Non-Perturbative Renormalization Group Flows in Two-Dimensional Quantum Gravity

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Abstract

Recently a block spin renormalization group approach was proposed for the dynamical triangulation formulation of two-dimensional quantum gravity. We use this approach to examine non-perturbatively a particular class of higher derivative actions for pure gravity.

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I. INTRODUCTION

Dynamically triangulated random surfaces provide a lattice representation of two-dimensional quantum gravity [1,2]. Both in the continuum and on a simplicial lattice the usual Einstein action based on the Ricci scalar is a topological invariant. Thus the simplest action for the lattice theory at fixed volume and genus can then be taken as zero.

In principle, it is possible to add other operators to this lattice action which are consistent with the underlying symmetries of the model - here reparametrization invariance. The lattice action would then take the form $S = \sum_i \beta_i O_i$ where $\{O_i\}$ are a set of generic operators with associated coupling constants $\{\beta_i\}$. For example, it is natural to consider operators which are the lattice analogues of higher derivative terms – integrals of powers of the scalar curvature. In general these actions may then possess one or more critical points $\{\beta_i^c\}$ in the coupling constant space where it may be possible to construct continuum limits for the model.

The usual theory of two dimensional quantum gravity is constructed about the special point $\beta_i = 0$. Perturbation theory then indicates that the higher operators are all irrelevant in the renormalization group sense – that is the long distance continuum physics of models with β_i non-zero is identical to that at the fixed point $\beta_i = 0$. Unfortunately, perturbation theory can tell us nothing, in principle, about the existence and properties of other fixed points situated in regions of the parameter space where any of the β_i are not small. To probe such regions a nonperturbative procedure is required. For conventional statistical mechanical models the block spin renormalization group is one such technique [3]. In this technique, a local kernel is used to construct an effective theory with a carefully controlled change of scale which allows the calculation of critical couplings and critical exponents.

Such a block spin formalism has recently been developed for dynamical triangulations and applied to two-dimensional quantum gravity coupled to Ising spins [4]. In contrast, a heuristic renormalization group inspired approach has been advocated in [5]. In this paper, we apply the block spin renormalization group approach to pure quantum gravity. The aim is to explore the fixed point structure of the lattice model when a particular class of higher derivative operator is included in the action. Specifically, we use an action $S = \alpha \sum_i \ln q_i$ where q_i is the coordination number of a site. Such a term consists of an infinite series of powers of the curvature and arises naturally when we couple the theory to scalar fields. We show that the approach does indeed yield an appropriate fixed point and present results which give strong evidence for the *nonperturbative* irrelevance of such higher order curvature terms.

II. BLOCK SPIN RENORMALIZATION GROUP

The details of the algorithm are given in [4]. Here, we just summarize the approach. The traditional way of implementing the renormalization group within a numerical simulation is to generate a sequence of lattice field configurations which are distributed according to the usual Boltzmann weight. Each of these is then progressively coarsened in some way which preserves the long distance physics. Corresponding to each initial fine lattice configuration a succession of 'blocked' lattices is thus generated. Typically, the fields on each 'blocked' lattice are determined by the fields of the lattice at one less blocking level. By examining the flows of expectation values of a set of operators and their correlators as a function of blocking level, it is then possible to extract the critical couplings and critical exponents.

The choice of an apt 'blocking' transformation is a very important issue. For the case of random triangulations, the lattice itself is the dynamical object. We thus require an algorithm for replacing a given random mesh with a succession of coarsened descendents with approximately the same long distance features. The most natural way to measure distance in this context is by defining all lattice links to have length unity. The distance between any two points is then taken as the geodesic length between them – the length (in lattice units) of the shortest path connecting them on the lattice.

In order for the blocking algorithm to be apt it must be able to replace a given mesh by one with a subset of the nodes triangulated in such a way that the *relative lengths of blocked geodesics reflect the underlying geodesic structure.* That is, like a metric, the blocked triangulation tells us which points are near and which are far apart and this must accurately reflect the situation on the underlying lattice. It appears to be a hard problem to give a rule which when applied to an arbitrary random lattice accomplishes this task. Our method, however, relies on a simple, local, iterative procedure to generate the coarsened lattices.

Suppose, by some method, it has been possible to generate a set of blockings of a given triangulation. In order to generate a Monte Carlo sample, the fine lattice (blocking level zero) is then updated using the stochastic link flip algorithm. In order that the coarsened lattices reflect the new fine lattice it is necessary to perform block link flips according to some suitable rule. This rule then ensures that they 'follow' the parent lattice as it is updated. Denote a generic lattice at blocking level k by T_k and its successor at level $k+1$ by T_{k+1}. Thus, any rule which specifies when to flip links in T_k in response to flips of the links in T_{k-1} provides a definition of the blocking transformation. An apt rule appears to be to flip a block link in T_k whenever that would connect two points that are closer (on the lattice T_{k-1}) than the two currently linked. This process is iterated recursively to generate a tower of blocked lattices for each fine lattice. This block rule ensures that a given block lattice is determined from its 'parent' at one less blocking level in such a way that the relative distance of blocked nodes is preserved.

There are two convenient ways to choose the original lattice and its blocked form. One is to start with a regular lattice and to choose distinct subsets of points (those corresponding to a usual square lattice blocking) that can obviously be triangulated in a regular way. The other is to start with a triangulation that is viewed as the block lattice and to add as many points as desired to produce the bare lattice. Updating the block lattice with a number of block link sweeps then relaxes the block lattice.

The Monte Carlo cycle thus begins with an update sweep of the fine lattice followed by a number of applications of the block link update rule (typically five to ten block link sweeps) at each blocking level.

Any expectation values computed on a blocked lattice can be viewed as coming from an effective action. There is a sequence of effective actions that corresponds to the sequence of blocking levels. If the original action is critical (and if the renormalization group transformation is apt), this sequence converges to a fixed point. Such should be the case for dynamical triangulations with action equal to zero. Such should also be the case if any irrelevant term is added to the action. In this case, the sequence should converge, not just to any fixed point, but to the same fixed point obtained without the irrelevant terms.

In practice, although the effective actions converge to a fixed point when the theory is critical, the expectation values obtained on the block lattices do not. This is because each renormalization group transformation reduces the size of the lattice and hence increases the finite size effects. A single sequence of blocking levels with their corresponding expectation values will not display convergence toward a fixed point. However, two sequences, beginning with bare lattices of different volumes can do this. The trick is to choose the bare lattice of one of the sequences to have the same volume as the first blocked level of the other sequence. In this way, expectation values can be compared on lattices with the same volume (and therefore the same finite size effects) but with different numbers of iterations of the renormalization group transformation. Since the finite size effects are identical, any difference in expectation values can only be due to a difference in effective actions. As the renormalization group transformation is iterated and the actions flow toward a fixed point, the difference in effective actions should rapidly decrease yielding a progressively smaller difference in expectation values.

III. RESULTS

Our first goal, then, is to implement the block spin renormalization group transformation described above on dynamical triangulations with an action equal to zero and to see if the matching procedure just outlined produces pairs of expectation values that are increasingly close as the blocking level is increased. Seven operators are used in this study. The first six are all powers or correlations of the coordination number at a site (q_i) minus six (its flat-space, regular lattice value) and are all normalized by the number of links. The first is the nearest neighbor correlation:

$$O_1 = \sum_{<ij>} (q_i - 6)(q_j - 6).$$

The second is the correlation between the nodes conjugate to a link (the nodes that the link would join if it were flipped):

$$O_2 = \sum_{<ij>} (q_{i'} - 6)(q_{j'} - 6)$$

(where i' and j' represent the nodes where the ends of the flipped link would go). The third is the product of the first two:

$$O_3 = \sum_{<ij>} (q_i - 6)(q_j - 6)(q_{i'} - 6)(q_{j'} - 6).$$

The fourth, fifth, and sixth are the second, third, and fourth powers of the coordination number minus six:

$$O_4 = \sum_i (q_i - 6)^2, \quad O_5 = \sum_i (q_i - 6)^3, \quad O_6 = \sum_i (q_i - 6)^4.$$

Finally, the seventh operator is the maximum coordination number of the lattice:

$$O_7 = \max(q_i)$$

Lattices were used with 9, 36, 144, 576, and 2304 nodes which allowed for up to four iterations of the blocking transformation. The results shown correspond to 1×10^5 bare lattice sweeps. Table 1 shows the expectation values at all blocking levels starting from the largest lattice. There is a great variation in the expectation values as a function of block level and it is not at all obvious that they are approaching a fixed point. The matching can be seen in table 2 which compares the seven expectation values after three and four iterations of the blocking transformation on lattices such that the final number of nodes is nine. They match fairly well, an indication that the effective theory is near its fixed point. Figure 1 uses expectation value differences of O_7 to give a graphical representation of the approach to the fixed point.

Now consider a perturbation of this scenario using the action

$$S = \alpha \sum_i \ln(q_i)$$

If this term is irrelevant, the sequence of expectation values generated by iterating the blocking transformation should approach those generated with $S = 0$ at large blocking levels, even if the expectation values differ a great deal at the lower blocking levels. Table 3 shows the data in the case of $\alpha = -1$. Figure 2, using O_7 again, gives a graphical representation of this data along with data for $\alpha = +1$. The fact that the expectation value differences approach zero as the blocking level increases confirms that $\ln(q)$ is indeed an irrelevant operator. The results are similar for much larger α. Figure 3 shows the analogous data for $\alpha = \pm 10$. At a true fixed point, all of the expectation values should match, not just one. Figures 4 and 5 give the expectation value differences for O_1 at the same values of α as in figures 2 and 3 respectively. Matching is demonstrated for this operator as well.

In [1] there is evidence that for negative enough α there is a phase transition to some crumpled state. Such a transition is not visible in perturbation theory [6]. If such a transition exists, one would expect the expectation values to flow to a set of values different from those obtained with $S = 0$. We find that while at negative values of α the expectation values on the bare lattice start looking dramatically different from those at $S = 0$ (for instance the value of O_7 increases by more than an order of magnitude) the renormalization group trajectories flow to the same point within statistics.

Thus, the renormalization group scheme used here gives no evidence for a phase transition. It is possible that there is such a transition and that either the particular renormalization group transformation used here is not "apt" for that transition or the expectation value differences of the blocked operators are smaller than our errors. The statistical uncertainty of the $\alpha = -10$ data at the highest level of blocking is from three to five times larger (depending on the operator) than that of any of the other values of α considered in this paper. In this regard, it should be noted that the effects of Ising matter at the critical point on expectation values in the gravitational sector are too small to be detected with current statistics. However, the effects of matter on the gravitational sector are notoriously small for this formulation of quantum gravity whereas the higher derivative term can clearly have a strong effect. It may be that there is a transition that is nearby in the space of theories possibly of a higher order of multicriticality. To see such a fixed point would require tuning of additional couplings.

To summarize, we have presented results concerning the fixed point structure of the dynamical triangulation model for two dimensional quantum gravity. These have been obtained using an adaptation of the Monte Carlo renormalization group to the situation where the lattice itself carries the dynamical degrees of freedom. We have studied a class of higher derivative operator and given evidence that such an operator is truly irrelevant outside of perturbation theory. We see no evidence for new fixed points or equivalently new phase transitions in the lattice model. It is possible however, that other choices of higher derivative operator might indeed show new structure [7]. The technique used in this paper can easily be applied to other actions as well.

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List of Figures

1. The difference between expectation values of the maximum coordination number, O_7, computed on lattices of the same size but for systems that differ (by one) in the number of times they have been blocked. The blocking level listed is that of the system that has been blocked the most. The original action is zero.

2. The difference between expectation values of the maximum coordination number, O_7, computed with two different actions as a function of the blocking level. The diamonds represent expectation values obtained with $S = \alpha \sum_i \ln(q_i)$ minus those obtained with $S = 0$ when $\alpha = +1$ while the squares represent the analogous results for $\alpha = -1$.

3. This figure is like figure two except that the squares represent $\alpha = +10$ and the crosses represent $\alpha = -10$. Crosses are missing for levels zero and one because the data is off scale by more than an order of magnitude.

4. This is the same plot as figure 2 except that O_1 is used instead of O_7.

5. This is the same plot as figure 3 except that O_1 is used instead of O_7. Again, some of the $\alpha = -10$ data is off scale.

TABLES

operator	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
O_1	1.499(1)	4.99(5)	4.33(6)	1.12(5)	-0.367(7)
O_2	2.611(2)	6.68(5)	8.3(2)	4.2(1)	0.35(2)
O_3	-4.63(2)	10.9(5)	-2(1)	0(1)	0.19(8)
O_4	3.500(1)	5.01(2)	5.42(8)	3.91(6)	0.70(1)
O_5	22.25(2)	56.5(8)	70(3)	26(1)	-0.58(3)
O_6	328.7(7)	1400(30)	1900(200)	370(20)	3.8(1)
O_7	29.87(3)	35.2(2)	28.8(4)	17.4(2)	7.89(1)

TABLE I. Expectation values of seven operators at all blocking levels beginning with a 2304 node lattice. The action is zero.

operator	$V = 2304$	$V = 576$
O_1	-0.367(7)	-0.363(4)
O_2	0.35(2)	0.34(1)
O_3	0.19(8)	0.27(4)
O_4	0.70(1)	0.677(5)
O_5	-0.58(3)	-0.52(1)
O_6	3.8(1)	3.54(5)
O_7	7.89(1)	7.881(6)

TABLE II. The expectation value of the seven operators on two lattices which have been blocked three and four times, respectively. The original volumes were chosen so that the final blocked lattices have the same number of nodes (nine).

operator	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
O_1	2.295(2)	5.10(6)	4.43(6)	1.12(5)	-0.367(8)
O_2	4.138(4)	7.16(8)	8.4(2)	4.5(2)	0.34(2)
O_3	-10.93(5)	11.1(6)	-2(2)	0(1)	0.31(8)
O_4	4.360(2)	5.16(3)	5.47(5)	4.04(8)	0.694(8)
O_5	38.68(7)	66(1)	71(2)	28(1)	-0.56(2)
O_6	727(3)	2000(70)	1900(100)	416(30)	3.71(7)
O_7	36.79(5)	37.5(3)	29.1(2)	17.9(2)	7.89(1)

TABLE III. Expectation values of seven operators at all blocking levels beginning with a 2304 node lattice. The action is $S = -\sum_i \ln(q_i)$