx\ DA\ D\bar{A}\ e^{-S[A,\bar{A},\lambda,x]}, \tag{7.1}$$

where $S[A, \bar{A}, \lambda, x]$ is given in (2.20). In a flat worldsheet background the two transformations of (7.1) leading to models A and B are valid for any isometry group G. The field theories described by A and B are equivalent, *i.e.*, there exists an exact mapping between the Hilbert spaces of the two theories which preserves the spectrum. The correspondence between the various local operators in the two theories is more involved. In particular, the relation between local energy-momentum tensors A and B and, therefore, the coupling to a non-flat background in the two models, is the subject of this section.

We will use two methods which should be equivalent:

a) Duality transformation in a curved worldsheet background.

b) Ward identities involving the two energy-momentum tensors calculated with the action in a flat background.

Consider (7.1) in a curved background. In the conformal gauge, classically the conformal factor does not appear. Through the measure, however, a dependence on the conformal factor may appear. This was discussed in section 2. It was shown that in a curved background the theory (7.1) does not lead to model A. If we want to get exactly model A we should add to (7.1) the anomalous piece (2.32).[6]

Equivalently, by taking functional derivatives with respect to the worldsheet metric and gauge field, we conclude that the energy-momentum tensor of model A should have an anomalous correlator with the current coupled to the gauge field when calculated with the flat background action.

As was remarked in section 2, for the "pure" Weyl anomaly, *i.e.*, the contributions to the term $\sqrt{h}R^{(2)}\frac{1}{\Box}R^{(2)}$ in the effective action, there is no multiplicative anomaly. As a consequence

[6]In section 2, this term was included in the theory from the beginning, since we were careful to keep the term $\exp J$ in the measure of Z (2.19).

we expect that once the current anomaly $\nabla A\frac{1}{\Box}R^{(2)}$ is cancelled by an appropriate counter term, the A- and B-models, if conformal, will have the same central charge.

We present now an independent calculation, confirming the conclusions above.

We consider for action (7.1) the topologically conserved current [7]

$$J_{a\alpha} = \epsilon_{\alpha\beta}\partial^\beta \lambda_a. \tag{7.2}$$

In the following, we choose "a" to be a quasi-Abelian direction. Using the equations of motion, $J_{a\alpha}$ is given by

$$J_{a\alpha} = \frac{\delta S}{\delta A^{a\alpha}} + \epsilon_{\alpha\beta} f^b_{ac} A^{c\beta}\lambda_b. \tag{7.3}$$

We want to express the topological current of model B, $J_{a\alpha}$, in terms of the variables of model A. The variables λ_b can be calculated by adding to S a source term, $\lambda_b K^b$, and taking the derivative of S with respect to the source:

$$\lambda_a(\zeta) = \frac{\delta S}{\delta K^a(\zeta)} = \int d\zeta' \frac{\delta S}{\delta A^b_\alpha(\zeta')}\frac{\delta A^b_\alpha(\zeta')}{\delta K^a(\zeta)}. \tag{7.4}$$

To first order in K, the gauge field is given by:

$$A^a_\alpha(\zeta) = (g^{-1}\partial_\alpha g)^a + \Big[h^{-1}(\zeta)\frac{1}{\Box_{\zeta\zeta'}}\epsilon_{\alpha\beta}\partial^\beta (gKg^{-1})_{\zeta'} g(\zeta)\Big]^a, \tag{7.5}$$

where $g^{-1}\partial_\alpha g = A_\alpha$. Using (7.5) the final result for the current is:

$$\begin{aligned} J_{a\alpha}(\zeta) &= \frac{\delta S}{\delta A^{a\alpha}(\zeta)} + \epsilon_{\alpha\beta} f^b_{ac} A^{c\beta}(\zeta) \times \\ &\times M_b{}^d(g^{-1}(\zeta))\frac{1}{\Box_{\zeta\zeta'}}\epsilon_{\gamma\delta}\partial^{\delta'}\Big[\frac{\delta S}{\delta A^{\gamma e}(\zeta')} M_d{}^e(g(\zeta'))\Big]. \end{aligned} \tag{7.6}$$

Here $M(g)$ is the adjoint representation (2.33). If S has a quadratic dependence on $A^{a\alpha}$, the second term in (7.6) contains

$$\epsilon^{\gamma\delta}\partial_\delta (gg^{-1}\partial_\gamma g g^{-1}) = \epsilon^{\gamma\delta}[g^{-1}\partial_\gamma g, g^{-1}\partial_\delta g], \tag{7.7}$$

which does not have a component in a quasi-Abelian direction. It follows that up to terms cubic in the fields,

$$J_{a\alpha}(\zeta) = \frac{\delta S}{\delta A^{a\alpha}(\zeta)}. \tag{7.8}$$

In particular, for the model based on the group G_V in section 6, $J_{a\alpha}$ has the form:

$$J_\alpha = E_{11}(t)\partial_\alpha \phi. \tag{7.9}$$

Considering at one-loop level the correlator with the energy-momentum tensor we obtain a non-zero, anomalous contribution from the term

$$T_{\alpha\beta} = E_{11}(\partial_\alpha\chi_1\partial_\beta\chi_1 + \partial_\alpha\chi_2\partial_\beta\chi_2) \tag{7.10}$$

[7] The conservation equation of this current is the "dual Bianchi identity" presented in ref. [3].

in the presence of the interaction term

$$L_{int} = -2E_{11}\phi(\partial_\alpha\chi_1\partial^\alpha\chi_1 + \partial_\alpha\chi_2\partial^\alpha\chi_2). \tag{7.11}$$

On the other hand, if we calculate directly the correlator in model B, since the current (7.2) is topological in terms of the new local field λ_a, it cannot be anomalous: the derivative and ϵ-tensors can be taken outside the correlator for which dimensional regularization can be used. We conclude that the energy-momentum tensors defined by (7.1) for models A and B differ by their correlators to the topological current (7.2). This current is the source of the gauge field $g^{-1}\partial_\alpha g$ and, therefore, we reproduce this way the anomaly contribution in eq. (2.32).

8 On localizing the mixed anomaly term

Let us write down the formal equality

$$\begin{aligned} &\int Ds\ Dt\ Dx\ e^{-\frac{1}{2\pi}\int d^2z\left(S_0[x]+2\partial s\bar\partial t+Q(x)\frac{1}{\Box}\sqrt{h}R^{(2)}\right)} \\ =\ &\int Du\ Dv\ Dx\ e^{-\frac{1}{2\pi}\int d^2z\left(S_0[x]+2\partial u\bar\partial v+4vQ(x)+\frac{1}{4}u\sqrt{h}R^{(2)}\right)}, \end{aligned} \tag{8.1}$$

where $\Box = 2\partial\bar\partial$. (The models are not necessarily the same.) One can show this equality by integrating out s, t on the left-hand side, and comparing with the integration over u, v on the right-hand side of eq. (8.1). Some choices of $Q(x)$ are interesting. For example, if $S_0[x]$ is a sigma model with beta-functions β_G, β_B, then with the choice

$$Q(x) = -\frac{1}{2}(\beta_{G_{\mu\nu}} + \beta_{B_{\mu\nu}})\partial x^\mu\bar\partial x^\nu, \tag{8.2}$$

the model $S_0[x] + 2\partial s\bar\partial t + Q(x)\frac{1}{\Box}\sqrt{h}R^{(2)}$ is manifestly independent of σ, to leading order, which implies that the model $S_0[x] + 2\partial u\bar\partial v + 4vQ(x) + \frac{1}{4}u\sqrt{h}R^{(2)}$ is conformally invariant to leading order. (For a discussion about higher order corrections, including a dilaton beta-function in (8.2), see section 6.) Similar models were discussed in ref. [15].

Moreover, if $S_0 = S[A, \bar A, \lambda, x]$ (2.20), and if we choose

$$Q(A, \bar A) = \frac{1}{4}(trT_a)(\partial\bar A^a + \bar\partial A^a), \tag{8.3}$$

then, using eq. (8.1), one finds that

$$\begin{aligned} &\int Dx\ D\lambda\ DA\ D\bar A\ Ds\ Dt\ e^{-\frac{1}{2\pi}\int d^2z\, S_a[x,A,\lambda,s,t]} \\ =\ &\int Dx\ D\lambda\ DA\ D\bar A\ Du\ Dv\ e^{-\frac{1}{2\pi}\int d^2z\, S_b[x,A,\lambda,u,v]}, \end{aligned} \tag{8.4}$$

where

$$\begin{aligned} S_a &= S[A,\bar A,\lambda,x] + S_{nonlocal}[A,\bar A,\sigma] + \frac{1}{2\pi}\int d^2z\ 2\partial s\bar\partial t, \\ S_b &= S[A,\bar A,\lambda,x] \\ &+ \frac{1}{2\pi}\int d^2z\left(2\partial u\bar\partial v + v(trT_a)(\partial\bar A^a + \bar\partial A^a) + \frac{1}{4}u\sqrt{h}R^{(2)}\right). \end{aligned} \tag{8.5}$$

Here $S[A,\bar{A},\lambda,x]$ and $S_{nonlocal}[A,\bar{A},\sigma]$ are given in eqs. (2.20) and (4.1), respectively. Now, integrating out λ on the left-hand side of eq. (8.4), we recover the A-model in section 3, with an additional decoupled null kinetic term $\partial s\bar{\partial}t$. On the other hand, integrating out $A,\bar{A}$ on the right-hand side of eq. (8.4), we get a *local* dual theory with action

$$
\begin{aligned}
S_b[\lambda,x,u,v] &= \frac{1}{2\pi}\int d^2z\,\Big(2\partial u\bar{\partial}v-(\Phi-u-ln\det N)\partial\bar{\partial}\sigma \\
&+ (\partial\lambda_a-\partial x^i F^L_{ia}+trT_a\partial v)N^{ab}(\bar{\partial}\lambda_b+F^R_{bj}\bar{\partial}x^j-trT_b\bar{\partial}v) \\
&+ F_{ij}\partial x^i\bar{\partial}x^j\Big)\,.
\end{aligned}
\tag{8.6}
$$

Comparing to $S_B[\lambda,x,\sigma]$ in eq. (4.3), we see that there is an additional null kinetic term, $\partial u\bar{\partial}v$, the term leading to non-locality, $trT_a\partial\sigma$, has been replaced by $trT_a\partial v$, and the dilaton Φ is shifted to $\Phi-u$. Therefore, the additional null coordinates u,v provide a localization of the B-model in a higher dimensional sigma model.

Finally, we should mention that another way to localize the mixed anomaly term, within a larger sigma model framework, is to bosonize the ghosts in eq. (2.21). By integrating out the gauge field in (2.19), one will obtain a local sigma model action.

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9 Appendix A

In this Appendix we prove the second equality in (2.32), namely

$$ln(\det M(g)\det M(\bar{g}))=trT_a\Big(\frac{1}{\partial}A^a+\frac{1}{\bar{\partial}}\bar{A}^a\Big). \tag{9.1}$$

If all the generators are traceless, there is nothing to prove. Suppose not all of them are traceless. We choose a basis for the generators T_a such that

$$tr\;T_1\neq 0\;,\qquad tr\;T_a=0\;,\qquad a=2,...,D=\dim G\;. \tag{9.2}$$

This is always possible by redefining $T_a\to T_a-(tr\;T_a/tr\;T_1)T_1$. Therefore, we want to prove that

$$ln\;\det\;M(g)=(tr\;T_1)\frac{1}{\partial}A^1. \tag{9.3}$$

Recall that $A = g^{-1}\partial g$, and we choose to parametrize g by

$$g(\alpha_a) = g_2 g_1 \ , \qquad g_1 = e^{\alpha^1 T_1} \ , \quad g_2 = e^{\sum_{a=2}^{D} \alpha^a T_a} \ . \tag{9.4}$$

Recall also that $M_a{}^b(g) T_b = g T_a g^{-1} \ \Rightarrow \ M_a{}^b(g) tr\ T_b = tr(g T_a g^{-1}) = tr\ T_a$, which implies

$$M_1{}^1(g) = 1 \ , \qquad M_a{}^1(g) = 0 \quad for \quad a \neq 1. \tag{9.5}$$

Let us now find A^1 in terms of α:

$$\begin{aligned} A^1 &= tr(A\tilde{T}^1) \quad = \quad tr(g^{-1}\partial g \tilde{T}^1) \\ &= tr({g_1}^{-1}\partial g_1 \tilde{T}^1) + tr({g_1}^{-1}{g_2}^{-1}\partial g_2 g_1 \tilde{T}^1) \\ &= \partial\alpha^1 \ tr(T_1\tilde{T}^1) + ({g_2}^{-1}\partial g_2)^a tr({g_1}^{-1} T_a g_1 \tilde{T}^1) \\ &= \partial\alpha^1 + ({g_2}^{-1}\partial g_2)^a M_a{}^1({g_1}^{-1}) = \partial\alpha^1. \end{aligned} \tag{9.6}$$

Here we have used eqs. (2.4), (2.33), (9.4), and in the last equality we have used eq. (9.5). Moreover,

$$ln\ \det\ M(g) = ln\ \det\ M(g_1) + ln\ \det\ M(g_2) = tr\ ln\ M(g_1) = tr\ T_1 \alpha^1 \ . \tag{9.7}$$

Here we have used eq. (9.4), and the fact that $tr\ T_a = 0$ for $a \neq 1$ implies $\det\ M(g_2) = 1$. Inserting $\alpha_1 = (1/\partial) A^1$ (9.6) in (9.7) proves eq. (9.3).

10 Appendix B

In this Appendix we study the conformal factor-dependent terms which may appear when Gaussian integration over the A-variables is performed in (4.2). The origin of these terms is the implicit dependence on the conformal factor in the measure of A [10]. This can be made explicit by expressing the vector field $A^a, \bar{A}^{\bar{b}}$ in terms of scalar variables $y^a, \bar{y}^{\bar{b}}$:

$$A^a \equiv \partial y^a, \qquad \bar{A}^{\bar{b}} \equiv \bar{\partial}\bar{y}^{\bar{b}}. \tag{10.1}$$

Since the x and λ variables do not participate in this effect, and the linear terms in A do not contribute (we can complete the square in $A, \bar{A}$), in order to find the σ-dependence of the A measure (to one loop), we can replace $S[A, \bar{A}, \lambda, x] + S_{nonlocal}[A, \bar{A}]$ in (4.2),(2.20),(2.32) by the action

$$S[y, \bar{y}, x] = \frac{1}{2\pi} \int d^2z \, [E_{ab}(x)\partial y^a \bar{\partial}\bar{y}^{\bar{b}} + \partial x \bar{\partial} x], \tag{10.2}$$

where x is a single field with flat metric. The variation with respect to the conformal factor in the A-measure is obtained by calculating the trace of the energy-momentum tensor of (10.2). This is given by the general beta-function equations [14],

$$T_{z\bar{z}} = \frac{1}{2}(R_{IJ} - \frac{1}{4}H^2_{IJ} - \frac{1}{2}\nabla^K H_{IJK})\partial X^I \bar{\partial} X^J, \tag{10.3}$$

where X^I in our case include $x, y^a, \bar{y}^{\bar{b}}$, and R, H are the Ricci curvature and antisymmetric field strength calculated from the metric and torsion. From eq. (10.2) we identify the metric G:

$$G_{xx} = 1, \qquad G_{a\bar{b}} = G_{\bar{b}a} = \frac{E_{ab}}{2}, \tag{10.4}$$

and torsion B:

$$B_{a\bar{b}} = -B_{\bar{b}a} = \frac{E_{ab}}{2}, \tag{10.5}$$

all other components being zero. Using the special (quasi-Kählerian) form of G and B and their independence of $y^a, \bar{y}^{\bar{b}}$, eq. (10.3) is easily calculated:

$$\begin{aligned} 2T_{z\bar{z}} &= \frac{\partial^2 ln\sqrt{\det G}}{\partial x^2}\partial x\bar{\partial}x + \frac{1}{2}\frac{\partial}{\partial x}(ln\sqrt{\det G})\frac{\partial E_{ab}}{\partial x}\partial y^a\bar{\partial}\bar{y}^{\bar{b}} \\ &= \partial\bar{\partial}(ln\sqrt{\det G}) \;=\; \partial\bar{\partial}(ln\ \det E), \end{aligned} \tag{10.6}$$

where in the second equality we used the equation of motion, and in the last step we used the special form of G (10.4). To get eq. (10.6) it is convenient to use:

$$\begin{aligned} R_{ij} &= \frac{\partial^2 ln\sqrt{g}}{\partial x^i\partial x^j} - \frac{\partial}{\partial x^k}\Gamma^k_{ij} + \Gamma^m_{ik}\Gamma^k_{mj} - \Gamma^m_{ij}\frac{\partial ln\sqrt{g}}{\partial x^m}, \\ g^{ij}\frac{\partial g_{ij}}{\partial x^m} &= 2\frac{\partial ln\sqrt{g}}{\partial x^m} \end{aligned} \tag{10.7}$$

Therefore, the conformal factor dependence produces a dilaton field $\Phi(x)$ for the x variables:

$$\Phi(x) = ln\ \det E. \tag{10.8}$$

Since the result in (10.6) is given in terms of the worldsheet variables, the result clearly is not dependent on the number of x variables.

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