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Differential Calculus on Quantum Groups: Constructive Procedure

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Abstract

A brief review of the construction and classifiaction of the bicovariant differential calculi on quantum groups is given.

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1 Preliminaries, notation

The aim of the present lecture is to give, following the paper [1], a procedure for an explicit construction of bicovariant differential calculi on quantum groups (see also [2],[15]). We shall also mention some recent classification results [13]. Our interest will lie mainly in the underlying ideas and we shall omit technical details in our exposition. The interested reader can consult, for these, the original papers or some nice review papers [3], [41]. We shall use the summation rule and the standard notation for the result of coproduct $\Delta a = \sum a_{(1)} \otimes a_{(2)}$. The antipode will be denoted by S and the counit by e. We assume that the reader is familiar with the Faddeev–Reshetikhin–Takhtajan (FRT) [4] approach to quantum groups and with the general theory of bicovariant differential calculi on quantum groups due to Woronowicz [5] (the nice paper [14] may also be helpful in this connection). In this lecture we shall show a direct relation between these two constructions.

We shall restrict ourselves to the q-deformations of the classical simple series A_{n-1}, B_n, C_n and D_n as they were introduced in [4]. Nevertheless, as will be easily seen, the procedure works in any case whenever the corresponding quantum group and its dual Hopf algebra can be obtained through the FRT construction [4]. We shall use its general formulation due to S. Majid [43]. In particular it means [17] that we are starting from some R-matrix $\mathcal{R}$ ($N^2 \times N^2$–matrix solution of the Yang-Baxter equation) and we associate a bialgebra $A(\mathcal{R})$ to it. The algebra of $A(\mathcal{R})$ is generated by the matrix of generators $T = (t_{ij})_{i,j=1}^{N}$ modulo the relation [4]

$$\mathcal{R} T_1 T_2 = T_2 T_1 \mathcal{R}. \tag{1}$$

The coalgebra structure is the standard one induced by the matrix comultiplication. The notation

$$\Delta T = T \dot{\otimes} T \tag{2}$$

will be used for it. The counit is given by

$$e(T) = I, \tag{3}$$

with I the unit matrix.

Taking a suitable quotient, the bialgebra $A(\mathcal{R})$ is usually made into a Hopf algebra, which will be denoted as A. The starting R-matrix is assumed to be such, that the Hopf algebra A thus obtained is a dual-quasitriangular one [44]. Particularly, we assume that $\mathcal{R}^{-1}$ and $\tilde{\mathcal{R}} = ((\mathcal{R}^{t_2})^{-1})^{t_2}$ exist, where t_2 means the transposition in the second matrix factor, and that $\mathcal{R}$ can be extended to a functional (denoted for a while by the same symbol $\mathcal{R}$) $\mathcal{R} : A \otimes A \to \mathbb{C}$, such that it obeys

$$\mathcal{R}(t_{ij}, t_{kl}) = \mathcal{R}_{ik,jl}, \qquad \mathcal{R}(St_{ij}, t_{kl}) = \mathcal{R}^{-1}_{ik,jl}, \qquad \mathcal{R}(t_{ij}, St_{kl}) = \tilde{\mathcal{R}}_{ik,jl} \tag{4}$$

and

$$\mathcal{R}(ab, c) = \sum \mathcal{R}(a, c_{(1)})\mathcal{R}(a, c_{(2)}), \qquad \mathcal{R}(a, bc) = \sum \mathcal{R}(a_{(1)}, c)\mathcal{R}(a_{(2)}, b). \tag{5}$$

The standard Jimbo's R-matrices [6] fulfil the above requirements and the resulting Hopf algebras are the algebras of quantized fuctions $Fun_q(G)$ of [4] corresponding to the classical simple series.

The above–mentioned properties of the R-matrix can be used in this case to introduce a Hopf algebra $U(\mathcal{R})$ dual to the Hopf algebra $Fun_q(G)$ generated by two matrices of generators $L^{\pm} = (l^{\pm}_{ij})^N_{i,j=1}$ of [4]. The following commutation relations take place

$$\mathcal{R}_{21} L^{\pm}_1 L^{\pm}_2 = L^{\pm}_2 L^{\pm}_1 \mathcal{R}_{21}, \tag{6}$$
$$\mathcal{R}_{21} L^{+}_1 L^{-}_2 = L^{-}_2 L^{+}_1 \mathcal{R}_{21}.$$

The comultiplication on matrices $L^{\pm}$ is again the matrix one

$$\Delta L^{\pm} = L^{\pm} \dot{\otimes} L^{\pm} \tag{7}$$

and the counit is given by

$$e(L^{\pm}) = I. \tag{8}$$

The pairing between Hopf algebras $Fun_q(G)$ and $U(\mathcal{R})$ is given on generators as

$$(L^{\pm}, T) = \mathcal{R}^{\pm}, \tag{9}$$

where $\mathcal{R}^{+} = \mathcal{R}_{21}$ and $\mathcal{R}^{-} = \mathcal{R}^{-1}$. For more details about the matrices $L^{\pm}$, we refer the reader to the above–mentioned paper [4], which describes also the relation between the Hopf algebra $U(\mathcal{R})$ and the quantized enveloping algebra $U_h(g)$ of Drinfeld and Jimbo [8], [7].

Concerning the bicovariant differential calculi on $A = Fun_q(G)$, let us start with the following fact, proved by Woronowicz [5]. Let us assume that we are given a family of functionals $F = (f_{ij})^k_{i,j=1} \in U \equiv U(\mathcal{R})$, $k \in \mathbb{N}$, such that

$$\Delta F = F \dot{\otimes} F \tag{10}$$

and

$$e(F) = I, \tag{11}$$

and a family of quantum functions $R = (R_{ij})^k_{i,j=1} \in A$, such that

$$\Delta R = R \dot{\otimes} R \tag{12}$$

and

$$e(R) = I. \tag{13}$$

Besides, matrices R and F are supposed to satisfy the following compatibility condition

$$R_{ij}(a * f_{ih}) = (f_{ji} * a) R_{hi}, \tag{14}$$

for all j, h and any $a \in A$. Here we used the notation

$$x * a = \sum a^{(1)}(x, a^{(2)}), \qquad a * x = \sum (x, a^{(1)}) a^{(2)}$$

for $x \in U$, $a \in A$. Let us now assume a free left module Γ over A generated by elements ω_i, $i = 1, 2, ..., k$, and let us introduce the right multiplication by elements of A and the left δ_L and right coaction δ_R of A on Γ by the following formulae

$$(a_i\omega_i)b = a_i(f_{ij} * b)\omega_j, \tag{15}$$

$$\delta_L(a_i\omega_i) = \Delta(a_i)(1 \otimes \omega_i), \tag{16}$$

$$\delta_R(a_i\omega_i) = \Delta(a_i)(\omega_j \otimes R_{ji}). \tag{17}$$

A theorem of Woronowicz says that the triple $(\Gamma, \delta_L, \delta_R)$ is a bicovariant bimodule and, vice versa, that any bicovariant bimodule is of this form. Elements ω_i, $i = 1, 2..., k$ form a basis in the linear subspace ${}_{inv}\Gamma \subset \Gamma$ of all left-invariant elements of Γ. We know from the general theory of Woronowicz that the space Γ of all one-forms on the quantum group A and the whole exterior algebra $\Gamma^\wedge$ over A are naturally equipped with a structure of a bicovariant bimodule.

It is an almost obvious fact that having two bicovariant bimodules Γ_1 and Γ_2 we can construct their tensor product $\Gamma_1 \otimes \Gamma_2$, which is again a bicovariant bimodule. The linear basis in ${}_{inv}(\Gamma_1 \otimes \Gamma_2)$ can be chosen as $\omega_{ij} = \omega_i \otimes \omega_j$. In this basis we have $R_{ij,kl} = R^1_{ik}R^2_{jl}$ and $f_{ij,kl} = f^1_{ik}f^2_{jl}$.

2 Construction of differential calculi

It is enough to consider only the first-order differential calculi on A because we know that a given first-order differential calculus on A can be uniquely extended to an exterior differential calculus on A [5].

Let us now assume the vector corepresentation of A given by the matrix of the generators $T = (t_{ij})^N_{i,j=1}$. A comparison of (12) and (13) with (2) and (3) suggests to try to find a bicovariant bimodule such that $R = T$. It is easily seen that we can choose both $F = S(L^\pm)^t$ in this case. The conditions (10) and (11) are satisfied due to (7) and (8) and the antipode properties. The compatibility condition (14) is sufficient (because of (10)) to check only for $a = t_{ij}$. In this case, (14) is equivalent to (1). The situation is quite similar for the choice $R = S(T)^t$. In this case we can take $F = L^\pm$ by the same reasoning. We denote the bicovariant bimodules thus obtained as Γ_1, Γ_2, Γ_1^c and Γ_2^c according to the choices

$$\Gamma_1 : \ R = T, \qquad F = S(L^+)^t, \tag{18}$$

$$\Gamma_2 : \ R = T, \qquad F = S(L^-)^t, \tag{19}$$

$$\Gamma_1^c : \ R = S(T)^t, \qquad F = L^-, \tag{20}$$

$$\Gamma_2^c : \ R = S(T)^t, \qquad F = L^+. \tag{21}$$

Now we take the tensor product $\Gamma = \Gamma_1 \otimes \Gamma_1^c$ to get a new bicovariant bimodule. For other choices of tensor products ($\Gamma_1^c \otimes \Gamma_1, \Gamma_2 \otimes \Gamma_2^c$ and $\Gamma_2^c \otimes \Gamma_2$) all that follows is quite analogous, or it leads to the trivial differential calculi ($da = 0$, for all $a \in A$)

in the cases $\Gamma_1 \otimes \Gamma_2^c, \Gamma_1^c \otimes \Gamma_2, \Gamma_2 \otimes \Gamma_1^c$ and $\Gamma_2^c \otimes \Gamma_1$. According to the general theory Γ can be described as follows. Let $(\omega_{ij})_{i,j=1}^N$ be a basis for ${}_{inv}\Gamma$. Right multiplication is given by

$$\omega_{ij} a = ((id \otimes S(l_{ki}^+) l_{jl}^-)\Delta a)\omega_{kl} \tag{22}$$

and the right coaction by

$$\delta_R(\omega_{ij}) = \omega_{kl} \otimes t_{ki} S(t_{jl}). \tag{23}$$

Our choice of the bicovariant bimodule Γ is motivated by the particular form of the coaction (23). It follows that the linear space ${}_{inv}\Gamma$ contains a bi-invariant element $\tau = \sum \omega_{ii}$, which can be used to define a derivative on A. For $a \in A$ we set

$$da = \tau a - a\tau. \tag{24}$$

It can be easily checked that such a derivative has all properties stated in [5]. The bi-invariance of τ is essential for the differential calculus to be a bicovariant one. In the case of real forms $Fun(G_q, \varepsilon_i)$ introduced in [4] the *-structure on ${}_{inv}\Gamma$ is given by

$$\omega_{ij}^* = -\varepsilon_i \varepsilon_j \omega_{ji}. \tag{25}$$

According to the rules of [5] the derivative (24) and the *-structure extend uniquely to the whole $\Gamma^\wedge$. The result for the derivative is

$$d\theta = \tau \wedge \theta - (-1)^k \theta \wedge \tau, \tag{26}$$

where k is the degree of a homogeneous element $\theta \in \Gamma^\wedge$.

Thus (24) defines a *-calculus. In the case of $SU_q(2)$, we get in this way the $4D_+$ calculus of Woronowicz [9],[10]. For $a \in A$ we have

$$da = ((id \otimes (S(l_{ki}^+) l_{il}^- - \delta_{kl} e)\Delta a)\omega_{kl}. \tag{27}$$

Let us denote by

$$\chi_{ij} = S(l_{ik}^+) l_{ki}^- - \delta_{ij} e, \tag{28}$$

or more compactly

$$\chi = S(L^+) L^- - Ie, \tag{29}$$

the matrix of left-invariant vector fields χ_{ij} on A. The "commutators" ($[\chi', \chi] = \sum S(\chi_{(1)})\chi' \chi_{(2)}$) among the elements χ_{ij} of the basis dual to the ω_{ij} can now be obtained directly from relations (6) between the functionals $l_{ij}^\pm$ or from the fact that $d^2(a) = 0$ for any $a \in \mathcal{A}$. We employ the notation $\lambda_{ijkl,mnop} = (S(l_{oi}^+) l_{jp}^-)(t_{mk} S(t_{ln}))$ (λ's are easily expressed with the help of matrices $\mathcal{R}$ and $\tilde{\mathcal{R}}$). We have

$$[\chi_{ij}, \chi_{kl}] = \chi_{ij}\chi_{kl} - \lambda_{mnop,ijkl}\chi_{mn}\chi_{op} = -\delta_{kl}\chi_{ij} + \lambda_{ssmn,ijkl}\chi_{mn}. \tag{30}$$

In a more compact notation

$$\mathcal{R}_{21}^{-1}\chi_1 \mathcal{R}_{12}^{-1}\chi_2 - \chi_2 \mathcal{R}_{21}^{-1}\chi_1 \mathcal{R}_{12}^{-1} = \chi_2 \mathcal{R}_{21}^{-1}\mathcal{R}_{12}^{-1} - \mathcal{R}_{21}^{-1}\mathcal{R}_{12}^{-1}\chi_2. \tag{31}$$

For the real forms the *-structure implies

$$\chi_{ij}^* = \varepsilon_i \varepsilon_j \chi_{ji} \tag{32}$$

on left-invariant vector fields χ_{ij}.

Let us note that the set of projections $P(dt_{mn}) = ((S(l^+_{ki})l^-_{il} - \delta_{kl}e), t_{mn})\omega_{kl}$ of the differentials of generators t_{mn} to the space ${}_{inv}\Gamma$ can be chosen as another basis of ${}_{inv}\Gamma$.

Now we can describe a direct relation between the enveloping algebra generated by $(\chi_{ij})_{i,j=1}^N$ and the algebra U of [4] generated by functionals $(l^{\pm}_{ij})_{i,j=1}^N$. Let us introduce matrix $L = (l_{ij})_{i,j=1}^N$, $L = S(L^+)L^-$. A similar matrix $L^+S(L^-)$ was introduced in [11] and investigated in [12]. The upper and lower triangular matrices $S(L^+)$ and L^- can be constructed from L by its decomposition into triangular parts as described in [12]. In this sense the enveloping algebra generated by χ's and the algebra U of FRT are equivalent. Let us now discuss very briefly the classical limit. We have $R = 1 + \hbar r + ...$, where $q = e^{\hbar}$ and r is the corresponding classical r-matrix, $L^{\pm} = 1 + \hbar\eta^{\pm} + ...$, with $\eta^{\pm}$ matrices of generators for the corresponding Lie algebra g. The matrix elements of $\chi = (\eta^- - \eta^+)$ are no longer linearly independent in this limit, and the linear space spanned by these is just the Lie algebra g. As a result the classical differential calculus on the group G is obtained as a quotient.

3 Some remarks on the classification of differential calculi

Let us mention that for the differential calculi described in the previous section the dimension of the space of left-invariant forms ${}_{inv}\Gamma$ is N^2, where N is the dimension of the vector representation of the corresponding classical simple group G. So it appears higher, as in the classical case. This is the price we have to pay for the bicovariance of the corresponding differential calculi. Nevertheless, these differential calculi are quite natural and they contain the classical differential calculus on the corresponding group G in the limit. Moreover, the assumption of bicovariance is a natural one and, besides, it is also technically important. There is no general theory of left-(right-) invariant differential calculi only.

Another difference, contrary to the classical case, appears. In general there are many non-isomorphic differential calculi on a given quantum group, and up to now there is a lack of a functorial method to construct a natural one.

Two differential calculi (Γ_1, d_1) and (Γ_2, d_2) are assumed to be isomorphic if there is a bimodule isomorphism $\phi : \Gamma_1 \to \Gamma_2$ and $\phi \circ d_1 = d_2$. Motivated in the previous section, the following question arises. How many non-isomorphic bicovariant differential calculi exist, such that the space of left- invariant forms ${}_{inv}\Gamma$ is spanned by the (not necessary linearly independent) left-invariant forms $P(t_{ij})$, there exist for a given quantum group A corresponding to the one of the classical simple groups? This was investigated in [13], q not being a root of unity and under the assumption $\dim {}_{inv}\Gamma \geq 2$. The discussion differs for the case of $SL(N)$ and for the remaning classical simple series B, C, D. The cases $SL(2)$ and $Sp(2)$ should also be treated separately.

In the $SL(N)$ case the result is obtained by the following generalization of Section. 1. There still is an ambiguity in the discussion done there, which amounts to replacing the standard R-matrix for the $A_{N-1} = SL(N)$ case of [4] by a new one, $\mathcal{R}_p$, which differs by only a factor p (equal to some of the N-th root of unity) from the standard one. This new R-matrix can be used in the FRT construction, which now leads us to new matrices of generators $L_p^{\pm}$ (the matrix T remains of course the same) and we can repeat the construction of the differential calculi as above, now starting with the bicovariant bimodules ${}_{p_k}\Gamma^{+} = {}_{p}\Gamma_1 \otimes {}_{p'}\Gamma_1^c$ and ${}_{p_k}\Gamma^{-} = {}_{p}\Gamma_2 \otimes {}_{p'}\Gamma_2^c$, where p_k, $k = 0, 1, ..., N-1$ is such N-th root of unity that $pp' = p_k$. The indices p and p' mean that the matrices $L_p^{\pm}$ or $L_{p'}^{\pm}$ were used in the construction of the corresponding bimodules. The resulting tensor products depend only on k. The rest of the construction is the same as in the Section 1. So we obtain $2N$ differential calculi in the $SL(N)$ case, corresponding to various choices of the signs $\pm$ and of the integer $k =$. It can be shown [13] that, except for a finite number of values of q, this list exhausts all non-equivalent bicovariant differential calculi on $SL_q(N)$, $N > 2$, q not being a root of unity, such that the space of left-invariant forms ${}_{inv}\Gamma$ is spanned by the left-invariant forms $P(dt_{ij})$ and $\dim {}_{inv}\Gamma \geq 2$. The exceptional values of q are also discussed in [13]. Only the two calculi corresponding to $k = 0$ contain the ordinary classical calculus in the limit $q \to 1$ as a quotient. In the remaining case of $SL_q(2)$, the calculi corresponding to the same k but to different signs $\pm$ can be easily shown to be isomorphic and the only non-isomorphic calculi are the $4D_+$ and $4D_-$ calculi of Woronowicz, as was shown in [38] .

A similar discussion can be done also in the remaining cases, corresponding to the simple classical groups of the types B_n, C_n, D_n [13]. We set $N = 2n + 1$ in the case B_n, and $N = 2n$ in the cases C_n and D_n. To formulate the result we need the differential calculus based on the bimodule $\Gamma = \Gamma_1 \otimes \Gamma_1^c$ and also an additional one, which is described as follows. It is again a tensor product $\Gamma' = \Gamma_1 \otimes \Gamma_0$, where the bimodule Γ_0 is constructed by setting $R = S(T)^t$ and $F = L^0$. The matrix of functionals L^0 is defined as

$$(L^0, a) = (-1)^k (L^-, a),$$

for $a \in A$ a homogeneous polynomial of order k in the generators t_{ij}. The remaining part of the construction of the corresponding differential calculus is the same as in Section 1. Let $N \geq 3$ and assume that q is not a root of unity. Except for finitely many values of q, the above-described differential calculi are the only non-isomorhipc bicovariant differential calculi of dimension N^2 [13]. The missing case of $Sp_q(2)$ is already covered by the discussion of the $SL_{q^2}(2)$, which is isomorphic to $Sp_q(2)$. Again, only the calculus (Γ, d) contains the ordinary classical calculus in the limit $q \to 1$ as a quotient.

4 Remarks

Let us mention other examples of quantum groups for which the construction of Section 1 works. The examples include the quantum group $GL_q(n)$ and its more-parametric modifications (see [18], [21],[22], [42],[40], [41], and many others), " "complex" quantum groups (real forms of the dual to the Drinfeld's quantum double) corresponding to the classical simple groups [23], [24], [25], the inhomogeneous quantum groups [26],[37],[19], [41], etc. For applications and subsequent developement, let us mention for instance the q-deformed BRST complex [20], [27], the q-deformed standard complex [28], the representation theory of quantized enveloping algebras [30], quantum mechanics on quantum spaces [29], the q-deformd gravity [26], the Cartan calculus on quantum groups [33], [31], etc. For alternative approaches, see e.g. [32], [17], [22], [34], [35], [36], [39]. It is almost impossible to mention all the papers related to the subject and we apologize to those authors whose work has been omitted.

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