

CERN-TH-7302/94

GEF-Th-6/1994

A FUNCTIONAL AND LAGRANGIAN FORMULATION OF TWO DIMENSIONAL TOPOLOGICAL GRAVITY*

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ABSTRACT

We reconsider two-dimensional topological gravity in a functional and lagrangian framework. We derive its Slavnov-Taylor identities and discuss its (in)dependence on the background gauge. Correlators of reparamerization invariant observables are shown to be globally defined forms on moduli space. The potential obstruction to their gauge-independence is the non-triviality of the line bundle on moduli space \mathcal{L}_x , whose first Chern-class is associated to the topological invariants of Mumford, Morita and Miller.

HEP-TH-9406096

CERN-TH 7302/94

GEF-Th 6/1994

June 1994

* This work is partially supported by the ECRP, contract SC1-CT92-0789.

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1. Introduction

Two-dimensional topological gravity [1] is locally trivial as any topological field theory and its "physical" content reduces to the coordinates of the moduli space of the "world-sheet" Riemann surfaces. It is natural to consider these variables of infrared nature. However, it has been shown [2] that the short distances play a crucial and unexpected rôle in topological gravity. It turns out that, for a suitable parametrization of the moduli space, almost all the correlation functions of the known "observables" of the theory can be made to vanish at the interior points of this space. Their only non-trivial contributions come from contact terms with the possible nodes of the "world-sheet", that is from the boundary of moduli space. This looks very much like an ultraviolet phenomenon.

To have a better understanding of this double nature of the correlation functions it is necessary to use a local formulation of the theory. This motivates the present paper, which seeks to define a field functional framework for two-dimensional topological gravity.

With the same purpose in mind, we have discussed and identified in [3] the field structure of a class of "observables" that are covariant under reparametrizations of the "world-sheet" and that correspond to the known relevant operators of the theory. To compute their correlation functions it is necessary to identify a non-degenerate action for the theory. The existence of $6g - 6$ moduli parametrizing the possible inequivalent complex structures of surfaces of genus g automatically induces the existence of the same number of zero-modes for each antighost field. It is therefore necessary to introduce a suitable gauge fixing for these zero-modes.

After the introduction of the suitable gauge fixing, the theory is characterized by a family of Slavnov-Taylor identities exhibiting its BRS invariance. We show that, in the case of BRS invariant operators that are independent of the antighost zero-modes, these identities signify that the correlation functions are closed forms on the moduli space. To understand if these correlation functions are globally defined we study their properties under arbitrary variations of the gauge slice. We show that in general such variations change the correlation functions by terms which are locally exact forms on the moduli space. However these terms vanish in the case of modular transformations constant on moduli space. From this we conclude that the correlation functions are globally defined forms on moduli space when one chooses gauge slices whose transition functions are modular transformations. We also point out that a potential obstruction to the background gauge independence of such globally defined forms is the non-triviality of the line bundle \mathcal{L}_x associated to the topological invariants of Mumford, Morita and Miller [4]-[7].

The paper is thus organized: in the following section 2, we revisit the known observables of the theory; Section 3 contains the construction of the Lagrangian; the Slavnov Taylor identities are discussed in Section 4; and the variations of the gauge-slice in section 5.

2. The Observables

Two-dimensional topological gravity [1] is a topological quantum field theory characterized by the following BRS transformation laws:

$$\begin{aligned} s g_{\mu\nu} &= \mathcal{L}_c g_{\mu\nu} + \psi_{\mu\nu} \\ s \psi_{\mu\nu} &= \mathcal{L}_c \psi_{\mu\nu} - \mathcal{L}_\gamma g_{\mu\nu}, \end{aligned} \quad (1)$$

where $g_{\mu\nu}$ is the two-dimensional metric, $\psi_{\mu\nu}$ is the gravitino field, c^μ is the ghost vector field and γ^μ is the superghost vector field. \mathcal{L}_c and \mathcal{L}_γ denote the action of infinitesimal diffeomorphisms with parameters c^μ and γ^μ respectively.

A class of observables local in the fields $g_{\mu\nu}, \psi_{\mu\nu}, c^\mu$ and γ^μ can be constructed [2],[3] starting from the Euler two-form

$$\sigma^{(2)} = \frac{1}{2} \sqrt{g} R \epsilon_{\mu\nu} dx^\mu \wedge dx^\nu, \quad (2)$$

where R is the two-dimensional scalar curvature and $\epsilon_{\mu\nu}$ is the antisymmetric numeric tensor defined by $\epsilon_{12} = 1$. Since s and the exterior differential d on the two-dimensional world-sheet commute among themselves, the two-form in Eq.(2) gives rise to the descent equations:

$$\begin{aligned} s\sigma^{(2)} &= d\sigma^{(1)} \\ s\sigma^{(1)} &= d\sigma^{(0)} \\ s\sigma^{(0)} &= 0. \end{aligned} \quad (3)$$

The zero-form $\sigma^{(0)}$ and the one-form $\sigma^{(1)}$ are computed to be

$$\begin{aligned} \sigma^{(0)} &= \sqrt{g} \epsilon_{\mu\nu} \left[\frac{1}{2} c^\mu c^\nu R + c^\mu D_\rho (\psi^{\nu\rho} - g^{\nu\rho} \psi_\sigma^\sigma) + D^\mu \gamma^\nu - \frac{1}{4} \psi_\rho^\mu \psi^{\nu\rho} \right] \\ \sigma^{(1)} &= \sqrt{g} \epsilon_{\mu\nu} [c^\nu R + D_\rho (\psi^{\nu\rho} - g^{\nu\rho} \psi_\sigma^\sigma)] dx^\mu. \end{aligned} \quad (4)$$

The Euler form, being a topological invariant, is locally d -exact,

$$\sigma^{(2)} = d\omega^{(1)}. \quad (5)$$

This, together with the descent equations (3), implies that

$$\begin{aligned}\sigma^{(1)} &= d\omega^{(0)} + s\omega^{(1)} \\ \sigma^{(0)} &= s\omega^{(0)}.\end{aligned}\tag{6}$$

Thus $\sigma^{(0)}$ is locally BRS trivial. However, since $\omega^{(1)}$ cannot be chosen to be a globally defined 1-form, it follows from Eqs. (6) that $\omega^{(0)}$ cannot be chosen to be a globally defined scalar field either. This means that $\sigma^{(0)}$ is a non-trivial class in the cohomology of s acting on the space of the *reparametrization covariant* tensor fields. One can verify explicitly that such cohomology is in one-to-one correspondence with the *semi-relative* state BRS cohomology defined on the state space of the theory quantized on the infinite cylinder in the conformal gauge. In fact, choosing a complex structure μ on the two-dimensional surface, with $(ds)^2 \sim |dz + \mu d\bar{z}|^2$, one obtains

$$\begin{aligned}\omega^{(1)} &= \frac{2}{\Theta} \left[\partial\mu - \mu\bar{\partial}\bar{\mu} + \frac{1}{2}(\bar{\nabla} - \mu\nabla)\ln\Theta \right] d\bar{z} - c.c. \\ \omega^{(0)} &= \nabla c^z + \bar{c}^{\bar{z}}\omega_{\bar{z}}^{(1)} + \frac{\bar{\mu}\psi_z}{\Theta} - c.c.,\end{aligned}\tag{7}$$

where $\Theta \equiv 1 - \mu\bar{\mu}$, $\nabla \equiv \partial - \mu\bar{\partial}$, c^z and ψ_z are the holomorphic components of the ghost field and of the traceless part of the gravitino fields, and *c.c.* denotes the complex conjugate expression in which all quantities are substituted with their barred expressions. In the (super)-conformal gauge,

$$\omega^{(0)} = \partial c - \bar{\partial}\bar{c},\tag{8}$$

and, at the level of states:

$$\omega^{(0)}(0)|0\rangle = c_0^-|0\rangle,\tag{9}$$

where $|0\rangle$ is the $SL(2, C)$ invariant conformal vacuum. Thus the state created by the operator $\sigma^{(0)}(0)$ — the so-called “dilaton” state — is non-trivial in the cohomology of the BRS operator acting on the space of states annihilated by b_0^- :

$$\sigma^{(0)}|0\rangle = s(c_0^-|0\rangle).\tag{10}$$

This is known, in the operator formalism, as the semi-relative BRS state cohomology [8], [9].

Since the superghosts γ^μ are commutative, one can build an infinite tower of cohomologically non-trivial operators by taking arbitrary powers of $\sigma^{(0)}$:

$$\sigma_n^{(0)} \equiv (\sigma^{(0)})^n\tag{11}$$

with $n = 0, 1, \dots$. The corresponding 2-forms

$$\sigma_n^{(2)} = n(\sigma_n^{(0)})^{n-1}\sigma^{(2)} + \frac{n(n-1)}{2}(\sigma_n^{(0)})^{n-2}\sigma_n^{(1)} \wedge \sigma_n^{(1)} \quad (12)$$

all belongs in the s -cohomology modulo d on the space of the reparametrization covariant tensor fields.

3. The Lagrangian

In order to evaluate correlators of observables σ_n , the choice of a lagrangian is required. The theory being topological, the choice of a lagrangian amounts to fixing the gauge.

Let \mathcal{M}_g be the moduli space of two-dimensional Riemann surfaces of a given genus g , and let $m = (m^i)$, with $i = 1, \dots, 6g - 6$, be local coordinates on \mathcal{M}_g . Fixing the gauge means choosing a background metric $\bar{g}_{\mu\nu}(x; m)$ for each gauge equivalence class of metrics corresponding to the point m of \mathcal{M}_g .

It is convenient to decompose $\bar{g}_{\mu\nu}$ as follows:

$$\bar{g}_{\mu\nu}(x; m) \equiv \sqrt{\hat{g}}\hat{g}_{\mu\nu}(x; m) \equiv e^{\hat{\varphi}}\hat{g}_{\mu\nu}(x; m), \quad (13)$$

with $\det(\hat{g})_{\mu\nu} = 1$. $\hat{g}_{\mu\nu}$ is given by the analogous definition for $g_{\mu\nu}$. We also introduce the the tensor density,

$$\hat{\psi}^{\mu\nu} \equiv \sqrt{\hat{g}}(\psi^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\psi^\sigma_\sigma),$$

in correspondence with the traceless part of the gravitino field. $\bar{g}_{\mu\nu}$ defines a gauge-slice on the field space whose associated lagrangian reads as follows:

$$\mathcal{L} = s \left[\frac{1}{2}b_{\mu\nu}(\hat{g}^{\mu\nu} - \hat{g}^{\mu\nu}) + \frac{1}{2}\beta_{\mu\nu}(\hat{\psi}^{\mu\nu} - d_P\hat{g}^{\mu\nu}) + \chi\partial_\mu(\hat{g}^{\mu\nu}\partial_\nu(\varphi - \bar{\varphi})) \right]. \quad (14)$$

In Eq.(14) we have introduced the ‘‘exterior derivative’’ operator

$$d_P \equiv p^i \frac{\partial}{\partial m^i},$$

where p^i are the anti-commuting supermoduli, with $i = 1, \dots, 6g - 6$, the superpartners of the commuting moduli m^i . $b_{\mu\nu}$, $\beta_{\mu\nu}$ and χ are the anti-ghost fields, with ghost numbers -1 , -2 and 0 respectively. Their BRS transformation laws are given by

$$\begin{aligned} s b_{\mu\nu} &= \Lambda_{\mu\nu}, & s \Lambda_{\mu\nu} &= 0 \\ s \beta_{\mu\nu} &= L_{\mu\nu}, & s L_{\mu\nu} &= 0 \\ s \chi &= \mathcal{L}_c \chi + \pi, & s \pi &= \mathcal{L}_c \pi - \mathcal{L}_\gamma \chi, \end{aligned} \quad (15)$$

where $\Lambda_{\mu\nu}$, $L_{\mu\nu}$ and π are Lagrangian multipliers. The Lagrangian in Eq.(14) written out in extended form reads:

$$\begin{aligned}\mathcal{L} = & \frac{1}{2}\Lambda_{\mu\nu}(\hat{g}^{\mu\nu} - \hat{\hat{g}}^{\mu\nu}) + \frac{1}{2}L_{\mu\nu}(\hat{\psi}^{\mu\nu} - d_P\hat{\hat{g}}^{\mu\nu}) \\ & - \frac{1}{2}b_{\mu\nu}\mathcal{L}_c\hat{g}^{\mu\nu} - \frac{1}{2}\beta_{\mu\nu}\mathcal{L}_\gamma\hat{g}^{\mu\nu} \\ & + \frac{1}{2}\hat{\psi}^{\mu\nu}[(\mathcal{L}_c\beta)_{\mu\nu} + b_{\mu\nu} + 2\partial_\mu\chi\partial_\nu(\varphi - \bar{\varphi})] \\ & + \pi\partial_\mu(\hat{g}^{\mu\nu}\partial_\nu(\varphi - \bar{\varphi})) - \chi\partial_\mu(\hat{g}^{\mu\nu}\partial_\nu\psi'),\end{aligned}\tag{16}$$

where

$$\psi' \equiv \bar{D}_\sigma c^\sigma + \frac{1}{2}\psi^\sigma_\sigma.\tag{17}$$

Eq.(16) shows the standard form of a topological Lagrangian that, however, in the present case is degenerate due to the presence of $6g-6$ moduli. Indeed, $\beta^{(i)} \equiv \int \beta_{\mu\nu} \frac{\partial}{\partial m^i} \hat{g}^{\mu\nu}$, define $6g-6$ zero-modes of the antighost field β .

To remove this degeneracy we must introduce further gauge fixing terms. The natural way of generating these terms is to extend the definition of the BRS generator s , assuming the following BRS transformation laws for the moduli and super-moduli:

$$\begin{aligned}s m^i &= C^i, & s C^i &= 0 \\ s p^i &= -\Gamma^i, & s \Gamma^i &= 0,\end{aligned}\tag{18}$$

where C^i and Γ^i are respectively anti-commuting and commuting Lagrange multipliers.

After this extension of the BRS transformations four new terms appear in the lagrangian:

$$\frac{1}{2}\beta_{\mu\nu}d_\Gamma\hat{g}^{\mu\nu} + \frac{1}{2}b_{\mu\nu}d_C\hat{g}^{\mu\nu} + \frac{1}{2}\beta_{\mu\nu}d_Pd_C\hat{g}^{\mu\nu} + \chi\partial_\mu(\hat{g}^{\mu\nu}\partial_\nu d_C\bar{\varphi}),\tag{19}$$

where the notation $d_C \equiv C^i \frac{\partial}{\partial m^i}$ and $d_\Gamma \equiv \Gamma^i \frac{\partial}{\partial m^i}$ has been introduced. The first term in Eq.(19) is the wanted zero mode fixing term. The second term generates the compensating term for its determinant.

After this second gauge fixing, integrating out the Lagrangian multipliers $\Lambda_{\mu\nu}$, $L_{\mu\nu}$, π' and χ forces the metric and the gravitino field to take their background values,

$$\begin{aligned}\hat{g}^{\mu\nu} &\rightarrow \hat{\hat{g}}^{\mu\nu}, & \varphi &\rightarrow \bar{\varphi}, & \hat{\psi}^{\mu\nu} &\rightarrow d_P\hat{\hat{g}}^{\mu\nu} \\ \frac{1}{2}\psi^\sigma_\sigma + \bar{D}_\sigma c^\sigma &\rightarrow d_C\bar{\varphi},\end{aligned}\tag{20}$$

and the lagrangian becomes

$$\begin{aligned}\mathcal{L}' = & \frac{1}{2}(-b_{\mu\nu}\mathcal{L}_c\hat{\hat{g}}^{\mu\nu} - \beta_{\mu\nu}\mathcal{L}_\gamma\hat{\hat{g}}^{\mu\nu} + d_P\hat{\hat{g}}^{\mu\nu}(\mathcal{L}_c\beta)_{\mu\nu} \\ & + b_{\mu\nu}(d_C\hat{\hat{g}}^{\mu\nu} - d_P\hat{\hat{g}}^{\mu\nu}) + \beta_{\mu\nu}d_\Gamma\hat{\hat{g}}^{\mu\nu} + \beta_{\mu\nu}d_Pd_C\hat{\hat{g}}^{\mu\nu}).\end{aligned}\tag{21}$$

In the following we will repeatedly make use of the fact that, when the observables do not contain the anti-ghost zero modes $b^{(i)}$, integrating them out introduces into the correlators the factor

$$\prod_{i=1}^{6g-6} \delta(\mathcal{C}^i - p^i). \quad (22)$$

If moreover there are no antighost zero modes $\beta^{(i)}$ and no antighost fields $b_{\mu\nu}$ in the observables, one can integrate out $\beta^{(i)}$ as well. This produces a further factor

$$\prod_{i=1}^{6g-6} \delta(\Gamma^i). \quad (23)$$

4. B.R.S. Identities

Let us now consider expectation values of observables $O_{n_\alpha}(\Phi) \equiv \int \sigma_{n_\alpha}^{(2)}$, where by Φ we denote collectively all the quantum fields but the moduli and the super-moduli m^i and p^i . It is useful to consider functional averages in which one integrates only with respect to the quantum fields Φ :

$$\begin{aligned} Z_{\{n_\alpha\}}(m^i, p^i) &\equiv \int [d\Phi] e^{-S(\Phi; m^i, p^i)} \prod_{\alpha} O_{n_\alpha}(\Phi) \\ &\equiv \langle \prod_{\alpha} O_{n_\alpha}(\Phi) \rangle. \end{aligned} \quad (24)$$

Because of ghost number conservation, $Z_{\{n_\alpha\}}$ is a monomial of the anti-commuting super-moduli:

$$Z_{\{n_\alpha\}}(m^i, p^i) = Z_{i_1 \dots i_N}(m^{i_1} p^{i_1} \dots p^{i_N}), \quad (25)$$

where N is the total ghost number of the observables O_{n_α} :

$$N = \sum_{\alpha} (\text{ghost} \# O_{n_\alpha}) = 2 \sum_{\alpha} (n_\alpha - 1). \quad (26)$$

Under a reparametrization $\tilde{m}^i = \tilde{m}^i(m)$ of the local coordinates m^i on the moduli space \mathcal{M}_g , the supermoduli transform as follows:

$$\tilde{p}^i = \frac{\partial \tilde{m}^i}{\partial m^j} p^j.$$

One can therefore identify the anti-commuting supermoduli with the differentials on the moduli space, i.e. $p^i \rightarrow dm^i$. Correspondingly, the function $Z_{\{n_\alpha\}}(m^i, p^i)$ of the moduli

and super-moduli can be thought of as a N-form on the moduli space \mathcal{M}_g , at least *locally* on \mathcal{M}_g . The question of whether or not such local form extends to a globally defined form on \mathcal{M}_g , will be addressed shortly.

Assume for the moment that form $Z_{\{n_\alpha\}}(m)$ is globally defined on \mathcal{M}_g . Whenever the following ghost number selection rule is satisfied,

$$N = 2 \sum_{\alpha} (n_{\alpha} - 1) = 6g - 6, \quad (27)$$

$Z_{\{n_\alpha\}}(m)$ defines a measure on \mathcal{M}_g which can be integrated to produce some number. The collection of these numbers encode the gauge-invariant content of two-dimensional topological gravity.

It is easy to show that the action of BRS operator s on the quantum fields Φ translates into the action of the exterior differential $d_P \equiv p^i \partial_i$ on the forms $Z_{\{n_\alpha\}}(m)$. More precisely, one can prove the following BRS identities:

$$\begin{aligned} (i) \quad s O_{n_\alpha}(\Phi) = 0 &\Rightarrow d_P Z_{\{n_\alpha\}}(m) = 0 \\ (ii) \quad O_{n_\alpha} = s X_{n_\alpha}(\Phi) &\Rightarrow Z_{\{n_\alpha\}}(m) = d_P W(m) \end{aligned} \quad (28)$$

where $W(m) \equiv \langle X_{n_\alpha} \prod_{\beta \neq \alpha} O_{n_\beta} \rangle$, with the provision that neither O_{n_α} nor X_{n_α} contains the anti-ghost zero modes and the antighost field b .

The proof of (i), for example, goes as follows:

$$\begin{aligned} d_P Z_{\{n_\alpha\}} &= \langle -d_P S \prod_{\alpha} O_{n_\alpha}(\Phi) \rangle \\ &= \langle -d_C S \prod_{\alpha} O_{n_\alpha}(\Phi) \rangle \\ &= \langle \sum_{\Phi} s \Phi \frac{\delta S}{\delta \Phi} \prod_{\alpha} O_{n_\alpha}(\Phi) \rangle \\ &= \langle s \prod_{\alpha} O_{n_\alpha}(\Phi) \rangle = 0, \end{aligned} \quad (29)$$

since, under the stated conditions, one can perform the substitutions $\mathcal{C}^i \rightarrow p^i$ and $\Gamma^i \rightarrow 0$ (see Eqs.(22) and (23)). The proof of (ii) is analogous.

5. (In)dependence on the gauge and global properties

We would like to study the dependence of $Z_{\{n_\alpha\}}(m)$ on the gauge-fixing function $\bar{g}_{\mu\nu}(x; m)$ which specifies for each point m of the moduli space \mathcal{M}_g a representative two-dimensional metric.

Given the gauge-slice corresponding to $\bar{g}_{\mu\nu}(x; m)$, a different gauge choice corresponding to $\bar{g}'_{\mu\nu}(x; m)$ will be related to $\bar{g}_{\mu\nu}$ by a diffeomorphism $\xi^\mu(x; m)$ depending on m :

$$\bar{g}'_{\mu\nu}(x; m) = \frac{\partial \xi^\rho}{\partial x^\mu} \frac{\partial \xi^\sigma}{\partial x^\nu} \bar{g}_{\rho\sigma}(x; m). \quad (30)$$

Consider for simplicity the case of an infinitesimal diffeomorphism $\xi^\mu \sim x^\mu + v^\mu(x; m)$, for which

$$\bar{g}'_{\mu\nu}(x; m) \sim \bar{g}_{\mu\nu}(x; m) + (\mathcal{L}_v \bar{g})_{\mu\nu}(x; m). \quad (31)$$

Denoting by W_v the action of the infinitesimal diffeomorphism in Eq.(31) on the form $Z_{\{n_\alpha\}}$, one has

$$\begin{aligned} W_v Z_{\{n_\alpha\}} &= \left\langle \int s \frac{1}{2} [b_{\mu\nu}(\mathcal{L}_v \hat{g})^{\mu\nu} + \beta_{\mu\nu} d_P(\mathcal{L}_v \hat{g})^{\mu\nu} + 2\chi \partial_\mu(\hat{g}^{\mu\nu} \partial_\nu \mathcal{L}_v \bar{\varphi})] \prod_\alpha O_{n_\alpha} \right\rangle \\ &= \left\langle s \int \frac{1}{2} [b_{\mu\nu}(\mathcal{L}_v \hat{g})^{\mu\nu} - (\mathcal{L}_v \beta)_{\mu\nu} d_P \hat{g}^{\mu\nu} + \beta_{\mu\nu} (\mathcal{L}_{d_P v} \hat{g})^{\mu\nu} + 2\chi \partial_\mu(\hat{g}^{\mu\nu} \partial_\nu \mathcal{L}_v \bar{\varphi})] \prod_\alpha O_{n_\alpha} \right\rangle \\ &= \left\langle s \int I_v S \prod_\alpha O_{n_\alpha} \right\rangle, \end{aligned} \quad (32)$$

where $\mathcal{L}_{d_P v} \equiv p^i \mathcal{L}_{\partial_i v}$ and where we introduced the operator I_v acting on the fields Φ as follows:

$$\begin{aligned} I_v c^\mu &= v^\mu(x; m) \\ I_v \gamma^\mu &= -d_P v^\mu(x; m) \\ I_v \Phi &= 0 \quad \text{for the other fields.} \end{aligned} \quad (33)$$

By considering the anticommutator of I_v and s ,

$$\hat{\mathcal{L}}_v = \{s, I_v\}, \quad (34)$$

one obtains an operator $\hat{\mathcal{L}}_v$ acting on a generic field supermultiplet $\Phi \equiv (\phi, \hat{\phi})$ as follows:

$$\begin{aligned} \hat{\mathcal{L}}_v \phi &= \mathcal{L}_v \phi \\ \hat{\mathcal{L}}_v \hat{\phi} &= \mathcal{L}_v \hat{\phi} + \mathcal{L}_{d_P v} \phi. \end{aligned} \quad (35)$$

In terms of I_v and $\hat{\mathcal{L}}_v$, the Ward identity in Eq.(32) becomes:

$$\begin{aligned} W_v Z_{\{n_\alpha\}} &= \langle s I_v \prod_{\alpha} O_{n_\alpha} \rangle = \langle \hat{\mathcal{L}}_v \prod_{\alpha} O_{n_\alpha} \rangle \\ &= d_P \langle I_v \prod_{\alpha} O_{n_\alpha} \rangle, \end{aligned} \tag{36}$$

where in the last line we used (ii) of Eq.(28). Eq.(36) is the fundamental equation to understand both the global properties of $Z_{\{n_\alpha\}}$ and its gauge dependence.

Let us consider first the question of the global definition of $Z_{\{n_\alpha\}}$. What makes this property not obvious is that $\bar{g}_{\mu\nu}(x; m)$ cannot be chosen to be a continuous function of m . In fact $\bar{g}_{\mu\nu}(x; m)$ is a section of the bundle over \mathcal{M}_g defined by the space of two-dimensional metrics of given genus. This bundle is not trivial, and therefore it does not admit a global section. On some coordinate patch on \mathcal{M}_g , $\bar{g}_{\mu\nu}$ must jump to a $\bar{g}'_{\mu\nu}$ related to $\bar{g}_{\mu\nu}$ by a diffeomorphism $\xi^\mu(x; m)$ as in Eq.(30). From Eq.(36) it follows that, for a generic local section $\bar{g}_{\mu\nu}$, $Z_{\{n_\alpha\}}$ jumps as well, and hence is not globally defined. However, the bundle in question is non-trivial only because of modular transformations. This means that it is possible to choose sections $\bar{g}_{\mu\nu}$ whose transition functions $\xi^\mu(x; m)$ are m -independent modular transformations.

Eq.(35) implies that for infinitesimal diffeomorphisms $v^\mu(x; m)$ which are independent on m , the operator $\hat{\mathcal{L}}_v$ reduces to usual diffeomorphisms, i.e.

$$\partial_i v^\mu = 0 \Rightarrow \hat{\mathcal{L}}_v = \mathcal{L}_v. \tag{37}$$

Thus, under such diffeomorphisms and for observables which are reparametrization invariant, $Z_{\{n_\alpha\}}$ is invariant:

$$W_v Z_{\{n_\alpha\}} = \langle \mathcal{L}_v \prod_{\alpha} O_{n_\alpha} \rangle = 0. \tag{38}$$

It is not difficult to see that this remains true for finite, m -independent diffeomorphisms ξ^μ .

In conclusion, the Ward identity in Eq.(36) ensures that if we consider sections $\bar{g}_{\mu\nu}$ whose transition functions are m -independent modular transformations and restrict ourselves to reparametrization invariant observables, the corresponding $Z_{\{n_\alpha\}}$ are globally defined forms on the moduli space \mathcal{M}_g . For the same reason, observables like $\sigma^{(0)} = s \omega^{(0)}$ which are locally BRS trivial, give rise to forms $\langle \sigma^{(0)} \dots \rangle = d_P \langle \omega^{(0)} \dots \rangle$ which are locally but not globally exact on \mathcal{M}_g . This is the reason why the relevant BRS cohomology

is the BRS cohomology acting on the space of reparametrization covariant tensor fields. As we have seen, this cohomology precisely corresponds to the semi-relative b_0^- BRS state cohomology of the operator formalism.

Eq.(36) also implies that two different sections $\bar{g}_{\mu\nu}$ and $\bar{g}'_{\mu\nu}$, whose transition functions are modular transformations, give rise to globally defined forms $Z_{\{n_\alpha\}}$ and $Z_{\{n_\alpha\}}'$ which differ by a locally exact form on \mathcal{M}_g ,

$$Z_{\{n_\alpha\}} - Z_{\{n_\alpha\}}' = d_P \langle I_v \prod_\alpha O_{n_\alpha} \rangle, \quad (39)$$

where $v = (v^\mu(m; x))$ is the infinitesimal diffeomorphism relating $\bar{g}_{\mu\nu}$ to $\bar{g}'_{\mu\nu}$. Gauge independence of the theory would require that the integral over \mathcal{M}_g of the form in the r.h.s. of Eq. (39) vanishes. Thus, the question of independence on the background metric $\bar{g}_{\mu\nu}$ is the one of the global definition of the form $\langle I_v \prod_\alpha O_{n_\alpha} \rangle$.

The vector field $v^\mu(m; x)$, for a fixed point x on the world-sheet, is a section of the complex line bundle \mathcal{L}_x on \mathcal{M}_g , whose fiber is the cotangent space of the world-sheet at x . This bundle is non-trivial, and its first Chern-class $c_1(\mathcal{L}_x)$ is related to the topological invariants of Mumford, Morita and Miller [4]-[7]. It follows that $c_1(\mathcal{L}_x)$ is precisely the obstruction to choose a globally defined $v^\mu(m; x)$. Since $v^\mu(m; x)$ is not globally defined, the form in the r.h.s. of Eq.(39) is potentially not globally exact.

In this sense, the non-trivial content of two-dimensional topological gravity is captured by the ‘‘anomaly’’ Equation (36). The work of Verlinde and Verlinde [2] suggests that there exists a suitable redefinition of the functional measure of two-dimensional gravity at the boundary of the moduli space \mathcal{M}_g which restores background gauge independence while preserving BRS invariance. We hope to come back to this issue in the future.

Acknowledgements

It is a pleasure to thank R. Stora for interesting discussions.

This work is carried out in the framework of the European Community Research Programme ‘‘Gauge theories, applied supersymmetry and quantum gravity’’, with a financial contribution under contract SC1-CT92-0789.

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