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INSTANTON EFFECTS IN MATRIX MODELS AND STRING EFFECTIVE LAGRANGIANS

RAM BRUSTEIN

Theory Division, CERN

CH-1211 Geneva 23, Switzerland

MICHAEL FAUX^{*)} and BURT A. OVRUT^{*)}

Department of Physics

University of Pennsylvania

Philadelphia, Pa 19104-6396

ABSTRACT

We perform an explicit calculation of the lowest order effects of single eigenvalue instantons on the continuous sector of the collective field theory derived from a $d = 1$ bosonic matrix model. These effects consist of certain induced operators whose exact form we exhibit.

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1. Introduction

Recently, it has been shown that matrix models [1] allow the construction of space-time Lagrangians valid to all orders in the string coupling parameter, at least for noncritical strings propagating in $d = 2$ dimensions. These Lagrangians are derived using the techniques of collective field theory [2, 3]. All order Lagrangians have been constructed, using these techniques, for both the $d = 1$ bosonic matrix model [4] and also for the $d = 1, \mathcal{N} = 2$ supersymmetric matrix model [5]. There are two remarkable features of these constructions. First, when interactions are included to all orders, the induced coupling blows up at finite points in space and delineates a zone of strong coupling. This is to be contrasted with the lowest order theory, where the coupling only diverges at spatial infinity. Secondly, since these all-order Lagrangians are derived from matrix models, they contain additional non-perturbative information which is directly accessible and computable. The existence of these new non-perturbative aspects of the theory relies on the observation that the matrix models contain two distinct sectors. The first of these is the so-called continuous sector, which consists of a continuous distribution of matrix eigenvalues. The second sector consists of discrete eigenvalues, which are distinguishable from the continuum eigenvalues. The classical configurations of the matrix model include time-dependent instanton solutions in which the discrete eigenvalues tunnel between two continuous eigenvalue sectors. In this paper we perform an explicit calculation of the leading order effects of such single eigenvalue instantons on the effective theory derived from a $d = 1$ bosonic matrix model. These consists of a set of induced operators, whose exact form we compute and exhibit.

This work is particularly relevant for the following reason. It is conjectured that, in the supersymmetric case, the same instantons described in this paper, and their associated bosonic and fermionic zero modes, provide a mechanism for supersymmetry breaking in the associated $d = 2$ effective superstring theory. It is presumed that the discrete nature of the single eigenvalues allows a novel circumvention of a particular no-go theorem, based on Witten's index, relevant to non-perturbative supersymmetry breaking in $d > 1$ dimensions. The calculation in this paper is a necessary preliminary to the explicit calculation of this effect, which will be discussed in a forthcoming paper [6]. Non-perturbative effects due to

single eigenvalue instantons were also discussed elsewhere [7, 8, 9]

This paper is structured as follows.

In section 2 we describe the distinct sectors of the matrix model in some detail. We compute the equation of motion for both the continuous sector and for the discrete sector and we analyse the mutual interaction between these two. We then compute and exhibit the complete set of single eigenvalue instanton solutions valid to lowest order in a small coupling constant.

In section 3, we integrate out the single eigenvalue instantons in a dilute-gas approximation. This then gives rise to a collective field theory which has the instanton effects incorporated.

2. Bosonic Matrix Models

A $d = 1$ bosonic matrix model has a time-dependent $N \times N$ Hermitian matrix, $M(t)$, as its fundamental variable. Its dynamics are described by the Lagrangian

$$L(\dot{M}, M) = \frac{1}{2} \text{Tr} \dot{M}^2 - V(M). \quad (2.1)$$

The potential is taken to be polynomial,

$$V(M) = \sum_{n=0}^{\infty} a_n \text{Tr} M^n, \quad (2.2)$$

where the a_n are real coupling parameters. The mass dimension of M is $\frac{1}{2}$ and the a_n have positive mass dimension $(n + 2)/2$. As $N \rightarrow \infty$, if the a_n are tuned simultaneously and appropriately, the associated partition function describes an ensemble of oriented two-dimensional Riemann surfaces, including contributions at all genus. It is argued that, in this limit, the model describes a string propagating in two space-time dimensions. For this to be so, it is necessary that the a_n scale as $N^{1-n/2}$ for large N . We will, henceforth, assume that the coupling parameters scale in this manner. It follows that, in the large N limit, all terms in (2.2) with $n \geq 3$ become negligibly small. Furthermore, the $n = 1$ term can be shifted away and the remaining terms in the potential written as

$$V(M) = \text{Tr}(NV_0 \cdot 1 - \frac{1}{2}\omega^2 M^2), \quad (2.3)$$

where 1 is the $N \times N$ unit matrix. The parameters V_0 and ω each have mass dimension one, and are arbitrary. In (2.3) the scaling behavior of the coefficients has been made explicit. The Lagrangian, (2.1), is invariant under the global $U(N)$ transformation $M \rightarrow \mathcal{U}^\dagger M \mathcal{U}$, where \mathcal{U} is an arbitrary $N \times N$ unitary matrix. The set of states which do not transform under \mathcal{U} comprise the $U(N)$ -singlet sector of the quantized theory. It can be shown that the physics of this singlet sector is described equivalently by a theory involving only the N eigenvalues, $\lambda_i(t)$, of the matrix $M(t)$ with the following Lagrangian,

$$L[\lambda] = \sum_{i=1}^N \left\{ \frac{1}{2} \dot{\lambda}_i^2 - (V_0 - \frac{1}{2} \omega^2 \lambda_i^2) - \frac{1}{2} \sum_{j \neq i} \frac{1}{(\lambda_i - \lambda_j)^2} \right\}. \quad (2.4)$$

The eigenvalues are first restricted to lie in the interval $-\frac{L}{2} \leq \lambda_i \leq \frac{L}{2}$ for any i . When we take the limit $N \rightarrow \infty$, we will simultaneously take $L \rightarrow \infty$. In this limit, over a given range, l , to be made explicit below, there exist two possibilities. If n represents the number of eigenvalues within this range, then the average density is given by $\rho = n/l$. In the limit $N \rightarrow \infty, L \rightarrow \infty$, ρ can remain small, and the eigenvalues populate the region sparsely. We refer to this situation as a “low density” or “discrete” distribution of eigenvalues over the region l . In the second case, ρ becomes very large or infinite, and the eigenvalues populate the region densely. In this case, the eigenvalues can be aggregated into a “collective field” which describes their physics en-masse. We refer to this second case as a “high density” or “continuous” distribution of eigenvalues. We begin by studying the continuous case.

2.1 Collective Field Theory

As yet, N and L remain finite. We introduce a continuous real parameter, x , constrained to lie in the interval $-\frac{L}{2} \leq x \leq \frac{L}{2}$, and over this line segment define a collective field,

$$\partial_x \varphi(x, t) = \sum_{i=1}^N \delta(x - \lambda_i(t)). \quad (2.5)$$

Since the λ_i have mass dimension $-\frac{1}{2}$, the parameter x also has mass dimension $-\frac{1}{2}$. The delta function has the inverse dimensionality of its argument, which is $+\frac{1}{2}$. Thus, since ∂_x is also a dimension $+\frac{1}{2}$ operator, it follows that $\varphi(x, t)$ is dimensionless. It follows from (2.5)

that

$$\int_{x_0}^{x_0+l} dx \partial_x \varphi(x, t) = n, \quad (2.6)$$

where n is the number of eigenvalues in the range l . Thus, $\varphi' = \partial_x \varphi$ is the eigenvalue density. In the range l , φ' has n degrees of freedom. Provided that $n/l \rightarrow \infty$ as $N \rightarrow \infty, L \rightarrow \infty$, the average density of eigenvalues then becomes infinite, and, modulo some technical subtleties irrelevant to this discussion, the field φ becomes an unconstrained, ordinary two dimensional field. In effect, φ' ceases to be a sum over delta functions and becomes a continuous eigenvalue density. It can be shown, in this case, that the eigenvalue Lagrangian, (2.4), may be rewritten in terms of the collective field as follows,

$$L[\varphi] = \int dx \left\{ \frac{\dot{\varphi}^2}{2\varphi'} - \frac{\pi^2}{6} \varphi'^3 - \left(V_0 - \frac{\omega^2}{2} x^2 \right) \varphi' \right\}. \quad (2.7)$$

The associated action is given by $S[\varphi] = \int dt L[\varphi]$. This expression describes the physics over all ranges of x where the eigenvalue density is large. The limits on the $\int dx$ integral are set accordingly. In this subsection, we restrict our attention to such a continuous sector. In the next subsection we will discuss the incorporation of a discrete sector into the theory. Since our interest is in the quantum theory, henceforth we will consider only the Euclidean version of the action, which is given by

$$S_E[\varphi] = \int dx dt \left\{ \frac{\dot{\varphi}^2}{2\varphi'} + \frac{\pi^2}{6} \varphi'^3 + \left(V_0 - \frac{\omega^2}{2} x^2 \right) \varphi' \right\}. \quad (2.8)$$

The equation of motion, obtained by varying (2.8) is

$$\partial_t \left(\frac{\dot{\varphi}}{\varphi'} \right) - \frac{1}{2} \partial_x \left\{ \frac{\dot{\varphi}^2}{\varphi'^2} + \pi^2 \varphi'^2 - \omega^2 x^2 \right\} = 0. \quad (2.9)$$

The static solution is obtained by taking $\dot{\varphi} = 0$, so that (2.9) reduces to

$$\partial_x \left\{ \pi^2 \varphi'^2 - \omega^2 x^2 \right\} = 0. \quad (2.10)$$

The most general solution to this equation is the following,

$$\hat{\varphi}'_0(x) = \frac{\omega}{\pi} \sqrt{x^2 - A^2}, \quad (2.11)$$

where A^2 is a constant which can be negative, zero, or positive. Since (2.8) only involves derivative couplings, however, the equation of motion, (2.10), is not sufficient to extremize

the action. This is because the action depends on the value of A^2 , which is undetermined by (2.10). In order to determine A^2 we must compute the action using (2.11) and find the value which represents the true extremum. Inserting (2.11) into (2.8), we find, for finite L , over a finite Euclidean time duration, $-\frac{T}{2} \leq t \leq \frac{T}{2}$, that

$$S_E[\tilde{\varphi}_0] = -\frac{\omega^3 T}{96\pi} \left\{ (L^2 + 2A^2 - 24\frac{V_0}{\omega^2})L\sqrt{L^2 - 4A^2} - 24A^2(A^2 - 4\frac{V_0}{\omega^2}) \ln\left(\frac{L + \sqrt{L^2 - 4A^2}}{2|A|}\right) \right\}. \quad (2.12)$$

This function is minimized, for all values of L, T, V_0 , and ω when

$$A^2 = 2V_0/\omega^2. \quad (2.13)$$

This determines A^2 in terms of the two parameters, V_0 and ω , but its sign remains undetermined. For convenience we will write V_0 as $\frac{1}{2}\omega^2 A^2$. In Figure 1, we plot (2.11) for the three cases $A^2 < 0, A^2 = 0$, and $A^2 > 0$. Superimposed on this plot is the “potential”, $V = \frac{1}{2}\omega^2(A^2 - x^2)$ which multiplies the linear φ' term in (2.7).

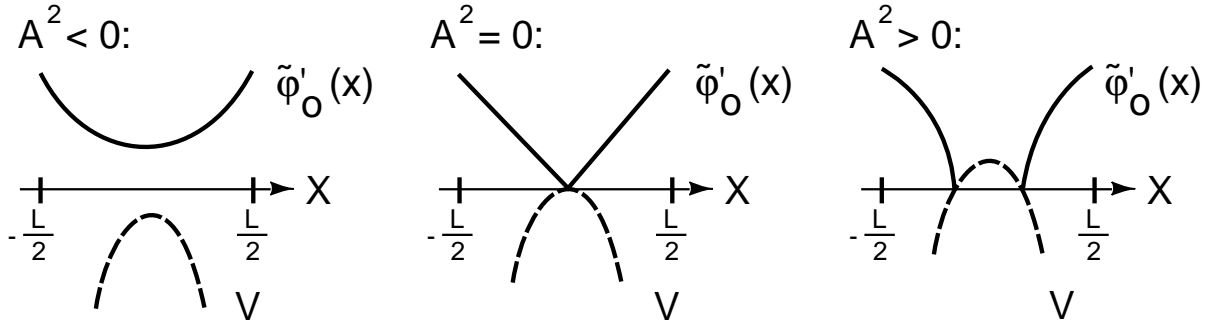


Figure 1. The potential and classical solution for different values of A^2 .

For the cases $A^2 \leq 0$, the eigenvalues continuously populate all values. That is, the range l over which there is a continuous distribution of eigenvalues is given by $-\frac{L}{2} \leq x \leq \frac{L}{2}$. The case $A^2 > 0$, however, leaves a region, $|x| < A$, which is not continuously occupied by eigenvalues, where a discrete sector may be accommodated. In this case the range l over which there is a continuous distribution of eigenvalues is given by $-\frac{L}{2} \leq x \leq -A$ and $A \leq x \leq \frac{L}{2}$. Our interest in this paper is to develop a technique for systematically incorporating discrete

eigenvalue dynamics into the collective field theory. We therefore restrict attention, for the remainder of this paper, to the case $A^2 > 0$.

Since φ' is now a continuous density of eigenvalues, we may use (2.6) to determine the approximate location of the first eigenvalues in the continuum; that is, those two eigenvalues closest to $x = \pm A$. We focus on the region $x \geq A$. There is an identical discussion regarding the opposite region, $x \leq -A$. Given (2.11), the first eigenvalue must live somewhere in the region $A \leq x \leq A + \epsilon_x$, where ϵ_x is determined by the following relation,

$$\begin{aligned} 1 &= \frac{\omega}{\pi} \int_A^{A+\epsilon_x} dx \sqrt{x^2 - A^2} \\ &= \frac{\omega A^2}{2\pi} \left\{ \frac{x}{A} \sqrt{\left(\frac{x}{A}\right)^2 - 1} - \ln\left(\frac{x}{A} + \sqrt{\left(\frac{x}{A}\right)^2 - 1}\right) \right\} \Bigg|_{x=A}^{x=A+\epsilon_x}. \end{aligned} \quad (2.14)$$

We make the important assumption that $\epsilon_x \ll A$. After some algebra, Eq.(2.14) then becomes

$$\frac{1}{2} \left(\frac{3\pi}{\omega A^2} \right)^{2/3} = \frac{\epsilon_x}{A} + \mathcal{O}\left(\left(\frac{\epsilon_x}{A}\right)^2\right). \quad (2.15)$$

For consistency, this requires that $(\omega A^2)^{-1} \ll 1$. This small dimensionless number will be central to much of the ensuing analysis, so we give it a special name,

$$g = \frac{1}{\omega A^2} \ll 1. \quad (2.16)$$

Since φ'_0 increases monotonically as x becomes larger than A , it is reasonable to assume that the first eigenvalue actually has a value nearer to $x = A + \epsilon_x$ rather than nearer to $x = A$. At any rate, it is clear that the first eigenvalue does not live precisely at the value $x = A$. This distinction will prove a necessary and important regulator on quantities which we will encounter. For definiteness, we assume henceforth that the first eigenvalue in the static continuum has a value $x = A + \epsilon_x$, where

$$\epsilon_x = \frac{1}{2} (3\pi g)^{2/3} A \quad (2.17)$$

and g is a small, dimensionless number, which, in the present context, parameterizes the width of the discrete region as well as our ignorance regarding the ‘‘graininess’’ of eigenvalues near the edge of the continuous distribution, when we adopt a collective field point of view.

2.2 Discrete Eigenvalue Dynamics

We now turn our attention to the region $|x| \leq A$. We assume, in addition to a continuum of eigenvalues λ_i for $i = 1$ to N , that there exists an additional discrete eigenvalue, which we denote λ_0 . There are then $N + 1$ total eigenvalues, and the Euclidean version of Lagrangian (2.4) now reads

$$L_E = \sum_{i=0}^N \left\{ \frac{1}{2} \dot{\lambda}_i^2 + \left(V_0 - \frac{1}{2} \omega^2 \lambda_i^2 \right) + \frac{1}{2} \sum_{j \neq i} \frac{1}{(\lambda_i - \lambda_j)^2} \right\}. \quad (2.18)$$

Note that the index i now runs over the $N + 1$ values from 0 to N . What do we mean by a discrete eigenvalue? It was shown in the previous section that the separation of the continuum eigenvalues nearest to $\pm A$ is of order ϵ_x . As long as $-A \leq \lambda_0 \leq A$, and

$$A - |\lambda_0| \gg \epsilon_x, \quad (2.19)$$

the eigenvalue λ_0 is truly distinct from the continuum and, hence, discrete. Assuming that λ_0 satisfies (2.19), it is useful to rewrite this Lagrangian by separating the λ_0 contribution from the contribution due to the continuum eigenvalues, as follows,

$$\begin{aligned} L_E &= \frac{1}{2} \dot{\lambda}_0^2 + \left(V_0 - \frac{1}{2} \omega^2 \lambda_0^2 \right) + \sum_{i \neq 0} \frac{1}{(\lambda_0 - \lambda_i)^2} \\ &+ \sum_{i=1}^N \left\{ \frac{1}{2} \dot{\lambda}_i^2 + \left(V_0 - \frac{1}{2} \omega^2 \lambda_i^2 \right) + \frac{1}{2} \sum_{j \neq i} \frac{1}{(\lambda_i - \lambda_j)^2} \right\}. \end{aligned} \quad (2.20)$$

As above, we may now rewrite this expression using the definition (2.5). We thus obtain

$$\begin{aligned} L_E[\lambda_0; \varphi] &= \frac{1}{2} \dot{\lambda}_0^2 + \frac{1}{2} \omega^2 (A^2 - \lambda_0^2) + \int dx \frac{\varphi'}{(x - \lambda_0)^2} \\ &+ \int dx \left\{ \frac{\dot{\varphi}^2}{2\varphi'} + \frac{\pi^2}{6} \varphi'^3 + \frac{1}{2} \omega^2 (A^2 - x^2) \varphi' \right\}. \end{aligned} \quad (2.21)$$

The third term in this expression represents the mutual interaction of the discrete eigenvalue with the continuum eigenvalues, which are collectively described using the field φ . We obtain the Euclidean equations of motion for λ_0 and for φ by variation of (2.21). Respectively, these are found to be

$$\ddot{\lambda}_0 + \omega^2 \lambda_0 + \int dx \frac{\varphi'}{(\lambda_0 - x)^3} = 0 \quad (2.22)$$

$$\partial_t\left(\frac{\dot{\varphi}}{\varphi'}\right) - \frac{1}{2}\partial_x\left\{\frac{\dot{\varphi}^2}{\varphi'^2} + \pi^2\varphi'^2 - \omega^2x^2 + \frac{2}{(\lambda_0 - x)^2}\right\} = 0. \quad (2.23)$$

We consider first the φ equation. We proceed to show, even in the presence of a nontrivial, but discrete, $\lambda_0(t)$, that the static background, $\dot{\varphi}'_0$, derived above is still a valid solution to leading order in ϵ_x . In order that $\dot{\varphi}'_0$ remains a valid solution, it must be so that the last term on the left hand side of (2.23) is negligible with respect to the two which precede it. We can then consistently neglect the time-dependent part of (2.23) as well. Since $\pi^2\dot{\varphi}'_0 - \omega^2x^2 = -\omega^2A^2$, this requirement is that

$$\omega^2A^2 \gg (\lambda_0 - x)^{-2}. \quad (2.24)$$

Furthermore, since λ_0 satisfies (2.19), $\epsilon_x^{-2} \geq (\lambda_0 - x)^{-2}$ and from (2.16) and (2.17) we derive $\epsilon_x^{-2} \approx \omega^2A^2 \cdot g^{1/3}$. Therefore, condition (2.24) requires simply that $\omega^2A^2 \gg \epsilon_x^{-2}$ or, equivalently

$$g^{1/3} \ll 1, \quad (2.25)$$

which we have already assumed. With this discussion in mind, we regard (2.11) as the static solution to (2.23), despite the presence of an additional discrete eigenvalue. We discuss below exactly how it is that such a discrete eigenvalue can arise.

Next, we turn our attention to the λ_0 equation, (2.22). This is the Euclidean equation of motion,

$$\ddot{\lambda}_0 - V'_{eff}(\lambda_0) = 0, \quad (2.26)$$

where

$$V_{eff}(\lambda_0) = -\frac{1}{2}\omega^2\lambda_0^2 + \widehat{V}(\lambda_0). \quad (2.27)$$

In this expression, \widehat{V} is the mean field interaction of λ_0 with all of the continuum eigenvalues,

$$\widehat{V}(\lambda_0) = \left(\int_{-L/2}^{-A} dx + \int_A^{L/2} dx\right) \frac{\dot{\varphi}'_0}{(\lambda_0 - x)^2}. \quad (2.28)$$

Using (2.11) and (2.13), we can compute this function for finite L . Ignoring an irrelevant constant term, and using (2.16), the full effective potential, in the limit $L \rightarrow \infty$ is then found to be

$$V_{eff}(\lambda_0) = \frac{\omega}{2g} \left\{ -\left(\frac{\lambda_0}{A}\right)^2 + 4g \frac{(\lambda_0/A)}{\sqrt{1 - (\lambda_0/A)^2}} \tan^{-1}\left(\frac{(\lambda_0/A)}{\sqrt{1 - (\lambda_0/A)^2}}\right) \right\}. \quad (2.29)$$

This function is plotted in Figure 2 for three different values of g .

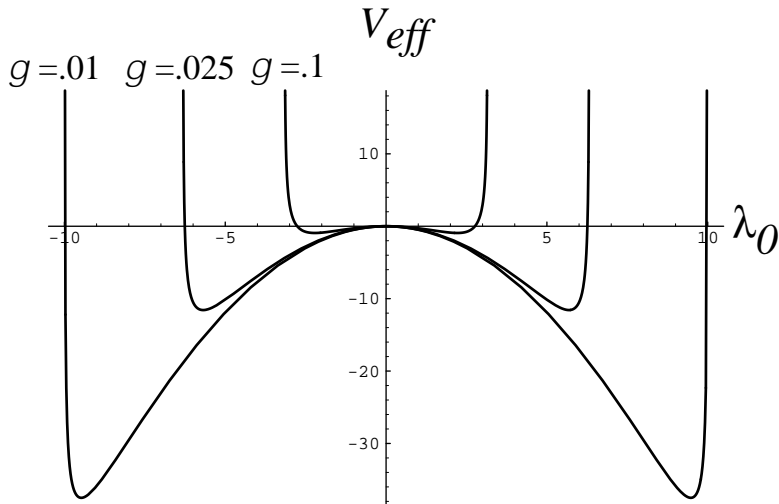


Figure 2. Effective Potential for $g = .1, .025$ and $.01$. V_{eff} and λ_0 are in units in which $\omega = 1$.

It is clear from the figure that the effect of the second term in (2.29), is to turn the potential over near $\lambda_0 = \pm A$, where it adds infinite confining walls. For small values of g , the minima of (2.29) occur at $\lambda_0 = \pm(A - 3\sqrt{3}\epsilon_x)$ to leading order in ϵ_x . However, we must be careful. Recall that λ_0 is a discrete eigenvalue, and $V_{eff}(\lambda_0)$ is well defined, only if λ_0 satisfies the condition (2.19). It is clear that these minima do not satisfy this condition and, hence lie outside the range of validity of our approximation. The actual situation is the following. As we have said, eigenvalue λ_0 is discrete and separated from the continuum, and $V_{eff}(\lambda_0)$ is well defined, provided λ_0 satisfies (2.24); that is, if λ_0 is sufficiently far from $\pm A$. However, when λ_0 approaches $\pm A$ to within order ϵ_x it, in effect, enters the continuum. This is because its separation from the first eigenvalues of the continuum is of the same order as the “graininess” of the continuum discussed previously. Under these circumstances, all eigenvalues, including λ_0 , must be treated as a continuum using a single collective field with action (2.8). It follows that there is only one equation of motion, the φ equation given in (2.9), whose static solution is shown in (2.11). Thus, the true equilibrium positions for λ_0 are at $\lambda_0 = \pm(A + \epsilon_x)$ rather than at $\lambda_0 = \pm(A - 3\sqrt{3}\epsilon_x)$ given above. To conclude, λ_0 can be treated as discrete, and $V_{eff}(\lambda_0)$ is well defined, for λ_0 sufficiently far from $\pm A$. When λ_0 approaches $\pm A$ to within order ϵ_x it is absorbed into the continuum, and disappears as a discrete entity. Of course,

this process can be reversed. It is possible for the first eigenvalue of the continuum to “leak” out and become a discrete eigenvalue λ_0 . We will return to such processes below.

This being said, we would like to find both static and time-dependent solutions for the Euclidean λ_0 equation of motion (2.26). As will become clear in the next section, we need only do this to lowest order; that is, to order ϵ_x^0 . In this case, we may take λ_0 as discrete, and $V_{eff}(\lambda_0)$ as well defined, for all values of λ_0 in the range $-A \leq \lambda_0 \leq A$. Of course, $V_{eff}(\lambda_0)$ must now be evaluated in the limit that $g \rightarrow 0$. This limiting case is given by $V_{eff}(\lambda_0) = -\frac{1}{2}\omega^2\lambda_0^2$ for $-A < \lambda_0 < A$. At $\lambda_0 = \pm A$, though, the potential turns over abruptly and becomes infinite confining walls, as discussed above. As $g \rightarrow 0$ the minima of the potential occur at $\lambda_0 = \pm A$, where the potential obtains cusps, which do not have well defined derivatives. For any finite value of g , however, the derivative vanishes at the minima of the potential. It is appropriate then, in the $g \rightarrow 0$ limit, to take $V'_{eff}(\pm A) = 0$. Hence, in this limit we can replace (2.26) by

$$\begin{aligned}\ddot{\lambda}_0 + \omega^2\lambda_0 &= 0 & ; & \quad -A < \lambda_0 < A \\ \ddot{\lambda}_0 &= 0 & ; & \quad \lambda_0 = \pm A.\end{aligned}\tag{2.30}$$

We also impose the following boundary conditions, $\lambda_0(t \rightarrow -\infty) = \pm A$ and, independently, $\lambda_0(t \rightarrow +\infty) = \pm A$. There are two static solutions to (2.30) which satisfy this boundary condition,

$$\widehat{\lambda}_{0\pm} = \pm A.\tag{2.31}$$

A simple time-dependent solution is given by

$$\widehat{\lambda}_0^{(+)}(t; t_1) = \begin{cases} -A & ; \quad t < t_1 - \frac{\pi}{2\omega} \\ +A \sin \omega(t - t_1) & ; \quad t_1 - \frac{\pi}{2\omega} \leq t \leq t_1 + \frac{\pi}{2\omega} \\ +A & ; \quad t > t_1 + \frac{\pi}{2\omega} \end{cases},\tag{2.32}$$

where t_1 is arbitrary. The solution (2.32) describes an eigenvalue which rolls (tunnels) from $-A$ to $+A$ over a time interval of duration $\frac{\pi}{\omega}$, centered at an arbitrary time t_1 . This solution is shown pictorially in Figure 3.

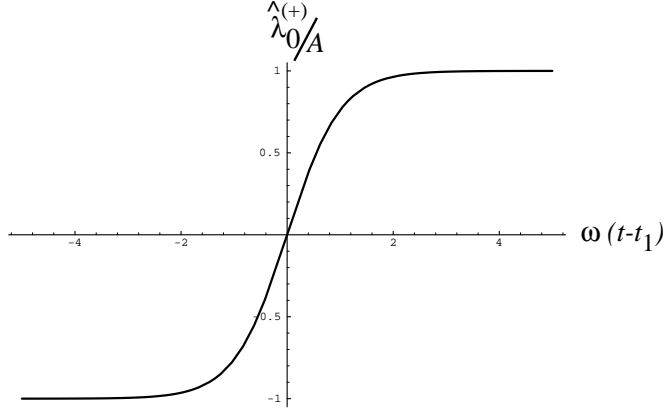


Figure 3. The “kink” solution, $\hat{\lambda}_0^{(+)}$

We refer to this solution as a “kink”. Its mirror image is also a valid solution,

$$\hat{\lambda}_0^{(-)}(t; t_1) = \begin{cases} +A & ; \quad t < t_1 - \frac{\pi}{2\omega} \\ -A \sin \omega(t - t_1) & ; \quad t_1 - \frac{\pi}{2\omega} \leq t \leq t_1 + \frac{\pi}{2\omega} \\ -A & ; \quad t > t_1 + \frac{\pi}{2\omega} \end{cases} , \quad (2.33)$$

It describes an eigenvalue which rolls from $+A$ to $-A$. It is referred to as an “anti-kink” and is shown pictorially in figure 4.

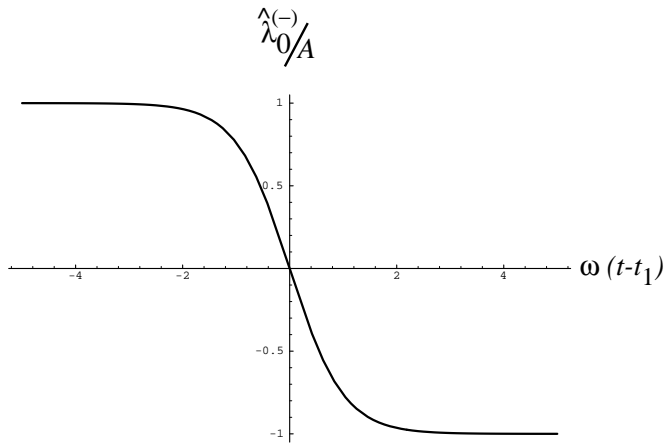


Figure 4. The “antikink” solution, $\hat{\lambda}_0^{(-)}$

Before discussing more general solutions, it is necessary that we make a few clarifying remarks. As discussed above, when $\lambda_0 = \pm A$ it is absorbed into the continuum and does not

have a distinct identity. In the kink solution $\widehat{\lambda}_0^{(+)}$ presented above, it is only at $t = t_1 - \frac{\pi}{2\omega}$ that the last eigenvalue in the continuum, located at $-A$, separates and leaks into the region between $-A$ and A . At $t = t_1 + \frac{\pi}{2\omega}$ the eigenvalue is then reabsorbed into the continuum at $+A$. Similar comments apply to the antikink solution $\widehat{\lambda}_0^{(-)}$. As we will see, it is useful to rephrase these solutions in such a way that λ_0 only exists during the interval $t_1 - \frac{\pi}{2\omega} \leq t \leq t_1 + \frac{\pi}{2\omega}$. That is, during the intervals $t < t_1 - \frac{\pi}{2\omega}$ and $t > t_1 + \frac{\pi}{2\omega}$ we refrain from calling any eigenvalue λ_0 , since all eigenvalues are then a part of the continuum collective field φ . We therefore rewrite the kink and antikink solutions as follows,

$$\lambda_0^{(\pm)} = \pm A \sin \omega(t - t_1) \quad ; \quad t_1 - \frac{\pi}{2\omega} \leq t \leq t_1 + \frac{\pi}{2\omega}, \quad (2.34)$$

which we depict graphically in Figure 5.

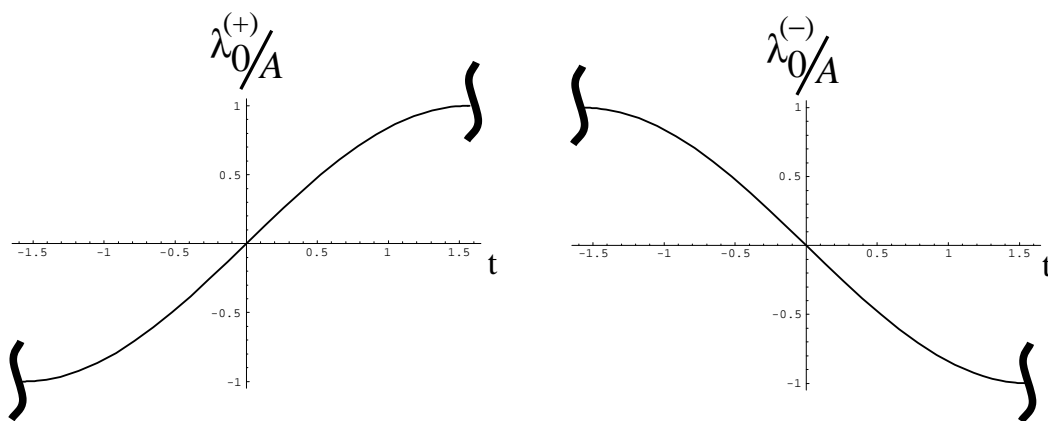


Figure 5. The modified “kink” and “antikink” solutions, $\lambda_0^{(\pm)}$. t is in units in which $\frac{\pi}{2\omega} = 1$.

In these graphical representations, the bars at the ends of the kinks and antikinks symbolize the emission or reabsorption of the eigenvalue into the continuum. The reason why we make this refinement will become clear presently.

There exist more general solutions than those which we have already discussed, in which the identity of λ_0 is a more complex and subtle issue. It is possible, for example, that a

kink, which ends with eigenvalue λ_0 attaching to the continuum at $+A$, could be followed, at some later time, by an antikink, in which the eigenvalue λ_0 separates from the continuum at $+A$, rolls to $-A$ and then reattaches there. Such a kink-antikink sequence, which we denote $\lambda_0^{(+-)}$, would satisfy the Euclidean equation of motion, (2.30). It is also possible, however, that a kink, which ends with the eigenvalue λ_0 attaching to the continuum at $+A$, could be followed, at some later time, by another kink in which a different eigenvalue detaches from the continuum at $-A$, traverses the region between $-A$ and $+A$, and then reattaches to the continuum at $+A$ immediately next to the eigenvalue involved in the first kink. This kink-kink sequence, which we denote $\lambda_0^{(++)}$, also satisfies (2.30). There are thus $2^2 = 4$ solutions which involve two distinct kinks,

$$\begin{aligned}
\lambda_0^{(++)} &= \begin{cases} +A \sin \omega(t - t_1) & ; \quad t_1 - \frac{\pi}{2\omega} \leq t \leq t_1 + \frac{\pi}{2\omega} \\ +A \sin \omega(t - t_2) & ; \quad t_2 - \frac{\pi}{2\omega} \leq t \leq t_2 + \frac{\pi}{2\omega} \end{cases} \\
\lambda_0^{(+-)} &= \begin{cases} +A \sin \omega(t - t_1) & ; \quad t_1 - \frac{\pi}{2\omega} \leq t \leq t_1 + \frac{\pi}{2\omega} \\ -A \sin \omega(t - t_2) & ; \quad t_2 - \frac{\pi}{2\omega} \leq t \leq t_2 + \frac{\pi}{2\omega} \end{cases} \\
\lambda_0^{(-+)} &= \begin{cases} -A \sin \omega(t - t_1) & ; \quad t_1 - \frac{\pi}{2\omega} \leq t \leq t_1 + \frac{\pi}{2\omega} \\ +A \sin \omega(t - t_2) & ; \quad t_2 - \frac{\pi}{2\omega} \leq t \leq t_2 + \frac{\pi}{2\omega} \end{cases} \\
\lambda_0^{(--)} &= \begin{cases} -A \sin \omega(t - t_1) & ; \quad t_1 - \frac{\pi}{2\omega} \leq t \leq t_1 + \frac{\pi}{2\omega} \\ -A \sin \omega(t - t_2) & ; \quad t_2 - \frac{\pi}{2\omega} \leq t \leq t_2 + \frac{\pi}{2\omega} \end{cases} \quad (2.35)
\end{aligned}$$

In all four cases $t_2 \geq t_1 + \frac{\pi}{\omega}$, but both t_1 and t_2 are otherwise arbitrary. We depict the four solutions (2.35) graphically in Figure 6.

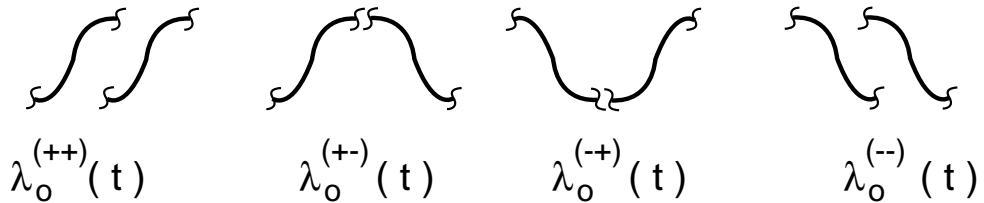


Figure 6. The four solutions (2.35).

One might ask whether it is possible for the second eigenvalue to detach from the continuum while the first eigenvalue is still discrete; that is for $t_1 \leq t_2 < t_1 + \frac{\pi}{\omega}$. In fact, there do exist solutions in which two eigenvalues detach from the same side of the continuum in

quick succession; that is, within a time interval less than $\frac{\pi}{\omega}$. The existence of such solutions and their exact form is actually inconsequential. This is because the probability of such a sequence is further suppressed by the instanton density, which is proportional to $e^{-\frac{\pi}{2g}}$. For sufficiently small g , this probability is negligibly small and we may therefore consistently ignore these solutions. We therefore remove the restriction on t_1, t_2 . An arbitrary solution consists of q events which are randomly distributed between kinks and antikinks, where $0 \leq q < \infty$. We refer to any such q -event solution as a q -instanton. For a given q there are 2^q distinct instanton configurations. For example, for $q = 3$, one solution consists of three consecutive kinks, which we denote $\lambda_0^{(+++)}$. A solution which consists of a kink followed by two antikinks is denoted $\lambda_0^{(+-)}$. Clearly, for $q = 3$, there are $2^3 = 8$ such solutions. Generically, we denote the 2^q q -instantons as $\lambda_0^{(q)}$. There are q zero modes associated with each $\lambda_0^{(q)}$. These correspond to the arbitrary times t_1, \dots, t_q , where $t_q \geq t_{q-1} + \frac{\pi}{\omega} \cdots \geq t_1 + \frac{\pi}{\omega}$, when the kinks or antikinks occur. Once again, we ignore all cases where several eigenvalues are simultaneously discrete, since the effect of these solutions is negligible.

This concludes our analysis of the discrete eigenvalue solutions. In the next section, we take these solutions as background solutions which we expand around when performing the path integral associated with the theory. We integrate the instantons out of the path integral and arrive at an effective theory for the collective field φ which has the instanton effects incorporated explicitly in terms of induced operators.

3. Integration Over Instantons

The partition function associated with the theory discussed above can be written as a sum over different q -instanton sectors,

$$Z = \sum_{q=0}^{\infty} Z_q \quad (3.1)$$

where, schematically,

$$Z_q = \int [d\varphi] \int [d\lambda_0]_q e^{-S[\lambda_0; \varphi]}. \quad (3.2)$$

In this expression the symbol $[d\lambda_0]_q$ indicates that λ_0 is expanded around $\lambda_0^{(q)}$. For notational convenience we have suppressed a subscript E on the action, but it is assumed throughout this section that we are in euclidean space. We proceed to define equation (3.2) in more

precise terms. First of all, remember that $\lambda_0^{(q)}$ generically represents all the 2^q instanton solutions which each have q single eigenvalue kinks-antikinks. Therefore, more specifically,

$$Z_q = \sum_{\{k_i\}} Z_{k_1 \dots k_q}, \quad (3.3)$$

where $k_i = \pm$, the summation is over all 2^q possible sets $\{k_1 \dots k_q\}$, and

$$Z_{k_1 \dots k_q} = \int [d\varphi] \int [d\lambda_0]_{k_1 \dots k_q} e^{-S[\lambda_0; \varphi]}. \quad (3.4)$$

The symbol $[d\lambda_0]_{k_1 \dots k_q}$ indicates that λ_0 is expanded around $\lambda_0^{(k_1 \dots k_q)}$ defined in section 2. Thus, $Z_2 = Z_{++} + Z_{+-} + Z_{-+} + Z_{--}$, and so on. In order to clarify the remaining factors in (3.2) we will focus on an example.

3.1 Calculation of Z_+

Consider Z_+ , the contribution to the partition function coming from single kink configurations. The correct expression is given by

$$Z_+ = \int [d\varphi] \int [d\lambda_0]_+ e^{-S[\lambda_0; \varphi]}, \quad (3.5)$$

where, as discussed in section 2, we expand $\lambda_0(t)$ around the solution $\hat{\lambda}_0^{(+)}(t; t_1)$, which we repeat here for convenience,

$$\hat{\lambda}_0^{(+)}(t; t_1) = \begin{cases} -A & ; \quad t < t_1 - \frac{\pi}{2\omega} \\ +A \sin \omega(t - t_1) & ; \quad t_1 - \frac{\pi}{2\omega} \leq t \leq t_1 + \frac{\pi}{2\omega} \\ +A & ; \quad t > t_1 + \frac{\pi}{2\omega} \end{cases}. \quad (3.6)$$

The parameter t_1 is a zero mode of this solution, and an integration over t_1 is implied within $\int [d\lambda_0]_+$. To make this explicit, we write

$$\lambda_0(t) = \hat{\lambda}_0^{(+)}(t; t_1) + \tilde{\lambda}(t). \quad (3.7)$$

Thus, extracting a t_1 integration, we obtain

$$\int [d\lambda_0]_+ = \sqrt{\frac{\pi}{2g}} \cdot \int_{-\infty}^{\infty} dt_1 \int [d\tilde{\lambda}'] \quad (3.8)$$

where $f[d\tilde{\lambda}']$ indicates integration over all functions orthogonal to $\dot{\lambda}_0^{(+)}$; that is, all functions $\tilde{\lambda}'(t)$ such that

$$\int_{-\infty}^{\infty} dt \tilde{\lambda}'(t) \dot{\lambda}_0^{(+)}(t; t_1) = 0. \quad (3.9)$$

The factor $\sqrt{\frac{\pi}{2g}}$ in (3.8) is a jacobian. We may now rewrite equation (3.5) as follows,

$$Z_+ = \sqrt{\frac{\pi}{2g}} \int dt_1 \int [d\varphi] \int [d\tilde{\lambda}'] e^{-S[\lambda_0; \varphi]}. \quad (3.10)$$

Recall that when $t < t_1 - \frac{\pi}{2\omega}$ and when $t > t_1 + \frac{\pi}{2\omega}$ the eigenvalue λ_0 is not discrete, but is actually part of the continuum. We therefore define $\hat{\varphi}$ as the continuous collective field with $N + 1$ eigenvalues $\lambda_0, \lambda_1, \dots, \lambda_N$, in order to distinguish it from φ , which has only N eigenvalues, $\lambda_1, \dots, \lambda_N$. The limit $N \rightarrow \infty$ is assumed in both cases. Hence, when $t < t_1 - \frac{\pi}{2\omega}$ and when $t > t_1 + \frac{\pi}{2\omega}$ we can write $f[d\varphi] f[d\tilde{\lambda}'] = f[d\hat{\varphi}]$ and $S[\lambda_0; \varphi] = S_\varphi[\hat{\varphi}]$, where $S_\varphi[\hat{\varphi}]$ is the euclidean collective field action given in equation (2.8), here expressed as a function of $\hat{\varphi}$ rather than φ . Thus, Z_+ can be factored as

$$\begin{aligned} Z_+ = & \sqrt{\frac{\pi}{2g}} \int dt_1 \left\{ \int [d\hat{\varphi}] \exp \left[-S_\varphi[\hat{\varphi}] \right]_{-\infty}^{t_1 - \frac{\pi}{2\omega}} \right. \\ & \times \int [d\varphi] \int [d\tilde{\lambda}'] \exp \left[-S[\lambda_0; \varphi] \right]_{t_1 - \frac{\pi}{2\omega}}^{t_1 + \frac{\pi}{2\omega}} \\ & \left. \times \int [d\hat{\varphi}] \exp \left[-S_\varphi[\hat{\varphi}] \right]_{t_1 + \frac{\pi}{2\omega}}^{\infty} \right\}. \end{aligned} \quad (3.11)$$

where $S \Big|_a^b = \int_a^b dt L$ and the functional integrals cover functions defined only during the time intervals specified in the associated integrands. It is useful to convert the remaining $f[d\varphi]$ integration in (3.11) into a $f[d\hat{\varphi}]$ integration. We proceed to do this. Let us denote the middle factor in (3.11) by

$$z_+(t_1) = \int [d\varphi] \int [d\tilde{\lambda}'] \exp \left\{ -S[\lambda_0; \varphi] \right\}_{t_1 - \frac{\pi}{2\omega}}^{t_1 + \frac{\pi}{2\omega}}, \quad (3.12)$$

where $\lambda_0(t) = \lambda_0^{(+)}(t; t_1) + \tilde{\lambda}'(t)$, t_1 is taken as a constant parameter, and from (2.21) we have

$$S[\lambda_0; \varphi] = S_\varphi[\varphi] + \int dt \left\{ \frac{1}{2} \dot{\lambda}_0^2 + \frac{1}{2} \omega^2 (A^2 - \lambda_0^2) + \int dx \frac{\varphi'}{(x - \lambda_0)^2} \right\}. \quad (3.13)$$

We now consider the following representation of the number 1,

$$1 = \frac{\int [d\lambda_\emptyset]_+ \exp\left\{-S_\emptyset[\lambda_\emptyset; \varphi] \Big|_{t_1 - \frac{\pi}{2\omega}}^{t_1 + \frac{\pi}{2\omega}}\right\}}{\int [d\lambda_\emptyset]_+ \exp\left\{-S_\emptyset[\lambda_\emptyset; \varphi] \Big|_{t_1 - \frac{\pi}{2\omega}}^{t_1 + \frac{\pi}{2\omega}}\right\}}, \quad (3.14)$$

where

$$S_\emptyset[\lambda_\emptyset; \varphi] = \int dt \left\{ \frac{1}{2} \dot{\lambda}_\emptyset^2 + \frac{1}{2} \omega^2 (A^2 - \lambda_\emptyset^2) + \int dx \frac{\varphi'}{(x - \lambda_\emptyset)^2} \right\} \quad (3.15)$$

and λ_\emptyset is a dummy eigenvalue expanded around a particular background $\lambda_\emptyset^{(+)}$ given by

$$\lambda_\emptyset^{(+)}(t; t_1) = \begin{cases} -A & ; \quad t_1 - \frac{\pi}{2\omega} \leq t < t_1 \\ +A & ; \quad t_1 < t \leq t_1 + \frac{\pi}{2\omega} \end{cases}. \quad (3.16)$$

Thus, we define

$$\lambda_\emptyset(t) = \lambda_\emptyset^{(+)}(t; t_1) + \hat{\lambda}(t). \quad (3.17)$$

Taylor expanding (3.15) in $\hat{\lambda}$, we find

$$S_\emptyset[\lambda_\emptyset; \varphi] = \int dt \left\{ \int dx \frac{\varphi'}{(x - \lambda_\emptyset^{(+)})^2} + \frac{1}{2} \hat{\lambda} \mathcal{O}_0 \hat{\lambda} + \mathcal{O}(\hat{\lambda}^3) \right\}, \quad (3.18)$$

where

$$\mathcal{O}_0 = -\partial_t^2 - \omega^2 + \int dx \frac{6\varphi'}{(x - \lambda_\emptyset^{(+)})^4}. \quad (3.19)$$

Note that, to lowest order in ϵ_x , the contribution linear in $\hat{\lambda}$ vanishes. This is because $\lambda_\emptyset = \pm A$ satisfies the static equation of motion derived from (3.15) to order ϵ_x^0 , as discussed in the paragraph which follows equation (2.29). In this paper we will restrict ourselves to the semi-classical approximation. In this approximation we may replace φ' in operator \mathcal{O}_0 with $\tilde{\varphi}'_0$ given in equation (2.11). Furthermore, since t_1 is a constant, it follows that $[d\lambda_\emptyset]_+ = [d\hat{\lambda}]$. It is then straightforward to show that

$$\int [d\lambda_\emptyset]_+ \exp\left\{-S_\emptyset[\lambda_\emptyset; \varphi] \Big|_{t_1 - \frac{\pi}{2\omega}}^{t_1 + \frac{\pi}{2\omega}}\right\} = \frac{1}{\sqrt{\det \mathcal{O}_0}} \exp\left\{-\int_{t_1 - \frac{\pi}{2\omega}}^{t_1 + \frac{\pi}{2\omega}} dt \int dx \frac{\varphi'}{(x - \lambda_\emptyset^{(+)})^2}\right\} \quad (3.20)$$

where

$$\det \mathcal{O}_0 = \left[\int [d\hat{\lambda}] \exp\left\{-\frac{1}{2} \int_{t_1 - \frac{\pi}{2\omega}}^{t_1 + \frac{\pi}{2\omega}} dt \hat{\lambda} \mathcal{O}_0 \hat{\lambda}\right\} \right]^{-2}. \quad (3.21)$$

Inserting (3.20) in the denominator of (3.14) we find that

$$1 = \sqrt{\det \mathcal{O}_0} \int [d\lambda_\emptyset]_+ \exp \left\{ -S_\emptyset[\lambda_\emptyset; \varphi] \Big|_{t_1 - \frac{\pi}{2\omega}}^{t_1 + \frac{\pi}{2\omega}} + \int_{t_1 - \frac{\pi}{2\omega}}^{t_1 + \frac{\pi}{2\omega}} dt \int dx \frac{\varphi'}{(x - \lambda_\emptyset^{(+)})^2} \right\}. \quad (3.22)$$

Inserting (3.22) into (3.12), we then obtain

$$z_+(t_1) = \sqrt{\det \mathcal{O}_0} \int [d\varphi] \int [d\lambda_\emptyset]_+ \int [d\tilde{\lambda}'] \times \exp \left\{ -(S[\lambda_0; \varphi] + S_\emptyset[\lambda_\emptyset; \varphi]) \Big|_{t_1 - \frac{\pi}{2\omega}}^{t_1 + \frac{\pi}{2\omega}} + \int_{t_1 - \frac{\pi}{2\omega}}^{t_1 + \frac{\pi}{2\omega}} dt \int dx \frac{\varphi'}{(x - \lambda_\emptyset^{(+)})^2} \right\}. \quad (3.23)$$

Now, using (3.13) and (3.15) we find

$$S[\lambda_0; \varphi] + S_\emptyset[\lambda_\emptyset; \varphi] = S_\varphi[\varphi] + \int dt \left\{ \frac{1}{2} \dot{\lambda}_\emptyset^2 + \frac{1}{2} \omega^2 (A^2 - \lambda_\emptyset^2) + \int dx \frac{\varphi'}{(x - \lambda_\emptyset)^2} \right\} + \int dt \left\{ \frac{1}{2} \dot{\lambda}_0^2 + \frac{1}{2} \omega^2 (A^2 - \lambda_0^2) + \int dx \frac{\varphi'}{(x - \lambda_0)^2} \right\}. \quad (3.24)$$

However, rewriting the collective field φ in terms of eigenvalues the first two terms of the last expression may be rewritten as follows,

$$\begin{aligned} S_\varphi[\varphi] + \int dt \left\{ \frac{1}{2} \dot{\lambda}_\emptyset^2 + \frac{1}{2} \omega^2 (A^2 - \lambda_\emptyset^2) + \int dx \frac{\varphi'}{(x - \lambda_\emptyset)^2} \right\} \\ = \sum_{i=1}^N \left\{ \frac{1}{2} \dot{\lambda}_i^2 + \frac{1}{2} \omega^2 (A^2 - \lambda_i^2) + \frac{1}{2} \sum_{j \neq i, \emptyset} \frac{1}{(\lambda_j - \lambda_i)^2} \right\} \\ + \frac{1}{2} \dot{\lambda}_\emptyset^2 + \frac{1}{2} \omega^2 (A^2 - \lambda_\emptyset^2) + \sum_{j \neq \emptyset} \frac{1}{(\lambda_j - \lambda_\emptyset)^2} \\ = \int dt \sum_{i=\emptyset}^N \left\{ \frac{1}{2} \dot{\lambda}_i^2 + \frac{1}{2} \omega^2 (A^2 - \lambda_i^2) + \frac{1}{2} \sum_{j \neq i} \frac{1}{(\lambda_j - \lambda_i)^2} \right\} \\ = S_\varphi[\hat{\varphi}]. \end{aligned} \quad (3.25)$$

Thus, λ_\emptyset replaces the missing eigenvalue in φ . Note that λ_\emptyset is expanded around (3.16) which is exactly the classical configuration of the missing eigenvalue. It follows that, over the range $t_1 - \frac{\pi}{2\omega} \leq t \leq t_1 + \frac{\pi}{2\omega}$,

$$\int [d\varphi] \int [d\lambda_\emptyset]_+ = \int [d\hat{\varphi}]. \quad (3.26)$$

Using (3.24), (3.25) and (3.26) in equation (3.23) we obtain

$$z_+(t_1) = \sqrt{\det \mathcal{O}_0} \int [d\hat{\varphi}] \exp \left\{ -S_\varphi[\hat{\varphi}] \Big|_{t_1 - \frac{\pi}{2\omega}}^{t_1 + \frac{\pi}{2\omega}} \right\} \\ \times \int [d\tilde{\lambda}'] \exp \left\{ -(S_0[\lambda_0] + S_I[\lambda_0; \varphi]) \Big|_{t_1 - \frac{\pi}{2\omega}}^{t_1 + \frac{\pi}{2\omega}} \right\}, \quad (3.27)$$

where

$$S_0[\lambda_0] = \int dt \left\{ \frac{1}{2} \dot{\lambda}_0^2 + \frac{1}{2} \omega^2 (A^2 - \lambda_0^2) \right\} \quad (3.28)$$

and

$$S_I[\lambda_0; \varphi] = \int dt \int dx \left\{ \frac{\varphi'}{(x - \lambda_0)^2} - \frac{\varphi'}{(x - \lambda_0^{(+)})^2} \right\}. \quad (3.29)$$

We would now like to perform the $\int [d\tilde{\lambda}']$ integration in (3.27). First, recall that $\lambda_0(t) = \lambda_0^{(+)}(t; t_1) + \tilde{\lambda}(t)$. Inserting this into (3.28) and (3.29), we obtain

$$S_0[\lambda_0] + S_I[\lambda_0; \varphi] = S_0[\lambda_0^{(+)}] + S_I[\lambda_0^{(+)}; \varphi] + \frac{1}{2} \int dt \tilde{\lambda} \mathcal{O}_1 \tilde{\lambda} + \mathcal{O}(\tilde{\lambda}^3), \quad (3.30)$$

where

$$\mathcal{O}_1 = -\partial_t^2 - \omega^2 + \int dx \frac{6\varphi'}{(x - \lambda_0^{(+)})^4}. \quad (3.31)$$

As before, the term linear in $\tilde{\lambda}$ vanishes since $\lambda_0^{(+)}$ is a solution of the equation of motion to order ϵ_x^0 . Furthermore, since we want to work to lowest order in \hbar only, we may replace φ' in operator \mathcal{O}_1 by $\tilde{\varphi}'_0$ given in (2.11). Therefore

$$\int [d\tilde{\lambda}'] \exp \left\{ -(S_0[\lambda_0] + S_I[\lambda_0; \varphi]) \Big|_{t_1 - \frac{\pi}{2\omega}}^{t_1 + \frac{\pi}{2\omega}} \right\} \\ = \frac{1}{\sqrt{\det' \mathcal{O}_1}} \exp \left\{ -(S_0[\lambda_0^{(+)}] + S_I[\lambda_0^{(+)}; \varphi]) \Big|_{t_1 - \frac{\pi}{2\omega}}^{t_1 + \frac{\pi}{2\omega}} \right\} \quad (3.32)$$

where

$$\det' \mathcal{O}_1 = \left[\int [d\tilde{\lambda}'] \exp \left\{ -\frac{1}{2} \int_{t_1 - \frac{\pi}{2\omega}}^{t_1 + \frac{\pi}{2\omega}} dt_1 \tilde{\lambda} \mathcal{O}_1 \tilde{\lambda} \right\} \right]^{-2}. \quad (3.33)$$

The action of the single-kink instanton is easily evaluated. It is given by

$$S_0[\lambda_0^{(+)}] \Big|_{t_1 - \frac{\pi}{2\omega}}^{t_1 + \frac{\pi}{2\omega}} = \frac{\pi}{2g}. \quad (3.34)$$

Using the above results in (3.27) we find

$$z_+(t_1) = \sqrt{\frac{\det \mathcal{O}_0}{\det' \mathcal{O}_1}} e^{-\frac{\pi}{2g}} \int_{-\infty}^{\infty} dt_1 \int [d\hat{\varphi}] \exp \left\{ -(S_\varphi[\hat{\varphi}] + S_I[\lambda_0^{(+)}; \varphi]) \Big|_{t_1 - \frac{\pi}{2\omega}}^{t_1 + \frac{\pi}{2\omega}} \right\}. \quad (3.35)$$

Using equations (3.35) and (3.12) in equation (3.11) then yields

$$\begin{aligned} Z_+ &= \mathcal{M} \int_{-\infty}^{\infty} dt_1 \int [d\hat{\varphi}] \exp \left\{ -S_\varphi[\hat{\varphi}] \Big|_{-\infty}^{\infty} \right\} \\ &\quad \times \exp \left\{ -S_I[\lambda_0^{(+)}; \varphi] \Big|_{t_1 - \frac{\pi}{2\omega}}^{t_1 + \frac{\pi}{2\omega}} \right\}, \end{aligned} \quad (3.36)$$

where

$$\mathcal{M} = \sqrt{\frac{\pi}{2g}} \sqrt{\frac{\det \mathcal{O}_0}{\det' \mathcal{O}_1}} e^{-\frac{\pi}{2g}}. \quad (3.37)$$

In Appendix A we explicitly compute this quantity. We find, for “reasonable” values of g , that $\mathcal{M} \approx \omega \sqrt{\frac{\pi}{2g}} e^{-\frac{\pi}{2g}}$ where the constant of proportionality is $\mathcal{O}(1)$. In the remainder of this paper, for simplicity, we will set this constant to unity. In this case

$$\mathcal{M} = \omega \sqrt{\frac{\pi}{2g}} e^{-\frac{\pi}{2g}}. \quad (3.38)$$

The scale \mathcal{M} is an important quantity since it sets the scale of all nonperturbative effects in the theory. At this point, the distinction between $\hat{\varphi}$ and φ becomes immaterial, so we will henceforth omit the hat on $\hat{\varphi}$. We will also suppress the limits $-\infty$ and ∞ on $S_\varphi[\varphi]$ and on the $\int dt_1$ integration, and we will abbreviate S_I as follows

$$S_I[\lambda_0^{(+)}; \varphi] \Big|_{t_1 - \frac{\pi}{2\omega}}^{t_1 + \frac{\pi}{2\omega}} = S_I^{(+)}[\varphi; t_1]. \quad (3.39)$$

Equation (3.36) can then be rewritten more concisely as

$$Z_+ = \mathcal{M} \int dt_1 \int [d\varphi] e^{-S_\varphi[\varphi]} e^{-S_I^{(+)}[\varphi; t_1]}. \quad (3.40)$$

This concludes the example calculation of Z_+ .

3.2 Calculation of Z

When we perform the same analysis on an arbitrary $Z_{k_1 \dots k_q}$, as we did above on the case Z_+ , we arrive at the following general result,

$$Z_{k_1 \dots k_q} = \mathcal{M}^q \prod_{i=1}^q \int_{>} dt_i \int [d\varphi] e^{-S_\varphi[\varphi]} \prod_{j=1}^q e^{-S_I^{(k_j)}[\varphi; t_j]}, \quad (3.41)$$

where $k_i = \pm$, $\prod_i \int_{>} dt_i$ is an ordered set of nested integrals,

$$\prod_{i=1}^q \int_{>} dt_i = \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_1 \int_{t_1 + \frac{\pi}{\omega}}^{\frac{T}{2}} dt_2 \cdots \int_{t_{q-1} + \frac{\pi}{\omega}}^{\frac{T}{2}} dt_q, \quad (3.42)$$

where $T \rightarrow \infty$, and $S_I^{(\pm)}$ is a generalization of (3.29),

$$S_I^{(\pm)}[\varphi; t_j] = \int_{t_j - \frac{\pi}{2\omega}}^{t_j + \frac{\pi}{2\omega}} dt \int dx \left\{ \frac{\varphi'(x, t)}{(x - \lambda_0^{(\pm)}(t - t_j))^2} - \frac{\varphi'(x, t)}{(x - \lambda_\emptyset^{(\pm)}(t - t_j))^2} \right\}. \quad (3.43)$$

Using (3.1), (3.3) and (3.41) we now find that

$$Z = \int [d\varphi] e^{-S_\varphi[\varphi]} \sum_{q=0}^{\infty} \frac{\mathcal{M}^q}{q!} \prod_{i=1}^q \int_{>} dt_i \sum_{\{k_i\}} \prod_{j=1}^q e^{-S_I^{(k_j)}[\varphi; t_j]}. \quad (3.44)$$

Notice that we do not let any pair of t_i 's come within $\Delta t = \frac{\pi}{\omega}$ of each other. As discussed in section 2, the reason for this is that the probability for configurations in which any pair of t_i 's are within this range is negligibly small. Such configurations, in which two kinks or anti-kinks overlap include complicated instanton-instanton interactions. As in more traditional instanton calculations these interactions are of strength proportional to the square of the instanton fugacity, $(e^{-\frac{\pi}{2g}})^2$ and therefore add negligible correction. We will therefore ignore them. This is a dilute gas approximation. The practical consequence of this is to remove the restriction on the range of the t_i 's. The integrand is then completely symmetric under $t_i \leftrightarrow t_j$ for any i and j . We may therefore replace the ordered (and unrestricted) $\int_{>} dt_i$ integrals with unordered integrals provided we insert a factor of $1/q!$ to compensate for overcounting. Thus, we may rewrite (3.44) as follows,

$$\begin{aligned} Z &= \int [d\varphi] e^{-S_\varphi[\varphi]} \sum_{q=0}^{\infty} \frac{1}{q!} \mathcal{M}^q \prod_{i=1}^q \int dt_i \sum_{\{k_i\}} \prod_{j=1}^q e^{-S_I^{(k_j)}[\varphi; t_j]} \\ &= \int [d\varphi] e^{-S_\varphi[\varphi]} \sum_{q=0}^{\infty} \frac{1}{q!} \left\{ \mathcal{M} \int dt_1 \left(e^{-S_I^{(+)}[\varphi; t_1]} + e^{-S_I^{(-)}[\varphi; t_1]} \right) \right\}^q. \end{aligned} \quad (3.45)$$

The sum over q is now an exponential, so that

$$Z = \int [d\varphi] e^{-S_{eff}[\varphi]}, \quad (3.46)$$

where

$$S_{eff}[\varphi] = S_\varphi[\varphi] + \Delta S[\varphi] \quad (3.47)$$

is the effective action with the q -instanton effects systematically incorporated, and

$$\Delta S[\varphi] = \mathcal{M} \int dt_1 \left\{ e^{-S_I^{(+)}[\varphi; t_1]} + e^{-S_I^{(-)}[\varphi; t_1]} \right\} \quad (3.48)$$

is the associated change in the action, where \mathcal{M} and $S_I^{(\pm)}$ are given in (3.37) and (3.43) respectively. Equation (3.48) is the change in the collective field action due to the presence of q -instantons, in the limit of small g . Note that this expression is not a two dimensional integral over a local density.

We should express the collective field theory, and any instanton-induced operators, in terms of canonically propagating fields. We begin the following subsection with the identification of the canonical theory, and proceed to re-analyze the above results in this more appropriate framework.

3.3 Canonical Theory

So far in this paper we have studied the collective field theory expressed in terms of the field φ . By examining equation (2.8), however, we discover that φ does not have a canonically normalized kinetic energy. We also find that the collective field Lagrangian is neither Lorentz invariant nor translationally invariant. The first of these problems is solved, in part, by expanding φ around the solution to the euclidean field equation $\tilde{\varphi}_0$ given in (2.11). Thus, we define

$$\varphi(x, t) = \tilde{\varphi}_0(x) + \frac{1}{\sqrt{\pi}} \zeta(x, t). \quad (3.49)$$

As discussed at length elsewhere, a canonical kinetic energy is obtained by expressing the Lagrangian in terms of a new spatial coordinate τ defined by the following relation,

$$\tau'(x) = \frac{1}{\pi} (\tilde{\varphi}'_0(x))^{-1}. \quad (3.50)$$

Note that τ has mass dimension -1 , which is the appropriate mass dimension for a spatial coordinate, whereas x has mass dimension $-\frac{1}{2}$. Expressing the euclidean collective field action (2.8) in terms of $\zeta(\tau, t)$, we find, in the absence of instanton effects, that

$$S_\zeta[\zeta] = \int dt \int d\tau \left\{ \frac{1}{2} (\dot{\zeta}^2 + \zeta'^2) - \frac{1}{2} \frac{g(\tau) \dot{\zeta}^2 \zeta'}{1 + g(\tau) \zeta'} + \frac{1}{6} g(\tau) \zeta'^3 - \frac{1}{3} \frac{1}{g(\tau)^2} \right\}, \quad (3.51)$$

where $g(\tau)$ is a space dependent coupling parameter, which we define below, and the τ integration is over the limits $-\infty < \tau \leq \tau_0 + \frac{\sigma}{2}$ and $\tau_0 + \frac{\sigma}{2} \leq \tau < \infty$, where τ_0 and σ are independent integration constants which arise when solving (3.50). The reason why there are two integration constants rather than one, given that (3.50) is a first-order differential equation, is that we must solve (3.50) independently over the two separate regions $-\infty < x \leq A$ and $A \leq x < \infty$. The region $-A < x < A$, where there is no continuous collective field theory, is the low density region. In τ space, this region is given by $\tau_0 - \frac{\sigma}{2} < \tau < \tau_0 + \frac{\sigma}{2}$, so that τ_0 is the center of the low density region and σ is the width. The coupling parameter, defined over $-\infty < \tau \leq \tau_0 - \frac{\sigma}{2}$ and $\tau_0 + \frac{\sigma}{2} \leq \tau < \infty$, is given by $g(\tau) = (\pi^{3/2} \tilde{\varphi}_0(x))^{-1}$, and is found to be

$$g(\tau) = 4\sqrt{\pi} \frac{g}{\omega} \frac{\frac{1}{\kappa} e^{-2\omega|\tau-\tau_0|}}{(1 - \frac{1}{\kappa} e^{-2\omega|\tau-\tau_0|})^2}, \quad (3.52)$$

where κ is a dimensionless number,

$$\kappa = \exp(-\omega\sigma), \quad (3.53)$$

which relates the width, σ , of the low density region in τ space to the natural length scale in the matrix model, $1/\omega$. Notice that the coupling parameter blows up as $\tau \rightarrow \tau_0 \pm \frac{\sigma}{2}$; that is, at the boundaries of the low density region.

We would now like to express the change in the effective action due to the q -instanton effects, equation (3.48), in terms of the canonical variable $\zeta(\tau, t)$. Since $S_I^{(\pm)}$ is linear in φ , it follows that

$$S_I^{(\pm)}[\varphi; t_1] = S_I^{(\pm)}[\tilde{\varphi}_0] + \frac{1}{\sqrt{\pi}} S_I^{(\pm)}[\zeta; \tau_0, t_1]. \quad (3.54)$$

The τ_0 dependence in the last term of this equation will be made clear presently. From (3.43), we find

$$S_I^{(\pm)}[\zeta; \tau_0, t_1] = \int_{t_1 - \frac{\pi}{2\omega}}^{t_1 + \frac{\pi}{2\omega}} dt \int d\tau \left\{ \frac{\zeta'(\tau, t)}{(x(\tau) - \lambda_0^{(\pm)}(t - t_1))^2} - \frac{\zeta'(\tau, t)}{(x(\tau) - \lambda_\emptyset^{(\pm)}(t - t_1))^2} \right\}, \quad (3.55)$$

where the prime now means differentiation with respect to τ , and where

$$x(\tau) = \begin{cases} -A \cosh\{\omega(\tau - \tau_0 + \sigma/2)\} & ; \tau \leq \tau_0 - \sigma/2 \\ +A \cosh\{\omega(\tau - \tau_0 - \sigma/2)\} & ; \tau \geq \tau_0 + \sigma/2 \end{cases}. \quad (3.56)$$

This last expression is found by integrating (3.50) to obtain $\tau(x)$ and then inverting the result to obtain $x(\tau)$. This function depends explicitly on τ_0 . This explains why there is an explicit τ_0 in equations (3.54) and (3.55). It is straightforward to compute $S_I^{(\pm)}[\tilde{\varphi}_0]$, given (2.11), (3.43), and the definitions (2.34) and (3.16). We emphasize, however, that one must include the cutoff ϵ_x in the lower limit of the x integration in this expression. We find

$$S_I^{(\pm)}[\tilde{\varphi}_0] = -2^{3/2} \sqrt{\frac{A}{\epsilon_x}} + \ln \sqrt{\frac{A}{\epsilon_x}} + \mathcal{O}\left(\frac{\epsilon_x}{A}\right). \quad (3.57)$$

As discussed above, ϵ_x is the size of the inter-eigenvalue separation near the edge of the continuum and so provides the natural regulator for expressions such as (3.57). From (2.17) it follows that, to lowest order in g , $e^{-S_I^{(\pm)}[\tilde{\varphi}_0]} \approx g^{1/3} e^{\mathcal{O}(g^{1/3})}$. The constant of proportionality in this expression is $\mathcal{O}(1)$. In the remainder of this paper, for simplicity, we will set this constant to unity. In this case

$$e^{-S_I^{(\pm)}[\tilde{\varphi}_0]} = g^{1/3} e^{\mathcal{O}(g^{1/3})}. \quad (3.58)$$

Since all x -space integrations are cut-off at a distance ϵ_x from the edge of the low density region; that is, at $|x| = A + \epsilon_x$, it follows that all τ space integrals must be cut-off as well at a value ϵ_τ . Specifically, in (3.55) and in all other expressions in this paper which include a $\int d\tau$ integration, the following is implied,

$$\int d\tau = \int_{-\infty}^{\tau_0 - \frac{\sigma}{2} - \epsilon_\tau} d\tau + \int_{\tau_0 + \frac{\sigma}{2} + \epsilon_\tau}^{\infty} d\tau. \quad (3.59)$$

The value of ϵ_τ is simple to obtain. We require that

$$\begin{aligned} x\left(\tau - \frac{\sigma}{2} - \epsilon_\tau\right) &= -A - \epsilon_x \\ x\left(\tau + \frac{\sigma}{2} + \epsilon_\tau\right) &= A + \epsilon_x. \end{aligned} \quad (3.60)$$

Using (3.56) and (2.17) it follows, to leading order in g , that

$$\epsilon_\tau = \frac{1}{\omega\sqrt{2}} (3\pi g)^{1/3}. \quad (3.61)$$

Now, using (3.58), substituting (3.54) into (3.48), and using (3.38), we find that

$$\Delta S[\zeta] = \omega g^{-1/6} e^{-\frac{\pi}{2g}} \int dt_1 \left\{ e^{-S_I^{(+)}[\zeta; \tau_0, t_1]} + e^{-S_I^{(-)}[\zeta; \tau_0, t_1]} \right\}. \quad (3.62)$$

To recap our results so far, the partition function for the collective field theory, including the instanton effects, expressed in terms of the canonical field ζ is

$$Z = \int [d\zeta] e^{-S_{eff}[\zeta]}, \quad (3.63)$$

where

$$S_{eff}[\zeta] = S_\zeta[\zeta] + \Delta S[\zeta], \quad (3.64)$$

$S_\zeta[\zeta]$ is given in (3.51), $\Delta S[\zeta]$ is given in (3.62), and the functions $S_I^{(\pm)}[\zeta; \tau_0, t_1]$ which appear in (3.62) are given in (3.55). Equation (3.62) is a significant result. Concisely, it is the induced change in the canonical collective field theory which results from the systematic inclusion of instanton effects. As we will demonstrate in the next subsection, equation (3.62) includes operators higher order in g . We will also demonstrate that this result includes nonlocal interactions. We will address each of these two issues and conclude the following subsection by exposing a more useful form for the induced action, as a two-dimensional integral over a density function expressed consistently to lowest order in g .

3.4 Lowest Order Induced Action as an Integral Over a Local Density

We begin by focussing on $S_I^{(\pm)}$. It is useful to separate these into two pieces,

$$S_I^{(\pm)} = S_{<I}^{(\pm)} + S_{>I}^{(\pm)}, \quad (3.65)$$

where $S_{<I}^{(\pm)}$ includes the contribution coming from the region $\tau < \tau_0 - \frac{\sigma}{2}$, and $S_{>I}^{(\pm)}$ includes the contribution coming from the region $\tau > \tau_0 + \frac{\sigma}{2}$. Using (3.43), we may write these as follows

$$S_{<I}^{(\pm)}[\zeta; \tau_0, t_1] = \int_{t_1 - \frac{\pi}{2\omega}}^{t_1 + \frac{\pi}{2\omega}} dt \int_{-\infty}^{\tau_0 - \frac{\sigma}{2} - \epsilon\tau} d\tau \mathcal{J}_{<}^{(\pm)}(\tau - \tau_0 + \frac{\sigma}{2}, t - t_1) \zeta'(\tau, t) \quad (3.66)$$

and

$$S_{>I}^{(\pm)}[\zeta; \tau_0, t_1] = \int_{t_1 - \frac{\pi}{2\omega}}^{t_1 + \frac{\pi}{2\omega}} dt \int_{\tau_0 + \frac{\sigma}{2} + \epsilon\tau}^{\infty} d\tau \mathcal{J}_{>}^{(\pm)}(\tau - \tau_0 - \frac{\sigma}{2}, t - t_1) \zeta'(\tau, t), \quad (3.67)$$

where

$$\mathcal{J}_{<}^{(\pm)}(\tau - \tau_0 + \frac{\sigma}{2}, t - t_1) = \frac{1}{(x(\tau - \tau_0 + \frac{\sigma}{2}) - \lambda_0^{(\pm)}(t - t_1))^2} - \frac{1}{(x(\tau - \tau_0 + \frac{\sigma}{2}) - \lambda_\emptyset^{(\pm)}(t - t_1))^2} \quad (3.68)$$

and

$$\mathcal{J}_{>}^{(\pm)}(\tau - \tau_0 - \frac{\sigma}{2}, t - t_1) = \frac{1}{(x(\tau - \tau_0 - \frac{\sigma}{2}) - \lambda_0^{(\pm)}(t - t_1))^2} - \frac{1}{(x(\tau - \tau_0 - \frac{\sigma}{2}) - \lambda_\emptyset^{(\pm)}(t - t_1))^2} \quad (3.69)$$

By shifting the arguments of the integration in (3.66) and (3.67) as $\tau \rightarrow \tau + \tau_0 \mp \frac{\sigma}{2}$ respectively and $t \rightarrow t + t_1$, we then obtain

$$S_{<I}^{(\pm)}[\zeta; \tau_0, t_1] = \int_{-\frac{\pi}{2\omega}}^{\frac{\pi}{2\omega}} dt \int_{-\infty}^{-\epsilon\tau} d\tau \mathcal{J}_{<}^{(\pm)}(\tau, t) \zeta'(\tau + \tau_0 - \frac{\sigma}{2}, t + t_1). \quad (3.70)$$

and

$$S_{>I}^{(\pm)}[\zeta; \tau_0, t_1] = \int_{-\frac{\pi}{2\omega}}^{\frac{\pi}{2\omega}} dt \int_{\epsilon\tau}^{\infty} d\tau \mathcal{J}_{>}^{(\pm)}(\tau, t) \zeta'(\tau + \tau_0 + \frac{\sigma}{2}, t + t_1). \quad (3.71)$$

We now Taylor expand ζ' in (3.70) and (3.71) around the spacetime points $(\tau_0 - \frac{\sigma}{2}, t_1)$ and $(\tau_0 + \frac{\sigma}{2}, t_1)$ respectively; that is, in $S_{<I}^{(\pm)}$ we expand ζ' around the rightmost edge of the continuous region of spacetime to the left of the low density region, and in $S_{>I}^{(\pm)}$ we expand ζ' around the leftmost edge of the continuous spacetime region to the right of the low density region. Doing this, we derive the following expression,

$$\zeta'(\tau + \tau_0 \mp \frac{\sigma}{2}, t + t_1) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!n!} \tau^m t^n \partial_{\tau_0}^m \partial_{t_1}^n \zeta'(\tau_0 \mp \frac{\sigma}{2}, t_1). \quad (3.72)$$

Substituting this into (3.70) and (3.71), we immediately find

$$\begin{aligned} S_{<I}^{(\pm)}[\zeta; \tau_0, t_1] &= \sum_{mn} \frac{1}{\omega^{m+n+1}} h_{<mn}^{(\pm)} \partial_{\tau_0}^m \partial_{t_1}^n \zeta'(\tau_0 - \frac{\sigma}{2}, t_1), \\ S_{>I}^{(\pm)}[\zeta; \tau_0, t_1] &= \sum_{mn} \frac{1}{\omega^{m+n+1}} h_{>mn}^{(\pm)} \partial_{\tau_0}^m \partial_{t_1}^n \zeta'(\tau_0 + \frac{\sigma}{2}, t_1), \end{aligned} \quad (3.73)$$

where

$$h_{<mn}^{(\pm)} = \frac{\omega^{m+n+1}}{m!n!} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dt \int_{-\infty}^{-\epsilon\tau} d\tau \mathcal{J}_{<}^{(\pm)}(\tau, t) \tau^m t^n. \quad (3.74)$$

and

$$h_{>mn}^{(\pm)} = \frac{\omega^{m+n+1}}{m!n!} \int_{-\frac{\pi}{2\omega}}^{\frac{\pi}{2\omega}} dt \int_{\epsilon\tau}^{\infty} d\tau \mathcal{J}_{>}^{(\pm)}(\tau, t) \tau^m t^n. \quad (3.75)$$

As we show explicitly in Appendix B, these are computable, finite, dimensionless numbers. Furthermore, due to the symmetry properties of $\mathcal{J}_{<}^{(\pm)}$ and $\mathcal{J}_{>}^{(\pm)}$, it follows that

$$h_{<mn}^{(+)} = (-)^{m+n} h_{>mn}^{(+)} = h_{>mn}^{(-)} = (-)^{m+n} h_{<mn}^{(-)} \equiv h_{mn}. \quad (3.76)$$

Thus,

$$\begin{aligned}
S_{<I}^{(+)}[\zeta; \tau_0, t_1] &= \frac{1}{\omega} h_{00} \zeta'(\tau_0 - \frac{\sigma}{2}, t_1) + \frac{1}{\omega^2} h_{01} \zeta''(\tau_0 - \frac{\sigma}{2}, t_1) + \frac{1}{\omega^2} h_{10} \dot{\zeta}'(\tau_0 - \frac{\sigma}{2}, t_1) + \dots, \\
S_{>I}^{(+)}[\zeta; \tau_0, t_1] &= \frac{1}{\omega} h_{00} \zeta'(\tau_0 + \frac{\sigma}{2}, t_1) - \frac{1}{\omega^2} h_{01} \zeta''(\tau_0 + \frac{\sigma}{2}, t_1) - \frac{1}{\omega^2} h_{10} \dot{\zeta}'(\tau_0 + \frac{\sigma}{2}, t_1) + \dots,
\end{aligned} \tag{3.77}$$

and

$$\begin{aligned}
S_{<I}^{(-)}[\zeta; \tau_0, t_1] &= \frac{1}{\omega} h_{00} \zeta'(\tau_0 - \frac{\sigma}{2}, t_1) - \frac{1}{\omega^2} h_{01} \zeta''(\tau_0 - \frac{\sigma}{2}, t_1) - \frac{1}{\omega^2} h_{10} \dot{\zeta}'(\tau_0 - \frac{\sigma}{2}, t_1) + \dots, \\
S_{>I}^{(-)}[\zeta; \tau_0, t_1] &= \frac{1}{\omega} h_{00} \zeta'(\tau_0 + \frac{\sigma}{2}, t_1) + \frac{1}{\omega^2} h_{01} \zeta''(\tau_0 + \frac{\sigma}{2}, t_1) + \frac{1}{\omega^2} h_{10} \dot{\zeta}'(\tau_0 + \frac{\sigma}{2}, t_1) + \dots,
\end{aligned} \tag{3.78}$$

It is convenient to adopt the following notation,

$$\zeta_{\pm} \equiv \zeta(\tau_0 \pm \frac{\sigma}{2}, t_1). \tag{3.79}$$

Since $S_I^{(\pm)} = S_{<I}^{(\pm)} + S_{>I}^{(\pm)}$, it then follows that

$$\begin{aligned}
S_I^{(+)} &= \frac{1}{\omega} h_{00} (\zeta'_- + \zeta'_+) + \frac{1}{\omega^2} h_{01} (\zeta''_- - \zeta''_+) + \frac{1}{\omega^2} h_{10} (\dot{\zeta}'_- - \dot{\zeta}'_+) + \frac{1}{\omega^3} h_{11} (\dot{\zeta}''_- + \dot{\zeta}''_+) + \dots \\
S_I^{(-)} &= \frac{1}{\omega} h_{00} (\zeta'_- - \zeta'_+) - \frac{1}{\omega^2} h_{01} (\zeta''_- - \zeta''_+) - \frac{1}{\omega^2} h_{10} (\dot{\zeta}'_- - \dot{\zeta}'_+) + \frac{1}{\omega^3} h_{11} (\dot{\zeta}''_- + \dot{\zeta}''_+) + \dots.
\end{aligned} \tag{3.80}$$

Using equations (3.74), (3.75), and (3.76), it is straightforward to compute the coefficients h_{mn} and we do it explicitly in Appendix B. Note that due to the cutoff ϵ_τ in (3.74) and (3.75), these coefficients depend, in general, on g . We find, for instance, to leading order in g , that

$$\begin{aligned}
h_{00} &= -\frac{4\sqrt{2}}{9} \\
h_{10} &= -\left(\frac{8\pi g}{9}\right)^{1/3} \\
h_{01} &= -\frac{\pi\sqrt{2}}{9}.
\end{aligned} \tag{3.81}$$

In general, the h_{mn} are found to have the following g dependence,

$$h_{mn} \sim \begin{cases} g^{m/3} & ; \quad m \leq 3 \\ g & ; \quad m > 3 \end{cases} \quad (3.82)$$

Note, from (3.80) and (3.82), that, as the first index of h_{mn} increases, that the corresponding terms in $S_I^{(\pm)}$ depend on higher powers of g . However, none of h_{0n} have g dependence for any value of n . We proceed to analyze the relative impact of these terms on generic N -point functions. By putting (3.80) back into (3.62) we can find all relevant interaction vertices. These are obtained by Taylor expanding the exponentials in (3.62). For instance, we obtain the quadratic vertices $\frac{1}{\omega^2} h_{00}^2 \zeta'_- \zeta'_-$ and $\frac{1}{\omega^3} h_{00} h_{10} \zeta'_- \zeta''_-$ where, as discussed above, $h_{00} \sim 1$ and $h_{10} \sim g^{1/3}$. It is clear that the effect of the second vertex, containing $h_{00} h_{10}$, on any N -point function, is suppressed by a factor $g^{1/3} p/w$, where p is a characteristic momentum, when compared with effects arising solely from the first vertex containing h_{00}^2 . This is true at tree level. At the quantum level, there may be some subtleties to this argument which we will not discuss in this paper. Similar considerations apply to all other induced operators, involving higher h_{mn} . It can thus be shown, provided

$$p \lesssim \omega, \quad (3.83)$$

that, when working to leading order in g , we can consistently drop all but the h_{0n} terms in (3.80). Now, of the terms which remain, as n increases, the corresponding terms in $S_I^{(\pm)}$ depend on higher derivatives of ζ . Thus, the effect of any vertex, containing h_{0n} , on any N -point function, is suppressed by a factor $(p/\omega)^n$, relative to effects arising from vertices containing only h_{00} . If we further restrict momenta, such that

$$p \ll \omega, \quad (3.84)$$

we can then consistently neglect all but the h_{00} terms in (3.80). This results in a vast simplification of the final result, so we will assume this approximation. It would be completely straightforward, however, to lift the restriction (3.84), and only require (3.83). One would then have to keep all h_{0n} terms in (3.80). It follows, from (3.80), to the order of approximation given in (3.84), that $S_I^{(+)} = S_I^{(-)}$, and therefore that (3.62) collapses to a single

exponential. Plugging (3.80) into (3.62) and using (3.79) and (3.81), we then find, to leading order in g ,

$$\Delta S[\zeta] = 2\omega g^{-1/6} e^{-\frac{\pi}{2g}} \int dt_1 \exp\left\{\frac{4\sqrt{2}}{3\omega} \left(\zeta'(\tau_0 + \frac{\sigma}{2}, t_1) + \zeta'(\tau_0 - \frac{\sigma}{2}, t_1)\right)\right\}. \quad (3.85)$$

Note however that equation (3.85) includes nonlocal interactions, since it involves contributions coming from ζ' evaluated simultaneously at $\tau_0 - \frac{\sigma}{2}$ and also at $\tau_0 + \frac{\sigma}{2}$. This is not surprising though, since we have arrived at this result by integrating over single eigenvalue instantons, which link effects on the left-hand side of the low-density region with effects on the the right-hand side of this region, and because there is a finite separation between these two sectors. One may wish to find some further approximation which would render the effective theory local. This can be done as follows. Provided we consider momenta which satisfy (3.84), and provided also that $\omega \lesssim \frac{1}{\sigma}$, the effective width of the low density region as seen by any field will be essentially zero. We therefore Taylor expand $\zeta'(\tau_0 \pm \frac{\sigma}{2}, t_1)$ around the point (τ_0, t_1) , thereby taking

$$\frac{1}{\omega} \zeta'(\tau_0 \pm \frac{\sigma}{2}, t_1) = \frac{1}{\omega} \zeta'(\tau_0, t_1) \pm \frac{\sigma\omega}{2\omega} \zeta''(\tau_0, t_1) + \dots \quad (3.86)$$

Then, in a manner identical to the previous discussion, we find that the contributions coming from vertices which involve σ are always suppressed by $(\sigma\omega)p/\omega$, where p is a characteristic momentum. Note that, since we now assume $\omega \lesssim \frac{1}{\sigma}$, the factor $(\sigma\omega)$ is $\lesssim \mathcal{O}(1)$. So, provided that

$$p \ll \omega \lesssim \frac{1}{\sigma}, \quad (3.87)$$

we may write the lowest order instanton-induced change in the collective field action approximately, in local form, as follows,

$$\Delta S[\zeta] = 2\omega g^{-1/6} e^{-\frac{\pi}{2g}} \int dt e^{-\frac{2\sqrt{2}}{3\omega} \zeta'(\tau_0, t)}. \quad (3.88)$$

We have dropped the subscript “1” on t_1 because it is now superfluous. This result can be written as a two-dimensional integral over a density $\Delta S = \int dt d\tau \Delta \mathcal{L}$, where

$$\Delta \mathcal{L} = 2\omega g^{-1/6} e^{-\frac{\pi}{2g}} \delta(\tau - \tau_0) e^{-\frac{2\sqrt{2}}{3\omega} \zeta'(\tau, t)}. \quad (3.89)$$

This is the final result of our calculation.

4. Conclusion

We have presented a detailed analysis of the interplay between the continuous and discrete sectors of a $d = 1$ bosonic matrix model, and have performed an explicit and complete calculation of the single eigenvalue instantons in the theory. In addition we have derived the precise form of the lowest order operators which are induced in the theory when the instantons are integrated out. The relevant fact which we have demonstrated is that the nonperturbative aspects of the collective field theory can be isolated, and their leading order effects systematically incorporated. This calculation is an essential preliminary for an interesting analysis involving the $d = 1, \mathcal{N} = 2$ supersymmetric matrix model. The supersymmetric case will be presented in a forthcoming paper[6].

Appendix A: Calculation of \mathcal{M}

In this Appendix we compute \mathcal{M} , the mass scale characteristic of nonperturbative effects in the collective field theory. In section 3 we found that

$$\mathcal{M} = \sqrt{\frac{\pi}{2g}} \sqrt{\frac{\det \mathcal{O}_0}{\det' \mathcal{O}_1}} e^{-\frac{\pi}{2g}}, \quad (\text{A.1})$$

which was first stated as equation (3.37). The first factor in this expression is a functional jacobian which results from the extraction of the zero mode t_0 from the $[d\lambda_0]$ functional measure, and the last term is a fugacity factor. The middle term includes the quantum effects involving $\tilde{\lambda}$, the fluctuations of λ_0 around the instanton background. It is the result of performing the $\int [d\tilde{\lambda}]$ integration. The operators \mathcal{O}_0 and \mathcal{O}_1 are given in (3.19) and (3.31) respectively. The first operator is given by

$$\mathcal{O}_0 = -\partial_t^2 - \omega^2 + \int dx \frac{3\check{\varphi}'_0(x)}{(x - \lambda_\emptyset^{(+)})^4}. \quad (\text{A.2})$$

But

$$\begin{aligned} \int dx \frac{3\check{\varphi}'_0(x)}{(x - \lambda_\emptyset)^4} &= \frac{6\omega}{\pi} \int_{A+\epsilon_x}^{\infty} dx \frac{\sqrt{x^2 - A^2}}{(x - A)^4} \\ &= \frac{6\omega}{\pi} \frac{1}{15A^2} \left\{ 1 + 6\sqrt{2} \left(\frac{A}{\epsilon_x} \right)^{\frac{5}{2}} \left(1 + \mathcal{O}\left(\frac{\epsilon_x}{A}\right) \right) \right\} \\ &\approx \frac{32}{5(3\pi^4)^{2/3}} \frac{\omega^2}{g^{2/3}}. \end{aligned} \quad (\text{A.3})$$

The approximation made in the last line of (A.3) is valid provided $g \ll 1$, which we always assume. Therefore we can reexpress \mathcal{O}_0 as follows,

$$\mathcal{O}_0 = -\partial_t^2 + \alpha\omega^2, \quad (\text{A.4})$$

where

$$\alpha = \frac{32}{5(3\pi^4 g)^{2/3}} - 1. \quad (\text{A.5})$$

The second operator is given by

$$\mathcal{O}_1 = -\partial_t^2 - \omega^2 + \int dx \frac{3\check{\varphi}'_0(x)}{(x - \lambda_0^{(+)})^4}. \quad (\text{A.6})$$

where $\lambda_0^{(+)} = A \sin \omega(t - t_1)$. For convenience, for this particular calculation, we define $\gamma = \sin \omega(t - t_1)$. Thus,

$$\begin{aligned} \int dx \frac{3\check{\varphi}'_0(x)}{(x - \lambda_0^{(+)})^4} &= \int dx \frac{3\check{\varphi}'_0(x)}{(x - \gamma A)^4} \\ &= \frac{3\omega}{\pi A^2 (1 - \gamma^2)^{5/2}} \left\{ (1 + \gamma) + \left(\tan^{-1} \frac{\gamma}{\sqrt{1 - \gamma^2}} - \frac{\pi}{2} \right) \right\} \\ &\approx \frac{\gamma}{(1 - \gamma^2)^{5/2}} \cdot \omega^2 g \\ &\ll \omega^2. \end{aligned} \quad (\text{A.7})$$

The last line is true provided γ isn't too close to one, which we assume since this operator only applies when acting on a discrete eigenvalue. So, for a crude but reasonable calculation, we can neglect the last term in (A.6) relative to the pure ω^2 term. Therefore, we take

$$\mathcal{O}_0 = -\partial_t^2 - \omega^2. \quad (\text{A.8})$$

We find the spectrum associated with both \mathcal{O}_0 and \mathcal{O}_1 by solving the eigenvalue problem,

$$\mathcal{O}_i \lambda_n = \omega_n^{(i)2} \lambda_n, \quad (\text{A.9})$$

where $i = 0$ or 1 and the $\lambda_n(t)$ are defined over $-\frac{\pi}{2\omega} \leq t \leq \frac{\pi}{2\omega}$, satisfy the boundary condition $\lambda_n(t = \pm \frac{\pi}{2\omega}) = 0$, and are orthonormal,

$$\int_{-\frac{\pi}{2\omega}}^{\frac{\pi}{2\omega}} dt \lambda_n(t) \lambda_m(t) = \delta_{nm}. \quad (\text{A.10})$$

Both operators have the same set of eigenfunctions, which are

$$\lambda_n(t) = \sqrt{\frac{2\omega}{\pi}} \sin\left\{n\omega\left(t - \frac{\pi}{2\omega}\right)\right\}. \quad (\text{A.11})$$

It is easily seen that

$$\omega_n^{(0)2} = (n^2 + \alpha)\omega^2 \quad (\text{A.12})$$

and

$$\omega_n^{(1)2} = (n^2 - 1)\omega^2. \quad (\text{A.13})$$

Thus,

$$\begin{aligned} \det \mathcal{O}_0 &= \prod_{n=1}^{\infty} (n^2 + \alpha)\omega^2 \\ \det' \mathcal{O}_1 &= \prod_{n=2}^{\infty} (n^2 - 1)\omega^2. \end{aligned} \quad (\text{A.14})$$

In the \det' case, we have removed the zero eigenvalue $\omega_1^{(1)2}$. This gives the following result,

$$\sqrt{\frac{\det \mathcal{O}_0}{\det' \mathcal{O}_1}} = \omega \sqrt{(1 + \alpha) \prod_{n=2}^{\infty} \left(\frac{n^2 + \alpha}{n^2 - 1}\right)}, \quad (\text{A.15})$$

where α is given in (A.5).

Now, in order that we respect assumptions made in section 2, specifically equation (2.16), we must take $g \ll 1$. However, since the factor $\exp(-\frac{\pi}{2g})$ which appears in (A.1) rapidly becomes incredibly small as g becomes smaller than .01, where it has a value $\sim 10^{-68}$, we consider a “reasonable” range of g to be between .01 and .1. In this way we consider circumstances in line with our assumptions but which don’t allow such a suppression of instanton effects as to make them physically uninteresting. We point out that for $g = .1$ and .05, $\exp(-\frac{\pi}{2g})$ is $\sim 10^{-7}$ and $\sim 10^{-14}$ respectively. Now, we have evaluated (A.15) numerically for various reasonable values of g , and we find that for $g = .1$, .05, and .01, that equation (A.15) becomes 1.03ω , 1.49ω , and 4.56ω respectively. Since these values are all ω times a factor of $\mathcal{O}(1)$, and since it is difficult to obtain a more compact closed-form expression for equation (A.15) which is valid over the “reasonable” range of g , it is useful, over this range of g , to simply take

$$\sqrt{\frac{\det \mathcal{O}_0}{\det' \mathcal{O}_1}} \approx \omega, \quad (\text{A.16})$$

which we will do for definiteness. Using (A.1), we then arrive at the following result

$$\mathcal{M} \approx \omega \sqrt{\frac{\pi}{2g}} e^{-\frac{\pi}{2g}}. \quad (\text{A.17})$$

To conclude, in this Appendix we have shown, regardless of these concerns, that for small values of g , the characteristic nonperturbative mass scale in the collective field theory is as given in equation (A.17).

Appendix B: Calculation of h_{mn}

In this appendix we calculate the leading order behaviour of the coefficients h_{mn} . From Eq.(3.76) we see that it is enough to calculate

$$h_{>mn}^{(+)} = \frac{\omega^{m+n+1}}{m!n!} \int_{-\frac{\pi}{2\omega}}^{\frac{\pi}{2\omega}} dt \int_{\epsilon_\tau}^{\infty} d\tau \mathcal{J}_{>}^{(+)}(\tau, t) \tau^m t^n. \quad (\text{B.1})$$

First, rescale $q = \omega t$, that simply takes away $n + 1$ powers of ω and sets the integration boundaries to $-\pi/2, +\pi/2$ then substitute the expression for $\mathcal{J}_{>}^{(+)}$ from Eq.(3.71) into Eq.(B.1)

$$h_{mn} = \frac{\omega^m}{m!n!} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dq \int_{\epsilon_\tau}^{\infty} d\tau \left[\frac{1}{(x(\tau - \tau_0 - \frac{\sigma}{2}) - \lambda_0^{(+)}(q))^2} - \frac{1}{(x(\tau - \tau_0 - \frac{\sigma}{2}) - \lambda_\emptyset^{(+)}(q))^2} \right] \tau^m q^n. \quad (\text{B.2})$$

Now, substitute the expression for $x(\tau - \tau_0 - \frac{\sigma}{2})$ and rescale $r = \omega\tau$ to obtain

$$h_{mn} = \frac{1}{m!n!} \frac{1}{\omega} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dq \int_{\epsilon_r}^{\infty} dr \left[\frac{1}{(A \cosh r - \lambda_0^{(+)}(q))^2} - \frac{1}{(A \cosh r - \lambda_\emptyset^{(+)}(q))^2} \right] r^m q^n, \quad (\text{B.3})$$

where, using Eq.(3.62), $\epsilon_r = \frac{1}{\sqrt{2}}(3\pi g)^{1/3}$. Note that ϵ_r is a dimensionless number. Then substitute the expressions for $\lambda_0^{(+)}(q)$, $\lambda_\emptyset^{(+)}(q)$ to obtain

$$\begin{aligned} h_{mn} &= \frac{g}{m!n!} \left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dq \int_{\epsilon_r}^{\infty} dr \frac{1}{(\cosh r - \sin(q))^2} r^m q^n \right. \\ &\quad - \int_{-\frac{\pi}{2}}^0 dq \int_{\epsilon_r}^{\infty} dr \frac{1}{(\cosh r + 1)^2} r^m q^n \\ &\quad \left. - \int_0^{\frac{\pi}{2}} dq \int_{\epsilon_r}^{\infty} dr \frac{1}{(\cosh r - 1)^2} r^m q^n \right]. \end{aligned} \quad (\text{B.4})$$

where we used $g = \frac{1}{\omega A^2}$. From Eq.(B.4) we can see that h_{mn} are finite dimensionless numbers for all m, n .

To estimate the leading g dependence in the small g limit to the coefficients h_{mn} note that the main contribution to h_{mn} , for $m \leq 3$, comes from regions that are close to the lower boundary of integration. In this region, the third term in the previous equation is always much larger than the first two terms. We can then write, to leading order in g ,

$$\begin{aligned} h_{mn} &= -\frac{g}{m!n!} \int_0^{\frac{\pi}{2}} dq \int_{\epsilon_r}^{\infty} dr \frac{1}{(\cosh r - 1)^2} r^m q^n \\ &= -\frac{g(\pi/2)^{n+1}}{m!(n+1)!} \int_{\epsilon_r}^{\infty} dr \frac{1}{(\cosh r - 1)^2} r^m. \end{aligned} \quad (\text{B.5})$$

The leading g dependence of the coefficients, for $m \leq 3$, can therefore be calculated to be

$$h_{mn} \approx g \epsilon_r^{m-3} = g^{m/3}. \quad (\text{B.6})$$

For $m > 3$, all the terms in Eq.(B.4) are equally important and, to leading order, their g -dependence is give by the overall factor of g . The actual results for the first few coefficients are give by

$$\begin{aligned} h_{00} &= -\frac{4\sqrt{2}}{9} \\ h_{10} &= -\left(\frac{8\pi g}{9}\right)^{1/3} \\ h_{01} &= -\frac{\pi\sqrt{2}}{9} \end{aligned} \quad (\text{B.7})$$

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