The topological B model as a twisted spinning particle

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Abstract

The B–twisted topological sigma model coupled to topological gravity is supposed to be described by an ordinary field theory: a type of holomorphic Chern–Simons theory for the open string, and the Kodaira–Spencer theory for the closed string. We show that the B model can be represented as a *particle* theory, obtained by reducing the sigma model to one dimension, and replacing the coupling to topological gravity by a coupling to a twisted one-dimensional supergravity. The particle can be defined on *any* Kähler manifold—it does not require the Calabi–Yau condition—so it may provide a more generalized setting for the B model than the topological sigma model.

The one-loop partition function of the particle can be written in terms of the Ray–Singer torsion of the manifold, and agrees with that of the original B model. After showing how to deform the Kähler and complex structures in the particle, we prove the independence of this partition function on the Kähler structure, and investigate the origin of the holomorphic anomaly. To define other amplitudes, one needs to introduce interactions into the particle. The particle will then define a field theory, which may or may not be the Chern–Simons or Kodaira–Spencer theories.

1 Introduction

Witten showed that by twisting an $N = (2, 2)$ sigma model one obtains a topological theory. In fact, depending on the relative sign of the $U(1)$ charges used to twist the theory, one obtains two topological theories: the A and the B models. From a world-sheet point of view the two theories are very similar, and "mirror symmetry" relates the A model on one manifold to the B model on its mirror [1]. This has proven to be very useful for Calabi–Yau calculations, since the basic observables of A models correspond to Kähler deformations of the manifold [2], while those of the B models correspond to complex-structure deformations.

Despite this apparent similarity of the A and B model, they have many basic differences. One such difference is that the B twisting is chiral, so the theory has a world-sheet Lorentz anomaly if the target space is not a Calabi–Yau manifold, which has a vanishing first Chern class, c_1 [1]. The A model, on the other hand, can be defined on any Kähler space*, with the restriction to being Calabi–Yau necessary only for conformal invariance. Thus the Calabi–Yau condition seems far more basic in the B model, and mirror symmetry can apparently only exist in the (physically relevant) Calabi–Yau case.

A more surprising difference is that the target space interpretations of the two theories seem to be completely different. The bosonic part of both theories, coming from the original $(2, 2)$ sigma model, has the form

$$\mathcal{S} = \int \mathrm{d}^2 z \left(t^a \, k^{(a)}_{\mu \bar{\mu}} \, \partial_z X^\mu \partial_{\bar{z}} \bar{X}^{\bar{\mu}} + \bar{t}^a \, k^{(a)}_{\mu \bar{\mu}} \, \partial_{\bar{z}} X^\mu \partial_z \bar{X}^{\bar{\mu}} \right) \ , \tag{1}$$

where X^μ is a complex coordinate on the target space, and the $k^{(a)}_{\mu \bar{\mu}}$'s are a normalized basis of Kähler metrics. The t^a's are coordinates on the moduli space of the complexified Kähler deformations of the theory; thus the target-space metric is $G_{\mu \bar{\mu}} = Re(t^a) \, k^{(a)}_{\mu \bar{\mu}}$, and the antisymmetric target-space tensor on the space is $B_{\mu \bar{\mu}} = Im(t^a) \, k^{(a)}_{\mu \bar{\mu}}$. In Witten's original study of the A model he showed that the (supersymmetrized) $\bar{t}$ term is BRST exact [2]. He then argued that one could study the theory in the formal limit $\bar{t} \to \infty$, which enforces the condition that maps from the worldsheet to the target space must be holomorphic. The t^a term then reduces to the instanton number of the map, and the A model simply "counts" holomorphic maps or, more generally, evaluates the Euler characteristic of the moduli space of such maps. Recently Bershadsky, Cecotti, Ooguri and Vafa (BCOV) discovered that when the A model is coupled to topological gravity—which is necessary if one wants interesting loop amplitudes—there is a BRST anomaly

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*They can actually be defined on any almost-complex space, although one only has two BRST symmetries in the Kähler case, and only there is the model a twisting of a $(2, 2)$ sigma model [2].

in the theory, so that it actually does depend on $\bar{t}^\dagger$ [4]. However, one can still study the "traditional" A model with "base-point" $\bar{t} \to \infty$, where the theory does have a topological target-space interpretation.

The B model is far less well understood. In it, Kähler deformations are BRST exact [1], so one can study it in the limit where both t and $\bar{t}$ become infinite. (More physically, one takes the large volume limit, where $V \propto t + \bar{t} \to \infty$.) It thus appears that the B model is concerned only with constant maps, and evaluates only quantities depending on the classical geometry of the target space. However, the situation is again complicated by the coupling of the theory to topological gravity. In this case BCOV showed that there is essentially no anomalous dependence on t or $\bar{t}$. However, the coupling means that one must integrate amplitudes over the moduli space of world-sheet Riemann surfaces. As was noticed by Witten, this means that the suppression of the action due to the large target-space volume V can be counteracted by being near to a degeneration of the Riemann surface [5]. This is most easily seen in the hamiltonian quantization of the theory, which is relevant for the one-loop amplitude. Thus, consider a world-sheet torus, with a coordinate σ ranging from 0 to 1 along the string, and a coordinate t (not to be confused with the $t^{(a)}$'s!) ranging from 0 to T, measuring the proper time along the worldsheet. In the large volume limit $V \to \infty$ the antisymmetric tensor $B_{\mu\bar{\mu}}$ becomes irrelevant, and the action (1) reduces to

$$\mathcal{S} \to \int \mathrm{d}t\, d\sigma\; G_{\mu\bar{\mu}} \left(\dot{\bar{X}}^{\bar{\mu}} \dot{X}^{\mu} + \bar{X}'^{\bar{\mu}} X'^{\mu} \right) \; . \tag{2}$$

The first term is proportional to V/T, whereas the second goes like VT. Thus in the limit $V \to \infty$, there is a contribution to the path integral when $T \to \infty$ with T/V constant. Such a torus is conformally equivalent to a circle—the worldline of a particle, instead of a string—and one sees that in this limit the amplitude is dominated by particle-like excitations. The straightforward generalization of this argument to arbitrary genus amplitudes shows that the partition function of the B model, which can be calculated in the limit $V \to \infty$, is dominated by configurations where the string worldsheet collapses to a Feynman-diagram-like structure, so it is natural to assume that B models should be calculable as (relatively) ordinary field theories.

The rest of this paper is organized as follows: In the next section we present the $N = (2, 2)$ sigma model reduced to one dimension and its symmetries. In section 3 we discuss how to couple the theory to gravity, to obtain a "spinning particle". In section 4 we perform the path integral of this particle on the circle, in the case when the target space is a complex one torus. (This example will be considered repeatedly throughout

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†See also ref. [3], where this is related to the non-holomorphicity of threshold corrections in the string.

the paper, being both tractable and very instructive.) We connect the appearance of the holomorphic anomaly in this case to a conflict with modular invariance. Using the torus result, we derive the partition function on an arbitrary Kähler manifold in section 5 using hamiltonian quantization, and relate it to the Ray–Singer torsion on the manifold. We find that the Hilbert space of the particle is described in terms of (p, q) forms, which is different from the $(0, q)$ forms in $\wedge^p T(M)$ in Witten's cohomology calculations, except on Calabi–Yau manifolds. In section 6, we discuss how to vary the Kähler and complex structures of the manifold in the particle case. We derive the Kodaira–Spencer equation, and also find an apparently necessary auxiliary condition that complex-structure variations must satisfy! In section 7, we show the independence of the particle on the Kähler structure, and in section 8 we investigate the holomorphic anomaly. Finally, we present our conclusions, and close with some speculations on the further development of the particle and its associated field theory.

2   A particle theory for the B model

We have given the argument as to why the B–string should be describable as a field theory: the next step is to find it! There is no completely deductive procedure for constructing a string field theory from a two-dimensional description of a string. Generally, the only clue is that the equation of motion of the field theory should correspond to the allowed states and deformations of the string. For the open string, Witten argued that the appropriate field theory should have the general structure of a Chern–Simons theory [5]. This was partially based on the structure of open string field theory [6], and partially on knowing that to preserve the BRST symmetry, one can couple the string only to connections with vanishing $(0, 2)$ curvature [5]. For the closed string, BCOV constructed a "Kodaira–Spencer" field theory [7]. This was based on the fact that the observables in the B model are the deformations of the complex structure [1], so the string equation of motion should give the Kodaira–Spencer equation [8], which describes such deformations. The purpose of our work is to give a relatively deductive derivation of the particle—as opposed to field theory—interpretation of the B–string. Such a particle theory is directly capable of giving only the propagators and partition function of the theory, and interactions will later need to be incorporated in order to reproduce the Feynman diagrams of the field theory and the string.

Temporarily setting aside the coupling to topological gravity, the particle action can be derived by taking the string action of the B model on the torus, and dimensionally reducing the theory to a circle. (Of course, at this point, one needs the full two-dimensional

action [1], including all the fermionic terms.) However, in practice it is better to construct the particle action directly, rather than by dimensional reduction. This is because the notion of spin becomes irrelevant in one dimension, so the fermionic fields become scalars. Thus the dimensional reductions of the A [2] or B [1] models are the same as that of the untwisted $(2, 2)$ sigma model [9], and since the distinction between left- and right-moving fields is also lost in one dimension, the $U(1)_L \otimes U(1)_R$ symmetry of the sigma model is enhanced to a $U(2)$ symmetry in the particle. Knowing this, one is lead uniquely to the usual so-called $N = 1$ one-dimensional sigma model. This sigma model can be written on any Riemann manifold, where it has an $O(2)$ symmetry. When the target space is Kähler, the symmetry is enhanced to a $U(2)$. Introducing a dimensionful coupling $\hbar$, its action can be written*:

$$ \mathcal{S} = \frac{1}{\hbar} \int \mathrm{d}t \; G_{\mu\bar{\mu}} \, \dot{\bar{X}}^{\bar{\mu}} \, \dot{X}^{\mu} + i \, \chi^{*i}_{\mu} \, \dot{\chi}^{\mu}_{i} + i \, \chi^{*i}_{\mu} \, \Gamma^{\mu}_{\rho\sigma} \, \dot{X}^{\rho} \chi^{\sigma}_{i} - \frac{1}{2} \, R_{\mu\bar{\mu}\nu\bar{\nu}} \, \chi^{\mu}_{j} \, \chi^{*j\bar{\mu}} \, \chi^{\nu}_{i} \, \chi^{*i\bar{\nu}} \; , \qquad (3) $$

with the $U(2)$ acting manifestly on the i and j indices of the fermionic fields. In Kähler space the action is invariant under two complex global supersymmetries, with parameters α_i. Formally taking α_i and its complex conjugate α^{*i} to be independent, the α_i transformations are given by:

$$ \delta \, \bar{X}^{\bar{\mu}} = -i \, \alpha_i \, \chi^{*i\bar{\mu}} \qquad \delta \, X^{\mu} = 0 $$
$$ \delta \, \chi^{\mu}_{i} = \dot{X}^{\mu} \, \alpha_i \qquad\qquad \delta \, \chi^{*i}_{\mu} = 0 \; , \qquad (4) $$

and the α^{*i} transformations by the complex conjugate of (4)†. This action is indeed the particle version of the B model closed string, before coupling to topological gravity. In particular, its partition function $\mathrm{Tr}(-1)^F$ is simply the Euler number of the target space.

In the case of the open string, one sees that χ_2 and χ^{*2} have antiperiodic boundary conditions, so they must be dropped in the particle limit. The particle version of the open B–string [5] is therefore simply the $U(1)$ truncation of (3). This should describe a type of Chern–Simons field theory, whose solutions are connections A in the Chan–Paton gauge group with vanishing $(0, 2)$ curvature [5]. To write the theory with these background fields included, one needs to introduce Wilson lines of the improved pullback of the connection $\Phi^*(A) - i \, \eta^{\bar{\mu}} \, F_{\bar{\mu}\nu} \, \rho^{\nu}$ into the path-integral of the string or the particle [5]. If one rewrites

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*The transcription from the B model of ref. [1] to our notation is $\rho_t \rightarrow \chi_1$, $\rho_x \rightarrow \chi_2$, $\eta \rightarrow \chi^{*1}$ and $\theta \rightarrow \chi^{*2}$. From section 5 on we shall return to the notation of [1], except for replacing $\rho_t \rightarrow \rho$ and $\rho_x \rightarrow \tilde{\rho}$.

†The action is hermitian, up to integrations by parts, and so should be completely symmetrical with respect to the "starred" and "unstarred" fields. Our choice to lower the space-time indices on the spinors χ^*'s obscures this symmetry. In particular, the α^{*i} transformations on the spinors is somewhat complicated (see eq. (9)).

these Wilson lines in terms of an integral over "boundary fermions" [10], one remains with a description of the theory in terms of a particle action.

3   Coupling to gravity

So far we have given simple arguments leading to a unique action for the matter B model. However, there is no deductive procedure for coupling the theory to topological gravity. In fact, even the original B sigma-model action has not been explicitly coupled to topological gravity. The only indication of this coupling is given by the form of the amplitudes of the theory. BCOV argued, by analogy to the form of ghost insertions in the bosonic string, that the one-loop partition function of the theory coupled to topological gravity should be given by [4]:

$$\mathcal{F}_1 = \frac{1}{2} \int_{\mathcal{M}} \frac{d^2\tau}{\tau_2} \operatorname{Tr} (-1)^F F_L F_R \, q^{H_L} \, \bar{q}^{H_R} \, , \qquad (5)$$

which is their "generalized index" [11, 12]*. The integration over τ comes, as usual, from writing the model on an arbitrary curved worldsheet, and integrating over the metric, modulo diffeomorphisms and Weyl transformations. The analogous coupling of the particle to the one-dimensional *einbein* e can be found either by gauge fixing the two-dimensional metric to $g_{\alpha\beta} \rightarrow \operatorname{diag}(e^2, 1)$, or by inserting *einbeins* to make the action invariant under one-dimensional diffeomorphisms. Note that we do not have the Liouville modes that are crucial in two-dimensional topological gravity [2, 13, 14]. This means [14] that we will not find gravitational descendants in our approach.

At this stage, the only effect of the other fields of the topological gravity is to give the insertions of F_L and F_R into the partition function. It is natural, but incorrect, to attempt to introduce these other fields by gauging all the global symmetries of the action, *i.e.* the $U(2)$ symmetry and the four supersymmetries. (This does give a new type of Kähler spinning particle theory, which is interesting in its own right [15].) To find the correct procedure, it is useful to track the appearance of these symmetries from the original B model, in order to decide which of them to gauge. As we have stated before, the diagonal subgroup of the $U(2)$ comes from the $U(1)_L$ and $U(1)_R$ in the original theory; the rest of the group is not a symmetry of the string. The two supersymmetries of eq. (4) come from the left- and right-handed BRST symmetries of the B model, while the complex-conjugate supersymmetries come from the sigma-model symmetries generated by the G_{zz} and $G_{\bar{z}\bar{z}}$

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*We should perhaps note here that $\mathcal{F}_1$ is infinite, because of the zero-modes of the hamiltonian. Only differences or derivatives of $\mathcal{F}_1$ are well defined.

of the two $N = 2$ superconformal algebras. (Since spin no longer has any meaning, there is no distinction between a supersymmetry and a BRST invariance in the particle.) In the original sigma model one would not gauge the BRST symmetries or the $U(1)$'s, so we shall gauge only diffeomorphisms and the two "G" symmetries[†]. This asymmetrical choice means that the theory is no longer unitary, and that we have finally distinguished the B model from the untwisted $(2,2)$ sigma model!

Proceeding to gauge the $\alpha^{*\,i}$ symmetries with gravitini $\psi^{*\,i}$, we obtain our action

$$
\mathcal{S} = \frac{1}{\hbar} \int \mathrm{d}t \, \frac{1}{e} \, G_{\mu\bar{\mu}} \, \dot{\bar{X}}^{\bar{\mu}} \left(\dot{X}^{\mu} + i\,\psi^{*\,i}\,\chi_i^{\mu} \right) + i\,\chi_{\mu}^{*\,i}\,\dot{\chi}_i^{\mu} + i\,\chi_{\mu}^{*\,i}\,\Gamma_{\rho\sigma}^{\mu}\,\dot{X}^{\rho}\,\chi_i^{\sigma}
$$
$$
- \frac{e}{2} \, R_{\mu\bar{\mu}\nu\bar{\nu}} \, \chi_j^{\mu}\,\chi^{*\,j\bar{\mu}}\,\chi_i^{\nu}\,\chi^{*\,i\bar{\nu}} \ . \tag{6}
$$

Note that, unlike the ungauged theory, $\mathcal{S}$ can be written only on Kähler manifolds. However, since particle theories never have local anomalies, being field theories in an odd number of dimensions, *there is no reason to impose the vanishing of c_1, so one is not restricted to Calabi–Yau manifolds.*

The action $\mathcal{S}$ is clearly diffeomorphism invariant, and has a manifest global $U(2)$. In the gauged theory, the two supersymmetries of (4) become

$$
\begin{aligned}
\delta\,e &= -i\,\alpha_i\,\psi^{*\,i} & \delta\,\psi^{*\,i} &= 0 \\
\delta\,\bar{X}^{\bar{\mu}} &= -i\,\alpha_i\,\chi^{*\,i\bar{\mu}} & \delta\,X^{\mu} &= 0 \\
\delta\,\chi_i^{\mu} &= \frac{1}{e}\left(\dot{X}^{\mu} + i\,\psi^{*\,j}\,\chi_i^{\mu} \right)\alpha_i & \delta\,\chi_{\mu}^{*\,i} &= 0 \ .
\end{aligned} \tag{7}
$$

These can be recognized as the remnants of the BRST transformations of the B model [1], and of topological gravity [14]. The two local supersymmetries with some spinor indices raised or lowered are given by

$$
\begin{aligned}
\delta\,\psi^{*\,i} &= \dot{\alpha}^{*\,i} & \delta\,e &= 0 \\
\delta\,X^{\mu} &= -i\,\alpha^{*\,i}\,\chi_i^{\mu} & \delta\,\bar{X}^{\bar{\mu}} &= 0 \\
\delta\,\chi^{*\,i\bar{\mu}} &= \frac{1}{e}\,\dot{\bar{X}}^{\bar{\mu}}\,\alpha^{*\,i} & \delta\,\chi_{i\bar{\mu}} &= 0 \ .
\end{aligned} \tag{8}
$$

The transformations of the spinors with the original indices are somewhat more complicated:

$$
\delta\,\chi_{\mu}^{*\,i} = \frac{1}{e}\,G_{\mu\bar{\mu}}\,\dot{\bar{X}}^{\bar{\mu}}\,\alpha^{*\,i} - i\,\Gamma_{\mu\sigma}^{\rho}\,\alpha^{*\,j}\,\chi_j^{\sigma}\,\chi_{\rho}^{*\,i} \qquad \delta\,\chi_i^{\mu} = i\,\Gamma_{\rho\sigma}^{\mu}\,\alpha^{*\,j}\,\chi_j^{\sigma}\,\chi_i^{\rho} \ , \tag{9}
$$

[†]A similar coupling of the B sigma model to topological gravity was considered in [16].

and include the noncovariant looking terms involving the Christoffel symbols that usually appear in spinor transformations in supersymmetric sigma models.

To get the "open string" particle, one should truncate eqs. (6–9) to the $U(1)$ case.

4   The particle on a complex torus

Having written a particle action for the B model, we would now like to evaluate its partition function. As usual, the diffeomorphisms on the circle can be gauge-fixed by setting $\dot{e} = 0$, up to constant translations and the $\mathbb{Z}_2$ symmetry of inverting the circle. Thus the path integral over the metric reduces to an integral over the length T of the circle, with measure [17]

$$\frac{1}{2} \int_0^\infty \frac{\mathrm{d}T}{T} T^s \; . \tag{10}$$

Here we have introduced a proper time regulator T^s for the ultraviolet and infrared infinities of the theory. In this zeta-function-like regularization one first writes all quantities as meromorphic functions in s, and then lets $s \to 0$ discarding all poles.

One would similarly like to choose the gauge $\dot{\psi}^{*\,i} = 0$ for the gravitini. Unfortunately here one runs into a difficulty: As with the diffeomorphisms, the gauge does not fix the local supersymmetry transformations (8) with constant $\alpha^{*\,i}$'s. These are analogues of conformal Killing spinors in fermionic strings, and they are difficult to deal with, since one must divide the path integral by the volume of the transformation group, which is zero. Usually one is saved from having to perform the calculation by an overcompensation of fermion-matter zero modes in the numerator of the integral, and the only previous case in which such a calculation was really needed is in the one-loop amplitude of the $N = 2$ string [18]. In that string one has to integrate over $U(1)$ moduli, so the calculation could be done by considering the theory with twisted boundary conditions. We would like to perform a similar trick in our case, and regularize the zero-mode infinity by twisting the boundary conditions of the model to $\psi^{*\,i}(T) = \exp(i\,\theta_i)\,\psi^{*\,i}(0)$ and $\chi_i^\mu(T) = \exp(-i\,\theta_i)\,\chi_i^\mu(0)$. These boundary conditions respect all the symmetries of the action, and for non-zero θ_i's one no longer has zero modes of the supersymmetries.

For the moment, let us consider a particle moving on a D complex-dimensional target-space torus. This case is anyway interesting, and is the only example in which the path-integral can be carried out explicitly, the gauge-fixed action being quadratic. Bearing in mind the transformations of eq. (8), the path integral over a gravitino modulo

the local supersymmetry gives the superJacobian

$$\text{sdet}_\theta \, i\partial_t = \left(\det_\theta \, i\partial_t\right)^{-1} , \tag{11}$$

where the index on the "det" is to remind us of the shifted boundary conditions of the fermions. Thus, integrating over the *einbein*, the gravitini and the matter fermions, the partition function on the D–torus reduces to[*]

$$\mathcal{F}_{\theta_i} = \frac{1}{2} \int_0^\infty \frac{\mathrm{d}T}{T^{1-s}} \left(\det_{\theta_1} i\partial_t\right)^{D-1} \left(\det_{\theta_2} i\partial_t\right)^{D-1} \int \mathcal{D}X \, \mathcal{D}\bar{X} \, \mathrm{e}^{-\int_0^T dt \, G_{\mu\bar\mu} \, \dot{\bar{X}}^{\bar\mu} \dot{X}^\mu} . \tag{12}$$

To evaluate the determinants, one must first square them, to obtain the positive-definite operator $-\partial_t^2$, which has eigenfunctions $\Psi_n^{(\theta)} \sim \mathrm{e}^{i(2\pi n + \theta)\, t/T}$ and eigenvalues $(2\pi n + \theta)^2/T^2$. The determinant is proportional to the product formula for sine's, and one can fix the proportionality constant using the zeta-function regularized result for the periodic case: $\det' \left(-\partial_t^2\right) = T^2$ [19]. The result is[†]:

$$\det_\theta^2 i\partial_t = -4 \sin^2\left(\theta/2\right) \xrightarrow{\theta \to 0} -\theta^2 . \tag{13}$$

If $D > 1$ the path integral (12) vanishes as we return to periodic boundary conditions. For the one-torus, it is very reasonable to argue that the regularized partition function can be defined as the periodic limit of (12), it being completely independent of the boundary conditions.

At this point we could return to the general problem of the particle on an arbitrary manifold. However, the full evaluation of the partition function on the torus is illuminating in its own right, so we shall first finish this calculation. The path integral over X and $\bar{X}$ is standard [19]. The usual constant and non-zero modes give a factor of $VT/\pi \, \det'\left(-\partial_t^2\right) = V/T$. All the interesting physics comes from the zero modes of X. If one considers a target-space torus with complex structure σ, these are given by all possible windings of the world-line of the particle around the torus:

$$X_{n,m} = (n + m\sigma)\frac{t}{T} . \tag{14}$$

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[*]We set $\hbar \to 1$ from now on.

[†]Particle theories can have global anomalies. For example, the "$N = 1/2$" particle is anomalous unless the target space is a spin manifold [20,21]. The fact that $\det_\theta i\partial_t$ is periodic only up to a sign is an indication of such an anomaly in $U(1)$–invariant quantum mechanics theories [22]. In our case there is never any anomaly: the initial sign of the path integral is ambiguous at the starting point in field space, but the sign can then be uniquely fixed over the entire space by considering the case when the two θ_i's are equal—there then being an even number of determinants.

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(This can be compared to the case of the A model on the torus, for which the interesting modes are the instantons that exist when the world-sheet and target-space tori have equivalent complex structures [4].) One thus has

$$
\begin{aligned}
\mathcal{F} &= \frac{1}{2} \int_0^\infty \frac{\mathrm{d}T}{T^{2-s}} \frac{V}{\pi} \sum_{n,m} \mathrm{e}^{-V \frac{|n+m\sigma|^2}{\sigma_2 T}} \\
&= \frac{1}{2} \left(\frac{V}{\pi} \right)^s \int_0^\infty \mathrm{d}\tilde{T} \, \tilde{T}^{-s} \sum_{n,m} \mathrm{e}^{-\pi \tilde{T} \frac{|n+m\sigma|^2}{\sigma_2}} .
\end{aligned}
\tag{15}
$$

To evaluate the integral, one first splits it into two regions: $0 \leq \tilde{T} \leq 1$, and $\tilde{T} \geq 1$. Because of the analytic continuation, the large $\tilde{T}$ integral is finite as $s \to 0$. To examine the behaviour of the integral for small $\tilde{T}$, note that the sum has the form of a heat kernel. (The reason for this will be clear from the hamiltonian quantization in section 5). At small times the heat kernel behaves as $1/\tilde{T}$ (see eq. (26)), resulting in a $-1/s$ pole from the integral. Aside from this pole, the integral is well behaved as $s \to 0$. Now, one can formally interchange the order of the summation and integration to get

$$
\mathcal{F} = \frac{1}{2} \left(\frac{V}{\pi} \right)^s \Gamma(1-s) \sum_{n,m}{}' \left(\frac{\sigma_2}{\pi |n+m\sigma|^2} \right)^{1-s} .
\tag{16}
$$

Then

$$
\partial_\sigma \mathcal{F} = \frac{1}{4\pi i} \left(\frac{V}{\pi} \right)^s \Gamma(2-s) \sum_{n,m}{}' \left(\frac{\sigma_2}{\pi |n+m\sigma|^2} \right)^{-s} \frac{1}{(n+m\sigma)^2} .
\tag{17}
$$

These expressions have several interesting features, which can be understood from basic principles, and which in fact allow one to evaluate $\partial_\sigma \mathcal{F}$ without having to do any calculation. First, while $\mathcal{F}$ has a residual dependence on V, because of the pole as $s \to 0$, $\partial_\sigma \mathcal{F}$ depends only on the complex structure. $\mathcal{F}$ is explicitly modular invariant, as it should be. This means that $\partial_\sigma \mathcal{F}$ transforms under modular transformations with weight 2 (in our normalization). $\mathcal{F}$ diverges as $\sigma_2 \to \infty$—this is the mirror of the large volume behaviour [4] in the A model—but the divergence is soft enough to make $\partial_\sigma \mathcal{F}$ finite in this limit. Finally, taking the limit $s \to 0$ naively in (17), one sees that $\partial_\sigma \mathcal{F}$ appears to be holomorphic. In fact these features are mutually incompatible, since they would mean that $\partial_\sigma \mathcal{F}$ would be a holomorphic modular form of weight 2, and no such object exists [23]! Clearly, because of the BRST anomaly in the theory [4], what must give way is the holomorphicity of $\partial_\sigma \mathcal{F}$. In fact, the regularized summation in (17) gives the nonholomorphic quantity [23]

$$
\hat{G}_2(\sigma) = -4\pi i \, \partial_\sigma \log \eta(\sigma) - \frac{\pi}{\sigma_2} .
\tag{18}
$$

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Integrating (17), recalling the pole term in (15), and using the symmetry between σ and $\bar{\sigma}$ in (15), we find[‡]

$$\mathcal{F} = -\frac{1}{2} \log \left(V \sigma_2 \, \eta^2(\sigma) \, \eta^2(\bar{\sigma}) \right) \ , \tag{19}$$

up to an infinite additive constant. The complex structure dependence of (19) agrees with BCOV [4].

We have seen that on the one-torus the anomalous dependence of $\partial_\sigma \mathcal{F}$ on $\bar{\sigma}$ can be traced to a conflict between modular covariance and holomorphicity, and that $\partial_\sigma \mathcal{F}$ can be determined uniquely in this case using only general properties. It would be interesting if this interpretation of the holomorphic anomaly could be generalized to other target spaces, but the theory of modular forms on Calabi–Yau spaces is apparently undeveloped.

5 Hamiltonian quantization and the Ray–Singer torsion

After this long digression, we can return to the problem of finding $\mathcal{F}$ on a general target space. We have argued that one can regulate the infinity coming from the volume of the space of constant supersymmetries by examining the theory with twisted boundary conditions. The *einbein* and the gravitini can then be completely gauged away, reducing the problem to that of calculating the partition function of the ungauged particle (3) on a circle with period T; the only remnant of the supergravity fields being that one should integrate over T with the measure (10), and that one should insert the regulated Jacobian coming from (11) and (13). The partition function on the circle can then be calculated in the hamiltonian formalism, giving

$$\mathcal{F} = -\frac{1}{2} \lim_{\theta_i \to 0} \int_0^\infty \frac{\mathrm{d}T}{T^{1-s}} \frac{1}{\theta_1 \theta_2} \operatorname{Tr} (-1)^F e^{i\theta_1 F_1} e^{i\theta_2 F_2} e^{-HT} \ . \tag{20}$$

Here H is the hamiltonian corresponding to eq. (3), and the twisted boundary conditions on the spinors χ_i^μ are implemented by the insertion of $e^{i\theta_i F_i}$, F_i being the appropriate fermion number operator. Expanding in the θ's, in order to carry out the limit, the leading term proportional to the Euler number is regulated to zero. The $1/\theta_i$ terms vanish by CPT, so one is left with our final expression:

$$\mathcal{F} = \frac{1}{2} \int_0^\infty \frac{\mathrm{d}T}{T^{1-s}} \operatorname{Tr} (-1)^F F_1 F_2 \, e^{-HT} \ . \tag{21}$$

[‡]In their evaluation of threshold corrections in string theories, Dixon, Kaplunovsky and Louis essentially calculated the one-loop partition function $\mathcal{F}_1$ of the $N = 2$ string on a one-torus [3]. Their $\mathcal{F}_1$ is symmetric between the Kähler and complex structure—a reflection of mirror symmetry—and can be regarded as the full partition function of both the A and the B sigma models. Eq. (15) corresponds to the case of their "degenerate maps".

This expression is clearly very similar to the index $\mathcal{F}_1$ of (5), which was postulated by BCOV to be the one-loop partition function of the B sigma model coupled to topological gravity in [4]. However, it should be borne in mind that $\mathcal{F}_1$ is defined over the Hilbert space of the string, whereas $\mathcal{F}$ is defined over the much smaller Hilbert space of the particle. In [7] $\mathcal{F}_1$ was evaluated in the B model by considering the large volume limit, and was seen to be related to the Ray–Singer torsion of the manifold. In our case, we can evaluate $\mathcal{F}$ directly.

There are several issues that need to be addressed in the hamiltonian quantization of the theory. The first is the choice of canonical variables: following Witten [1], we would like Q^1, "the BRST operator" of the theory, to act as the Dolbeault operator. In order to do this, it turns out to be necessary to drop the manifest $U(2)$ symmetry of the action, and to take as the canonically conjugate variables:

$$
\begin{array}{ccc}
X^\mu & \leftrightarrow & P_\mu \\
\bar{X}^{\bar{\mu}} & \leftrightarrow & \bar{P}_{\bar{\mu}} \\
\rho_{\bar{\mu}} \equiv \chi_{1\bar{\mu}} & \leftrightarrow & \eta^{\bar{\mu}} \equiv \chi^{*1\bar{\mu}} \\
\tilde{\rho}^\mu \equiv \chi_2^\mu & \leftrightarrow & \theta_\mu \equiv \chi_\mu^{*2} \; .
\end{array}
$$

Next one has to fix some operator orderings: The supersymmetry charges Q^i and $\bar{Q}_i$ and the hamiltonian H are determined classically by varying the gauged action (6) with respect to the supergravity fields. In order for the $(2,2)$ supersymmetry algebra to close, these operators must be ordered as

$$
\begin{aligned}
Q^1 &= \eta^{\bar{\mu}} \bar{P}_{\bar{\mu}} & \bar{Q}_1 &= G^{\mu\bar{\mu}} \rho_{\bar{\mu}} \left(P_\mu + i\, \Gamma_{\mu\rho}^\sigma\, \tilde{\rho}^\rho \theta_\sigma \right) \\
Q^2 &= G^{\mu\bar{\mu}} \theta_\mu \left(\bar{P}_{\bar{\mu}} + i\, \Gamma_{\bar{\mu}\bar{\rho}}^{\bar{\sigma}} \eta^{\bar{\rho}} \rho_{\bar{\sigma}} \right) & \bar{Q}_2 &= \tilde{\rho}^\mu P_\mu \; ,
\end{aligned}
\tag{22}
$$

with

$$
H = \left\{ Q^1, \bar{Q}_1 \right\} = \left\{ Q^2, \bar{Q}_2 \right\} \; .
\tag{23}
$$

This ordering leads us to a different interpretation for the Hilbert space of the particle than that of Witten's for the B model [1]. He took $\eta^{\bar{\mu}}$ and θ_μ to be creation operators, so that in the large-volume limit, the states of the theory were equivalent to the space of $(0,q)$ forms in $\wedge^p T(M)$, or simply the antisymmetric tensors $A_{\bar{\mu}_1 \cdots \bar{\mu}_q}^{\mu_1 \cdots \mu_p}$. Here we see that for Q^1 to represent $-i\bar{\partial}$, $\eta^{\bar{\mu}}$ should indeed be a creation operator. However, for $\bar{Q}_1$ to then be the geometrical operator $-i\bar{\partial}^\dagger$, $\tilde{\rho}^\mu$ and not θ_μ must be the other creation operator! (The theory is then symmetrical under combined complex conjugation and multiplication by σ_1.) This means that *the B model is a theory with operators acting on the (p,q) forms of a Kähler manifold*. Of course, on a Calabi–Yau manifold one can always use the holomorphic tensor $\Omega_{\mu_1 \cdots \mu_n}$ to convert between holomorphic vectors and forms.

With this interpretation of the fermionic operators, F_1 and F_2 simply give q and p, respectively[*], and the hamiltonian of the theory becomes $\{-i\bar{\partial}\,,\,-i\bar{\partial}^\dagger\} = -\nabla^2_{p,q}$ —the (negative of the) laplacian acting on the (p,q) forms of the target space. Thus

$$
\begin{aligned}
\mathcal{F} &= \frac{1}{2} \sum_{p,q} (-1)^{p+q}\, p\, q\, \int_0^\infty \frac{\mathrm{d}T}{T^{1-s}}\, \mathrm{Tr}\, \mathrm{e}^{\nabla^2_{p,q}\, T} \\
&\equiv \frac{1}{2} \sum_{p,q} (-1)^{p+q}\, p\, q\, \Gamma(s)\, \zeta\left(s, \nabla^2_{p,q}\right)\,,
\end{aligned}
\tag{24}
$$

where we have introduced the zeta-function of the laplacian $\zeta\left(s, \nabla^2_{p,q}\right)$. If $\nabla^2_{p,q}$ had no zero modes, one would have $\zeta(0, \nabla^2_{p,q}) = 0$, so that $\Gamma(s)\,\zeta(0, \nabla^2_{p,q}) \to \zeta'(0, \nabla^2_{p,q})$. $\mathcal{F}$ would then be precisely the sum, weighted by $(-1)^p\, p$, of the logarithms of the Ray–Singer torsions $\log T_p$ [24]. With zero modes one has extra infinite and anomalous pieces, as we saw in the torus case in (19), but these drop from $\partial_\sigma \mathcal{F}$. (Note that the zeta-function regularization is crucial in the formula (24) for the Ray–Singer torsion. $\mathcal{F}$ is formally the sum of the logarithms of the eigenvalues of H, but this sum is highly divergent. This will also be true for the string. In a field theory approach, dimensional regularization will play the role of the zeta-function regularization.) When the target space is a Calabi–Yau manifold, BCOV argued that their index $\mathcal{F}_1$ is also given by (24) (up to regularization issues) [4]. Having obtained the same answer, we have an *a posteriori* justification of the formal arguments we used in deriving the particle action (6).

Returning again to the one-torus, we can now explicitly evaluate $\mathcal{F}$ in the hamiltonian formalism. The eigenvectors H are

$$
\Psi_{n,m} = \frac{1}{\sqrt{\sigma_2}}\, \mathrm{e}^{\pi \frac{n+m\bar{\sigma}}{\sigma_2} z}\, \mathrm{e}^{-\pi \frac{n+m\sigma}{\sigma_2} \bar{z}}\,,
$$

with eigenvalues

$$
\frac{\pi^2}{V}\, \frac{|n+m\sigma|^2}{\sigma_2}\,.
$$

Substituting these into (24), and noting that p and q range from 0 to 1, one gets

$$
\begin{aligned}
\mathcal{F} &= \frac{1}{2} \int_0^\infty \frac{\mathrm{d}T}{T^{1-s}} \sum_{n,m} \mathrm{e}^{-\frac{\pi^2 T}{V} \frac{|n+m\sigma|^2}{\sigma_2}} \\
&= \frac{1}{2} \left(\frac{V}{\pi}\right)^s \int_0^\infty \frac{\mathrm{d}T}{T}\, T^s \sum_{n,m} \mathrm{e}^{-\pi T \frac{|n+m\sigma|^2}{\sigma_2}}\,,
\end{aligned}
\tag{25}
$$

---

[*]The F's are determined only up to signs and additive normal ordering constants. Since the analogue of $\mathcal{F}$ without both fermion number insertions vanishes, these ambiguities can change $\mathcal{F}$ only by an sign. As we noted previously, the signs of the the fermionic determinants used in the derivation of $\mathcal{F}$ are also ambiguous, and we determine the overall sign of $\mathcal{F}$ by comparison to the lagrangian result (15).

to be compared to our previous expressions in (15). To evaluate (25) one needs the Poisson resummation formula for the heat kernel [24]:

$$\sum_{n,m} e^{-\pi T \frac{|n+m\sigma|^2}{\sigma_2}} = \frac{1}{T} \sum_{n,m} e^{-\frac{\pi}{T} \frac{|n+m\sigma|^2}{\sigma_2}} \ ! \tag{26}$$

(This somewhat surprising formula might have been expected from target-space modular invariance.) As a result, one finds that the integral in (25) is invariant under the interchange $s \leftrightarrow (1-s)$, giving a typical zeta-function identity $\Gamma(s)\zeta(s,\nabla^2_{p,q}) = \Gamma(1-s)\zeta(1-s,\nabla^2_{p,q})$. Using this identity, (25) agrees perfectly with the Lagrangian calculation of (15).

Finally, hamiltonian quantization in the open case leads to the immediate analogue of (24)

$$\mathcal{F} = \frac{1}{2} \sum_q (-1)^q \, q \, \Gamma(s) \, \zeta(s,\nabla^2_{0,q}) \ . \tag{27}$$

This theory becomes much more interesting if one introduces background gauge fields, by inserting Wilson lines in some representation R of the Chan–Paton group into the path integral. The string carries group indices on both of its ends, so string states are in (some subset of) the $R \otimes R$ representation. Particle states are simply in R. In general, the path integral with Wilson lines cannot be easily evaluated. However, when one implements the Wilson loop insertions locally in the particle using boundary fermions [10], a hamiltonian quantization of the theory shows that one simply replaces the laplacian operator by the appropriate covariantized laplacian. It is important that, as was pointed out by Witten for the B string, one can introduce only gauge fields for which the associated field strength has a vanishing $(0,2)$ component [5]. This means that the states of the theory are in a *holomorphic* vector bundle E. The Ray–Singer torsion of the bundle can then be defined, and the partition function is given by $\mathcal{F} = \log T_0(E)$.

6   Deforming the Kähler and complex structure of the manifold

So far, we have considered the particle defined on a manifold with a fixed Kähler and complex structure. It is interesting to also consider how the theory can be deformed. In [1] the topological B model was varied using the two-form operators in the BRST cohomology of the model. Later, this was generalized to include "anti-topological" and "mixed" deformations (see [7]). We do not wish to use these arguments, which are based upon how one varies topological or twisted $N = 2$ superconformal theories. Instead, we simply look for all transformations of the theory that preserve the supersymmetry

algebra. Since $(2,2)$ sigma models can only be defined on Kähler spaces, this reduces to the question of how to deform Kähler spaces.

The simplest deformation is to keep the complex structure fixed, and to change the Kähler metric. Such a deformation can be carried out explicitly in the particle. Under an infinitesimal change $g_{\mu\bar{\mu}} \to g_{\mu\bar{\mu}} + h_{\mu\bar{\mu}}$, one sees from (22) that Q^1 and $\bar{Q}_2$ are clearly invariant, whereas $\bar{Q}_1$ and Q^2 change by

$$
\begin{aligned}
\delta\bar{Q}_1 &= \left[\bar{Q}_2\,,\,X\right] \\
\delta Q^2 &= -\left[Q^1\,,\,X\right]\,,
\end{aligned}
\tag{28}
$$

with

$$
X = h^{\mu\bar{\mu}}\rho_{\bar{\mu}}\,\theta_\mu\,.
\tag{29}
$$

In order for the $N = 2$ supersymmetry algebra to be preserved, X must commute with $\bar{Q}_1$ and Q^2. This happens *iff* the 2–form $h_{\mu\bar{\mu}}$ is closed which, of course, is the case for a deformation of the Kähler form. Using (23) and (28), one sees that the hamiltonian is changed by

$$
\delta H = \left\{Q^1\,,\,\left[\bar{Q}_2\,,\,X\right]\right\}\,.
\tag{30}
$$

Note that whereas in the sigma model one has a *complexified* Kähler structure, so that there are two independent variations like (28) and (30), here there is only one such variation.

The only other variation one can make on a Kähler space is to deform its complex structure. This gives a finite-dimensional space of transformations. They are harder to carry out explicitly in the particle, since the complex structure appears only implicitly in the Lagrangian (6) and the supersymmetry operators (22). However, knowing that under changes of complex structure ∂ and $\bar{\partial}$ mix, one is led, in analogy to (28), to try the transformations:

$$
Q^1 \to Q^1 - \left[\bar{Q}_2\,,\,Y\right] \qquad Q^2 \to Q^2 + \left[\bar{Q}_1\,,\,Y\right]\,,
\tag{31}
$$

with the $\bar{Q}$'s unchanged. Because of the equality of the two expressions for H in (23), the same Y must appear in both transformations. One also has the complex conjugate transformations:

$$
\bar{Q}_2 \to \bar{Q}_2 + \left[Q^1\,,\,\bar{Y}\right] \qquad \bar{Q}_1 \to \bar{Q}_1 - \left[Q^2\,,\,\bar{Y}\right]\,,
\tag{32}
$$

with the Q's unchanged. By counting dimensions and the two fermion numbers, one sees that Y and $\bar{Y}$ take the form

$$
\begin{aligned}
Y &= A^{\mu}_{\bar{\mu}}\,\eta^{\bar{\mu}}\theta_\mu \\
\bar{Y} &= \bar{A}^{\bar{\mu}}_{\mu}\,\rho_{\bar{\mu}}\tilde{\rho}^\mu\,,
\end{aligned}
\tag{33}
$$

with $A_{\bar\mu}^\mu$ and $\bar A_\mu^{\bar\mu}$ some tensors on the target space.

We shall now concentrate on the $A_{\bar\mu}^\mu$ transformations; the case of the $\bar A_\mu^{\bar\mu}$ transformations follows by complex conjugation. In order for the modified Q^1 to be nilpotent, one sees that, infinitesimally, $\bar\partial A = 0$. This condition is Witten's statement that the Y in (33) should be a zero-form in the BRST cohomology of the theory, which he identified as generating (holomorphic) changes of the complex structure. Under the transformation (31), the hamiltonian changes by

$$H \to H + \left\{ \bar Q_2 , \left[\bar Q_1 , Y \right] \right\} , \tag{34}$$

which is the two-form operator corresponding to Y. In fact, it was noticed in [25] that Q^1 remains nilpotent under *finite** transformations (31), as long as $A_{\bar\mu}^\mu$ satisfies the full Kodaira–Spencer equation for the variation of complex structures: [8]

$$\bar\partial A^\mu + A^\nu \partial_\nu A^\mu = 0 . \tag{35}$$

(This equation has been written thinking of A^μ as a one form. If one writes A as well as a vector field, (35) can be written more geometrically as $\bar\partial A + 1/2\,[A\,,\,A] = 0$.) In view of this, we see that eqs. (31) with (33) do indeed represent a (finite) holomorphic change of the complex structure of the manifold. We have therefore found all the possible deformations of the theory.

It is still necessary to check that the modified theory satisfies the full $(2,2)$ supersymmetry algebra. It is easy to see that all the anticommutators involving $\bar Q_i$'s close, taking into account the fact that the hamiltonian has been modified (34). The remaining conditions are that Q^2 be nilpotent, and that it anticommute with Q^1. Repeatedly using the Jacobi identity, one finds that this occurs if A satisfies the auxiliary condition

$$\mathcal{D}^{[\nu} A_{\bar\mu}^{\mu]} + A_{\bar\rho}^{[\nu} \mathcal{D}^{\bar\rho} A_{\bar\mu}^{\mu]} = 0 ; \tag{36}$$

here the brackets indicate antisymmetrization, and $\mathcal{D}$ is the (raised) covariant derivative on the manifold. The geometrical reason for this condition is that in a Kähler space $\{\bar\partial, \bar\partial^\dagger\} = \{\partial, \partial^\dagger\} = \nabla^2$, so complex-structure deformations of $\bar\partial$ and $\partial^\dagger$ must be related. Eq. (36) states that the deformed $\partial^\dagger$ is nilpotent and anticommutes with $\bar\partial$. There is also clearly a nice symmetry between the Kodaira–Spencer equation and (36). However, we

---

*Note, however, that the transformations (31) and (32) are incompatible. Thus, one can only have a finite holomorphic deformation if one keeps the antiholomorphic part of the complex structure fixed. This agrees with the well-known result that the space of complex-structure deformations is not affine. Using (31) for finite transformations means that we are using "canonical coordinates" [7] on the moduli space of complex structures.

have not encountered this equation in the literature. (Of course it would have arisen in the discussion on the B sigma model in [1, 25] if the extra BRST operator of the theory would have been considered.)

In general, equations like the Kodaira–Spencer equation and (36) are difficult to solve, and it is even difficult to know when they have solutions. We have been able to show that *in the infinitesimal case*, one can use the diffeomorphism invariance of the theory $A \to A + \bar{\partial}\,\xi$ to solve (36) if the manifold is Calabi–Yau or if $H^{(0,2)}$ of the manifold vanishes[†]. However, from our derivation, whenever there is a complex-structure deformation on a Kähler manifold, one should *always* be able to represent it by a solution A of the Kodaira–Spencer equation that also satisfies (36).

7 Independence of the particle on the Kähler structure

Using BRST invariance, Witten argued that the B model should not depend on the Kähler structure of the target space. (This is also essentially true of the Ray–Singer torsion [24].) In [7], BCOV showed (for genus $g > 1$) that this result remains true despite the presence of BRST anomalies[*]. Using the results of the previous section, we can reproduce the argument of BCOV for the simpler case of the particle. First, consider the variation of $\mathcal{F}$ under an infinitesimal change of complex structure (31). Expanding the "0–form" generator Y in (33) as $Y = t^i\,Y_i$, and substituting the variation of the hamiltonian (34) into our expression for $\mathcal{F}$ (21), one gets

$$\partial_{t^i}\mathcal{F} = -\frac{1}{2}\int_0^\infty \mathrm{d}T\,T^s\,\mathrm{Tr}\,(-1)^F\,\bar{Q}_1\,\bar{Q}_2\,Y_i\,\mathrm{e}^{-HT}\ . \tag{37}$$

Unlike $\mathcal{F}$ itself, $\partial_{t^i}\mathcal{F}$ is finite and well-defined, and since the $\bar{Q}$'s come from the G's of the sigma model, it has the traditional form of a topological one-point function on a torus [14].

We would now like to deform the Kähler metric in (37). Recalling from (28) that this is done by changing $\bar{Q}_1$ and H, with $\delta\bar{Q}_1 = \left[\bar{Q}_2\,,X\right]$, one obtains

$$\begin{aligned}
\partial_{t^i}\,\delta\mathcal{F} = -\frac{1}{2}\int_0^\infty \mathrm{d}T\,T^s\ \mathrm{Tr}\,(-1)^F\cdot\ \Big(&\bar{Q}_2\,X\,\bar{Q}_2\,Y_i\,\mathrm{e}^{-HT} \\
&-\int_0^T \mathrm{d}t\,\bar{Q}_1\,\bar{Q}_2\,Y_i\,\mathrm{e}^{-Ht}\left\{Q^1,\left[\bar{Q}_2\,,X\right]\right\}\,\mathrm{e}^{-H(T-t)}\Big)\ .
\end{aligned} \tag{38}$$

[†]We would like to thank Ori Ganor for helpful discussions on this point.

[*]At genus one, the calculation is complicated by the fact that one has to work with derivatives of $\mathcal{F}_1$, rather than with $\mathcal{F}_1$ itself.

Our calculation is now similar to that of the holomorphic anomaly in [12], except that our quantities are nicely regularized. First, we move Q^1 in the second term around the trace. Since Y_i is in the "BRST cohomology", Q^1 moves through everything except for $\bar{Q}_1$, with which it anticommutes to give a factor of H. This gives us

$$
\begin{aligned}
\partial_{t^i}\,\delta\mathcal{F} &= \frac{1}{2}\int_0^\infty \mathrm{d}T\,T^s\,\frac{\mathrm{d}}{\mathrm{d}T}\,\mathrm{Tr}\int_0^T \mathrm{d}t\,(-1)^F \bar{Q}_2\,Y_i\,\mathrm{e}^{-H\,t}\,\bar{Q}_2\,X\,\mathrm{e}^{-H\,(T-t)} \\
&= -\frac{s}{2}\int_0^\infty \frac{\mathrm{d}T}{T^{1-s}}\int_0^T \mathrm{d}t\,\mathrm{Tr}\,(-1)^F \bar{Q}_2\,Y_i\,\mathrm{e}^{-H\,t}\,\bar{Q}_2\,X\,\mathrm{e}^{-H\,(T-t)}\,.
\end{aligned}
\tag{39}
$$

Because of the explicit factor of s, $\partial_{t^i}\,\delta\mathcal{F}$ vanishes unless the T integration gives a pole as $s \to 0$. This will occur *iff* the t integral is finite and nonzero as $T \to 0$ or $T \to \infty$.

As $T \to \infty$, being careful to keep all contributions, the integral tends to

$$
\int_0^{T/2} \mathrm{d}t\,\mathrm{Tr}\,(-1)^F \left(\bar{Q}_2\,Y_i\,\mathrm{e}^{-H\,t}\,\bar{Q}_2\,X P + \bar{Q}_2\,Y_i\,P\,\bar{Q}_2\,X\,\mathrm{e}^{-H\,t} \right)\,,
\tag{40}
$$

P being the projection onto the $H = 0$ sector of the theory. This sector is supersymmetric, so it is annihilated by $\bar{Q}_2$. Cycling P to be next to $\bar{Q}_2$, one sees that (40) vanishes, so there is no contribution to $\partial_{t^i}\,\delta\mathcal{F}$ from the t integral at large T. For small T, it appears to be clear that the t integral vanishes. The only subtlety is that the heat kernel diverges as $1/T^d$ for small times—d being the *complex* dimension of spacetime. (Such divergences gives rise to the contact terms that appear in the string derivation of the anomaly [4, 7].) Using the fact that the product of a local operator times the heat kernel can be written as a Laurent series in T [26], and knowing that $\partial_{t^i}\,\delta\mathcal{F}$ is finite, one can see that the integral indeed vanishes in this limit. (This can also be checked explicitly in the torus case.) Thus $\partial_{t^i}\,\delta\mathcal{F} = 0$.

A similar argument shows that $\partial_{\bar{t}_i}\,\delta\mathcal{F}$ also vanishes. Thus, as we saw in the torus case (19), the partition function of the particle depends on the Kähler structure of the metric only by a trivial additive factor, independent of the complex structure.

8   The holomorphic anomaly

The holomorphic anomaly is derived in a very similar manner. One now wants to substitute the antiholomorphic variation of the complex structure of the $\bar{Q}$'s and of H

coming from eq. (32) into $\partial_{t^i} \mathcal{F}^*$. This gives

$$
\begin{aligned}
\bar{\partial}_{\bar{t}_i} \partial_{t^i} \, \mathcal{F} = -\frac{1}{2} \int_0^\infty \mathrm{d}T \, T^s \; \mathrm{Tr} \, (-1)^F \cdot \bigg(& \\
\left(\bar{Q}_1 \, [\, Q^1 , \, \bar{Y}_{\bar{i}} \,] \, Y_i + [\, \bar{Y}_{\bar{i}} , \, Q^2 \,] \, \bar{Q}_2 \, Y_i \right) \, \mathrm{e}^{-H\,T} & \\
- \int_0^T \mathrm{d}t \, \bar{Q}_1 \, \bar{Q}_2 \, Y_i \, \mathrm{e}^{-H\,t} \left\{ Q^2 , \left[Q^1 , \bar{Y}_{\bar{i}} \right] \right\} \, \mathrm{e}^{-H\,(T-t)} \bigg) \, . &
\end{aligned}
\tag{41}
$$

Again cycling the Q's around the trace and integrating by parts, this simplifies to

$$
\begin{aligned}
\bar{\partial}_{\bar{t}_i} \partial_{t^i} \, \mathcal{F} = -\frac{s}{2} \int_0^\infty \frac{\mathrm{d}T}{T^{1-s}} \cdot & \\
\mathrm{Tr} \, (-1)^F \left(\bar{Y}_{\bar{i}} \, Y_i \, \mathrm{e}^{-H\,t} - \int_0^T \mathrm{d}t \, H \, Y_i \, \mathrm{e}^{-H\,t} \, \bar{Y}_{\bar{i}} \, \mathrm{e}^{-H\,(T-t)} \right) & \\
= -\frac{s\,(1-s)}{2} \int_0^\infty \frac{\mathrm{d}T}{T^{2-s}} \int_0^T \mathrm{d}t \, \mathrm{Tr} \, (-1)^F \, Y_i \, \mathrm{e}^{-H\,t} \, \bar{Y}_{\bar{i}} \, \mathrm{e}^{-H\,(T-t)} \, . &
\end{aligned}
\tag{42}
$$

Again the explicit s factor can be canceled only from a logarithmic divergence in the T integral. This means that the anomaly is given by one-half of the coefficient of the term in the t integral linear in T, evaluated between infinity and zero. (Equivalently, one-half of the constant in T piece of the second line of (42).) As $T \to \infty$, one gets

$$
\bar{\partial}_{\bar{t}_i} \partial_{t^i} \, \mathcal{F} = \frac{1}{2} \, \mathrm{Tr} \, (-1)^F \, Y_i \, P \, \bar{Y}_{\bar{i}} \, P \, ,
\tag{43}
$$

which gives us the easier part of the anomaly.

The contribution to the anomaly from $T \to 0$ is again subtle. In the closed string case, it comes from contact interactions between the analogues of Y_i and $\bar{Y}_{\bar{i}}$. Here, it should come from the small-time expansion of the heat kernel. Using the same sort of argument we had previously, one sees that one can get a finite contribution to the anomaly from the constant terms in the small-time expansions of $Y_i \, \mathrm{e}^{-H\,T}$ and $\bar{Y}_{\bar{i}} \, \mathrm{e}^{-H\,T}$ in (42). Unfortunately, we have not not yet been able to calculate these terms. We do, however, know the answer! As we have noted before, the partition function is the sum, weighted by $(-1)^p \, p$, of the logarithms of the Ray–Singer torsion $\wedge^p T^*$. The holomorphic (Quillen) anomaly of the Ray–Singer torsion of any holomorphic vector bundle V has been calculated [27], giving

$$
\partial \, \bar{\partial} \, \log T(V) = \frac{1}{2} \partial \, \bar{\partial} \sum_q (-1)^q \, d_q + \pi i \int_M \mathrm{Td}(T) \, \mathrm{Ch}(V) \bigg|_{(1,1)} \, .
\tag{44}
$$

---

*In [12] and [4] the terms coming from varying the $\bar{Q}$'s were not considered. As we have argued, they should be important in shifting the "basepoint" of the theory.

(See [7] for an explanation of the symbols.) Therefore the small T behaviour of (42) must give

$$- \pi i \int_M \mathrm{Td}(T) \sum_p (-1)^p p \, \mathrm{Ch} \left(\wedge^p T^* \right) . \qquad (45)$$

9   Conclusions

Witten showed that the partition function of the B model coupled to topological gravity is dominated by world-sheets that collapse to one-dimensional "nets" [5]. Since these nets can be produced by Feynman diagrams, he argued that one should be able to describe the B models by ordinary field theories. It is obvious that on such world sheets the sigma model dimensionally reduces to a one-dimensional sigma model. The difficulty is how to represent the coupling to the topological gravity. Here we have suggested that this coupling should be replaced by a twisted coupling of the sigma model to a one-dimensional supergravity. Thus the topological sigma model is reduced to the supersymmetric twisted spinning particle of eq. (6). By performing a hamiltonian quantization of the theory one can see explicitly—after clearing up a few subtleties—that the one-loop partition function of the spinning particle (21) agrees with that of the B string. Note that because the particle can be defined for any complex structure, the partition function is more like a generating function, since it effectively describes all the amplitudes of the theory at one loop. The particle has the advantage that, at least at one loop, it gives a more general description of the B model than the two-dimensional sigma model approach. This is because there are no local anomalies in particle theories, so the particle can be defined on any Kähler manifold, and not only when the Calabi–Yau condition is satisfied. It is natural to speculate that the particle on a non-Calabi–Yau manifold is the mirror of a nonconformal topological A model.

A somewhat surprising feature that arises from the hamiltonian quantization is that the Hilbert space of the particle is naturally described by (p, q) forms. This is despite the fact that the BRST cohomology of the B string corresponds to $(0, q)$ forms in $\wedge^p T(M)$ [1]. We may note that in particle theories one does not have a one-to-one correspondence between allowed deformations and states in the theory. Thus the deformations of the theory correspond to changes of the complex structure of the Kähler manifold, and are indeed described by the tensors $A^\mu_{\bar\mu}$'s, and their complex conjugates.

While from a physics viewpoint we would be disappointed if the particle could not be generalized to give other amplitudes, from the mathematical point of view it is already interesting that we can write a particle model that can describe Ray–Singer torsion on a

manifold. (One would hope that the anomaly derivation in section 8 can be completed, so that the particle description would already show some practical use at this stage.) This is analogous to the one-dimensional sigma models written to calculate index theorems [21, 28]. The partition function of the B model is written in terms of a particular sum of the Ray–Singer torsion over holomorphic p–forms [7]. One can find interesting alternative theories by gauging various subgroups of the global $U(2)$ symmetry of the particle [15]. In particular, if one gauges the $U(1)$ generated by $(1 - \sigma_3)$, and adds an appropriately normalized Chern–Simons term [22], one can get a description of the Ray–Singer torsion on $\wedge^p T^*$ for any particular p. We have also seen that in the particle description of the open string—obtained by truncating (6) to a $U(1)$ action—one can couple the theory to appropriate background gauge fields [5] to get the Ray–Singer torsion of E, a holomorphic vector bundle in some representation of the gauge group.

We have stressed that the holomorphic anomaly on the one-torus can be seen as arising from a conflict between holomorphicity and modular invariance, and have speculated about a possible generalization of this to other manifolds. We have also derived an auxiliary condition to the Kodaira–Spencer equation, which appears to be necessary for variations of the complex structure of a Kähler manifold. It is not clear to us whether or not this last result is surprising or obvious.

10   Comments on possible field theories

So far, all our discussion has been of the particle at one loop. In that case the only remnants of the gravitini are the insertions of the fermion number operators F_i in the partition function (21). Therefore one might suspect that the success of the particle in reproducing the one-loop partition function does not necessarily imply that the particle action (6) is correct. One indication that we have the right coupling to gravity is that at higher loop amplitudes one would expect "zero-modes" of the gravitini to give rise to insertions of the local supersymmetry currents, as occurs in two-dimensional theories coupled to topological gravity [14]. These currents in the particle are those appropriate to the B model.

If one wants to proceed to arbitrary amplitudes, one will have to do one of two things: either to calculate the path integral of the particle on nets, or to write the relevant field theory. At least in principle it is easy to calculate string amplitudes on complicated Riemann surfaces. In particle theories one has the fundamental problem of having to introduce interactions at the vertices. In our case we do have a very natural geometrical candidate for such an n–point vertex: one simply takes the forms corresponding to the

states of the n particles, and integrates their product over the manifold. In general, one might expect that only the 3-point vertex will be needed in the theory, since the theory is at least somewhat topological (and since the Kodaira–Spencer equation is quadratic). However, it is not obvious that this will be the case, and the only check will be to see whether the interacting particle can generate the genus g holomorphic anomaly of BCOV [7].

In general, it is not easy to find a string field theory from a string. One first needs to know the space of the string fields, then the theory's kinetic operator and finally its interactions. Normally one knows that the string field should describe the space of states of the first quantized gauge-fixed string, including its ghosts. The linearized equations of motion and gauge invariances of the field theory then come from the BRST operator of the string. Although such an approach was very successful in describing the open string [6], there can be complications to this method, as witnessed in the difficulties in the construction of the closed-string field theory [29]. An alternative approach is to note that the beta-functions of the string coupled to background fields should be the low-energy equations of motion of the field theory. In the more topological theories, such as the various $N = 2$ strings [30] and the B strings [5, 7], one may hope that these equations, which for some reason are always quadratic, could be exact.

In the case of the particle, things are even more difficult. As we have already stated, the first problem is that the above two approaches do not match. The Hilbert space of the particle is the space of forms, and the constraints from varying the supergravity fields in (6) are that ∂, $\bar{\partial}^\dagger$ and ∇^2 all vanish on these forms. This means that the BRST cohomology in the Hilbert space of the particle should be equivalent to the de Rham cohomology of the target space, although things might be more complicated because of the commuting supersymmetry ghost system. On the other hand, the deformations of the theory are described by the fields $A_{\bar{\mu}}^{\mu}$ satisfying the Kodaira–Spencer equation (35) and our auxiliary equation (36). These spaces are only compatible on Calabi–Yau manifolds. An alternative statement of the problem is that in string theories the legitimate conformally invariant vertex operators are (generally) equivalent to the states in the BRST cohomology. In particle theories, there is no constraint from conformal invariance, and one has a free choice of vertex operators [17].

With all these caveats, the most obvious possibility for a field theory of the B string is still to take the $A_{\bar{\mu}}^{\mu}$'s as the basic fields of the theory, and to choose an action whose equation of motion gives the Kodaira–Spencer equation. This gives the "Kodaira–Spencer field theory" of BCOV [7]. However, having the correct classical equations of motion may not be enough to fix the full field theory, and one must be very careful to know that one

is working on the correct Hilbert space. As an example of this, the $N = 2$ closed string has an analogous "Plebanski action" [30]. However, there appears to be a discrepancy between the one-loop three-point function calculated from the string, and that calculated from the field theory [31]. The fact that the Kodaira–Spencer action in [7] is nonlocal may be a warning sign that one does not yet have the correct Hilbert space. The final check will again be whether or not the field theory can reproduce the genus g anomaly equations of the B model.

Acknowledgments

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