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THE STRENGTH OF NON-PERTURBATIVE EFFECTS IN MATRIX MODELS AND STRING EFFECTIVE LAGRANGIANS

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ABSTRACT

We present a summary of the results of an explicit calculation of the strength of non-perturbative interactions in matrix models and string effective Lagrangians. These interactions are induced by single eigenvalue instantons in the $d = 1$ bosonic matrix model. A well defined approximation scheme is used to obtain induced operators whose exact form we exhibit. We briefly discuss the possibility that similar instantons in a supersymmetric version of the theory may break supersymmetry dynamically.

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Recently, it has been shown that matrix models [1] allow the construction of space-time Lagrangians valid to all orders in the string coupling parameter, at least for noncritical strings propagating in $d = 2$ dimensions. These Lagrangians are derived using the techniques of collective field theory [2, 3]. All order Lagrangians have been constructed, using these techniques, for both the $d = 1$ bosonic matrix model [4] and also for the $d = 1, \mathcal{N} = 2$ supersymmetric matrix model [5]. There are two remarkable features of these constructions. First, when interactions are included to all orders, the induced coupling blows up at finite points in space and delineates a zone of strong coupling. This is to be contrasted with the lowest order theory, where the coupling only diverges at spatial infinity. Secondly, since these all-order Lagrangians are derived from matrix models, they contain additional non-perturbative information which is directly accessible and computable. The existence of these new non-perturbative aspects of the theory relies on the observation that the matrix models contain two distinct sectors. The first of these is the so-called continuous sector, which consists of a continuous distribution of matrix eigenvalues. The second sector consists of discrete eigenvalues, which are distinguishable from the continuum eigenvalues. The classical configurations of the matrix model include time-dependent instanton solutions in which the discrete eigenvalues tunnel between two continuous eigenvalue sectors. We perform an explicit calculation of the leading order effects of such single eigenvalue instantons on the effective theory derived from a $d = 1$ bosonic matrix model. These consists of a set of induced operators, whose exact form we compute and exhibit. The results presented here are a summary of the results contained in [6]. All calculations are presented in painful details there.

We conjecture that, in the supersymmetric case, the same instantons described in this talk, and their associated bosonic and fermionic zero modes, provide a mechanism for supersymmetry breaking in the associated $d = 2$ effective superstring theory. It is presumed that the discrete nature of the single eigenvalues allows a novel circumvention of some no-go theorems, based on Witten's index, relevant to non-perturbative dynamical supersymmetry breaking in $d > 1$ dimensions. The present calculation is a necessary preliminary ingredient to the explicit calculation of this effect, which we are pursuing at these very moments and hope to report on soon [7]. Non-perturbative effects due to single eigenvalue instantons and their implications were also discussed elsewhere [8, 9, 10, 11]. Recently, an interesting complementary approach was suggested [12].

A $d = 1$ bosonic matrix model has a time-dependent $N \times N$ Hermitian matrix, $M(t)$, as its fundamental variable. Its dynamics are described by the Lagrangian

$$L(\dot{M}, M) = \frac{1}{2} Tr \dot{M}^2 - V(M). \tag{1}$$

The potential is taken to be polynomial,

$$V(M) = \sum_{n=0}^{\infty} a_n Tr M^n, \tag{2}$$

As $N \to \infty$, if the a_n are tuned simultaneously and appropriately, the associated partition function describes an ensemble of oriented two-dimensional Riemann surfaces, including contributions at all genus. It is argued that, in this limit, the model describes a string propagating in two space-time dimensions. In the large N limit, the potential may be written as

$$V(M) = Tr(NV_0 \cdot \mathbf{1} - \frac{1}{2}\omega^2 M^2), \tag{3}$$

where $\mathbf{1}$ is the $N \times N$ unit matrix. The parameters V_0 and ω each have mass dimension one, and are arbitrary. In (3) the scaling behavior of the coefficients has been made explicit. The Lagrangian, (1), is invariant under the global $U(N)$ transformation $M \to \mathcal{U}^\dagger M \mathcal{U}$, where $\mathcal{U}$ is an arbitrary $N \times N$ unitary matrix. The set of states which do not transform under $\mathcal{U}$ comprise the $U(N)$-singlet sector of the quantized theory. It can be shown that the physics of this singlet sector is described equivalently by a theory involving only the N eigenvalues, $\lambda_i(t)$, of the matrix $M(t)$ with the following Lagrangian,

$$L[\lambda] = \sum_{i=1}^{N} \{ \frac{1}{2}\dot{\lambda}_i^2 - (V_0 - \frac{1}{2}\omega^2 \lambda_i^2) - \frac{1}{2}\sum_{j \neq i} \frac{1}{(\lambda_i - \lambda_j)^2} \}. \tag{4}$$

The eigenvalues are first restricted to lie in the interval $-\frac{L}{2} \leq \lambda_i \leq \frac{L}{2}$ for any i. When we take the limit $N \to \infty$, we will simultaneously take $L \to \infty$. In this limit, over a given range, l, to be made explicit below, there exist two possibilities. If n represents the number of eigenvalues within this range, then the average density is given by $\rho = n/l$. In the limit

$N \to \infty, L \to \infty$, ρ can remain small, and the eigenvalues populate the region sparsely. We refer to this situation as a "low density" or "discrete" distribution of eigenvalues over the region l. In the second case, ρ becomes very large, and the eigenvalues populate the region densely. In this case, the eigenvalues can be aggregated into a "collective field" which describes their collective motion. We refer to this second case as a "high density" or"continuous" distribution of eigenvalues. We begin by studying the continuous case.

We introduce a continuous real parameter, x, constrained to lie in the interval $-\frac{L}{2} \leq x \leq \frac{L}{2}$, and over this line segment define a collective field,

$$\partial_x \varphi(x,t) = \sum_{i=1}^{N} \delta(x - \lambda_i(t)). \tag{5}$$

It follows from (5) that

$$\int_{x_0}^{x_0+l} dx \partial_x \varphi(x,t) = n, \tag{6}$$

where n is the number of eigenvalues in the range l. Thus, $\varphi' = \partial_x \varphi$ is the eigenvalue density. In the range l, φ' has n degrees of freedom. Provided that $n/l \to \infty$ as $N \to \infty, L \to \infty$, the average density of eigenvalues then becomes infinite, and, modulo some technical subtleties irrelevant to this discussion, the field φ becomes an unconstrained, ordinary two dimensional field. In effect, φ' ceases to be a sum over delta functions and becomes a continuous eigenvalue density. It can be shown, in this case, that the eigenvalue Lagrangian, (4), may be rewritten in terms of the collective field as follows,

$$L[\varphi] = \int dx \{ \frac{\dot{\varphi}^2}{2\varphi'} - \frac{\pi^2}{6} \varphi'^3 - (V_0 - \frac{\omega^2}{2} x^2) \varphi' \}. \tag{7}$$

The associated action is given by $S[\varphi] = \int dt L[\varphi]$. This expression describes the physics over all ranges of x where the eigenvalue density is large. The limits on the $\int dx$ integral are set accordingly. Since our interest is in the quantum theory, henceforth we will consider only the Euclidean version of the action, which is given by

$$S_E[\varphi] = \int dx dt \{ \frac{\dot{\varphi}^2}{2\varphi'} + \frac{\pi^2}{6} \varphi'^3 + (V_0 - \frac{\omega^2}{2} x^2) \varphi' \}. \tag{8}$$

The equation of motion, obtained by varying (8) is

$$\partial_t (\frac{\dot{\varphi}}{\varphi'}) - \frac{1}{2} \partial_x \left\{ \frac{\dot{\varphi}^2}{\varphi'^2} + \pi^2 \varphi'^2 - \omega^2 x^2 \right\} = 0. \tag{9}$$

The static solution is obtained by taking $\dot{\varphi} = 0$, so that (9) reduces to

$$\partial_x \left\{ \pi^2 \varphi'^2 - \omega^2 x^2 \right\} = 0. \tag{10}$$

The solution to this equation is the following,

$$\tilde{\varphi}_0'(x) = \frac{\omega}{\pi}\sqrt{x^2 - A^2}, \tag{11}$$

where A^2 is a positive constant. Additional analysis reveals that

$$A^2 = 2V_0/\omega^2. \tag{12}$$

Since φ' is now a continuous density of eigenvalues, we may use (6) to determine the approximate location of the first eigenvalues in the continuum; that is, those two eigenvalues closest to $x = \pm A$. We focus on the region $x \geq A$. There is an identical discussion regarding the opposite region, $x \leq -A$. Given (11), the first eigenvalue must live somewhere in the region $A \leq x \leq A + \epsilon_x$, where ϵ_x is determined by the following relation,

$$\begin{aligned} 1 &= \frac{\omega}{\pi}\int_A^{A+\epsilon_x} dx \sqrt{x^2 - A^2} \\ &= \frac{\omega A^2}{2\pi}\left\{ \frac{x}{A}\sqrt{(\frac{x}{A})^2 - 1} - \ln(\frac{x}{A} + \sqrt{(\frac{x}{A})^2 - 1}) \right\}\Bigg|_{x=A}^{x=A+\epsilon_x}. \end{aligned} \tag{13}$$

We make the important assumption that $\epsilon_x << A$. After some algebra, Eq.(13) then becomes

$$\frac{1}{2}(\frac{3\pi}{\omega A^2})^{2/3} = \frac{\epsilon_x}{A} + \mathcal{O}\left((\frac{\epsilon_x}{A})^2 \right). \tag{14}$$

For consistency, this requires that $(\omega A^2)^{-1} << 1$. This small dimensionless number will be central to much of the ensuing analysis, so we give it a special name,

$$g = \frac{1}{\omega A^2} << 1. \tag{15}$$

It is clear that the first eigenvalue does not live precisely at the value $x = A$. This distinction will prove a necessary and important regulator on quantities which we will encounter. For definiteness, we assume henceforth that the first eigenvalue in the static continuum has a value $x = A + \epsilon_x$, where

$$\epsilon_x = \frac{1}{2}(3\pi g)^{2/3} A \tag{16}$$

and g is a small, dimensionless number, which, in the present context, parameterizes the width of the discrete region as well as our ignorance regarding the "graininess" of eigenvalues near the edge of the continuous distribution, when we adopt a collective field point of view. We now turn our attention to the region $|x| \leq A$. We assume, in addition to a continuum of eigenvalues λ_i for $i = 1$ to N, that there exists an additional discrete eigenvalue, which we denote λ_0. There are then $N+1$ total eigenvalues, and the Euclidean version of Lagrangian (4) now reads

$$L_E = \sum_{i=0}^{N} \{ \frac{1}{2}\dot{\lambda}_i^2 + (V_0 - \frac{1}{2}\omega^2\lambda_i^2) + \frac{1}{2}\sum_{j\neq i} \frac{1}{(\lambda_i - \lambda_j)^2} \}. \tag{17}$$

Note that the index i now runs over the $N+1$ values from 0 to N. What do we mean by a discrete eigenvalue? The separation of the continuum eigenvalues nearest to $\pm A$ is of order ϵ_x. As long as $-A \leq \lambda_0 \leq A$, and

$$A - |\lambda_0| >> \epsilon_x, \tag{18}$$

the eigenvalue λ_0 is truly distinct from the continuum and, hence, discrete. Assuming that λ_0 satisfies (18), it is useful to rewrite this Lagrangian by separating the λ_0 contribution from the contribution due to the continuum eigenvalues, as follows,

$$\begin{aligned} L_E &= \frac{1}{2}\dot{\lambda}_0^2 + (V_0 - \frac{1}{2}\omega^2\lambda_0^2) + \sum_{i\neq 0} \frac{1}{(\lambda_0 - \lambda_i)^2} \\ &+ \sum_{i=1}^{N} \{ \frac{1}{2}\dot{\lambda}_i^2 + (V_0 - \frac{1}{2}\omega^2\lambda_i^2) + \frac{1}{2}\sum_{j\neq i} \frac{1}{(\lambda_i - \lambda_j)^2} \}. \end{aligned} \tag{19}$$

As above, we may now rewrite this expression using the definition (5). We thus obtain

$$\begin{aligned} L_E[\lambda_0;\varphi] &= \frac{1}{2}\dot{\lambda}_0^2 + \frac{1}{2}\omega^2(A^2 - \lambda_0^2) + \int dx \frac{\varphi'}{(x-\lambda_0)^2} \\ &+ \int dx \{ \frac{\dot{\varphi}^2}{2\varphi'} + \frac{\pi^2}{6}\varphi'^3 + \frac{1}{2}\omega^2(A^2 - x^2)\varphi' \}. \end{aligned} \tag{20}$$

The third term in this expression represents the mutual interaction of the discrete eigenvalue with the continuum eigenvalues, which are collectively described using the field φ. We obtain the Euclidean equations of motion for λ_0 and for φ by variation of (20). Respectively, these are found to be

$$\ddot{\lambda}_0 + \omega^2 \lambda_0 + \int dx \frac{\varphi'}{(\lambda_0 - x)^3} = 0 \tag{21}$$

$$\partial_t(\frac{\dot{\varphi}}{\varphi'}) - \frac{1}{2}\partial_x \left\{ \frac{\dot{\varphi}^2}{\varphi'^2} + \pi^2 \varphi'^2 - \omega^2 x^2 + \frac{2}{(\lambda_0 - x)^2} \right\} = 0. \tag{22}$$

We consider first the φ equation. It is possible to show, even in the presence of a nontrivial, but discrete, $\lambda_0(t)$, that the static background, $\tilde{\varphi}_0'$, derived above is still a valid solution to leading order in ϵ_x.

Next, we turn our attention to the λ_0 equation, (21). This is the Euclidean equation of motion,

$$\ddot{\lambda}_0 - V_{eff}'(\lambda_0) = 0, \tag{23}$$

where

$$V_{eff}(\lambda_0) = \frac{\omega}{2g} \left\{ -(\frac{\lambda_0}{A})^2 + 4g \frac{(\lambda_0/A)}{\sqrt{1 - (\lambda_0/A)^2}} \tan^{-1}(\frac{(\lambda_0/A)}{\sqrt{1 - (\lambda_0/A)^2}}) \right\}. \tag{24}$$

The effect of the second term in (24), is to turn the potential over near $\lambda_0 = \pm A$, where it adds infinite confining walls. The eigenvalue, λ_0 can be treated as discrete, and $V_{eff}(\lambda_0)$ is well defined, for λ_0 sufficiently far from $\pm A$. When λ_0 approaches $\pm A$ to within order ϵ_x it is absorbed into the continuum, and disappears as a discrete entity. Of course, this process can be reversed. It is possible for the first eigenvalue of the continuum to "leak" out and become a discrete eigenvalue λ_0. We will return to such processes below.

This being said, we would like to find both static and time-dependent solutions for the Euclidean λ_0 equation of motion (23). In the small g limit we can replace (23) by

$$\begin{aligned} \ddot{\lambda}_0 + \omega^2 \lambda_0 &= 0 \quad ; \quad -A < \lambda_0 < A \\ \ddot{\lambda}_0 &= 0 \quad ; \quad \lambda_0 = \pm A. \end{aligned} \tag{25}$$

We also impose the following boundary conditions, $\lambda_0(t \to -\infty) = \pm A$ and, independently, $\lambda_0(t \to +\infty) = \pm A$. There are two static solutions to (25) which satisfy this boundary condition,

$$\widehat{\lambda}_{0\pm} = \pm A. \tag{26}$$

A simple time-dependent solution is given by

$$\widehat{\lambda}_0^{(+)}(t;t_1) = \begin{cases} -A & ; \quad t < t_1 - \frac{\pi}{2\omega} \\ +A\sin\omega(t-t_1) & ; \quad t_1 - \frac{\pi}{2\omega} \leq t \leq t_1 + \frac{\pi}{2\omega} \\ +A & ; \quad t > t_1 + \frac{\pi}{2\omega} \end{cases} , \tag{27}$$

where t_1 is arbitrary. The solution (27) describes an eigenvalue which rolls (tunnels) from $-A$ to $+A$ over a time interval of duration $\frac{\pi}{\omega}$, centered at an arbitrary time t_1. We refer to this solution as a "kink". Its mirror image is also a valid solution,

$$\widehat{\lambda}_0^{(-)}(t;t_1) = \begin{cases} +A & ; \quad t < t_1 - \frac{\pi}{2\omega} \\ -A\sin\omega(t-t_1) & ; \quad t_1 - \frac{\pi}{2\omega} \leq t \leq t_1 + \frac{\pi}{2\omega} \\ -A & ; \quad t > t_1 + \frac{\pi}{2\omega} \end{cases} , \tag{28}$$

It describes an eigenvalue which rolls from $+A$ to $-A$. It is referred to as an "anti-kink".

Taking into account the fact that, when at $\pm A$, the discrete eigenvalue gets reabsorbed in the continuum, we may rewrite the kink and antikink solutions as follows,

$$\lambda_0^{(\pm)} = \pm A\sin\omega(t-t_1) \quad ; \; t_1 - \frac{\pi}{2\omega} \leq t \leq t_1 + \frac{\pi}{2\omega}, \tag{29}$$

There exist more general solutions than those which we have already discussed, in which the identity of λ_0 is more complex. It is possible, for example, that a kink, which ends with eigenvalue λ_0 attaching to the continuum at $+A$, could be followed, at some later time, by an antikink, in which the eigenvalue λ_0 separates from the continuum at $+A$, rolls to $-A$ and then reattaches there. Such a kink-antikink sequence, which we denote $\lambda_0^{(+-)}$, would satisfy the Euclidean equation of motion, (25). It is also possible, however, that a kink, which ends with the eigenvalue λ_0 attaching to the continuum at $+A$, could be followed, at some later time, by another kink in which a different eigenvalue detaches from the continuum at $-A$, traverses the region between $-A$ and $+A$, and then reattaches to the continuum at $+A$ immediately next to the eigenvalue involved in the first kink. This kink-kink sequence, which we denote $\lambda_0^{(++)}$, also satisfies (25). There are thus $2^2 = 4$ solutions which involve two distinct kinks,

$$\lambda_0^{(++)} = \begin{cases} +A\sin\omega(t-t_1) & ; \quad t_1 - \frac{\pi}{2\omega} \leq t \leq t_1 + \frac{\pi}{2\omega} \\ +A\sin\omega(t-t_2) & ; \quad t_2 - \frac{\pi}{2\omega} \leq t \leq t_2 + \frac{\pi}{2\omega} \end{cases}$$

$$
\begin{aligned}
\lambda_0^{(+-)} &= \begin{cases} +A\sin\omega(t-t_1) & ; \quad t_1-\frac{\pi}{2\omega}\le t\le t_1+\frac{\pi}{2\omega} \\ -A\sin\omega(t-t_2) & ; \quad t_2-\frac{\pi}{2\omega}\le t\le t_2+\frac{\pi}{2\omega} \end{cases} \\
\lambda_0^{(-+)} &= \begin{cases} -A\sin\omega(t-t_1) & ; \quad t_1-\frac{\pi}{2\omega}\le t\le t_1+\frac{\pi}{2\omega} \\ +A\sin\omega(t-t_2) & ; \quad t_2-\frac{\pi}{2\omega}\le t\le t_2+\frac{\pi}{2\omega} \end{cases} \\
\lambda_0^{(--)} &= \begin{cases} -A\sin\omega(t-t_1) & ; \quad t_1-\frac{\pi}{2\omega}\le t\le t_1+\frac{\pi}{2\omega} \\ -A\sin\omega(t-t_2) & ; \quad t_2-\frac{\pi}{2\omega}\le t\le t_2+\frac{\pi}{2\omega} \end{cases}
\end{aligned} \tag{30}
$$

In all four cases $t_2 \ge t_1 + \frac{\pi}{\omega}$, but both t_1 and t_2 are otherwise arbitrary. An arbitrary solution consists of q events which are randomly distributed between kinks and antikinks, where $0 \le q < \infty$. For a given q there are 2^q distinct instanton configurations. Generically, we denote the 2^q instantons as $\lambda_0^{(q)}$. There are q zero modes associated with each $\lambda_0^{(q)}$. These correspond to the arbitrary times $t_1, ..., t_q$, where $t_q \ge t_{q-1} \cdots \ge t_1$, when the kinks or antikinks occur. We ignore all cases where several eigenvalues are simultaneously discrete, since the effect of these solutions is negligible.

The partition function associated with the theory discussed above can be written as a sum over different instanton sectors,

$$
Z = \sum_{q=0}^{\infty} Z_q \tag{31}
$$

where, schematically,

$$
Z_q = \int [d\varphi] \int [d\lambda_o]_q e^{-S[\lambda_0;\varphi]}. \tag{32}
$$

In this expression the symbol $[d\lambda_0]_q$ indicates that λ_0 is expanded around $\lambda_0^{(q)}$. For notational convenience we have suppressed a subscript E on the action, but it is assumed throughout this section that we are in euclidean space. We proceed to define equation (32) in more precise terms. First of all, remember that $\lambda_0^{(q)}$ generically represents all the 2^q instanton solutions which each have q single eigenvalue kinks-antikinks. Therefore, more specifically,

$$
Z_q = \sum_{\{k_i\}} Z_{k_1\cdots k_q}, \tag{33}
$$

where $k_i = \pm$, the summation is over all 2^q possible sets $\{k_1 \cdots k_q\}$, and

$$
Z_{k_1\cdots k_q} = \int [d\varphi] \int [d\lambda_0]_{k_1\cdots k_q} e^{-S[\lambda_0;\varphi]}. \tag{34}
$$

The symbol $[d\lambda_0]_{k_1\cdots k_q}$ indicates that λ_0 is expanded around $\lambda_0^{(k_1\cdots k_q)}$. Thus, $Z_2 = Z_{++} + Z_{+-} + Z_{-+} + Z_{--}$, and so on. After some lengthy analysis, using a dilute gas approximation, we arrive at the following general result

$$\begin{aligned} Z &= \int [d\varphi] e^{-S_\varphi[\varphi]} \sum_{q=0}^{\infty} \frac{1}{q!} \mathcal{M}^q \prod_{i=1}^{q} \int dt_i \sum_{\{k_i\}} \prod_{j=1}^{q} e^{-S_I^{(k_j)}[\varphi;t_j]} \\ &= \int [d\varphi] e^{-S_\varphi[\varphi]} \sum_{q=0}^{\infty} \frac{1}{q!} \Big\{ \mathcal{M} \int dt_1 \Big(e^{-S_I^{(+)}[\varphi;t_1]} + e^{-S_I^{(-)}[\varphi;t_1]} \Big) \Big\}^q . \end{aligned} \tag{35}$$

The sum over q is now an exponential, so that

$$Z = \int [d\varphi] e^{-S_{eff}[\varphi]}, \tag{36}$$

where

$$S_{eff}[\varphi] = S_\varphi[\varphi] + \Delta S[\varphi] \tag{37}$$

is the effective action with the instanton effects systematically incorporated, and

$$\Delta S[\varphi] = \mathcal{M} \int dt_1 \Big\{ e^{-S_I^{(+)}[\varphi;t_1]} + e^{-S_I^{(-)}[\varphi;t_1]} \Big\} \tag{38}$$

is the associated change in the action. The action $S_I^{(\pm)}$ is given by

$$S_I^{(\pm)}[\varphi; t_j] = \int_{t_j - \frac{\pi}{2\omega}}^{t_j + \frac{\pi}{2\omega}} dt \int dx \Big\{ \frac{\varphi'(x,t)}{(x - \lambda_0^{(\pm)}(t - t_j))^2} - \frac{\varphi'(x,t)}{(x - \lambda_\emptyset^{(\pm)}(t - t_j))^2} \Big\}. \tag{39}$$

where

$$\lambda_\emptyset^{(\pm)}(t; t_1) = \begin{cases} \mp A \quad ; \quad t_1 - \frac{\pi}{2\omega} \le t < t_1 \\ \pm A \quad ; \quad t_1 < t \le t_1 + \frac{\pi}{2\omega} \end{cases} . \tag{40}$$

The quantity $\mathcal{M}$ is a dimensionful parameter that sets the basic strength for induced non-perturbative interactions

$$\mathcal{M} = \omega \sqrt{\frac{\pi}{2g}} e^{-\frac{\pi}{2g}}. \tag{41}$$

So far, we have studied the collective field theory expressed in terms of the field φ. By examining equation (8), however, we discover that φ does not have a canonically normalized kinetic energy. We also find that the collective field Lagrangian is neither Lorentz invariant nor translation invariant. The first of these problems is solved, in part, by expanding φ around the solution to the euclidean field equation $\tilde{\varphi}_0$ given in (11). Thus, we define

$$\varphi(x,t) = \tilde{\varphi}_0(x) + \frac{1}{\sqrt{\pi}}\zeta(x,t). \tag{42}$$

As discussed at length elsewhere, a canonical kinetic energy is obtained by expressing the Lagrangian in terms of a new spatial coordinate τ defined by the following relation,

$$\tau'(x) = \frac{1}{\pi}(\tilde{\varphi}_0'(x))^{-1}. \tag{43}$$

Note that τ has mass dimension -1, which is the appropriate mass dimension for a spatial coordinate, whereas x has mass dimension $-\frac{1}{2}$. Expressing the euclidean collective field action (8) in terms of $\zeta(\tau,t)$, we find, in the absence of instanton effects, that

$$S_\zeta[\zeta] = \int dt \int d\tau \left\{ \frac{1}{2}(\dot{\zeta}^2 + \zeta'^2) - \frac{1}{2}\frac{\mathcal{g}(\tau)\dot{\zeta}^2\zeta'}{1+\mathcal{g}(\tau)\zeta'} + \frac{1}{6}\mathcal{g}(\tau)\zeta'^3 - \frac{1}{3}\frac{1}{\mathcal{g}(\tau)^2} \right\}, \tag{44}$$

where $\mathcal{g}(\tau)$ is a space dependent coupling parameter, which we define below, and the τ integration is over the limits $-\infty < \tau \leq \tau_0 + \frac{\sigma}{2}$ and $\tau_0 + \frac{\sigma}{2} \leq \tau < \infty$, where τ_0 and σ are independent integration constants which arise when solving (43). The reason why there are two integration constants rather than one, given that (43) is a first-order differential equation, is that we must solve (43) independently over the two separate regions $-\infty < x \leq A$ and $A \leq x < \infty$. The region $-A < x < A$, where there is no continuous collective field theory, is the low density region. In τ space, this region is given by $\tau_0 - \frac{\sigma}{2} < \tau < \tau_0 + \frac{\sigma}{2}$, so that τ_0 is the center of the low density region and σ is the width. The coupling parameter, defined over $-\infty < \tau \leq \tau_0 - \frac{\sigma}{2}$ and $\tau_0 + \frac{\sigma}{2} \leq \tau < \infty$, is given by $\mathcal{g}(\tau) = (\pi^{3/2}\tilde{\varphi}_0(x))^{-1}$, and is found to be

$$\mathcal{g}(\tau) = 4\sqrt{\pi}\frac{g}{\omega}\frac{\frac{1}{\kappa}e^{-2\omega|\tau-\tau_0|}}{(1-\frac{1}{\kappa}e^{-2\omega|\tau-\tau_0|})^2}, \tag{45}$$

where κ is a dimensionless number,

$$\kappa = \exp(-\omega\sigma), \tag{46}$$

which relates the width, σ, of the low density region in τ space to the natural length scale in the matrix model, $1/\omega$. Notice that the coupling parameter blows up as $\tau \to \tau_0 \pm \frac{\sigma}{2}$; that is, at the boundaries of the low density region.

We would now like to express the change in the effective action due to the instanton effects, equation (38), in terms of the canonical variable $\zeta(\tau,t)$. Since $S_I^{(\pm)}$ is linear in φ, it follows that

$$S_I^{(\pm)}[\varphi; t_1] = S_I^{(\pm)}[\tilde{\varphi}_0] + \frac{1}{\sqrt{\pi}} S_I^{(\pm)}[\zeta; \tau_0, t_1]. \tag{47}$$

The τ_0 dependence in the last term of this equation will be made clear presently. From (39), we find

$$S_I^{(\pm)}[\zeta; \tau_0, t_1] = \int_{t_1-\frac{\pi}{2\omega}}^{t_1+\frac{\pi}{2\omega}} dt \int d\tau \left\{ \frac{\zeta'(\tau,t)}{(x(\tau) - \lambda_0^{(\pm)}(t-t_1))^2} - \frac{\zeta'(\tau,t)}{(x(\tau) - \lambda_\emptyset^{(\pm)}(t-t_1))^2} \right\}, \tag{48}$$

where the prime now means differentiation with respect to τ, and where

$$x(\tau) = \begin{cases} -A\cosh\{\omega(\tau - \tau_0 + \sigma/2)\} & ; \tau \le \tau_0 - \sigma/2 \\ +A\cosh\{\omega(\tau - \tau_0 - \sigma/2)\} & ; \tau \ge \tau_0 - \sigma/2 \end{cases} . \tag{49}$$

This last expression is found by integrating (43) to obtain $\tau(x)$ and then inverting the result to obtain $x(\tau)$. This function depends explicitly on τ_0. This explains why there is an explicit τ_0 in equations (47) and (48). It is straightforward to compute $S_I^{(\pm)}[\tilde{\varphi}_0]$ and we find

$$S_I^{(\pm)}[\tilde{\varphi}_0] = -2^{3/2}\sqrt{\frac{A}{\epsilon_x}} + \ln\sqrt{\frac{A}{\epsilon_x}} + \mathcal{O}(\frac{\epsilon_x}{A}). \tag{50}$$

As discussed above, ϵ_x is the size of the inter-eigenvalue separation near the edge of the continuum and so provides the natural regulator for expressions such as (50). From (16) it follows that, to lowest order in g

$$e^{-S_I^{(\pm)}[\tilde{\varphi}_0]} = g^{1/3} e^{\mathcal{O}(g^{1/3})}. \tag{51}$$

Since all x-space integrations are cut-off at a distance ϵ_x from the edge of the low density region; that is, at $|x| = A + \epsilon_x$, it follows that all τ space integrals must be cut-off as well at a value ϵ_τ. Specifically, in (48) and in all other expressions which include a $\int d\tau$ integration, the following is implied,

$$\int d\tau = \int_{-\infty}^{\tau_0 - \frac{\sigma}{2} - \epsilon_\tau} d\tau + \int_{\tau_0 + \frac{\sigma}{2} + \epsilon_\tau}^{\infty} d\tau. \tag{52}$$

The value of ϵ_τ is simple to obtain. We require that

$$\begin{aligned} x(\tau - \frac{\sigma}{2} - \epsilon_\tau) &= -A - \epsilon_x \\ x(\tau + \frac{\sigma}{2} + \epsilon_\tau) &= A + \epsilon_x. \end{aligned} \tag{53}$$

Using (49) and (16) it follows, to leading order in g, that

$$\epsilon_\tau = \frac{1}{\omega\sqrt{2}}(3\pi g)^{1/3}. \tag{54}$$

Now, using (51), substituting (47) into (38), and using (41), we find that

$$\Delta S[\zeta] = \omega g^{-1/6} e^{-\frac{\pi}{2g}} \int dt_1 \Big\{ e^{-S_I^{(+)}[\zeta;\tau_0,t_1]} + e^{-S_I^{(-)}[\zeta;\tau_0,t_1]} \Big\}. \tag{55}$$

Equation (55) is a significant result. Concisely, it is the induced change in the canonical collective field theory which results from the systematic inclusion of instanton effects. A lengthy analysis allows us to calculate from Eq.(55) the induced action as an integral over a local density. Skipping a lot of details we simply state the results

$$\begin{aligned} S_I^{(+)} &= \frac{1}{\omega} h_{00}(\zeta'_- + \zeta'_+) + \frac{1}{\omega^2} h_{01}(\zeta''_- - \zeta''_+) + \frac{1}{\omega^2} h_{10}(\dot{\zeta}'_- - \dot{\zeta}'_+) + \frac{1}{\omega^3} h_{11}(\dot{\zeta}''_- + \dot{\zeta}''_+) + \cdots \\ S_I^{(-)} &= \frac{1}{\omega} h_{00}(\zeta'_- + \zeta'_+) - \frac{1}{\omega^2} h_{01}(\zeta''_- - \zeta''_+) - \frac{1}{\omega^2} h_{10}(\dot{\zeta}'_- - \dot{\zeta}'_+) + \frac{1}{\omega^3} h_{11}(\dot{\zeta}''_- + \dot{\zeta}''_+) + \cdots . \end{aligned} \tag{56}$$

where

$$h_{mn} = \frac{\omega^{m+n+1}}{m!n!} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dt \int_{-\infty}^{-\epsilon_\tau} d\tau \mathcal{J}(\tau, t) \tau^m t^n, \tag{57}$$

$$\zeta_\pm \equiv \zeta(\tau_0 \pm \frac{\sigma}{2}, t_1) \tag{58}$$

and

$$\mathcal{J}(\tau - \tau_0 + \frac{\sigma}{2}, t - t_1) = \frac{1}{(x(\tau - \tau_0 + \frac{\sigma}{2}) - \lambda_0^{(\pm)}(t - t_1))^2} - \frac{1}{(x(\tau - \tau_0 + \frac{\sigma}{2}) - \lambda_\emptyset^{(\pm)}(t - t_1))^2} \tag{59}$$

It is straightforward to compute the coefficients h_{mn}. We find, for instance, to leading order in g, that

$$\begin{aligned} h_{00} &= -\frac{4\sqrt{2}}{9} \\ h_{10} &= -(\frac{8\pi g}{9})^{1/3} \\ h_{01} &= -\frac{\pi\sqrt{2}}{9}. \end{aligned} \tag{60}$$

In general, the h_{mn} are found to have the following g dependence,

$$h_{mn} \sim \begin{cases} g^{m/3} & ; \quad m \leq 3 \\ g & ; \quad m > 3 \end{cases} \tag{61}$$

Note, from (56) and (61), that, as the first index of h_{mn} increases, that the corresponding terms in $S_I^{(\pm)}$ depend on higher powers of g. However, none of h_{0n} have g dependence for any value of n. We proceed to analyze the relative impact of these terms on generic N-point functions. By putting (56) back into (55) we can find all relevant interaction vertices. These are obtained by Taylor expanding the exponentials in (55). For instance, we obtain the quadratic vertices $\frac{1}{\omega^2} h_{00}^2 \zeta_-' \zeta_-'$ and $\frac{1}{\omega^3} h_{00} h_{10} \zeta_-' \zeta_-''$ where, as discussed above, $h_{00} \sim 1$ and $h_{10} \sim g^{1/3}$. It is clear that the effect of the second vertex, containing $h_{00}h_{10}$, on any N-point function, is suppressed by a factor $g^{1/3}p/w$, where p is a characteristic momentum, when compared with effects arising solely from the first vertex containing h_{00}^2. This is true at tree level. At the quantum level, there may be some subtleties to this argument which we will not discuss. Similar considerations apply to all other induced operators, involving higher h_{mn}. It can thus be shown, provided

$$p \lesssim \omega, \tag{62}$$

that, when working to leading order in g, we can consistently drop all but the h_{0n} terms in (56). Now, of the terms which remain, as n increases, the corresponding terms in $S_I^{(\pm)}$ depend on higher derivatives of ζ. Thus, the effect of any vertex, containing h_{0n}, on any N-point function, is suppressed by a factor $(p/\omega)^n$, relative to effects arising from vertices containing only h_{00}. If we further restrict momenta, such that

$$p << \omega, \tag{63}$$

we can then consistently neglect all but the h_{00} terms in (56). This results in a vast simplification of the final result, so we will assume this approximation. It would be completely straightforward, however, to lift the restriction (63), and only require (62). One would then have to keep all h_{0n} terms in (56). It follows from (56), that, to leading order in g,

$$\Delta S[\zeta] = 2\omega g^{-1/6} e^{-\frac{\pi}{2g}} \int dt_1 \exp\left\{ \frac{4\sqrt{2}}{3\omega} \left(\zeta'(\tau_0 + \frac{\sigma}{2}, t_1) + \zeta'(\tau_0 - \frac{\sigma}{2}, t_1) \right) \right\}. \tag{64}$$

Note however that equation (64) includes nonlocal interactions, since it involves contributions coming from ζ' evaluated simultaneously at $\tau_0 - \frac{\sigma}{2}$ and also at $\tau_0 + \frac{\sigma}{2}$. This is not surprising though, since we have arrived at this result by integrating over single eigenvalue instantons, which link effects on the left-hand side of the low-density region with effects on the the right-hand side of this region, and because there is a finite separation between these two sectors. One may wish to find some further approximation which would render the effective theory local. This can be done as follows. Provided we consider momenta which satisfy (63), and provided also that $\omega \lesssim \frac{1}{\sigma}$, the effective width of the low density region as seen by any field will be essentially zero. We therefore Taylor expand $\zeta'(\tau_0 \pm \frac{\sigma}{2}, t_1)$ around the point (τ_0, t_1), thereby taking

$$\frac{1}{\omega}\zeta'(\tau_0 \pm \frac{\sigma}{2}, t_1) = \frac{1}{\omega}\zeta'(\tau_0, t_1) \pm \frac{\sigma\omega}{2\omega}\zeta''(\tau_0, t_1) + \cdots. \tag{65}$$

Then, in a manner identical to the previous discussion, we find that the contributions coming from vertices which involve σ are always suppressed by $(\sigma\omega)p/\omega$, where p is a characteristic momentum. Note that, since we now assume $\omega \lesssim \frac{1}{\sigma}$, the factor $(\sigma\omega)$ is $\lesssim \mathcal{O}(1)$. So, provided that

$$p << \omega \lesssim \frac{1}{\sigma}, \tag{66}$$

we may write the lowest order instanton-induced change in the collective field action approximately, in local form, as follows,

$$\Delta S[\zeta] = 2\omega g^{-1/6} e^{-\frac{\pi}{2g}} \int dt e^{-\frac{2\sqrt{2}}{3\omega}\zeta'(\tau_0, t)}. \tag{67}$$

We have dropped the subscript "1" on t_1 because it is now superfluous. This result can be written as a two-dimensional integral over a density $\Delta S = \int dt d\tau \Delta\mathcal{L}$, where

$$\Delta\mathcal{L} = 2\omega g^{-1/6} e^{-\frac{\pi}{2g}} \delta(\tau - \tau_0) e^{-\frac{2\sqrt{2}}{3\omega}\zeta'(\tau, t)}. \tag{68}$$

This is the final result of our calculation.

References

[1] D. J. Gross and N. Miljkovic, *Phys.Lett.* **B238** (1990) 217;
P. Ginsparg and J. Zinn-Justin, *Phys. Lett.* **B240** (1990) 333 ;
E. Brezin, V. Kazakov, Al. Zamolodchikov, *Nucl. Phys.* **B338** (1990) 673.

[2] S. R. Das and A. Jevicki, *Mod. Phys. Lett.***A5** (1990) 1639.

[3] A. Jevicki, Brown preprint, BROWN-HET-918 (1993).

[4] R. Brustein and B. Ovrut, *Phys. Lett.* **B309** (1993) 45;
R. Brustein and B. Ovrut, preprint, UPR-523T (1992);
R. Brustein and B. Ovrut, Talk presented at 26th International Conference on High Energy Physics (ICHEP 92), Dallas, TX, 6-12 Aug 1992.

[5] R. Brustein, M. Faux and B. Ovrut, CERN preprint, CERN-TH.7013/93, Talk presented at the international Europhysics Conference on High Energy Physics, Marseille, France, July 22-28, 1993;
R. Brustein, M. Faux and B. Ovrut, CERN preprint, CERN-TH.7017/93, to appear in it Nucl. Phys. **B** (1994);
R. Brustein, M. Faux and B. Ovrut, CERN preprint, CERN-TH.7051/93, Talk presented at the International Workshop on Supersymmetry and Unification of Fundamental Interactions (SUSY 93), Boston, MA, Mar. 29 - Apr. 1, 1993.

[6] R. Brustein, M. Faux and B. Ovrut, CERN/Pennsylvania preprint,
CERN-TH.7301/94 / UPR-608T (1994).

[7] R. Brustein, M. Faux and B. Ovrut, Work in progress.

[8] S. H. Shenker, talk presented at the Cargese Workshop on Random Surfaces, Quantum Gravity and Strings, Cargese, France, May 28 - Jun 1, 1990.

[9] A. Dabholkar, *Nucl. Phys.* **B368** (1992) 293.

[10] J. Lee and P. Mende, *Phys. Lett.* **B312** (1993) 433.

[11] A. Dahr, G. Mandal and S. R. Wadia, *Int. J. Mod. Phys.* **A8** (1993) 3811.

[12] J. Polchinski, Santa Barbara preprint, NSF-ITP-94-73 (1994).