Particle Accelerators, 1994, Vol. 48, pp. 135–168
Reprints available directly from the publisher
Photocopying permitted by license only

ANALYTIC CRITERIA FOR STABILITY OF BEAM-LOADED RADIO-FREQUENCY SYSTEMS

SHANE KOSCIELNIAK

TRIUMF, 4004 Wesbrook Mall, Vancouver, B.C., Canada V6T 2A3

(Received 31 March 1994; in final form 24 June 1994)

This paper presents the instability analysis of a beam-loaded radio-frequency system with beam phase-loop and cavity tuning-loop for both accelerating and non-accelerating beams. In addition, the case of rf voltage-proportional feedback around the cavity is presented. The symbolic manipulation program SMP was used to expand and simplify the Routh determinantal conditions for a fifth order characteristic polynomial. In some cases, the conditions have easy physical interpretations and it is possible to give an analytic criterion for the threshold beam current. However, for the most part, the Routh conditions lead to simultaneous quadratic conditions on the beam current and loop gains. Finally, SMP was used to study the case of dipole-quadrupole mode coupling for an accelerating beam.

KEY WORDS: Beam loading, coherent instability, collective effects, high-power beams, instabilities, radio-frequency devices

1 INTRODUCTION

The topic of beam loaded radio-frequency systems dates back to Robinson's paper[2] of 1964. However, apart from the work of Pedersen[3] there has been little substantial advance in obtaining analytic stability criteria for realistic rf systems with a multitude of feedback loops. The stumbling block is one of mathematical complexity. The analysis proceeds as follows: (i) write the matrix equation for the system vector, (ii) expand the matrix determinant to find the characteristic polynomial, and (iii) apply the Routh[1] criteria to the coefficients of the polynomial. The order n of the characteristic polynomial scales roughly as the matrix dimension. The size of the polynomial scales roughly[a] with the square of the matrix dimension; and by size we mean the number of terms in each of the polynomial coefficients a_n. The largest member of the Routh criteria grows as the polynomial size to the power of $n-1$; and so a small change in the matrix dimension can result in an enormous increase in complexity. The case of beam and cavity alone, studied by Robinson, results in a fourth order polynomial and is tractable by hand-working. In general, adding a control loop with a single time constant will raise the order by one; so introducing two loops yields a sixth order polynomial, and (by human standards) there results "an astronomical number of

[a] We have assumed a sparse non-diagonal matrix. If the matrix is diagonal, then the scaling is linear.

terms" to contemplate and simplify. However, using a computer tool such as the symbolic manipulation program SMP[4] to keep track of all the terms the problem becomes feasible, though still not facile because the task of simplification requires substantial human skills of pattern recognition. Indeed, the author estimates that 80% of his time was spent attempting to reduce and simplify the Routh conditions.

In this paper, we concentrate on 5^{th} order systems; and briefly mention some results for 4^{th} and 6^{th} order characteristic polynomials. Though not all of the results are new, the presentation given here is more complete and more detailed than any other source the present author is aware of.

The initial part of this document, Sections 2, 3 and 4 is concerned with establishing the dynamical equations for the cavity, the beam and the control loops, respectively. In Section 5 we record the generic Routh criteria up to sixth order. In the remaining Sections 6 through 14, we build up and examine ever more complex systems using the components described in the preliminary Sections 2 through 4.

2 BEAM LOADING THEORY

The purpose of this section is to develop a model for the cavity response to small amplitude and phase modulations of the driving current vectors. The disposition of steady state phasors is as shown in Figure 1. We adopt the notation introduced in Reference 5. The cavity voltage is $\mathbf{V}(t)e^{j\omega t}$ and the total current driving the cavity is $\mathbf{I}_T(t)e^{j\omega t}$, where t indicates time and ω is the drive angular frequency. Bold face type will indicate complex quantities, and ordinary type face will denote pure scalars. We shall employ dot notation for time derivatives. The cavity fundamental resonance is modelled as a parallel resonance LCR circuit. Let $\Omega_{\text{res}} = 1/\sqrt{LC}$ be the cavity resonance frequency and $\alpha = \Omega_{\text{res}}/(2Q) = 1/(2RC)$ be the cavity half-bandwidth. As noted in Reference 5, the dynamical effect of sweeping the cavity resonance frequency is insignificant provided $\alpha \gg \mid \dot{\Omega}/\Omega \mid$ which is always the case. Hence the voltage and current obey the equation:

$$\ddot{\mathbf{V}} + 2(\alpha + j\omega)\dot{\mathbf{V}} + (\Omega^2 - \omega^2 + 2j\alpha\omega)\mathbf{V} = 2\alpha R\,(\dot{\mathbf{I}}_T + j\omega\mathbf{I}_T)\ . \tag{1}$$

We choose to write the voltage and current as the sum of steady state parts $\mathbf{V}^0 = V^0 e^{j\psi_V}$ and $\mathbf{I}_T^0 = I_T^0 e^{j\psi_T}$, and small time dependent perturbations. We shall use ψ to denote steady state phases and ϕ perturbation phases.

2.0.1. Steady state The steady state components obey the relation:

$$\frac{\left(\Omega^2 - \omega^2 + 2j\alpha\omega\right)}{2\alpha j\omega} = \frac{I_T^0}{I_V^0}\exp j(\psi_T - \psi_V) \qquad \text{where} \quad I_V^0 = \frac{V^0}{R}\ . \tag{2}$$

Let $\Psi = \psi_V - \psi_T$ be the phase difference of response and drive phasors. We compare real and imaginary parts and divide to eliminate the current moduli,

and so obtain: $\tan\Psi = \dfrac{(\Omega^2 - \omega^2)}{2\alpha\omega}$ which relation defines the tuning angle Ψ.

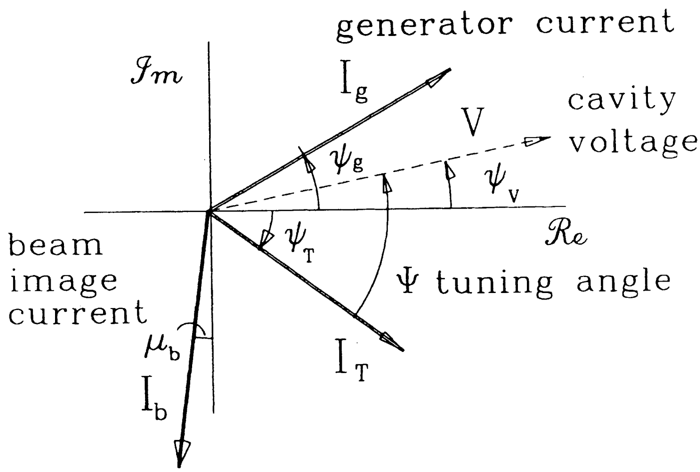

FIGURE 1: Steady-state phasor diagram for beam loaded cavity.

To further define the circumstance, we must specify the steady state generator current $\mathbf{I}_g^0 = I_g^0 e^{j\psi_g}$ and beam image current $\mathbf{I}_b^0 = I_b^0 e^{j\psi_b}$ which sum to form the total current $\mathbf{I}_T^0$. The beam current is approximately 90° out of phase with the cavity voltage; the precise relation depends on the synchronous phase angle μ_b. We adopt the convention $\mu_b = 0$ for a non-accelerating beam. Hence $\psi_b = \pm(\pi/2 + \mu_b)$, and the negative sign applies below transition energy and the positive above. With these assignments, the steady state relation between current vectors is:

$$\left(\Omega^2 - \omega^2 + 2j\alpha\omega\right) I_V^0 e^{j\psi_V} = 2\alpha j\omega \left[I_g^0 e^{j\psi_g} - jI_b^0 e^{-j\mu_b}\right] . \tag{3}$$

We divide throughout by $2j\alpha\omega \times I_V^0 e^{j\psi_V}$, and introduce the dimensionless current ratios $Y_g = I_g^0/I_V^0$ and $Y_b = I_b^0/I_V^0$. The steady state relation becomes:

$$1 - j\ \tan\Psi = Y_g e^{j(\psi_g - \psi_V)} - jY_b e^{-j(\mu_b + \psi_V)} . \tag{4}$$

We set $\psi_V = 0$ and then compare real and imaginary parts:

$$\mathcal{R}e \Rightarrow\ 1 = Y_g \cos\psi_g - Y_b \sin\mu_b\ , \tag{5a}$$

$$\mathcal{I}m \Rightarrow\ \tan\Psi = Y_b \cos\mu_b - Y_g \sin\psi_g . \tag{5b}$$

2.0.2. Non steady state We now find the relation between current and voltage vectors for small perturbations. Let us assume the $\exp(j\omega t)$ time dependence, and introduce the "slow approximation" $\ddot{\mathbf{V}} \ll \omega\dot{\mathbf{V}}$ and $\dot{\mathbf{I}}_T \ll \omega\mathbf{I}_T$. Since the radio-frequency $\omega/2\pi$ is usually many MHz whereas the perturbation frequencies are usually a few kHz, the approximation is a good one. We allow for a varying cavity resonance frequency $\Omega(t) = \Omega_0 + \Delta\Omega(t)$. The cavity response is modelled by:

$$[1/\alpha - j/\omega]\dot{\mathbf{V}} + [1 - j\tan\Psi - j\Omega_0\Delta\Omega/(\alpha\omega)]\mathbf{V} = R\,\mathbf{I}_T(t)\,. \quad (6)$$

We introduce the perturbation vectors $\mathbf{e}$ as follows: $\mathbf{V} = \mathbf{V}^0(1+\mathbf{e}_V)$ and $\mathbf{I}_T = \mathbf{I}_T^0(1+\mathbf{e}_T)$. We subtract the steady state equation $[1 - j\tan\Psi]\mathbf{V}^0 = R\,\mathbf{I}_T^0$ and divide throughout by $\mathbf{V}^0$, to obtain:

$$[1/\alpha - j/\omega]\,\dot{\mathbf{e}}_V + [1 - j\tan\Psi]\,\mathbf{e}_V - \left(\mathbf{I}_T^0/\mathbf{I}_V^0\right)\mathbf{e}_T - j\Omega_0\Delta\Omega(t)/[\alpha\omega] = 0\,. \quad (7)$$

We assume $\dot{\mathbf{V}} \ll \omega\mathbf{V}$, note that $\Omega_0\Delta\Omega/(\alpha\omega) \approx \Delta\Omega/\alpha$ to first order, and replace α^{-1} with τ_c the cavity time constant, and so find:

$$\tau_c\,\dot{\mathbf{e}}_V + [1 - j\tan\Psi]\,\mathbf{e}_V - (I_T^0/I_V^0)e^{-j\Psi}\mathbf{e}_T - j\tau_c\Delta\Omega(t) = 0\,. \quad (8)$$

To proceed further, explicit expressions for the constituents of $\mathbf{I}_T$ are required.

$$\text{Let}\quad \mathbf{I}_T^0(1+\mathbf{e}_T) = \mathbf{I}_g^0(1+\mathbf{e}_g) + \mathbf{I}_b^0(1+\mathbf{e}_b)\,,$$

$$\text{then}\quad \mathbf{I}_T^0\mathbf{e}_T = I_g^0 e^{j\psi_g}\mathbf{e}_g - jI_b^0 e^{-j\mu_b}\mathbf{e}_b\,. \quad (9)$$

The dimensionless components z_r and ϕ_r of the perturbation vector $\mathbf{e}_r = (z_r + j\phi_r)$ model amplitude and phase modulations, respectively. We substitute for $\mathbf{e}_r$, replace time derivatives with the Laplace operator s, and compare real and imaginary parts to find, respectively,

$$z_V(1+s\tau_c) + \phi_V\tan\Psi + Y_g\left(\phi_g\sin\psi_g - z_g\cos\psi_g\right) + Y_b\left(z_b\sin\mu_b - \phi_b\cos\mu_b\right) = 0\,, \quad (10a)$$

$$\phi_V(1+s\tau_c) - z_V\tan\Psi - Y_g\left(\phi_g\cos\psi_g + z_g\sin\psi_g\right) + Y_b\left(z_b\cos\mu_b + \phi_b\sin\mu_b\right) = \tau_c\Delta\Omega\,. \quad (10b)$$

Note, until we choose a definite value for the steady state generator phase, ψ_g, there is no direct relation between $\tan\Psi$ and $(Y_b\,,\mu_b)$.

2.1 *RF feedback around the cavity*

In this section we show how including a voltage proportional feedback around the cavity modifies the cavity equations. This type of feedback (with small, precise loop delay), as discussed by Boussard,[6] requires a high power summing junction since it is the entire rf signal which is fed back. The terminology "rf feedback" is preferred over "fast feedback" which might be confused with "fast phase loop".

2.1.1. Steady state The steady state generator current $\mathbf{I}_g^0 = I_g^0 e^{j\psi_g}$ is the sum of the demand current $\mathbf{I}_d^0 = I_d^0 e^{j\psi_d}$ and the feedback current $\mathbf{I}_f^0 = I_f^0 e^{j\psi_f}$. Together with $\mathbf{I}_b^0$ they sum to form the total current $\mathbf{I}_T^0$. We consider the case of a delayless voltage proportional feedback, so $\mathbf{I}_f^0 = -he^{j\psi_h} I_V^0 e^{j\psi_V}$ where h is the gain. With this assignment, the steady state relation between current vectors is:

$$\left(\Omega^2 - \omega^2 + 2j\alpha\omega\right) I_V^0 e^{j\psi_V} = 2\alpha j\omega \left[I_d^0 e^{j\psi_d} - jI_b^0 e^{-j\mu_b} - he^{j\psi_h} I_V^0 e^{j\psi_V} \right] . \tag{11}$$

We divide throughout by $2j\alpha\omega \times I_V^0 e^{j\psi_V}$, and introduce the dimensionless current ratio $Y_d = I_d^0 / I_V^0$. The steady state relation becomes:

$$1 - j \tan \Psi = Y_d e^{j(\psi_d - \psi_V)} - jY_b e^{-j(\mu_b + \psi_V)} - he^{j\psi_h} . \tag{12}$$

We set $\psi_V = 0$ and then compare real and imaginary parts:

$$\mathcal{R}e \Rightarrow \quad 1 = Y_d \cos \psi_d - Y_b \sin \mu_b - h \cos \psi_h , \tag{13a}$$

$$\text{and } \mathcal{I}m \Rightarrow \quad \tan \Psi = Y_b \cos \mu_b - Y_d \sin \psi_d + h \sin \psi_h . \tag{13b}$$

2.1.2. Non steady state We now find the relation between current and voltage vectors for small perturbations. The cavity response is modelled by:

$$\tau_c \dot{\mathbf{e}}_V + [1 - j \tan \Psi]\mathbf{e}_V - (I_T^0/I_V^0)e^{-j\Psi}\mathbf{e}_T - j\tau_c \Delta\Omega(t) = 0 . \tag{14}$$

To proceed further, explicit expressions for the constituents of $\mathbf{I}_T$ are required.

$$\text{Let} \quad \mathbf{I}_T^0(1 + \mathbf{e}_T) = \mathbf{I}_d^0(1 + \mathbf{e}_d) + \mathbf{I}_b^0(1 + \mathbf{e}_b) + \mathbf{I}_f^0(1 + \mathbf{e}_f) , \tag{15a}$$

$$\text{then} \quad \mathbf{I}_T^0 \mathbf{e}_T = I_d^0 e^{j\psi_d} \mathbf{e}_d - jI_b^0 e^{-j\mu_b} \mathbf{e}_b - he^{j\psi_h} I_V^0 \mathbf{e}_f . \tag{15b}$$

We substitute the perturbation vectors $\mathbf{e}_r = (z_r + j\phi_r)$, note that $\mathbf{e}_f = \mathbf{e}_V$, and compare real and imaginary parts to find, respectively,

$$\begin{aligned} &z_V(1 + s\tau_c) + \phi_V \tan \Psi + Y_d\,(\phi_d \sin \psi_d - z_d \cos \psi_d) \\ &+Y_b\,(z_b \sin \mu_b - \phi_b \cos \mu_b) + h\,(z_V \cos \psi_h - \phi_V \sin \psi_h) = 0 , \end{aligned} \tag{16a}$$

$$\begin{aligned} &\phi_V(1 + s\tau_c) - z_V \tan \Psi - Y_d\,(\phi_d \cos \psi_d + z_d \sin \psi_d) \\ &+Y_b\,(z_b \cos \mu_b + \phi_b \sin \mu_b) + h\,(\phi_V \cos \psi_h + z_V \sin \psi_h) = 0 . \end{aligned} \tag{16b}$$

Note, setting $h = 0$ will recover the situation of no rf feedback around the cavity, and in this case the demand and generator currents are equal so that subscript d can be replaced by g. Note, until we choose a definite value for the steady state demand phase ψ_d there is no direct relation between $\tan \Psi$ and (Y_b, μ_b). If the tuning changes, then $\tau_c \Delta\Omega(t)$ should be added to the right hand side of Equation (16b) for imaginary parts.

3 BEAM EQUATIONS

We shall give equations for rigid bunch dipole and quadrupole oscillations in the limit of small amplitudes.

3.1 *Dipole motion*

We suppose the ideal cavity drive frequency ω_0 is synchronous with a particle travelling with the equilibrium momentum demanded by the guide magnetic field. However, as a result of modulations, the cavity phase may advance or lag the ideal phase ($\Phi_0 = \int \omega_0 dt$) by an amount ϕ_V. Likewise, the beam centroid may differ from the ideal phase (Φ_0) by an amount ϕ_b. The energy given to the beam is the vector dot product of cavity voltage and beam current.

3.1.1. Steady state If the beam image current has phase ψ_b, then the beam current has phase $\psi_b' = \psi_b + \pi$. Suppose there is a single cavity with peak voltage V^0. The energy gain per turn is:

$$\Delta E_s = eV^0 \cos\left[\psi_b' - \psi_V\right] = eV^0 \sin \mu_b \ . \tag{17}$$

3.1.2. Non steady state Suppose the cavity has relative phase and amplitude modulations ϕ_V and z_V. The energy gain of the beam centroid is:

$$\Delta E_b = eV^0(1 + z_V)\cos\left[(\psi_b' + \phi_b) - (\psi_V + \phi_V)\right] \ . \tag{18}$$

We subtract the steady state energy gain, Equation (17), to find the energy deviation δE:

$$\delta E = eV^0(1 + z_V)[\sin \mu_b \cos(\phi_b - \phi_V) - \cos \mu_b \sin(\phi_b - \phi_V)] - eV^0 \sin \mu_b \ . \tag{19}$$

We expand to first order and take the Laplace transform, to obtain:

$$s\,\delta E = K_1 \left[z_V \sin \mu_b + (\phi_V - \phi_b) \cos \mu_b\right] \ . \tag{20}$$

Because of the energy deviation, the phase error ϕ_b will advance at the rate: $s\,\phi_b = K_2 \delta E$. The product of constants is

$$K_1 \times K_2 = \Omega_s^2 = f_\infty^2 \left| \frac{1}{\gamma_t^2} - \frac{1}{\gamma_s^2} \right| \frac{eV^0}{2\pi h_s E_s} \ , \tag{21}$$

where the symbols follow those in Reference 7. Note that we have chosen to define Ω_s, the synchrotron frequency, *sans* the usual $\cos\mu_b$ term.

3.2 Quadrupole motion

In this section we derive the envelope equation for a bunch which has elliptical equi-density contours in the longitudinal phase plane. The equations presented follow from Reference 8. We start from the phase equation for a general particle, with index i.

$$\ddot{\phi}_i = \Omega_s^2 \{(\phi_V \cos\mu_b + z_V \sin\mu_b) - [(1+z_V)\cos\mu_b - \phi_V \sin\mu_b] \times \phi_i\} \; . \tag{22}$$

Let $\phi_i = A\beta(t)\cos\psi(t)$ with the constraint $\partial\psi/\partial t = 1/\beta^2$. We use the chain rule to evaluate the derivative and find $\ddot{\phi}_i = A\cos\psi\,[\ddot{\beta} - 1/\beta^3]$. Hence:

$$A\,[\ddot{\beta} - 1/\beta^3] = -\,\Omega_s^2\,[(1+z_V)\cos\mu_b - \phi_V \sin\mu_b] \times A\beta$$
$$+\;\Omega_s^2[\phi_V\cos\mu_b + z_V\sin\mu_b]/\cos\psi\;.$$

We make the assignments $\cos\psi = \pm 1$ (as occur at the extrema of the phase oscillations) and sum the two resulting equations so as to eliminate the terms in z_V and ϕ_V. Finally, we note that on the envelope $A\beta = \Theta$ is the bunch half-length. Hence the envelope equation is:

$$\frac{d^2\Theta}{dt^2} - \frac{A^4}{\Theta^3} = -\Omega_s^2\,[\cos\mu_b(1+z_V) - \phi_V \sin\mu_b] \times \Theta\;. \tag{23}$$

Let $\Theta = \Theta_0 + \theta$ be the sum of a steady state part Θ_0 and a small perturbation $\theta(t)$.

3.2.1. Steady state We set z_V and the time derivatives to zero, and recover an expression for the amplitude $A^2 = \Theta_0^2\,\Omega_s\sqrt{\cos\mu_b}$ which is an invariant of motion.

3.2.2. Non steady state The equation for small oscillations is:

$$\ddot{\theta} = -\Omega_s^2\cos\mu_b\left[[(1+z_V) + \phi_V\tan\mu_b](\Theta_0+\theta) - \Theta_0^4/(\Theta_0+\theta)^3\right]\;. \tag{24}$$

This is Taylor expanded to first order in θ, and becomes:

$$\ddot{\theta} + 4\Omega_s^2\cos\mu_b \times \theta = \Theta_0\Omega_s^2[\phi_V\sin\mu_b - z_V\cos\mu_b]\;. \tag{25}$$

Let us introduce the variable δW, then the Laplace transform of the envelope oscillation can be derived from:

$$s\,\theta = \Omega_s^2\,\delta W \qquad \text{and} \qquad s\,\delta W = -4\cos\mu_b \times \theta - \Theta_0\,[z_V\cos\mu_b - \phi_V\sin\mu_b]\;. \tag{26}$$

To complete our description of the beam coupling to the cavity, we must find the relation between variations of the bunch length and amplitude modulation of the beam current. We shall use the first order approximation

$$z_b + F_0 \times \theta = 0 , \tag{27}$$

which is reported in Reference 9. The form factor F_0 depends on the bunch shape and the equilibrium bunch half-length Θ_0. Let the normalized bunch shape be written $\lambda(\Theta_0, x)$ where x is the rf phase in radians. Under the conditions λ is symmetric about $x = 0$ and $\lambda = 0$ at $x = \Theta_0$ we find that

$$F_0(\Theta_0) = -\int_0^{\Theta_0} \frac{\partial \lambda}{\partial \Theta_0} \cos(x) dx \Big/ \int_0^{\Theta_0} \lambda \cos(x) dx . \tag{28}$$

For the family of functions $\lambda = (\Theta_0^2 - x^2)^\alpha$ the integrals reduce to:

$$F_0(\Theta_0) = \frac{(2\alpha + 1)}{\Theta_0} - \frac{J_{\alpha-1/2}(\Theta_0)}{J_{\alpha+1/2}(\Theta_0)} \qquad \text{with } \alpha > 0 , \tag{29}$$

and where J_n are Bessel functions whose order (integer or fraction) depend on α. In the limit of short bunches, $\Theta_0 < 1$, we obtain the example cases: $\alpha = 1/2$ then $F_0 \approx \Theta_0/4$, $\alpha = 1$ then $F_0 \approx \Theta_0/5$.

4 CONTROL LOOPS

We give the Laplace transforms of the feedback equations. The Laplace operator variable s is equivalent to a first differential in the time domain.

4.1 Beam phase and radial loops

If there is an rf feedback around the cavity, this loop modifies the demand phase ϕ_d; otherwise the loop modifies the generator phase ϕ_g. The loop acts by modifying the instantaneous cavity drive frequency according to the following equation.

$$\Delta\omega(s) = (\omega - \omega_0) = s\phi_g = \frac{K_p}{(1 + s\tau_p)} \times (\phi_b - \phi_V) \quad - \quad \frac{K_r}{(1 + s\tau_r)} \times \delta E . \tag{30}$$

In the limit $\tau_p \to 0$ we obtain "fast response" for the phase-loop. The phase open loop gain is K_p, and the radial open loop gain is K_r.

4.2 Beam quadrupole loop

This loop is intended to damp oscillations of the bunch length. If there is an rf feedback around the cavity, this loop modifies the demand amplitude z_d; otherwise the loop modifies the generator amplitude z_g.

$$z_g = \frac{sK_a}{(1 + s\tau_a)} \times \theta \ . \tag{31}$$

If $K_a < 0$ this loop will anti-damp the oscillations.

4.3 Tuning loop

This loop endeavours to bring the generator current and gap voltage vectors parallel (that is in-phase) by modifying the cavity resonance frequency.

$$\tau_c \Delta\Omega_{\text{res}} = \frac{K_t}{(s + \omega_t)} \times (\phi_g - \phi_V) \ . \tag{32}$$

Often the corner frequency ω_t is very small, and we can approximate the system by a pure integrator (that is $\omega_t = 0$). In the absence of any other loops or beam-cavity interaction, the loop will reduce the phase error to zero (that is $\phi_g = \phi_V$) provided the gain K_t is positive.

5 ROUTH DETERMINANTS

The zeros of the characteristic polynomial are the poles of the system transfer function. Consequently, if the system response contains only self-damped oscillations, then the zeros of the characteristic must all lie in the left half of the complex plane. A necessary condition is for the coefficients of s^n to be greater than zero. In addition, the Routh criteria [RH(i) > 0 for $i = 1, 2, \ldots n$] for combinations of the coefficients must be satisfied.

Assume the characteristic polynomial is of the form:

$$a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \cdots + a_n s^n \ ,$$

then there are the following Routh-Hurwitz stability conditions for orders $n = 2, 3, \ldots 6$.

<u>2nd order polynomial,</u>

RH(1): a_2
RH(2): a_1
RH(3): a_0

<u>3rd order polynomial,</u>

RH(1): a_3
RH(2): a_2
RH(3): $-a_0 a_3 + a_1 a_2$
RH(4): a_0

<u>4th order polynomial,</u>

RH(1): a_4
RH(2): a_3
RH(3): $-a_1 a_4 + a_2 a_3$
RH(4): $-\left[a_0 a_3^2 + a_1 (a_1 a_4 - a_2 a_3)\right]$
RH(5): a_0

5th order polynomial,
RH(1): a_5
RH(2): a_4
RH(3): $-a_2a_5 + a_3a_4$
RH(4): $-[a_1a_4^2 + a_2(a_2a_5 - a_3a_4) - a_0a_4a_5]$
RH(5): $-[a_0(a_2a_5 - a_3a_4)^2 - (a_0a_5 - a_1a_4)(a_1a_4^2 + a_2(a_2a_5 - a_3a_4) - a_0a_4a_5)]$
RH(6): a_0
6th order polynomial,
RH(1): a_6
RH(2): a_5
RH(3): $-a_3a_6 + a_4a_5$
RH(4): $-[a_2a_5^2 + a_3(a_3a_6 - a_4a_5) - a_1a_5a_6]$
RH(5): $-[a_1(a_3a_6 - a_4a_5)^2 - (a_1a_6 - a_2a_5)(a_2a_5^2 + a_3(a_3a_6 - a_4a_5) - a_1a_5a_6) + a_0a_5^2(a_3a_6 - a_4a_5)]$
RH(6): $(a_0a_5^2 + a_1(a_3a_6 - a_4a_5))[a_1(a_3a_6 - a_4a_5)^2 - (a_1a_6 - a_2a_5)(a_2a_5^2 + a_3(a_3a_6 - a_4a_5) - a_1a_5a_6) + a_0a_5^2(a_3a_6 - a_4a_5)] - a_0a_5(a_2a_5^2 + a_3(a_3a_6 - a_4a_5) - a_1a_5a_6)^2$
RH(7): a_0

6 CAVITY AND BEAM DIPOLE MODE

This is the case originally treated by Robinson.[2]

6.0.1. Steady state relations The model assumes that the generator (or power tube) current is maintained by an ideal feed-forward according to:

$$Y_g \cos \psi_g = 1 + Y_b \sin \mu_b \quad ; \quad Y_g \sin \psi_g = Y_b \cos \mu_b - \tan \Psi \ . \tag{33}$$

6.0.2. Determinantal matrix The vector equation $\mathbf{M} \times \mathbf{x} = \mathbf{0}$, where $\mathbf{x} = (z_v, \phi_v, \phi_b, \delta E)$, implies:

$$\begin{vmatrix} z1 & z2 & -C_bY_b & 0 \\ -z2 & z1 & S_bY_b & 0 \\ 0 & 0 & s & -\Omega_s^2 \\ -S_b & -C_b & C_b & s \end{vmatrix} = 0 \ . \tag{34}$$

Here $z1 = 1 + s\tau_c \ ; \ z2 = \tan \Psi \ ; \ C_b = \cos(\mu_b) \ ; \ S_b = \sin(\mu_b)$.

6.0.3 Characteristic polynomial

$$\Omega_s^2[\cos \mu_b(1 + \tan^2 \Psi) - Y_b \tan \Psi] + 2\Omega_s^2 \cos(\mu_b)\tau_c \, s + [(1 + \tan^2 \Psi) + (\Omega_s\tau_c)^2 \cos \mu_b]s^2 + 2\tau_c \, s^3 + \tau_c^2 \, s^4 \ . \tag{35}$$

6.1 Routh determinants

RH(1), RH(2) and RH(3) imply $\tau_c \geq 0$.
RH(4): $\tan\Psi \geq 0$, hence $\Psi \geq 0$.
If RH(4) < 0, and the cavity is detuned in the wrong sense, then there is an instability with: $\mathcal{I}m[s] \approx \pm\Omega_s \cos\mu_b$.
RH(5): $\cos\mu_b(1+\tan^2\Psi) - Y_b \tan\Psi > 0$ implies the Robinson limit:

$$Y_b < 2\cos\mu_b \Big/ \sin 2\Psi \; . \tag{36}$$

If RH(5) < 0, then there is an instability with $\mathcal{I}m[s] = 0$. In this case phase-focusing of the bunch centroid is lost, and the bunch simply wanders.
Note, the Robinson criterion RH(5) is general[b] not because we have substituted a general expression for the generator current, but rather because no expression at all is needed for the generator.
Note, substituting the matched generator condition ($\psi_g = 0$ or $\tan\Psi = Y_b C_b$) in RH(5) gives the special case stability limit: $Y_b < 1/\sin\mu_b$.

7 CAVITY WITH IDEAL RF FEEDBACK, AND BEAM DIPOLE MODE

7.0.1. Steady state relations The generator (or power tube) current $\mathbf{I}_g$ is the sum of the demand $\mathbf{I}_d$ and the feedback current $\mathbf{I}_f = -(h/R)\mathbf{V}$. The ideal feedback phasing is $\psi_h = 0$. The generator conditions (Y_g and ψ_g) are set according to Equation (33).

The demand current is set by an ideal feed-forward, according to

$$Y_d \cos\psi_d = Y_g \cos\psi_g + h \quad ; \quad Y_d \sin\psi_d = Y_g \sin\psi_g \; . \tag{37}$$

Because there are no feedback loops, neither $\mathbf{I}_g$ nor $\mathbf{I}_d$ explicitly enter the matrix elements. However, as seen from the demand vector, the cavity shunt resistance and cavity time constant are reduced by a factor $1/(1+h)$ where h is the feedback gain.

7.0.2. Determinantal matrix The vector equation $\mathbf{M}\times\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = (z_v, \phi_v, \phi_b, \delta E)$, implies:

$$\begin{vmatrix} z1 & z2 & -C_b Y_b & 0 \\ -z2 & z1 & S_b Y_b & 0 \\ 0 & 0 & s & -\Omega_s^2 \\ -S_b & -C_b & C_b & s \end{vmatrix} = 0 \; , \tag{38}$$

with $z1 = (1+h) + s\tau_c$; $z2 = \tan\Psi$; $C_b = \cos(\mu_b)$; $S_b = \sin(\mu_b)$.

[b] General in the sense that we may pick any combination of Ψ and Y_b provided that the generator is correctly adjusted.

7.0.3. Characteristic polynomial

$$\Omega_s^2 \left\{\cos\mu_b[(1+h)^2+\tan^2\Psi] - Y_b\tan\Psi\right\} + 2\Omega_s^2\cos\mu_b(1+h)\tau_c\, s + \\ + \left[(1+h)^2+\tan^2\Psi+(\Omega_s\tau_c)^2\cos\mu_b\right] s^2 + 2(1+h)\tau_c\, s^3 + \tau_c^2 s^4 \,. \tag{39}$$

7.1 Routh determinants

RH(1), RH(2) and RH(3) imply $\tau_c \geq 0$.
RH(4): $\tan\Psi \geq 0$, hence $\Psi \geq 0$.
RH(5): $\cos\mu_b\left[(1+h)^2+\tan^2\Psi\right] - Y_b\tan\Psi > 0$ implies:

$$Y_b < \frac{2\cos\mu_b}{\sin 2\Psi}\left[1+h(2+h)\cos^2\Psi\right] = \cos\mu_b\left[\frac{2}{\sin 2\Psi}+\frac{h(h+2)}{\tan\Psi}\right]. \tag{40}$$

Note, from Equation (40) that the rf feedback becomes less efficient at large tuning angles. For example, if $h \gg 1$ and $\cos\Psi \simeq 1/h$ then the approximate condition: $Y_b < 4\cos\mu_b/\sin 2\Psi$ results, which is only a factor 2 better than the Robinson limit. In general, to obtain a reasonable advantage from rf feedback we need $h \gg 1/\cos\Psi$.

Note, substituting the matched generator condition ($\psi_g = 0$ and $\tan\Psi = Y_b C_b$) in RH(5) gives the special case: $Y_b < (1+h)/\sin\mu_b$, which clearly shows the intensity limit is raised. In this case $\psi_d = 0$, so demand current and voltage are in-phase.

Note, that apart from the substitutions $\tau_c \Rightarrow \tau_c/(1+h)$, $\tan\Psi \Rightarrow \tan\Psi/(1+h)$, $Y_b \Rightarrow Y_b/(1+h)$, the characteristic polynomial is identical with the case for no rf feedback around the cavity, Equation (35). This indicates that the effect of the feedback is to increase the cavity bandwidth, reduce the relative detuning, and reduce the relative beam-loading ratio I_b^0/I_V^0.

8 CAVITY WITH RF FEEDBACK MIS-PHASED, AND BEAM DIPOLE MODE

Mis-phasing of the cavity rf feedback can occur quite naturally, either through error or because of the loop delay τ_d. It is usual when setting up the feedback to put the drive frequency ω equal to the resonance frequency Ω and then to adjust the phase advance around the loop to be 0° or 180° depending on whether negative or positive feedback is used. Let us assume negative feedback, then $\Omega_{\text{res}}\tau_d = 2\pi n$ where $n = 0, 1, 2, \ldots$ is an integer. Suppose now that beam is introduced, and that the cavity is detuned to compensate the reactive beam-loading. The drive and resonance frequencies are no longer equal, and so there will a residual phase advance $\psi_h = (\Omega_{\text{res}} - \omega)\tau_d$. If the feedback gain h has not already been pushed to the delay-limited value, then it is possible to adjust the feedback phasing with beam current present and so reduce (or eliminate) the error ψ_h. However, this is not done in practise because the beam-current magnitude may vary on a pulse-to-pulse basis.

8.0.1. Steady state relations The generator (or power tube) current $\mathbf{I}_g$ is the sum of the demand $\mathbf{I}_d$ and the feedback current $\mathbf{I}_f = -(1/R)\mathbf{h} \times \mathbf{V}$ where $\mathbf{h} = he^{j\psi_h}$ is complex. Generator is set according to Equation (33).

The demand is set by an ideal feed-forward, according to

$$Y_d \cos\psi_d = Y_g \cos\psi_g + h\cos\psi_h \;\; ; \;\; Y_d \sin\psi_d = Y_g \sin\psi_g + h\sin\psi_h \; .$$

Note, we shall treat ψ_d as the dependent variable; and so the mis-phasing ψ_h does not alter the optimal tuning condition $\tan\Psi = Y_b \cos\mu_b$.

8.0.2. Determinantal matrix The vector equation $\mathbf{M} \times \mathbf{x} = \mathbf{0}$, where $\mathbf{x} = (z_v, \phi_v, \phi_b, \delta E)$, implies:

$$\begin{vmatrix} z1 & z2 & -C_b Y_b & 0 \\ -z2 & z1 & S_b Y_b & 0 \\ 0 & 0 & s & -\Omega_s^2 \\ -S_b & -C_b & C_b & s \end{vmatrix} = 0 \, , \tag{41}$$

with $z1 = (1+h\cos\psi_h)+s\tau_c$; $z2 = \tan\Psi - h\sin\psi_h$; $C_b = \cos(\mu_b)$; $S_b = \sin(\mu_b)$.

8.0.3. Characteristic polynomial

$$\begin{aligned} &\Omega_s^2 \left\{ \cos\mu_b [1 + h^2 + \tan^2\Psi + 2h(\cos\psi_h - \sin\psi_h \tan\Psi)] - Y_b(\tan\Psi - h\sin\psi_h) \right\} + \\ &+ 2\Omega_s^2 \cos\mu_b (1 + h\cos\psi_h)\tau_c \, s + [1 + h^2 + \tan^2\Psi + 2h(\cos\psi_h - \sin\psi_h \tan\Psi) + \\ &\qquad + (\Omega_s \tau_c)^2 \cos\mu_b] s^2 + 2(1 + h\cos\psi_h)\tau_c \, s^3 + \tau_c^2 s^4 \, . \end{aligned} \tag{42}$$

8.1 Routh determinants

RH(1): $\tau_c^2 \geq 0$.
RH(2): $\tau_c\,(1 + h\cos\psi_h) > 0$ implies two conditions: (i) $\tau_c > 0$ and (ii) $\cos\psi_h > -1/h$. When the second condition is exceeded, we have positive rather than negative rf feedback; hence the mis-phasing must not be too great.
RH(4): $\tan\Psi - h\sin\psi_h \geq 0$.
This condition implies that part of the tuning diagram becomes inaccessible, depending on the sign of ψ_h.

RH(3): $(1 + h^2 + \tan^2 \Psi) + 2h(\cos\psi_h - \sin\psi_h \tan\Psi) \geq 0$.

This condition is automatically satisfied when RH(4) and RH(2) are satisfied, as is found by substitution.

RH(5) implies:

$$Y_b < \cos\mu_b \frac{\left[1 + h^2 + \tan^2 \Psi + 2h\left(\cos\psi_h - \sin\psi_h \tan\Psi\right)\right]}{(\tan\Psi - h\sin\psi_h)} . \tag{43}$$

If $\psi_h > 0$ and RH(2) and RH(4) are satisfied, then the Robinson limit can be exceeded almost indefinitely; substitute $\tan\Psi = h\sin\psi_h + \varepsilon$, then $Y_b < (1 + h\cos\psi_h)^2/\varepsilon$ to first order in the small quantity ε. What happens here is that the demand (i.e. set-point at input to the high power summing junction) is adjusted so as to partially cancel the steady state beam image current. At the stability limit RH(4)=0, the demand quadrature component is $Y_d \sin\psi_d = Y_b \cos\mu_b$ and so exactly cancels the beam current component. Of course, this cancellation requires an ideally accurate feed-forward setting of the demand current phase and is not completely practical, particularly as the beam current must be measured.

Alternatively, we may make the tuning angle Ψ the dependent variable, and discover what is the consequence of setting the demand current phase equal to zero ($\psi_d = 0$). We find that the detuning is modified: $\tan\Psi = Y_b \cos\mu_b + h\sin\psi_h$. If we substitute for $\tan\Psi$ into RH(5), then $Y_b < (1 + h\cos\psi_h)/\sin\mu_b$ which shows the stability limit is degraded compared with the case of ideal feedback ($\psi_h = 0$).

9 CAVITY, BEAM DIPOLE MODE, AND FAST BEAM PHASE-LOOP

The model of Section 6 is supplemented with a beam phase-loop which is intended to damp bunch dipole oscillations. We shall assume that the phase-loop has the response of a pure integrator, that is $\phi_g = (K_p/s) \times (\phi_b - \phi_v)$. We call this loop fast, because it amounts to proportional (i.e. no lag) feedback to the input of a variable frequency source. Note, ϕ_g is the deviation of the generator phase from the steady state set-point ψ_g.

9.0.1. Steady state relations The model assumes that the generator (or power tube) current is maintained according to:

$$Y_g C_g = Y_g \cos\psi_g = 1 + Y_b \sin\mu_b \quad ; \quad Y_g S_g = Y_g \sin\psi_g = Y_b \cos\mu_b - \tan\Psi .$$

Because we have added a control loop, we are forced to insert these explicit forms for the generator components into the determinantal matrix.

9.0.2. Determinantal matrix The vector equation $\mathbf{M} \times \mathbf{x} = \mathbf{0}$, where $\mathbf{x} = (z_v, \phi_v, \phi_g, \phi_b, \delta E)$, implies:

$$\begin{vmatrix} z1 & z2 & S_g Y_g & -C_b Y_b & 0 \\ -z2 & z1 & -C_g Y_g & S_b Y_b & 0 \\ 0 & K_p & s & -K_p & 0 \\ 0 & 0 & 0 & s & -\Omega_s^2 \\ -S_b & -C_b & 0 & C_b & s \end{vmatrix} = 0 ,$$

with $z1 = 1 + s\tau_c$; $z2 = \tan\Psi$; $C_b = \cos(\mu_b)$; $S_b = \sin(\mu_b)$.

9.0.3. *Characteristic polynomial*

$$\Omega_s^2[\cos\mu_b(1+\tan^2\Psi) + K_p\tau_c \sin\mu_b(Y_b\cos\mu_b - \tan\Psi) - Y_b\tan\Psi] +$$
$$+ [K_p(1+\tan^2\Psi) + K_pY_b(\sin\mu_b - \cos\mu_b\tan\Psi) + 2\Omega_s^2\cos(\mu_b)\tau_c]\, s +$$
$$+ \left[(1+\tan^2\Psi) + K_p\tau_c(1+Y_b\sin\mu_b) + (\Omega_s\tau_c)^2\cos\mu_b\right] s^2 + 2\tau_c s^3 + \tau_c^2 s^4 . \quad (45)$$

A necessary condition for stability is that the coefficient of s^1 should be greater than zero, and this implies

$$Y_b < \frac{\sec^2\Psi + 2\Omega_s^2(\tau_c/K_p)\cos\mu_b}{\cos\mu_b(\tan\Psi - \tan\mu_b)} , \quad (46)$$

from which we conclude $\tan\Psi \geq \tan\mu_b$ and $K_p > 0$. In the limit $\mu_b \to 0$ we find an expression which resembles the Robinson limit:

$$Y_b < \frac{2}{\sin 2\Psi}\left[1 + 2(\Omega_s\cos\Psi)^2(\tau_c/K_p)\right] = \frac{2}{\sin 2\Psi} + \frac{2\Omega_s^2\tau_c}{K_p\tan\Psi} . \quad (47)$$

However, in most cases this limit is subordinate to RH(5) below.

9.1 *Routh determinants*

RH(1) and RH(2) imply $\tau_c \geq 0$.
RH(3): $2 + K_p\tau_c[\cos 2\Psi + Y_b\cos\Psi\sin(\Psi+\mu_b)] > 0$.
For $K_p > 0$ and $(\Psi + \mu_b) > 0$, condition RH(3) is satisfied automatically for all tuning angles $\Psi \leq \pi/4$. For tuning angles greater than 45°, the term $\cos 2\Psi$ becomes negative, and stability is not guaranteed unless $K_p\tau_c \leq 2$. For example, consider the case $\mu_b = 0$; then RH(3) implies

$$Y_b > \frac{-2}{\sin 2\Psi}\left[\cos 2\Psi + \frac{2}{K_p\tau_c}\right] \quad \text{when } \Psi > 0 . \quad (48)$$

Though this condition gives the minimum beam load ratio Y_b, the maximum value will be constrained by RH(5) below.

Condition RH(3) also allows a domain of stability with $\Psi+\mu_b < 0$; the damping provided by the phase-loop can overcome (to a limited extent) the instability caused by detuning in the wrong sense ($\Omega_{\rm res} < \omega_{\rm drive}$ below transition energy). In the limit $K_p\tau_c \gg 1$, we obtain the approximate stability limit:

$$Y_b < \frac{\cos 2\Psi}{\cos\Psi \sin|\Psi+\mu_b|} \qquad \text{when } \Psi+\mu_b < 0.$$

$$\text{or} \quad Y_b < 2/\tan|2\Psi| \qquad \text{when } \mu_b \to 0 \quad \text{and } \Psi < 0\,. \tag{49}$$

Note, negative detuning does not conflict with RH(5) below.
RH(5): $\cos\mu_b(1+\tan^2\Psi) + K_p\tau_c \sin\mu_b Y_g \sin\psi_g - Y_b\tan\Psi > 0$.
When $\Psi > 0$ this implies a slight modification to the Robinson limit:

$$Y_b < \frac{2\cos\mu_b}{\sin 2\Psi}\left[1 + K_p\tau_c\cos^2\Psi \tan\mu_b\,(Y_g\sin\psi_g)\right] = \frac{2\cos\mu_b}{\sin 2\Psi} + \frac{K_p\tau_c(Y_g\sin\psi_g)\sin\mu_b}{\tan\Psi}\,. \tag{50}$$

The additional stability (when $\mu_b\times\psi_g > 0$) arises from arranging the steady state generator current to partially oppose the steady state beam image current; this has the effect of slightly reducing the vector-geometric cross-coupling between amplitude and phase modulations. However, this violates the matched generator condition, and implies that the tube must deliver reactive power.
RH(4):

$$\begin{aligned} 0 \le\, & 2K_p\sec^2\Psi[\sec^2\Psi + Y_b(\sin\mu_b - \cos\mu_b\tan\Psi) + \\ & + (\Omega_s\tau_c)^2\cos\mu_b\times(\cos 2\Psi + \tan\mu_b\sin 2\Psi)] + \\ & +\; 2K_p(\Omega_s\tau_c)^2\cos\mu_b Y_b(\cos\mu_b\tan\Psi - \sin\mu_b) + \\ & +\; \tau_c K_p^2[(1+Y_b\sin\mu_b)^2 - (Y_g\sin\psi_g\tan\Psi)^2] \;+\; 4\Omega_s^2\tau_c Y_b\tan\Psi\,. \end{aligned} \tag{51}$$

A sufficient condition for RH(4) > 0 is $\tan\Psi = \tan\mu_b$; which agrees with the coefficient of s^1 being positive. Alternatively, we may substitute $\tan\Psi = Y_b\cos\mu_b$ (or $\psi_g = 0$) and so find RH(4) > 0 at all points on the matched generator curve. Finally, we note that $\mu_b = 0$, $\tan\Psi < 1/\tan\psi_g$ and RH(5) > 0 are sufficient conditions for RH(4) > 0.

To conclude, the domain of stability for positive tuning angles is determined by the beam load values Y_b that satisfy both RH(3) and RH(5). For the case $K_p\tau_c \gg 1$ and $\mu_b = 0$, this domain is given by:

$$-2\cos 2\Psi < Y_b \sin 2\Psi < 2 \,. \tag{52}$$

Evidently, as $\Psi \to 90°$, Y_b becomes limited to a very narrow band of stable values.

9.2 *Slow beam phase-loop*

We briefly consider the effect of introducing a time constant τ_p into the phase-loop, as is inevitable in reality. Thus the phase-loop is modelled by:

$$s\phi_g = (\phi_b - \phi_v) \times K_p/(1 + s\tau_p)\,.$$

The substitution $K_p \Rightarrow K_p/(1 + s\tau_p)$ into the previous characteristic equation gives a polynomial of order five. Assuming $K_p \geq 0$, then the requirement that the coefficient of s^1 is positive implies

$$Y_b < \frac{K_p \sec^2\Psi + \Omega_s^2 \cos\mu_b \left[2\tau_c + \tau_p \sec^2\Psi\right]}{K_p(\cos\mu_b \tan\Psi - \sin\mu_b) + \Omega_s^2 \tau_p \tan\Psi} \tag{53}$$

from which we conclude

$$\tan\Psi \geq \frac{\sin\mu_b}{\cos\mu_b + \Omega_s^2\tau_p/K_p}\,.$$

The Routh determinants RH(1) and RH(2) imply $\tau_c \geq 0$ and $\tau_p \geq 0$, while RH(3) constitutes a constraint on the phase-loop gain:

$$K_p < \frac{2[(\tau_p \tan\Psi)^2 + (\tau_p + \tau_c)^2]}{\tau_c^2 \tau_p (1 + Y_b \sin\mu_b)}\,. \tag{54}$$

The determinants RH(4) and RH(5) are somewhat intractable, containing decades of monomial terms. However, in the special case $\mu_b = 0,\ K_p = \Omega_s,\ \tau_c = \tau_p = 1/\Omega_s$ the conditions reduce to:

RH(4): $2(1 + \tan^2\Psi + \tan^4\Psi) + 15Y_b > 0$

RH(5): $(-27 - 45\tan^2\Psi + 6\tan^6\Psi) + Y_b \tan\Psi(204 + 93\tan^2\Psi - 6\tan^4\Psi)$
$\quad - Y_b^2 75\tan^2\Psi > 0$

which suggests that RH(3) is only a necessary condition for stability. The last determinant RH(6) is identical with RH(5) of the previous case $\tau_p = 0$.

10 CAVITY, BEAM DIPOLE MODE, AND TUNING LOOP

We assume that a feedforward (or program) accomplishes the bulk of the cavity tuning. We do not assume that the tuning program is perfect, and hence the matched generator condition $\psi_g = 0$ is not required. However, we do suppose that the power tube operates in a linear fashion when required to deliver reactive power; which typically implies $|\tan\psi_g| \leq 1$.

The tuning loop response for small oscillations about the program set-point is modelled as a pure integrator: $\tau_c\, \Delta\Omega_{\text{res}} = (K_t/s) \times (\phi_g - \phi_V)$. However, since there are no other loops present, $\phi_g = 0$ for all time. This model was studied by Pedersen[3] for the non-accelerating beam case, and here it will be generalized to $\mu_b \neq 0$ as occurs during acceleration.

10.0.1. Steady state relations The generator (or power tube) current $\mathbf{I}_g$ is set by an ideal feed-forward according to Equation (33).

10.0.2. Determinantal matrix The vector equation $\mathbf{M} \times \mathbf{x} = \mathbf{0}$, where $\mathbf{x} = (z_v, \phi_v, \tau_c \Delta\Omega_{\text{res}}, \phi_b, \delta E)$, implies:

$$\begin{vmatrix} z1 & z2 & 0 & -C_b Y_b & 0 \\ -z2 & z1 & -1 & S_b Y_b & 0 \\ 0 & K_t & s & 0 & 0 \\ 0 & 0 & 0 & s & -\Omega_s^2 \\ -S_b & -C_b & 0 & C_b & s \end{vmatrix} = 0\,, \tag{55}$$

$$\text{with} \quad z1 = 1 + s\tau_c \quad ; \quad z2 = \tan\Psi \quad ; \quad C_b = \cos(\mu_b) \quad ; \quad S_b = \sin(\mu_b)\ .$$

10.0.3. Characteristic polynomial

$$\begin{aligned} &\Omega_s^2 \cos\mu_b K_t (1 - Y_b \sin\mu_b) + \Omega_s^2 [\cos\mu_b (1 + \tan^2\Psi + \tau_c K_t) - Y_b \tan\Psi] s + \\ &+ [K_t + 2\Omega_s^2 \cos(\mu_b)\tau_c] s^2 + [1 + \tan^2\Psi + \tau_c K_t + (\Omega_s \tau_c)^2 \cos\mu_b]\, s^3 + \\ &\qquad + 2\tau_c\, s^4 + \tau_c^2\, s^5\ . \end{aligned} \tag{56}$$

A necessary condition for stability is that all coefficients of the characteristic polynomial be greater than or equal to zero, else there are roots in the right hand side of the complex plane. Examination of the coefficient of s^1 (when $\Psi > 0$) implies the condition:

$$Y_b < \frac{2\cos\mu_b}{\sin 2\Psi}\left(1 + K_t \tau_c \cos^2\Psi\right) \quad = \quad \cos\mu_b \left[\frac{2}{\sin 2\Psi} + \frac{K_t \tau_c}{\tan\Psi}\right] \qquad \text{if } \Psi > 0\ . \tag{57}$$

However, this condition is subordinate to RH(5).

10.1 Routh determinants

RH(1) and RH(2) imply $\tau_c \geq 0$.
RH(3): $2\sec^2\Psi + K_t \tau_c \geq 0$ implies a lower (negative) limit on K_t but which is subordinate to RH(6) below.
RH(4): $K_t (2\sec^2\Psi + K_t \tau_c) + Y_b\, \Omega_s^2 \tau_c\, [4\tan\Psi - K_t \tau_c \sin 2\mu_b] \geq 0$.

For the case $\mu_b = 0$ and $\Psi \geq 0$ this condition is satisfied automatically. However, Routh condition RH(4) turns out to be subordinate to RH(5):

$$- (\Omega_s \tau_c)^2 Y_b \times \Big\{ Y_b \Omega_s^2 \tau_c (\tau_c \sin 2\mu_b - 4 \tan \Psi)^2 - 8\Omega_s^2 \cos \mu_b \tau_c \tan \Psi \sec^2 \Psi +$$
$$+ 2K_t \times [\sec^2 \Psi - (\Omega_s \tau_c)^2 \cos \mu_b](2 \tan \Psi - \sin 2\mu_b \sec^2 \Psi) + \tau_c K_t^2 [2 \tan \Psi +$$
$$+ (\Omega_s \tau_c)^2 \cos \mu_b \sin 2\mu_b - 3 \sin 2\mu_b \sec^2 \Psi] - \tau_c^2 K_t^3 \sin 2\mu_b \Big\} > 0 \,. \tag{58}$$

This expression for $Y_b(K_t)$ can be solved for the beam current Y_b, and is found to factor:

$$Y_b < [0.5 K_t \sin 2\mu_b (\sec^2 \Psi + \tau_c K_t) - K_t \tan \Psi + \Omega_s^2 \cos(\mu_b) \tau_c \times$$
$$\times \; (2 \tan \Psi - 0.5 K_t \tau_c \sin 2\mu_b)](2 \sec^2 \Psi + K_t \tau_c) \, /$$
$$\Omega_s^2 \tau_c (2 \tan \Psi - 0.5 \tau_c K_t \sin 2\mu_b)^2 \,. \tag{59}$$

Because the beam current (Y_b) is positive, we find a quadratic constraint on the tuning loop gain:

$$K_t \{\sin 2\Psi + \sin 2\mu_b [(\Omega_s \tau_c \cos \Psi)^2 \cos \mu_b - 1]\} +$$
$$- \tau_c K_t^2 \cos^2 \Psi \sin 2\mu_b \; - \; 2\Omega_s^2 \cos(\mu_b) \tau_c \sin 2\Psi < 0 \,. \tag{60}$$

The above expression (59) is general and applies to accelerating buckets. We now show how the expressions simplify for a non-accelerating beam, to demonstrate the correspondence with Pedersen 3. Let us compare RH(4) and RH(5) in the limit $\mu_b \to 0$, and for the cases $\Psi > 0$ and $\Psi < 0$.

10.1.1. Tuning angle positive RH(4) is satisfied automatically, but we find RH(5) requires

$$Y_b < \frac{(2\Omega_s^2 \tau_c - K_t)(2 \sec^2 \Psi + K_t \tau_c)}{4\Omega_s^2 \tau_c \tan \Psi} = \left(1 - \frac{K_t}{2\Omega_s^2 \tau_c}\right) \left[\frac{2}{\sin 2\Psi} + \frac{K_t \tau_c}{\tan \Psi}\right] , \tag{61}$$

and conclude that the tuning loop gain is limited to $K_t < 2\Omega_s^2 \tau_c$ when $\Psi > 0$. Indeed, this instability regime where $Y_b \ll 1$, $\Psi > 0$ and $K_t > 2\Omega_s^2 \tau_c$ has been experimentally observed in the CERN PS Booster.[3] Note also that for $K_t > 0$ and $\mu_b = 0$, RH(5) is a more severe constraint than the condition on the coefficient of s^1.

10.1.2. Tuning angle negative

RH(4) implies
$$Y_b < K_t \frac{(2 + K_t\tau_c \cos^2\Psi)}{2\Omega_s^2\tau_c \sin|2\Psi|}$$

whereas RH(5) implies
$$Y_b < (K_t - 2\Omega_s^2\tau_c)\frac{(2 + K_t\tau_c \cos^2\Psi)}{2\Omega_s^2\tau_c \sin|2\Psi|}$$

and so RH(5) constitutes the more severe constraint. Note, when $\Psi < 0$ the tuner gain must satisfy $K_t > 2\Omega_s^2\tau_c$.

The gain condition, for +ve and –ve tuning angles, can be summarized $(K_t - 2\Omega_s^2\tau_c) \times \Psi < 0$.

RH(6): Examination of the coefficient of s^0 implies the condition $K_t > 0$ when $Y_b \sin\mu_b < 1$ as is required by RH(5).

11 CAVITY, BEAM DIPOLE MODE, TUNING LOOP, AND FAST BEAM PHASE-LOOP

We supplement the previous model with the ideal (or fast) phase-loop; $s\phi_g = K_p(\phi_b - \phi_v)$. We note that because $s\tau_c\Delta\Omega_{\text{res}} = K_t(\phi_g - \phi_v)$ there is the possibility for cross-coupling to the tuning loop through the cavity-voltage phase-perturbation ϕ_g.

11.0.1. Steady state relations The generator (or power tube) current I_g is set as follows:

$$Y_g C_g = Y_g \cos\psi_g = 1 + Y_b \sin\mu_b \quad ; \quad Y_g S_g = Y_g \sin\psi_g = Y_b \cos\mu_b - \tan\Psi \ . \tag{62}$$

The presence of the phase-loop implies we must substitute explicit expressions for the generator into the determinantal matrix.

11.0.2. Determinantal matrix The vector equation $\mathbf{M} \times \mathbf{x} = \mathbf{0}$, where $\mathbf{x} = (z_v, \phi_v, \tau_c\Delta\Omega_{\text{res}}, \phi_g, \phi_b, \delta E)$, implies:

$$\begin{vmatrix} z1 & z2 & 0 & S_gY_g & -C_bY_b & 0 \\ -z2 & z1 & -1 & -C_gY_g & S_bY_b & 0 \\ 0 & K_t & s & -K_t & 0 & 0 \\ 0 & K_p & 0 & s & -K_p & 0 \\ 0 & 0 & 0 & 0 & s & -\Omega_s^2 \\ -S_b & -C_b & 0 & 0 & C_b & s \end{vmatrix} = 0 \ ,$$

with $z1 = 1 + s\tau_c \quad ; \quad z2 = \tan\Psi \quad ; \quad C_b = \cos(\mu_b) \quad ; \quad S_b = \sin(\mu_b)$.

11.0.3. Characteristic polynomial

$$\Omega_s^2 \cos\mu_b K_t(1 - Y_b \sin\mu_b) + \Big\{\Omega_s^2[\cos\mu_b(1 + \tan^2\Psi + \tau_c K_t) - Y_b \tan\Psi] +$$
$$+ K_p[K_t + \Omega_s^2 \tau_c \sin\mu_b + (Y_b \cos\mu_b - \tan\Psi)]\Big\} s + \Big\{K_t + 2\Omega_s^2 \cos(\mu_b)\tau_c +$$
$$+ K_p[(1 + \tan^2\Psi) + \tau_c K_t + Y_b(\sin\mu_b - \cos\mu_b \tan\Psi)]\Big\} s^2 + [1 + \tan^2\Psi +$$
$$+ \tau_c K_t + (\Omega_s \tau_c)^2 \cos\mu_b + \tau_c K_p(1 + Y_b \sin\mu_b)] s^3 + 2\tau_c s^4 + \tau_c^2 s^5 . \tag{64}$$

The coefficients of s^1 and s^2 in the characteristic polynomial have the possibility to change sign (when $\Psi > 0$), and a necessary condition for stability is that they be greater than zero. The coefficient of s^1 is automatically positive on and below the matched generator curve $\tan\Psi = Y_b \cos\mu_b$ provided that $\mu_b \leq 45°$. Alternatively for the general case $\psi_g \neq 0$, we find the beam current limit:

$$Y_b < \frac{2\cos\mu_b(1 + \tau_c K_t \cos^2\Psi) + K_p[2K_t \cos^2\Psi/\Omega_s^2 - \tau_c \sin\mu_b \sin 2\Psi]}{\sin 2\Psi - K_p \tau_c \cos^2\Psi \sin 2\mu_b}$$

$$\text{for} \quad \Psi > 0\text{ , or} \qquad Y_b < \frac{2}{\sin 2\Psi} + \frac{K_t(\tau_c + K_p/\Omega_s^2)}{\tan\Psi} \qquad \text{when} \quad \mu_b \to 0 . \tag{65}$$

The coefficient of s^2 is automatically positive if $\tan\Psi \leq \tan\mu_b$; alternatively, we find the limit

$$Y_b < \frac{K_p + \cos^2\Psi[2\Omega_s \tau_c \cos\mu_b + K_t(1 + \tau_c K_p)]}{K_p \cos\Psi \sin(\Psi - \mu_b)}$$

$$\text{or} \qquad Y_b < \frac{2}{\sin 2\Psi} + \frac{2\Omega_s^2 \tau_c + K_t(1 + \tau_c K_p)}{K_p \tan\Psi} \qquad \text{when} \quad \mu_b = 0 . \tag{66}$$

11.1 Routh determinants

RH(1) and RH(2) imply $\tau_c \geq 0$.
RH(3) factors and simplifies to:

$$2 + \tau_c K_p \cos 2\Psi + Y_b \tau_c K_p \cos\Psi \sin(\Psi + \mu_b) + \tau_c K_t(1 - K_p \tau_c)\cos^2\Psi \geq 0.$$

This condition is reminiscent of RH(3) in the case of a beam loaded cavity with phase-loop, Section 9. Let us first consider positive detuning $\Psi > 0$. For the case $\mu_b = 0$ and $K_p \tau_c \gg 1$, we find the approximate condition:

$$Y_b > \frac{K_t \tau_c}{\tan|\Psi|} - \frac{2}{\tan|2\Psi|}\,, \tag{67}$$

which is easily satisfied for $\Psi \leq 45°$, but which must be approached with care for larger tuning angles.

Now consider negative detuning, that is $\Psi + \mu_b < 0$. Solving for Y_b we find the beam current limit:

$$Y_b < \frac{\cos 2\Psi + 2/(\tau_c K_p) + (1 - \tau_c K_p)(K_t/K_p)\cos^2\Psi}{\cos\Psi \sin|\Psi + \mu_b|} \quad \text{when } \Psi + \mu_b < 0\,. \tag{68}$$

In the limit $\tau_c K_p \gg 1$, we obtain the approximate condition:

$$Y_b < \frac{\cos 2\Psi - K_t \tau_c \cos^2\Psi}{\cos\Psi \sin|\Psi + \mu_b|} \quad \text{or} \quad Y_b < \frac{2}{\tan|2\Psi|} - \frac{K_t\tau_c}{\tan|\Psi|} \quad \text{when } \mu_b = 0\,. \tag{69}$$

By comparison of Equation (69) with Equation (49), it is seen that the tuning loop slightly reduces the stability domain when $\Psi < 0$.

We should also like RH(3) to be satisfied in the low current limit ($Y_b \to 0$); and for the case $K_p \tau_c \gg 1$ this implies the approximate condition:

$$K_t \tau_c < 1 - \tan^2\Psi \leq 1\,. \tag{70}$$

The fourth Routh determinant simplifies under the substitution[4] $\tan\Psi \Rightarrow Y_b \cos\mu_b$, as occurs when the generator is matched. One finds the condition RH(4):

$$\begin{aligned}
&\left\{2(K_p + K_t) + \tau_c(K_p^2 + K_t^2) + 2\Omega_s^2\tau_c^2 K_p \cos\mu_b (1 - \tau_c K_t) - \tau_c^3 K_p^2 K_t^2\right\} + \\
&\quad + 2Y_b\left[K_p(1 + \tau_c K_p + \tau_c K_t) + (\Omega\tau_c)^2(K_p - K_t)\cos\mu_b\right]\sin\mu_b + \\
&\quad + Y_b^2 \times \left[2\cos^2\mu_b(K_p + K_t) + \tau_c(K_p^2 \sin^2\mu_b + 4\Omega_s^2\cos\mu_b) + \right. \\
&\quad \left. + 2\tau_c K_p K_t \cos^2\mu_b\right] + Y_b^3 K_p \cos\mu_b \sin 2\mu_b > 0\,.
\end{aligned} \tag{71}$$

We note that a sufficient condition is that the coefficients Y_b^0, Y_b^1, Y_b^2, Y_b^3 be greater than zero. Only the coefficients of Y_b^0 (the constant term) and Y_b^1 have the possibility to change sign; and so, by inspection, sufficient conditions for RH(4)>0 are $\tau_c K_t \leq 1$ and $K_p \geq K_t$, as is found by substitution.

In the special case $\mu_b = 0$ we find the quadratic condition:

$$2(K_p + K_t) + \tau_c \left(K_p^2 + K_t^2\right) - \tau_c^3 K_p^2 K_t^2 + 2\Omega_s^2 \tau_c^2 K_p (1 - \tau_c K_t) + \\ + Y_b^2 \left[2(K_p + K_t) + \tau_c 4\Omega_s^2 + 2\tau_c K_p K_t\right] > 0 \,, \tag{72}$$

and this turns out to be subordinate to RH(5).

The fifth Routh determinant has many decades of monomial terms, and so we are led to consider the special case $\tan\Psi = Y_b \cos\mu_b$, as occurs on the matched generator curve. This results in a fifth order polynomial in Y_b. However, for a non-accelerating beam the condition $\mu_b = 0$ reduces the system to a quadratic in Y_b^2.

$$K_p \left[K_t (K_t + K_p - K_p \tau_c K_t) + \Omega_s^2 (1 - \tau_c K_t)\right] \left[2 + \tau_c (K_p + K_t) \right. \\ \left. + 2\Omega_s^2 \tau_c^2 (1 + \tau_c K_t) + \tau_c^2 K_p K_t\right] + Y_b^2 \left[2K_p K_t^2 + 2K_p^2 K_t + 2\tau_c K_p^2 K_t^2 \right. \\ + \Omega_s^2 \left(2K_p - 2K_t - \tau_c K_t^2 + 5\tau_c K_p K_t + 3\tau_c^2 K_p K_t^2\right) \\ \left. + 2\Omega_s^4 \tau_c \times (2 + \tau_c K_t)\right] - Y_b^4\, 2\Omega_s^2 K_t > 0 \,. \tag{73}$$

The allowed domain of Y_b will be maximized when the coefficients of Y_b^0 and Y_b^2 are greater than zero. By inspection, we note that $K_t \tau_c \leq 1$ and $K_p \geq K_t$ is a sufficient condition for coefficients Y_b^0 and Y_b^2 both to be positive.

Considering the equations in a less restrictive manner, we observe that the coefficient of Y_b^0 in RH(4) and the coefficients of Y_b^0 and Y_b^2 in RH(5) are quadratic binomials in K_p and K_t which can be solved for either K_p or K_t. For instance, if $\Omega_s \leq K_p \leq 1/\tau_c$ then there is no upper limit on the gain K_t.

RH(6): $1 - Y_b \sin\mu_b > 0$ imposes a further constraint on the beam current, which is the same as the no-loop case for a matched generator.

12 CAVITY WITH IDEAL RF FEEDBACK, BEAM DIPOLE MODE, AND TUNING LOOP, AND FAST BEAM PHASE-LOOP

We supplement the previous model with ideal voltage proportional feedback around the cavity, that is $\mathbf{I}_f = -(h/R)\mathbf{V}$. The phase-loop model is $s\phi_d = K_p(\phi_b - \phi_v)$, where ϕ_d is the demand phase at input to the summing junction. The tuning loop model is $s\tau_c \Delta\Omega_{\text{res}} = K_t(\phi_g - \phi_v)$. We should prefer to write our equations so that control loops couple to the demand values, that is entirely in terms of z_d and ϕ_d; and so z_g and ϕ_g must be eliminated. Now the instantaneous relation between current vector perturbations is:

$$\mathbf{I}_g^0 \mathbf{e}_g = \mathbf{I}_d^0 \mathbf{e}_d - (h/R)\mathbf{e}_v V^0$$

where $\mathbf{e}_g = (z_g + j\phi_g)$, $\mathbf{e}_d = (z_d + j\phi_d)$ and $\mathbf{e}_v = (z_v + j\phi_v)$. We have the steady state relation: $\mathbf{I}_d^0 = \mathbf{I}_g^0 + \mathbf{I}_V^0$, and so $\mathbf{I}_g^0$ can be written in terms of the steady state beam current and cavity parameters. We substitute the statics relations, and compare real and imaginary parts:

$$\mathcal{R}e \Rightarrow \quad Y_g C_g z_g - Y_g S_g \phi_g = (Y_g C_g + h) z_d - Y_g S_g \phi_d - h z_v ,$$

$$\mathcal{I}m \Rightarrow \quad Y_g S_g z_g + Y_g C_g \phi_g = Y_g S_g z_d + (Y_g C_g + h) \phi_d - h \phi_v .$$

Eliminating z_g, substituting $\varepsilon = (Y_b C_b - \tan \Psi)$ and subtracting a multiple of ϕ_v we find:

$$\left[(1 + Y_b S_b)^2 + \varepsilon^2\right] (\phi_g - \phi_v) = \left[(1 + Y_b S_b)^2 + \varepsilon^2 + h(1 + Y_b S_b)\right] (\phi_d - \phi_v) + h\varepsilon(z_v - z_d) .$$

To first order in ε we have:

$$(\phi_g - \phi_v) = \frac{(1 + h + Y_b S_b)}{(1 + Y_b S_b)} (\phi_d - \phi_v) + \frac{h\varepsilon}{(1 + Y_b S_b)^2} (z_v - z_d) . \tag{74a}$$

In the limit $Y_b S_b < 1$ and $h\varepsilon < 1$ we obtain the approximate relation: $(\phi_g - \phi_v) = (1 + h)(\phi_d - \phi_v)$, and this is valid on and close to the matched generator curve. Since there is no amplitude loop, $z_d = 0$ for all time; and so it becomes possible to write the matrix coefficient completely in terms of ϕ_d. In a similar fashion we may eliminate ϕ_g, and obtain z_g to first order in ε:

$$(z_g - z_v) = \frac{(1 + h + Y_b S_b)}{(1 + Y_b S_b)} (z_d - z_v) + \frac{h\varepsilon}{(1 + Y_b S_b)^2} (\phi_d - \phi_v) . \tag{74b}$$

We shall also choose to write the cavity equations as if the cavity is driven by $\mathbf{I}_d$ and $\mathbf{I}_b$; this has the effect of causing the substitution $z1 \Rightarrow (1 + h) + s\tau_c$.

12.0.1. Steady state relations The generator (or power tube) current $\mathbf{I}_g = \mathbf{I}_d + \mathbf{I}_f$ is the sum of the demand $\mathbf{I}_d$ and feedback $\mathbf{I}_f$ currents. The demand current $\mathbf{I}_d$ is set as follows:

$$Y_d \cos \psi_d = C_d Y_d = (1 + h) + Y_b \sin \mu_b \quad ; \quad Y_d \sin \psi_d = S_d Y_d = Y_b \cos \mu_b - \tan \Psi .$$

The presence of the phase-loop implies we must substitute explicit expressions for the demand current into the matrix coefficients. The feedback current is $\mathbf{I}_f = -(h/R)\mathbf{V}$.

12.0.2. Determinantal matrix The vector equation $\mathbf{M} \times \mathbf{x} = \mathbf{0}$, where $\mathbf{x} = (z_v, \phi_v, \tau_c \Delta\Omega_{\text{res}}, \phi_d, \phi_b, \delta E)$, implies:

$$\begin{vmatrix} z1 & z2 & 0 & S_d Y_d & -C_b Y_b & 0 \\ -z2 & z1 & -1 & -C_d Y_d & S_b Y_b & 0 \\ 0 & K_t(1+h) & s & -K_t(1+h) & 0 & 0 \\ 0 & K_p & 0 & s & -K_p & 0 \\ 0 & 0 & 0 & 0 & s & -\Omega_s^2 \\ -S_b & -C_b & 0 & 0 & C_b & s \end{vmatrix} = 0 \ . \qquad (75)$$

12.1 Routh determinants

It is found that the characteristic polynomial is identical with that of the previous Section 11, except with the substitutions:

$$\tau_c \Rightarrow \frac{\tau_c}{1+h} \ , \qquad \tan\Psi \Rightarrow \frac{\tan\Psi}{1+h} \ , \qquad Y_b \Rightarrow \frac{Y_b}{1+h} \ , \qquad (76)$$

made throughout the expression. This being so, we can take over all the results of the previous section regarding the positive definite nature of the polynomial and regarding the Routh-Hurwitz determinants. For example RH(6) becomes

$$1 - \frac{Y_b \sin\mu_b}{(1+h)} > 0 \ , \qquad (77)$$

which indicates that the stability limit is enhanced by a factor $(1+h)$, just as for the case of rf feedback only and matched generator.

13 CAVITY, BEAM DIPOLE MODE, AND SLOW TUNING LOOP, AND FASE BEAM PHASE-LOOP

We choose to model the tuning loop response as a single pole, or lag with time constant τ_t; that is

$$\tau_c \Delta\Omega_{\mathrm{res}} = K_t(\phi_g - \phi_v)/(1 + s\tau_t) \ . \qquad (78)$$

The phase-loop model is unchanged: $s\phi_g = K_p(\phi_b - \phi_v)$. Once more, there is the possibility for cross-coupling between phase and tuning loops through the cavity phase ϕ_g.

13.0.1. Steady state relations The generator current $\mathbf{I_g}$ is set as follows:

$$Y_g \cos\psi_g = Y_g C_g = 1 + Y_b \sin\mu_b \quad ; \quad Y_g \sin\psi_g = Y_g S_g = Y_b \cos\mu_b - \tan\Psi \ .$$

The presence of the phase-loop implies we must substitute explicit expressions for the generator into the determinantal matrix. The absence of an amplitude loop implies that the generator amplitude is set by an ideal feed-forward.

13.0.2. Determinantal matrix The vector equation $\mathbf{M} \times \mathbf{x} = \mathbf{0}$, where $\mathbf{x} = (z_v, \phi_v, \tau_c \Delta\Omega_{\mathrm{res}}, \phi_g, \phi_b, \delta E)$, implies:

$$\begin{vmatrix} z1 & z2 & 0 & S_g Y_g & -C_b Y_b & 0 \\ -z2 & z1 & -1 & -C_g Y_g & S_b Y_b & 0 \\ 0 & K_t & 1+s\tau_t & -K_t & 0 & 0 \\ 0 & K_p & 0 & s & -K_p & 0 \\ 0 & 0 & 0 & 0 & s & -\Omega_s^2 \\ -S_b & -C_b & 0 & 0 & C_b & s \end{vmatrix} = 0 \,, \tag{79}$$

with $z1 = 1 + s\tau_c \quad ; \quad z2 = \tan\Psi \quad ; \quad C_b = \cos(\mu_b) \quad ; \quad S_b = \sin(\mu_b)$.

13.0.3. Characteristic polynomial The size of the largest Routh determinant scales as the 4$^{\mathrm{th}}$ power of the size of the polynomial. Replacing the tuner integral control with lag control will increase the length of the polynomial by roughly 60%, and octuple the number of terms in RH(5). To avoid this, we shall consider the case $\tau_c \ll \tau_t$ and neglect τ_c as compared with τ_t whenever both terms are multiplied by the same factor[c]. Employing this strategy, the characteristic polynomial reduces to the approximate form:

$$\begin{aligned}
&\Omega_s^2 \Big[\cos\mu_b \Big(1 + \tan^2\Psi + K_t\Big) - \tau_c K_p \sin\mu_b \tan\Psi - Y_b \tan\Psi \\
&+ 0.5 Y_b \sin 2\mu_b (\tau_c K_p - K_t)\Big] + \Big\{K_p \Big(1 + \tan^2\Psi + K_t - \Omega_s^2 \tau_c \tau_t \sin\mu_b \tan\Psi\Big) \\
&+ \Omega_s^2 \cos\mu_b \Big[\tau_t (1 + \tan^2\Psi) + \tau_c K_t\Big] + Y_b \big[K_p (\sin\mu_b - \cos\mu_b \tan\Psi) \\
&+ \Omega_s^2 \tau_t (0.5 \tau_c K_p \sin 2\mu_b - \tan\Psi)\big]\Big\} \, s + \Big[(1 + K_p \tau_t)(1 + \tan^2\Psi) \\
&+ K_t (1 + \tau_c K_p) + Y_b K_p \tau_t (\sin\mu_b - \cos\mu_b \tan\Psi) + 2\Omega_s^2 \tau_c \tau_t \cos\mu_b\Big] \, s^2 \\
&+ \Big[\tau_t (1 + \tan^2\Psi) + \tau_c (K_t + K_p \tau_t) + (\Omega_s \tau_c)^2 \tau_t \cos\mu_b + Y_b \tau_c K_p \tau_t \sin\mu_b\Big] \, s^3 \\
&+ 2\tau_c \tau_t \, s^4 + \tau_c^2 \tau_t \, s^5 \,.
\end{aligned} \tag{80}$$

The finite gain K_t of the tuning loop at dc ($s = 0$) strongly modifies the form of the polynomial as compared with the case of a pure integral control which has infinite gain at dc. It should also be noted that K_t is now dimensionless, whereas previously (Section 11) it had the dimension of frequency.

[c] In fact, we have also neglected $2\tau_c$ compared with τ_t.

The coefficients of s^1 and s^2 in the characteristic polynomial have the possibility to change sign (when $\Psi > 0$), and a necessary condition for stability is that they be greater than zero.

The coefficient of s is positive if the beam current satisfies:

$$Y_b < \frac{2K_p\left(1+K_t\cos^2\Psi\right)+\Omega_s^2\left[2\cos\mu_b\left(\tau_t+\tau_c K_t\cos^2\Psi\right)-\tau_c K_p\tau_t\sin 2\Psi\sin\mu_b\right]}{2K_p\cos\Psi\sin(\Psi-\mu_b)+\Omega_s^2\tau_t\left(\sin 2\Psi-\tau_c K_p\cos^2\Psi\sin 2\mu_b\right)}$$

when $\Psi > 0$,

$$\text{or}\quad Y_b < \frac{2}{\sin 2\Psi}+\frac{K_t}{\tan\Psi}\frac{\left(K_p+\Omega_s^2\tau_c\right)}{\left(K_p+\Omega_s^2\tau_t\right)} \quad \text{if} \quad \mu_b = 0\,. \tag{81}$$

However, this condition is subordinate to RH(6), below.

The coefficient of s^2 is positive if $\tan\Psi \le \tan\mu_b$, otherwise we find a limit on the beam current:

$$Y_b < \frac{1+K_p\tau_t+\cos^2\Psi\left[K_t+\tau_c K_p(1+K_t)+2\Omega_s^2\cos(\mu_b)\tau_c\tau_t\right]}{K_p\cos\Psi\left[\tau_t\sin(\Psi-\mu_b)-\tau_c\cos\Psi\sin\mu_b\right]}$$

when $\Psi > 0$,

$$\text{or}\quad Y_b < \frac{2}{\sin 2\Psi}\left[1+\frac{1}{K_p\tau_t}\right]+\frac{1}{\tan\Psi}\left[\frac{K_t}{K_p\tau_t}+\frac{\tau_c}{\tau_t}(1+K_t)+\frac{2\Omega_s^2\tau_c}{K_p}\right] \quad \text{if} \quad \mu_b = 0\,. \tag{82}$$

13.1 Routh determinants

RH(1) and RH(2) imply $\tau_c \ge 0$ and $\tau_t \ge 0$.
RH(3): $\tau_c\tau_t[2\tau_t^2\sec^2\Psi+\tau_c K_t\tau_t(1-\tau_c K_p)+\tau_c K_p\tau_t^2\cos 2\Psi\sec^2\Psi+Y_b\tau_c K_p\tau_t^2(\sin\mu_b+\cos\mu_b\tan\Psi)] > 0$.
For positive detuning $\Psi > 0$ and $K_p > 0$, this condition is fairly easy to satisfy except in the cases $\tau_c K_p > 1$ and $\Psi > 45°$. There are several inferences that can be made, but first let us write RH(3) in a more convenient form:

$$\frac{2}{\tau_c K_p}(1+\tan^2\Psi)-K_t\frac{\tau_c}{\tau_t}+(1-\tan^2\Psi)+Y_b\cos\mu_b(\tan\mu_b+\tan\Psi) > 0\,, \tag{83}$$

where we have assumed $\tau_c K_p \gg 1$. Firstly, from a practical standpoint, the system should be stable in the limit of zero beam current, and this leads to a limit on the tuner gain K_t:

$$K_t\frac{\tau_c}{\tau_t} < 1 \;\text{ when }\; Y_b = 0\;,\;\; \tan^2\Psi \ll 1 \;\text{ and }\; \tau_c K_p \gg 1\,. \tag{84}$$

For the case of large detuning ($\tan^2\Psi \gg 1$) and $Y_b \to 0$ we find that $K_t < 0$ is required, but this conflicts with other Routh conditions. RH(3) can also be construed as a limitation on the beam current:

$$Y_b \cos\mu_b(\tan\mu_b + \tan\Psi) > K_t\frac{\tau_c}{\tau_t} + (\tan^2\Psi - 1) - \frac{2(1+\tan^2\Psi)}{\tau_c K_p} . \quad (85)$$

For the case $\tan^2\Psi \gg 1$ and $\mu_b = 0$ we find the approximate condition

$$Y_b > \tan\Psi\left[1 - \frac{2}{\tau_c K_p}\right] + \left[K_t\frac{\tau_c}{\tau_t} - 1\right]\frac{1}{\tan\Psi} \approx \tan\Psi \text{ with } \Psi > 45^\circ . \quad (86)$$

In light of this, the matched generator condition $\tan\Psi = Y_b C_b$ is not a sufficient condition for RH(3)> 0 .

The condition RH(3) > 0 also allows a domain of stability with negative detuning, $\Psi + \mu_b < 0$, provided the beam current is not too large:

$$Y_b < \frac{\tau_t(2 + \tau_c K_p\cos 2\Psi) + \tau_c K_t\cos^2\Psi(1 - \tau_c K_p)}{\tau_c K_p \tau_t \cos\Psi \sin|\Psi + \mu_b|} \quad \text{when} \quad \Psi + \mu_b < 0 . \quad (87)$$

For the case of a non-accelerating beam, $\mu_b = 0$, the condition simplifies to:

$$Y_b < \frac{2}{\tan|2\Psi|} + \frac{4}{\tau_c K_p \sin|2\Psi|} + \frac{K_t}{\tau_t \tan|\Psi|}\left[\frac{1}{K_p} - \tau_c\right] ,$$

and in the limit $\tau_c K_p \gg 1$ this becomes: $$Y_b < \frac{2}{\tan|2\Psi|} - \frac{\tau_c}{\tau_t}\frac{K_t}{\tan|\Psi|} . \quad (88)$$

RH(4): after factoring away $\tau_c\tau_t$, this condition contains 46 monomial terms. Substituting the matched generator condition, $\tan\Psi \Rightarrow Y_b\cos\mu_b$, reduces the system to 29 terms; and making use of the condition $\tau_c \ll \tau_t$ reduces the system still further.
RH(4) becomes:

$$\begin{aligned}
&\Big\{2\tau_t + \tau_t^2 K_p(2 + \tau_c K_p) + 2K_t\left(\tau_t - \tau_c^2 K_p\right) + \tau_c K_t^2\left(1 - \tau_c^2 K_p^2\right) \\
&+ 2(\Omega_s\tau_c)^2 K_p \tau_t \cos\mu_b \times (\tau_t - \tau_c K_t)\Big\} + Y_b\big\{2\tau_t[K_p(\tau_t + \tau_c K_t) \\
&+ \tau_c K_p^2 \tau_t + (\Omega_s\tau_c)^2\cos\mu_b(K_p\tau_t - K_t)]\sin\mu_b\big\} \\
&+ Y_b^2\tau_t\Big[2\cos^2\mu_b(2 + K_t + K_p\tau_t + \tau_c K_p K_t) \\
&+ \tau_c K_p^2\tau_t\sin^2\mu_b + 4\Omega_s^2\cos(\mu_b)\tau_c\tau_t\Big] \\
&+ Y_b^3 K_p \tau_t^2\cos\mu_b\sin 2\mu_b + Y_b^4 2\tau_t\cos^4\mu_b > 0 .
\end{aligned} \quad (89)$$

For a non-accelerating beam, this becomes a quadratic in Y_b^2 :

$$\Big\{2\tau_t + \tau_t^2 K_p(2+\tau_c K_p) + 2K_t(\tau_t - \tau_c^2 K_p) + \tau_c K_t^2(1-\tau_c^2 K_p^2) + 2(\Omega_s\tau_c)^2 K_p\tau_t(\tau_t - \tau_c K_t)\Big\} + + Y_b^2\tau_t\Big[2\left(2+K_t+K_p\tau_t+\tau_c K_p K_t\right) + 4\left(\Omega_s^2\tau_c\tau_t + 1\right)\Big] + Y_b^4 2\tau_t > 0\,. \tag{90}$$

The domain of stability will be maximized when the coefficient of Y_b^0 (the term independent of Y_b) is positive.

RH(5): After substituting the matched generator condition, $\psi_g = 0$, the fifth Routh determinant still consists of 251 monomial terms. However, after substituting $\mu_b = 0$ and factoring, the system collapses to 71 terms, and making extensive use of the condition $\tau_c \ll \tau_t$ reduces the condition still further. We find the following quadratic condition in Y_b^2 :

$$\begin{aligned}
&K_p\Big\{2\tau_t\left(1+\Omega_s^2\tau_t^2\right) + K_t^2\Big[2\tau_t - \tau_c^2K_p(2+\tau_cK_p)\\
&+ (\Omega_s\tau_c)^2\tau_t\left(1-\Omega_s^2\tau_c^2 - 2\tau_cK_p\right)\Big] + \tau_cK_p^2\tau_t^2 + K_p\tau_t^2\left(2+\Omega_s^2\tau_c\tau_t\right)\\
&+ K_t\tau_t\Big[4 + K_p\tau_t(2+\tau_cK_p) - \Omega_s^2\tau_c\tau_t\left(1-2\tau_cK_p - \Omega_s^2\tau_c^2\right)\Big]\\
&+ (\Omega_s\tau_t)^2\tau_c\left(2\Omega_s^2\tau_c\tau_t - 7\right) + (\tau_cK_t)^3\Big[\Omega_s^2(1-\tau_cK_p) - K_p^2\Big]\Big\} +\\
&Y_b^2\tau_t\times\Big\{K_p\Big[4 + K_t(6+2K_t) + \Omega_s^2\left(2\tau_t^2 + 3\tau_c^2K_t^2 + 5\tau_cK_t\tau_t\right)\Big]\\
&+ 2K_p^2\left(\tau_t + \tau_cK_t^2 + K_t\tau_t\right) + 4\Omega_s^4\tau_c\tau_t^2 - (\Omega_sK_t)^2\tau_c - 2\Omega_s^2K_t\tau_t\left(1-\Omega_s^2\tau_c^2\right)\Big\}\\
&+ Y_b^4\,2\tau_t\left(K_p + K_pK_t - \Omega_s^2K_t\tau_t\right) > 0\,.
\end{aligned} \tag{91}$$

Under the conditions $K_t < \tau_t/\tau_c$ and $K_p > \Omega_s^2\tau_t$ RH(5) is positive for all Y_b, though there may be an upper bound on K_p. The conditions on K_t and K_p do not conflict with RH(4) > 0.

RH(6): $(K_t + \sec^2\Psi)\cos\mu_b - (Y_b + \tau_cK_p\sin\mu_b)\tan\Psi + 0.5Y_b(\tau_cK_p - K_t) > 0$.
Solving for Y_b we find the beam current is limited to:

$$Y_b < \frac{2\cos\mu_b(1+K_t\cos^2\Psi) - \tau_cK_p\sin 2\Psi\sin\mu_b}{\sin 2\Psi + \cos^2\Psi\sin 2\mu_b(K_t - \tau_cK_p)} \quad \text{when} \quad \Psi > 0$$

$$\text{or} \quad Y_b < \frac{2}{\sin 2\Psi} + \frac{K_t}{\tan\Psi} \quad \text{if} \quad \mu_b = 0\,. \tag{92}$$

14 CAVITY, BEAM DIPOLE MODE AND BEAM QUADRUPOLE MODE

Robinson type stability for dipole-quadrupole mode coupling has been investigated by Wang,[10] using a Sacherer style formalism and for the case of a non-accelerating beam. In this section we employ the equivalent circuit model and generalize to the case of an accelerating beam. However, for ease of exposition we shall consider the Routh determinants for $\mu_b = 0$ before going on to the case $\mu_b \neq 0$.

14.0.1. Steady state relations The generator current $\mathbf{I}_g$ is set by an ideal feed forward as per Equation (33).

14.0.2. Determinantal matrix The vector equation $\mathbf{M} \times \mathbf{x} = \mathbf{0}$, where $\mathbf{x} = (z_v, \phi_v, z_b, \phi_b, \delta E, \theta, \delta W)$, implies:

$$\begin{vmatrix} z1 & z2 & S_b Y_b & -C_b Y_b & 0 & 0 & 0 \\ -z2 & z1 & C_b Y_b & S_b Y_b & 0 & 0 & 0 \\ 0 & 0 & 0 & s & -\Omega_s^2 & 0 & 0 \\ -S_b & -C_b & 0 & C_b & s & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s & -\Omega_s^2 \\ C_b\Theta_0 & -S_b\Theta_0 & 0 & 0 & 0 & 4C_b & s \\ 0 & 0 & 1 & 0 & 0 & F_0 & 0 \end{vmatrix} = 0\,, \tag{93}$$

with $z1 = 1 + s\tau_c$; $z2 = \tan\Psi$; $C_b = \cos(\mu_b)$; $S_b = \sin(\mu_b)$; $F_0 = F_0(\Theta_0)$.

14.0.3. Characteristic polynomial

$$\begin{aligned} &\Omega_s^4 \left[\cos\mu_b 4(\cos\mu_b \sec^2\Psi - Y_b\tan\Psi) + Y_b F_0\Theta_0(Y_b - \cos\mu_b\tan\Psi)\right] + \\ &+ \Omega_s^2\left[5\cos\mu_b\sec^2\Psi + 4(\tau_c\Omega_s\cos\mu_b)^2 - Y_b\tan\Psi(1 + F_0\Theta_0)\right]s^2 + \\ &+ 8(\Omega_s^2\cos\mu_b)^2\tau_c s + 10\tau_c\Omega_s^2\cos\mu_b\, s^3 + \\ &+ \left[\sec^2\Psi + 5(\Omega_s\tau_c)^2\cos\mu_b\right]s^4 + 2\tau_c s^5 + \tau_c^2 s^6\,. \end{aligned} \tag{94}$$

14.1 Non-accelerating beam

For the case $\mu_b = 0$ the characteristic polynomial reduces to:

$$\Omega_s^4\left[4\left(\sec^2\Psi - Y_b\tan\Psi\right) + Y_b F_0\Theta_0(Y_b - \tan\Psi)\right] + \Omega_s^2\left[5\sec^2\Psi + 4(\tau_c\Omega_s)^2\right.$$

$$\left.-Y_b\tan\Psi(1+F_0\Theta_0)\right]s^2 + 8\Omega_s^4\tau_c\,s + 10\tau_c\Omega_s^2\,s^3 + \left[\sec^2\Psi + 5(\Omega_s\tau_c)^2\right]s^4 + 2\tau_c\,s^5 + \tau_c^2\,s^6$$

The coefficient of s^2 has the possibility to change sign when $\Psi > 0$, and this implies a limit on the beam current:

$$Y_b < \frac{2}{(1+F_0\Theta_0)}\left[\frac{5}{\sin 2\Psi} + \frac{2(\Omega_s\tau_c)^2}{\tan\Psi}\right].$$

However, this condition is superseded by RH(5), RH(6) and RH(7).

14.1.1. Routh determinants

RH(1), RH(2) and RH(3) all imply $\tau_c \geq 0$.
RH(4): $(2\Omega_s\tau_c)^2(1+F_0\Theta_0)\tan\Psi > 0$ implies the tuning angle must be positive ($\Psi > 0$), because $F_0 \geq 0$.
RH(5) factors and can be manipulated to the form:

$$\tan\Psi(1+4F_0\Theta_0) + Y_b\left[F_0\Theta_0 - \sin^2\Psi(1+F_0\Theta_0)^2\right] > 0\,. \tag{95}$$

There is no constraint for tuning angles $\sin\Psi \leq \sqrt{F_0\Theta_0}/(1+F_0\Theta_0)$; and in this range, condition RH(6) supersedes. In the cases (i) $F_0\Theta_0 \ll 1$ and (ii) large tuning angles $\sin\Psi \approx 1$ we find the two approximate conditions:

$$\text{(i)}\quad Y_b < \frac{2}{\sin 2\Psi}\times\frac{(1+4F_0\Theta_0)}{(1+F_0\Theta_0)^2}\,,\qquad \text{(ii)}\quad Y_b < \frac{\tan\Psi(1+4F_0\Theta_0)}{1+F_0\Theta_0(1+F_0\Theta_0)}\,. \tag{96}$$

Case (i) resembles the Robinson limit, and both conditions are close in numerical value to the Robinson limit for large tuning angles $\Psi > 1$ radian.
RH(6) factors to: $(3\tan\Psi + Y_b)(3\tan\Psi - Y_bF_0\Theta_0) > 0$ from which we conclude: $Y_b < 3\tan\Psi/(F_0\Theta_0)$ when $\Psi > 0$ and $\Theta_0 > 0$. This is a severe constraint for small tuning angles and long bunches. This instability regime has been observed in computer simulations reported in Reference 11.
RH(7):

$$4\sec^2\Psi - Y_b\tan\Psi(4+F_0\Theta_0) + Y_b^2F_0\Theta_0 > 0 \text{ or } (\tan\Psi - Y_b)(\tan\Psi - Y_bF_0\Theta_0/4) + 1 > 0.$$

The presence of the Y_b^2 term in the quadratic will favourably modify the stability as compared with the Robinson limit. However, for small tuning angles this condition is subordinate to

RH(6). For large tuning angles $\tan\Psi \gg 1$, we apply a single Newton-Raphson iteration to find the approximate condition:

$$Y_b < \frac{2}{\sin 2\Psi} \times \frac{(1+2F_0\Theta_0)}{(1+F_0\Theta_0/4)^2} \,. \tag{97}$$

It should be noted that $F_0\Theta_0$ is usually less than unity, so the numerator dominates and the stability limit is enhanced by including the quadrupole mode in the analysis.

To summarize, the effect of dipole-quadrupole mode coupling is to dramatically alter the small tuning angle stability. However, for large tuning angles the usual Robinson limit applies with only slight modification.

14.1.2. Matched generator We may substitute the condition $\psi_g = 0$ and find the beam current limit for operation on the matched generator curve $Y_b = \tan\Psi$.

The only conditions which are not trivial to satisfy are: the coefficient of s^2 greater than zero, and RH(5) and RH(6) greater than zero. RH(6) reduces to $3 - F_0\Theta_0 > 0$, and is a sufficient condition for stability. Since $F_0\Theta_0 \leq 1$ in most cases, RH(6) is satisfied. Hence we recover the same result as for the case of dipole motion only: for a non-accelerating beam and a driven narrow-band cavity with resistive loading of the generator there is no longitudinal instability.

14.2 Accelerating beam

We consider the condition $\mu_b > 0$, in which case the coefficients of s^1 and s^3 are positive. From the characteristic polynomial, Equation (94) we note that the coefficient of s^2 may change sign when $\Psi > 0$; this implies a beam current limit,

$$Y_b < \frac{\cos\mu_b\left[5\sec^2\Psi + 4(\Omega_s\tau_c)^2\cos\mu_b\right]}{(1+F_0\Theta_0)\tan\Psi} \,, \tag{98}$$

but the condition is, however, subordinate to the Routh determinants below.

14.2.1. Routh determinants
RH(1), RH(2) and RH(3) imply $\tau_c \geq 0$.
RH(4) simplifies to: $4\tan\Psi(\Omega_s\tau_c)^2 Y_b(1+F_0\Theta_0) > 0$; and we recover the condition $\Psi > 0$.
RH(5) simplifies to:

$$8\Omega_s^4\tau_c^3 Y_b \left\{\frac{\cos\mu_b\tan\Psi(1+4F_0\Theta_0)}{\cos^2\Psi} + \frac{F_0\Theta_0 Y_b}{\cos^2\Psi} - Y_b\tan^2\Psi(1+F_0\Theta_0)^2\right\} > 0 \,, \tag{99}$$

which implies the beam current condition:

$$Y_b < \frac{\cos\mu_b\tan\Psi(1+4F_0\Theta_0)}{\sin^2\Psi(1+F_0\Theta_0)^2 - F_0\Theta_0} \,. \tag{100}$$

The last expression usually provides a small modification to the Robinson limit.

RH(6) factors to:

$$32F_0\Theta_0\Omega_s^8\tau_c^5Y_b^2(Y_b+3\cos\mu_b\tan\Psi)(3\cos\mu_b\tan\Psi-F_0\theta_0Y_b)/\cos^2\Psi>0\ . \quad (101)$$

We suppose $F_0 \geq 0$, so this leaves the condition $Y_b < 3\cos\mu_b\tan\Psi/F_0\Theta_0$ which poses a severe constraint to the maximum beam current at small tuning angles unless the bunches are short.

RH(7) factors to:

$$\Omega_s^4\left[\frac{4\cos^2\mu_b}{\cos\Psi^2}-Y_b\tan\Psi\cos\mu_b\,[4+F_0\Theta_0]+F_0\Theta_0Y_b^2\right]>0\ , \quad (102)$$

which simplifies to

$$Y_b\,[1+(F_0\Theta_0/4)]<\frac{2\cos\mu_b}{\sin 2\Psi}+\frac{F_0\Theta_0Y_b^2}{4\cos\mu_b\tan\Psi}\ . \quad (103)$$

The term in Y_b^2 in this quadratic will favourably modify the stability compared with the Robinson limit. However, for small tuning angles conditions RH(5) and RH(6) may supersede RH(7).

15 CONCLUSION

Analytic criteria for the stability of a beam-loaded rf cavity with beam phase-loop and cavity tuning-loop both modelled as ideal integrators have been derived. Few simple results comparable to the Robinson criteria are obtainable for these multi-parameter systems. In fact, due to the size of the 5^{th} Routh determinant (up to two hundred monomial terms for the accelerating beam case) the abridged criteria reported herein are incomplete in that they are not both necessary and sufficient[e]. Thus substitution of numerical system parameters, or the use of graphical methods may still be required to establish absolute stability bounds.

ACKNOWLEDGEMENTS

I should like to thank Ron Balden for his endless patience in helping me master some of the intricacies of SMP,[f] and for his valuable assistance in suggesting subtle expansion and simplification strategies that are less likely to exhaust the computer virtual memory allocated to SMP.

[e] Of course, the complete criteria are available in very lengthy SMP output files.

[f] SMP is the direct predecessor of the much more widespread symbolic manipulation/mathematical computation system Mathematica, the chief designer of both systems being Stephen Wolfram. Development work on SMP effectively ceased in 1988 with the advent of Mathematica. The fundamental similarities in design philosophy (e.g. the central role of pattern matching) between SMP and Mathematica allows (unlike Mathematica's current major active competitors Maple and MACSYMA) a fairly straightforward translation of existing SMP work into Mathematica.

REFERENCES

1. E. Routh: Advanced Part of "Dynamics of a System of Rigid Bodies", Vol.II (Macmillan & Co., London, 1930).
2. K. Robinson: Stability of beam in radio-frequency system; Internal Report CEAL-1010, Feb. 1964, Cambridge Electron Accelerator.
3. F. Pedersen: Beam loading effects in the CERN PS Booster, IEEE Trans. Nucl. Sci. Vol. NS-22, No. 3, June 1975, p. 1906–1090.
4. SMP: A Symbolic Manipulation Program; Inference Corporation, Los Angeles, California.
5. S. Koscielniak: Injection transient compensation in the Collector; TRIUMF TRI-DN-91-K166.
6. D. Boussard: Control of cavities with high beam loading; CERN/SPS/85-31 (ARF).
7. C. Bovet *et al.*, A selection of formulae and data useful for the design of A.G. synchrotrons; CERN/MPS-SI/Int. DL/70/4.
8. L. Teng and W. Lee: Beam bunch length matching at transition; Fermilab report NAL/FN-223, 1971.
9. S. Koscielniak: A general theory of beam loading; TRIUMF TRI-DN-89-K25.
10. Tai-Sen Wang: Bunches beam longitudinal mode coupling and Robinson type instabilities, Particle Accelerators Vol.34, p.105, 1990.
11. S. Koscielniak: Computer simulations of beam loading; TRIUMF TRI-DN-88-34.