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Cancellation of Renormalon Ambiguities in the Heavy Quark Effective Theory

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Abstract

Recently, it has been shown that the concept of the pole mass of a heavy quark becomes ambiguous beyond perturbation theory, because of the presence of infrared renormalons. We argue that the predictions of heavy quark effective theory, whose construction is based on the pole mass, are free of such ambiguities. In the $1/m_Q$ expansion of physical quantities, infrared and ultraviolet renormalons compensate each other between coefficient functions and matrix elements. We trace the appearance of these compensations for current-induced exclusive heavy-to-heavy and heavy-to-light transitions, and for inclusive decays of heavy hadrons. In particular, we show that the structure of the heavy quark expansion is not obscured by renormalons, and none of the predictions of heavy quark effective theory are invalidated.

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1 Introduction

For heavy quarks, the non-relativistic bound-state picture suggests the notion of the pole mass $m_Q^{\rm pole}$ defined, order by order in perturbation theory, as the position of the singularity in the renormalized quark propagator. The pole mass is gauge invariant, infrared finite, and renormalization-scheme independent [1]. In the context of perturbation theory, it is thus a meaningful "physical" parameter. Once non-perturbative effects are taken into account, however, this concept needs to be generalized, since in reality there is no pole in the quark propagator because of confinement. Recently, it has been shown that signals for such non-perturbative effects can be found in the asymptotic behaviour of perturbation theory itself. The presence of infrared renormalons in the perturbative series that relates the pole mass to a mass defined at short distances leads to an unavoidable ambiguity of order $\Lambda_{\rm QCD}$ in the definition of $m_Q^{\rm pole}$ [2, 3]. The appearance of renormalons signals that perturbation theory is incomplete without the inclusion of non-perturbative corrections. In fact, much of the non-perturbative structure of a theory can be inferred from a study of the singularities of correlation functions after Borel transformation with respect to the coupling constant. The application of this approach to QCD was pioneered by 't Hooft [4]. The positions of the singularities on the positive real axis signal the magnitude of non-perturbative corrections. In turn, the structure of non-perturbative corrections implies constraints for the structure of infrared renormalons [5]–[7].

The existence of a "physical" definition of the mass of a heavy quark, which agrees with the pole mass up to terms of order $\Lambda_{\rm QCD}$, plays a crucial role in the construction of the heavy quark effective theory (HQET) [8]–[17], which by now has become the main theoretical tool used to analyze the properties and decays of hadrons containing a heavy quark. In view of the intrinsic ambiguity in the definition of the pole mass, the question arises whether the HQET is an inconsistent effective theory, whose predictions are plagued by renormalon ambiguities. The purpose of this paper is to demonstrate that this is not the case. Renormalons enter HQET predictions because one tries to separate perturbative and non-perturbative (as opposed to short- and long-distance) effects into coefficient functions and matrix elements. Infrared renormalons appear in the coefficient functions since soft loop momenta give a non-negligible contribution to the Feynman integrals which appear in their calculation. Similarly, ultraviolet renormalons appear in the matrix elements because of the power divergence of Feynman integrals in the HQET. In this paper, we argue that in predictions for physical quantities such as weak decay amplitudes, renormalon ambiguities cancel between coefficient functions and matrix elements.[1] A generic HQET prediction for

[1]For the case of the heavy quark two-point function this cancellation has been demon-

a physical quantity $A(m_Q)$ is of the form

$$A(m_Q) = C_0(m_Q/\mu)\, M_0(\mu) + \frac{1}{m_Q}\, C_1(m_Q/\mu)\, M_1(\mu) + \dots \,. \qquad (1)$$

We will trace the cancellation of renormalons explicitly to subleading order in $1/m_Q$, by showing that the infrared renormalon in C_0 cancels against an ultraviolet renormalon in the matrix element M_1, so that the sum of the two terms on the right-hand side is unambiguous. In more complicated processes such as flavour-changing transitions between two heavy hadrons of different velocity, the way in which these cancellations take place is rather non-trivial. However, that they take place should not be a surprise. In fact, the appearance of renormalons could be avoided if in the construction of the HQET one would follow the idea of Wilson's operator product expansion (OPE) [18] literally [2, 3]. The OPE is not designed to separate perturbative and non-perturbative effects, but to disentangle the physics on different distance scales. This is not accomplished when one uses dimensional regularization in the calculation of the coefficient functions. Instead, one should introduce a hard factorization scale $\mu < m_Q$ by cutting out momenta $k < \mu$ from the Feynman diagrams which determine the Wilson coefficients, and attribute these contributions to the matrix elements. In practice, this procedure is impracticable and awkward, but it would eliminate the infrared renormalons from the coefficient functions and the ultraviolet renormalons from the matrix elements. What is important is that in the HQET such a program could be implemented without changing the transformation property of the effective Lagrangian under the spin-flavour symmetry [19]. Hence, the structure of the predictions obtained using the HQET remains unaffected. This implies that renormalons enter the usual (practical) form of the HQET in such a way that they do not spoil the relations imposed by heavy quark symmetry [20] and the equation of motion (including the vanishing of certain $1/m_Q$ corrections at zero recoil [14]), and they do not increase the number of hadronic form factors that appear in a given order of the $1/m_Q$ expansion.

Let us note in passing that in the lattice formulation of the HQET (or indeed in any regularization scheme with a dimensionful cut-off) one encounters ultraviolet divergences which behave as powers of the ultraviolet cutoff (i.e. inverse powers of the lattice spacing). These power divergences are due to the mixing of higher dimensional operators with lower dimensional ones. They become more severe as higher-order terms in the $1/m_Q$ expansion are calculated. The presence of power divergences, and the fact that they are likely to imply the existence of non-perturbative effects, and hence to require non-perturbative subtractions, was explained in Ref. [21]. The close connection between the presence of power divergences and that of ultraviolet

strated in Ref. [2].

renormalons in matrix elements of higher dimensional operators in the HQET was pointed out in Ref. [2], and will become apparent below. Techniques for the non-perturbative subtraction of the power divergences in lattice simulations are being developed [22]; a brief outline of the approach can be found in Ref. [23].

The outline of the paper is as follows: In Sect. 2, we briefly discuss the appearance of renormalons in the asymptotics of perturbation theory and their relation to singularities in the Borel transform of correlation functions. To obtain a renormalon calculus which is convenient for explicit calculations, we follow Refs. [2, 7] and consider QCD in the limit of a large number of light quark flavours. We then recall some of the reasoning behind the usual construction of the HQET and show in which way it is affected by infrared renormalons in the pole mass of the heavy quark. In Sects. 3 and 4, we study the cancellation of renormalons in exclusive heavy-to-heavy and heavy-to-light decay processes. In the first type of decays, the symmetries of the effective theory imply a set of non-trivial consistency conditions, which relate the infrared renormalons in the coefficient functions of bilinear heavy quark currents to the infrared renormalon in the pole mass. We derive the exact form of these relations, which are independent of any unknown hadronic matrix element. We then check them to order $1/N_f$. We also show with an explicit calculation that a sum rule recently derived by Shifman et al. [24], which has been used to put a bound on the hadronic form factor that enters the extraction of $|V_{cb}|$ from semileptonic decays, cannot be correct, due to a mismatch of infrared and ultraviolet renormalons. In Sect. 5, we show that renormalon contributions cancel in inclusive, current-induced decays of hadrons containing a heavy quark. This proves a conjecture of Bigi et al. [3], although we do not agree on the details of the cancellation. In Sect. 6, we summarize our results and give some conclusions.

2 Renormalons and the Construction of the HQET

Given a perturbative series for some quantity $F(\alpha_s)$ in terms of the coupling constant $\alpha_s(\mu)$ renormalized at some scale μ,

$$F(\alpha_s) = \sum_{n=0}^{\infty} F_n \left(\frac{\beta_0}{4\pi}\,\alpha_s(\mu)\right)^n , \tag{2}$$

where $\beta_0 = 11 - \frac{2}{3}N_f$ is the first coefficient of the β-function, we define the Borel transform $\widetilde{F}(u)$ of $F(\alpha_s)$ by

$$\widetilde{F}(u) = F_0\,\delta(u) + \sum_{n=0}^{\infty} \frac{1}{n!}\,F_{n+1}\,u^n . \tag{3}$$

If the series is Borel summable, the function $F(\alpha_s)$ can be reconstructed from its Borel transform using the integral relation

$$F(\alpha_s) = \int\limits_0^\infty \mathrm{d}u\, \exp\left(- \frac{4\pi u}{\beta_0\, \alpha_s(\mu)} \right) \widetilde{F}(u)\,. \tag{4}$$

However, if the coefficients F_n in (2) develop a factorial divergence for large n, the Borel transform $\widetilde{F}(u)$ can have singularities on the integration contour, and the naïve Borel summation fails. In such a case, the result of the integration depends on a regularization (or resummation) prescription, and $F(\alpha_s)$ is not uniquely defined in terms of $\widetilde{F}(u)$.

In QCD, one source of divergence in the expansion coefficients of a perturbative series is related to higher-order diagrams in which a virtual gluon line with momentum k is dressed by a number of fermion, gluon and ghost loops.[2] Effectively, this introduces the running coupling constant $g_s(k)$ at the vertices. Since the coupling constant increases for low momenta because of asymptotic freedom, the insertion of additional bubbles drives the gluon line to increasingly softer momentum, i.e. the infrared region in Feynman integrals becomes more important. When the running coupling constant is expressed in terms of a fixed coupling constant renormalized at some large scale μ, using

$$\alpha_s(k) \simeq \frac{\alpha_s(\mu)}{1 - \dfrac{\beta_0}{4\pi}\,\alpha_s(\mu)\,\ln\dfrac{\mu^2}{k^2}} = \sum_{n=0}^\infty\, [\alpha_s(\mu)]^{n+1} \left(\frac{\beta_0}{4\pi}\, \ln\frac{\mu^2}{k^2} \right)^n , \tag{5}$$

the appearance of powers of large logarithms leads to a factorial divergence in the expansion coefficients F_n in (2). Associated with this are renormalon singularities in the Borel transform $\widetilde{F}(u)$.

In our case, the renormalon singularities will occur as single poles on the real axis in the Borel plane. Poles on the positive real axis, which arise from the low-momentum region of Feynman diagrams, are called infrared renormalons.[3] Let us denote the positions of these poles by u_i and their residues by r_i, so that

$$\widetilde{F}(u) = \sum_i \frac{r_i}{u - u_i} + \dots\,, \tag{6}$$

where the ellipses represent terms that are regular for $u > 0$. For the calculation of the inverse Borel transform from (4), we may write the pole

[2]We hasten to add that in a non-abelian theory bubble summation is not a gauge-invariant procedure. This is one of the reasons why we will have to use a large-N_f expansion to obtain a consistent renormalon calculus, see below.

[3]Similarly, poles on the negative real axis arise from the high-momentum region and are called ultraviolet renormalons.

denominators in terms of a principle value and a δ-function contribution:

$$\frac{1}{u-u_i} \to \mathrm{P}\,\frac{1}{u-u_i} + \eta_i\,\delta(u-u_i)\,. \tag{7}$$

Here, η_i is a complex number which depends on the regularization prescription. For instance, one may choose one of the following regularizations (with $\delta \to +0$)

$$\begin{aligned} \frac{u-u_i+\kappa\delta}{(u-u_i)^2+\delta^2} &\quad\to\quad \eta_i = \kappa\pi\,, \\ \frac{1}{u-u_i \mp i\delta} &\quad\to\quad \eta_i = \pm i\pi\,. \end{aligned} \tag{8}$$

One may also choose the principal value prescription itself, in which case $\eta_i = 0$. We write the regularized form of the Borel transform as

$$\widetilde{F}(u) = \widetilde{F}_{\mathrm{reg}}(u) + \sum_i \eta_i\,r_i\,\delta(u-u_i)\,, \tag{9}$$

where by definition $\widetilde{F}_{\mathrm{reg}}(u)$ contains the pole terms regularized with a principle value prescription, and all ambiguity resulting from the freedom to use a different prescription resides in the δ-function contributions. The inverse Borel transformation then leads to

$$F(\alpha_s) = F_{\mathrm{reg}}(\alpha_s) + \sum_i \eta_i\,r_i\,\exp\left(-\frac{4\pi u_i}{\beta_0\,\alpha_s(\mu)}\right) \simeq F_{\mathrm{reg}}(\alpha_s) + \sum_i \eta_i\,r_i \left(\frac{\Lambda_{\mathrm{QCD}}}{\mu}\right)^{2u_i}, \tag{10}$$

where $F_{\mathrm{reg}}(\alpha_s)$ is the inverse Borel transform of $\widetilde{F}_{\mathrm{reg}}(u)$. In the last step, we have used the one-loop expression

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0\,\ln(\mu^2/\Lambda_{\mathrm{QCD}}^2)} \tag{11}$$

for the running coupling constant.[4] These definitions make explicit the fact that terms which depend on the regularization prescription are exponentially small in the coupling constant, i.e. they have the form of power corrections. The leading asymptotic behaviour is determined by the nearest infrared renormalon pole at $u = u_1$. We define the renormalon ambiguity ΔF as the coefficient of η_1:

$$\Delta F = r_1\,\exp\left(-\frac{4\pi u_1}{\beta_0\,\alpha_s(\mu)}\right) \simeq r_1 \left(\frac{\Lambda_{\mathrm{QCD}}}{\mu}\right)^{2u_1}. \tag{12}$$

It is a measure of the intrinsic ambiguity in the quantity F arising from the necessity to regularize the divergent behaviour of perturbation theory in large

[4]Note that the last relations in (10) and (12) become exact in the large-N_f limit.

orders. It is the purpose of this paper to trace how these leading (in powers of $\Lambda_{\rm QCD}/\mu$) ambiguities cancel in HQET predictions for physical quantities.

Although the appearance of renormalons can hardly be doubted on physical grounds, a rigorous proof of their existence does not exist even in field theories that are much simpler than QCD. For this reason, various forms of large-N expansions have become the state-of-the-art approach to study renormalon singularities. In QCD, one uses $1/N_f$ as an expansion parameter, where N_f is the number of light quark flavours. In the large-N_f limit, the insertions of fermion loops in a gluon propagator are the only higher-order contributions that have to be retained in the perturbative expansion, since they involve powers of $N_f\,\alpha_s = O(N_f^0)$. Unfortunately, QCD in the large-N_f limit is not an asymptotically free theory; the first coefficient of the β-function becomes negative for $N_f > 33/2$. However, it is believed that although the $1/N_f$ expansion is not adequate to describe the dynamics of QCD, it can still be used to locate the position of the renormalon poles in the Borel plane. In other words, the hope is that tracing the fermionic contribution to the β-function one gets the remaining contributions for free, and that using the correct value of β_0 in (4) gives the right result. Although there exists no proof of this assertion, we will accept it as a working hypothesis.

In the large-N_f limit, the summation of bubbles can be performed directly on the gluon propagator. In Landau gauge, and after renormalization of the fermion loops, the Borel transform of the resummed propagator takes the form [2, 7]

$$\widetilde{D}_{ab}^{\mu\nu}(k,u) = i\delta_{ab}\left(\frac{e^C}{\mu^2}\right)^{-u}\frac{k^\mu k^\nu - g^{\mu\nu}k^2}{(-k^2)^{2+u}}\,, \tag{13}$$

where μ is the renormalization scale, and C is a scheme-dependent constant. In the $\overline{\rm MS}$ scheme, $C=-5/3$. Consider now an arbitrary correlation function without external gluons. To order $1/N_f$, all its dependence on the coupling constant α_s comes from diagrams containing one resummed gluon propagator. The Borel transform of such diagrams is simply obtained by using the Borel transformed propagator (13) instead of the usual propagator.

Following Beneke and Braun [2], let us then consider the structure of infrared renormalons in the pole mass and on-shell wave-function renormalization of a heavy quark. In terms of the self-energy $\Sigma(\not p)$, one has

$$m_Q^{\rm pole} = m_Q + \Sigma(\not p)\Big|_{\not p = m_Q^{\rm pole}}\,, \qquad 1 - Z_Q^{-1} = \frac{\partial\Sigma(\not p)}{\partial\not p}\Big|_{\not p = m_Q^{\rm pole}}\,, \tag{14}$$

where m_Q is the bare mass. In general, these are complicated implicit equations. However, since the self-energy is of order $1/N_f$ with respect to the bare mass, one can replace $m_Q^{\rm pole}$ by m_Q on the right-hand side, thereby neglecting terms of order $1/N_f^2$. We work in Landau gauge and use dimensional regularization. By evaluating the diagram depicted in Fig. 1, we obtain for the Borel transform of the self-energy the relations

$$\begin{aligned}
\widetilde{\Sigma}(\not p, u)\Big|_{\not p = m_Q} &= \frac{C_F\, m_Q}{\beta_0}\,(d-1)\,e^{-Cu}\,(4\pi)^{2-d/2}\left(\frac{m_Q}{\mu}\right)^{d-4-2u} \\
&\quad \times (d-2-2u)\,\frac{\Gamma(2-\frac{d}{2}+u)\,\Gamma(d-3-2u)}{\Gamma(d-1-u)} + O(N_f^{-2})\,, \\
\frac{\partial\widetilde{\Sigma}(\not p, u)}{\partial \not p}\Big|_{\not p = m_Q} &= -\frac{(1+u)}{m_Q}\,\widetilde{\Sigma}(\not p, u)\Big|_{\not p = m_Q} + O(N_f^{-2})\,,
\end{aligned} \tag{15}$$

where $C_F = 4/3$, and d denotes the number of space-time dimensions. For generic u, one can evaluate these expressions for $d = 4$. The positions of renormalons are determined by the Γ-functions in the numerator. There are infrared renormalons at positive half-integer values of u, as well as ultraviolet renormalons at negative integer values of u. For $u = 0$, the self-energy and its derivative are ultraviolet divergent in $d = 4$ dimensions. One can subtract the ultraviolet divergence by subtracting the pole at $u = 0$ after setting $d = 4$ [2]. This determines the renormalized Borel transform up to a scheme-dependent function $R(u)$, which is entire in the Borel plane if a renormalization scheme with analytic counterterms (such as $\overline{\rm MS}$) is employed. The result is

$$\begin{aligned}
\widetilde{m}_Q^{\rm pole}(u) = m_Q^{\rm R}\bigg\{\delta(u) + \frac{C_F}{\beta_0}\bigg[6\,e^{-Cu}\left(\frac{\mu}{m_Q}\right)^{2u}(1-u)\,\frac{\Gamma(u)\,\Gamma(1-2u)}{\Gamma(3-u)} - \frac{3}{u} + R_m(u)\bigg] \\
+ O(N_f^{-2})\bigg\}\,,
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
\widetilde{Z}_Q^{\rm R}(u) = \delta(u) + \frac{C_F}{\beta_0}\bigg[-6\,e^{-Cu}\left(\frac{\mu}{m_Q}\right)^{2u}(1-u^2)\,\frac{\Gamma(u)\,\Gamma(1-2u)}{\Gamma(3-u)} + \frac{3}{u} + R_Z(u)\bigg] \\
+ O(N_f^{-2})\,,
\end{aligned} \tag{17}$$

where $m_Q^{\rm R}$ is the renormalized mass. The first expression has been derived in Ref. [2]. Note that to order $1/N_f$ the choice of m_Q in the parentheses on the right-hand side is arbitrary. The functions $R_m(u)$ and $R_Z(u)$ depend on the renormalization scheme specified by the superscript "R". In the $\overline{\rm MS}$ scheme, one has $R_m(u) = -5/2 + O(u)$ and $R_Z(u) = 11/2 + O(u)$. For the discussion of renormalon singularities these functions are irrelevant. The asymptotic behaviour of the perturbative expansions for $m_Q^{\rm pole}$ and Z_Q is determined by the nearest infrared renormalon pole, which is located at $u = 1/2$. According to (12), it leads to intrinsic ambiguities given by

$$\begin{aligned}
\Delta m_Q^{\rm pole} &= -\frac{2C_F}{\beta_0}\,e^{-C/2}\,\Lambda_{\rm QCD} + O(N_f^{-2})\,, \\
\Delta Z_Q &= \frac{3C_F}{\beta_0}\,e^{-C/2}\,\frac{\Lambda_{\rm QCD}}{m_Q} + O(N_f^{-2})\,.
\end{aligned} \tag{18}$$

Note that the product $e^{-C/2}\,\Lambda_{\rm QCD}$ is scheme-independent.

After this lengthy introduction into the problem, let us now turn to the construction of the HQET [8]–[11]. A heavy quark interacting with light degrees of freedom inside a hadron is almost on-shell. It is then natural to split its momentum into a "large" and a "small" piece according to $p_Q = m_Q\,v + k$, where v is the velocity of the hadron, and m_Q is some choice of the heavy quark mass discussed in detail below. For the moment let us just require that the components of the residual momentum k are much smaller than m_Q. One then proceeds by introducing a velocity-dependent heavy quark field $h_v(x)$, which is related to the original field $Q(x)$ by

$$h_v(x) = \exp(i m_Q\, v\cdot x)\,\frac{(1+\not v)}{2}\,Q(x)\,. \tag{19}$$

The effective Lagrangian for h_v reads [10, 11, 15]

$$\mathcal{L}_{\rm eff} = \bar h_v\,(iv\cdot D - \delta m)\,h_v + \dots\,, \tag{20}$$

where δm is the residual mass term for the heavy quark in the effective theory. It appears since there is a freedom in the choice of the expansion parameter m_Q in (19). One can show that in physical matrix elements only the combination $(m_Q+\delta m)$ appears, i.e. different choices of m_Q are compensated by different values of δm [15]. The ellipses in (20) represent terms that contain additional powers of iD^μ/m_Q or $\delta m/m_Q$. If one arranges things in such a way that the components of k and δm are of order $\Lambda_{\rm QCD}$ and independent of m_Q, this construction provides a systematic expansion in powers of $\Lambda_{\rm QCD}/m_Q$. Moreover, the leading terms in the effective Lagrangian (20) are then invariant under a spin-flavour symmetry group. To this end, the heavy quark mass m_Q used in the field redefinition (19) must be a "physical" mass such as the pole mass, the mass of the lightest hadron that contains the heavy quark, or any other definition that differs from the pole mass by an amount of order $\Lambda_{\rm QCD}$. The residual mass term is given by $\delta m = m_Q^{\rm pole} - m_Q$; i.e. if one chooses the pole mass to construct the HQET, the residual mass vanishes, and to any finite order in perturbation theory the effective heavy quark propagator has a pole at $k=0$. However, from our previous considerations we know that there is an intrinsic ambiguity of order $\Lambda_{\rm QCD}$ in the definition of the pole mass, once non-perturbative effects are taken into account. Hence, if one wants to write down the Lagrangian of the HQET without specifying a particular Borel summation prescription, one can do this for the price of an ambiguous residual mass term [2]. To be specific, let us construct the HQET using the heavy quark mass defined with a principle value prescription to regularize the poles in the Borel plane [cf. (10)]. It then follows that

$$\delta m = \eta_1\,\Delta m_Q^{\rm pole} = -\eta_1\,\frac{2C_F}{\beta_0}\,e^{-C/2}\,\Lambda_{\rm QCD} + O(N_f^{-2})\,. \tag{21}$$

The ambiguity associated with the definition of the pole mass shows up in the form of an ambiguous parameter in the effective Lagrangian (20). At first sight this may seem a problem: How can one derive unambiguous predictions from a Lagrangian that contains an ambiguous parameter? The answer is that the effective theory has to be matched onto the full theory at some large momentum scale. In this process there appear coefficient functions multiplying the operators of the HQET. The ambiguous residual mass term is required to cancel ambiguities in these coefficient functions. The important point to note is that δm is independent of m_Q and thus does not break the flavour symmetry of the effective Lagrangian. The way in which the residual mass enters the $1/m_Q$ expansion has been investigated in Ref. [15].

Let us come back, at this point, to the original formulation of Wilson's OPE [18], in which renormalons never appear. Introducing a hard factorization scale μ in the construction of the HQET would yield a residual mass term of the form $\delta m \sim \mu\,\alpha_s(\mu)$. Likewise, hadronic matrix elements in the effective theory as well as the Wilson coefficient functions would have a power-like dependence on μ, in such a way that the factorization scale disappears from the final predictions for physical quantities. This is the content of the renormalization-group equation. In this formulation, the parameters of the theory are not plagued by ambiguities, but they depend on the arbitrary parameter μ. Moreover, the precise form of this dependence (for instance, the coefficient of the $\mu\,\alpha_s(\mu)$ term in δm) depends on how exactly the hard cutoff is implemented in Feynman diagrams. Hence, there is a similar arbitrariness in the definition of these parameters as in the case of the practical form of the OPE, which contains renormalons. Finally, we note that chosing $\mu = m_Q$ as a factorization scale (as it was proposed in Refs. [3, 25]) breaks the flavour symmetry of the effective Lagrangian (through the residual mass term) and is thus not a viable choice in processes that involve more than one heavy quark flavour.

3 Heavy-to-Heavy Transition Matrix Elements

In this section we investigate how renormalons appear in the hadronic matrix elements that describe current-induced transitions between two hadrons containing heavy quarks with masses m_1 and m_2. These matrix elements are of the form $\langle H_2(v_2)|\,\bar Q_2\,\Gamma\,Q_1\,|H_1(v_1)\rangle$, where v_1 and v_2 denote the velocities of the hadrons. The quantum numbers of the light degrees of freedom are assumed to be the same in the initial and final state, but are otherwise arbitrary. We will restrict ourselves to the cases of vector or axial vector currents ($\Gamma = \gamma^\mu$ or $\gamma^\mu\gamma_5$). This covers semileptonic weak decays such as $\bar B \to D^{(*)}\ell\,\bar\nu$ and $\Lambda_b \to \Lambda_c\,\ell\,\bar\nu$.

In the HQET, the currents which mediate these transitions obey an expansion in a series of local operators multiplied by coefficient functions. These functions depend upon the heavy quark masses, the renormalization scale, and the hadron velocity product $w = v_1 \cdot v_2$. A particular property of heavy-to-heavy transitions is that the coefficients of the operators of dimension four are all related to the coefficients of the dimension-three operators [26]. The reason for this is an invariance of the effective theory under reparametrization of the heavy quark momentum [16]. As an example, we give the exact form of the expansion of the vector current to order $1/m_Q$ [26]:

$$\begin{aligned}
\bar{Q}_2\,\gamma^\mu Q_1 \to\ & C_1^V \left\{ \bar{h}_{v_2}\gamma^\mu h_{v_1} + \frac{1}{2m_1}\,\bar{h}_{v_2}\gamma^\mu i\not{\mathcal{D}}_1\,h_{v_1} - \frac{1}{2m_2}\,\bar{h}_{v_2}\,i\overleftarrow{\not{\mathcal{D}}_2}\,\gamma^\mu h_{v_1} \right\} \\
&+ \frac{\partial C_1^V}{\partial w} \left\{ \frac{1}{m_1}\,\bar{h}_{v_2}\gamma^\mu i v_2\!\cdot\!\mathcal{D}_1\,h_{v_1} - \frac{1}{m_2}\,\bar{h}_{v_2}\,i v_1\!\cdot\!\overleftarrow{\mathcal{D}_2}\,\gamma^\mu h_{v_1} \right\} \\
&+ C_2^V \left\{ \bar{h}_{v_2}\,v_1^\mu\,h_{v_1} + \frac{1}{2m_1}\,\bar{h}_{v_2}\,v_1^\mu\,i\not{\mathcal{D}}_1\,h_{v_1} - \frac{1}{2m_2}\,\bar{h}_{v_2}\,i\overleftarrow{\not{\mathcal{D}}_2}\,v_1^\mu\,h_{v_1} + \frac{1}{m_1}\,\bar{h}_{v_2}\,i\mathcal{D}_1^\mu\,h_{v_1} \right\} \\
&+ \frac{\partial C_2^V}{\partial w} \left\{ \frac{1}{m_1}\,\bar{h}_{v_2}\,v_1^\mu\,i v_2\!\cdot\!\mathcal{D}_1\,h_{v_1} - \frac{1}{m_2}\,\bar{h}_{v_2}\,i v_1\!\cdot\!\overleftarrow{\mathcal{D}_2}\,v_1^\mu\,h_{v_1} \right\} \\
&+ C_3^V \left\{ \bar{h}_{v_2}\,v_2^\mu\,h_{v_1} + \frac{1}{2m_1}\,\bar{h}_{v_2}\,v_2^\mu\,i\not{\mathcal{D}}_1\,h_{v_1} - \frac{1}{2m_2}\,\bar{h}_{v_2}\,i\overleftarrow{\not{\mathcal{D}}_2}\,v_2^\mu\,h_{v_1} - \frac{1}{m_2}\,\bar{h}_{v_2}\,i\overleftarrow{\mathcal{D}_2^\mu}\,h_{v_1} \right\} \\
&+ \frac{\partial C_3^V}{\partial w} \left\{ \frac{1}{m_1}\,\bar{h}_{v_2}\,v_2^\mu\,i v_2\!\cdot\!\mathcal{D}_1\,h_{v_1} - \frac{1}{m_2}\,\bar{h}_{v_2}\,i v_1\!\cdot\!\overleftarrow{\mathcal{D}_2}\,v_2^\mu\,h_{v_1} \right\} \\
&+ O\Big(\frac{1}{m_1^2}, \frac{1}{m_2^2}, \frac{1}{m_1 m_2}\Big)\,,
\end{aligned} \tag{22}$$

where $C_i^V = C_i^V(m_1/\mu, m_2/\mu, w)$. A similar expansion with coefficients C_i^A, and with γ_5 inserted after whatever object carries the Lorentz index μ, holds for the axial vector current. The symbols $\mathcal{D}_i$ represent combinations of a gauge-covariant derivative and the residual mass term. They are defined as [15]

$$i\mathcal{D}_1^\mu = iD^\mu - \delta m\, v_1^\mu\,, \qquad i\overleftarrow{\mathcal{D}_2^\mu} = i\overleftarrow{D^\mu} + \delta m\, v_2^\mu\,. \tag{23}$$

We will show below that the coefficient functions $C_i^{V,A}$ of the dimension-three operators contain infrared renormalons at $u = 1/2$, corresponding to power behaviour of order $1/m_1$ or $1/m_2$. In order for the physical heavy-to-heavy transition amplitudes to be unambiguous, we have to require that these renormalons be compensated by ultraviolet renormalons in HQET matrix elements of dimension-four operators. This requirement is analogous to the renormalization-group equation in Wilson's OPE. The complete set of dimension-four operators consists of the local current operators in (22) as well as operators containing the time-ordered product of a dimension-three operator with a $1/m_Q$ insertion from the effective Lagrangian [14], which at this order is given by [8, 11]

$$\mathcal{L}_{\rm eff} = \bar h_v\, iv\cdot\mathcal{D}\, h_v + \frac{1}{2m_Q}\,\bar h_v\,(i\mathcal{D})^2 h_v + C_{\rm mag}(m_Q/\mu)\,\frac{g_s}{4m_Q}\,\bar h_v\,\sigma_{\mu\nu}G^{\mu\nu} h_v + \dots\,. \tag{24}$$

However, a cancellation of renormalon ambiguities can only occur between terms that have the structure of matrix elements of local dimension-three operators. In other words, only the matrix elements of dimension-four operators that can mix with lower dimensional operators can contain ultraviolet renormalons.

In heavy-to-heavy transitions, the ultraviolet renormalons in the matrix elements of the local dimension-four operators can be related to the infrared renormalon in the pole mass. Using the equation of motion of the HQET, $iv_1\!\cdot\!\mathcal{D}_1\, h_{v_1} = 0$, as well as an integration by parts, one can show that [14, 15]

$$\begin{aligned}
\langle H_2(v_2)|\,\bar h_{v_2}\,\Gamma\, i\mathcal{D}_1^\alpha\, h_{v_1}\,|H_1(v_1)\rangle &= \frac{\bar\Lambda}{w+1}\,\langle H_2(v_2)|\,\bar h_{v_2}\,\Gamma\,(w\, v_1^\alpha - v_2^\alpha)\, h_{v_1}\,|H_1(v_1)\rangle + \dots\,,\\
-\langle H_2(v_2)|\,\bar h_{v_2}\, i\overleftarrow{\mathcal{D}_2^\alpha}\,\Gamma\, h_{v_1}|H_1(v_1)\rangle &= \frac{\bar\Lambda}{w+1}\langle H_2(v_2)|\,\bar h_{v_2}(w\, v_2^\alpha - v_1^\alpha)\,\Gamma\, h_{v_1}|H_1(v_1)\rangle + \dots\,,
\end{aligned} \tag{25}$$

where Γ may be an arbitrary Dirac matrix. The ellipses represent terms that vanish upon contraction with $v_{1\alpha}$ or $v_{2\alpha}$. These terms cannot be written in the form of matrix elements of local operators. Hence, as explained above, they must be free of renormalons. As an example, consider the case of the ground-state pseudoscalar and vector mesons. There, the matrix elements of local dimension-three operators can be parametrized in terms of the Isgur-Wise function [20]:

$$\langle M_2(v_2)|\,\bar h_{v_2}\,\Gamma\, h_{v_1}\,|M_1(v_1)\rangle = -\xi(w,\mu)\,\mathrm{Tr}\Big\{\overline{\mathcal{M}}_2(v_2)\,\Gamma\,\mathcal{M}_1(v_1)\Big\}\,, \tag{26}$$

where $\mathcal{M}(v)$ are the tensor wave functions defined in Ref. [12]. The matrix elements of local dimension-four operators can be written as [14, 15]

$$\begin{aligned}
\langle M_2(v_2)|\,\bar h_{v_2}\Gamma\, i\mathcal{D}_1^\alpha h_{v_1}\,|M_1(v_1)\rangle &= -\frac{\bar\Lambda}{w+1}\,\xi(w,\mu)\,\mathrm{Tr}\Big\{\overline{\mathcal{M}}_2(v_2)\,(w\, v_1^\alpha - v_2^\alpha)\,\Gamma\,\mathcal{M}_1(v_1)\Big\}\\
&\quad + \xi_3(w,\mu)\,\mathrm{Tr}\bigg\{\bigg(\gamma^\alpha - \frac{v_1^\alpha + v_2^\alpha}{w+1}\bigg)\,\overline{\mathcal{M}}_2(v_2)\,\Gamma\,\mathcal{M}_1(v_1)\bigg\}\,.
\end{aligned} \tag{27}$$

Note that the Feynman rules of the HQET imply that there cannot appear Dirac matrices next to Γ under the trace with the meson wave functions. Obviously, the structure of the trace associated with the function $\xi_3(w,\mu)$ is different from the structure of the trace in (26). It follows that $\xi_3(w,\mu)$ does not contain an ultraviolet renormalon at $u = 1/2$. Let us now come back to the terms shown explicitly in (25). They have the structure of matrix elements of the local dimension-three operators. For instance, in the case of the second operator on the right-hand side in (22) one has $\Gamma = \gamma^\mu\gamma_\alpha$, and between the heavy quark spinors one can replace $(w\, v_1^\alpha - v_2^\alpha)\,\gamma^\mu\gamma_\alpha$ by $(w+1)\gamma^\mu - 2v_2^\mu$. The parameter

$$\bar\Lambda = m_{H_i} - m_i - \delta m = m_{H_i} - m_i^{\rm pole}\,; \qquad i = 1,2 \tag{28}$$

denotes the asymptotic value of the difference between the hadron and heavy quark pole masses, which is flavour-independent. Note that this parameter is independent of the choice of the expansion parameter m_Q used in the construction of the HQET [15]. However, because of its dependence on the pole mass it does contain an ultraviolet renormalon [2]. Using (21), we find that the corresponding ambiguity in $\bar\Lambda$ is given by

$$\Delta\bar\Lambda = -\Delta m_Q^{\rm pole} = \frac{2C_F}{\beta_0}\, e^{-C/2}\,\Lambda_{\rm QCD} + O(N_f^{-2})\,. \tag{29}$$

Next consider the matrix elements of the operators containing the time-ordered product of two local operators. An insertion of the kinetic operator $(1/2m_Q)\,\bar h_v\,(i\mathcal{D})^2 h_v$ into a matrix element of a local dimension-three operator does not affect the transformation properties under the Lorentz group and heavy quark spin symmetry. The effect of such an insertion is simply a multiplicative renormalization of the original matrix element. We define a function $K(w,\mu)$ by[5]

$$\begin{aligned}
&\langle H_2(v_2)|\, i\int {\rm d}^4x\, {\rm T}\Big\{\bar h_{v_1}(x)\,(i\mathcal{D}_1)^2 h_{v_1}(x), \bar h_{v_2}(0)\,\Gamma\, h_{v_1}(0)\Big\}\,|H_1(v_1)\rangle \\
&= \langle H_2(v_2)|\, i\int {\rm d}^4x\, {\rm T}\Big\{\bar h_{v_2}(x)\,(i\mathcal{D}_2)^2 h_{v_2}(x), \bar h_{v_2}(0)\,\Gamma\, h_{v_1}(0)\Big\}\,|H_1(v_1)\rangle \\
&= K(w,\mu)\,\langle H_2(v_2)|\,\bar h_{v_2}(0)\,\Gamma\, h_{v_1}(0)\,|H_1(v_1)\rangle\,.
\end{aligned} \tag{30}$$

Clearly, these time-ordered products can mix with the dimension-three operators under renormalization. This is obvious if a dimensionful regulator is employed. But even in dimensional regularization, it can be seen from the renormalization-group equation for the function $K(w,\mu)$, which contains an inhomogeneous term proportional to $\bar\Lambda$ [27]:

$$\mu\,\frac{\rm d}{{\rm d}\mu}\,K(w,\mu) = -2\bar\Lambda\,(w-1)\,\frac{\partial}{\partial w}\,\gamma_{\rm hh}(w)\,. \tag{31}$$

[5] For meson decays, the form factor $K(w,\mu)$ is usually written as $K(w,\mu) = 2\chi_1(w,\mu)/\xi(w,\mu)$ [14], where $\xi(w,\mu)$ is the Isgur-Wise function [20].

Here $\gamma_{\rm hh}(w)$ denotes the velocity-dependent anomalous dimension of the currents in the effective theory [12]:

$$\begin{aligned}\gamma_{\rm hh}(w) &= \frac{C_F\,\alpha_s}{\pi}\Big[w\,r(w)-1\Big] + O(\alpha_s^2)\,;\\ r(w) &= \frac{1}{\sqrt{w^2-1}}\,\ln\left(w+\sqrt{w^2-1}\right).\end{aligned} \tag{32}$$

From (31), it follows that the function $K(w,\mu)$ contains an ultraviolet renormalon at $u = 1/2$. Let us denote its renormalon ambiguity by $\Delta K(w)$. Vector current conservation implies that $K(w,\mu)$ must vanish at zero recoil ($w = 1$) [14], and this requires that

$$\Delta K(1) = 0\,. \tag{33}$$

Finally, we note that insertions of the chromo-magnetic operator change the transformation properties of matrix elements in such a way that there is no mixing with matrix elements of lower dimensional operators [14]. Hence, the HQET functions that parameterize these matrix elements are free of ultraviolet renormalons. This is again special to the case of heavy-to-heavy transitions, where the spin symmetry applies to both the initial and final state.

We can now equate the infrared and ultraviolet renormalon ambiguities to derive a set of conditions that have to be fulfilled in order to obtain unambiguous predictions for the physical heavy-to-heavy transition form factors. Separating the terms associated with different Lorentz structures, we obtain from (22), (25), (29) and (30):

$$\begin{aligned}\Delta C_1 &= \frac{\Delta m_{\rm pole}}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\left\{\frac{w\pm1}{w+1}\,C_1 + 2(w-1)\,\frac{\partial C_1}{\partial w}\right\} - \frac{\Delta K}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)C_1\,,\\ \Delta C_2 &= \frac{\Delta m_{\rm pole}}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\left\{\frac{w\pm1}{w+1}\,C_2 + 2(w-1)\,\frac{\partial C_2}{\partial w}\right\} + \frac{\Delta m_{\rm pole}}{m_1}\,\frac{w\mp1}{w+1}\,C_2\\ &\quad - \frac{\Delta m_{\rm pole}}{m_2}\,\frac{1}{w+1}\,(C_1\pm C_2+C_3) - \frac{\Delta K}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)C_2\,,\\ \Delta C_3 &= \frac{\Delta m_{\rm pole}}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\left\{\frac{w\pm1}{w+1}\,C_3 + 2(w-1)\,\frac{\partial C_3}{\partial w}\right\} + \frac{\Delta m_{\rm pole}}{m_2}\,\frac{w\mp1}{w+1}\,C_3\\ &\quad \mp \frac{\Delta m_{\rm pole}}{m_1}\,\frac{1}{w+1}\,(C_1\pm C_2+C_3) - \frac{\Delta K}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)C_3\,.\end{aligned} \tag{34}$$

We use a short-hand notation where we omit the superscript V or A on the coefficient functions, and where upper (lower) signs refers to the coefficients of the vector (axial vector) current. In total, there are thus six relations. We find it useful to solve for ΔK using the relation for ΔC_1^A, and to eliminate ΔK from the remaining relations. This leads to

$$\Delta K = \Delta m_{\rm pole}\,(w-1)\left\{\frac{1}{w+1} + 2\,\frac{\partial}{\partial w}\,\ln C_1^A\right\} - 2\left(\frac{1}{m_1}+\frac{1}{m_2}\right)^{-1}\frac{\Delta C_1^A}{C_1^A}\,, \tag{35}$$

as well as

$$
\begin{aligned}
\frac{1}{\Delta m_{\rm pole}}\left(\frac{\Delta C_1^V}{C_1^V}-\frac{\Delta C_1^A}{C_1^A}\right) &= \left(\frac{1}{m_1}+\frac{1}{m_2}\right)\left\{\frac{1}{w+1}+(w-1)\,\frac{\partial}{\partial w}\,\ln\frac{C_1^V}{C_1^A}\right\}, \\
\frac{1}{\Delta m_{\rm pole}}\left(\Delta C_2-\frac{C_2}{C_1}\,\Delta C_1\right) &= -\frac{1}{m_2}\,\frac{1}{w+1}\,(C_1\pm C_2+C_3)+\frac{1}{m_1}\,\frac{w\mp 1}{w+1}\,C_2 \\
&\quad +\left(\frac{1}{m_1}+\frac{1}{m_2}\right)(w-1)\,C_2\,\frac{\partial}{\partial w}\,\ln\frac{C_2}{C_1}, \\
\frac{1}{\Delta m_{\rm pole}}\left(\Delta C_3-\frac{C_3}{C_1}\,\Delta C_1\right) &= \mp\frac{1}{m_1}\,\frac{1}{w+1}\,(C_1\pm C_2+C_3)+\frac{1}{m_2}\,\frac{w\mp 1}{w+1}\,C_3 \\
&\quad +\left(\frac{1}{m_1}+\frac{1}{m_2}\right)(w-1)\,C_3\,\frac{\partial}{\partial w}\,\ln\frac{C_3}{C_1}. \qquad (36)
\end{aligned}
$$

Equation (35) determines the structure of the ultraviolet renormalon in the hadronic form factor $K(w,\mu)$ in terms of the infrared renormalon in the pole mass and in the coefficient function C_1^A. Since we are not able to calculate the hadronic form factor $K(w,\mu)$ from first principles (not even using a $1/N_f$ expansion), we cannot check this relation, but we can use it to compute ΔK given a calculation of $\Delta m_{\rm pole}$ and ΔC_1^A. The remaining five relations in (36) form a set of consistency conditions involving only the infrared renormalons in the pole mass and in the coefficient functions. Unless these conditions are satisfied, a compensation of infrared and ultraviolet renormalons is not possible. The existence of such relations is non-trivial and is a consequence of the strong constraints imposed by heavy quark symmetry on the structure of the weak decay form factors in heavy-to-heavy transitions.

The above results are exact to all orders in $1/N_f$. In the large-N_f limit, they simplify since the renormalon ambiguities are of order $1/N_f$, and we can use the fact that $C_i=\delta_{i1}+O(1/N_f)$. This leads to

$$
\Delta K=\frac{w-1}{w+1}\,\Delta m_{\rm pole}-2\left(\frac{1}{m_1}+\frac{1}{m_2}\right)^{-1}\Delta C_1^A+O(N_f^{-2}), \qquad (37)
$$

and

$$
\begin{aligned}
\Delta C_1^V-\Delta C_1^A &= \frac{\Delta m_{\rm pole}}{w+1}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)+O(N_f^{-2}), \\
\Delta C_2^{V,A} &= -\frac{1}{w+1}\,\frac{\Delta m_{\rm pole}}{m_2}+O(N_f^{-2}), \\
\Delta C_3^{V,A} &= \mp\frac{1}{w+1}\,\frac{\Delta m_{\rm pole}}{m_1}+O(N_f^{-2}). \qquad (38)
\end{aligned}
$$

Let us now check these relations with an explicit calculation of the asymptotic behaviour of the coefficient functions to order $1/N_f$. The coefficients are obtained by comparing matrix elements in the HQET with matrix elements in the full theory at some reference scale μ. This matching procedure is independent of the external states, and it is most economic to evaluate the matrix elements with on-shell quark states. If one uses dimensional regularization, all loop diagrams in the HQET vanish, i.e. the coefficient functions are simply given by the on-shell vertex functions of the full theory [8, 17]. Hence, the Borel-transformed coefficient functions are obtained by evaluating the diagram shown in Fig. 2 supplemented by wave-function renormalization. Setting $d=4$ in the final result, we obtain

$$\begin{aligned}
\widetilde{C}_1^{V,A}(u) &= \delta(u) - \frac{3C_F}{\beta_0}\,e^{-Cu}\left[\left(\frac{\mu}{m_1}\right)^{2u} + \left(\frac{\mu}{m_2}\right)^{2u}\right](1-u^2)\,\frac{\Gamma(u)\,\Gamma(1-2u)}{\Gamma(3-u)} \\
&\quad + \frac{C_F}{\beta_0}\,e^{-Cu}\left(\frac{\mu^2}{m_1 m_2}\right)^{u}\frac{\Gamma(u)\,\Gamma(1-2u)}{\Gamma(2-u)}\left\{2\Big[(1+u)\,w \pm u\Big]\,F_{11}^{1+u}\right. \\
&\qquad\qquad \left. + u\left(\frac{m_1}{m_2}\,F_{21}^{1+u} + \frac{m_2}{m_1}\,F_{12}^{1+u}\right) + 2\frac{(1-u)(1-2u)}{(2-u)}\,F_{11}^{u}\right\} \\
&\quad + \frac{C_F}{\beta_0}\left\{-\frac{2}{u}\Big[w\,r(w)-1\Big] + R_{V,A}(u)\right\} + O(N_f^{-2})\,, \\
\widetilde{C}_2^{V,A}(u) &= -\frac{2C_F}{\beta_0}\,e^{-Cu}\left(\frac{\mu^2}{m_1 m_2}\right)^{u}\frac{\Gamma(1+u)\,\Gamma(1-2u)}{\Gamma(2-u)} \\
&\quad \times\left\{F_{12}^{1+u} - \frac{(1-2u)}{3(2-u)}\left(F_{22}^{1+u} \pm \frac{2m_1}{m_2}\,F_{31}^{1+u}\right)\right\} + O(N_f^{-2})\,, \\
\widetilde{C}_3^{V,A}(u) &= \mp\frac{2C_F}{\beta_0}\,e^{-Cu}\left(\frac{\mu^2}{m_1 m_2}\right)^{u}\frac{\Gamma(1+u)\,\Gamma(1-2u)}{\Gamma(2-u)} \\
&\quad \times\left\{F_{21}^{1+u} - \frac{(1-2u)}{3(2-u)}\left(F_{22}^{1+u} \pm \frac{2m_2}{m_1}\,F_{13}^{1+u}\right)\right\} + O(N_f^{-2})\,,
\end{aligned} \tag{39}$$

where

$$F_{ab}^{c} = \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)}\int\limits_0^1 \mathrm{d}x\,x^{a-1}(1-x)^{b-1}\left[x^2\,\frac{m_1}{m_2} + (1-x)^2\,\frac{m_2}{m_1} + 2x(1-x)\,w\right]^{-c}. \tag{40}$$

The terms in the last row in $\widetilde{C}_1^{V,A}(u)$ come from a renormalization of the ultraviolet divergences for $u=0$. The coefficient of the $1/u$ pole is proportional to the one-loop coefficient of the velocity-dependent anomalous dimension in (32). The detailed form of the entire function $R_{V,A}(u)$ is irrelevant for our discussion. We note that a check of the complicated expressions (39) is provided by an expansion around $u=0$, from which we recover the one-loop results for the coefficient functions derived in Ref. [28].

It is a simple exercise to extract the residues of the renormalon poles at $u=1/2$ from the above expressions. The relevant parameter integrals are given by

$$F_{12}^{3/2} = \frac{2}{w+1}\sqrt{\frac{m_1}{m_2}}\,, \qquad F_{21}^{3/2} = \frac{2}{w+1}\sqrt{\frac{m_2}{m_1}}\,, \qquad F_{11}^{3/2} = \frac{1}{2}\,(F_{12}^{3/2} + F_{21}^{3/2})\,. \tag{41}$$

Using (12) and (18), we then compute the corresponding renormalon ambiguities in the coefficient functions. We find that the consistency conditions (38) are indeed satisfied. In particular, we note that

$$\begin{aligned} \Delta C_1^V &= \frac{\Delta m_{\rm pole}}{4}\,\frac{3w+1}{w+1}\left(\frac{1}{m_1}+\frac{1}{m_2}\right) + O(N_f^{-2})\,, \\ \Delta C_1^A &= \frac{\Delta m_{\rm pole}}{4}\,\frac{3(w-1)}{w+1}\left(\frac{1}{m_1}+\frac{1}{m_2}\right) + O(N_f^{-2})\,, \end{aligned} \tag{42}$$

so that the difference obeys the first relation given in (38). Moreover, for the ultraviolet renormalon ambiguity in the function $K(w,\mu)$ we obtain from (37)

$$\Delta K(w) = -\frac{w-1}{w+1}\,\frac{\Delta m_{\rm pole}}{2} + O(N_f^{-2})\,. \tag{43}$$

Note that $\Delta K(w)$ vanishes for $w=1$, as required by vector current conservation [see (33)].

We have emphasized above that the appearance of renormalons does not obscure the structure of the heavy quark expansion. In particular, Luke's theorem [14], which concerns the vanishing of first-order power corrections at zero recoil, remains unaffected. Let us illustrate this important fact with the example of the meson decay form factor $h_{A_1}(w)$ defined by [17]

$$\langle D^*(v_2,\epsilon)|\,\bar c\,\gamma^\mu\gamma_5 b\,|\bar B(v_1)\rangle = \sqrt{m_B m_{D^*}}\,(w+1)\,h_{A_1}(w)\,\epsilon^{*\mu} + \dots\,, \tag{44}$$

where $w = v_1\cdot v_2$, and ϵ denotes the polarization vector of the D^*-meson. This form factor plays a crucial role in the extraction of $|V_{cb}|$ from the extrapolation of the $\bar B\to D^*\ell\,\bar\nu$ decay rate to zero recoil. One obtains [29]

$$\lim_{w\to 1}\frac{1}{\sqrt{w^2-1}}\,\frac{\mathrm{d}\Gamma(\bar B\to D^*\ell\,\bar\nu)}{\mathrm{d}w} = \frac{G_F^2}{4\pi^3}\,(m_B-m_{D^*})^2\,m_{D^*}^3\,|V_{cb}|^2\,|h_{A_1}(1)|^2\,. \tag{45}$$

The important point is that $h_{A_1}(1)$ is protected by Luke's theorem against first-order power corrections [14]. It follows that

$$h_{A_1}(1) = \eta_A + O(1/m_Q^2)\,; \qquad \eta_A = C_1^A(w=1)\,, \tag{46}$$

where we use m_Q as a generic notation for m_c or m_b. The presence of an infrared renormalon at $u=1/2$ in the short-distance coefficient η_A would spoil this non-renormalization theorem. However, from (37) and (33) it follows that the infrared renormalon at $u=1/2$ in C_1^A vanishes at zero recoil. Our explicit result (42) confirms this to order $1/N_f$. Thus, the theoretical uncertainty in the determination of $|V_{cb}|$ is of order $1/m_Q^2$; it is not affected by a renormalon ambiguity of order $1/m_Q$. Note, however, that the expression for $\widetilde{C}_1^A(u)$ in (39) contains a renormalon pole at $u = 1$, which does not vanish at zero recoil. The corresponding ambiguity in η_A is given by

$$\Delta\eta_A = \frac{C_F}{2\beta_0}\, e^{-C}\, \Lambda_{\rm QCD}^2 \left(\frac{1}{m_1} + \frac{1}{m_2}\right)^2 + O(N_f^{-2})\,. \tag{47}$$

This infrared renormalon must be compensated by an ultraviolet renormalon in the terms of order $1/m_Q^2$ in (46). For completeness, let us also study the renormalization of the vector current at zero recoil. There, the relevant combination of coefficient functions is

$$\eta_V = C_1^V(w=1) + C_2^V(w=1) + C_3^V(w=1)\,. \tag{48}$$

We find that the leading renormalon pole in the Borel transform of η_V is located at $u = 1$. From its residue, we obtain

$$\Delta\eta_V = \frac{C_F}{2\beta_0}\, e^{-C}\, \Lambda_{\rm QCD}^2 \left(\frac{1}{m_1} - \frac{1}{m_2}\right)^2 + O(N_f^{-2})\,. \tag{49}$$

Note that $\Delta\eta_V$ vanishes in the limit $m_1 = m_2$, in which the vector current is conserved and not renormalized, and hence $\eta_V = 1$.

As a second example, we demonstrate the cancellation of renormalon ambiguities in the ratio of the vector form factor $h_V(w)$ defined by

$$\langle D^*(v_2,\epsilon)|\, \bar c\, \gamma^\mu b\, |\bar B(v_1)\rangle = \sqrt{m_B m_{D^*}}\, h_V(w)\, \epsilon^{\mu\nu\alpha\beta}\, \epsilon_\nu^*\, v_{2\alpha} v_{1\beta} \tag{50}$$

and the axial form factor $h_{A_1}(w)$. Including power corrections of order $1/m_c$ (and neglecting those of order $1/m_b$ and higher), one finds that [27]

$$\frac{h_V(w)}{h_{A_1}(w)} = \frac{C_1^V(w)}{C_1^A(w)} \left\{ 1 + \frac{\bar\Lambda}{m_c} \left[\frac{1}{w+1} + (w-1)\, \frac{\partial}{\partial w}\, \ln\frac{C_1^V(w)}{C_1^A(w)} \right] + \dots \right\}. \tag{51}$$

Using the first relation in (36), we see that the infrared renormalon of order $1/m_c$ in the ratio C_1^V/C_1^A is precisely compensated by the ultraviolet renormalon of the term proportional to $\bar\Lambda$.

We conclude this section by pointing out an important implication of our result (47). Recently, it has been claimed that one can derive a sum rule for the form factor $h_{A_1}(1)$, from which it is possible to obtain a bound for the non-perturbative corrections of order $1/m_Q^2$ in (46). The sum rule reads [24]

$$h_{A_1}^2(1) + \dots = \eta_A^2 - \frac{\lambda_2}{3m_c^2} + \frac{\lambda_1 + 3\lambda_2}{4} \left(\frac{1}{m_c^2} + \frac{1}{m_b^2} + \frac{2}{3 m_c m_b} \right) + O(1/m_Q^3)\,, \tag{52}$$

where the ellipses represents positive contributions from transitions into excited states. The parameters λ_1 and λ_2 are defined in terms of the B-meson matrix elements of the kinetic and the chromo-magnetic operator in the effective Lagrangian (24). In principle, these HQET parameters could contain ultraviolet renormalons. However, since λ_2 is proportional to the mass splitting between B and B^* mesons, it is protected from renormalons. Moreover, from the expansion of the meson mass m_B in powers of $1/m_b$ (this extends (28) to order $1/m_b$ [30])

$$m_B = m_b^{\rm pole} + \bar\Lambda - \frac{\lambda_1 + 3\lambda_2}{2m_b} + O(1/m_b^2)\,, \tag{53}$$

and from the fact that to order $1/N_f$ the Borel transform of the pole mass given in (16) does not contain an infrared renormalon pole at $u=1$, it follows that λ_1 does not contain an ultraviolet renormalon[6] (at least) to order $1/N_f$ [2]. We conclude that the non-perturbative corrections in (52) do not contain the ultraviolet renormalons required to cancel the infrared renormalon in the perturbative coefficient η_A^2. Hence, there must be something wrong with the sum rule. Either the short-distance correction on the right-hand side is not given by η_A^2, or there must be additional terms of order $1/m_Q^2$ to compensate the renormalon in η_A^2. Therefore, the numerical implications derived from this sum rule in Ref. [24] should be taken with caution.

4 Heavy-to-Light Transition Matrix Elements

We have seen in the previous section that the appearance of renormalons in heavy-to-heavy transition matrix elements is to a large extent constrained by the symmetries and equation of motion of the HQET, which apply to both the initial and final hadron states. As a consequence, ultraviolet renormalons enter the HQET matrix elements of dimension-four operators only through the parameter $\bar\Lambda$ and a single function $K(w,\mu)$. Since the ultraviolet renormalon in $\bar\Lambda$ is related to the infrared renormalon in the pole mass, it is possible to derive the consistency relations (36), which determine the infrared renormalon poles in the coefficient functions independently of any unknown hadronic matrix element.

It is well-known that in heavy-to-light transitions there are fewer constraints imposed by heavy quark symmetry. In particular, most (if not all) form factors appearing at order $1/m_Q$ mix with lower dimensional operators under renormalization. Examples are provided by the $1/m_Q$ expansions for meson decay constants [31] and the semileptonic $\bar B\to\pi\,\ell\,\bar\nu$ decay form factors

[6]Note that even if λ_1 and λ_2 would contain ultraviolet renormalons, the mass dependence of the power corrections in (52) would not match with the mass dependence of the infrared renormalon pole in η_A as given in (47).

[32]. Therefore, it is not possible to derive consistency relations analogous to (36) in this case. The best one can achieve is to deduce the structure of ultraviolet renormalons in the hadronic form factors of the HQET from a calculation of the infrared renormalons in the coefficient functions and the pole mass. We shall discuss this for the simplest case of meson decay constants.

Consider heavy-to-light transition matrix elements of the form $\langle X|\,\bar q\,\Gamma\,Q\,|H(v)\rangle$, where $\Gamma=\gamma^\mu$ or $\gamma^\mu\gamma_5$, $H(v)$ is a heavy hadron with velocity v, and X is some light final state. For simplicity, we set the mass of the light quark to zero and use a regularization scheme with anticommuting γ_5. This leads to a simple relation between the coefficient functions appearing in the expansion of the vector and axial vector currents [15]:

$$\begin{aligned}\bar q\,\gamma^\mu Q &\to C_1(m_Q/\mu)\,\bar q\,\gamma^\mu h_v + C_2(m_Q/\mu)\,\bar q\,v^\mu h_v + O(1/m_Q)\,,\\ \bar q\,\gamma^\mu\gamma_5\,Q &\to C_1(m_Q/\mu)\,\bar q\,\gamma^\mu\gamma_5\,h_v - C_2(m_Q/\mu)\,\bar q\,v^\mu\gamma_5\,h_v + O(1/m_Q)\,.\end{aligned} \tag{54}$$

These coefficients can be calculated in analogy to the previous section. For their Borel transforms, we obtain

$$\begin{aligned}\widetilde C_1(u) &= \delta(u)+\frac{C_F}{\beta_0}\left\{-3\,e^{-Cu}\left(\frac{\mu}{m_Q}\right)^{2u}\left(1+\frac{u}{3}-u^2\right)\frac{\Gamma(u)\,\Gamma(1-2u)}{\Gamma(3-u)}\right.\\ &\qquad\left.+\frac{3}{2u}+R(u)\right\}+O(N_f^{-2})\,,\\ \widetilde C_2(u) &= \frac{4C_F}{\beta_0}\,e^{-Cu}\left(\frac{\mu}{m_Q}\right)^{2u}\frac{\Gamma(1+u)\,\Gamma(1-2u)}{\Gamma(3-u)}+O(N_f^{-2})\,,\end{aligned} \tag{55}$$

where $R(u)=5/4+O(u)$ in the $\overline{\rm MS}$ scheme. We have checked that from an expansion around $u=0$ one recovers the known one-loop expressions for the coefficient functions given in Ref. [33]. From the residues of the poles at $u=1/2$, it is straightforward to compute the renormalon ambiguities

$$\begin{aligned}\Delta C_1 &= -\frac{11}{12}\,\frac{\Delta m_{\rm pole}}{m_Q}+O(N_f^{-2})\,,\\ \Delta C_2 &= \frac{2}{3}\,\frac{\Delta m_{\rm pole}}{m_Q}+O(N_f^{-2})\,.\end{aligned} \tag{56}$$

To see how renormalons cancel in physical quantities, let us consider the $1/m_Q$ expansion for pseudoscalar and vector meson decay constants in the HQET. It reads [31]

$$\begin{aligned}f_P\sqrt{m_P} &= \Big[C_1(m_Q/\mu)+C_2(m_Q/\mu)\Big]\,F(\mu)\left\{1+\frac{1}{m_Q}\left[G_1(\mu)-b(m_Q/\mu)\,\frac{\bar\Lambda}{6}\right]\right.\\ &\qquad\left.+\frac{6}{m_Q}\left[C_{\rm mag}(m_Q/\mu)\,G_2(\mu)-B(m_Q/\mu)\,\frac{\bar\Lambda}{12}\right]\right\}+O(1/m_Q^2)\,,\end{aligned}$$

$$f_V\sqrt{m_V} = C_1(m_Q/\mu)\,F(\mu)\left\{1+\frac{1}{m_Q}\left[G_1(\mu)-b(m_Q/\mu)\,\frac{\bar\Lambda}{6}\right]\right.$$
$$\left.-\frac{2}{m_Q}\left[C_{\rm mag}(m_Q/\mu)\,G_2(\mu)-B(m_Q/\mu)\,\frac{\bar\Lambda}{12}\right]\right\}+O(1/m_Q^2)\,, \quad (57)$$

where $C_{\rm mag}(m_Q/\mu) = 1 + O(\alpha_s)$ is the coefficient of the chromo-magnetic operator in the effective Lagrangian (24), while $B(m_Q/\mu) = 1 + O(\alpha_s)$ and $b(m_Q/\mu) = O(\alpha_s)$ are coefficients that appear at order $1/m_Q$ in the expansion of the currents [31]. $F(\mu), G_1(\mu)$, and $G_2(\mu)$ are hadronic parameters, which are independent of m_Q. Both $G_1(\mu)$ and $G_2(\mu)$ mix with lower dimensional operators and contain ultraviolet renormalons, as can be seen from the renormalization-group equations [31]

$$\mu\,\frac{\rm d}{{\rm d}\mu}\,G_1(\mu) = \frac{\bar\Lambda}{6}\,\mu\,\frac{\rm d}{{\rm d}\mu}\,b(m_Q/\mu)\,,$$
$$\mu\,\frac{\rm d}{{\rm d}\mu}\Big[C_{\rm mag}(m_Q/\mu)\,G_2(\mu)\Big] = \frac{\bar\Lambda}{12}\,\mu\,\frac{\rm d}{{\rm d}\mu}\,B(m_Q/\mu)\,. \quad (58)$$

Requiring that in (57) the infrared renormalons in the coefficient functions cancel against the ultraviolet renormalons in $\bar\Lambda$ and $G_i(\mu)$, we obtain the relations

$$\frac{\Delta C_1+\Delta C_2}{C_1+C_2} = -\frac{\Delta m_{\rm pole}}{6m_Q}\,(b+3B)-\frac{1}{m_Q}\Big(\Delta G_1+6C_{\rm mag}\,\Delta G_2\Big)\,,$$
$$\frac{\Delta C_1}{C_1} = -\frac{\Delta m_{\rm pole}}{6m_Q}\,(b-B)-\frac{1}{m_Q}\Big(\Delta G_1-2C_{\rm mag}\,\Delta G_2\Big)\,. \quad (59)$$

To order $1/N_f$, they simplify to

$$\Delta G_1 = -\frac{m_Q}{4}\Big(4\Delta C_1+\Delta C_2\Big)+O(N_f^{-2}) = \frac{3\Delta m_{\rm pole}}{4}+O(N_f^{-2})\,,$$
$$\Delta G_2 = -\frac{\Delta m_{\rm pole}}{12}-\frac{m_Q}{8}\,\Delta C_2+O(N_f^{-2}) = -\frac{\Delta m_{\rm pole}}{6}+O(N_f^{-2})\,. \quad (60)$$

This determines the ultraviolet renormalon ambiguities in the hadronic parameters $G_i(\mu)$. The situation encountered here is general for heavy-to-light transitions; since there are always at least two hadronic parameters that contain ultraviolet renormalons, it is not possible to derive a consistency condition for the infrared renormalons in the coefficient functions C_1 and C_2. However, the residues of the renormalons in the coefficient functions determine in a unique way the residues of the ultraviolet renormalon poles in the hadronic parameters of the HQET.

5 Inclusive Decays of Heavy Hadrons

After the analysis of exclusive transitions, we will now consider current-induced inclusive decays of hadrons containing a heavy quark. Examples are the semileptonic decays $\bar{B} \to X_q\,\ell\,\bar{\nu}$ and $\Lambda_b \to X_q\,\ell\,\bar{\nu}$, where $q = c$ or u, as well as the rare decay $\bar{B} \to X_s\,\gamma$. The flavour-changing current relevant to semileptonic decays is $\bar{q}\,\gamma^\mu(1-\gamma_5)\,b$. For the penguin-induced transitions, it is of the form $\bar{q}\,\sigma^{\mu\nu}(1\pm\gamma_5)\,b$. The inclusive decay distributions can be calculated in powers of $1/m_b$ using an OPE for the transition amplitude [34]–[40]

$$T(v,p) = -i\int \mathrm{d}^4x\, e^{-ip\cdot x}\,\langle H(v)|\,\mathrm{T}\left\{\bar{b}(x)\,\Gamma_1\,q(x),\bar{q}(0)\,\Gamma_2\,b(0)\right\}|H(v)\rangle\,, \quad (61)$$

where $H(v)$ denotes the decaying b-flavoured hadron with velocity v, p is the momentum carried by the current (in the cases above, the total lepton or photon momentum, respectively), and Γ_i are abbreviations for the appropriate Dirac matrices. The OPE is constructed by performing a phase redefinition [cf. (19)]

$$b_v(x) = \exp(-im_b\,v\cdot x)\,b(x) = h_v(x) + O(1/m_b) \quad (62)$$

to pull out the leading dependence of the fields on the heavy quark mass. The next step is to write $T(v,p)$ as a sum of coefficient functions multiplying local, higher dimensional operators. The coefficients are determined by evaluating the diagrams shown in Fig. 3, where the momentum of the b-quark is as usual written in the form $p_b = m_b\,v + k$. The residual momentum k is equivalent to a derivative acting on the rescaled heavy quark field b_v.

We will evaluate the contributions in the OPE including terms of order $1/m_b$ and $1/N_f$. In general, the equation of motion can be used to relate all terms of order $1/m_b$ to the residual mass term in the HQET Lagrangian, which is itself of order $1/N_f$. Hence, it will be sufficient to evaluate the $1/m_b$ corrections at tree level. Let us then start with the discussion of the tree diagram in Fig. 3. It gives

$$T_{\mathrm{tree}} = \langle\,\Gamma_1\,\frac{1}{m_b\,\not{v} - \not{p} - m_q + i\not{D}}\,\Gamma_2\,\rangle\,, \quad (63)$$

where we use the short-hand notation $\langle H(v)|\,\bar{b}_v\,\Gamma\,b_v\,|H(v)\rangle \equiv \langle\,\Gamma\,\rangle$. The tree diagram contains the propagator of the q-quark in the background field of the light degrees of freedom in the decaying hadron. To proceed, we expand the propagator as

$$\frac{1}{m_b\,\not{v} - \not{p} - m_q + i\not{D}} = \frac{1}{m_b\,\not{v} - \not{p} - m_q} - \frac{1}{m_b\,\not{v} - \not{p} - m_q}\,i\not{D}\,\frac{1}{m_b\,\not{v} - \not{p} - m_q} + \dots\,, \quad (64)$$

where the second term is of order $1/m_b$ relative to the first term. The forward matrix element of any local operator $\bar{b}_v\,\Gamma_\alpha\,iD^\alpha\,b_v$ containing a single covariant derivative can be evaluated, up to $1/m_b$ corrections, using (62) together with the equation of motion $iv\cdot D\,h_v=\delta m\,h_v$, where δm is the residual mass term. It follows that

$$\langle\,\Gamma_\alpha\,iD^\alpha\,\rangle=\delta m\,\langle\,\Gamma_\alpha\,v^\alpha\,\rangle+\dots\,, \tag{65}$$

where Γ_α denotes an arbitrary Dirac matrix, and the ellipses represent terms that are suppressed by one power of $1/m_b$. Applying this relation, and resumming the expanded propagator (64), we find

$$T_{\rm tree}=\langle\,\Gamma_1\,\frac{1}{(m_b+\delta m)\,\not{v}-\not{p}-m_q}\,\Gamma_2\,\rangle+\dots\,, \tag{66}$$

where the ellipses represent terms of order $1/m_b^2$ relative to the leading term. We observe that, as in the case of exclusive decays, the residual mass term always appears together with the HQET expansion parameter m_b in the combination $m_b^{\rm pole}=m_b+\delta m$, i.e. it is the pole mass that enters the tree-level expression for the transition amplitude. The infrared renormalon in the pole mass leads to an ambiguity given by

$$\Delta T_{\rm tree}=\langle\,\Gamma_1\,\frac{1}{m_b\,\not{v}-\not{p}-m_q}\,(-\Delta m_{\rm pole}\,\not{v})\,\frac{1}{m_b\,\not{v}-\not{p}-m_q}\,\Gamma_2\,\rangle\,. \tag{67}$$

To see how this renormalon is cancelled, let us now turn to the calculation of the radiative corrections depicted in Fig. 3. We study the Borel transform of the transition amplitude to order $1/N_f$ using the resummed gluon propagator (13). In the calculation, we only keep terms that have a renormalon pole at $u=1/2$. We obtain:

$$\begin{aligned}
\widetilde{T}_{\rm vertex}(u)&=\frac{8C_F}{\beta_0}\,e^{-C/2}\,\mu\,\Gamma(1-2u)\,\langle\Gamma_1\frac{1}{m_b\,\not{v}-\not{p}-m_q}\,\not{v}\,\frac{1}{m_b\,\not{v}-\not{p}-m_q}\,\Gamma_2\rangle+\dots,\\
\widetilde{T}_{\rm box}(u)&=-\frac{4C_F}{\beta_0}\,e^{-C/2}\,\mu\,\Gamma(1-2u)\,\langle\,\Gamma_1\,\frac{1}{m_b\,\not{v}-\not{p}-m_q}\,\not{v}\,\frac{1}{m_b\,\not{v}-\not{p}-m_q}\,\Gamma_2\,\rangle\\
&\quad+\frac{6C_F}{\beta_0}\,e^{-C/2}\,\frac{\mu}{m_b}\,\Gamma(1-2u)\,\langle\,\Gamma_1\,\frac{1}{m_b\,\not{v}-\not{p}-m_q}\,\Gamma_2\,\rangle+\dots\,,\\
\widetilde{T}_{\rm WFR}(u)&=-\frac{6C_F}{\beta_0}\,e^{-C/2}\,\frac{\mu}{m_b}\,\Gamma(1-2u)\,\langle\,\Gamma_1\,\frac{1}{m_b\,\not{v}-\not{p}-m_q}\,\Gamma_2\,\rangle+\dots\,.
\end{aligned} \tag{68}$$

The ellipses represent terms that are regular at $u=1/2$, and terms of order $1/N_f^2$. Note that there is no renormalon contribution from the renormalization of the q-quark propagator. Moreover, the renormalon poles with residues proportional to the tree diagram cancel between the box graph and wave-function renormalization. For the sum of all loop contributions, we find

$$\widetilde{T}_{\rm loops}(u)=\frac{4C_F}{\beta_0}\,e^{-C/2}\,\mu\,\Gamma(1-2u)\,\langle\,\Gamma_1\,\frac{1}{m_b\,\not{v}-\not{p}-m_q}\,\not{v}\,\frac{1}{m_b\,\not{v}-\not{p}-m_q}\,\Gamma_2\,\rangle+\dots\,. \tag{69}$$

From the residue of the pole at $u = 1/2$, we obtain for the renormalon ambiguity

$$\Delta T_{\rm loops} = \langle\, \Gamma_1\, \frac{1}{m_b\, \rlap/v - \rlap/k - m_q}\, \Delta m_{\rm pole}\, \rlap/v\, \frac{1}{m_b\, \rlap/v - \rlap/k - m_q}\, \Gamma_2 \,\rangle + O(N_f^{-2})\,. \qquad (70)$$

As expected, the sum of all contributions in the OPE for the transition amplitude is free of renormalon ambiguities:

$$\Delta T = \Delta T_{\rm tree} + \Delta T_{\rm loops} = 0\,. \qquad (71)$$

That this cancellation occurs was conjectured by Bigi et al. in Ref. [3], however without presenting an explicit calculation. In fact, it was claimed that infrared renormalons only appear in the vertex corrections and mass renormalization, but not in the box diagram. Our calculation shows that this is not correct.[7] Nevertheless, we confirm that the cancellation occurs when all diagrams are taken into account.

The situation encountered here is special in that to order $1/m_b$ there do not appear non-perturbative corrections when the pole mass is used in the OPE of the transition amplitude. Hence, at this order there are no ultraviolet renormalons. What we have demonstrated above is a cancellation of infrared renormalons. Consider, as an example, the total decay rate for the process $\bar B\to X_u\,\ell\,\bar\nu$. It can be calculated from the imaginary part of the transition amplitude. Neglecting the mass of the u-quark, one obtains the well-known result [41]

$$\Gamma(\bar B\to X_u\,\ell\,\bar\nu) = \frac{G_F^2\,|\,V_{ub}|^2}{192\pi^3}\, C(m_b)\, \Big\{ 1 + O(1/m_b^2) \Big\}\,, \qquad (72)$$

where

$$C(m_b) = (m_b^{\rm pole})^5 \left\{ 1 - \frac{2\alpha_s(m_b)}{3\pi} \left(\pi^2 - \frac{25}{4} \right) + \dots \right\}. \qquad (73)$$

We have shown that the infrared renormalon at $u = 1/2$ in the pole mass is cancelled by an infrared renormalon in the perturbative series. It is possible to eliminate these renormalons explicitly by introducing a heavy quark mass $m_b^{\rm R}$ renormalized at short distances instead of using the pole mass [3, 25]. As long as $m_b^{\rm R}$ differs from $m_b^{\rm pole}$ by a multiplicative factor $Z[\alpha_s(m_b)]$, this substitution does not induce $1/m_b$ corrections to the decay rate (72). For instance, we may work in the $\overline{\rm MS}$ scheme and use the running mass $\overline{m}_b(\mu)$ evaluated at $\mu = m_b$. This leads to

$$C(m_b) = [\overline{m}_b(m_b)]^5 \left\{ 1 - \frac{2\alpha_s(m_b)}{3\pi} \left(\pi^2 - \frac{65}{4} \right) + \dots \right\}. \qquad (74)$$

[7] We note that the renormalon contributions in the individual diagrams are the same in all covariant gauges.

The perturbative series in this expression does no longer contain a renormalon at $u = 1/2$. Note, however, that at some higher order in the $1/m_b$ expansion there will appear ultraviolet renormalons in the non-perturbative corrections to the decay rate (72). Correspondingly, the coefficient $C(m_b)$ must contain infrared renormalons at larger values of the Borel parameter u, which cannot be eliminated by introducing the renormalized mass $m_b^{\rm R}$.

6 Summary and Conclusions

We have investigated the appearance of renormalons in the HQET by considering the $1/m_Q$ expansion for exclusive heavy-to-heavy and heavy-to-light transitions, as well as for inclusive decays of heavy hadrons. We have argued that, in general, infrared renormalons in the coefficient functions of HQET operators are compensated by ultraviolet renormalons in the matrix elements of higher dimensional operators, and we have identified which of the HQET matrix elements contain such ultraviolet renormalons. In the case of heavy-to-heavy transitions, the symmetries and the equation of motion of the effective theory lead to five consistency relations among the infrared renormalons in the pole mass and the coefficient functions. We have checked that these relations are satisfied to next-to-leading order in an expansion in powers of $1/N_f$.

The most important, though not surprising, result of our analysis is that the appearance of renormalons does not alter the structure of the heavy quark expansion, and does not invalidate any of the predictions derived using the HQET. In particular, Luke's theorem, as well as relations between weak decay form factors, remain valid. In this sense, there is no "renormalon problem" in the HQET. However, as in any OPE it is true that some of the dimensionful hadronic parameters describing the non-perturbative corrections in the heavy quark expansion have an intrinsic uncertainty of order $\Lambda_{\rm QCD}^n$. An example is provided by the mass parameter $\bar\Lambda$. In the practical form of the OPE, in which dimensional regularization is employed in the calculation of the coefficient functions, ambiguities arise from the necessity to specify a resummation prescription to regulate the divergent asymptotic behaviour of perturbation theory. In the literal form of Wilson's OPE, they arise from the introduction of a hard factorization scale μ. The hadronic parameters of the effective theory then exhibit a power-like dependence on μ, in a way that depends on how the cutoff is implemented. In both cases, to define these parameters precisely would require one to fix terms in the coefficient functions that are exponentially small in the coupling constant. As long as one works with truncated perturbative expressions for the Wilson coefficients, the errors due to the truncation are parametrically larger than power corrections. This type of ambiguity is inherent in any OPE and as such cannot be avoided. In this context, we note that the introduction of a short-distance mass instead of the pole mass, which was proposed in Refs. [3, 25], does in general not help to eliminate renormalons. An exception is the case of inclusive decays of heavy hadrons, where this procedure eliminates the leading infrared renormalons. On the other hand, such a choice of the heavy quark mass destroys the flavour symmetry of the effective Lagrangian of the HQET and is thus unattractive, at least in processes that involve more than one heavy quark flavour.

Finally, we like to point out that our somewhat formal investigation of renormalons can serve for tests of HQET calculations. In some cases, the requirement that a compensation of infrared and ultraviolet renormalons occurs leads to non-trivial relations. An example is provided by the consistency conditions (36) for heavy-to-heavy transitions. Using a similar argument, we could show with an explicit calculation that a sum rule derived by Shifman et al. [24], which has been used to put a bound on the hadronic form factor that enters the extraction of $|V_{cb}|$ from semileptonic decays, must be incorrect. This sum rule relates a physical observable to a theoretical expression in which infrared renormalons in a coefficient function do not match with ultraviolet renormalons in non-perturbative parameters. This expression has an intrinsic ambiguity and thus cannot be complete. A further investigation of what goes wrong with the argument presented in Ref. [24] is necessary before any useful phenomenological bound can be derived.

While this paper was in writing, we became aware of a preprint by Beneke et al. [42], who demonstrate the cancellation of infrared renormalons in inclusive decay rates. Their results agree with our Sect. 5.

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Figures

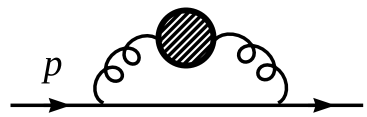

Figure 1: Borel transform of the heavy quark self-energy to order $1/N_f$. The resummed gluon propagator (13) is denoted by the dashed bubble.

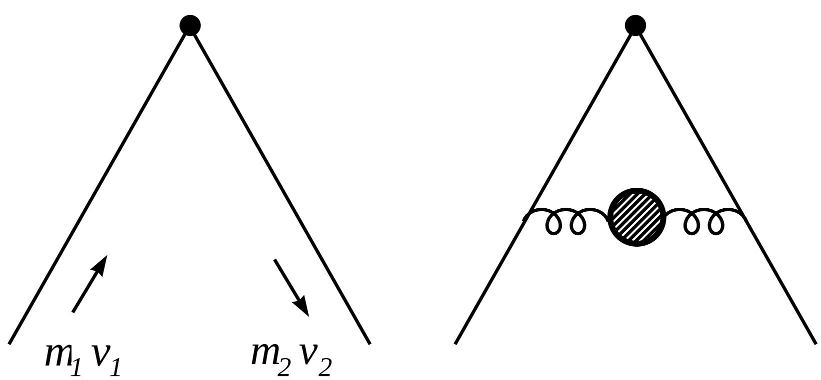

Figure 2: Vertex contribution to the matching calculation of the coefficient functions of heavy-heavy currents.

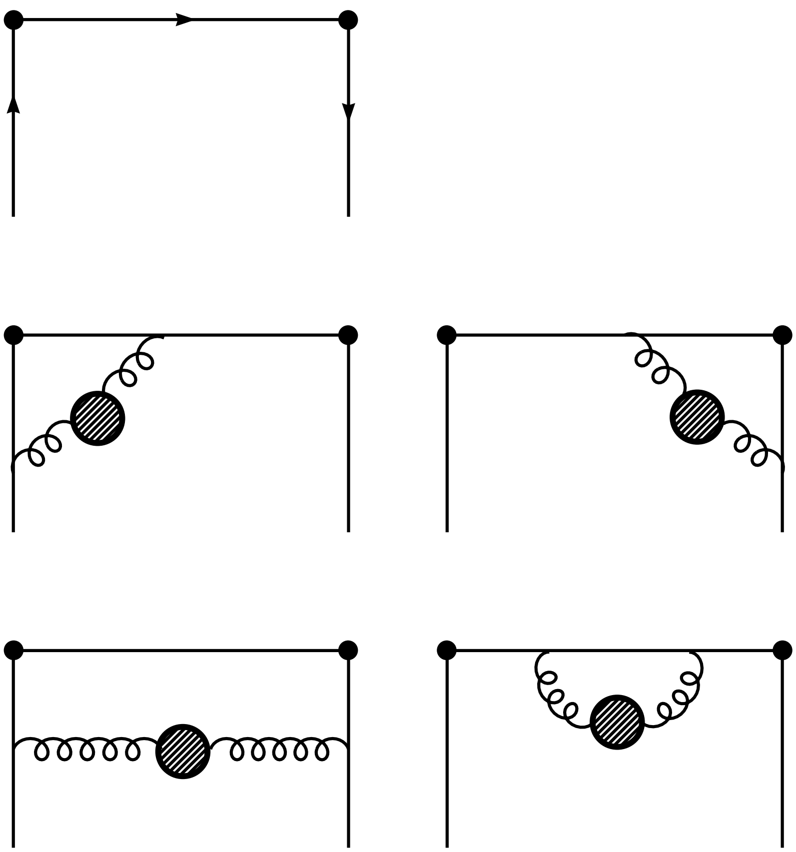

Figure 3: Tree-level contribution and radiative corrections to the transition amplitude.