CERN-TH.7391/94

SPATIAL GEOMETRY OF HAMILTONIAN GAUGE THEORIES

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ABSTRACT

The Hamiltonians of $SU(2)$ and $SU(3)$ gauge theories in 3+1 dimensions can be expressed in terms of gauge invariant spatial geometric variables, i.e., metrics, connections and curvature tensors which are simple local functions of the non-Abelian electric field. The transformed Hamiltonians are local. New results from the same procedure applied to the $SU(2)$ gauge theory in 2+1 dimensions are also given.

Talk presented at the Conference
QCD '94
Montpellier, France, 7-13 July 1994

CERN-TH.7391/94
August 1994

We outline a formalism which contains a rather new approach to non-perturbative dynamics of the gluon sector of QCD. What is achieved is a formally exact transformation of the Hamiltonian on the physical subspace of states obeying the Gauss law constraint. The new Hamiltonian is local and is expressed in terms of gauge invariant spatial geometric variables, i.e., a dynamical metric $G_{ij}(x)$ which is a simple function of the non-Abelian electric field $E^{ai}(x)$ and the Christoffel connection Γ^i_{jk} and curvature-tensor $R^i_{jk\ell}$ computed from G_{ij} by the standard formulas of Riemannian geometry. For gauge group $SU(2)$ the underlying geometry is purely Riemannian, and the six gauge-invariant variables contained in G_{ij} are essentially all that are required. For gauge group $SU(3)$ there is a more complicated metric-preserving geometry with torsion, and the torsion tensors are expressed in terms of a set of 16 gauge-invariant variables. The Hamiltonian we find is admittedly complicated and has some strange features. But it also has some physical features, and I am moderately optimistic that physical and geometric insight can be combined so that results of physical interest can be drawn from the formalism.

We start with an observation about the basic equations of canonical Hamiltonian dynamics in $A^a_0 = 0$ gauge with conjugate variables $A^a_i(x)$, the non-Abelian vector potential, and $E^{ai}(x)$. The equal-time commutation relations, the Gauss law constraint, the non-Abelian magnetic field, and the Hamiltonian are

$$[A^a_i(x), E^{bj}(x')] = i\delta^{ab}\delta^j_i\delta^{(3)}(x-x') \tag{1}$$

$$G^a(x)\psi = \frac{1}{g}(\partial_i E^{ai} + gf^{abc}A^b_i E^{ci})\psi = 0 \tag{2}$$

$$B^{ai}(x) = \epsilon^{ijk}[\partial_j A^a_k + \frac{1}{2}gf^{abc}A^b_j A^c_k] \tag{3}$$

$$H = \frac{1}{2}\int d^3x\delta_{ij}[E^{ai}E^{aj} + B^{ai}B^{aj}] \tag{4}$$

We observe that (1)-(3) are covariant under spatial diffeomorphisms of the initial value manifold $\mathbf{R}^3$, that is coordinate transformation $x^i \to y^\alpha(x^i)$, $i, \alpha = 1, 2, 3$, with the transformation rules

$$A^a_i(x) \quad\to\quad A^a_\alpha(y) = \frac{\partial x^i}{\partial y^\alpha}A^a_i(x)$$

covariant vector

$$E^{ai}(x) \quad\to\quad E^{a\alpha}(y) = \left|\frac{\partial x}{\partial y}\right|\frac{\partial y^\alpha}{\partial x^i}E^{ai}(x)$$

contravariant vector density (5)

These rules are quite natural, since $A^a = A^a_i dx$ is a one-form, and E^{ai} is usually realized as $E^{ai}(x) = -i\delta/\delta A^a_i(x)$. The magnetic field is also a contravariant vector density. The Hamiltonian (4) is not invariant under the diffeomorphisms (5) because the Cartesian metric δ_{ij} appears. Nevertheless, we shall be guided in our work by the idea of preserving the diffeomorphic covariance of the canonical formalism.

We now summarize the recent preprint [1] in which these ideas are implemented in the electric field representation [2-4] of non-Abelian gauge theories, with state functionals $\psi[E^{ai}]$ and the potential $A_i^a = i\delta/\delta E^{ai}$. The constraint (2) can be expressed as

$$G^a(x)\psi = \left(\frac{1}{g}\partial_i E^{ai} - if^{abc}E^{bi}\delta/\delta E^{ci}\right)\psi \tag{6}$$

The second term, which we call $\bar{G}^a(x)$, is a local group rotation operator. If the constraint were simply $\bar{G}^a(x)\psi[E] = 0$, then we could easily find a broad class of states which satisfy it, namely wave functionals which depend on the local invariants formed from E^{ai}. For example, the second rank tensor density $\varphi^{ij} = E^{ai}E^{aj}$ is gauge invariant for any group, and for $SU(2)$ its six components constitute an essentially complete set of local invariants. For $SU(3)$ one must add the ten components of the third rank tensor density $\varphi^{ijk} = d^{abc}E^{ac}E^{bj}E^{ck}$.

The first key step in our work is to perform a unitary transformation [2] to eliminate the unwanted term in (6). We write

$$\psi[E] = \exp(i\Omega[E]/g)F[E] \tag{7}$$

and try to find a phase $\Omega[E]$ such that

$$G^a(x)\exp(i\Omega[E]/g)F[E] = \exp(i\Omega[E]/g)\bar{G}^a(x)F[E] \tag{8}$$

This leads to the two requirements on $\Omega[E]$

1. its gauge variation is
$$\delta\Omega[E] = \int d^3x\theta^a(x)\partial_i E^{ai}(x)\ ,$$
2. it is invariant under diffeomorphisms.

For gauge group $SU(2)$, these requirements are satisfied by

$$\Omega[E] = \frac{1}{2}\int d^3x\epsilon^{abc}E^{ai}E^{bj}\partial_i E_j^c \tag{9}$$

where E_j^c is the matrix inverse of E^{ai}. For a general group there is a local expression of similar structure, but E_j^c is replaced by a quantity $R_j^c = (M^{-1})_{j\ k}^{c\ d}E^{dk}$ where $M^{cj\ dk}$ is 3 dim $G\times$ 3 dim G direct product matrix which is a quadratic function of E^{ai}.

Unitary transformation of the operators of the theory gives

$$\bar{E}^{ai} \equiv \exp(i\Omega[E]/g)E^{ai}\exp(-i\Omega[E]/g) = E^{ai} \tag{10}$$

$$\begin{aligned}\bar{A}_i^a &\equiv \exp(i\Omega[E]/g)A_i^a\exp(-i\Omega(E)/g)\\ &\equiv i\frac{\delta}{\delta E^{ai}} + \frac{1}{g}\omega_i^a(x)\end{aligned} \tag{11}$$

$$\omega_i^a(x) \equiv -\frac{\delta\Omega}{\delta E^a(x)} \tag{12}$$

The second key step in our approach is to realize that, as an immediate consequence of 1. and 2. above, $\omega_i^a(x)$ transforms as a covariant vector under diffeomorphisms and as a gauge connection. So $\omega_i^a(x)$ is a composite gauge connection constructed as a local function of $E^{bj}(x)$ and $\partial_i E^{bj}(x)$. It is $\omega_i^a(x)$ which contains the geometric information in our approach, which we now explore for the case of gauge group $SU(2)$.

If we introduce the quantity $e_i^a(x)$ related to E^{ai} by

$$E^{ai} = \frac{1}{2}\epsilon^{ijk}\epsilon^{abc}e_j^b e_k^c \tag{13}$$

then $\omega_i^a = 1/2\epsilon^{abc}\omega_i^{bc}$ is exactly the dual of the Riemannian spin connection on a three-manifold with frame (dreibein) e_i^a. This has the important implication that a Riemannian spatial geometry underlies $SU(2)$ gauge theory. It was probably guaranteed that the approach would generate some geometry, but this could have been more complicated than Riemannian, perhaps with torsion or even non-metricity. The electric field is a geometric quantity, a densitized inverse dreibein, and it satisfies a condition of covariant constancy

$$\nabla_i E^{ak} \equiv \partial_i E^{ak} + \Gamma'^k_{ij} + \epsilon^{abc}\omega_i^b E^{ck} \equiv 0 \tag{14}$$

where

$$\begin{aligned} \Gamma'^k_{ij} &= -\frac{1}{2}\delta_j^k \partial_i \ln\det G \Gamma_i^k(G) \\ G_{ij} &= (\det\varphi)^{1/2}(\varphi^{-1})_{ij} \end{aligned} \tag{15}$$

is the standard Christoffel connection plus a $\partial_i \ln\det G$ term necessary because E^{ak} is a density.

The geometrization of gauge theory means that any locally gauge invariant quantity can be expressed in terms of φ^{ij} or G_{ij} (it is matter of convenience which of these tensor variables is used). Let us show how this is done for the unitary transformed "expectation value" of the Hamiltonian

$$\text{“} < F|H|F > \text{”} = \frac{1}{2}\int d^3x \delta_{ij} \quad \left[\bar{E}^{ai}\bar{E}^{aj}F^*F + (\bar{B}^{ai}F)^*(\bar{B}^{aj}F)\right] \tag{16}$$

The electric energy density simply involves the Cartesian trace of φ^{ij}. To work out the magnetic terms substitute (11) in the unitary transform of (3). On a general wave functional $F[E]$ one obtains

$$\bar{B}^{ai}(x)F[E] = \Big(\frac{1}{g}\hat{B}^{ai} + i\epsilon^{ijk}\hat{D}_j\frac{\delta}{\delta E^{ak}} - \frac{1}{2}g\epsilon^{ijk}f^{abc}\frac{\delta}{\delta E^{bj}}\ \frac{\delta}{\delta E^{ck}}\Big)F[E] \tag{17}$$

where $\hat{D}_j$ is a gauge covariant derivative with composite connection ω_j^b and $\hat{B}^{ai}$ is the magnetic field of ω.

We now impose the Gauss law constraint by letting $F \to F[\varphi^{ij}]$. Using the chain rule to convert $\delta/\delta E$ to $\delta/\delta\varphi$, the previous expression becomes

$$\bar{B}^{ai}(x)F[\varphi] = 2\{\frac{1}{g}E^{ap}(R_p^i - \frac{1}{2}\delta_p^i R) + i\epsilon^{ijk}E^{ap}\nabla_j\frac{\delta}{\delta\varphi^{kp}} - g\epsilon^{ijk}\epsilon^{pqr}E_r^a \det E\frac{\delta}{\delta\varphi^{jq}}\frac{\delta}{\delta\varphi^{kr}}\}F[\varphi] \tag{18}$$

Note that through the chain rule and (14), the connection terms necessary to make $\nabla_j \delta / \delta\varphi^{kp} F[\varphi]$ a spatial covariant derivative automatically appear, and that $\hat{B}^{ai}$ can be expressed as the electric field contracted with the Einstein tensor of the spatial geometry. One may now see that all gauge indices in (18) cancel in the Hamiltonian (16) because of $E^{ap}E^{aq} = \varphi^{pq}$, etc., so the Hamiltonian can be expected entirely in terms of gauge invariant geometric variables! (Actually, we have oversimplified the present discussion beginning in (13), where we effectively assumed that $\det E(x)$ is non-negative. Incorporation of both signs of $\det E(x)$ causes some complication for which we refer readers to [1]. We have also dropped certain operator ordering $\delta(0)$ terms which are treated in [1].)

The Hamiltonian has several unusual features, which also appear in the non-geometric treatments of [2-3].

1) Non-perturbative $1/g^2$ and $1/g$ terms appear as a consequence of the unitary transformation used to simplify Gauss' law. It is then far from clear how to do perturbative calculations to check the short-distance properties of the transformed theory. But these terms may be a virtue, since they are a consequence of the exact treatment of the non-Abelian gauge invariance.

2) The Hamiltonian is non-polynomial in φ^{ij} or G_{ij}. It contains imaginary terms and terms up to fourth order in functional derivatives.

3) There are singularities in H when $\det E = \sqrt{\det \varphi} = 0$, which can be traced back to the fact that the $SU(2)$ phase (9) requires the inverse matrix. We take the view that these singularities are the gauge theory analogue of the angular momentum barrier in central force quantum mechanics. Our recent work suggests that this energy barrier operates in the following way. For a configuration $\varphi^{ij}(x)$ for which $\det \varphi$ vanishes on a two-surface, the energy density contains singular factors such that $\int d^3x$ in (16) diverges unless $F[\varphi^{ij}]$ itself vanishes. Since the singularities are one way in which the gauge theory Hamiltonian in gauge-invariant variables differs from that of φ^4 theory, one may speculate that the singularities are a clue to the special dynamical features of gauge theories at low energy. It is this that we are now studying.

Finally we state that the spatial geometry of the $SU(3)$ theory was also discussed in [1]. Although results are not as explicit as for $SU(2)$, one can see that the spatial geometry of $SU(3)$ is more complicated. There is a covariantly constant dynamical metric, but there is torsion of both conventional and novel type.

Very recently, Bauer and Freedman have applied the same geometrical ideas to $SU(2)$ gauge theory in 2+1 dimensions. The gauge coupling g carries dimension, $[g^2] = 1$, and is therefore "pulled out" in front of the Lagrangian. The potential $A_i^a(x)$ and electric field $E^{aj}(x)$ both have dimension one. The magnetic field is a scalar density

$$B^a(x) = \epsilon^{ij}[\partial_i A_j^a + \frac{1}{2}\epsilon^{abc}A_i^b A_j^c] \ , \tag{19}$$

and g disappears from the Gauss constraint (2), but appears in the energy density of (4) which becomes $[g^2E^2 + g^{-2}B^2]$. The gauge invariant tensor density $\varphi^{ij} = E^{ai}E^{aj}$ and the dynamical metric tensor are simply related by $\varphi^{ij} = \epsilon^{ik}\epsilon^{j\ell}G_{k\ell}$ and have dimension two.

The coupling g also disappears from the unitary transformation (7), and we find that the phase $\Omega[E]$ which satisfies the requirements 1. and 2. is

$$\Omega[E] = \int d^2x \epsilon^{abc} E^{ai} E^{bj} (\varphi^{-1})_{jk} \partial_i E^{ck} \tag{20}$$

Physical states obeying the transformed gauge constraint can be taken as functionals $F[G_{ij}]$ of configurations of positive semi-definite symmetric tensors $G_{ij}(x)$ on the plane.

We need a basis of vectors for the adjoint representation of $SU(2)$ to obtain information on the spatial geometry from the composite connection ω_i^a which is the variational derivative (12) of (20). We use the basis e^{a1}, e^{a2} and e^a defined by

$$\begin{aligned} e^{ai}(x) &\equiv \frac{1}{\sqrt{G}} E^{ai} \qquad i = 1, 2 \\ \epsilon^{abc} e^{bi} e^{cj} &\equiv \frac{\epsilon^{ij}}{\sqrt{G}}\, e^a \end{aligned} \tag{21}$$

The gauge covariant derivative $\hat{D}_i e^{ak}$ can be expanded in the basis as

$$\hat{D}_i e^{ak} \equiv -\Gamma_{ij}^k e^{aj} - T_i^k e^a \tag{22}$$

This expression is the analogue of (14).

Even without the specific form of ω_i^a, one can show from (22) that Γ_{ij}^k transforms under diffeomorphisms of the plane as a connection which is metric compatible, while T_i^k is a tensor. At this stage, Γ_{ij}^k could have an anti-symmetric part, a possible torsion tensor. However, when the specific form of ω_i^a is inserted in (22), one finds after detailed calculation that T_i^k vanishes and that Γ_{ij}^k is the symmetric Christoffel connection. This is the first simplification of the 2 + 1 dimensional case – the underlying geometry is two-dimensional Riemannian, although the Lie algebra is three-dimensional and the general expansion (22) suggests torsion. (A frame and expansion analogous to (21)-(22) occur in the case of $SU(3)$ in 3 + 1 dimensions, and torsions do not vanish.)

The final step is to work out the transformed Hamiltonian. Here there is another simplification: due to the (partial) orthogonality of the frame (21), terms with an explicit imaginary i cancel, and expectation values can be written as the sum of three real positive terms,

$$
\begin{aligned}
\langle F|H|F\rangle \;=\; & \frac{1}{2}\int d^2x \int [dG_{ij}]\Big\{ g^2\delta^{ij}G_{ij}F^*F \\
& + \frac{4G_{k\ell}}{g^2}\,(\nabla_i\,\frac{\delta F^*}{\delta G_{ik}})\,(\nabla_j\,\frac{\delta F}{\delta G_{j\ell}}) \\
& + \frac{\det G}{g^2}\,\left|\frac{1}{2}RF + 2\hat\epsilon_{ik}\hat\epsilon_{j\ell}\,\frac{\delta^2 F}{\delta G_{ij}\delta G_{k\ell}}\right|^2\Big\}
\end{aligned} \tag{23}
$$

where $\nabla_j(\delta F/\delta G_{j\ell})$ is the Riemannian covariant derivative, R is the scalar curvature, and $\hat\epsilon_{ik} = (\pm 1, 0)$. The functional measure is simply

$$
[dG_{ij}] = \prod_x \prod_{i\le j} dG_{ij}(x) \tag{24}
$$

The Hamiltonian (23) is considerably simpler than the three-dimensional case (16). Yet it has the same qualitative features, so it should be useful for pilot studies of gauge field dynamics.

References

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