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Bunched beam transfer matrices in single and double RF systems

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Abstract

Longitudinal bunched beam transfer functions are considered for the interaction of the beam with an arbitrary accelerating RF system. The matrices which connect amplitude and phase of beam current modulations with amplitude and phase modulation of the RF voltage are obtained for a symmetric potential well. The case of long bunches in a non-accelerating bucket and the case of short bunches in an accelerating bucket, both for single and dual (bunch-lengthening mode) RF systems are analysed in more detail.

An exact analytical solution is found for nonlinear motion in bunch-lengthening mode as created by a double RF system with frequency ratio equal 2 (storage regime).

It is shown that Landau damping is lost when the synchrotron frequency as a function of oscillation amplitude has a flat region (a point with zero derivative) inside the bunch but outside the bunch centre (as can be obtained in a double RF system).

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Contents

1 Introduction.

A general consideration of the beam-cavity interaction can be split into two main parts:

-reaction of the cavity to the perturbations on the beam, which can be described in many cases using the notion of impedance function and

-response of the beam to the perturbation of the accelerating voltage, which can be expressed in terms of the beam transfer functions.

Analysis of the stability of the system based on an equivalent circuit model or the Vlasov equation includes both these parts and leads to the dispersion equation [1]-[3].

When the RF system becomes more complex (multi-harmonics cavities, different loops, feedback systems) it is convenient to consider the response of the beam separately and then use it in the more complete model of the beam-cavity interaction [4]. One of the possible ways to analyse the stability of the beam-cavity system is to use matrix formalism. Then the elements of the beam transfer matrix (BTM) give the amplitude and phase modulation of harmonics of the beam current as a response to the amplitude and phase modulation of the external voltage.

Below we present the longitudinal response of the bunched beam to the voltage modulation in single and double RF systems using beam transfer matrices. We shall consider short bunches in the accelerating regime and long bunches in the storage regime. These cases, which correspond to the symmetric potential well, allow a simplification of the general expressions, so that only even or odd modes of bunch oscillations give contributions to the matrix elements. No intensity effects are included.

Bunched-beam transfer functions for single RF systems were discussed in [5]-[14]. Measurements of amplitude and phase of the beam transfer functions at low and high intensities can give information about the incoherent frequency spread within the bunch, the coherent frequency shift and the coupling impedance of the machine.

The longitudinal stability of the system in the presence of higher RF harmonics was considered in [15]-[17]. Double RF systems operating in bunch-lengthening mode (blm) [18] have recently been used in many accelerators to improve beam stability by increasing the synchrotron frequency spread or by producing flat bunches. The analytical solution, found in [18], for nonlinear motion in a quartic potential well is a good approximation for short bunches. In the present paper an exact analytical solution is found for nonlinear motion in a double RF system (blm) with frequency ratio equal to 2 (storage regime). This solution describes the nonmonotonic dependence of the frequency of synchrotron oscillations on their amplitude which is important for longer bunches. Indeed, it was found in [16] that in a double RF system, for sufficiently long bunches, the threshold of the coupled bunch mode instability equals zero. Below we consider the nature of this phenomenon.

2 Main equations and definitions

The longitudinal motion of particles can be described by the equations:

$$\frac{d\phi}{dt} = \frac{h\omega_0\eta}{\beta^2 E_s}u, \tag{1}$$

$$\frac{du}{dt} = \frac{q\omega_0}{2\pi}[V(\phi) - V_s], \tag{2}$$

where

q is the electrical charge of the particle,

$f_0 = \omega_0/(2\pi)$ is the revolution frequency,

$\eta = 1/\gamma_t^2 - 1/\gamma^2$,

$\phi = \varphi - \phi_s$ is the phase deviation of the particle with respect to the synchronous phase ϕ_s of the main RF system with harmonic number h;

$u = E - E_s$, E is the energy of the particle, V_s is the voltage seen by the synchronous particle with energy E_s.

In the general case with more than one RF system, the resulting voltage which affects the particles can be written in the form:

$$V(\phi) = \sum_{n=1}^{N} V_n \sin(g_n\phi + g_n\phi_n). \tag{3}$$

Here $g_n = h_n/h$, $V_1\sin(\phi + \phi_1)$ represents the main RF voltage, $\phi_1 \equiv \phi_s$ is its stable phase, $V_s \equiv V(0)$, $h_1 \equiv h$, V_n and $h_n f_0$ are the voltage and frequency applied to the n-th RF system, ϕ_n is stable phase relative to the h_n-th harmonic waveform, N being the number of RF systems involved. We suppose here that harmonics of the higher frequency RF systems are not necessary an integer multiple of the fundamental.

The Hamiltonian of the system has the form:

$$H = \frac{h\omega_0\eta}{2\beta^2 E_s}u^2 - \frac{q\omega_0}{2\pi}\int_0^{\phi}[V(\phi') - V_s]\,d\phi'. \tag{4}$$

Now let us consider the system with perturbations so that the voltage affecting the particle is $V(\phi) + \tilde{V}(\phi, t)$. Then the equation of synchrotron motion is:

$$\ddot{\phi} + \frac{\omega_{s0}^2}{V_1\cos\phi_s}[V(\phi) - V_s] = -\frac{\omega_{s0}^2}{V_1\cos\phi_s}\tilde{V}(\phi), \tag{5}$$

where ω_{s0} is the synchrotron frequency of small oscillations in a single RF system with peak voltage V_1 and synchronous phase ϕ_s:

$$\omega_{s0}^2 = -\frac{h\omega_0^2\eta q V_1\cos\phi_s}{2\pi\beta^2 E_s}. \tag{6}$$

Note that the system of differential equations (1), (2) which supposes that the RF voltage is distributed uniformly over the ring, can be used only for the analysis of processes slow compared with the revolution period. It means that the frequency of modulation Ω should satisfy the condition $\Omega \ll \omega_0$.

In a co-rotating coordinate system, ϕ corresponds to the azimuth of the particle measured in RF radian units of the main RF system, from the position of the synchronous particle, $\phi = h(\omega_0 t - \theta)$. Here θ is angular azimuth around the ring in the laboratory frame.

Quantities varying in azimuthal coordinate and time can be presented in a circular machine as wave-like states, so that the perturbation of the voltage in our model can be written in the form

$$\tilde{V}(\phi, t) = \sum_{k=-\infty}^{\infty} \int_{-\infty-i\sigma}^{\infty-i\sigma} \tilde{V}_k(\omega) e^{-i\frac{k}{h}\phi + i\omega t} d\omega, \tag{7}$$

where

$$\tilde{V}_k(\omega) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_0^{\infty} \tilde{V}(\phi, t) e^{i\frac{k}{h}\phi - i\omega t} d\phi \, dt \tag{8}$$

Above we used the Fourier transformation in the coordinate ϕ and the Laplace (or one-sided Fourier) transformation in the time domain. Initial conditions (the perturbation is absent for $t < 0$) are satisfied for vanishingly small $\sigma > 0$.

The deviation of the perturbed distribution function $\mathcal{F}(u, \phi, t)$ from its equilibrium

$$f(u, \phi, t) = \mathcal{F}(u, \phi, t) - \mathcal{F}_0(u, \phi) \tag{9}$$

can be connected with the perturbation in voltage through the Vlasov equation

$$\frac{df}{dt} = -\dot{u}\frac{\partial \mathcal{F}_0}{\partial u}, \tag{10}$$

where full time derivative should be taken along unperturbed phase trajectories (see, for example [8]). The solution of equation (10) can be written in the form

$$f(u, \phi, t) = -\frac{q\omega_0}{2\pi} \int_0^t \frac{\partial \mathcal{F}_0}{\partial u} \tilde{V}(\phi, t') dt', \tag{11}$$

where the coordinate of the particle ϕ, at the moment t, is defined by the coordinate ϕ' at the moment t', $\phi = \Phi(\phi', t - t')$, using the equations of motion.

Integration of (11) over u gives the beam current. Expansion of the beam current in azimuthal harmonics $j_p(t)$ allows the beam response to the perturbation of the voltage to be represented as

$$j_p(t) = \sum_{k=-\infty}^{\infty} \int_0^t G_{pk}(t - t') \tilde{V}_k(t') dt'. \tag{12}$$

Using the convolution theorem we can transform expression (12) to the frequency domain:

$$j_p(\omega) = \sum_{k=-\infty}^{\infty} G_{pk}(\omega) \tilde{V}_k(\omega), \tag{13}$$

where $G_{pk}(\omega)$ is the beam transfer function to be defined below.

3 Beam transfer functions for an arbitrary RF voltage

In this section to obtain the transfer function of the beam for an arbitrary RF voltage we shall follow the approach developed in Ref.[2].

Let us introduce new variables, r and ψ, which correspond respectively to the amplitude and phase of the synchrotron oscillations. For the cases of single and double RF systems considered later it is convenient to present them in the form

$$r = \frac{1}{\sqrt{2}}[\frac{\dot{\phi}^2}{2\omega_{s0}{}^2} + W(\phi, \phi_s)]^{1/2}, \tag{14}$$

$$\psi = sgn(\eta u)\frac{\omega_s(r)}{2\omega_{s0}}\int_{\phi_{max}}^{\phi}\frac{d\phi'}{\sqrt{r^2 - W(\phi', \phi_s)}} \tag{15}$$

with $\phi_{max} = \phi_{max}(r)$, where

$$W(\phi, \phi_s) = \frac{1}{V_1 \cos\phi_s}\int_0^{\phi}[V(\phi') - V_s]\, d\phi' \tag{16}$$

and $\omega_s(r)$ is the frequency of the nonlinear synchrotron oscillations in the system.

In the absence of perturbations the equations of motion (1)-(2) in the new variables have the form:

$$\dot{r} = 0, \qquad \dot{\psi} = \omega_s(r), \tag{17}$$

and the distribution function in longitudinal phase space is a function only of the Hamiltonian of the system H or variable r, $\mathcal{F} = \mathcal{F}_0(r)$.

With a perturbation of the system the first of the equations (17) becomes

$$\dot{r} = -\frac{\dot{\phi}\tilde{V}(\phi, t)}{4rV_1\cos\phi_s}. \tag{18}$$

In these variables the Vlasov equation can be written in the form

$$\frac{\partial f}{\partial t} + \dot{r}\frac{\partial \mathcal{F}_0}{\partial r} + \dot{\psi}\frac{\partial f}{\partial \psi} = 0. \tag{19}$$

If we take into account that the solution of equation (19) for the Fourier harmonic

$$f(r, \psi, \omega) = \frac{1}{2\pi}\int_0^{\infty} f(r, \psi, t)e^{-i\omega t}\, dt \tag{20}$$

should be periodic in ψ:

$$f(r, \psi, \omega) = \sum_{m=-\infty}^{\infty} f_m(r, \omega)e^{-im\psi}, \tag{21}$$

where

$$f_m(r, \omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(r, \psi, \omega)e^{im\psi}d\psi, \tag{22}$$

then we have

$$f(r,\psi,\omega) = i \sum_{m=-\infty}^{\infty} \frac{\dot{r}_m \frac{d\mathcal{F}_0}{dr} e^{-im\psi}}{\omega - m\omega_s(r)}. \tag{23}$$

Here we used the notation

$$\dot{r}_m = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_0^{\infty} \dot{r}(\phi,t) e^{im\psi - i\omega t} d\psi\, dt. \tag{24}$$

Then after substitution of expression (18) for the function $\dot{r}(\phi,t)$, we find

$$\dot{r}_m = -\frac{1}{8\pi r V_1 \cos\phi_s} \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} \dot{\phi} \tilde{V}_k(\omega) e^{im\psi - i\frac{k}{h}\phi} d\psi. \tag{25}$$

Integration of (25) by parts gives

$$\dot{r}_m = -\sum_{k=-\infty}^{\infty} \frac{\omega_s(r)\tilde{V}_k(\omega) m h}{4 r V_1 k \cos\phi_s} I^*_{mk}. \tag{26}$$

Here $k \neq 0$.

The function I_{mk} defined as

$$I_{mk}(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\frac{k}{h}\phi(r,\psi) - im\psi} d\psi. \tag{27}$$

was introduced for the first time in [2]. As can be seen it has the following properties which we shall use later:

$$I_{-mk} = I_{mk}, \qquad I_{m-k} = I^*_{mk}. \tag{28}$$

For a symmetric potential well, so that $\phi(\pi - \psi) = -\phi(\psi)$, we also have

$$I^*_{mk} = (-1)^m I_{mk}. \tag{29}$$

Now using (26) we can write for a perturbation in the distribution function (23)

$$f(r,\psi,\omega) = -i \frac{\omega_s(r) h}{4 r V_1 \cos\phi_s} \frac{d\mathcal{F}_0}{dr} \sum_{k=-\infty}^{\infty} \frac{\tilde{V}_k(\omega)}{k} \sum_{m=-\infty}^{\infty} \frac{m I^*_{mk} e^{-im\psi}}{\omega - m\omega_s(r)}. \tag{30}$$

Deviations of the distribution function and beam current from their equilibriums are connected by the relation

$$j(\phi,\omega) = \frac{N_e q c}{8\pi R S \omega_{s0}^2} \int_{-\infty}^{\infty} f(r,\psi,\omega) d\dot{\phi}, \tag{31}$$

where N_e is the total number of particles and R is the average radius of the machine.

The normalisation factor S is defined as

$$S = \int_0^{r_{max}} \frac{\mathcal{F}_0(r) r dr}{\omega_s(r)}. \tag{32}$$

Here we used the fact that $d\phi d\dot{\phi} = [4\omega_{s0}^2/\omega_s(r)] r dr d\psi$.

Expansion of the beam current perturbation in azimuthal harmonics gives

$$j(\phi,\omega) = \sum_{p=-\infty}^{\infty} j_p(\omega)e^{-i\frac{p}{h}\phi}. \tag{33}$$

Then for the p-th azimuthal harmonic of the perturbation in beam current we get

$$j_p(\omega) = \frac{I_0}{2\pi S}\int_0^{r_{max}}\int_{-\pi}^{\pi}\frac{f(r,\psi,\omega)e^{i\frac{p}{h}\phi}}{\omega_s(r)}rdrd\psi, \tag{34}$$

where $I_0 = N_e qc/(2\pi R)$ is the average beam current in the machine.

After substitution of the expression (30) into (34) we have as a result [2]:

$$j_p(\omega) = \sum_{k=-\infty}^{\infty} G_{pk}(\omega)\tilde{V}_k(\omega). \tag{35}$$

The elements of this beam transfer matrices are defined as

$$G_{pk}(\omega) = -i\frac{I_0}{4S}\frac{h}{kV_1\cos\phi_s}\sum_{m=-\infty}^{\infty} m\int_0^{r_{max}}\frac{d\mathcal{F}_0}{dr}\frac{I^*_{mk}(r)I_{mp}(r)dr}{\omega - i\sigma - m\omega_s(r)}, \tag{36}$$

where the integration contour in (36) is chosen to satisfy the initial conditions.

Using one of the properties (28) of the function I_{mk}, we can also transform the expression above to another form:

$$G_{pk}(\omega) = -i\frac{I_0}{2S}\frac{h}{kV_1\cos\phi_s}\sum_{m=1}^{\infty} m^2\int_0^{r_{max}}\frac{d\mathcal{F}_0}{dr}\frac{I^*_{mk}(r)I_{mp}(r)\omega_s(r)dr}{(\omega - i\sigma)^2 - m^2\omega_s^2(r)}. \tag{37}$$

This expression for the beam transfer function obtained for a single bunch can be generalised to the system of M bunches (see [13]), so that an additional summation over all bunches appears. However, for the case of equidistant and identical bunches, we can use for the BTF the same expression with M times higher intensity, also remembering that $G_{pk}(\omega) \neq 0$ only if $p = k + nM$, where $n = 0, 1...$

For a single bunch or nonidentical bunches voltage modulation leads to perturbations of the beam current at all multiples of the revolution frequency. Below we consider the beam response at the azimuthal harmonics which correspond to the frequencies of RF systems involved in acceleration. These harmonics are important for the analysis of the beam-loading problems. We shall use the general expression for the BTF to obtain the dependence of phase and amplitude of the beam current modulation on the voltage amplitude and phase modulation. Functions $I_{mk}(r)$ are defined by the type of potential well considered and should be calculated for particular cases.

4 Beam current modulation

The beam current distribution in the system without perturbations

$$I(\phi) = \frac{I_0}{4S\omega_{s0}^2}\int_{-\infty}^{\infty}\mathcal{F}_0(r)d\dot{\phi} \tag{38}$$

can be expanded in a series of azimuthal harmonics

$$I(\phi) = \sum_{p=-\infty}^{\infty} I_p e^{-i\frac{p}{h}\phi}, \tag{39}$$

where the amplitude of the p-th azimuthal harmonic of the unperturbed beam current is defined by the expression

$$I_p = \frac{I_0}{2\pi S}\int_0^{r_{max}}\int_{-\pi}^{\pi}\frac{\mathcal{F}_0(r)e^{i\frac{p}{h}\phi(r,\psi)}rdrd\psi}{\omega_s(r)}. \tag{40}$$

For symmetric potential wells, $I_p = I_{-p}$ and $\mathrm{Im} I_p = 0$. Then with modulation of the voltage, phase or amplitude, applied, the beam current becomes

$$I(\phi) + j(\phi, t) = \sum_{p=-\infty}^{\infty}[I_p + j_p(t)]e^{-i\frac{p}{h}\phi} = \sum_{p=-\infty}^{\infty}\sqrt{[I_p + \Re j_p(t)]^2 + [\Im j_p(t)]^2}e^{-i(\frac{p}{h}\phi-\gamma_p)}, \tag{41}$$

where

$$\tan\gamma_p = \frac{\Im j_p}{I_p + \Re j_p}. \tag{42}$$

The response of the beam to a modulation in amplitude or phase of the voltage can be described as a change in amplitude and phase of the azimuthal harmonics of the beam current:

$$I(\phi) + j(\phi, t) = \sum_{p=-\infty}^{\infty}[I_p + \Delta I_p(t)]e^{-i\frac{p}{h}(\Phi-\phi_s-\Delta\phi_p^b)}, \tag{43}$$

where for $\mid j_p \mid \ll I_p$ from (41) and (42) we get

$$\Delta I_p(t) = \Re j_p(t), \qquad \Delta\phi_p^b(t) = \frac{h}{p}\frac{\Im j_p(t)}{I_p} \tag{44}$$

in the time domain and

$$\Delta I_p(\omega) = \frac{1}{2}[j_p(\omega) + j_{-p}(\omega)], \qquad \Delta\phi_p^b(\omega) = \frac{h}{p}\frac{[j_p(\omega) - j_{-p}(\omega)]}{2iI_p} \tag{45}$$

in the frequency domain.

To find the amplitude and phase modulation of the beam current let us consider the sum over k in expression (37). Using the properties (28) of the function I_{mk} and the fact that for the symmetric potential well $I_{mk}^* = (-1)^m I_{mk}$, we get

$$\sum_{k=-\infty}^{\infty}\frac{\tilde{V}_k(\omega)I_{mk}^*}{k} = \sum_{k=1}^{\infty}\frac{I_{mk}[\tilde{V}_k(-1)^m - \tilde{V}_{-k}]}{k}. \tag{4}$$

For the expression in square brackets we have

$$[\tilde{V}_k(-1)^m - \tilde{V}_{-k}] = \begin{cases} -(\tilde{V}_k + \tilde{V}_{-k}), & \text{if m - odd} \\ (\tilde{V}_k - \tilde{V}_{-k}), & \text{if m - even.} \end{cases} \tag{47}$$

Now we can rewrite (35) in the form

$$j_p(\omega) = -i\frac{I_0}{V_1\cos\phi_s}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} M_m^{pk}(\omega)\begin{cases} -(\tilde{V}_k + \tilde{V}_{-k}), & \text{if m - odd} \\ (\tilde{V}_k - \tilde{V}_{-k}), & \text{if m - even} \end{cases} \tag{48}$$

where

$$M_m^{pk}(\omega) = \frac{m^2 h}{2Sk}\int_0^{r_{max}} \frac{d\mathcal{F}_0}{dr}\frac{I_{mp}(r)I_{mk}(r)\omega_s(r)dr}{(\omega - i\sigma)^2 - m^2\omega_s^2(r)}. \tag{49}$$

Taking into account the fact that

$$M_m^{-pk}(\omega) = (-1)^m M_m^{pk}(\omega) \tag{50}$$

we obtain expressions

$$j_p(\omega) + j_{-p}(\omega) = -i\frac{I_0}{V_1\cos\phi_s}\sum_{k=1}^{\infty}\sum_{m=2}^{\infty} M_m^{pk}(\omega)[1+(-1)^m][\tilde{V}_k(\omega) - \tilde{V}_{-k}(\omega)] \tag{51}$$

and

$$j_p(\omega) - j_{-p}(\omega) = i\frac{I_0}{V_1\cos\phi_s}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} M_m^{pk}(\omega)[1-(-1)^m][\tilde{V}_k(\omega) + \tilde{V}_{-k}(\omega)], \tag{52}$$

which define with the help of relations (45) the beam current amplitude and phase modulation.

Now we should consider the values of V_k and V_{-k} in the case of amplitude or phase modulation of RF voltage.

5 RF voltage amplitude and phase modulation

To define the elements of the beam transfer matrix we should have as input the amplitude and phase modulation of the voltage in a multi-harmonic RF system.

For small enough amplitudes of modulation ΔV_n and $\Delta\phi_n$ (so that we need take into account only linear terms) we can present the perturbation in the voltage as

$$\tilde{V}(\phi,t) = \sum_{n=1}^{N}\{[\Delta V_n(t)\sin[g_n(\phi+\phi_n)] + \Delta\phi_n(t)g_nV_n\cos[g_n(\phi+\phi_n)]\}, \tag{53}$$

where $g_n = h_n/h$.

According to the definition (8) of $\tilde{V}_k(\omega)$ we get

$$\tilde{V}_k(\omega) = \frac{1}{2}\sum_n \delta_{k,h_n} e^{-ig_n\phi_n}[i\Delta V_n(\omega) + \Delta\phi_n(\omega)g_nV_n], \tag{54}$$

where

$$\Delta V_n(\omega) = \frac{1}{2\pi}\int_0^{\infty}\Delta V_n(t)e^{-i\omega t}dt, \tag{55}$$

$$\Delta\phi_n(\omega) = \frac{1}{2\pi}\int_0^{\infty}\Delta\phi_n(t)e^{-i\omega t}dt \tag{56}$$

and $\delta_{k,n}$ is Kronecker's symbol.

Then

$$\tilde{V}_k(\omega) + \tilde{V}_{-k}(\omega) = \sum_n \delta_{k,h_n}[\Delta V_n(\omega)\sin(g_n\phi_n) + \Delta\phi_n(\omega) g_n V_n \cos(g_n\phi_n)] \tag{57}$$

and

$$\tilde{V}_k(\omega) - \tilde{V}_{-k}(\omega) = i\sum_n \delta_{k,h_n}[\Delta V_n(\omega)\cos(g_n\phi_n) - \Delta\phi_n(\omega) g_n V_n \sin(g_n\phi_n)]. \tag{58}$$

6 Beam transfer matrices for a symmetric potential well

After substitution of the expressions (57) and (58) found above, into (51) and (52) the amplitude and phase modulation of p-th harmonic of beam current may be written as:

$$\Delta\phi_p^b(\omega) = \frac{h}{pI_p}\frac{I_0}{V_1\cos\phi_s}\sum_{k,m=1}^{\infty} M_m^{pk}(\omega)a_m \sum_n \delta_{k,h_n}[\Delta V_n(\omega)\sin(g_n\phi_n) + \Delta\phi_n(\omega)g_n V_n\cos(g_n\phi_n)] \tag{59}$$

$$\Delta I_p(\omega) = \frac{I_0}{V_1\cos\phi_s}\sum_{k,m=1}^{\infty} M_m^{pk}(\omega)b_m \sum_n \delta_{k,h_n}[\Delta V_n(\omega)\cos(g_n\phi_n) - \Delta\phi_n(\omega)g_n V_n\sin(g_n\phi_n)] \tag{60}$$

where coefficients

$$a_m = \frac{[1-(-1)^m]}{2}, \quad b_m = \frac{[1+(-1)^m]}{2}. \tag{61}$$

Expressions (59) and (60) give the bunched beam transfer functions for symmetric potential well. In the general case of an accelerating beam or of phase shift between the two RF systems for a non-accelerating beam, so that the accelerating bucket is not symmetric in ϕ, both odd and even m-modes will contribute to the modulation of the beam current amplitude and phase.

Matrix element $M_m^{pk}(\omega)$ defined by (49) is proportional to the dispersion integral and contains information about the amplitude and phase of the beam response (beam current phase and amplitude modulation) with respect to the excitation (modulation of the RF voltage amplitude or phase). Indeed the dispersion integral can be split into its principal value ($\mathcal{P}$) and the residue at the pole according to the Dirac formula

$$\lim_{\sigma\to+0}\frac{1}{x \pm i\sigma} = \mathcal{P}[\frac{1}{x}] \mp i\pi\delta(x). \tag{62}$$

To apply this identity we shall expand the synchrotron frequency using the Taylor series around the point of resonance $r = r_0$, where $\omega = m\omega_s(r_0)$:

$$\omega_s(r) = \omega_s(r_0) + \omega_s'(r_0)(r-r_0) + \frac{1}{2}\omega_s''(r_0)(r-r_0)^2 + ... \tag{63}$$

For positive ω and a dependence $\omega_s(r)$ inside the bunch, such that $\omega_s'(r_0) = 0$ only if $d\mathcal{F}_0/dr = 0$, we can write

$$M_m^{pk}(\omega) = \frac{m^2 h}{2Sk}[\mathcal{P}\int_0^{r_{max}} \frac{d\mathcal{F}_0}{dr}\frac{I_{mk}(r)I_{mp}(r)\omega_s(r)dr}{\omega^2 - m^2\omega_s^2(r)} + i\frac{\pi}{2m^2}\frac{d\mathcal{F}_0}{dr}|_{r=r_0}\frac{I_{mk}(r_0)I_{mp}(r_0)}{|\omega_s'(r_0)|}]. \quad (64)$$

In ref.[16] from the fact that second term in expression (64) equals infinity when $\omega_s'(r_0) = 0$, it was concluded that the threshold intensity for instability of the bunch with nonmonotonic behaviour of the synchrotron frequency goes to zero. However, if $\omega_s'(r_0) = 0$, then to evaluate the dispersion integral we should use the first nonzero term in the expansion (63). The case when $\omega_s'(r_0) = 0$ and $\omega_s''(r_0) \neq 0$ can appear in the double RF system and will be considered later.

Matrix element $M_m^{pk}(\omega)$ is defined when function $I_{mk}(r)$ is known. Below we calculate this function for the single and double RF systems.

6.1 Calculation of functions $I_{mk}(r)$

For an arbitrary potential well the functions $I_{mk}(r)$, defined by expression (27), can also be calculated numerically using the following formula:

$$I_{mk}(r) = \frac{ik}{\pi m}\int_{\phi_{min}}^{\phi_{max}} e^{i\frac{k}{h}\phi}\sin m\psi(r,\phi)\, d\phi, \quad (65)$$

where the function $\psi = \psi(r,\phi)$ is given by (15) and $\phi_{min} = \phi_{min}(r)$, $\phi_{max} = \phi_{max}(r)$ are the solutions of equation (14) for $\dot{\phi} = 0$.

For a potential well symmetric in ϕ we can rewrite expression (27) in the form:

$$I_{mk} = b_m I_{mk}^c + i a_m I_{mk}^s, \quad (66)$$

where

$$I_{mk}^s(r) = \frac{1}{\pi}\int_0^{\pi} \sin\frac{k}{h}\phi(\psi,r)\cos m\psi\, d\psi, \quad (67)$$

$$I_{mk}^c(r) = \frac{1}{\pi}\int_0^{\pi} \cos\frac{k}{h}\phi(\psi,r)\cos m\psi\, d\psi, \quad (68)$$

and a_m and b_m were defined by (61).

There are a few important cases when an analytical expression for the function $I_{mk}(r)$ can be obtained as well. These are, for example, cases of short bunches in the single RF system and in the double RF system (blm).

6.1.1 Single RF system

As is well known, the analytical solution of the system of equations (14)-(15) for nonlinear motion in a single RF system exists only for a non-accelerating beam and can be expressed in terms of elliptic functions:

$$\sin\frac{\phi}{2} = r\mathrm{cd}(y|\mathrm{m}_1), \quad (69)$$

where $\mathrm{cd}(y) = \mathrm{cn}(y)/\mathrm{dn}(y)$, $\mathrm{cn}(y)$ and $\mathrm{dn}(y)$ are Jacobian elliptic functions with parameter $\mathrm{m}_1 = r^2$ and argument $y = [\omega_{s0}/\omega_s(r)]\psi$; the value of r varies from 0 to 1 inside the bucket. Then function I_{mk} can be evaluated numerically from expressions (66) - (68). For $k/h = 1$ function I_{mk} can be calculated after substitution of (69) into the trigonometric formulas for double argument.

The frequency of synchrotron oscillations is defined by

$$\omega_s(r) = \frac{\pi\omega_{s0}}{2\mathrm{K}(\mathrm{m}_1)}, \tag{70}$$

where $\mathrm{K}(\mathrm{m}_1)$ is the complete elliptic integral of the first kind with parameter $\mathrm{m}_1 = r^2$.

The case of an accelerating beam can be treated analytically only for short bunches when the potential well can be considered as quadratic: $W \sim \phi^2$. Then the solution of the equation (15), which we also can get as an approximation for small r from (69), is

$$\phi = 2r\cos\psi. \tag{71}$$

This allows function $I_{mk}(r)$ to be calculated analytically:

$$I_{mk}(r) = i^m J_m(2\frac{k}{h}r), \tag{72}$$

where $J_m(x)$ is the Bessel function of order m.

In Fig.1 we show results of the numerical calculation of $I_{mk}(r)$ using the exact analytical solution (69) valid for non-accelerating beam, together with the approximate solution (72) which can be applied for short bunches both in accelerating and storage regimes.

6.1.2 Bunch-lengthening mode in double RF system

To produce the bunch lengthening mode the parameters of the double RF system should satisfy the conditions [18]

$$\cos\phi_s = -\frac{V_2}{V_1}g_2\cos(g_2\phi_2), \tag{73}$$

$$\sin\phi_s = -\frac{V_2}{V_1}g_2^2\sin(g_2\phi_2). \tag{74}$$

For short bunches the potential well is quartic and we have from (14)

$$r = \frac{1}{2}[\frac{\dot{\phi}^2}{\omega_{s0}{}^2} + \frac{(g_2^2-1)\phi^4)}{12}]^{1/2}. \tag{75}$$

Below, as an example, we consider the double RF system with the often used ratio of radio frequencies $g_2 = h_2/h = 2$. Then for a non-accelerating beam, the potential well has a form $W \sim \sin^4(\phi/2)$ and a nonlinear solution of equation (15) can be written in the form:

$$\sin\frac{\phi}{2} = \frac{\sqrt{r}\mathrm{cn}(z|\mathrm{m}_2)}{\sqrt{1 - r\mathrm{sn}^2(z|\mathrm{m}_2)}}, \tag{76}$$

where the parameter of the Jacobian elliptic functions

$$m_2 = \frac{r+1}{2} \qquad (77)$$

varies from 1/2 to 1 inside the bucket ($0 \le r \le 1$), and the argument

$$z = \sqrt{2r}\frac{\omega_{s0}}{\omega_s(r)}\psi = \frac{2K(m_2)}{\pi}\psi. \qquad (78)$$

The frequency ω_{s0} is defined by expression (6).

Then function I_{mk}, as for the case of the nonlinear motion in the single RF system, can again be evaluated numerically from expressions (76) and (66) - (68). The results for $k/h = 1, 2$ are shown in Figs.2,3.

The frequency of synchrotron oscillations

$$\omega_s(r) = \frac{\pi\omega_{s0}}{K(m_2)}\sqrt{\frac{r}{2}} \qquad (79)$$

as a function of r is shown in Fig.4 together with the frequency $\omega_s(r)$ found in [18] for the case of a quartic potential well. This approximate solution can be obtained from (79) in the limit $r \to 0$ so that $m_2 = (r+1)/2 \sim 1/2$ and $K(m_2) \sim K(1/2) = 1.85407$.

For small r expression (76) gives the solution

$$\varphi = 2\sqrt{r}\,\mathrm{cn}(\frac{2K(\frac{1}{2})}{\pi}\psi|\frac{1}{2}) \qquad (80)$$

found in [18] for a quartic potential well. Then using the first term of the expansion into series

$$\mathrm{cn}(\frac{2K}{\pi}\psi) = \sum_{m=0}^{\infty} c_m \cos[(2m+1)\psi], \qquad (81)$$

where coefficients

$$c_m = \frac{2\sqrt{2}\pi}{K}\frac{e^{-\pi(m+1/2)}}{1+e^{-\pi(2m+1)}}, \qquad (82)$$

with

$$c_0 \simeq 0.955, \qquad c_1 \simeq 0.043,$$

we obtain for function $I_{mk}(r)$ in the case of short bunches the following approximation

$$I_{mk}(r) \simeq i^m J_m(\frac{k}{h}2c_0\sqrt{r}). \qquad (83)$$

This solution is also shown (dashed line) in Figs.2,3 for comparison.

6.2 BTM for a single RF system

In the case of short bunches in a single RF system, after substitution of the calculated function $I_{mk}(r)$ (72) into (59) and (60) we have for the first harmonic of the beam current perturbation

$$I_1\Delta\phi_1^b = I_0[\frac{\Delta V_1}{V_1}\tan\phi_s + \Delta\phi_1]S_1, \tag{84}$$

$$\Delta I_1 = -I_0[\frac{\Delta V_1}{V_1} - \Delta\phi_1\tan\phi_s]S_2, \tag{85}$$

where

$$S_1 = \sum_{m=1}^{\infty} a_m s_m, \qquad S_2 = \sum_{m=2}^{\infty} b_m s_m, \tag{86}$$

$$s_m = -\frac{m^2}{2S}\int_0^{r_{max}} \frac{d\mathcal{F}_0}{dr}\frac{J_m^2(2r)\omega_s(r)dr}{(\omega - \imath\sigma)^2 - m^2\omega_s^2(r)} \tag{87}$$

and the normalisation factor S is defined by expression (32). The value of r_{max} is related to the half bunch length ϕ_{max}, expressed in radians of first RF system, by $r_{max} = \sin(\phi_{max}/2)$.

We can rewrite expressions (84) and (85) in the matrix form:

$$\begin{bmatrix} \Delta I_1 \\ I_1\Delta\phi_1^b \end{bmatrix} = I_0 \begin{bmatrix} -S_2 & S_2\tan\phi_s \\ S_1\tan\phi_s & S_1 \end{bmatrix} \begin{bmatrix} \Delta V_1/V_1 \\ \Delta\phi_1 \end{bmatrix}. \tag{88}$$

For a non-accelerating beam with $\phi_s = 0$ matrix (88) becomes diagonal.

In Fig.5 the amplitude and phase of the elements of beam transfer matrix s_1 and s_2 calculated from the real and imaginary parts of the dispersion integral (64) for two different distribution functions are shown for a bunch with $r_{max} = 0.3$, so that the half bunch length $\phi_{max} \simeq 2r_{max} = 0.61$ (rad).

We consider the binomial family of distribution functions:

$$\mathcal{F}_0(r) \propto (1 - \frac{r^2}{r_{max}^2})^{\mu}, \ \ r \le r_{max}, \tag{89}$$

with $\mu \ge 1$. (For $\mu < 1$, $d\mathcal{F}_0/dr$ is infinite at the beam edge $r = r_{max}$). For large μ this distribution is close to the Gaussian.

With increasing bunch length and hence synchrotron frequency spread the amplitude of the element s_1 of the BTM decreases. This means that the system becomes less sensitive to perturbations and therefore more stable. The dependence of the peak amplitude of s_1 and s_2 on bunch length is shown in Figs.6(a,b). For long bunches to evaluate the dispersion integral we used values of $I_{mk}(r)$ calculated numerically (see Fig.1).

For bunches with small synchrotron frequency spread $\Delta\omega_s$, changing the modulation frequency can excite different multipoles inside the bunch separately. Simultaneous excitation of two different multipoles by a fixed modulation frequency is possible only for quite high frequencies which excite multipoles with

$$m > \frac{\omega_s(r_{max})}{\Delta\omega_s}. \tag{90}$$

However excitation becomes less efficient for large m due to the factor $I_{mk}(r)I_{mp}(r)$ in the dispersion integral. For example, for a bunch with $r_{max} = 0.6$ and $\phi_{max} = 1.3$, the peak amplitude of the element s_3 is $\sim$ 1000 times less than of s_1.

It is possible to simplify the expressions for the transfer matrix for modulation frequencies outside the synchrotron frequency band. Then we get

$$s_m \simeq \frac{m^2\omega_s^2}{\omega^2 - m^2\omega_s^2} F_m. \tag{91}$$

Here the factor F_m depends only on the stationary particle distribution in longitudinal phase space

$$F_m = -\frac{\int_0^{r_{max}} \frac{d\mathcal{F}_0}{dr} J_m^2(2r)\, dr}{2\int_0^{r_{max}} \mathcal{F}_0(r) r\, dr} \tag{92}$$

and differs by the factor 2 from the 'reduced form factor' defined in Ref.[19].

It can be shown by using the first term in the series expansion of the Bessel function $J_1(r)$, and then integration by parts, that in the short bunch limit

$$F_1 \simeq 1$$

and doesn't depend on the form of the stationary distribution function. This gives

$$s_1 \simeq \frac{\omega_s^2}{\omega^2 - \omega_s{}^2}. \tag{93}$$

However F_2, and therefore s_2, is already a function of the form of the distribution and the bunch length l_b. From (91) for s_2 we have

$$s_2 \simeq \frac{4\omega_s^2}{\omega^2 - 4\omega_s{}^2} F_2(l_b), \tag{94}$$

Let us consider a few examples of the distribution functions:

(1) $\mathcal{F}_0(r) \propto (1 - \frac{r^2}{r_{max}^2})$, $r \leq r_{max}$, where $r_{max} \simeq \phi_{max}/2 = l_b h/(4R)$,

$$F_2 \simeq r_{max}^2/6,$$

(2) $\mathcal{F}_0(r) \propto e^{-2\frac{r^2}{r_{max}^2}}$,

$$F_2 \simeq r_{max}^2/4.$$

Note that the approximate formulae (93) and (94) don't work for frequencies inside the synchrotron frequency band, where the dependence of the BTF on bunch length can be completely different. Indeed, as shown in Fig.6, the peak amplitude of s_1 strongly depends on bunch length whereas that of s_2 stays almost constant.

Keeping only the first term in the series over m in expressions (86), so that $S_1 \simeq s_1$ and $S_2 \simeq s_2$, we can present the transfer matrix in the form:

$$\begin{bmatrix} \Delta I_1 \\ I_1\Delta\phi_1^b \end{bmatrix} = I_0 \begin{bmatrix} -s_2 & s_2\tan\phi_s \\ s_1\tan\phi_s & s_1 \end{bmatrix} \begin{bmatrix} \Delta V_1/V_1 \\ \Delta\phi_1 \end{bmatrix}. \tag{95}$$

This expression can be compared, for example, with the transfer matrix obtained in Ref.[14] for a single RF system by a different approach.

6.3 BTM for a double RF system

We consider bunches accelerated in the symmetric potential well created by the double RF system with $h_2/h = 2$ (blm).

After substitution of the function $I_{mk}(r)$ defined by (83) into expression (64), we get for the amplitude and phase modulation of the first and second harmonics of the beam current (with respect to the fundamental one) the results which can be written in the matrix form:

$$\begin{bmatrix} \Delta I_1 \\ I_1 \Delta \phi_1^b \\ \Delta I_2 \\ I_2 \Delta \phi_2^b \end{bmatrix} = I_0 \begin{bmatrix} -D_2^{11} & D_2^{11} \tan\phi_s & D_2^{12}/2 & -D_2^{12}\tan\phi_s/2 \\ D_1^{11}\tan\phi_s & D_1^{11} & -D_1^{12}\tan\phi_s/4 & -D_1^{12} \\ -2D_2^{21} & 2D_2^{21}\tan\phi_s & D_2^{22} & -D_2^{22}\tan\phi_s \\ D_1^{21}\tan\phi_s & D_1^{21} & -D_1^{22}\tan\phi_s/4 & -D_1^{22} \end{bmatrix} \begin{bmatrix} \Delta V_1/V_1 \\ \Delta\phi_1 \\ \Delta V_2/V_2 \\ \Delta\phi_2 \end{bmatrix}. \tag{96}$$

where we used definitions similar to the case of one RF system:

$$D_1^{ln} = \sum_{m=1}^{\infty} a_m d_m^{ln}, \qquad D_2^{ln} = \sum_{m=2}^{\infty} b_m d_m^{ln}. \tag{97}$$

Here $n = h_n/h$ and coefficients $d_m^{ln} = d_m^{ln}(\omega)$ are defined as

$$d_m^{ln} = -\frac{m^2}{2Sln}\int_0^{r_{max}} \frac{d\mathcal{F}_0}{dr}\frac{J_m(2lc_0\sqrt{r})J_m(2nc_0\sqrt{r})\omega_s(r)dr}{(\omega - i\sigma)^2 - m^2\omega_s^2(r)}, \tag{98}$$

where $r_{max} = \sin^2(\phi_{max}/2)$. To get matrix (96) the relations (73) and (74) were also used. Note, that $D_m^{ln} = D_m^{nl}$ and the series over m contain only odd terms in D_1^{ln} and even in D_2^{ln} due to the definition (61) of the coefficients a_m and b_m.

Keeping only the first term in the series (97) will lead to a matrix equivalent to (96), with all capital D replaced by small d as was done for the case of single RF system. However in contrast to the single RF system, even for very short bunches a fixed modulation frequency excites simultaneously an infinite number of different multipoles due to the fact that the synchrotron frequency equals zero at the centre of the bunch (see Fig.4). Of course, the contribution from the higher multipoles is significantly reduced by the factors $I_{ml}(r)$, $I_{mn}(r)$ and the fact that $d\mathcal{F}_0/dr = 0$ at the centre of the bunch.

Amplitude and phase of the matrix elements d_1^{11}, d_3^{11} and d_1^{12} are shown in the Figs.7, 8 for different values of bunch length. For long bunches we consider only the case of a non-accelerating beam. Then in the matrix (96) matrix elements proportional to $\tan\phi_s$ go to zero.

Outside the synchrotron frequency band ($\omega \gg m\omega_s(r_{max})$) we have an asymptotic formula similar to (91) as found for the case of a single RF system:

$$d_m^{ln} \simeq \frac{21(nl)^{m-1}c_0^{2m}}{8[(m-1)!]^2}\frac{\omega_{s0}^2}{\omega^2}. \tag{99}$$

For $m = 1$ and $m = 2$ it gives

$$d_1^{ln} \simeq \frac{\omega_{s0}^2}{\omega^2}, \qquad d_2^{ln} \simeq 0.044\phi_{max}^2 nl\frac{4\omega_{s0}^2}{\omega^2}. \tag{100}$$

The dependence of peak amplitude of the elements d_1^{11}, d_2^{11} and d_1^{12} as a function of bunch length is shown in Figs.6(a,b) and Fig.9, in the first case together with functions calculated for single RF system. As can be seen in the graphs, in contrast to the single RF system, increasing the synchrotron frequency spread beyond some value does not improve stability of the system.

Fig.4 shows that for a bunch length more than some critical value, $\phi_{max} > \phi_{cr} = 2.035(rad)$, $(r_{max} > r_{cr} = \sin^2(\phi_{cr}/2) = 0.724)$, the dependence of the synchrotron frequency $\omega_s(r)$ is no longer monotonic. Therefore for a fixed modulation frequency the resonant condition $\omega = m\omega_s(r)$ is satisfied at two points, r_0 and r_1, simultaneously. Then the imaginary part of the dispersion integral contains a second term - a contribution from the pole at $r = r_1$ instead of the only one term written in (64). Both these poles and the corresponding contour of integration are shown in Fig.10.

Note that, for the accelerating regime, the critical value of bunch length can be calculated numerically [16] and it can be significantly smaller than in the storage regime. The critical bunch length decreases also with increasing frequency ratio g_2 in the double RF system.

If the modulation frequency is such that the resonant condition $\omega = m\omega_s(r_{cr})$ is satisfied, then to calculate the contribution from the pole $[\omega - m\omega_s(r_{cr})]$ we should use the next term in the expansion (63) and consider the integral

$$\int_0^{r_{max}} \frac{d\mathcal{F}_0}{dr} \frac{I_{mk}(r)I_{mp}(r)dr}{m\omega_s''(r_{cr})(r - r_{cr})^2 + i\sigma}. \tag{101}$$

This expression has two poles

$$r = r_{cr} \pm \sqrt{\frac{\sigma}{2m|\omega_s''(r_{cr})|}}(1 + i). \tag{102}$$

The contour of integration defined by the initial conditions doesn't allow the singularity to be avoided when $\sigma \to +0$ (this can be seen from Fig.10 for the case when the two poles coincide). As a result the amplitude of the matrix element in frequency domain at $\omega = m\omega_s(r_{cr})$ is infinite. Below we consider this case in more detail.

6.4 Landau damping in a system with nonmonotonic behaviour of synchrotron frequency

In this section we shall try to understand why the unintegrable singularity appears in the dispersion integral (101) when the synchrotron frequency as a function of oscillation amplitude has zero derivative inside the bunch but outside the bunch centre.

First, let's perform the back transformation of the transfer function to the time domain and consider the integral proportional to the element of the beam transfer matrix

$$g_m(t) = \int_{-\infty}^{\infty} \int_0^{r_{max}} \frac{R_m(r)e^{i\omega t}\, d\omega\, dr}{\omega - m\omega_s(r) - \imath\sigma} = -\imath \int_0^{r_{max}} R_m(r)e^{-\imath m\omega_s(r)t}\theta(t)\, dr, \qquad (103)$$

where $R_m(r) = d\mathcal{F}_0/dr I_{mk}(r)I_{mp}(r)$ (we don't specify values for k and p) and $\theta(t)$ is the unit step function.

The response of the beam to the modulation of the voltage according to (12) has a term proportional to the integral

$$I(t) = \int_0^t \sum_{m=-\infty}^{\infty} g_m(t-t')\tilde{V}(t')dt'. \qquad (104)$$

For a modulation voltage $\tilde{V}(t) \sim e^{-\imath\Omega t}$ we have

$$I(t) \sim \sum_{m=-\infty}^{\infty} \int_0^t \int_0^{r_{max}} R_m(r)e^{-\imath m\omega_s(r)(t-t')}e^{i\Omega t'}\, dt'\, dr. \qquad (105)$$

After integration over t' we obtain

$$I(t) \sim \sum_{m=-\infty}^{\infty} \int_0^{r_{max}} R_m(r)e^{-\imath m\omega_s(r)t}\frac{[e^{i(m\omega_s(r)-\Omega)t} - 1]}{i[m\omega_s(r) - \Omega]}\, dr. \qquad (106)$$

Far from resonance, for $|m\omega_s(r) - \Omega|t \gg 1$ we have an oscillating solution. Close to the resonance, where

$$|m\omega_s(r) - \Omega| \ll \frac{1}{t} \qquad (107)$$

we can expand the exponent in the square brackets and write

$$\frac{e^{i(m\omega_s(r)-\Omega)t} - 1}{i[m\omega_s(r) - \Omega]} \simeq t\,. \qquad (108)$$

Following the explanation of the nature of the Landau damping given in Ref.[9] we should say now that the band of resonance frequencies, for which we have a solution (108) growing with time, shrinks with time according to the condition (107). However, due to the integration over r in (105) it is important to consider the interval of resonant amplitudes $|r-r_0|$. Indeed, at resonance

$$\omega_s(r) - \Omega/m = \omega_s'(r_0)(r-r_0) + \frac{1}{2}\omega_s''(r_0)(r-r_0)^2 + ... \qquad (109)$$

Now we have two different situations. First, corresponding to the normal Landau damping, when $\omega_s'(r_0) \neq 0$ and instead of (107) we get

$$|r - r_0| \ll \frac{1}{t|\omega_s'(r_0)|m}. \qquad (110)$$

This means that the interval of resonant amplitudes decreases $\sim 1/t$. This ensures that the beam current modulation doesn't grow with time.

The situation is different if, at the point of resonance, $\omega_s'(r_0) = 0$, but $\omega_s''(r_0) \neq 0$ and $R_m(r_0) \neq 0$. Then condition (107) defines the width of the resonant region as

$$|r - r_0| \ll \frac{1}{\sqrt{t|\omega_s''(r_0)|m}}. \qquad (111)$$

This interval of resonant amplitudes shrinks with time only as $1/\sqrt{t}$. As a result the amplitude of the modulation in the beam grows $\sim \sqrt{t}$. It is quite a slow instability compared to the usual exponential growth, but nevertheless Landau damping in the system with zero derivative of synchrotron frequency inside the bunch (but outside the bunch centre) is lost. One of the possible ways this instability can develop is by an evolution of the particle distribution in the longitudinal phase space which leads to a stationary distribution with zero derivative of the distribution function $d\mathcal{F}_0/dr$ around the point $r = r_0$. "Strange" behaviour of long bunches in the double RF system has been observed experimentally [20], [21].

The previous analysis can also be extended to a system with a point where both first and second derivatives of the synchrotron frequency equal zero, so that dependence of the synchrotron frequency on amplitude is nevertheless monotonic inside the bunch. In this case we can get a modulation growing with time as $t^{2/3}$.

It is interesting to note that nonmonotonic dependence of synchrotron frequency on amplitude inside the bunch was also found in a single RF system, [12], when intensity effects were included. The double hump in the amplitude of the dipole beam transfer function was observed during measurements for high intensities.

It seems that this phenomenon should also be taken into account for the calculation of the thresholds of transverse mode coupling instability for long bunches in a double RF system, where similar dispersion integrals appear in the presence of synchrotron frequency spread [22].

7 Conclusions

Expressions for bunched beam transfer matrices were obtained in the case of a symmetric potential well. These results were applied to short bunches in an accelerating bucket and to long bunches in the storage regime, both for single and double RF systems.

An exact analytical solution was found for nonlinear motion in the bunch-lengthening mode created by double RF system with ratio of radio frequencies equals 2 (non-accelerating beam).

We have shown that Landau damping in a system with zero derivative of synchrotron frequency inside the bunch (but outside the bunch centre) is lost and that beam current modulation grows with time as $\sqrt{t}$. One of the examples of such a system is bunch-lengthening mode in a double RF system for bunch lengths more than some critical value.

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References

[1] K.W.Robinson, Stability of beam in radio-frequency system, Report No. CEAL-1010, Cambridge, Mass., 1964.

[2] A.N.Lebedev, Coherent synchrotron oscillations in the presence of the space charge, Atomnaya Energiya 25 (2), p.100, 1968.

[3] F.J.Sacherer, Method for computing bunched-beam instabilities, CERN/SI-BR/72-5, 1972.

[4] F.Pedersen, Beam loading effects in the CERN PS Booster, IEEE Trans. Nucl. Sci. NS-22, p.1906, 1975.

[5] F.Pedersen and F.Sacherer, Theory and performance of the longitudinal active damping system for the CERN PS Booster, IEEE Trans. Nucl. Sci. NS 24-3, 1977, p.1396.

[6] A.Hofmann, B.Zotter, Measurement of beam stability and coupling impedance by RF excitation, IEEE Trans. Nucl. Sci. Vol. NS-24, No 3, p.1487, 1977.

[7] J.Gareyte, Landau damping of the quadrupole mode in SPEAR II, SPEAR-207, 1977.

[8] S.Chattopadhyay, Some fundamental aspects of fluctuations and coherence in charged -particle beams in storage rings, CERN 84-11, 1984.

[9] J.Gareyte, Beam observation and the nature of instabilities, AIP Conf. Proc., v.1, p.430, 1987.

[10] D.Boussard, Schottky noise and beam transfer function diagnostics, CERN 87-03, 1987, p.416.

[11] D.Boussard, J.Gareyte, Measurement of the SPS coupling impedance, SPS Improvement report No. 181, 1980.

[12] J.M.Jowett et al, Beam response measurements at SPEAR, CERN/LEP-TH/88-34.

[13] V.I.Balbekov and S.V.Ivanov, Longitudinal beam instabilities in the proton synchrotrons, Proc. 13th Int. Conf. on High- Energy Acc., Novosibirsk, 1987, v.2, p.124.

[14] S.Koscielniak, A general theory of beam loading, Particle Accelerators, v.31, p.205, 1990.

[15] Y.Chin, Longitudinal stability limit for electron bunches in a double RF system, NIM, 215, p.501, 1983.

[16] V.I.Balbekov and S.V.Ivanov, Analysis of the methods of supression of longitudinal bunched beam instability by Landau damping, Atomnaya Energiya 62 (2), p.98, 1987.

[17] Tai-Sen Wang, Synchrotron beam-loading stability with a higher RF harmonic, Proc. of the 1993 PAC, p.3500.

[18] A.Hofmann and S.Myers, Beam dynamics in a double RF system, Proc. 11th Int. Conf. on High-Energy Accelerators, Geneva, 1980, p.610.

[19] Tai-Sen Wang, Bunched beam longitudinal mode coupling and Robinson type instabilities, Particle Accelerators v.34, p.105, 1990.

[20] J.M. Baillod et al, A Second Harmonic (6-16 MHz) RF System with Feedback-reduced Gap Impedance for Accelerating Flat-topped Bunches in the CERN PS Booster, IEEE Trans. Nucl. Sci., NS-30, p.3499, 1983.

[21] T.Linnecar, E.N.Shaposhnikova, Analysis of the voltage programs for the $Sp\bar{p}S$ double frequency RF system, CERN SL/Note 92-44 (RFS).

[22] Y.Chin, Transverse mode coupling instability in a double RF system, CERN SL/93-03.

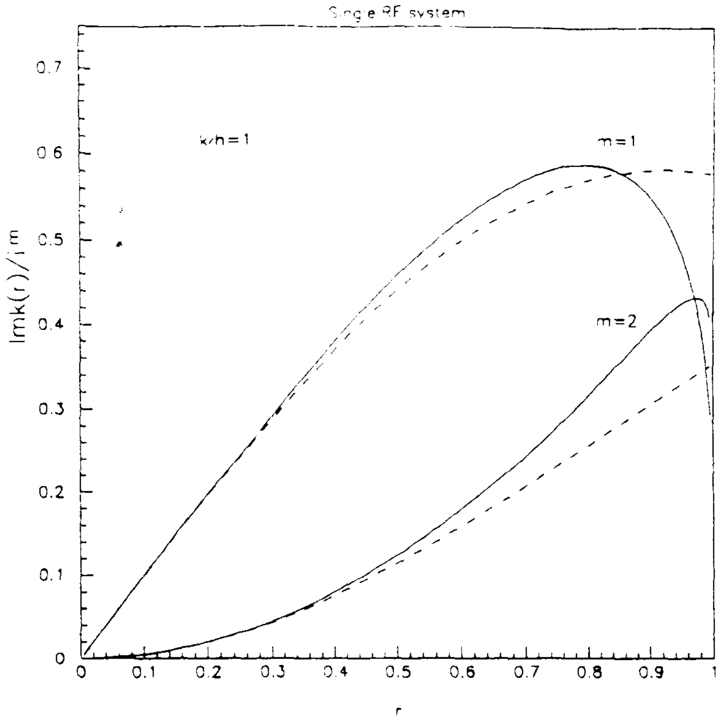

Figure 1: Function $I_{mk}(r)/i^m$ calculated for a single RF system using exact solution for non-accelerating beam (solid line) and approximate solution (dashed line) for parameters $m = 1, 2$ and $k/h = 1$.

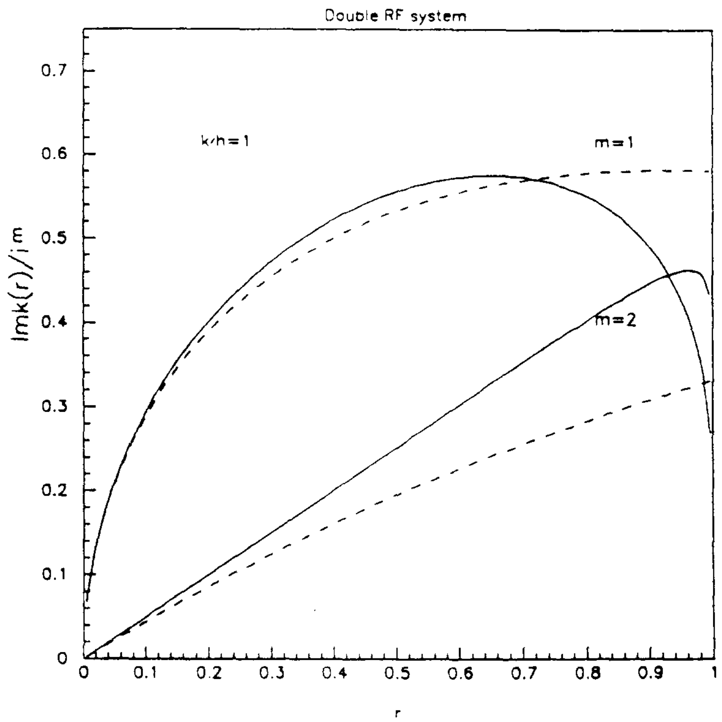

Figure 2: Function $I_{mk}(r)/i^m$ calculated for a double RF system (blm, $g_2 = 2$) using exact solution for non-accelerating beam (solid line) and approximate solution for short bunches (dashed line) for parameters $m = 1, 2$ and $k/h = 1$.

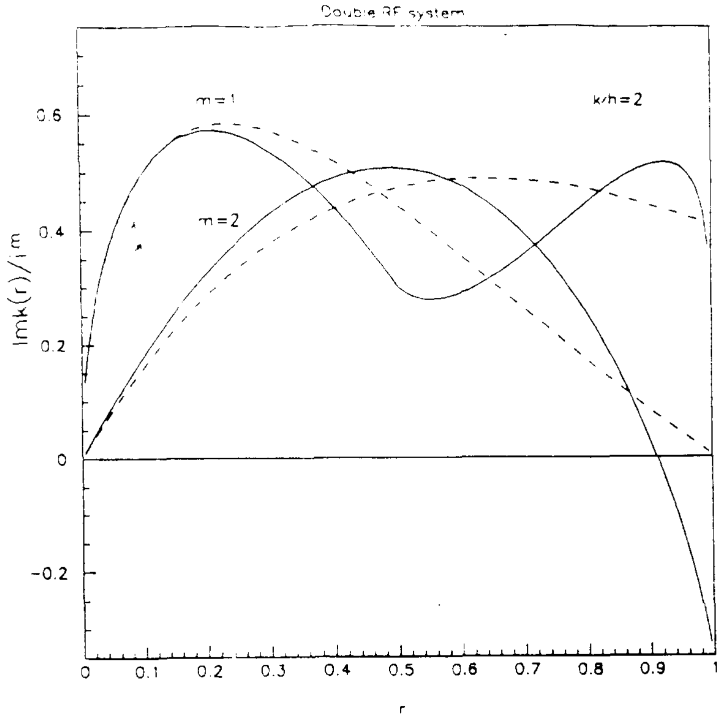

Figure 3: Function $I_{mk}(r)/i^m$ calculated for a double RF system (blm) using exact (solid line) and approximate (dashed line) solution for parameters $m = 1,2$ and $k/h = 2$.

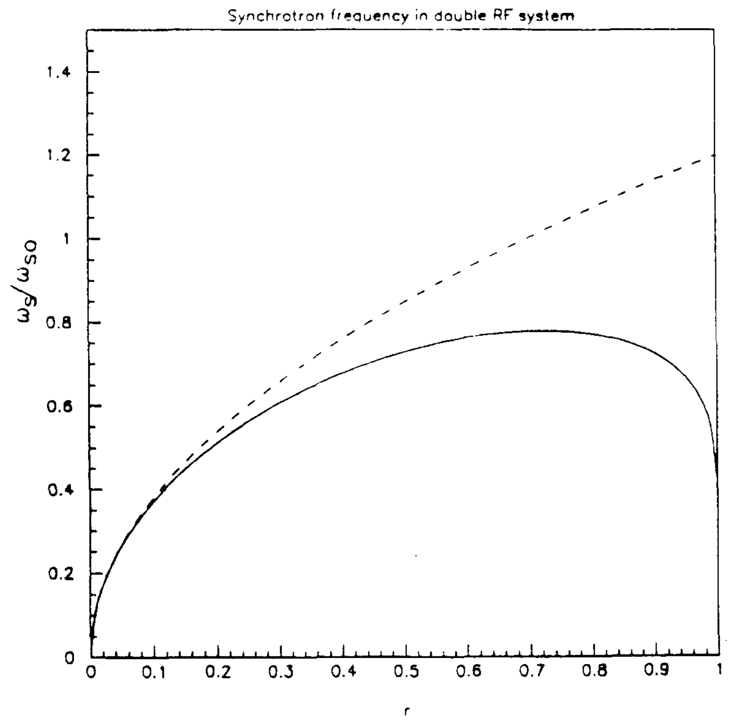

Figure 4: The frequency of synchrotron oscillations $\omega_s(r)$ in a double RF system (blm, $g_n = 2$): exact solution for non-accelerating beam (solid line) and approximate solution found for quartic potential well (dashed line).

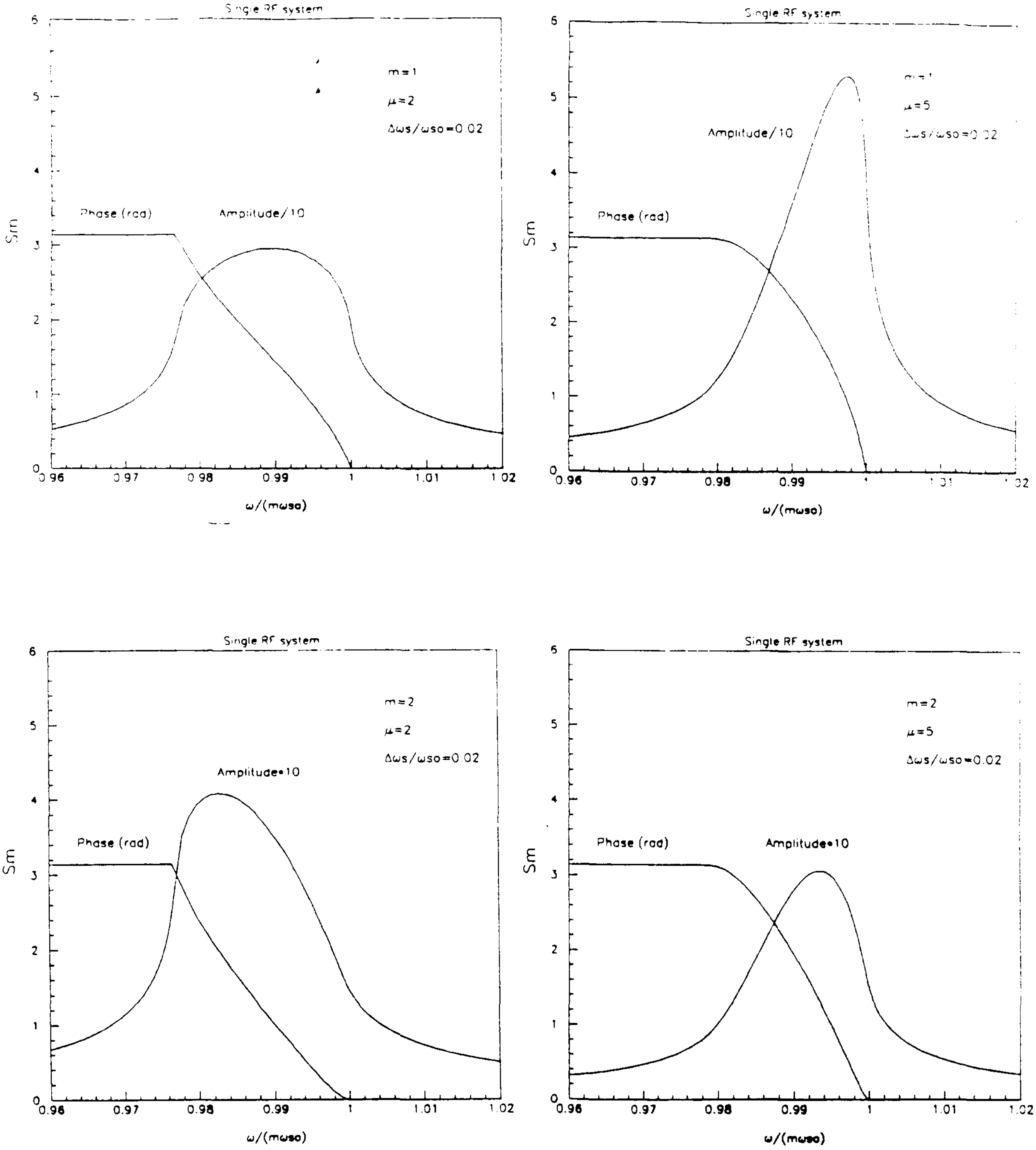

Figure 5: Amplitude and phase of the elements of the BTM $s_m(\omega)$ $(m = 1, 2)$ in a single RF system calculated for two types of particle distribution in longitudinal phase space (binomial with $\mu = 2, 5$) and half bunch length $\phi_{max} = 0.6$ (rad).

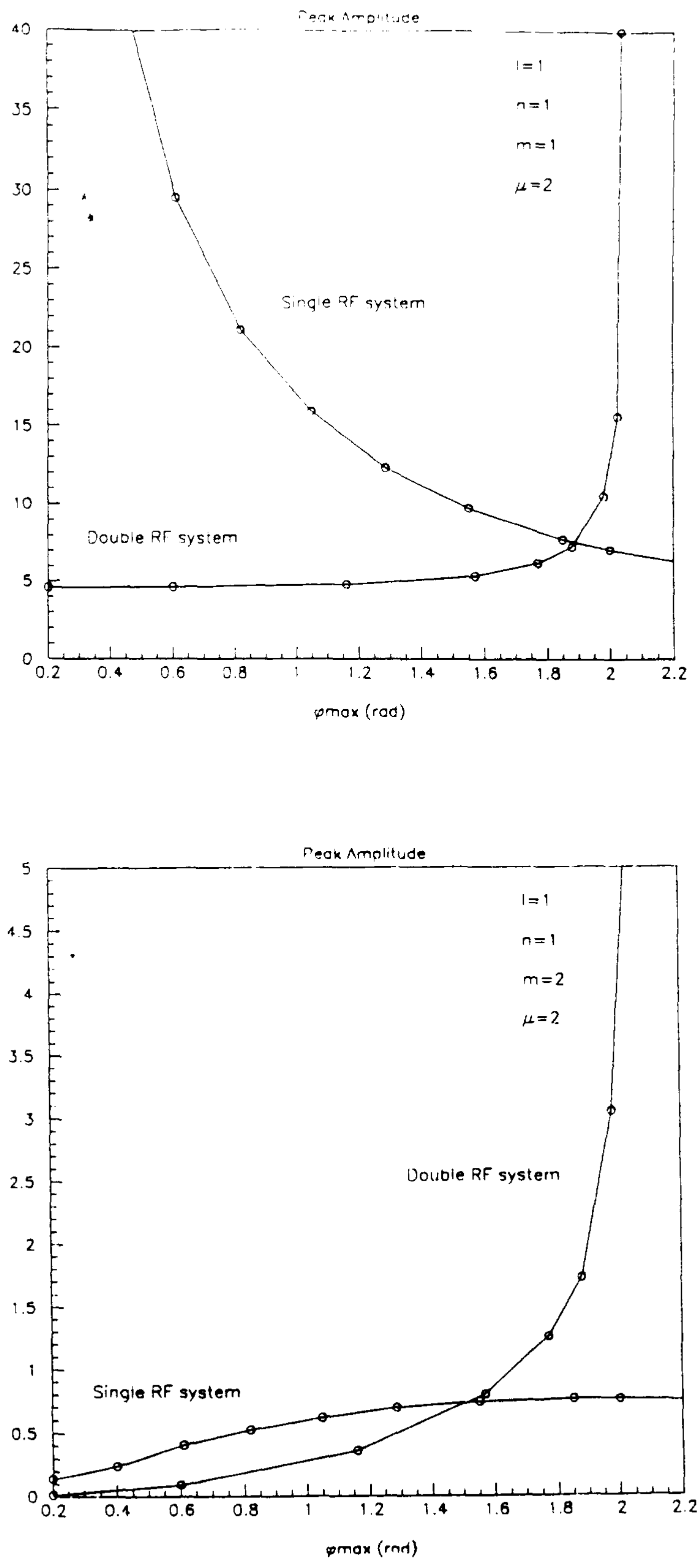

Figure 6: Dependence of the peak value of the amplitude of the BTM elements on half bunch length (in radians) calculated for modes $m = 1$ (a) and $m = 2$ (b) and $k/h = 1$ in single and double RF systems (blm) for a bunch with distribution function with $\mu = 2$.

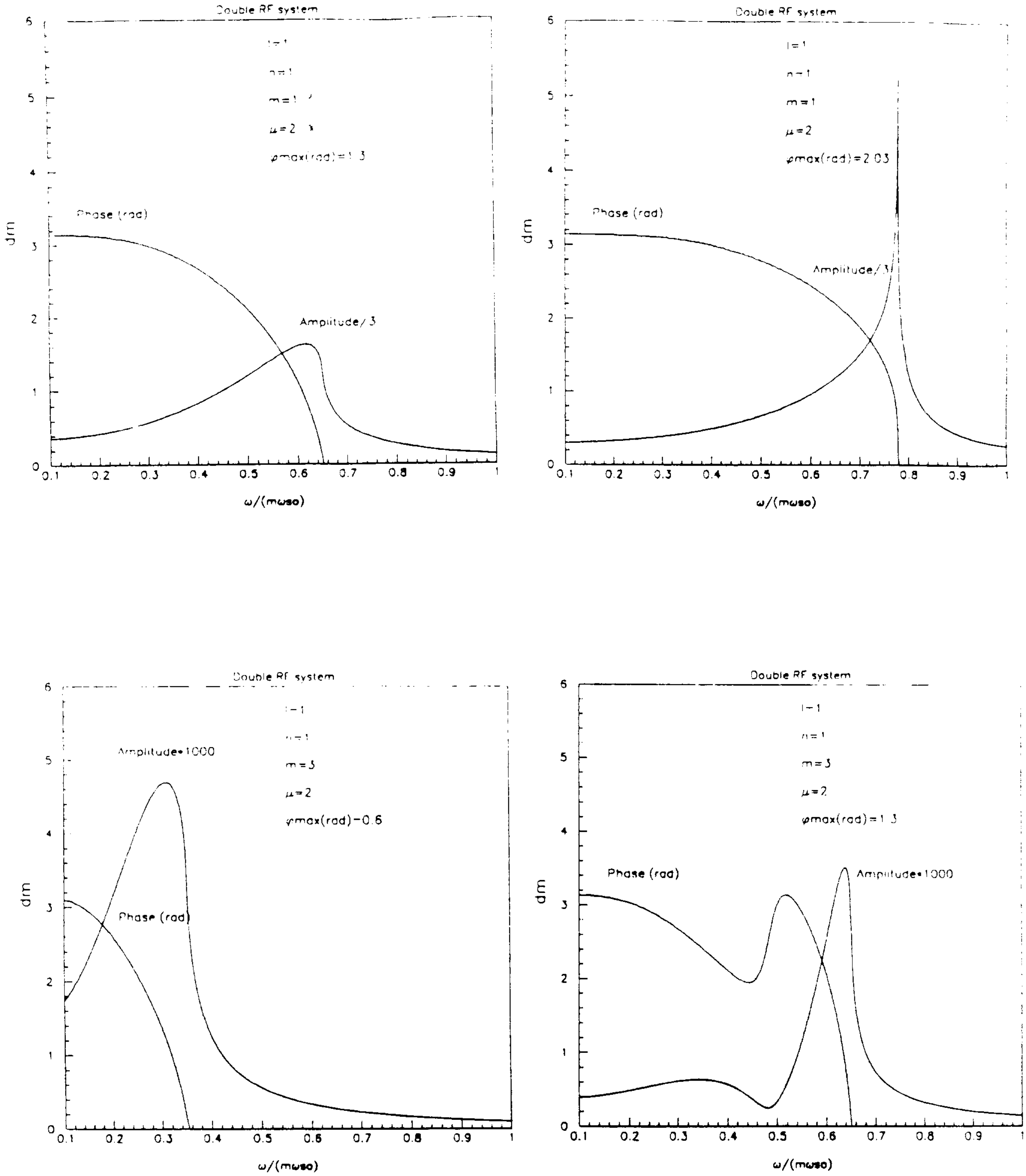

Figure 7: Amplitude and phase of the BTM elements $d_1^{11}(\omega)$ and $d_3^{11}(\omega)$ in a double RF system (blm, $g_2 = 2$) calculated for different bunch lengths.

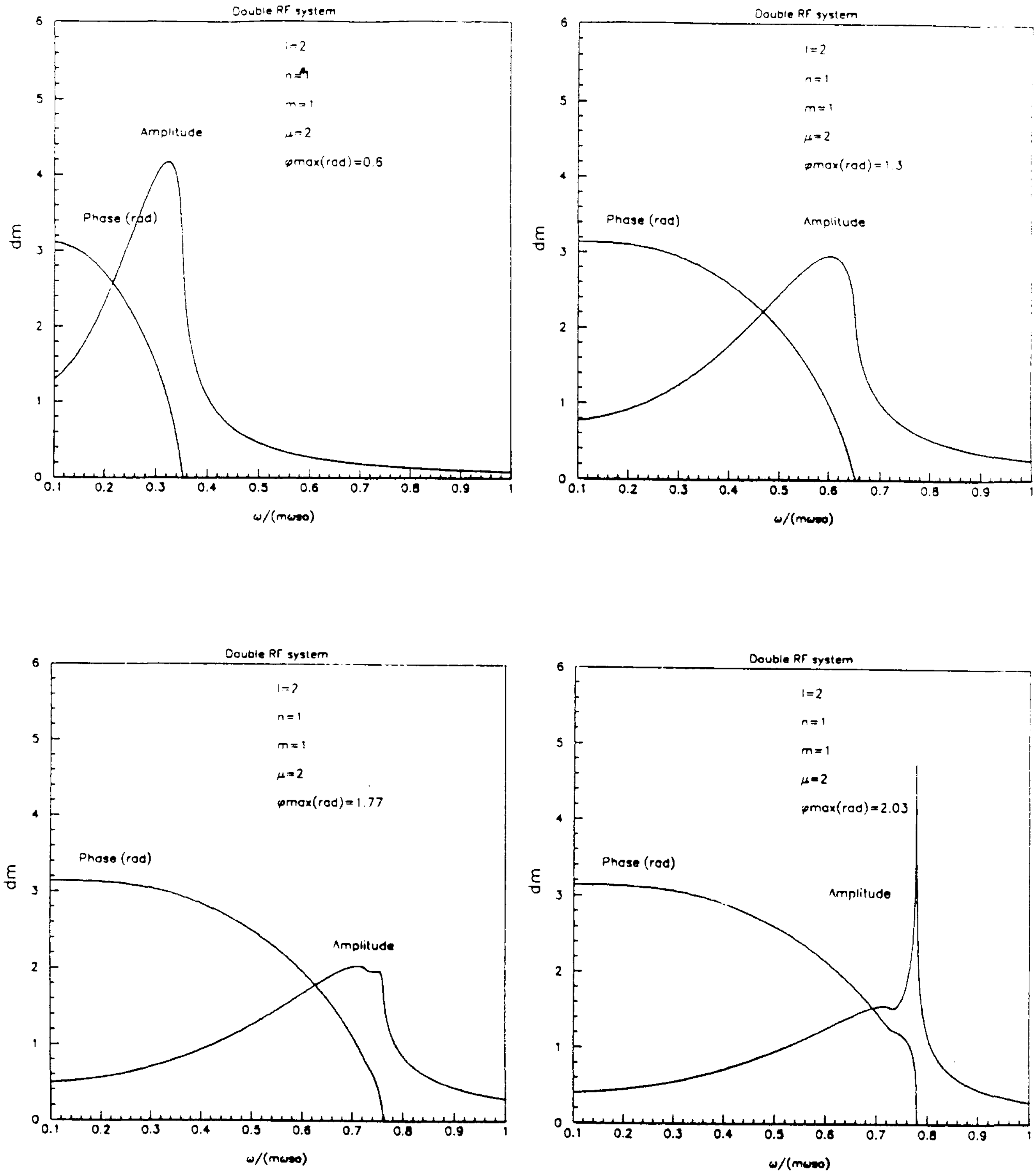

Figure 8: Amplitude and phase of the BTM elements $d_1^{12}(\omega)$ in a double RF system (blm, $g_2 = 2$) calculated for different bunch lengths.

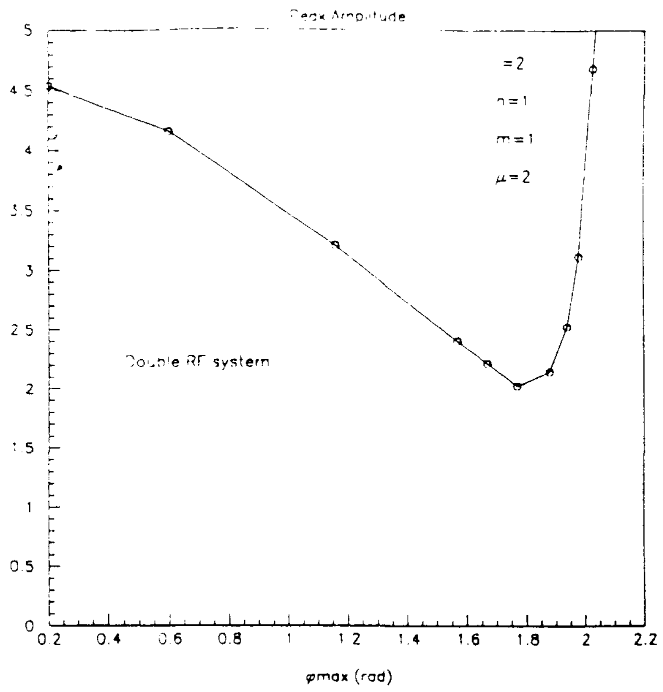

Figure 9: Dependence of the peak value of the amplitude of the BTM elements d_1^{12} on half bunch length (in radians) in a double RF system (blm, $g_2 = 2$) for distribution function with $\mu = 2$.

Figure 10: Integration contour for dispersion integral in the case of nonmonotonic dependence of synchrotron frequency on amplitude.