Integrability and duality in two-dimensional QCD

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Abstract

We consider bosonized QCD_2, and prove that after rewritting the theory in terms of gauge invariant fields, there exists an integrability condition valid for the quantum theory as well. Furthermore, performing a duality type transformation we obtain an appropriate action for the description of the strong coupling limit, which is still integrable. We also prove that the model displays a complicated set of constraints, restricting the dynamics of part of the theory, but which are necessary to maintain the positive metric Hilbert space.

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1. Introduction

In contrast to the case of Schwinger model, quantum chromodynamics of massless fermions in 1+1 dimensions can not be solved in terms of free fields. Several methods have been used in such case, some of them rendering useful results. We mention here the $1/N$ expansion introduced by 't Hooft[1], from which one obtains some information about the spectrum of the theory, and the computation of the exact fermion determinant[2] in terms of a Wess Zumino Witten model[3,4], by which one arrives at an equivalent bosonic action[2,5]. Several authors made afforts in the direction of solving such a difficult model[6], but an exact solution is still missing (see [7] for an extensive review).

Working in the light cone gauge ($A_- = 0$) and formulating the problem in terms of light cone variables 't Hooft obtained a non-linear equation for the fermion self energy, from which he could obtain the above mentioned information about the spectrum of the theory. This procedure is however ambiguous, as pointed out by Wu[8], and implies a tachyon for small bare fermion masses (therefore also in the massless fermions case), see also [9]. This situation clearly requires that a non-perturbative and explicitly gauge invariant approach should be used in order to obtain informations based on firm grounds. Some authors speculated that this situation was a sign of a possible non-trivial phase structure of the theory[26].

The model has also been extensively studied, especially in the absence of fermions, in relation to string theory. Indeed, string theory should be a model describing bound states in strong interactions. Therefore Wilson loops in this model should be described by the string approach as well[10]. In this sense it is natural to integrate out the fermions obtaining gauge invariant objects, which describe mesonic bound states with an infinite string attached to it. In such a case, the construction of fields such as $\Sigma = UV$ (see section 2) or $\widetilde{g} = UgV$, describing gauge invariant bound states is natural after all. The original content of the (gauge dependent) fermion fields can still be recovered from the source terms. Those are kept as a bookkeeping concerning the translation between original gauge dependent fields, and the bosonic formulation, which is non-local. In this sense, we also recall the construction of bilinear fermion fields[11] in the light cone gauge, which arises also from a WZW type action.

The fact that QCD_2 can be studied using non-perturbative methods is by itself a non-trivial statement. In the light cone gauge this can be motivated by the fact that bylinears in the fermion fields form a W_∞ algebra[11], which underlies integrable theories[12]. Here, we shall see that one obtains higher-conservation laws building an affine Lie algebra. We use methods based on the Polyakov Wiegman identity[2] extensively used in refs. [13] and [14], and the consequent duality transformations[14], which seem to have a widespread application in such a class of models[14,15]. Due to the underlying gauge invariance, imposition of the BRST condition in order to obtain the spectrum will be of crucial importance[14,16].

Having in mind the above motivations, we first rewrite, in section 2 the problem in terms of bosonic matrix variables. Fermionic Greens functions can be obtained from the sources, which have been kept in the process. However, the path integration is performed in terms of bosonic fields. We found a set of fields in terms of which the partition function factorizes, as a product of two conformal theories, and a third model corresponding to an off-critical perturbation of the WZW theory. Later we verify that the different sectors interact non-trivially due to the constraint structure. However we proceede with a semi classical reasoning, computing the equation of motion of the latter field, proving that it corresponds to an integrability condition of the theory. Notice that due to the fermionic integration, there are already, at this point, corrections of the order of $\hbar$. We verify that there is a non-trivial change of variables which leads to a dual formulation of the theory, in terms of fields which are appropriated to describe the strong coupling limit. Using the Poisson bracket structure in section 4, we verify that the higher conservation laws obey a Kac-Moody algebra. We also argue that the quantization of such higher conservation laws can be done by means of the introduction of renormalization factors for the current contribution. In section 5 we discuss the constraints arising from the structure of the gauge interaction, and subsequently (section 6) we abtain also second class constraints. In section 7 we discuss the consequence of the constraints for the dual theory, and later we argue about the Regge behaviour of the spectrum[1]. We still discuss the possibility of a conformally invariant type solution for the current, ending with some further conclusions.

2. Bosonization of two-dimensional QCD

We shall consider the QCD_2 lagrangian, given by the expression

$$\mathcal{L} = -\frac{1}{4}\mathrm{tr}\, F_{\mu\nu}F_{\mu\nu} + \overline{\psi} i \not{D}\psi \quad , \tag{2.1}$$

where $D_\mu = \partial_\mu - ieA_\mu$. We work out some results in the path integral formulation, but in a later stage consider also the canonical quantization, forcing us to use both, Euclidian and Minkowski spaces alternatively. The conventions are given in the appendix.

The partition function is

$$\mathcal{Z}\left[\overline{\eta},\eta,i_\mu\right] = \int \mathcal{D}\psi\mathcal{D}\overline{\psi}\mathcal{D}A_\mu\, \mathrm{e}^{-\int \mathrm{d}^2z\,\mathcal{L} - \int \mathrm{d}^2z\,(\overline{\eta}\psi + \overline{\psi}\eta + i_\mu A_\mu)} \quad , \tag{2.2}$$

where $\eta, \overline{\eta}$ are the external sources for the fermions $\overline{\psi}, \psi$ and i_μ the external source for the gauge field A_μ.

In order to obtain the bosonized version of the theory one has to rewrite the fermionic determinant $\det i\not{D}$ as a bosonic functional integral. To achieve such aim we consider the change of variables[2]

$$\begin{aligned} \overline{A} &= \frac{i}{e}V\overline{\partial}V^{-1} \quad , \\ A &= \frac{i}{e}U^{-1}\partial U \quad . \end{aligned} \tag{2.3}$$

The fermionic determinant is given up to factors of the free Dirac operator, to be discussed later in connection with the ghost system, by the expression

$$\det i\not{D} = \mathrm{e}^{\Gamma[UV]} \quad , \tag{2.4}$$

where the $\Gamma[g]$ is the Euclidian Wess-Zumino-Witten (WZW) functional, given by the expression

$$\Gamma[g] = \frac{1}{8\pi}\int \mathrm{d}^2z\, \partial_\mu g^{-1}\partial_\mu g - \frac{i}{4\pi}\epsilon^{\mu\nu}\int \mathrm{d}r \int \mathrm{d}^2z\, \hat{g}^{-1}\dot{\hat{g}}\hat{g}^{-1}\partial_\mu\hat{g}\hat{g}^{-1}\partial_\nu\hat{g} \quad , \tag{2.5}$$

where $\hat{g}(r,z,\overline{z})$ is the usual extension of $g(z\overline{z})$ to a space having the Euclidian two-dimensional space as a boundary, and $\hat{g}(1,z,\overline{z}) = g(z,\overline{z})$, $\hat{g}(0,z,\overline{z}) = 1$. The WZW functional obeys the Polyakov Wiegman identity[2]

$$\Gamma[UV] = \Gamma[U] + \Gamma[V] + \frac{1}{4\pi}\mathrm{tr}\int \mathrm{d}^2z\, U^{-1}\partial U\, V\overline{\partial}V^{-1} \quad , \tag{2.6}$$

which will be used extensively in this work. We find the seeked bosonized formulation taking advantage of the invariance of the Haar measure, and write the above fermion determinant as

$$\det i\,\not{D} \equiv \mathrm{e}^{-W(A)} = \int \mathcal{D}g\,\mathrm{e}^{-S_F[A,g]} \quad , \tag{2.7a}$$

where $S_F[A,g]$ is the equivalent of the fermionic action in terms of the bosonic variables and gauge field and reads

$$\begin{aligned} S_F[A,g] &= \Gamma[UgV] - \Gamma[UV] \\ &= \Gamma[g] + \frac{1}{4\pi}\int \mathrm{d}^2z\, \left[e^2A_\mu A_\mu - e^2 Ag\overline{A}g^{-1} - ieAg\overline{\partial}g^{-1} - ie\overline{A}g^{-1}\partial g\right] \quad , \end{aligned} \tag{2.7b}$$

The above equation was obtained by repeated use of the Polyakov Wiegman identity (2.6), respecting local gauge transformations. The external sources have been used in order to redefine the integration over the fermionic field as

$$\overline{\psi}i\,\not{D}\psi + \overline{\eta}\psi + \overline{\psi}\eta = \left(\overline{\psi} + \overline{\eta}(i\,\not{D})^{-1}\right)i\,\not{D}\left(\psi + (i\,\not{D})^{-1}\eta\right) - \overline{\eta}(i\,\not{D})^{-1}\eta \quad . \tag{2.8}$$

The change of variables (2.3) leads to a non-trivial Jacobian, but fortunately it is also the exponencial of the WZW functional, that is

$$\mathcal{D}A\mathcal{D}\overline{A} = \mathrm{e}^{c_V\Gamma[UV]}\mathcal{D}U\mathcal{D}V \quad , \tag{2.9}$$

where c_V is the quadratic Casimir, and a definite regularization respecting vector gauge invariance has been chosen. As we stressed after (2.3), this is again written up to a factor containing the free Dirac operator.

The non-linearity in the gauge field interaction can also be disentangled by means of the identity (see notation in the appendix)

$$\mathrm{e}^{\frac{1}{4}\int \mathrm{d}^2z\,\mathrm{tr}\,F_{\mu\nu}^2} = \int \mathcal{D}E\;\mathrm{e}^{\int \mathrm{d}^2z\,\left[\frac{1}{2}\mathrm{tr}\,E^2 + \frac{1}{2}\mathrm{tr}\,EF_{z\overline{z}}\right]} \quad , \tag{2.10}$$

where E is a matrix-valued field. Taking into account the informations above we arrive at

$$\mathcal{Z}\,[\overline{\eta},\eta,i_\mu] = \int \mathcal{D}E\mathcal{D}U\mathcal{D}V\mathcal{D}g\times$$
$$\times\, \mathrm{e}^{-\Gamma[UgV]+(c_V+1)\Gamma[UV]+\int \mathrm{d}^2z\, \mathrm{tr}\,[\frac{1}{2}E^2+\frac{1}{2}EF_{z\bar{z}}]-\int \mathrm{d}^2z\, i_\mu A_\mu+\int \mathrm{d}^2z\mathrm{d}^2w\, \overline{\eta}(z)(i\not{D})^{-1}(z,w)\eta(w)} \quad . \tag{2.11}$$

If we were considering massive fermions, we should include a term $m\mathrm{tr}\,(g+g^{-1})$ in the effective action[6]. However we shall avoid such a complication and only consider the massless case. We also have to deal with gauge fixing. In fact, the process of introducing ghosts is standard, and we suppose that the procedure is included above, until it is necessary to explicitely take into account the ghost degrees of freedom, which will be the case upon consideration of the spectrum, when the BRST condition has to be used. Up to that point our manipulations do not explicitely depend on the gauge fixing/ghost system, and we proceed without it (or else, keeping it behind our minds).

Defining the gauge invariant field $\widetilde{g} = UgV$, and using the invariance of the Haar measure, that is $\mathcal{D}g = \mathcal{D}\widetilde{g}$, we see that the $\widetilde{g}$ field decouples (allways up to BRST condition - see later) and we are left with

$$\mathcal{Z}\,[\overline{\eta},\eta,i_\mu] = \int \mathcal{D}\widetilde{g}\,\mathrm{e}^{-\Gamma[\tilde{g}]}\int \mathcal{D}E\mathcal{D}U\mathcal{D}V\mathcal{D}(ghosts)\times$$
$$\times\, \mathrm{e}^{(c_V+1)\Gamma[UV]+\int \mathrm{d}^2z\, \mathrm{tr}\,[\frac{1}{2}E^2+\frac{1}{2}EF_{z\bar{z}}]-S_{ghosts}-\int \mathrm{d}^2z\, i_\mu A_\mu+\int \mathrm{d}^2z\mathrm{d}^2w\, \overline{\eta}(z)(i\not{D})^{-1}(z,w)\eta(w)} \quad , \tag{2.12}$$

where A_μ variables are given in terms of the UV variables.

The presence of the gauge field strenght $F_{z\bar{z}}$ hinders further developments in the way it is presented above. However, in terms of the U and V variables we can write

$$\mathrm{tr}\,EF_{z\bar{z}} = \frac{i}{e}\mathrm{tr}\,UEU^{-1}\partial(\Sigma\overline{\partial}\Sigma^{-1}) \quad , \tag{2.13}$$

where $\Sigma = UV$. This will permit a complete separation of some variables. Indeed, Σ is a more natural candidate to represent the physical degrees of freedom, since U and V are not separately gauge invariant. In the way it is written, in eq. (2.13), we can redefine E taking advantage once more of the invariance of the Haar measure, in such a way that the effective action only depends on the combination $\Sigma = UV$, while U and V appear separately only in the source terms, which are gauge dependent, as they should, that is,

$$\begin{aligned} A &= \frac{i}{e}U^{-1}\partial U \quad , \\ \overline{A} &= \frac{i}{e}(U^{-1}\Sigma)\overline{\partial}(\Sigma^{-1}U) \quad . \end{aligned} \tag{2.14}$$

If we eventually choose the light cone gauge, $U=1$, $\overline{A} = \frac{i}{e}\Sigma\overline{\partial}\Sigma^{-1}$ and $A=0$. From the structure of (2.13), it is natural to redefine variables as $\widetilde{E}' = UEU^{-1}$, $\mathcal{D}E = \mathcal{D}\widetilde{E}'$. Notice that already at this point the E redefinition implies, in terms of the gauge potential, an infinite gauge tail, which captures the possible gauge transformations. It is also conveniente to make the rescaling $\widetilde{E}' = 2ie(c_V+1)\widetilde{E}$, with a constant Jacobian. In terms of the field $\widetilde{E}$, consider the change of variables

$$\partial\widetilde{E} = \frac{1}{4\pi}\beta^{-1}\partial\beta \quad , \quad \mathcal{D}\widetilde{E} = \mathrm{e}^{c_V\Gamma[\beta]}\mathcal{D}\beta \quad . \tag{2.15}$$

We use the identity (2.6) to transform the $\beta\Sigma$ interaction into terms which can be handled in a more appropriated fashion. Writting both steps separately we have

$$\begin{aligned}\mathcal{Z}\left[\overline{\eta},\eta,i_\mu\right] = &\int \mathcal{D}\widetilde{g}\,\mathrm{e}^{-\Gamma[\widetilde{g}]}\mathcal{D}U\mathcal{D}(ghosts)\,\mathrm{e}^{-S_{ghosts}}\int \mathcal{D}\Sigma\mathcal{D}\widetilde{E}\mathrm{e}^{(c_V+1)\Gamma[\Sigma]}\times \\ &\times \mathrm{e}^{(c_V+1)\mathrm{tr}\int \mathrm{d}^2z\partial\widetilde{E}\Sigma\overline{\partial}\Sigma^{-1} - 2e^2(c_V+1)^2\int\mathrm{d}^2z\mathrm{tr}\,\widetilde{E}^2 - \int \mathrm{d}^2z\, i_\mu A_\mu + \int \mathrm{d}^2z\mathrm{d}^2w\,\overline{\eta}(z)(i\not{D})^{-1}(z,w)\eta(w)}\end{aligned} \tag{2.16}$$

in such a way that after substitution of (2.15) and use of (2.6) for $\Gamma[\beta\Sigma]$, we arrive at

$$\begin{aligned}\mathcal{Z}\left[\overline{\eta},\eta,i_\mu\right] = &\int \mathcal{D}\widetilde{g}\,\mathrm{e}^{-\Gamma[\widetilde{g}]}\mathcal{D}U\mathcal{D}(ghosts)\,\mathrm{e}^{-S_{ghosts}}\int \mathcal{D}\Sigma\mathcal{D}\beta\times \\ &\times \mathrm{e}^{(c_V+1)\Gamma[\beta\Sigma] - \Gamma[\beta] - \frac{2e^2(c_V+1)^2}{(4\pi)^2}\mathrm{tr}\int \mathrm{d}^2z[\partial^{-1}(\beta^{-1}\partial\beta)]^2 - \int \mathrm{d}^2z i_\mu A_\mu + \int \mathrm{d}^2z\mathrm{d}^2w\overline{\eta}(z)(i\not{D})^{-1}(z,w)\eta(w)}.\end{aligned} \tag{2.17}$$

We define the (massive) parameter

$$\lambda = \frac{c_V+1}{2\pi}e \quad , \tag{2.18}$$

and the field $\widetilde{\Sigma} = \beta\Sigma$, in terms of which the partition function reads

$$\begin{aligned}\mathcal{Z}\left[\overline{\eta},\eta,i_\mu\right] = &\int \mathcal{D}\widetilde{g}\,\mathrm{e}^{-\Gamma[\widetilde{g}]}\mathcal{D}U\mathcal{D}(ghosts)\,\mathrm{e}^{-S_{ghosts}}\int \mathcal{D}\widetilde{\Sigma}\,\mathrm{e}^{(c_V+1)\Gamma[\widetilde{\Sigma}]}\times \\ &\times \int \mathcal{D}\beta\,\mathrm{e}^{-\Gamma[\beta] - \frac{\lambda^2}{2}\mathrm{tr}\int \mathrm{d}^2z[\partial^{-1}(\beta^{-1}\partial\beta)]^2}\,\mathrm{e}^{-\int \mathrm{d}^2z\, i_\mu A_\mu + \int \mathrm{d}^2z\mathrm{d}^2w\overline{\eta}(z)(i\not{D})^{-1}(z,w)\eta(w)} \quad ,\end{aligned} \tag{2.19}$$

where now $A = \frac{i}{e}U^{-1}\partial U \quad , \quad \overline{A} = \frac{i}{e}(U^{-1}\beta^{-1}\widetilde{\Sigma})\overline{\partial}(\widetilde{\Sigma}^{-1}\beta U)$.

Up to source terms, and the BRST constraints to be discussed later, the above generating functional factorizes in terms of a conformal theory for $\widetilde{g}$, representing a gauge invariant fermionic bound states degrees of freedom, a second conformal field theory for $\widetilde{\Sigma}$, representing some gauge condensate, and an off-critically perturbed conformal field theory for the β-field, which describes also a gauge field condensate, in view of the change of variables (2.15). The conformal field theory representing $\widetilde{\Sigma}$ has an action with a negative sign (see (2.19)). Therefore we have to carefully take into account the BRST constraints in order to arrive at a positive metric Hilbert space. Since we will study first the β-degrees of freedom, we leave this problem for a later section.

3. Integrable perturbation of the WZW theory and duality

We consider the perturbed WZW action

$$
\begin{aligned}
S &= \Gamma[\beta] + \frac{1}{2}\lambda^2 \mathrm{tr} \int \mathrm{d}^2 z \; \left[\partial^{-1}(\beta^{-1}\partial\beta)\right]^2 \quad , \\
&= \Gamma[\beta] + \frac{1}{2}\lambda^2 \Delta(\beta) \quad .
\end{aligned}
\tag{3.1}
$$

We will look for the Euler-Lagrange equations for β. It is not difficult to find the variations:

$$
\begin{aligned}
\delta\Gamma[\beta] &= \left[-\frac{1}{4\pi}\overline{\partial}(\beta^{-1}\partial\beta)\right]\beta^{-1}\delta\beta \quad , \\
\delta\Delta(\beta) &= 2\Big[\partial^{-1}(\beta^{-1}\partial\beta) + \left[\partial^{-2}(\beta^{-1}\partial\beta), (\beta^{-1}\partial\beta)\right]\Big]\beta^{-1}\delta\beta \quad .
\end{aligned}
\tag{3.2}
$$

Collecting the terms, we find it useful to define the current components

$$
\begin{aligned}
J^\beta &= \beta^{-1}\partial\beta \quad , \\
\overline{J}^\beta &= 4\pi\lambda^2\partial^{-2}J^\beta = 4\pi\lambda^2\partial^{-2}(\beta^{-1}\partial\beta) \quad ,
\end{aligned}
\tag{3.3}
$$

which summarize the β equation of motion as a zero curvature condition given by

$$
[\mathcal{D}, \overline{\mathcal{D}}] = [\partial - J^\beta, \overline{\partial} - \overline{J}^\beta] = \overline{\partial}J^\beta - \partial\overline{J}^\beta - [\overline{J}^\beta, J^\beta] = 0 \quad . \tag{3.4}
$$

This is the integrability condition for the Lax pair[17]

$$
\mathcal{D}_\mu M = 0 \quad , \quad \text{with} \quad \mathcal{D}_\mu = \partial_\mu - J^\beta_\mu \quad , \tag{3.5}
$$

where $J^\beta = J^\beta_1 + iJ^\beta_2$, $\overline{J}^\beta = J^\beta_1 - iJ^\beta_2$ and M is the monodromy matrix. This is not a Lax pair as in usual non-linear sigma models[18], where J^β_μ is a conserved current, in which case we obtain a conserved non-local charge from (3.4), as well as higher local and non-local conservation laws, derived from an extension of (3.4) in terms of an arbitrary spectral parameter[18]. However, in a certain extent, the situation is simpler in the present case, due to the rather unusual form of the currents (3.3), which permits to write the commutator appearing in (3.4) as a total derivative, in such a way that in terms of the current $\overline{J}^\beta$, we have

$$
\partial\left\{4\pi\lambda^2\overline{J}^\beta - \partial\overline{\partial}\,\overline{J}^\beta + [\overline{J}^\beta, \partial\overline{J}^\beta]\right\} = 0 \quad . \tag{3.6}
$$

Therefore the quantity

$$
\overline{I}^\beta(\overline{z}) = 4\pi\lambda^2\overline{J}^\beta(z,\overline{z}) - \partial\overline{\partial}\overline{J}^\beta(z,\overline{z}) + [\overline{J}^\beta(z,\overline{z}), \partial\overline{J}^\beta(z,\overline{z})] \quad , \tag{3.7}
$$

does not depend on z, and it is a simple matter to derive an infinite number of conservation laws from the above (see later in the Minkowskian formulation).

This means that two-dimensional QCD is an integrable system! Moreover, it corresponds to an off-critical perturbation of the WZW-action. If we write $\beta = \mathrm{e}^{i\phi} \sim 1 + i\phi$, we verify that the perturbing term corresponds to a mass term for ϕ. Later we will discuss in more detail this issue in the large N limit (for the $SU(N)$ theory). The next natural step is to obtain the algebra obeyed by (3.7), and its representation. However there is a difficulty presented by the non-locality of the perturbation. We now introduce a further auxiliary field defining a dual action, local in all fields, and representing the low energy scales of the theory, and later we return to the problem of finding the algebra obeyed by (3.7).

Consider the Δ-term of the action (3.1). We write the quadratic term in (3.1) introducing the integral over a Gaussian field $\partial\overline{C}$ as

$$\mathrm{e}^{-\frac{1}{2}\lambda^2\Delta} = \int \mathcal{D}\overline{C}\, \mathrm{e}^{\int \mathrm{d}^2 z\, \frac{1}{2}\mathrm{tr}\,(\partial\overline{C})^2 - \lambda \mathrm{tr} \int \mathrm{d}^2 z\, \overline{C}(\beta^{-1}\partial\beta)} \quad , \tag{3.8}$$

where the left hand side is readily obtained completing the square in the r.h.s.

Indeed, at this point we have two choices. We can turn to Minkowski space, and proceed with the canonical quantization of the action (3.1) with the non-local term substituted in terms of the $\overline{C}$ field dependent expression obtained in the exponent of the integrand of the r.h.s. of equation (3.8). Before that, motivated by the presence of the auxiliary vector field $\overline{C}$, we make again a change of variables of the type

$$\begin{aligned} \overline{C} &= \frac{1}{4\pi\lambda} W\overline{\partial}W^{-1} \quad , \\ \mathcal{D}\overline{C} &= \mathrm{e}^{c_V\Gamma[W]}\mathcal{D}W \quad , \end{aligned} \tag{3.9}$$

together with the now very frequently used identity (2.6) in order to find a dual action. We have for the β-partition function the expression

$$\mathcal{Z} = \int \mathcal{D}\beta\mathcal{D}W\, \mathrm{e}^{-\Gamma[\beta] + c_V\Gamma[W] - \frac{1}{4\pi}\int \mathrm{d}^2 z\, W\overline{\partial}W^{-1}\beta^{-1}\partial\beta + \int \mathrm{d}^2 z\, \frac{1}{2(4\pi\lambda)^2}[\partial(W\overline{\partial}W^{-1})]^2} \quad , \tag{3.10}$$

from which can separate the contribution $-\Gamma[\beta W] \equiv -\Gamma[\widetilde{\beta}]$; after such manoeuver we are left with

$$\mathcal{Z} = \int \mathcal{D}\widetilde{\beta}\, \mathrm{e}^{-\Gamma[\widetilde{\beta}]} \int \mathcal{D}W\, \mathrm{e}^{(c_V+1)\Gamma[W] + \frac{\mathrm{tr}}{2(4\pi\lambda)^2}\int \mathrm{d}^2 z\, [\partial(W\overline{\partial}W^{-1})]^2} \quad . \tag{3.11}$$

The dual action has now a coupling constant corresponding to the inverse of the initial charge. Therefore (3.11) is appropriated for the study of a strongly coupled limit. Notice that the procedure is, in a sense, familiar to the one used to obtain a dual action, where a non-dynamical field is introduced, and one eliminates the original dynamical and fields leaving the so called dual formulation. See the refs. [13], [14] and [15] for further details. We separate a further WZW-conformal piece, and we are left with a local massive action for W. The drawback is the fact that now W itself has an action with a negative sign. Naively it describes also massive excitations, although a complete description of the spectrum can only be obtained after disentangling the non-linear relations and imposing the BRST conditions.

For the sources, we have now to replace A in (2.19) by $\frac{i}{e}(U^{-1}W\widetilde{\beta}^{-1}\widetilde{\Sigma})\overline{\partial}(\widetilde{\Sigma}^{-1}\widetilde{\beta}W^{-1}U)$. We notice also here, that we have dual descriptions of two-dimensional QCD. In the first, valid in the perturbative region, for high energies, we find out a non-local perturbation of the WZW action. In terms of W the perturbation is local, but at the price of a negative sign in the naive kinetic term in the W action, which is appropriated to describe the low energy (strong coupling) regime of the theory. In spite of such different complementary descriptions, both models are integrable. In the weak coupling regime we found the conservation laws (3.3 - 3.6). In the case of the W-theory, it is not difficult to find the equations of motion, and again derive the similar relations for the quantity

$$\overline{I}^W(\overline{z}) = \frac{1}{4\pi}(c_V+1)\overline{J}^W(z,\overline{z}) + \frac{1}{(4\pi\lambda)^2}\partial\overline{\partial}\,\overline{J}^W(z,\overline{z}) + \frac{1}{(4\pi\lambda)^2}[\overline{J}^W(z,\overline{z}),\partial\overline{J}^W](z,\overline{z}) \quad , \tag{3.12}$$

with $\overline{J}^W = W\overline{\partial}W^{-1}$ and $\partial\overline{I}^W = 0$, i.e. $\overline{I}^W$ does not depend on z.

Therefore, after finding isomorphic higher charges for both formulations, we are motivated to find their corresponding algebras, and later quantize them.

4. Higher conservation laws and corresponding algebras

To obtain the algebra obeyed by the previously found conserved charges, it is easier to go to Minkowski space, proceed with the canonical quantization[19], obtaining first the Poisson algebra, and later the constraints and the quantum commutators of the model. In fact, from the computation of the fermion determinant, we have an effective bosonic action which already takes into account some quantum corrections, namely the fermionic loops have been summed up. Therefore, the Poisson brackets already have quantum corrections arising from fermionic loops. This fact minimizes the possibilities of anomalies in the full quantum definition of the charges[20]. As a matter of fact, we shall see that quantum corrections are restricted to the introduction of renormalization constants.

From the conventions described in the appendix we find the Minkowski space action

$$S = -(c_V+1)\Gamma_M[W] + \frac{1}{2(4\pi\lambda)^2}\int \mathrm{d}^2x\,[\partial_+[(W\partial_-W^{-1})]^2 \quad , \tag{4.1}$$

with the Minkowski space WZW functional given by

$$\Gamma_M[W] = \frac{1}{8\pi}\mathrm{tr}\int \mathrm{d}^2x\,\partial^\mu W^{-1}\partial_\mu W + \frac{1}{4\pi}\epsilon^{\mu\nu}\mathrm{tr}\int_0^1 \mathrm{d}r\int \mathrm{d}^2x\,\hat{W}^{-1}\dot{\hat{W}}\hat{W}^{-1}\partial_\mu\hat{W}\hat{W}^{-1}\partial_\nu\hat{W} \quad . \tag{4.2}$$

Due to the presence of higher derivatives in the above action, it is convenient to introduce an auxiliary field and rewritte it in the equivalent form

$$S = -(c_V+1)\Gamma_M[W] + \mathrm{tr}\,\frac{1}{2}\int \mathrm{d}^2x\,\left[-B^2 + \frac{1}{2\pi\lambda}\partial_+B\partial_-WW^{-1}\right] \quad , \tag{4.3}$$

where (4.1) is obtained completing the square in the B-term in (4.3). The momentum canonically conjugated to the variable W is

$$\begin{aligned}\Pi^W_{ij} &= \frac{\partial S}{\partial \partial_0 W_{ij}} = -\frac{1}{4\pi}(c_V+1)\partial_0 W^{-1}_{ji} - \frac{1}{4\pi}(c_V+1)A_{ji} + \frac{1}{4\pi\lambda}(W^{-1}\partial_+ B)_{ji} \quad , \\ &= \hat{\Pi}^W_{ij} - \frac{1}{4\pi}(c_V+1)A_{ji} \quad ,\end{aligned} \tag{4.4}$$

where the first term is obtained from the principal sigma model term in the WZW action, the second arises from the pure WZW term, and the third one from the interaction with the auxiliary field. It is convenient to separate the WZW contribution A_{ij} to the momentum, since the new variable $\widehat{\Pi}^W$ is local in the original fields. The treatment of the WZW term (second above) follows closely the one introduced in [19], see also [7]. An explicit form for A_{ij} cannot be obtained in terms of local fields, but we only need its derivatives, which are not difficult to obtain, that is[7,19]

$$F_{ij;kl} = \frac{\delta A_{ij}}{\delta W_{lk}} - \frac{\delta A_{kl}}{\delta W_{ji}} = \partial_1 W^{-1}_{il} W^{-1}_{kj} - W^{-1}_{il}\partial_1 W^{-1}_{kj} \quad , \tag{4.5}$$

in terms of which we have the Poisson bracket relation

$$\begin{aligned}\left\{\hat{\Pi}^W_{ij}(x), \hat{\Pi}^W_{kl}(y)\right\} &= -\frac{c_V+1}{4\pi}\left(\frac{\delta A_{lk}}{\delta W_{ij}} - \frac{\delta A_{ji}}{\delta W_{kl}}\right) \\ &= \frac{c_V+1}{4\pi}\left(\partial_1 W^{-1}_{jk} W^{-1}_{li} - \partial_1 W^{-1}_{li} W^{-1}_{jk}\right)\delta(x^1-y^1) \quad .\end{aligned}$$

The momentum associated with the B field is

$$\Pi^B_{ij} = -\frac{1}{4\pi\lambda}(W\partial_- W^{-1})_{ji} \quad . \tag{4.6}$$

We can now list the relevant field operators appearing in the definition of the conservation law (3.12), which we rewrite in Minkowski space as

$$\begin{aligned}I^W_- &= \frac{1}{4\pi}(c_V+1)J^W_- - \frac{1}{(4\pi\lambda)^2}\partial_+\partial_- J^W_- - \frac{1}{(4\pi\lambda)^2}[J^W_-, \partial_+ J^W_-] \quad , \\ \partial_+ I^W_- &= 0 \quad .\end{aligned} \tag{4.7}$$

In terms of phase space variables they are

$$\begin{aligned}J^W_- &= W\partial_- W^{-1} = -4\pi\lambda\widetilde{\Pi}_B \quad , \\ \partial_+ J^W_- &= -4\pi\lambda\partial_+\widetilde{\Pi}_B = 4\pi\lambda B \quad , \\ \partial_+\partial_- J^W_- &= (4\pi\lambda)^2[W\widetilde{\widetilde{\Pi}}^W - (c_V+1)\lambda\widetilde{\Pi}_B] - (4\pi\lambda)\lambda(c_V+1)W'W^{-1} - 8\pi\lambda B' \quad ,\end{aligned} \tag{4.8}$$

where the tilde means tranposition of the matrix indices. It is straighforward to compute the Poisson algebra. We have

$$\left\{I^W_{ij}(t,x), I^W_{kl}(t,y)\right\} = \left[I^W_{kj}\delta_{il} - I^W_{il}\delta_{kj}\right]\delta(x^1-y^1) - \alpha\delta^{il}\delta^{kj}\delta'(x^1-y^1) \tag{4.9}$$

where $\alpha = \frac{1}{2\pi}(c_V + 1)$. The current itself is a realization of the Kac-Moody algebra, since

$$
\begin{aligned}
\{ I^W_{ij}(t,x), J^W_{-kl}(t,y)\} &= (J^W_{-kj}\delta_{il} - J^W_{-il}\delta_{kj})\delta(x^1-y^1) + 2\delta_{il}\delta_{kj}\delta'(x^1-y^1) \quad , \\
\{ J^W_{ij}(t,x), J^W_{-kl}(t,y^1)\} &= 0 \quad .
\end{aligned}
\tag{4.10}
$$

We thus obtain a Kac-Moody algebra for I^W_-, and J^W_- is a representation of such an algebra, with a central extension. We shall return to this discussion later, after consideration of the quantization of the charge.

The Hamiltonian density can also be computed, and we arrive at the phase space expression

$$
\begin{aligned}
H_W =& \hat{\widetilde{\Pi}}^W W' + 4\pi\lambda \hat{\widetilde{\Pi}}^W \widetilde{\Pi}^B W - \widetilde{\Pi}^B B' - 4\pi\lambda^2(c_V+1)(\widetilde{\Pi}^B)^2 \\
& - 2(c_V+1)\lambda\widetilde{\Pi}^B W'W^{-1} + \frac{1}{4\pi}(c_V+1)(W'W^{-1})^2 + \frac{1}{2}B^2 \quad ,
\end{aligned}
\tag{4.11}
$$

where $B' = \partial_1 B$, $W' = \partial_1 W$; the above Hamiltonian can be rewritten in a quadratic form in terms of the currents, although in such a case we have also velocities, due to the appearance of the time derivatives:

$$
H_W = \alpha\left(J^W_1\right)^2 - \frac{1}{(4\pi\lambda)^2}\left[\partial^2_+ J^W_- J^W_+ + J^W_-\partial_-\partial_+ J^W_- - (\partial_+ J^W_-)^2\right] \quad , \tag{4.12}
$$

where $J^W_1 = \frac{1}{2}(J^W_+ - J^W_-)$ and $J^W_+ = W\partial_+ W^{-1}$. At this point we can compare the model with its β-formulation. In this case we have the action

$$
S = \Gamma_M[\beta] + \lambda \mathrm{tr}\int \mathrm{d}^2x\, C_-\beta^{-1}\partial_+\beta + \frac{1}{2}\mathrm{tr}\int \mathrm{d}^2x\,(\partial_+ C_-)^2 \quad , \tag{4.13}
$$

where C_- is the Minkowski space conterpart of $\overline{C}$ (see eq. (3.8)).

The canonical quantization proceeds straighforwardly, and the relevant phase space expressions are obtained for $\overline{J}^\beta$ in (3.3), which in Minkowski space, due to the C_- equation of motion reads

$$
J^\beta_- = 4\pi\lambda^2\partial_+^{-2}(\beta^{-1}\partial_+\beta) = 4\pi\lambda C_- \quad , \tag{4.14a}
$$

$$
\Pi_- = \partial_+ C_- \quad , \tag{4.14b}
$$

while β-momentum is given by

$$
\hat{\widetilde{\Pi}}^\beta_{ji} = \frac{1}{4\pi}\partial_0\beta^{-1}_{ji} + \lambda(C_-\beta^{-1})_{ji} \quad , \tag{4.14c}
$$

where the hat above Π^β means that we neglected the WZW contribution as before[25], and as a consequence

$$
\left\{\hat{\widetilde{\Pi}}^\beta_{ji}(t,x), \hat{\widetilde{\Pi}}^\beta_{lk}(t,y)\right\} = -\frac{1}{4\pi}\left(\partial_1\beta^{-1}_{jk}\beta^{-1}_{li} - \partial_1\beta^{-1}_{li}\beta^{-1}_{jk}\right)\delta(x-y) \tag{4.14d}
$$

From the definition of the canonical momentum associated with C_- we have

$$\partial_+ J_-^\beta = 4\pi\lambda\Pi_- \quad . \tag{4.15}$$

The conserved charge is (from (3.6) we change $\partial \to -\partial_-$, $\overline{\partial} \to \partial_+$, $J_-^\beta \to J_-^\beta$)

$$\begin{aligned} I_-^\beta &= 4\pi\lambda^2 J_-^\beta + \partial_+\partial_- J_-^\beta + [J_-^\beta, \partial_+ J_-^\beta] \quad , \\ \partial_+ I_-^\beta &= 0 \quad ; \end{aligned} \tag{4.16}$$

therefore the situation is analogous to the one we found previously interchanging the (B, Π_B) phase space variables with (Π_-, C_-) (noticing the exchanged order).

The Hamiltonian might be computed at this point. However we will postpone it to a later section, since we will have to compute it in terms of more appropriated currents, rendering the problem easier to be formulated in terms of the constraints, hidden in the gauge transformation properties.

We come now to the point where we are urged to consider the quantization of the symmetry current (4.16). Let us consider the problem in the β-language, since the short distance expansion depends on the high energy behavior of the theory, therefore, since the only massive scale is the coupling constant, we have to consider the weak coupling limit. The weak coupling limit is better described by the β-action. In such a case, we need the short distance expansion of the current $J_-^\beta = 4\pi\lambda^2 \partial_+^{-2}(\beta^{-1}\partial_+\beta)$ with itself. Since the short distance expansion is compatible with the weak coupling limit, where the theory is conformally invariant, Wilson expansions can be dealt as usually.

We consider the short distance expansion

$$[J_-^\beta(x), \partial_+ J_-^\beta(y)]\Big|_{y=x+\epsilon} = \sum_n a^{(n)}(\epsilon)\mathcal{O}^{(n)}(x) \quad , \tag{4.18}$$

aiming at a classification of $\mathcal{O}^{(n)}(x)$ according to its dimension[20]. It is in fact easier to start out of local objects, that is in terms of J_+^β

$$J_+^\beta = \frac{1}{4\pi\lambda^2}\partial_+^2 J_-^\beta = \beta^{-1}\partial_+\beta \quad , \tag{4.19}$$

and later act with antiderivative operators. Therefore we analyse the auxiliary operator product expansion

$$[J_+^\beta(x), J_+^\beta(y)]\Big|_{y=x+\epsilon} = \sum_n a_{J_+^\beta}^{(n)}(\epsilon)\mathcal{O}_{J_+^\beta}^{(n)}(x) \quad . \tag{4.20}$$

Let us suppose that the theory is local. The fact that the interaction contains an antiderivative will be taken into account subsequently. Along such premises the problem is very simple, and has been solved long ago[20], with the classification for possible operators $\mathcal{O}^{(n)}_{J^\beta_+}(x)$:

1. $\dim \mathcal{O}^{(n)} = 0$, no operator ,
2. $\dim \mathcal{O}^{(n)} = 1$, $\mathcal{O}^{(1)}_{J^\beta_+}(x) = \beta^{-1}\partial\beta$,
3. $\dim \mathcal{O}^{(n)} = 2$, $\mathcal{O}^{(2)}_{J^\beta_+}(x) = \partial(\beta^{-1}\partial\beta)$,
4. $\dim \mathcal{O}^{(n)} \geq 3$, operators with finite coefficients .

Therefore we find

$$[J^\beta_+(x), J^\beta_+(y)] = a^{(1)}(\epsilon)J^\beta_+(x) + a^{(2)}(\epsilon)\partial J^\beta_+(x) \quad , \tag{4.21}$$

where $\dim a^{(1)}(\epsilon) = 1$, therefore $a^{(1)}(\epsilon)$ is linearly divergent, that is, $a^{(1)}(\epsilon) \sim 1/\epsilon$, while $\dim a^{(2)} = 0$, and $a^{(2)}(\epsilon)$ is logarithmically divergent, that is $a^{(2)}(\epsilon) \sim \ln\epsilon$. Acting on the above Wilson expansion with ∂_x^{-2} we obtain the new expansion

$$[\partial_+^{-2} J^\beta_+(x), J^\beta_+(y)] = a^{(1)}(\epsilon)\partial_+^{-2} J^\beta_+(x) + \widetilde{a}^{(2)}(\epsilon)\partial_+^{-1} J^\beta_+(x) \quad , \tag{4.22}$$

where $\widetilde{a}^{(2)}(\epsilon)$ is $a^{(2)}(\epsilon)$ plus a possible $\partial^{-1}a^{(1)}(\epsilon)$ correction. We now act with ∂_ϵ^{-1}, obtaining

$$[\partial_+^{-2} J^\beta_+(x), \partial_+^{-1} J^\beta_+(y)] = \partial^{-1}\widetilde{a}^{(2)}(\epsilon)\partial_+^{-2} J^\beta_+(x) + \text{finite} \quad , \tag{4.23}$$

since $\partial^{-1}a^{(2)}$ is finite.

Let us now discuss the effect of the non-local term in the action. Its local version is given by an expression containing the C_- field (see [19]). Such a field has dimension zero, and its interaction contains a λ factor in the Lagrangian, namely

$$\mathcal{L}_{C_-} = \frac{1}{2}(\partial_+ C_-)^2 + \lambda C_- \beta^{-1}\partial_+\beta \quad . \tag{4.24}$$

Therefore C_- comes in a Wilson expansion accompannied by a λ-factor, and has, effectively, dimension 1. Moreover, it can not appear alone, since by Lorentz transformation it acquires a factor which is the inverse of the one required for the current, since it is the $(-)$ component of a vector, while the current is a $(+)$ component. Therefore it can appear at most with a logarithmically divergent coefficient in the $J^\beta_+(x)J^\beta_+(x+\epsilon)$ expansion, and is irrelevant to the present problem, due to the subsequent manipulations.

This discussion leads to the definition of the cut-off charges

$$Q^f_\delta = \int \mathrm{d}x^-\, f(x^-)\left\{\mathcal{Z}_\delta J^\beta_-(x) + [J^\beta_-(x), \partial J^\beta_-(x+\delta)]\right\} \quad , \tag{4.25}$$

with renormalized charge Q^f and renormalization constant $\mathcal{Z}_\delta$ respectively given by

$$Q^f = \lim_{\delta\to 0} Q^f_\delta \quad , \quad \mathcal{Z}_\delta = 1 - \partial^{-1}a^{(1)}(\delta) \quad . \tag{4.26}$$

Such a charge is finite, and

$$\frac{dQ^f_\delta}{dx^+} = \lim_{\delta\to 0}\int \mathrm{d}x^-\, f(x^-)\Big\{\mathcal{Z}_\delta \partial_+ J^\beta_- + \partial_+\big[\partial^{-1} a^{(1)} J^\beta_- + \mathcal{N}[J^\beta_-, \partial_+ J^\beta_-]\big]\Big\} \longrightarrow 0 \quad , \tag{4.27}$$

where $\mathcal{N}$ is a normal product prescription rendering the product in $[J^\beta_-, \partial_+ J^\beta_-]$ finite.

The infinite constant can also be interpreted as a charge renormalization. Due to the renormalization of the higher charge, we cannot give an interpretation to the field operator I^β_{ij} by itself, but only to an arbitrary linear combination involving the charge and the current. In any case, since I^β_{ij} is a right moving field operator, it is natural to assume, in view of the Poisson algebra (4.9) that it obeys an algebra given by[21,22]

$$I^\beta_{ij}(\overline{z}) I^\beta_{kl}(\overline{w}) = (I^\beta_{kj}\delta_{il} - I^\beta_{il}\delta_{kj})(\overline{w})\frac{1}{\overline{z}-\overline{w}} - \alpha\frac{\delta^{il}\delta^{kj}}{(\overline{z}-\overline{w})^2} \quad . \tag{4.28}$$

For J^β_{-ij} we are forced into a milder assumption. Indeed, since J^β_{-ij} is a representation of such an algebra with a central extension and commutes with itself, the equation $\partial_+ J^\beta_{-ij} = 0$ would be too simple to realize the whole problem we are considering. In such a case we would be left with unequal time commutators for the last equation (4.10). But in any case, since I^β_{ij} is a right moving field operator, the equal time requirement in the first equation (4.10) is also superfluous, and we get an operator product algebra of the type

$$I^\beta_{-ij}(\overline{z}) J^\beta_{-kl}(w,\overline{w}) = (J^\beta_{-kj}\delta_{ij} - J^\beta_{-il}\delta_{kj})(w,\overline{w})\frac{1}{\overline{z}-\overline{w}} + 2\frac{\delta^{il}\delta^{kj}}{(\overline{z}-\overline{w})^2} \quad , \tag{4.29}$$

where once again we turned to the Euclidian variables. The consequence is the fact that holomorphic derivatives of the current, are indeed primary fields. However the second equation in (4.10) can not be taken at arbitrary times, since J^β_- depends on both x^+ *and* x^-. Moreover, if J^β_- were purely right moving, the last equation, for unequal times would imply that it is a trivial operator.

Some conclusions may be drawn for J^β_-. As we stressed above, $\partial_+ J^\beta_-$ can not be zero*, in the full quantum theory, however, in view of (4.29), we conclude that left(-) derivatives of this current are primary fields[22], since

$$I^\beta_{ij}\partial^n_+ J^\beta_{-kl} = \frac{\partial^n_+ J^\beta_{-il}\delta_{kj} - \partial^n_+ J^\beta_{-kl}\delta_{il}}{\bar{z}-\bar{w}} \quad . \tag{4.30}$$

Therefore, we expect a Kac-Moody algebra for $I^\beta(z)$ and $\partial^n I^\beta$ should be primary fields, depending on parameters $\overline{z}$.

Such an underlining Kac-Moody structure is the most unexpected result in this paper, since it arose out of a non-linear relation obeyed by the current, which can be traced back

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* In the case J^β_- is left moving we expect further modifications of the commutators. See discussion in the conclusions.

to an integrability condition of the model. Moreover, the theory has an explicit mass term - although free massive fermionic theories as well as some off-critical perturbations of conformally invariant theories in two dimensions may contain affine Lie symmetry algebras.

The current itself is now a realization of such algebra in its right moving sector. Indeed, we have derived the algebra (4.28) from the Poisson structure.

5. The GKO construction

Gauged WZW theories provide a lagrangian realization of the GKO construction[16,23]. Deleting the "mass term" λ in (2.19) we have a gauged WZW theory as explicited in (2.7). The WZW functional is invariant under a $G\times G$ symmetry transformation given by

$$g(z,\overline{z}) \rightarrow \overline{G}(\overline{z})g(z,\overline{z})G(z) \quad . \tag{5.1}$$

In general one can gauge the anomaly free vector subgroup $H\subset G\times G$ by means of the addition of the term

$$\frac{1}{4\pi}\mathrm{tr}\int \mathrm{d}^2x\left[e^2A_+A_- - e^2a_+gA_-g^{-1} + ieA_-g^{-1}\partial_+g + ieA_+g\partial_-g^{-1}\right] \quad . \tag{5.2}$$

In the QCD_2 case, H corresponds to G.

Such a gauging procedure introduces constraints in the theory, as discussed by Karabali and Schnitzer[16]. In order to understand this point in more detail, we have to consider the effect of the ghost sector. In general ghosts are introduced considering a gauge fixing function $\mathcal{F}(A)$, and introducing a factor

$$\det\left(\frac{\partial\mathcal{F}}{\partial A_\mu}\frac{\partial A_\mu}{\partial\epsilon}\right)\delta\left(\mathcal{F}(A)\right) \tag{5.3}$$

in the partition function, where ϵ is the gauge parameter. However here if we are to render explicit the conformal content of the theory, it is more useful to represent all possible chiral determinants in terms of ghost integrals, such that the reparametrization invariance is also explicit and one can later verify that the gauge fixing procedure as outlined above, and which is more frequently used in the gauge field literature, is trivial in the sense that one is led to a unit Faddeev-Popov determinant.

Therefore ghosts are introduced writting determinants in terms of ghost systems, and decoupling them from the gauge fields by a chiral rotation, a procedure which is possible in two-dimensional space-time. This is equivalent to write all determinants as

$$\det D = \mathrm{e}^{c_V\Gamma[U]}(\det\partial)^{c_V} \quad , \quad \det\overline{D} = \mathrm{e}^{c_V\Gamma[U]}(\det\overline{\partial})^{c_V} \tag{5.4}$$

and substitute the free Dirac determinant in terms of ghosts as

$$
\begin{aligned}
(\det \partial)^{c_V} &= \int \mathcal{D}\overline{b}\mathcal{D}\overline{c}\, \mathrm{e}^{-\mathrm{tr} \int \mathrm{d}^2x\, \overline{b}\partial\overline{c}} \quad , \\
(\det \overline{\partial})^{c_V} &= \int \mathcal{D}b\mathcal{D}c\, \mathrm{e}^{-\mathrm{tr} \int \mathrm{d}^2x\, b\overline{\partial}c} \quad .
\end{aligned}
\tag{5.5}
$$

In fact the determinant of the Dirac operator does not factorize as in (5.4) because of the regularization ambiguity. At very step one has to assure vector current conservation. Such determinants cancel out by changing some of variables (as in (2.15)) but do not cancel in (2.19), from which we are led to the contribution

$$
\int \mathcal{D}\overline{b}\mathcal{D}b\mathcal{D}c\mathcal{D}\overline{c}\, \mathrm{e}^{-\mathrm{tr} \int \mathrm{d}^2x\, (b\overline{\partial}c + \overline{b}\partial\overline{c})} \quad . \tag{5.6}
$$

Although decoupled at the lagrangian level, such terms are essential due to subsequent constraints arising in the zero total conformal charge sector, leading to BRST constraints on physical states. Such constraints are obtained in a system of interacting conformally invariant sectors $(g, \Sigma, b, \overline{b}, c, \overline{c})$ described by the partition function.

$$
\mathcal{Z} = \int \mathcal{D}g\mathcal{D}\Sigma\mathcal{D}b\mathcal{D}\overline{b}\mathcal{D}c\mathcal{D}\overline{c}\, \mathrm{e}^{-k\Gamma[g] + (c_V+k)\Gamma[\Sigma] - \mathrm{tr} \int \mathrm{d}^2x\, (b\overline{\partial}c + \overline{b}\partial_+\overline{c})} \quad . \tag{5.7}
$$

One can couple the system to external gauge fields A^{ext} and $\overline{A}^{ext}$ as above, or equivalently by means of the minimal substitution[16] $\partial \to D^{ext} = \partial - ieA^{ext}$ and $\overline{\partial} \to \overline{D}^{ext} = \overline{\partial} - ie\overline{A}^{ext}$ with $A^{ext} = \frac{i}{e} U_{ext}^{-1} \partial U_{ext}$, and $\overline{A}^{ext} = \frac{i}{e} V_{ext} \overline{\partial} V_{ext}^{-1}$. The interaction of the fields from the WZW theory with such external gauge fields is equivalently obtained from (2.7b), that is[16]

$$
\begin{aligned}
-k\Gamma[g, A] &= -k\Gamma[U_{ext} g V_{ext}] + k\Gamma[U_{ext} V_{ext}] \\
(c_V + k)\Gamma[\Sigma, A] &= (c_V + k)\Gamma[U_{ext} \Sigma V_{ext}] - (c_V + k)\Gamma[U_{ext} V_{ext}] \\
-\mathrm{tr} \int \mathrm{d}^2x\, [b\overline{D}^{ext} c + \overline{b} D^{ext} \overline{c}] &= -\mathrm{tr} \int \mathrm{d}^2x\, [bV_{ext} \overline{\partial}(V_{ext}^{-1} c) + \overline{b} U_{ext}^{-1} \partial (U_{ext} \overline{c})] \quad ,
\end{aligned}
\tag{5.8}
$$

where k is the central charge . In our preceding discussion $k = 1$.

In the first two cases invariance of the Haar measure permits to change variables as

$$
\begin{aligned}
\widetilde{g} &= U_{ext} g V_{ext} \quad , \quad \mathcal{D}\widetilde{g} = \mathcal{D}g \quad , \\
\widetilde{\Sigma} &= U_{ext} \Sigma V_{ext} \quad , \quad \mathcal{D}\Sigma = \mathcal{D}\widetilde{\Sigma} \quad ,
\end{aligned}
\tag{5.9}
$$

while in the last case one can do a chiral rotation leaving back the free ghost system and a WZW term $c_V \Gamma[U_{ext} V_{ext}]$. Therefore, the $\Gamma[U_{ext} V_{ext}]$ term cancels due to the balance of central charges, and the partition function does not depend on the external gauge fields. This implies, in particular, that the functional derivative of the partition function with respect to the external gauge fields vanish, therefore

$$
\frac{\delta \mathcal{Z}(A^{ext}, \overline{A}^{ext})}{\delta A^{ext}} \Big|_{A^{ext}, \overline{A}^{ext} = 0} = 0 = \frac{\delta \mathcal{Z}(A^{ext}, \overline{A}^{ext})}{\delta \overline{A}^{ext}} \Big|_{A^{ext}, \overline{A}^{ext} = 0} \quad , \tag{5.10}
$$

which are equivalent, due to the minimal coupling, to the set of constraints

$$\left\langle kg^{-1}\partial g-(c_V+k)\Sigma^{-1}\partial\Sigma-4\pi[b,c]\right\rangle=0=\langle J_g+J_\Sigma+J_{ghost}\rangle \quad , \tag{5.11a}$$

as well as

$$\left\langle k\overline{\partial}gg^{-1}-(c_V+k)\overline{\partial}\Sigma\Sigma^{-1}-4\pi[\overline{b},\overline{c}]\right\rangle=0=\left\langle \overline{J}_g+\overline{J}_\Sigma+\overline{J}_{ghost}\right\rangle \quad . \tag{5.11b}$$

Each current above satisfies a Kac-Moody algebra with a corresponding central charge. One can build up a BRST charge Q as

$$Q=\sum :c^i_{-n}\left(J^i_{g_n}+J^i_{\Sigma_n}\right):-\frac{1}{2}if^{ijk}\sum :c^i_{-n}b^j_{-m}c^k_{n+m}: \quad , \tag{5.12}$$

where the i,j,k indices refer to the adjoint representation of the symmetry group, f^{ijk} the structure constants, and the mode expansion of the fields read

$$\begin{aligned} c^i &= \sum c^i_n z^{-n} \quad , \\ b^i &= \sum b^i_n z^{-n-1} \quad , \\ J^i_{g,\Sigma} &= \sum \left(J^i_{g,\Sigma}\right)_n z^{-n-1} \quad . \end{aligned} \tag{5.13}$$

The above charge is nilpotent: $Q^2=\frac{1}{2}\{Q,Q\}=0$. This implies that the above system is a set of first class constraints (indeed a similar set of constraints originates for $\overline{Q}$ resp. $\overline{J},\overline{b},\overline{c}$).

The stress tensor can be computed in terms of such currents, and we have three contributions, namely $T(z)=T_g(z)+T_\Sigma(z)+T^{ghost}(z)$, with the respective central charges $c_g=c(g,k)=\frac{2kd_g}{c_g+2k}$, $\quad c_\Sigma=c(H,-k-c_H)$, and $c_{ghost}=-2d_H$, where one supposes here that Σ takes values in $H\in G$. The total central charge, $c^{tot}=c_g+c_\Sigma+c_{ghost}$ coincides with that obtained from the GKO construction for coset space conformal theories, and the total energy tensor decomposes in terms of the GKO stress energy tensor and a residual piece, T', with zero central charge.

Representations of T' are thus trivial, and gauged WZW model is equivalent to the GKO construction of G/H conformal field theories. The physical subspace is generated by a product of matter and ghost sectors, obeying the equation

$$Q|\mathrm{phys}\rangle=0 \quad . \tag{5.14}$$

This solves also the problem of the sector with negative central charge, which should not be considered separately, being coupled through the BRST condition. Had we not such condition one would expect problems concerning negative metric states. Therefore one cannot consider each sector separately.

In the case of the inclusion of QCD_2 in such a scheme, we shall see that there are further constraints. Although the new constraints seem to be of the first class type when considered alone, there is a combination which is second class due to the cancelation of the ghost contribution. Therefore, in the case of QCD_2 we have to deal with Dirac quantization procedure of second class constraints![24]

However, we shall see that several interesting properties characteristic of the model, as well as part of the conformal structural relations still holds true, and QCD_2 problem can be undertood as an integrable perturbation of a (very simple) GKO construction of coset space conformal field theory. We will have a GKO construction of a very simple coset model to an off-critical perturbation of a WZW theory by means of second class constraints.

6. Coupling to external gauge fields and constraints

Consider the Minkowskian effective action

$$S_{eff} = \Gamma[\widetilde{g}] - (c_V + 1)\Gamma[\Sigma] + \Gamma[\beta] - \frac{1}{2}\lambda^2 \int \mathrm{d}^2 x\, [\partial_+^{-1}(\beta^{-1}\partial_+\beta)]^2 + S_{ghosts} \quad . \tag{6.1}$$

Let us start with by first coupling the fields $(\widetilde{g}, \Sigma, ghosts)$ to external gauge fields

$$A_-^{ext} = \frac{i}{e} V_{ext} \partial_- V_{ext}^{-1} \quad , \quad A_+^{ext} = \frac{i}{e} U_{ext}^{-1} \partial_+ U_{ext} \quad . \tag{6.1a}$$

We find

$$\begin{aligned} S_{eff}(A) = \Gamma[U_{ext}\widetilde{g}V_{ext}] - (c_V + 1)\Gamma[U_{ext}\Sigma V_{ext}] + \Gamma[U_{ext}\, ghosts\, V_{ext}] + \\ + [1 - (c_V + 1) + c_V]\Gamma[U_{ext} V_{ext}] \quad . \end{aligned} \tag{6.2}$$

Invariance of the Haar measure, and vanishing of the total central charge (i.e. vanishing coefficient of the last term above) tell us that the action does not depend on the external gauge fields. Nevertheless, the action can also be written as

$$\begin{aligned} S_{eff}(A) = S_{eff}(0) - \frac{1}{4\pi} A_+^{ext} \left[ie\widetilde{g}\partial_-\widetilde{g}^{-1} - ie(c_V + 1)\Sigma\partial_-\Sigma^{-1} + J_-(ghost)\right] \\ - \frac{1}{4\pi} A_-^{ext} \left[ie\widetilde{g}^{-1}\partial_+\widetilde{g} - ie(c_V + 1)\Sigma^{-1}\partial_+\Sigma + J_+(ghost)\right] + \mathcal{O}(A^2) \quad . \end{aligned} \tag{6.3}$$

Functionally differentiating the partition function once with respect to A_+^{ext} and separately with respect to A_-^{ext}, and putting $A_\pm^{ext} = 0$ we find the constraints

$$\begin{aligned} i\widetilde{g}\partial_-\widetilde{g}^{-1} - i(c_V + 1)\Sigma\partial_-\Sigma^{-1} + J_-(ghosts) \sim 0 \\ i\widetilde{g}^{-1}\partial_+\widetilde{g} - i(c_V + 1)\Sigma^{-1}\partial_+\Sigma + J_+(ghosts) \sim 0 \end{aligned} \tag{6.4}$$

leading to two BRS charges $Q^{(\pm)}$ as discussed by [16], which are nilpotent. Therefore we find two first class constraints.

The field $\overset{ext}{A_+}$ can also be coupled to the field β instead of $\widetilde{g}$, since the system $(\beta, \Sigma, ghosts)$ has also vanishing central charge. In such a case we have to disentangle the non-local interaction considering instead of the third and fourth terms in (6.1) the β-action

$$S(\beta) = \Gamma[\beta] + \int \mathrm{d}^2x\, \frac{1}{2}(\partial_+ C_-)^2 + \int \mathrm{d}^2x\, \lambda C_- \beta^{-1}\partial_+\beta \quad . \tag{6.5}$$

We make the minimal substitution $\partial_+ \to \partial_+ - ie\overset{ext}{A_+}$, repeating the previous arguments for the $(\beta, \Sigma, ghosts)$ system, and we arrive now at the constraint

$$\beta\partial_-\beta^{-1} + 4\pi\lambda\beta C_-\beta^{-1} - i(c_V + 1)\Sigma\partial_-\Sigma^{-1} + J_-(ghost) \sim 0 \quad . \tag{6.6}$$

One could naively expect that, repeating the previous arguments the system has a new set of first class constraints. But if we instead consider the equivalent system of the first set, together with the difference of the $(-)$ currents, namely

$$\Omega_{ij} = (\beta\partial_-\beta^{-1})_{ij} + 4\pi\lambda(\beta C_-\beta^{-1})_{ij} - (\widetilde{g}\partial_-\widetilde{g}^{-1})_{ij} \quad , \tag{6.7}$$

one readily verifies that the above constraint can not lead to a nilpotent BRST charge due to the absence of ghosts. Therefore, it must be treated as a second class constraint, defining the field C_-. The Poisson algebra obeyed by Ω_{ij} is

$$\begin{aligned} \{\Omega_{ij}(t,x), \Omega_{kl}(t,y)\} &= (\widetilde{\Omega}_{il}\delta_{kj} - \widetilde{\Omega}_{kj}\delta_{il})(t,x)\delta(x-y) + 2\delta_{il}\delta_{kj}\delta'(x-y) \\ \widetilde{\Omega} &= \widetilde{g}\partial_-\widetilde{g}^{-1} + \beta\partial_-\beta^{-1} + 4\pi\lambda\beta C_-\beta^{-1} \quad . \end{aligned} \tag{6.8}$$

Notice the change of sign in $\widetilde{\Omega}$. We can thus using the above define the undetermined velocities, and no further constraint is generated.

The fact that the theory possesses second class constraints is very annoying, since they can not be realized by the usual cohomology construction. Therefore, instead of building a convenient Hilbert space, one has to modify the dynamics, since the usual relation between Poisson brackets and commutators is replaced by the relation between Dirac brackets and commutators.

Nevertheles, as we will see, several nice structure unraveled so far remain, after such a harshening mutilation, untouched. Indeed, we shall see that there is a rather deep separation between the "right" currents, obeying equations analogous to those written so far, and the "left" currents, which will obey a modified dynamics, due to the second-class constraints.

As a consequence of the definition of the canonical momenta, eq. (4.14c) the constraints have a phase space formulation as

$$\Omega_{ij} = 4\pi(\beta\overset{\approx}{\hat{\Pi}}{}^{\beta})_{ij} + \partial_1\beta\beta^{-1} - 4\pi(\widetilde{g}\overset{\approx}{\hat{\Pi}}{}^{\tilde{g}})_{ij} - \partial_1\widetilde{g}\widetilde{g}^{-1} \quad , \tag{6.9}$$

which has been used to compute (6.8). Notice that the structure of the right hand side of the phase space expression is rather simple. Indeed, the C_- field just redefines the momentum associated with β, and the above constraints is analogous to the description of non-abelian chiral bosons[25], that is WZW theory with a constraint on a chiral current. It follows that the Poisson algebra is very simple. Indeed, one finds[25]

$$\begin{aligned}\{\Omega_{ij}(x),\Omega_{kl}(y)\} =&16\pi\delta_{il}\delta_{kj}\delta'(x^1-y^1)+4\pi[(4\pi\beta\tilde{\hat{\Pi}}^{\beta}+\beta'\beta^{-1}+4\pi\tilde{g}\tilde{\hat{\Pi}}^{\tilde{g}}+\tilde{g}'g^{-1})_{kj}\delta_{il}-\\ &-(4\pi\beta\tilde{\hat{\Pi}}^{\beta}+\beta'\beta^{-1}+4\pi\tilde{g}\tilde{\hat{\Pi}}^{\tilde{g}}+\tilde{g}'g^{-1})_{il}\delta_{kj}]\delta(x^1-y^1)\\ =&16\pi\delta_{il}\delta_{kj}\delta'(x^1-y^1)+8\pi[j_{-kj}\delta_{il}-j_{-il}\delta_{kj}]\delta(x^1-y^1)\quad ,\end{aligned} \tag{6.10}$$

where $j_-=4\pi\beta\tilde{\hat{\Pi}}^{\beta}+\beta'\beta^{-1}$ satisfies the Poisson algebra

$$\{j_{-ij},j_{-kl}\}=8\pi\delta_{il}\delta_{kj}\delta'(x-y)+4\pi(j_{-kj}\delta_{il}-j_{-il}\delta_{kj})\delta(x-y)\quad . \tag{6.11}$$

The above expression defines also the Q-matrix

$$Q_{ij;kl}=\{\Omega_{ij}(x),\Omega_{kl}(y)\}\Big|_{equal\quad time}\quad , \tag{6.12}$$

which is not a combination of constraints, therefore no further constraint is generated by the Dirac algorithm. The inverse of the Dirac matrix is not difficult to compute and we have the expression[25]

$$\begin{aligned}\left(Q^{-1}\right)_{ij;kl}=&\frac{1}{32\pi}\delta_{il}\delta_{kj}\epsilon(x)+\\ &+\frac{1}{64\pi}(\delta_{il}j_{jk}-\delta_{jk}j_{li})|x|+\\ &+\frac{1}{128\pi}(\delta_{ia}j_{jb}-\delta_{jb}j_{ai})(\delta_{al}j_{bk}-\delta_{bk}j_{la})\frac{1}{2}x^2\epsilon(x)+\\ &+\frac{1}{256\pi}(\delta_{ia}j_{jb}-\delta_{jb}j_{ai})(\delta_{ca}j_{bd}-\delta_{bd}j_{ac})(\delta_{lc}j_{dk}-\delta_{dk}j_{cl})\frac{1}{3}x^3\epsilon(x)+\cdots\quad ,\end{aligned} \tag{6.13}$$

where x is the space component of x^μ.

The next step consists in replacing the Poisson brackets by Dirac brackets. Thus we have to compute the Poisson brackets of the relevant quantities with the constraints. We use

$$\{A,B\}_{DB}=\{A,B\}_{PB}-\{A,\Omega_\alpha\}_{PB}Q^{-1}_{\alpha\beta}\{\Omega_\beta,B\}_{PB}\quad . \tag{6.14}$$

We will see that functions of

$$J_+^\beta=\beta^{-1}\partial_+\beta=-4\pi\tilde{\Pi}\beta+\beta^{-1}\beta'+4\pi\lambda C_-\quad , \tag{6.15}$$

commute with Ω_α, and their Dirac brackets coincide with their Poisson brackets.

Canonical quantization and Dirac formulation of the β sector is achieved by the formulae ($4.14a,b,c$), (4.15), from which we obtain

$$\frac{1}{4\pi}\beta^{-1}\partial_\pm\beta=-\tilde{\hat{\Pi}}^{\beta}\beta\pm\frac{1}{4\pi}\beta^{-1}\beta'+\lambda C_-\quad . \tag{6.18}$$

It is useful, in view of (6.8) to consider the combination

$$\frac{1}{4\pi}\partial_-\beta\beta^{-1} = -\beta\tilde{\hat{\Pi}}^{\beta} - \frac{1}{4\pi}\beta'\beta^{-1} + \lambda\beta C_-\beta^{-1} \quad , \tag{6.19}$$

or also, aiming at the expression of the constraint (6.8) which contains the C_- field, we have

$$\beta\partial_-\beta^{-1} + 4\pi\lambda C_-\beta^{-1} = -4\pi\beta\tilde{\hat{\Pi}}^{\beta} - \beta'\beta^{-1} \quad . \tag{6.20}$$

Thus, in terms of phase space variables the constraint is given by (6.9). Using the above phase space expressions we find

$$\begin{aligned} \left\{J_-^{\beta}, \Omega\right\} &= 0 \\ \left\{[J_-^{\beta}, \partial_+ J_-^{\beta}], \Omega\right\} &= \{[C_-, \Pi_-], \Omega\} = 0 \quad . \end{aligned} \tag{6.21}$$

For $\left\{\partial_+\partial_- J_-^{\beta}, \Omega\right\}$ we have first to compute

$$\begin{aligned} \partial_+\partial_- J_-^{\beta} &= \partial_+^2 J_-^{\beta} - 2(\partial_+ J_-^{\beta})' \quad , \\ &= 4\pi\lambda^2\beta^{-1}\partial_+\beta - 2(\Pi_-)' \quad . \end{aligned} \tag{6.22}$$

We use the fact that $\{\Pi'_-, \Omega\} = 0$ and we are left with

$$\beta^{-1}\partial_+\beta = -4\pi\tilde{\hat{\Pi}}^{\beta}\beta + \beta^{-1}\beta' + 4\pi\lambda C_- \quad . \tag{6.23}$$

Using now $\{C_-, \Omega\} = 0$ we have just to consider

$$j_{+ij} = \left(-4\pi\tilde{\hat{\Pi}}^{\beta}\beta + \beta^{-1}\beta'\right)_{ij} \quad . \tag{6.24}$$

However, since $\{j_+, j_-\} = 0$ we have $\{j_+, \Omega\} = 0$! As a conclusion, the Dirac algebra is the same as the Poisson algebra!! This is a non-trivial result, because it holds in spite of the fact that due to (6.9) the Dirac algebra obeyed by $\hat{\Pi}^{\beta}$ and β changes drastically, especially if we take into account the expression of the inverse Dirac matrix (6.13), which is non-local and has an infinite number of terms!

7. BRST constraints in the dual case

In the duality transformation relating the β and the W fields, we also find interesting relations arising out of the constraint structure of the theory. First let us perform a more detailed analysis of the ghost structure. Back to the transformations defined by (3.9) we have the factor of $(\det \partial_+ \ \det \partial_-)^{c_V}$ left out, which contributes as

$$\mathcal{Z}^{gh'} = \int \mathcal{D}b'_+ \mathcal{D}b'_- \mathcal{D}c' \mathcal{D}\overline{c}' \, \mathrm{e}^{-\mathrm{tr} \int \mathrm{d}^2 x \, (b'_+ \partial_- c' + b'_- \partial_+ \overline{c}')} \quad . \tag{7.1}$$

The coupling of a subset of fields to external gauge potential written in the form (6.1a) as described in section 6 can be made, and as usual. If such a set has a vanishing total central charge, the partition function does not depend on the gauge potential, and we are led to constraints again. With the partition function written in the W language as in (4.3), and taking into account all appropriate ghosts, we have various self commuting constraints. Some of them, such as

$$\begin{aligned} J_{\tilde{g}} - (c_V + 1) J_\Sigma + J_{ghost} \sim 0 \quad , \\ J_{\tilde{\beta}} - (c_V + 1) J_\Sigma + J_{ghost} \sim 0 \quad , \end{aligned} \tag{7.2}$$

are the same as before, with the advantage that now $\widetilde{\beta}$ is a pure WZW field, in such a way that it can be simply identified with $\widetilde{g}$, without further consequences. However, further constraints involving also the W field arise, such as

$$J_+^{\tilde{g}} - (c_V + 1) J_+^W + J_{+ghost} \sim 0 \quad , \tag{7.3}$$

in such a way that we have, as a consequence, the non-trivial second class constraint

$$J_+^\Sigma - J_+^W \sim 0 \quad , \tag{7.4}$$

or more explicitely

$$(c_V + 1)\Sigma^{-1}\partial_+\Sigma - (c_V + 1) W^{-1}\partial_+ W + \frac{1}{\lambda} W^{-1}\partial_+ B W = 0 \quad .$$

Above we proceeded as in the β formulation, but with the interaction of the A_-^{ext} field with the W, while in the (dual) β case we considered A_+^{ext}.

The phase space expression is given in the formula

$$\Omega^{W,\Sigma} = -\widetilde{\widehat{\Pi}}^W W + \frac{1}{4\pi} W^{-1} W' + \widetilde{\widehat{\Pi}}^\Sigma \Sigma - \frac{1}{4\pi}\Sigma^{-1}\Sigma' \sim 0 \quad , \tag{7.5}$$

and resembles the β-formulation (see (6.9)). However, as intriguing as it might appear, if we now substitute the B field from the constraint (7.4) back into the action we find a non-local term. This means that while in the β-formulation which is non-local at the begining we end up with a local action after substituting back the constraint, while in the

W-formulation, which is local at the begining we end up with a non-local action; another feature of duality in both formulations.

Keeping the Dirac algebra in mind, we substitute back the configuration space constraints into the action, maintaining the phase space structure. In such a case, using (6.7), and (2.6), we redefine $\beta g \equiv P$, $\beta = Pg^{-1}$, and find the effective action

$$\begin{aligned} S =& \Gamma[P] - \frac{1}{2\pi} g^{-1}\partial_+ g g^{-1}\partial_- g - \frac{1}{4\pi} P^{-1}\partial_+ P P^{-1}\partial_- P + \frac{1}{4\pi} P^{-1}\partial_+ P g^{-1}\partial_- g + \\ &+ \frac{1}{4\pi} P^{-1}\partial_- P g^{-1}\partial_+ g + \frac{1}{4\pi}\partial_- g g^{-1}\partial_+ P P^{-1} - \frac{1}{2\pi}\partial_- g g^{-1} P g^{-1}\partial_+ g g^{-1} P + \\ &+ \frac{1}{2(4\pi\lambda)^2}\left[\partial_+\left\{gP^{-1}g\partial_-(g^{-1}Pg^{-1})\right\}\right]^2 \quad . \end{aligned} \tag{7.6}$$

The equation of motion / conservation law (3.6) still holds, as previously proved. From action (7.6) we can find the equations of motion. Notice that the final action is a WZW theory off the critical point, a principal σ model, and current-current type interactions between them.

For the dual formulation a further interesting structure arises. The constraint is now

$$\partial_+ B = -\lambda(c_V + 1) W\Sigma^{-1}\partial_+(\Sigma W^{-1}) \quad . \tag{7.7}$$

Similarly to the above, we use (7.7) and (2.6) to introduce $S = W\Sigma$, replacing the W-field. Interesting enough, it is now the dual formulation which is non-local due to the presence of the B field. We arrive again to the WZW theory for S, a principal σ model term for Σ, current-current type interactions, and principal σ model terms for S. The latter are such that the (wrong) sign of the principal σ term in $\Gamma[S]$ changes, and we arrive at the WZW model with a relative minus sign, or $\Gamma[S^{-1}]$!

However, the standard procedure to deal with the constraints is to substitute the phase space expressions in the Hamiltonian. But in such case, the constraint (6.9) does not depend upon C_-, and leads just to a connection between the right moving current of the g sector, the left moving current being untouched by such a relation! Therefore, still in the present case where we witnessed the appearance of second class constraints, their main role was to assure the positive metric requirement, as we have seen by means of the change of sign of the WZW action in the dual formulation.

8. Spectrum

Having recognized the role played by the β-action, we pass to discuss the spectrum of the theory. The first and huge step towards understanding of the model was taken by 't Hooft, who used the Bethe-Salpeter equation in the large N limit to prove that the bound states form a Rege trajectory. By adopting the light cone-gauge ($A_- = 0$) and formulating the Feynman rules in terms of light cone coordinates, the non-linear integral equation

$$\Sigma(p) = -\frac{ie^2}{2\pi^2}\int \frac{\mathrm{d}k_-}{k_-^2}\int \mathrm{d}k_+ \frac{k_- + p_-}{M^2 + (k+p)^2 + (k_- + p_-)\Sigma(k+p) - i\epsilon} \quad , \tag{8.1}$$

was obtained for the fermion self energy $\Sigma(p)$, and the $i\epsilon$-description was used in order to perform the $\mathrm{d}k_+\mathrm{d}k_-$ integrals. The infrared problems are very serious (as we readily see from the above equations). Using an infrared cut-off λ in order to restrict the k_- integration to $k_- \geq \lambda$ one can perform the integral above. Since k_- scales as a boost, the procedure turns out to be Lorentz invariant. The quark poles are pushed to infinity as $\lambda \to \infty$ displaying in a clear fashion the confining properties of the theory. The solution is $\Sigma(p) = \frac{e^2}{\pi^2}\frac{1}{p_-}$, and this leads[1] to the above mentioned Regge behavior.

However, the procedure has been subjected to some criticism. In particular, T.T. Wu[8] pointed out that the principal value prescription is ambiguos due to its non-commutative nature. If not enough, the above solution for the self energy function implies a tachyon for small bare electron mass, as explicited in

$$S_F(p) = \frac{\not{p} + M + \frac{e^2}{2\pi}\frac{\gamma_-}{p_-}}{p^2 - M^2 + \frac{e^2}{\pi}} \quad . \tag{8.2}$$

In particular, the massless theory has such a tachyon pole!

T.T. Wu performed a Wick rotation working in the Euclidian space, and after rotating back to the Minkowki space found a different result for the fermion self energy, namely

$$\Sigma(p) = \frac{1}{p_-}\left\{M^2 - p^2 - \sqrt{(M^2 - p^2)^2 + \frac{4e^2}{\pi}p^2}\right\}^{1/2} \quad . \tag{8.3}$$

The anomalous branch cut reflects the fact that all rainbow ghaphs has been tested for the Schwinger model. However, the complex light cone gauge involves a non-unitarity transformation and the relation between the results remained unclear. By all means, there are works indicating that in the axial gauge $n^\mu A_\mu = 0$, $n^\mu n_\mu = -1$, it is inconsistent to use principal value prescription[9].

Some authors have even speculated that QCD_2 may exist in two distinct phases[26]. In the large N regime (weak coupling) the gluons remain massless, since fermion loops do not contribute. Such is the 't Hooft's phase. There would exist also a Higgs phase, as in $U(1)$ gauge interaction (Schwinger model) where the gauge field acquires a mass via the well known Higgs mechanism. In this case the $SU(N)$ symmetry would be broken to the maximal abelian subgroup of $SU(N)$.

Here we do not intend to provide a definite answer to such a complex question, but some directions may be outlined from the computation we performed. Indeed, we have appropriate formulation to deal separately with both regimes: the weak coupling regime described by the β-action may be discussed perturbatively. We will see that in the large

N limit the relevant mass parameter is the one defined by 't Hooft, and we arrive at a possibility of computing the exact mass spectrum, once the complicated constraint structure is disentangled.

In order to understand the question concerning the spectrum, we first have to know which is the mass of the simplest excitation, or the mass parameter characterizing the theory. We thus consider the action

$$S[\beta] = \Gamma[\beta] + \frac{1}{2}\lambda^2 \int \mathrm{d}^2x \left[\partial_+^{-1}(\beta^{-1}\partial_+\beta)\right]^2 \quad , \tag{8.4}$$

and write a background-quantum splitting for the β-field as

$$\beta = \beta_0 \mathrm{e}^{i\xi} \quad , \tag{8.5}$$

after which we have the background-quantum splitting of the action up to second order in the quantum field ξ. However we have to be careful since in the large N the second term is the zeroth order lagrangian, from which we suppose that the ξ field acquires a mass μ^2 to be computed. The WZW term splits as

$$\Gamma[\beta] = \Gamma[\beta_0] + \frac{1}{2}\int \mathrm{d}^2x \; \beta_0^{-1}\partial_\mu\beta_0 \, \xi \overleftrightarrow{\partial_\nu} \xi \, (g^{\mu\nu} + \epsilon^{\mu\nu}) \quad . \tag{8.6}$$

Using the fact that $\Gamma[\beta]$ is at the critical point, it is not difficult to compute the $\beta_0^{-1}\partial_\mu\beta_0$ two-point function at one loop order. We have the zeroth order contribution from the second term in (8.4), and the one loop contribution, which leads to the result.

$$\beta^{-1}\partial_+\beta \frac{\lambda^2}{p_+^2}\beta^{-1}\partial_+\beta - N\frac{p_\mu p_\nu}{p^2}(g^{\mu\rho}+\epsilon^{\mu\rho})(g^{\nu\sigma}+\epsilon^{\nu\sigma})F(p)\beta^{-1}\partial_\rho\beta\beta^{-1}\partial_\sigma\beta \quad , \tag{8.7}$$

where

$$F(p) = \frac{1}{2\pi}\sqrt{\frac{p^2-4\mu^2}{p^2}}\ln\frac{\sqrt{-p^2+4\mu^2}+\sqrt{-p^2}}{\sqrt{-p^2+4\mu^2}-\sqrt{-p^2}} - \frac{1}{\pi} \quad . \tag{8.8}$$

For $p^2 = \mu^2$, we find

$$\beta_0^{-1}\partial_+\beta_0 \, \beta_0^{-1}\partial_+\beta_0 \frac{1}{p_+^2}\left(\lambda^2 - 4N\mu^2 F(\mu^2)\right) \quad . \tag{8.9}$$

The zero of the two-point function contribution to the action is at

$$\mu^2 = fe^2N = f\left(e^{'t\,Hooft}\right)^2 \quad , \tag{8.10}$$

where f is a numerical constant, in accordance with 't Hooft's results.

The fact that the second term in (8.4) has an extra factor of N arises from the fact that the fermion loops are suppressed by a factor $1/N$. Since the fermion loops contribute with a WZW functional, while the λ term stems from the gauge field self interaction (see eqs. (2.17-19)) the factors of N are correct. Moreover, it is exactly the given assignement that is compatible with the planar expansion. Finally, we have to quote the fact that the 't Hooft's analysis for the bound state $\overline{\psi}\gamma_+\psi$ leads to a Bethe-Salpeter equation compatible with the previous results, the methods following closely 't Hooft's analysis.

A more detailed information about the spectrum of the theory can be obtained from the Hamiltonian formulation. From the action

$$S = \Gamma[\beta] + \int \mathrm{d}^2x \frac{1}{2}(\partial_+ C_-)^2 + \int \mathrm{d}^2x \lambda C_- \beta^{-1}\partial_+\beta \quad , \tag{8.11}$$

we obtain the canonical momenta

$$\begin{aligned} \tilde{\hat{\Pi}}^\beta &= \frac{1}{4\pi}\partial_0\beta^{-1} + \lambda C_-\beta^{-1} \quad , \\ \Pi_- &= \partial_+ C_- \quad , \end{aligned} \tag{8.12}$$

and the Hamiltonian density

$$H = \frac{1}{2}\Pi_-(\Pi_- - 2C') - 2\pi(\tilde{\hat{\Pi}}^\beta\beta)^2 - \frac{1}{8\pi}(\beta^{-1}\beta')^2 + 4\pi\lambda\tilde{\hat{\Pi}}^\beta\beta C_- - 2\pi\lambda^2 C_-^2 - \lambda\beta^{-1}\beta' C_- \, . \tag{8.13}$$

The important currents are

$$\begin{aligned} J_+^\beta &= \beta^{-1}\partial_+\beta = -4\pi\tilde{\hat{\Pi}}^\beta\beta + \beta^{-1}\beta' + 4\pi\lambda C_- \quad , \\ j_-^\beta &= \beta\partial_-\beta^{-1} + 4\pi\lambda\beta C_-\beta^{-1} = 4\pi\beta\tilde{\hat{\Pi}}^\beta + \beta'\beta^{-1} \quad , \end{aligned} \tag{8.14}$$

in terms of which the Hamiltonian density reads (notice that j_-^β is not related to J_-^β, eqs. (3.3) and (4.14a))

$$H_\beta = -\frac{1}{16\pi}\left[\left(J_+^\beta\right)^2 + \left(j_-^\beta\right)^2\right] - \frac{1}{2}\lambda J_+^\beta C_- + \pi\lambda^2 C_-^2 + \frac{1}{2}\Pi\left(\Pi_- - 2C'\right) \quad . \tag{8.15}$$

From the previously discussed constraint structure (6.9), the current j_- is related to the free right moving fermion current $j_-^g = g\partial_- g^{-1}$, and we will drop it in the discussion of the spectrum for β. Moreover, from the Sugawara construction of the Virasoro algebra, in terms of the Kac Moody generators, we know that the Sugawara piece $H_+ = \frac{-1}{16\pi}(J_+^\beta)^2$ acquires a factor $(c_V+1)^{-1} \overset{SU(N)}{=} (N+1)^{-1}$ in the quantum theory. The C_-^2 terms are not known, since the C_- equation of motion is not easily solvable. Nevertheless, in terms of C_- and its conjugate momentum the Hamiltonian is quadratic. If we take for granted that the zero mode term is just the squared momentum, and moreover neglecting the $C_- J_+$ interaction, the Hamiltonian eigenstates have masses abeying the Regge behavior

$$m^2 \sim n\mu^2 \quad . \tag{8.16}$$

Corrections to this equation can be obtained using a large N expansion for the field C_-, a procedure which is at least possible upon considering the large N limit of (8.11).

9. Conclusions

We have achieved several aims in the present work. The first concerns the issue of obtaining a bosonized version of QCD_2. In fact, this problem has been solved long ago[2,3,5]. Here we use those well known methods in order to rephrase this problem in terms of perturbations of a set of WZW models. Therefore features concerning integrability of the original theory are rendered much clearer, and an approach based on higher symmetries might be envisaged[27,28,29].

However QCD_2 is a very complex theory. From the different results obatained by 't Hooft, on the one hand and T.T. Wu on the other hand, several authors were led to support the idea that QCD_2 presents two phases, an unbroken weak coupling limit, as described by 't Hooft, with a mesonic spectrum described by a Regge trajectory, and a Higgs phase, corresponding to the break of $SU(N)$ symmetry to the maximal abelian subgroup of $SU(N)$. There is no sign of such spontaneous breakdown for vector like theories, but this may happen in chiral gauge theories as a consequence of the vacuum polarization. In order to be able to deal with such a problem, dual formulations valid for different regimes must be available. In this direction we found the alternatives presented by the β and W fields, the first being an alternative for the week coupling limit, where we gave arguments to suport 't Hooft's proof of the Regge behavior. However the W field formulation is more involved, and we could not draw any result based on firm ground. In any case, the integrability of both formulations seems to be assured by the existence of higher conservation laws, which are in fact very similar in both cases. Nevertheless, a proof of the quantum integrability has only been possible in the β formulation. Whether this is just a missing technical detail or a sign of some new physics can not be decided but by speculation.

The integrability of the theory is one of the strongest points in this work. There have been several signs, in the literature, pointing to the possible integrability of non-abelian gauge theories in two dimensions. Gorsky and Nekrasov[30] have studied the large N Calogero-type Hamiltonian systems and found interesting relations with two-dimensional Yang Mills theory. More recently Fadeev and Korchemski[31] found that the Lipatov model[32] is described by the spin zero limit of a spin system, which in turn is integrable. Our integrability conditions eqs. (3.4-7) and (3.12) are at the core of the integrability of the model, proving it. It would be interesting to translate such condition to Colagero-type hamiltonian systems, as well as to the Lipatov model, or still to the Verlinde's high energy description of strong interactions[33].

We have to point out that gauged WZW models contain a rather non-trivial set of constraints. As pointed out by [16], although there are non-interacting subsets of fields at the lagrangian level, the BRST constraints couple them. Such coupling is essential for the maintainance of positivity of Hilbert space, due to some wrong sign of a part of the WZW actions. In the present case some combinations of the constraints are second class, and the Dirac prodedure has to be used in full detail. However, as it turns out, there is a decoupling between the non-trivial sector described by the perturbed (off-critical) WZW theory, and the constrained sector and the integrability condition turns out to fulfill the same algebra for the Dirac as well as for the Poisson algebra. It turns out that such is the

Kac-Moody algebra, and one component of the current is a realization of the Kac-Moody algebra.

Using such spliting between the off-critical J_+ current and the constraint j_- current, we can write the hamiltonian in a convenient way, and related the square momentum eigenstates to the Sugawara hamiltonian eigenstates, supporting the Regge behavior obtained by 't Hooft in the large N limit. The same method does not seem to work in the W formulation.

There is also a solution to our problem, which is compatible with the classical structure of the current algebra, mentioned after eq. (4.28), namely

$$\partial_+ J_- = 0 \quad , \tag{9.1}$$

for both J^β and J^W. If this is the case, we have to modify the algebraic structure and impose (9.1) as a constraint, which in terms of the canonical fields reads

$$\Omega_1 = -\frac{1}{\lambda(c_V+1)} B \sim 0 \tag{9.2}$$

in the W case (for the β case one has to change $B \to \Pi_-$).

Time independence of such a constraint leads to a secondary constraint

$$\Omega_2 = j_- - (c_V+1)\lambda \widetilde{\Pi}_B - \frac{1}{2\pi\lambda} B' \sim \partial_+ \partial_- J_-^W \sim 0 \quad . \tag{9.3}$$

There are no further constraints, and the Dirac Q-matrix is

$$Q_{ij;kl} = \{\Omega_{ij}, \Omega_{kl}\}^{-1} = \begin{pmatrix} (j_{-il}\delta_{kj} - j_{-kj}\delta_{il})\delta(x^1-y^1) & \delta_{il}\delta_{kj}\delta(x^1-y^1) \\ -\delta_{il}\delta_{kj}\delta(x^1-y^1) & 0 \end{pmatrix} \quad . \tag{9.4}$$

In particular, for the conservation relations involving J_-, it leads to a Kac-Moody algebra. Moreover, using the constraints we identify I_- with the current itself! Such a semi classical reasoning misses the central term. The presence of a Kac-Moody algebra in the QCD_2 would be sufficiently astonishing, and we are not able, at the moment, to forsee neither its consequences, nor even whether such a possibility can indeed be realized in the present model, or speculate whether it may be so in some conformally invariant phase. We intend to plunge deeper into the algebraic structure of the model in a subsequent publication.

Appendix

In Minkowski space,

$$x^\mu = (x^0, x^1) \quad , \quad \partial_\pm = \partial_0 \pm \partial_1 \quad , \quad x^\pm = x^0 \pm x^1 \quad . \tag{A.1}$$

In Euclidian space

$$x_\mu = (x_1, x_2)\,,\, \overline{\partial} = \partial_1 - i\partial_2 \equiv \partial_-^E\,,\, \partial = \partial_1 + i\partial_2 \equiv \partial_+^E\,,\, z = x_1 - ix_2\,,\, \overline{z} = x_1 + ix_2\,. \tag{A.2}$$

In order to translate from one space to the other, we have $x_2 = ix_0$, implying (notice the importante $(-)$ sign!)

$$\overline{\partial} \longleftrightarrow -\partial_- \quad , \quad \partial \longleftrightarrow \partial_+ \quad . \tag{A.3}$$

Notice also that

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\overline{\partial} \quad , \quad \frac{\partial}{\partial z} = \frac{1}{2}\partial \quad . \tag{A.4}$$

With these conventions,

$$\begin{aligned} F_{\mu\nu}F_{\mu\nu} &= 2F_{12}^2 = -\frac{1}{2}F_{z\bar{z}}^2 \quad , \\ F_{z\bar{z}} &= -iF_{12} + iF_{21} = -2iF_{12} \quad . \end{aligned} \tag{A.5}$$

Path integrals are allways performed in Euclidian space, while in the canonical quantization we use the Minkowski version.

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