GEOMETRY AND DYNAMICS WITH TIME-DEPENDENT CONSTRAINTS

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Abstract

We describe how geometrical methods can be applied to a system with explicitly time-dependent second-class constraints so as to cast it in Hamiltonian form on its physical phase space. Examples of particular interest are systems which require time-dependent gauge fixing conditions in order to reduce them to their physical degrees of freedom. To illustrate our results we discuss the gauge-fixing of relativistic particles and strings moving in arbitrary background electromagnetic and antisymmetric tensor fields.

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1. Introduction

A Hamiltonian dynamical system can be described geometrically by a phase space manifold Γ (of dimension $2d$ say) equipped with a symplectic form ω and a Hamiltonian function H (see Abraham and Marsden (1978) and Arnol'd (1978)). The condition that ω is symplectic means that it is a non-degenerate closed two-form, so it can be used to introduce a Poisson bracket $\{\,,\}$ on Γ. The evolution of the system in time t is given by a particular set of trajectories on Γ, parametrized by t, such that Hamilton's equation

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \{f, H\} \tag{1}$$

holds for any time-dependent function f on Γ.

Since the seminal work of Dirac (1950,1958,1964) there has been intensive study of systems of this type which can be consistently constrained to some physical phase space manifold Γ^* (of dimension $2n$ say) which is embedded in Γ in a manner we now describe. In the most general case the embedding of Γ^* in Γ can depend on time and it must therefore be defined by a family of maps

$$\varphi_t : \Gamma^* \to \Gamma \;, \tag{2}$$

depending smoothly on t, each of which is a diffeomorphism onto its image $X_t \subset \Gamma$. We assume that each of the trajectories on Γ for which (1) holds has the property that it always lies in the subspaces X_t for each t, or else that it always lies in the complements of these spaces. It is clear that trajectories of the former type correspond exactly under the embedding (2) to trajectories on Γ^*, and one can attempt to reformulate the dynamics for this subclass of trajectories in a manner which is intrinsic to Γ^*.

We define ω^* on Γ^* at time t by pulling back ω using φ_t and we assume that this is also a symplectic form (albeit a time-dependent one in general). We can then use ω^* to introduce the Dirac bracket $\{\,,\}^*$ on Γ^*. The key issue which we shall address here is whether, for a given choice of embeddings (2), one can find a Hamiltonian function H^* such that Hamilton's equation

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \{f, H^*\}^* \tag{3}$$

holds for any time-dependent function f on Γ^*. When the embeddings (2) are independent of time, (3) follows easily from (1) with $H^* = H$. In the general case, however, the dynamics on Γ^* is determined not just by the dynamics on Γ but also by the time-dependence of the embeddings φ_t, and under these circumstances it is non-trivial to determine whether (3) holds for some function H^*.

The most compelling reason for studying this general situation is the fact that gauge choices with explicit time dependence are essential in order to restrict systems which are invariant under time-reparametrizations, such as the relativistic particle, string or general relativity, to their physical degrees of freedom (but see Henneaux *et al* (1992) for possible modifications of the action to allow other gauge choices). Here we summarize the solution of this problem given in Evans and Tuckey (1993) and we clarify some related issues. (We have recently learned that Mukunda (1980) has previously obtained results which are

locally equivalent to ours using an algebraic approach. Related work from the Lagrangian point of view appears in (Rañada 1994).) We then give some new examples, extending the treatment of the relativistic particle in a background field (Evans 1993) to the case of a string in an arbitrary antisymmetric tensor background.

2. Extended phase space and constrained dynamics

We define extended phase space to be $\bar{\Gamma} = \Gamma \times \mathbb{R}$, where the second factor is time. We can, in a natural way, regard H and ω as living on $\bar{\Gamma}$ (by pulling back using the projection map) and we define the contact form on $\bar{\Gamma}$ to be

$$\Omega = \omega + \mathrm{d}H \wedge \mathrm{d}t \ . \tag{4}$$

(In Evans and Tuckey (1993) Ω was called the Poincaré-Cartan two-form; in Abraham and Marsden (1978) Ω is introduced as an example of a contact structure.) Any trajectory on Γ parametrized by t is clearly equivalent to a trajectory on $\bar{\Gamma}$ with parameter s chosen such that $\mathrm{d}t/\mathrm{d}s$ is nowhere zero. Let V be the tangent vector to the trajectory on $\bar{\Gamma}$. Then Hamilton's equation (1) is precisely the condition

$$i(V)\Omega = 0 \tag{5}$$

(where $i(V)$ denotes interior multiplication of a form by the vector field V).

When the system is constrained we can similarly define extended physical phase space to be $\bar{\Gamma}^* = \Gamma^* \times \mathbb{R}$. The family of embeddings (2) is equivalent to the single embedding

$$\bar{\varphi} : \bar{\Gamma}^* \to \bar{\Gamma} \ , \qquad \bar{\varphi}(x, t) = (\varphi_t(x), t) \ , \tag{6}$$

which is a diffeomorphism onto its image $\bar{X} = \{(x, t) : x \in X_t, t \in \mathbb{R}\} \subset \bar{\Gamma}$ (assuming, as stated earlier, that φ_t varies smoothly with t). Define the form Ω^* on $\bar{\Gamma}^*$ to be the pull back of Ω using $\bar{\varphi}$. In general this has the structure

$$\Omega^* = \omega^* + (\mathrm{d}H + Y) \wedge \mathrm{d}t \tag{7}$$

for some one-form Y. (Here we use the fact that any time-dependent form on Γ^* can be regarded as a smooth form on $\bar{\Gamma}^*$; when the form is time-independent this reduces to pulling back using the projection map.)

Any solution of Hamilton's equation (1) which lies in $\bar{X}$ clearly corresponds to a trajectory in $\bar{\Gamma}^*$ with tangent vector V^* which satisfies

$$i(V^*)\,\Omega^* = 0 \ . \tag{8}$$

By comparison with (4) and (5) we see that Hamilton's equation (3) holds on Γ^* if and only if

$$Y = \mathrm{d}K \bmod \mathrm{d}t \qquad \text{and then} \qquad H^* = H + K \tag{9}$$

for some function K on $\bar{\Gamma}^*$. One can show that this holds locally (*ie.* in any contractible region on Γ^*) if and only if ω^* is independent of time on Γ^*, a fact which will prove useful later.

3

To discuss specific examples it is convenient to introduce on Γ local coordinates z^M, $M = 1, \ldots, 2d$. The subsets $X_t \subset \Gamma$ are defined by a set of time-dependent constraint functions $\psi^I(z^M, t)$, $I = 1, \ldots, 2(d-n)$, which are, in the language of Dirac (1950,1958,1964), second-class. The fact that these constraint functions are preserved in time is equivalent to our initial assumption that there exists a subset of trajectories confined to the subspaces X_t. The condition that the constraint functions are second-class is equivalent to our assumption that the form ω^* is symplectic.

If ξ^A, $A = 1, \ldots, 2n$, are local coordinates on Γ^* then the embeddings φ_t or $\bar{\varphi}$ allow us to regard the z^M as time-dependent functions of these variables on $\bar{X}$, and we have the explicit expression

$$Y = - \frac{\partial z^M}{\partial t} \frac{\partial z^N}{\partial \xi^A} \omega_{MN} \, \mathrm{d}\xi^A = - \omega_{MN} \frac{\partial z^M}{\partial t} \, \mathrm{d}z^N \quad \mathrm{mod} \ \mathrm{d}t \tag{10}$$

for the one-form appearing in (9). It is convenient in practice to specify φ_t or $\bar{\varphi}$ by giving explicit expressions for a set of functions $\xi^A(z^M, t)$, which we call physical variables; on restriction to $\bar{X}$ these functions define (the inverses of) these embeddings in terms of the local coordinates.

3. Remarks

Our result (9) establishes necessary and sufficient conditions for a family of embeddings φ_t, or a choice of physical variables $\xi^A(z^M, t)$, to result in a Hamiltonian time-evolution equation (3) on Γ^*. For a given set of constraint subspaces X_t, or equivalently a set of constraint functions $\psi^I(z^M, t)$, a family of embeddings or physical variables having this property always exists locally. This follows from Darboux's Theorem, which tells us that locally we can find embeddings φ_t or choose coordinates $(\xi^A) = (q^\alpha, p_\alpha)$ on Γ^* such that $\omega^* = \mathrm{d}q^\alpha \wedge \mathrm{d}p_\alpha$. Since this expression is manifestly independent of time on Γ^*, it satisfies the criterion which we gave following (9). On the other hand, there are clearly many embeddings or choices of physical variables for which (3) will not hold, as can be seen by performing an arbitrary time-dependent coordinate transformation to make ω^* time-dependent.

Our result does not tell us how to explicitly construct a set of embeddings or physical variables with the required property, and in general this remains an open problem. A partial result in this direction is case (B) of Evans (1991). This applies to a system with time-independent gauge symmetry generators which has imposed on it a set of time-dependent gauge-fixing conditions involving some subset of canonical variables which all commute under the Poisson bracket. It is worth pointing out that if these canonical variables are regarded as configuration space coordinates in some equivalent Lagrangian description, then the result in question can also be obtained by first gauge-fixing the Lagrangian and then passing to the Hamiltonian formalism. The new examples we shall present below lie outside the scope of case (B) of Evans (1991). Thus the result (9) is still useful for finding good sets of physical variables, even though it offers no general method for doing so.

Finally we emphasize that our main motivation for the work summarized here is the reduced phase space approach to the canonical quantisation of systems which require time-dependent gauge choices. Even at the classical level, a system whose time evolution is not

described by an equation of the form of (3) falls outside the realm of conventional Hamiltonian mechanics. In passing to the quantum theory, (3) becomes the Heisenberg equation of motion, which guarantees the existence of a unitary time evolution operator. In the absence of a classical evolution equation of the form of (3), Gitman and Tyutin (1990a) have given an alternative prescription for the Heisenberg quantum evolution equation, in which extra terms appearing on the right hand side of (3) are taken over. This approach is complicated by the difficulty in obtaining an explicitly unitary time evolution. The consideration of simple examples such as the relativistic particle in an arbitrary background electromagnetic field (Evans 1993) reveals that our approach can be much simpler – compare with Gitman and Tyutin (1990b), Gavrilov and Gitman (1993). Batalin and Lyakovich (1991) have also considered the quantization of systems with time-dependent Hamiltonian and constraints.

4. Examples

We shall now apply our result (9) to discuss the gauge-fixing of relativistic particles and strings moving in d-dimensional Minkowski space-time with background gauge fields. We shall take coordinates x^μ on Minkowski space-time which are either 'orthonormal' with $\mu = 0, \ldots, d-1$, or of 'light-cone' type with $\mu = +, -, 1, \ldots, d-2$, so that the flat metric has components $-g_{00} = g_{11} = \ldots = g_{d-1\,d-1} = g_{+-} = g_{-+} = 1$ and all others vanishing. (This means that $x^\pm = (x^{d-1} \pm x^0)/\sqrt{2}$ agreeing with the conventions of Evans (1993) but not Evans (1991).) It is useful to set up some conventions regarding indices which will allow us to deal with temporal and light-cone gauge conditions in a uniform way. Thus we shall let the single index n on any vector denote either 0 or $+$, and we shall label the remaining components by $a = 1, \ldots, d-1$ or $a = -, 1, \ldots, d-2$ respectively. We shall also find it useful to denote the 'transverse' components by $i = 1, \ldots, d-2$. In what follows the ranges of these indices will always be understood.

4.1 Relativistic particle in an electromagnetic field
A particle of mass m and charge e moving in an arbitrary electromagnetic field $A_\mu(x^\nu)$ can be described by the Lagrangian

$$L = -\left[m\sqrt{-\dot{x}^2} + eA_\mu(x^\nu)\,\dot{x}^\mu \right] . \tag{11}$$

Here $x^\mu(t)$ is the particle's trajectory, t is a parameter along the worldline, and $\dot{x} = \mathrm{d}x/\mathrm{d}t$. Introducing the canonical momentum $p_\mu = \partial L/\partial \dot{x}^\mu$ conjugate to x^μ, we have coordinates (x^μ, p_μ) on phase space, with Poisson bracket $\{x^\mu, p_\nu\} = \delta^\mu_\nu$. There is a single, first-class constraint

$$\phi = (p + eA)^2 + m^2 = 0 . \tag{12}$$

The Hamiltonian is $H = \lambda\phi$, where λ is an arbitrary (time-dependent) function on phase space.

Consider the class of gauge-fixing conditions of the form

$$x^n = f(p_\mu, t) , \tag{13}$$

5

where f can be any function of its arguments which defines a good gauge choice (we shall make no attempt to be more precise concerning this last point). We define a set of physical variables

$$(\xi^A) = (x^{a*}, p_a) \quad \text{where} \quad x^{a*} = x^a - \int dp_n \frac{\partial f}{\partial p_a} \tag{14}$$

(partial derivatives and integrals of f are to be understood in terms of the functional dependence given by (13)) and it is clear that in principle the equations (12), (13) and (14) allow us to express all quantities as functions of (ξ^A, t). We claim that the system can then be described by these physical variables together with a Hamiltonian

$$H^* = - \int dp_n \frac{\partial f}{\partial t} \;, \tag{15}$$

which is valid for any background gauge field A_μ and any function f.

The explicit restriction to physical phase space is of course very involved for a general background field. In principle we can substitute from (13) and (14) into (12) to find p_n as a function of the physical variables and time, and substitution of this result back into (13) and (14) then determines all the x^μ as functions of (ξ^A, t). Fortunately, it is not necessary to carry out this elimination explicitly in order to verify that our chosen physical variables do indeed satisfy the criterion (9) leading to the general expression for the Hamiltonian given above. This is because the explicit time dependence of x^μ enters only through p_n and f; and by using this fact it is easy to calculate from (10) that $Y = -\mathrm{d}(\int dp_n \, \partial f/\partial t) \bmod \mathrm{d}t$. Since the original Hamiltonian H vanishes when $\phi = 0$, the result follows.

Examples are:

$$x^n = t \quad \text{giving} \quad x^{a*} = x^a \,, \quad H^* = -p_n \,, \tag{16}$$

which reproduces the temporal and light-cone results of Evans (1993);

$$
\begin{aligned}
x^+ &= p^+ t \quad \text{giving} \quad x^{-*} = x^- - p^- t \,, \quad x^{i*} = x^i \,, \quad H^* = -p_+ p_- \;; \\
x^0 &= p^0 t \quad \text{giving} \quad x^{a*} = x^a \,, \quad H^* = \tfrac{1}{2} p_0^2 \,.
\end{aligned}
\tag{17}
$$

These expressions are deceptively simple in appearance because they represent very complicated functions of the physical variables in the case of a general background. It is interesting that the Hamiltonians have universal forms in terms of the original momenta, in the sense that the dependence on the background field enters only through these particular functions.

4.2 Relativistic closed string in an antisymmetric tensor field

A closed string moving in an arbitrary background antisymmetric tensor field $B_{\mu\nu}(x^\rho)$ can be described by the Lagrangian

$$L = - \int_0^{2\pi} d\sigma \left[\left((\dot{x}.x')^2 - (\dot{x})^2 (x')^2 \right)^{1/2} + B_{\mu\nu}(x^\rho) \dot{x}^\mu x'^\nu \right] \;. \tag{18}$$

Here $x^\mu(t, \sigma)$ describes the string's trajectory, t and $0 \leq \sigma \leq 2\pi$ parametrize the world-sheet, and $\dot{x} = \partial x/\partial t$, $x' = \partial x/\partial \sigma$. Introducing the momentum $p_\mu(\sigma) = \delta L/\delta \dot{x}^\mu(\sigma)$

6

conjugate to $x^\mu(\sigma)$ as usual, we have coordinates $(x^\mu(\sigma), p_\mu(\sigma))$ on phase space, with Poisson bracket $\{x^\mu(\sigma), p_\nu(\sigma')\} = \delta^\mu_\nu \delta(\sigma - \sigma')$. The only constraints are first-class and are given by

$$(p_\mu + B_{\mu\nu} x'^\nu)^2 + (x')^2 = 0 \ , \tag{19}$$

$$x' . p = 0 \ . \tag{20}$$

Again the Hamiltonian H is proportional to the constraints.

It is useful to introduce the position zero-mode and total momentum of the string by

$$X^\mu(t) = \frac{1}{2\pi} \int_0^{2\pi} d\sigma \, x^\mu(t, \sigma) \ , \qquad P_\mu(t) = \int_0^{2\pi} d\sigma \, p_\mu(t, \sigma) \ . \tag{21}$$

The factor of 2π ensures X^μ and P_μ are conjugate variables. One can then write a decomposition of the string fields

$$x^\mu = X^\mu + \tilde{x}^\mu \ , \qquad p_\mu = \tfrac{1}{2\pi} P_\mu + \tilde{p}_\mu \ , \tag{22}$$

where the tilded variables represent the oscillator degrees of freedom.

We consider the class of gauge-fixing conditions of the form

$$x^n = f(P_\mu, t) \ , \tag{23}$$

$$p^n = \tfrac{1}{2\pi} P^n \ , \tag{24}$$

where f is, as before, any function of its arguments which provides a good gauge-fixing condition. The gauge conditions and constraints allow for the complete elimination of one pair of string position and momentum variables corresponding to the direction in space-time labelled by n, and they allow also for the elimination of one additional set of string oscillators. We introduce a set of physical variables

$$(\xi^A) = (X^{a*}, P_a, \tilde{x}^i(\sigma), \tilde{p}_i(\sigma)) \quad \text{where} \quad X^{a*} = X^a - \int dP_n \frac{\partial f}{\partial P_a} \tag{25}$$

(comments similar to those following (14) apply) and with a little thought one can see that the equations (19), (20), (23), (24) indeed allow us in principle to express all the original variables in terms of (ξ^A, t). At this stage the analysis looks very like that for the particle, at least as far as the string zero mode and total momentum variables are concerned. For the case of a light-cone gauge $(n = +)$ this comparison is accurate: for the physical variables given above one can again calculate Y from (10) and deduce that (9) holds, yielding a Hamiltonian

$$H^* = - \int dP_n \frac{\partial f}{\partial t} \ , \tag{26}$$

which is valid for any background field $B_{\mu\nu}$ and any gauge-fixing function f. For the case of a temporal gauge condition $(n = 0)$, however, the physical variables written above do not satisfy (9) in general, unless the background has some special symmetry. The previous arguments break down because the solutions for the redundant oscillator variables $\tilde{x}^{d-1}$

and $\tilde{p}_{d-1}$ in terms of (ξ^A, t) can depend explicitly on time for a general background field. This difficulty is absent if, for example, the background vanishes, $B_{\mu\nu} = 0$, and then the physical variables and Hamiltonian written above hold for any function f.

Examples are:

$$x^n = t \quad \text{giving} \quad X^{a*} = X^a \;, \quad H^* = -P_n \;; \tag{27}$$

and

$$x^+ = P^+ t \quad \text{giving} \quad X^{-*} = X^- - P^- t \;, \quad X^{i*} = X^i \;, \quad H^* = -P_+ P_- \;;$$
$$x^0 = P^0 t \quad \text{giving} \quad X^{a*} = X^a \;, \quad H^* = \tfrac{1}{2} P_0^2 \;. \tag{28}$$

These last examples generalise previous light-cone and temporal gauge-fixing constructions (Goddard *et al* 1973, Scherk 1975, Goddard *et al* 1975).

Acknowledgments

References

Abraham R and Marsden J E, 1978, *Foundations of mechanics* Benjamin/Cummings, 1978

Arnol'd V I, 1978, *Mathematical methods of classical mechanics* Springer, 1978

Batalin I A and Lyakovich S L, 1991, in *Group theoretical methods in physics* Nova Science, 1991, vol. 2, p. 57

Dirac P A M, 1950, Canad. J. Math. **2** 129

Dirac P A M, 1958, Proc. Roy. Soc. **A246** 326

Dirac P A M, 1964, *Lectures on quantum mechanics* Academic, 1964

Evans J M, 1991, Phys. Lett. **B256** 245

Evans J M, 1993, Class. Quantum Grav. **10** L221

Evans J M and Tuckey P A, 1993, Int. J. Mod. Phys. **A8** 4055

Gavrilov S P and Gitman D M, 1993, Class. Quantum Grav. **10** 57

Gitman D M and Tyutin I V, 1990a, *Quantization of fields with constraints* Springer, 1990

Gitman D M and Tyutin I V, 1990b, Class. Quantum Grav. **7** 2131

Goddard P, Goldstone J, Rebbi C and Thorn C B, 1973, Nucl. Phys. **B56** 109

Goddard P, Hanson A J and Ponzano G, 1975, Nucl. Phys. **B89** 76

Henneaux M, Teitelboim C and Vergara J, 1992, Nucl. Phys. **B387** 391

Mukunda N, 1980, Phys. Script. **21** 801

Rañada M F, 1994, J. Math. Phys. **35** 748

Scherk J, 1975, Rev. Mod. Phys. **47** 123