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# IMPROVED ENVELOPE AND EMITTANCE DESCRIPTION OF PARTICLE BEAMS USING THE FOKKER-PLANCK APPROACH

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Beam dynamics calculations that are based on the Vlasov equation do not permit the treatment of stochastic phenomena such as intra-beam scattering. If the nature of the stochastic process can be regarded as a Markov process, we are allowed to use the Fokker-Planck equation to describe the change of the phase space volume the beam occupies. From the Fokker-Planck equation we derive equations of motion for the beam envelopes and for the rms-emittances. Compared to previous approaches based on Liouville's theorem, these equations contain additional terms that describe the temperature balancing within the beam. Our formalism is applied to the effect of intra-beam scattering relevant for beams circulating in storage rings near thermodynamical equilibrium. In this case, the Fokker-Planck coefficients can be treated as adiabatic constants of motion. Due to the simplified analysis based on 'beam moments', we obtain fairly simple equations that allow us to estimate the growth rate of the beam emittance.

KEY WORDS: Envelope equation, Fokker-Planck equation, storage ring, intra-beam scattering

## 1 INTRODUCTION

Analytical approaches in the optics of charged particle beams including self-forces of the beam usually assume that Liouville's theorem applies for the phase space distribution function $f$ in the 6-dimensional '$\mu$ space'. According to this theorem, the volume of an element of phase space populated by particles remains constant in time. From Liouville's theorem we can derive straightforwardly the equation of motion for $f$, commonly referred to as the Vlasov or the 'collisionless' Boltzmann equation in the general case. Although it is practically impossible to integrate the Vlasov equation directly, it may serve as the starting-point for analytical investigations under certain conditions. As an example, we quote the so-called 'Kapchinskij-Vladimirskij' model[1] for the distribution function $f$, which simplifies the Vlasov equation to yield a closed set of differential equations for the envelopes of an unbunched beam.

Nevertheless, we have to recall that Liouville's theorem applies only if the beam transformation can be regarded as a deterministic process. On the basis of Vlasov's equation, we are thus unable to describe the variety of stochastic phenomena of beam optics – such as

turbulent heating during the relaxation phase of free field energy, thermal equipartitioning within the beam, or the effect of intra-beam scattering in storage rings.

Recently, Bohn[2] made an attempt to extend the self-consistent Vlasov-Poisson treatment of beam optics. He included the Fokker-Planck equation in his analysis to describe the dilution of the phase space due to turbulent heating for a mismatched 'sheet' beam.

Since the Fokker-Planck equation is even more complicated than the Vlasov equation, attempts for a direct integration without referring to the 'sheet beam' model do not appear worth while. The complexity of the problem of directly integrating the Vlasov equation led to attempts to simplify the analytical description of beam optics. A largely accepted method has been presented by Lapostolle[3] and Sacherer[4] describing the beam optics in terms of 'root-mean-square' (rms) beam moments and their respective equations of motion. In this paper, we apply this idea to derive 'moment equations' to the Fokker-Planck equation.

We thus obtain additional moment terms that – at least in principle – allow us to describe any Markov process within the beam not yet covered by the Vlasov approach. The remaining task is to determine the Fokker-Planck coefficients for the non-Liouvillean process to be investigated.

In this article, we refer to the work of Jansen[5], who calculated these coefficients for the effect of intra-beam scattering for charged particle beams near thermodynamical equilibrium. Including collision effects is particularly important for highly charged heavy ions and high phase space densities. We restrict ourselves to cases where the effect of collisions is small compared with external forces or the smooth part of the self-fields. Hence we do not treat the beam's turbulent heating phase, but the long-term behaviour of 'relaxed' beams undergoing intra-beam scattering, which typically occurs for beams circulating in storage rings.

Prior to mathematical details, we discuss in section 2 the restrictions under which this Fokker-Planck approach is applicable to charged particle beams. Proceeding further, we will set up the extended moment equations for the frame of reference moving with the beam (section 3). These moment equations are transformed to the laboratory system in section 4, using the usual 'trace space' notation. We thus obtain the extended envelope and emittance equations (section 5). After discussing the concepts of 'excess field energy' and 'temperature' of ion beams in sections 6 and 7, respectively, we review in section 8 the relationship between the scattering-related Fokker-Planck coefficients and the beam parameters. In the following, we derive differential equations that describe the change of beam emittances due to a temperature balancing process. This is performed for unbunched beams in section 9 and for the more general cases of bunched beams and coasting beams in storage rings in section 10 and 11. Finally, we will discuss some numerical examples in section 12. The implications of our theory for beam transport devices such as quadrupole channels and storage rings will be outlined. In addition, it is shown that the nature of *numerical* emittance growth effects observed in computer simulations of charged particle beams can also be explained with the help of Fokker-Planck approach.

## 2 THE FOKKER-PLANCK APPROACH FOR CHARGED PARTICLE BEAMS

The 6-dimensional phase space distribution function $f = f(\vec{x}, \vec{p}; t)$ that represents a

charged particle beam satisfies Liouville's theorem

$$\frac{df}{dt} = 0 \,, \tag{1}$$

if the particles do not interact and if the time evolution of their coordinates follows Hamilton's canonical equations. For charged particle beams whose electric self-fields must be taken into account, Eq. (1) remains true if the space charge potential $V^{sc}$ is sufficiently smooth that it can be treated analogous to an external defocusing potential.

Writing (1) explicitly and inserting the equation of motion of a particle with the mass $m$ and charge $q$ under the influence of external focusing fields $\vec{F}^{ext}$ and the smooth part of the self-fields $\vec{E}^{sc}$

$$\dot{\vec{p}} = \vec{F}^{ext} + q\vec{E}^{sc} \,,$$

we directly obtain the Vlasov equation. The Poisson equation describes the relationship between the distribution function $f$ and the space-charge field $\vec{E}^{sc}$:

$$div\, \vec{E}^{sc} = -4\pi\, q \int f\, d^3p \,.$$

If we want to include irreversible effects in our analysis of beam optics, we must give up the strictly deterministic viewpoint of the Vlasov equation. Examples of such effects are turbulent heating due to mismatch, energy loss due to radiation damping, residual gas scattering, cooling, intra-beam scattering. All the processes are associated with a change of the 6-dimensional phase space volume the beam occupies, hence they may not be analyzed on the basis of the Vlasov approach.

The action of non-Liouvillean effects can be described mathematically replacing Eq. (1) by the Boltzmann equation:

$$\frac{df}{dt} = \left[\frac{\partial f}{\partial t}\right]_{NL} \,.$$

Explicitly, this equation can be written in its non-relativistic form as:

$$\frac{\partial f}{\partial t} + \vec{p}\cdot\vec{\nabla}_x f + \frac{1}{m}\left(\vec{F}^{ext} + q\vec{E}^{sc}\right)\cdot\vec{\nabla}_p f = \left[\frac{\partial f}{\partial t}\right]_{NL} \,. \tag{2}$$

The r.h.s. term thus formally describes the change of the beam's phase space volume due to the non-Liouvillean effects. If this term can be neglected, Eq. (2) reduces to the Vlasov equation.

Under certain conditions to be discussed below, the non-Liouvillean effects can be modelled by the Fokker-Planck equation (see References.[5], for example):

$$\left[\frac{\partial f}{\partial t}\right]_{NL} = -\sum_i \frac{\partial}{\partial p_i}\left\{F_i(\vec{p},t)\, f\right\} + m^2 \sum_{i,j} \frac{\partial^2}{\partial p_i \partial p_j}\left\{D_{ij}(\vec{p},t)\, f\right\} \,.$$

Herein $D_{ij}$ stands for the 'diffusion tensor' elements, whereas the $F_i$ are referred to as the 'drift vector' components.

The Fokker-Planck approach is based on the assumption that there exists a specific time interval $\Delta t$ such that the phase space distribution $f(t + \Delta t)$ at time $t + \Delta t$ only depends on $f(t)$ and *not* on $f$ at any earlier time. During this time interval $\Delta t$, any 'memory' of earlier events is cancelled. This means that the time interval $\Delta t$ covers a considerable number of events, e.g. particle-particle collisions if we want to analyse the effect of intra-beam scattering. During this interval, the phase space distribution function $f$ must remain approximately unchanged. In other words, the velocity changes of all particles due to the non-Liouvillean effect during successive time intervals must be statistically independent. A stochastic process that has this property is referred to as a 'Markov process'.

It can be concluded that the Fokker-Planck approach is only applicable if $f$ changes slowly compared with the typical time constant of the non-Liouvillean effect.

The most difficult part in our task to include non-Liouvillean effects in our analysis of beam optics is to determine the Fokker-Planck coefficients $D_{ij}$ and $F_i$. Especially the effect of turbulent heating of the beam due to mismatch – where the beam is far from thermodynamical equilibrium – implies that that these coefficients are explicitly time dependent. This makes it extremely complicated – or even impossible – to evaluate the Fokker-Planck coefficients on a rigorous basis, hence forces to apply heuristic models[2].

If we want to treat cases where the beam can be assumed to be near thermodynamical equilibrium, things become easier. In these cases, the Fokker-Planck coefficients can be taken to be adiabatic constants of motion. This is true for beam circulating in storage rings for times that are large compared to the internal beam dynamics' time scale (for example the betatron frequency).

Eq. (2) represents the self-consistent equation of motion for a phase space distribution $f$. It describes a charged particle beam under the influence of external focusing forces, the *smooth* part of the self-fields due to its macroscopic charge density function, and the non-Liouvillean effects of a Markov process within the beam, which may change the phase space volume the beam occupies.

## 3 THE EXTENDED MOMENT EQUATIONS

In order to circumvent a lengthy direct integration of Eq. (2), we switch from the total information on the beam that is contained in the phase space distribution function $f$ to physical quantities that contain less information on the beam but still can serve as measures for the beam quantities of interest. A usual way to simplify our description of beam dynamics is to derive equations of motion for the beam's second central moments from the equation of motion for $f$.

With $f$ as the normalized particle phase space distribution function, the beam moments can be defined by integrals over all phase space, $\tau$:

$$\begin{aligned}
\langle x_i^2 \rangle &= \int x_i^2 \, f \, d\tau \\
\langle x_i p_i \rangle &= \int x_i p_i \, f \, d\tau \\
\langle p_i^2 \rangle &= \int p_i^2 \, f \, d\tau \\
\langle x_i E_i \rangle &= \int x_i E_i \, f \, d\tau \\
\langle p_i E_i \rangle &= \int p_i E_i \, f \, d\tau \, .
\end{aligned}$$

We obtain – as an example – the time derivative of $\langle x_i^2 \rangle$ using

$$\frac{d}{dt}\langle x_i^2 \rangle = \int x_i^2 \frac{\partial f}{\partial t} d\tau \tag{3}$$

and inserting $\partial f/\partial t$ from Eq. (2) into (3). It is physically reasonable to assume that the phase space density $f$ as well as all its derivatives vanish at the boundary of the populated phase space. As a consequence, all solvable parts of the integrals evaluate to zero inserting the integration boundaries. The non-vanishing moment terms assemble for each phase space plane $i = 1, 2, 3$ to the following coupled set of equations:

$$\begin{aligned}
\tfrac{d}{dt}\langle x_i^2 \rangle - \tfrac{2}{m}\langle x_i p_i \rangle &= 0 \\
\tfrac{d}{dt}\langle x_i p_i \rangle - \tfrac{1}{m}\langle p_i^2 \rangle - \langle x_i F_i^{\text{ext}} \rangle - q \, \langle x_i E_i \rangle &= \langle x_i F_i \rangle \\
\tfrac{d}{dt}\langle p_i^2 \rangle - 2 \, \langle p_i F_i^{\text{ext}} \rangle - 2q \, \langle p_i E_i \rangle &= 2 \, \langle p_i F_i \rangle + 2m^2 \langle D_{ii} \rangle \, .
\end{aligned} \tag{4}$$

The left-hand terms of Eqs. (4) have been derived first by Sacherer[4], assuming that Liouville's theorem applies. The non-zero right-hand terms[6] originate in the Fokker-Planck equation that describe the non-Liouvillean effects. We observe that only the diagonal terms of the diffusion tensor appear in the moment equations. All terms of the right-hand side of (4) are proportional to the Fokker-Planck coefficients. As a consequence, they vanish if the non-Liouvillean effects can be neglected. The set of coupled differential equations then reduces to Sacherer's equations, as expected.

## 4 TRANSFORMATION TO THE LABORATORY SYSTEM

For circular structures, the dependency of the orbital angular velocity on the particle momentum is included defining an 'effective mass' $m_{z;b}$ for the longitudinal equation of motion. In the center of momentum frame moving with the beam, it is given by[7]:

$$m_{z;b} = \frac{m}{\eta \gamma^2} \, ,$$

with

$$\eta = \begin{cases} \gamma^{-2} - \gamma_t^{-2} & \text{for a circular orbit} \\ \gamma^{-2} & \text{for a straight motion.} \end{cases} \tag{5}$$

Herein, $\gamma_t$ denotes the *transition* $\gamma$. Thus, in the beam's frame of reference, we have $m_{z;b} \equiv m$ for a straight motion, whereas $m_{z;b} = m/(1 - \gamma^2/\gamma_t^2)$ in a ring structure. The definition of a constant 'effective mass' for the longitudinal beam dynamics is justified,

if the time scale of the non-Liouvillean effects is large compared to the particle's revolution time. If this is not true, one has to use a time dependent $\eta$ in place of the constant one. This replacement does not affect the structure of the beam moment equations.

Usually we may formulate the equation of motion (2) in the non-relativistic limit, if we refer to the beam's center of momentum frame. In this paper, we will restrict ourselves to these cases. On the other hand, we are interested in the beam properties as they occur in the laboratory system. We therefore must transform all quantities from the beam frame (index $b$) into the laboratory system (index $\ell$). In the latter system, the time scale is 'stretched' by the relativistic mass factor $\gamma$ compared to the particle's proper time. Following the usual 'trace space'[7] formulation of beam optics, we replace the proper time $t$ as the independent variable by the distance $s$ along the focusing structure measured in the laboratory system:

$$s \equiv (\Delta s)_\ell = c\beta \cdot (\Delta t)_\ell = c\beta\gamma \cdot (\Delta t)_b \equiv c\beta\gamma \cdot t \ .$$

We further introduce the 'trace space' coordinates via

$$x_1 \equiv x_b = x_\ell \quad , \quad x_2 \equiv y_b = y_\ell \quad , \quad x_3 \equiv z_b = \gamma\,(\Delta z)_\ell \equiv \gamma z_\ell \ ,$$

and

$$p_1 \equiv p_{x;b} = mc\beta\gamma \cdot x'_\ell \quad , \quad p_2 \equiv p_{y;b} = mc\beta\gamma \cdot y'_\ell \quad , \quad p_3 \equiv p_{z;b} = mc\beta\eta^{-1} z'_\ell \ .$$

The electric space charge fields as well as the external focusing forces transform according to:

$$E_{x;b} = \gamma^{-1} E_{x;\ell} \quad , \quad E_{y;b} = \gamma^{-1} E_{y;\ell} \quad , \quad E_{z;b} = E_{z;\ell} \ .$$

The drift vector components $F_i$ stand for the frictional forces that are part of the non-Liouvillean effects. To first order, the drift coefficients $F_i$ can be approximated by Stokes' law:

$$F_i = -\beta_{f;i} \cdot p_i \ ,$$

wherein the $\beta_{f;i}(t)$ are usually designated as 'dynamic friction coefficients'. With the external focusing force functions $k_{x,y,z}(s)$ that describe *linear* external focusing forces, the moment equations (4) become, skipping all indices $\ell$:

$$\begin{aligned}
&\tfrac{d}{ds}\langle x^2\rangle - 2\langle xx'\rangle && = 0 \\
&\tfrac{d}{ds}\langle xx'\rangle - \langle x'^2\rangle + k_x^2(s)\langle x^2\rangle - \tfrac{q}{mc^2\beta^2\gamma^3}\langle xE_x\rangle && = -\tfrac{\beta_{f;x}}{c\beta\gamma}\,\langle xx'\rangle \\
&\tfrac{d}{ds}\langle x'^2\rangle + 2k_x^2(s)\langle xx'\rangle - \tfrac{2q}{mc^2\beta^2\gamma^3}\langle x'E_x\rangle && = -\tfrac{2\beta_{f;x}}{c\beta\gamma}\,\langle x'^2\rangle + \tfrac{2\langle D_{xx}\rangle}{c^3\beta^3\gamma^3} \ .
\end{aligned} \tag{6}$$

The corresponding set of equations holds for the $y$-direction. For the $z$-direction, we obtain:

$$\begin{aligned}
\tfrac{d}{ds}\langle z^2\rangle - 2\langle zz'\rangle &= 0 \\
\tfrac{d}{ds}\langle zz'\rangle - \langle z'^2\rangle + k_z^2(s)\langle z^2\rangle - \tfrac{q\cdot\eta}{mc^2\beta^2\gamma}\langle zE_z\rangle &= -\tfrac{\beta_{f;z}}{c\beta\gamma}\langle zz'\rangle \\
\tfrac{d}{ds}\langle z'^2\rangle + 2k_z^2(s)\langle zz'\rangle - \tfrac{2q\cdot\eta}{mc^2\beta^2\gamma}\langle z'E_z\rangle &= -\tfrac{2\beta_{f;z}}{c\beta\gamma}\langle z'^2\rangle + \tfrac{2\langle D_{zz}\rangle}{c^3\beta^3\gamma^5}\,.
\end{aligned} \tag{7}$$

We conclude this section with the remark that the sets of moment equations (6) and (7) are not compatible with single particle equations of motion if the diffusion tensor elements $\langle D_{ii}\rangle$ cannot be neglected. This is not astonishing if we recall that the Fokker-Planck equation models a *stochastic* process, i.e. the evolution of a probability distribution. In contrast, the Vlasov approach is strictly *deterministic*, hence allows us – at least in principle – to trace separately the phase space motion of each individual particle due to all forces acting on it.

## 5 THE EXTENDED ENVELOPE AND RMS-EMITTANCE EQUATIONS

Following further Sacherer's formalism, we define the rms-emittance $\varepsilon_x(s)$ as

$$\varepsilon_x^2(s) = \langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2 \tag{8}$$

and combine the first and the second equation of (6) in order to set up a differential equation for $\sqrt{\langle x^2\rangle}$. This quantity is of particular importance in the study of charged particle beams, since it is proportional to the actual beam width in the $x$-direction:

$$\frac{d^2}{ds^2}\sqrt{\langle x^2\rangle} + \frac{\beta_{f;x}}{c\beta\gamma}\frac{d}{ds}\sqrt{\langle x^2\rangle} + k_x^2(s)\sqrt{\langle x^2\rangle} - \frac{q}{mc^2\beta^2\gamma^3}\frac{\langle xE_x\rangle}{\sqrt{\langle x^2\rangle}} - \frac{\varepsilon_x^2(s)}{\sqrt{\langle x^2\rangle^3}} = 0\,. \tag{9}$$

Comparing Eq. (9) with Sacherer's 'rms envelope equation'[4], we observe that one additional term containing the first order derivative of $\sqrt{\langle x^2\rangle}$ appears in our theory.

The 'dynamical friction coefficients' $\beta_{f;i}$ are always positive. Mathematically, this implies that all variations of the envelope function $\sqrt{\langle x^2\rangle}$ are damped. As a consequence for periodic structures, eventual mismatch oscillations of the beam envelopes fade away. In this sense, the 'dynamical friction' leads to a 'self-matching' of the beam to the periodic structure. In the laminar beam limit ($\varepsilon_x \to 0$), Eq. (9) represents a damped oscillator equation, which becomes linear for $E_x \propto x$. The amplitude of any solution function then decreases with the relaxation time $\tau_f$ of

$$\tau_f = 2/\beta_f\,.$$

This value thus provides us with the order of magnitude for the relaxation time of mismatch oscillations of the beam envelope, hence with the time the beam needs to match itself to a periodic focusing structure. We will discuss a numerical example in section 12.

In general, $\langle xE_x\rangle$ and $\varepsilon_x^2$ are not constants of motion but unknown functions of $s$. This accounts for the fact that Eq. (9) is not closed, hence cannot be used directly to evaluate

the envelope function. In the following, we will discuss the conditions under which Eq. (9) can be combined with additional differential equations in order to establish a closed set of equations that can be integrated.

On the basis of Eqs. (6), the derivative of the rms-emittance (8) is readily calculated to give

$$\frac{1}{\langle x^2\rangle}\frac{d}{ds}\varepsilon_x^2(s) = \frac{2q}{mc^2\beta^2\gamma^3}\left(\langle x'E_x\rangle - \frac{\langle xx'\rangle}{\langle x^2\rangle}\langle xE_x\rangle\right) - 2\left(\frac{\beta_{f;x}}{c\beta\gamma}\frac{\varepsilon_x^2(s)}{\langle x^2\rangle} - \frac{\langle D_{xx}\rangle}{c^3\beta^3\gamma^3}\right). \tag{10}$$

The equation for the $z$-direction may be written

$$\frac{1}{\langle z^2\rangle}\frac{d}{ds}\varepsilon_z^2(s) = \frac{2q\eta}{mc^2\beta^2\gamma}\left(\langle z'E_z\rangle - \frac{\langle zz'\rangle}{\langle z^2\rangle}\langle zE_z\rangle\right) - 2\left(\frac{\beta_{f;z}}{c\beta\gamma}\frac{\varepsilon_z^2(s)}{\langle z^2\rangle} - \frac{\langle D_{zz}\rangle}{c^3\beta^3\gamma^5}\right).$$

In the following section, we analyse the physical meaning of those moments that contain electric field components $E_i$. Thereafter, the terms related to the Fokker-Planck coefficients will be investigated. We will show that they describe the relaxation of temperature differences within the beam.

## 6 THE 'EXCESS FIELD ENERGY' OF A CHARGED PARTICLE BEAM

As has been shown previously[8,9,10], the sum of the beam moments $\langle x'E_x\rangle$, $\langle y'E_y\rangle$, and $\langle z'E_z\rangle$ can be correlated to the change of the electrostatic field energy $W$ constituted by all charges of the beam:

$$\frac{dW}{ds} = -Nq\left(\langle x'E_x\rangle + \langle y'E_y\rangle + \langle z'E_z\rangle\right). \tag{11}$$

If the charge distribution is *uniform*, the space charge field components $E_x$, $E_y$, and $E_z$ are *linear* functions of $x$, $y$, and $z$, respectively. We conclude from (11), that the derivative of the field energy $W_u$ of a uniform charge distribution can be expressed as

$$\frac{dW_u}{ds} = -Nq\left(\frac{\langle xx'\rangle}{\langle x^2\rangle}\langle xE_x\rangle + \frac{\langle yy'\rangle}{\langle y^2\rangle}\langle yE_y\rangle + \frac{\langle zz'\rangle}{\langle z^2\rangle}\langle zE_z\rangle\right).$$

We thus obtain for the difference between a real and a uniform beam:

$$\frac{d}{ds}(W - W_u) = -Nq\left(\langle x'E_x\rangle - \frac{\langle xx'\rangle}{\langle x^2\rangle}\langle xE_x\rangle + (\text{similar } y\text{- and } z\text{-terms})\right). \tag{12}$$

If we compare beams of the same rms-dimensions, we learn that the *minimum* field energy applies to the *uniform* density profile[9]. Consequently, $W - W_u$ defines the additional ('excess') field energy the beam possesses, if its charge density is not uniform. Since $W_u$ constitutes the minimum field energy of a beam of given size, at maximum the 'excess' part of the total field energy can be converted into thermal beam energy.

Once this phase space reorganization process has been completed, which means that a self-consistent phase space distribution possessing a specific equilibrium 'excess' field energy has been established, only fluctuations of $W - W_u$ can occur. We conclude that the change of the 'excess field energy' can be regarded as an onset effect that does not govern the long-term beam behaviour. Simulations[11,8] as well as theoretical investigations[12] have shown that the thermalization process takes place very rapidly within a few plasma periods.

## 7 THE TEMPERATURE OF A CHARGED PARTICLE BEAM

As usual in statistical physics, we relate the temperature of an ensemble of particles to their 'incoherent' motion. Since charged particle beams change their size while passing through a focusing device, we must subtract the 'coherent' part of the particle velocities from the total velocities in order to obtain physically reasonable expressions for the beam temperatures.

The temperatures in all spatial directions are not necessarily the same. We thus have to define them separately for each direction. In order to determine the temperature $T_x$, we must first calculate $g(x, x'; s)$ denoting the projection of the phase space distribution function $f$ onto the $x, x'$-subspace at $s$:

$$g(x, x'; s) = \int f \, dy \, dy' \, dz \, dz' \, .$$

The scaled 'coherent' velocity $\xi'(x; s)$ in the $x$-direction as a function of $x$ is given by:

$$\xi'(x; s) = \frac{\int x' g(x, x'; s) \, dx'}{\int g(x, x'; s) \, dx'} \, . \tag{13}$$

Now the laboratory $x$-temperature $kT_x$ in energy units can be defined as[7]:

$$kT_x = mc^2 \beta^2 \gamma \int (x' - \xi')^2 \, f \, d\tau \, ,$$

which is easily simplified to

$$kT_x = mc^2 \beta^2 \gamma \left( \langle x'^2 \rangle - \langle \xi'^2 \rangle \right) . \tag{14}$$

In general, it is not possible to calculate $\xi'(x; s)$ for a given distribution function $f$ in a closed analytical form. As an example that can be treated analytically, we quote the 'K-V' model[1] for the distribution function $f$ of an unbunched beam. It has the property that all projections of $f$ onto 2-dimensional subspaces lead to uniformly populated ellipses. For this case, the integrals contained in (13) are easily solved to give:

$$\xi'(x; s) = \frac{\langle x x' \rangle}{\langle x^2 \rangle} x \qquad \rightsquigarrow \qquad \langle \xi'^2 \rangle = \frac{\langle x x' \rangle^2}{\langle x^2 \rangle} \, . \tag{15}$$

For non-K-V distributions, we can use (15) only as an approximation for the 'coherent' share of the particle velocities. Calculating the temperature according to Eq. (14), the related error vanishes together with the 'coherent' motion, i.e. if $\langle \xi'^2 \rangle = 0$. This occurs repeatedly at positions, where the beam width in $x$-direction takes on a maximum or a

minimum value. We thus do not introduce a systematic error using (15) for all types of phase space distributions $f$.

Under these circumstances, the temperatures are correlated to the second order beam moments only. We thus obtain for the transverse temperatures in the laboratory system:

$$kT_x = \frac{\langle p_{x;\ell}^2 \rangle}{m\gamma} = mc^2\beta^2\gamma \cdot \frac{\varepsilon_x^2(s)}{\langle x^2 \rangle} .$$

In the same frame, the longitudinal temperature $T_z$ is given by

$$kT_z = \frac{1}{|\eta|} mc^2\beta^2\gamma \cdot \frac{\varepsilon_z^2(s)}{\langle z^2 \rangle} = mc^2\beta^2\gamma \cdot \frac{\bar{\varepsilon}_z^2(s)}{\langle z^2 \rangle} ,$$

defining $\bar{\varepsilon}_z(s) = \varepsilon_z(s)/\sqrt{|\eta|}$ as the 'effective longitudinal emittance'. In these definitions, we must take the absolute value of $\eta$ since the change of the longitudinal beam emittance due to particle-particle collisions occurs in a symmetric way above and below transition energy.

The equilibrium temperature $T$ is simply the average of the $x$-, $y$-, and $z$-temperatures:

$$\frac{kT}{mc^2\beta^2\gamma} = \frac{1}{3}\left( \frac{\varepsilon_x^2(s)}{\langle x^2 \rangle} + \frac{\varepsilon_y^2(s)}{\langle y^2 \rangle} + \frac{\bar{\varepsilon}_z^2(s)}{\langle z^2 \rangle} \right) . \tag{16}$$

If the longitudinal temperature need not to be taken into account, we can restrict ourselves to the average of the transverse temperatures. The transverse equilibrium temperature $T_\perp$ can then be defined as:

$$\frac{kT_\perp}{mc^2\beta^2\gamma} = \frac{1}{2}\left( \frac{\varepsilon_x^2(s)}{\langle x^2 \rangle} + \frac{\varepsilon_y^2(s)}{\langle y^2 \rangle} \right) . \tag{17}$$

Depending on the physics of our system, we have to decide whether to use (16) or (17) as the appropriate expression for the equilibrium beam temperature to be inserted into Eq. (10). For example, the dynamics of bunched beams as well as beams within dispersive systems require a complete 3-dimensional description. The case of an unbunched beam that passes through a non-dispersive system allows a simplified treatment. For this case, we can restrict ourselves to the use of (17) as the equilibrium beam temperature of the 'transverse' phase space.

## 8 THE FOKKER-PLANCK COEFFICIENTS FOR INTRA-BEAM SCATTERING

So far, the procedure of deriving extended moment equations has been performed formally by speaking in general of 'non-Liouvillean' effects to be described by the Fokker-Planck equation. In other words, up to now nothing has been said about the Fokker-Planck coefficients $\beta_{f;i}$ and $D_{ii}$. As an example of a physical effect that does not obey Liouville's theorem, we will treat the process of intra-beam scattering. In order to learn how the Fokker-Planck coefficients for the effect of intra-beam scattering are correlated to the physical beam properties, it is necessary to study the process of Coulomb scattering of an ensemble of

particles in detail. This has been done extensively by Jansen[5], treating the more general 'many body problem of particles interacting through an inverse square force law'. We therefore can restrict ourselves to results that apply in this context.

The method of evaluating the Fokker-Planck coefficients for a intra-beam scattering effects of a charged particle beam can be summarized as follows:

- item in the first step, the velocity changes of a test particle due to scattering from a single beam particle as a function of the test particle's initial velocity and impact parameter are calculated,
- second, the expression obtained in the first step is averaged over all possible impact parameters,
- finally, averaging over all particle velocities of the beam is performed. This means that the velocity distribution of the beam must be taken into account.

For simplicity, we assume that the *equilibrium* particle beam has an *isotropic* Maxwellian velocity distribution. If the actual beam is in a state not too far from this equilibrium state, it is justifiable to assume that friction as well as the diffusion processes are also isotropic. Then only one diffusion coefficient $D$ in conjunction with a single friction coefficient $\beta_f$ appears in our equations:

$$D \equiv \langle D_{xx} \rangle = \langle D_{yy} \rangle = \langle D_{zz} \rangle \quad , \quad \beta_f \equiv \beta_{f;x} = \beta_{f;y} = \beta_{f;z} \; .$$

Under these circumstances, $D$ turns out to be proportional to the 'dynamical friction coefficient' $\beta_f$, yielding the well-known Einstein[13] relation:

$$D = \beta_f \cdot \frac{kT_b}{m} = \beta_f \cdot \gamma \frac{kT_\ell}{m} \tag{18}$$

As the result of the averaging procedures, $\beta_f$ is given by[5,14]:

$$\beta_f = \frac{16\sqrt{\pi}}{3} \, n \, c \left( \frac{q^2}{m\gamma c^2} \right)^2 \cdot \left( \frac{m\gamma c^2}{2kT} \right)^{3/2} \cdot \ln \Lambda \; . \tag{19}$$

In these expressions, $kT$ denotes the equilibrium beam temperature in energy units, $m$ the particle rest mass, $q$ its charge, and $\gamma$ the relativistic mass factor, $n$ stands for the average particle density and $\ln \Lambda$ for the so-called Coulomb logarithm.

As usual for phenomena involving Coulomb scattering, we must establish a reasonable upper limit for the maximum impact parameter in order to keep $\Lambda$ finite. For non-neutralized systems, it is suggested by Jansen[5] to identify the maximum impact parameter with the average distance between the particles rather than with the Debye length. This leads to the following formula for $\Lambda$:

$$\Lambda = \frac{3}{2} \cdot \frac{kT}{q^2 \cdot n^{1/3}} \; .$$

Since $\beta_f$ depends only logarithmically on $\Lambda$, any result from our approach will not depend critically on the exact value of $\Lambda$. Inserting (18) into Eq. (10) yields

$$\frac{1}{\langle x^2\rangle}\frac{d}{ds}\varepsilon_x^2(s) = \frac{2q}{mc^2\beta^2\gamma^3}\left(\langle x'E_x\rangle - \frac{\langle xx'\rangle}{\langle x^2\rangle}\langle xE_x\rangle\right) - 2k_f\left(\frac{\varepsilon_x^2(s)}{\langle x^2\rangle} - \frac{kT}{mc^2\beta^2\gamma}\right), \quad (20)$$

with the abbreviation $k_f = \beta_f/c\beta\gamma$.

In the following sections, we insert the expressions (16) and (17) for the equilibrium temperatures into Eq. (20). This will lead to fairly simple growth rate formulae suitable to estimate the effect of intra-beam scattering on the beam emittance.

## 9 THE EMITTANCE EQUATION FOR UNBUNCHED BEAMS

Provided that the longitudinal and the transverse particle motion can approximately be treated as decoupled, it is convenient to define a 'transverse equilibrium temperature' $T_\perp$ according to (17). We insert this expression into Eq. (20):

$$\frac{1}{\langle x^2\rangle}\frac{d}{ds}\varepsilon_x^2(s) + k_f\left(\frac{\varepsilon_x^2(s)}{\langle x^2\rangle} - \frac{\varepsilon_y^2(s)}{\langle y^2\rangle}\right) = \frac{2q}{mc^2\beta^2\gamma^3}\left(\langle x'E_x\rangle - \frac{\langle xx'\rangle}{\langle x^2\rangle}\langle xE_x\rangle\right). \quad (21)$$

The terms appearing on the right-hand side are related to the change of 'excess field energy' (12), hence to the 'onset' effect that the beam particles rearrange to form a stationary density profile. If we want to analyse the 'long-term' beam behaviour, we can disregard these terms.

As has been shown by Sacherer[4], the moment $\langle xE_x\rangle$ contained in Eq. (9) can be expressed in terms of second order moments, in the case that the transverse charge density has elliptical symmetry:

$$\langle xE_x\rangle = \frac{I}{c\beta}\frac{\sqrt{\langle x^2\rangle}}{\sqrt{\langle x^2\rangle}+\sqrt{\langle y^2\rangle}}. \quad (22)$$

Computer simulations show that this is a reasonable assumption for unbunched beams confined by linear focusing forces – even if the space charge forces are non-linear.

With the help of (22), we can combine Eqs. (9) and (21) to establish a closed set of coupled differential equations for the 'long-term' values of envelopes and emittances containing solely second order beam moments:

$$\begin{aligned}
&\frac{1}{\langle x^2\rangle}\frac{d}{ds}\varepsilon_x^2(s) + k_f\left(\frac{\varepsilon_x^2(s)}{\langle x^2\rangle} - \frac{\varepsilon_y^2(s)}{\langle y^2\rangle}\right) &&= 0\\
&\frac{1}{\langle y^2\rangle}\frac{d}{ds}\varepsilon_y^2(s) + k_f\left(\frac{\varepsilon_y^2(s)}{\langle y^2\rangle} - \frac{\varepsilon_x^2(s)}{\langle x^2\rangle}\right) &&= 0\\
&\frac{d^2}{ds^2}\sqrt{\langle x^2\rangle} + k_f\frac{d}{ds}\sqrt{\langle x^2\rangle} + k_x^2(s)\sqrt{\langle x^2\rangle} - \frac{K/2}{\sqrt{\langle x^2\rangle}+\sqrt{\langle y^2\rangle}} - \frac{\varepsilon_x^2(s)}{\sqrt{\langle x^2\rangle^3}} &&= 0\\
&\frac{d^2}{ds^2}\sqrt{\langle y^2\rangle} + k_f\frac{d}{ds}\sqrt{\langle y^2\rangle} + k_y^2(s)\sqrt{\langle y^2\rangle} - \frac{K/2}{\sqrt{\langle x^2\rangle}+\sqrt{\langle y^2\rangle}} - \frac{\varepsilon_y^2(s)}{\sqrt{\langle y^2\rangle^3}} &&= 0
\end{aligned} \quad (23)$$

with $K$ denoting the dimensionless 'generalized perveance'

$$K = \frac{2qI}{mc^3\beta^3\gamma^3} .$$

In order to gain a better insight into the mathematical implications of these equations, we introduce $r_t$ as the ratio of the transverse beam temperatures

$$r_t(s) = \frac{T_x}{T_y} = \frac{\varepsilon_x^2(s)}{\langle x^2\rangle} \cdot \frac{\langle y^2\rangle}{\varepsilon_y^2(s)} , \tag{24}$$

and rewrite the first two equations of (23) in an equivalent form

$$\begin{aligned} \tfrac{d}{ds} \ln \varepsilon_x^2(s) &= k_f \left(r_t^{-1}(s) - 1\right) \\ \tfrac{d}{ds} \ln \varepsilon_y^2(s) &= k_f \left(r_t(s) - 1\right) . \end{aligned} \tag{25}$$

We observe that the rms-emittance in the $x$-direction increases as long as the temperature $T_x$ is smaller than $T_y$. At the same time, $\varepsilon_y$ decreases until a temperature balance in both transverse directions is achieved. These changes take place in the opposite direction as long as $T_x > T_y$. If we add Eqs. (25), we find a quantity that increases in any case:

$$\frac{d}{ds} \ln \varepsilon_x^2(s)\varepsilon_y^2(s) = k_f \cdot \frac{\left[1 - r_t(s)\right]^2}{r_t(s)} \geq 0 . \tag{26}$$

The product $\varepsilon_x^2(s)\varepsilon_y^2(s)$ can be regarded as the 4-dimensional transverse rms-emittance $\varepsilon_\perp^4(s)$. It increases as long as the transverse beam temperatures are not balanced. This indicates that the expression $\ln \varepsilon_x(s)\varepsilon_y(s)$ can serve as a measure for the transverse beam entropy[16].

We now formally integrate Eq. (26):

$$\frac{\varepsilon_x^2(s)\,\varepsilon_y^2(s)}{\varepsilon_x^2(0)\,\varepsilon_y^2(0)} = \exp\left( k_f \int_0^s \frac{\left[1 - r_t(z)\right]^2}{r_t(z)}\, dz \right) \geq 1 . \tag{27}$$

The inequality holds since $k_f$ as well as the integrand cannot take on negative values. It is not possible to solve Eq. (27) for $\varepsilon_x\varepsilon_y$ since the integrand itself is a function of the transverse emittances. This dependence cancels out, if the ratio of the transverse emittances comes close to unity

$$\varepsilon_x(s) \approx \varepsilon_y(s) .$$

We can apply this approximation in cases, where the main temperature differences have already relaxed and the resulting temperature fluctuations are due to variations of the transverse beam widths. With this restriction in mind, the temperature ratio (24) can be replaced by the 'envelope ratio':

$$r_t(s) \approx r_e(s) = \frac{\langle y^2\rangle}{\langle x^2\rangle} ,$$

yielding

$$\varepsilon_\perp^4(s) = \varepsilon_\perp^4(0) \cdot \exp\left( k_f \int_0^s \frac{\left[1 - r_e(z)\right]^2}{r_e(z)}\, dz \right) .$$

For periodic beam focusing systems, it is helpful to introduce the following dimensionless quantity $I_e$:

$$I_e = \frac{1}{S} \int_0^S \frac{\left[1 - r_e(z)\right]^2}{r_e(z)}\, dz , \tag{28}$$

with $S$ as the period length. In the following sections, we will refer to $I_e$ as the 'ellipticity' of the focusing system. Since the integrand cannot take on negative values, we find that $I_e \geq 0$. The integral depends solely on the course of the envelope ratio $r_e(s)$ which a matched beam defines while passing through one focusing period. Thus, $I_e$ is a characteristic of the specific focusing system only.

Reinserting the time as the independent variable, the emittance ratio $\varepsilon_\perp(t)/\varepsilon_\perp(0)$ increases in time as

$$\frac{\varepsilon_\perp(t)}{\varepsilon_\perp(0)} = \exp\left\{ \tfrac{1}{4}\, \beta_f\, I_e \cdot t \right\} .$$

Consequently, the $e$-folding time $\tau_{\text{ef}}$ for $\varepsilon_\perp(t)/\varepsilon_\perp(0)$ is given by

$$\tau_{\text{ef}}^{-1} = \tfrac{1}{4}\, \beta_f\, I_e . \tag{29}$$

We observe that the rate of the emittance growth due to intra-beam scattering depends, besides $\beta_f$, on the 'ellipticity' $I_e$ of the beam envelopes along the focusing structure. If $I_e$ does not vanish – as it is always the case in strong focusing systems – a scattering induced growth of the emittance can never be avoided.

Note that (29) is valid only as long as the particle density $n$, hence the 'dynamical friction coefficient' $\beta_f$, is not appreciably changed. This means that Eq. (29) provides us with the *instantaneous* emittance growth rate for given beam parameters.

In section 12, we will present the results of numerical calculations. Of course, the effect of emittance growth due to weak particle-particle collisions can be neglected for *linear* structures, since the time the beam resides within the structure is small compared to $\tau_{\text{ef}}$. Nevertheless, the relation (29) gives us an insight into the physical consequences of processes that involve statistically fluctuating forces – as they also occur due to simplifications in computer simulations of charged particle beams. We will shortly touch this subject in the last section.

## 10 THE EMITTANCE EQUATION FOR BUNCHED BEAMS

For the general 3-dimensional treatment, we insert (16) for the equilibrium temperature into Eq. (10):

$$\frac{1}{\langle x^2\rangle}\frac{d}{ds}\varepsilon_x^2(s) = \frac{2q}{mc^2\beta^2\gamma^3}\left(\langle x'E_x\rangle - \frac{\langle xx'\rangle}{\langle x^2\rangle}\langle xE_x\rangle\right) - \frac{2k_f}{3}\left(\frac{2\varepsilon_x^2(s)}{\langle x^2\rangle} - \frac{\varepsilon_y^2(s)}{\langle y^2\rangle} - \frac{\bar{\varepsilon}_z^2(s)}{\langle z^2\rangle}\right) \tag{30}$$

The corresponding equations can be written for $\varepsilon_y^2(s)$ and $\varepsilon_z^2(s)$. If we add these three equations, we can apply Eq. (12) in order to set up a differential equation that correlates the changes of the rms-emittances with the change of the 'excess' space charge field energy $W - W_u$ of a charged particle beam:

$$\frac{1}{\langle x^2\rangle}\frac{d\varepsilon_x^2(s)}{ds} + \frac{1}{\langle y^2\rangle}\frac{d\varepsilon_y^2(s)}{ds} + \frac{1}{\langle z^2\rangle}\frac{d\bar{\varepsilon}_z^2(s)}{ds} = \frac{-2}{mc^2\beta^2\gamma^3 N}\frac{d}{ds}\big[W(s) - W_u(s)\big]. \tag{31}$$

Note that all temperature terms contained in Eq. (30) cancel. For that reason, Eq. (31) has exactly the same form as the equations derived earlier[8,9,10], using the Vlasov equation as the starting-point.

As before, we may neglect the field energy terms if we want to analyse the long-term beam behaviour. Then Eq. (30) reduces to

$$\frac{1}{\langle x^2\rangle}\frac{d}{ds}\varepsilon_x^2(s) = -\frac{2k_f}{3}\left(\frac{2\varepsilon_x^2(s)}{\langle x^2\rangle} - \frac{\varepsilon_y^2(s)}{\langle y^2\rangle} - \frac{\bar{\varepsilon}_z^2(s)}{\langle z^2\rangle}\right), \tag{32}$$

again likewise for the $y$- and $z$-directions.

Together with the rms-envelope equations (9) for all the spatial coordinates, and the expressions for $\langle xE_x\rangle$, $\langle yE_y\rangle$, and $\langle zE_z\rangle$ for bunched beams[4], these differential equations form a closed set.

Eq. (32) can be written in an equivalent form in terms of temperature ratios, as defined in Eq. (24)

$$\frac{d}{ds}\ln\varepsilon_x^2(s) = \frac{2k_f}{3}\left(\frac{T_y(s)}{T_x(s)} + \frac{T_z(s)}{T_x(s)} - 2\right), \tag{33}$$

and similar for $\varepsilon_y^2$ and $\bar{\varepsilon}_z^2$. Depending on the actual temperature ratios, the expression in parentheses can be positive or negative. This means that according to the direction of the temperature flow within the beam, the rms-emittance $\varepsilon_x^2(s)$ can increase or decrease. In contrast, the product $\varepsilon_x^2\varepsilon_y^2\bar{\varepsilon}_z^2$ can only increase during the balancing process. This is verified, if we add (33) to the similar equations for $\varepsilon_y^2$ and $\bar{\varepsilon}_z^2$:

$$\frac{d}{ds}\ln\varepsilon_x^2(s)\varepsilon_y^2(s)\bar{\varepsilon}_z^2(s) = \frac{2k_f}{3}\left(\frac{(1-r_{xy})^2}{r_{xy}} + \frac{(1-r_{xz})^2}{r_{xz}} + \frac{(1-r_{yz})^2}{r_{yz}}\right) \geq 0\,. \tag{34}$$

For the sake of clarity, we used abbreviations for the temperature ratios, namely

$$r_{xy}(s) = \frac{T_y(s)}{T_x(s)} \quad , \quad r_{xz}(s) = \frac{T_z(s)}{T_x(s)} \quad , \quad r_{yz}(s) = \frac{T_z(s)}{T_y(s)}\,.$$

As already demonstrated in the previous section, Eq. (34) can be formally integrated:

$$\frac{\varepsilon_{\mathrm{tot}}^3(S)}{\varepsilon_{\mathrm{tot}}^3(0)} \equiv \frac{\varepsilon_x(S)\,\varepsilon_y(S)\,\bar{\varepsilon}_z(S)}{\varepsilon_x(0)\,\varepsilon_y(0)\,\bar{\varepsilon}_z(0)} = \exp\left\{\tfrac{1}{3}k_f S\,(I_{xy}+I_{xz}+I_{yz})\right\}, \tag{35}$$

with $I_{xy}$, $I_{xz}$, and $I_{yz}$ denoting the three possible integrals of the temperature ratio functions. For example, the dimensionless quantity $I_{xy}$ is given by:

$$I_{xy} = \frac{1}{S}\int_0^S \frac{\left[1-r_{xy}(s)\right]^2}{r_{xy}(s)}\,ds \geq 0 \qquad , \qquad r_{xy}(s) = \frac{\varepsilon_y^2(s)}{\langle y^2\rangle}\,\frac{\langle x^2\rangle}{\varepsilon_x^2(s)}\,. \tag{36}$$

From Eq. (35), the average $e$-folding time $\tau_{\mathrm{ef}}$ for the emittance ratio $\varepsilon_{\mathrm{tot}}(t)/\varepsilon_{\mathrm{tot}}(0)$ is calculated to give

$$\tau_{\mathrm{ef}}^{-1} = \tfrac{1}{9}\,\beta_f\,(I_{xy}+I_{xz}+I_{yz})\,. \tag{37}$$

We learn that a temperature balancing process – which is driven by the fluctuating component of the interaction potential – is always accompanied by an increase of the total beam emittance $\varepsilon_{\mathrm{tot}}$. If in a periodic system the temperature imbalance is restored periodically due to the specific beam handling, the integrals $I_{xy}$, $I_{xz}$, and $I_{yz}$ are positive, hence a repeated, not saturating growth of the emittance occurs.

## 11 THE EMITTANCE EQUATION FOR COASTING BEAMS IN STORAGE RINGS

If two beam particles circulate within a storage ring at different longitudinal velocities, they will take on any relative position along the ring circumference in the course of their lifetime within the ring. This means that we do not observe a correlation between the longitudinal locations of the beam particles and their respective velocities:

$$\langle zz'\rangle = 0 \qquad \rightsquigarrow \qquad \langle z^2\rangle = \text{const.}$$

In place of $z'$, we use in the following the particle's relative momentum deviation $\Delta p/p$ from the reference particle; $z'$ can be expressed in terms of the relative revolution time deviation as

$$z' = -\frac{\Delta\tau}{\tau}\,.$$

The ratio of the relative revolution time deviation to $\Delta p/p$ is given by the factor $-\eta$, which has already been defined in Eq. (5):

$$\eta = -\frac{\Delta\tau}{\tau}\Big/\frac{\Delta p}{p} \qquad \rightsquigarrow \qquad z' = \eta\,\frac{\Delta p}{p} \equiv \eta\,\delta\,.$$

All expressions containing the longitudinal emittance can then be replaced by:

$$\bar{\varepsilon}_z^2(s) = |\eta|^{-1}\varepsilon_z^2(s) = |\eta|\,\langle z^2\rangle\langle\delta^2\rangle\,.$$

The coupled set of equations (32) describing the emittance changes can now be written as:

$$\begin{aligned}
\frac{1}{\langle x^2\rangle}\frac{d}{ds}\varepsilon_x^2(s) &= -\frac{2k_f}{3}\left(\frac{2\varepsilon_x^2(s)}{\langle x^2\rangle} - \frac{\varepsilon_y^2(s)}{\langle y^2\rangle} - |\eta|\,\langle\delta^2\rangle\right) \\
\frac{1}{\langle y^2\rangle}\frac{d}{ds}\varepsilon_y^2(s) &= -\frac{2k_f}{3}\left(\frac{2\varepsilon_y^2(s)}{\langle y^2\rangle} - \frac{\varepsilon_x^2(s)}{\langle x^2\rangle} - |\eta|\,\langle\delta^2\rangle\right) \qquad (38) \\
|\eta|\,\frac{d}{ds}\langle\delta^2\rangle &= -\frac{2k_f}{3}\left(2|\eta|\,\langle\delta^2\rangle - \frac{\varepsilon_x^2(s)}{\langle x^2\rangle} - \frac{\varepsilon_y^2(s)}{\langle y^2\rangle}\right),
\end{aligned}$$

neglecting, as before, effects that change the 'excess field energy'. A similar set of equations has been derived and numerically integrated earlier by Seurer[16] on the basis of a temperature relaxation ansatz. If we add the Eqs. (38), all right-hand side terms cancel, yielding:

$$\frac{1}{\langle x^2\rangle}\frac{d}{ds}\varepsilon_x^2(s) + \frac{1}{\langle y^2\rangle}\frac{d}{ds}\varepsilon_y^2(s) + |\eta|\,\frac{d}{ds}\langle\delta^2\rangle = 0\,. \qquad (39)$$

Provided that we are allowed to treat the transverse beam sizes as adiabatic constants, Eq. (39) can be rewritten as:

$$\frac{d}{ds}\left[\frac{\varepsilon_x^2(s)}{\langle x^2\rangle} + \frac{\varepsilon_y^2(s)}{\langle y^2\rangle} + |\eta|\,\langle\delta^2\rangle\right] = 0\,.$$

Integration leads to the constant of motion that has been derived by Piwinski[17]:

$$\frac{\varepsilon_x^2(s)}{\langle x^2\rangle} + \frac{\varepsilon_y^2(s)}{\langle y^2\rangle} + |\eta|\,\langle\delta^2\rangle = \text{const.} \qquad (40)$$

For strong focusing rings, this adiabatic invariant does *not* exist, since we can no longer assume the transverse beam dimensions to stay approximately constant along the focusing period. For this case, we must integrate the coupled set of equations (38) together with the envelope equations of (23) in order to determine the change of both the transverse emittances and the momentum spread due to intra-beam scattering effects. Since a temperature imbalance within the beam is periodically reestablished by the strong transverse focusing, no state of equilibrium, as suggested by (40), is ever reached. In terms of our approach, this means that the temperature ratio integrals, $I_{xy}$, $I_{xz}$, and $I_{yz}$, as defined in (36), do not vanish. Strong focusing devices thus always 'generate' a positive $e$-folding time $\tau_{\text{ef}}$ for the total beam emittance $\varepsilon_{\text{tot}}(t)$:

$$\tau_{\text{ef}}^{-1} = \tfrac{1}{9}\,\beta_f\,(I_{xy} + I_{xz} + I_{yz})\,.$$

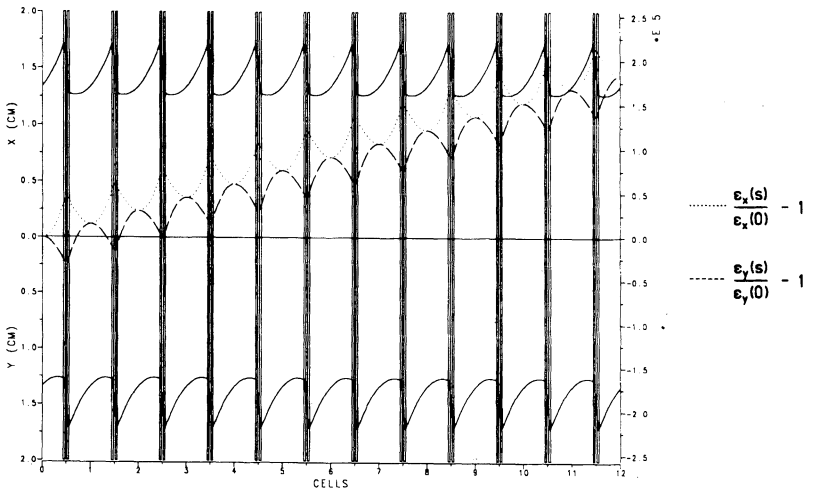


FIGURE 1: Envelope and emittance growth functions of a matched beam passing through a Periodic Quadrupole Channel at $\sigma_0=60°$, $\sigma=15°$. (The scale on the right-hand side applies to the dimensionless emittance growth functions.)

## 12 RESULTS OF NUMERICAL CALCULATIONS

### *12.1 Quadrupole Channels*

With quadrupole lenses, a net focusing effect can only be obtained if we combine quadrupoles of different focusing orientation in a doublet or other alternating gradient arrangements. As a consequence, it is not possible to keep a beam azimuthally round along a structure that contains quadrupoles. It follows that the integral (28) is always greater than zero, leading to a finite growth rate even for perfectly matched beams. Since the imbalance of the transverse beam temperatures is restored by each quadrupole, no equilibrium can ever be reached. We thus do not find a relaxation of the growth rate, but always a finite emittance growth gradient.

According to (29), the growth rate depends on the beam parameters, yielding a specific value of the 'dynamical friction coefficient' $\beta_f$ (19), and the lattice parameters, which are associated with specific value of the 'ellipticity' $I_e$.

An example of a low energy beam transport channel is plotted in Fig. 1. Using beam parameters as listed in Tab. 1 and matching the beam perfectly to the channel, we find a value of $I_e = 0.135$ for the 'ellipticity'. These settings lead to the $e$-folding time for $\varepsilon_\perp$ of $\tau_{ef} = 2.7$ s. During this space of time, the beam would pass $6.4 \times 10^5$ focusing periods.

TABLE 1: List of parameters for the Periodic Quadrupole Channel simulation

| | |
|---|---|
| ion | $^{40}Ar^{1+}$ |
| energy | 4.75 KeV/amu |
| period length $S$ | 4.0 m |
| zero current phase advance $\sigma_0$ | 60° |
| depressed phase advance $\sigma$ | 15° |
| beam current $I$ | 0.25 mA |
| initial rms emittances $\varepsilon_{x,y}(0)$ | $3.125\times10^{-6}$ m |
| ellipticity $I_e$ (matched beam) | 0.135 |
| friction coefficient $\beta_f$ | 11.1 $s^{-1}$ |
| emittance $e$-folding time $\tau_{ef}$ | 2.7 s |

The growth rate of the transverse emittances per period amounts to

$$\frac{\varepsilon_\perp(S)}{\varepsilon_\perp(0)} - 1 \approx 1.5 \times 10^{-6} .$$

*12.2 Storage Rings*

The emittance growth effect due to temperature imbalance is not negligible, if the time the beam resides within the focusing structure is of the order of the $e$-folding time (29) or more. This is the case in circular devices such as storage rings.

We base our calculations on the geometry and the beam data of the 'Experimental Storage Ring' (ESR)[18,19], which started operation at GSI in 1990.

As an approximation, we neglect the change of the transverse beam widths due to the ring dispersion. Due to these simplifications, we can calculate the envelope dynamics using the envelope equations contained in (23), together with the set of emittance equations (38). Of course, (38) can also be combined with a more sophisticated method for evaluating the envelope functions. Namely, the local ring dispersion $\alpha_p(s) = \Delta x/(\Delta p/p)$ should be taken into account in a refined model.

The cooled $^{20}Ne^{10+}$ beam with the parameters listed in Tab. 2 corresponds to $2.6 \times 10^9$ stored particles. With the values of $\beta_f = 0.131$ $s^{-1}$ for the 'dynamical friction coefficient' we obtain $\tau_{ef} = 7.0$ s as $e$-folding time for the total beam emittance $\varepsilon_{tot}(t)$, i.e. $1.2 \times 10^7$ turns. Note that due to the bending magnets, the average beam envelopes in the $x$- and $y$-directions no longer agree, as is the case for matched beams in symmetric quadrupole or solenoid channels. This leads to different emittance growth rates for the transverse directions (Fig. 2):

$$\frac{\varepsilon_x(S)}{\varepsilon_x(0)} - 1 \approx 5.6\times10^{-8} , \quad \frac{\varepsilon_y(S)}{\varepsilon_y(0)} - 1 \approx 4.3\times10^{-8} , \quad \frac{\langle(\Delta p/p)^2\rangle(S)}{\langle(\Delta p/p)^2\rangle(0)} - 1 \approx 3.0\times10^{-7} .$$

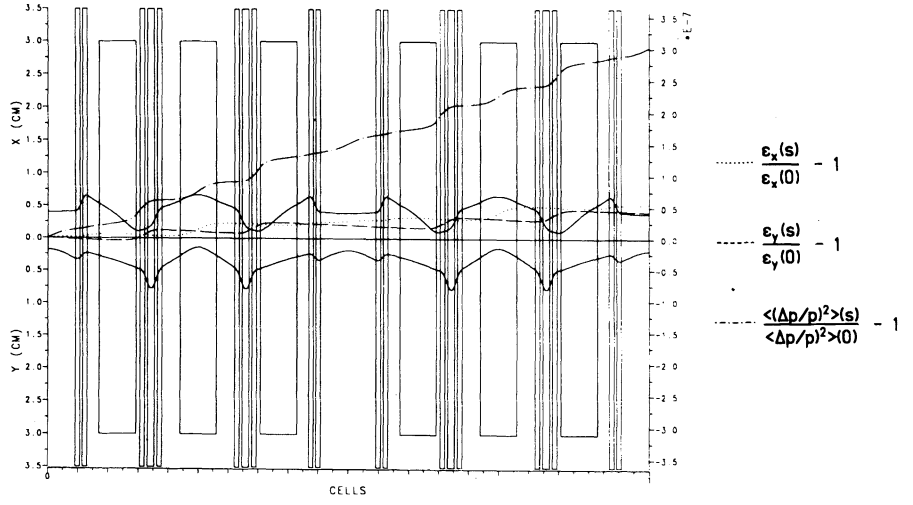


FIGURE 2: Envelope and emittance growth functions of a beam passing through the 'Experimental Storage Ring' (ESR) at $Q_h$=2.31 and $Q_v$=2.25. (The scale on the right-hand side applies to the dimensionless emittance growth functions.)

There is one conclusion we can draw at this point. According to Eq. (37), the $e$-folding frequency of the total beam emittance is proportional to the 'ellipticities' $I_{xy}$, $I_{xz}$, and $I_{yz}$, which are positive numbers that characterize the specific ring optics. They increase the more the matched beam lacks $x$-$y$ symmetry along the focusing period. Designing the ring optics to yield largely smooth envelope functions thus minimizes the intra-beam scattering induced emittance growth rate.

### *12.3 Computer Simulations of Charged Particle Beams*

The Fokker-Planck approach to investigate the behaviour of charged particle beams improves our capability not only to analyse effects observed in reality, but also to interpret the results of computer simulations of such beams.

The joint effect of all simplifications necessary to keep the computing time finite can be visualized in the way that an additional 'error field' is added to the 'true field' we would get in the ideal case. In other words, the calculated self-field $\vec{E}_c$ can be interpreted as the sum of the true self-field $\vec{E}_t$ generated by the real beam and a statistically fluctuating 'error field' $\vec{E}_f$:

$$\vec{E}_c = \vec{E}_t + \vec{E}_f \ .$$

TABLE 2: List of parameters for the ESR simulation

| | |
|---|---|
| ion | $^{20}Ne^{10+}$ |
| energy | 250 MeV/amu |
| period length $S$ | 108.36 m |
| horizontal tune $Q_h$ | 2.31 |
| vertical tune $Q_v$ | 2.25 |
| $\eta = \gamma^{-2} - \gamma_t^{-2}$ | 0.493 |
| beam current $I$ | 7 mA |
| initial rms emittances $\varepsilon_{x,y}(0)$ | $0.25 \times 10^{-6}$ m |
| initial rms momentum spread $\Delta p/p$ | $1.3 \times 10^{-4}$ |
| ellipticity $I_{xy}$ | 4.710 |
| ellipticity $I_{xz}$ | 2.689 |
| ellipticity $I_{yz}$ | 2.382 |
| friction coefficient $\beta_f$ | $0.1306\ s^{-1}$ |
| horiz. emittance $e$-folding time $\tau_{x,\mathrm{ef}}$ | 10.5 s |
| vert. emittance $e$-folding time $\tau_{y,\mathrm{ef}}$ | 13.7 s |
| long. emittance $e$-folding time $\tau_{z,\mathrm{ef}}$ | 1.9 s |
| total emittance $e$-folding time $\tau_{\mathrm{ef}}$ | 7.0 s |

If we describe the action of the fluctuating 'error field' $\vec{E}_f$ in the same way as above using the Fokker-Planck approach, we end up with the same moment equations (6) for the beam. The amplitude of the fluctuating error field is certainly much larger than field fluctuations due to the discreteness of the beam charges. We therefore use implicitly a much larger 'simulation friction coefficient' $\beta_f^{\mathrm{sim}}$, if we simulate a charged particle beam on the basis of 'representative particles' and a 'particle-in-cell' method to determine the self-fields. This leads to a much larger gradient in the emittance growth calculated in our simulations than we would expect in reality. If we simulate the beam transformation through the quadrupole channel described in subsection 12.1, we observe a continuous growth of the transverse beam emittance, as plotted in Fig. 3. In view of these simulation results, the growth rate per cell amounts to:

$$\frac{\varepsilon_\perp(S)}{\varepsilon_\perp(0)} - 1 \approx 4 \times 10^{-5} \,.$$

We thus find for the ratio of the real beam $\beta_f$ to the 'simulation friction coefficient' $\beta_f^{\mathrm{sim}}$ for this specific simulation example:

$$\frac{\beta_f}{\beta_f^{\mathrm{sim}}} \approx 1.5 \times 10^{-6} / 4 \times 10^{-5} \approx 0.04 \,.$$

$\beta_f^{\mathrm{sim}}$ depends in a rather complicated way on the number of simulation particles, the step width of the time integration, and the mesh size. For that reason, it seems to be impossible to derive an analytical expression for this ratio.

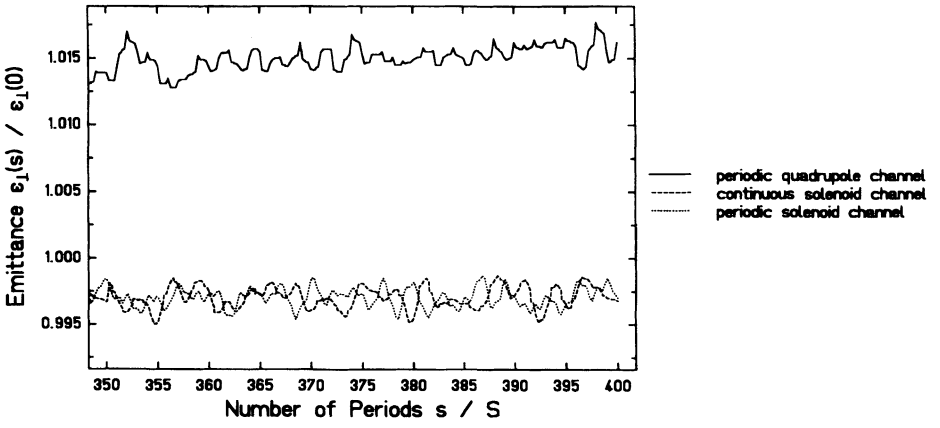


FIGURE 3: Emittance growth functions obtained by 'particle-in-cell' simulations of matched beams passing through different types of beam transport channels at $\sigma_0=60°$, $\sigma=15°$. The underlying beam parameters are listed in Tab. 1.

Qualitatively, the results of our simulations exhibit the effect anticipated by the Fokker-Planck approach: no emittance growth occurs if we transform matched beams through solenoid channels, since $I_e \equiv 0$ in these cases. In contrast, we observe a steady, non-saturating growth if we transform the equivalent beam through a periodic quadrupole channel. Regarding Eq. (29), this is the expected result: we cannot avoid emittance growth due to fluctuating self-fields, if the 'ellipticity' (28) does not vanish.

## 13 CONCLUSIONS

The Vlasov equation serves as the equation of motion for the phase space distribution function in the case that Liouville's theorem applies.

This is obviously not true for all processes that may come to effect during the lifetime of a charged particle beam. If we additionally want to investigate non-Liouvillean effects, we must give up the assumption that the total phase space volume occupied by the beam remains constant. The Fokker-Planck equation can be used to describe the phase space dilution process in case that the underlying physical effect can be regarded as a Markov process.

Applying Sacherer's[4] formalism to derive 'moment equations' to the Fokker-Planck rather than the Vlasov equation, hence to include non-Liouvillean effects in our 'moment' analysis, we obtain additional terms that describe

- the process of temperature balancing within the beam,
- the effect of 'damping' of mismatch oscillations.

This rather general approach has been applied to the effect of intra-beam scattering. In case that the diffusion as well as the friction processes within the beam can be regarded as approximately isotropic, the scattering effects are related to a single coefficient $\beta_f$, which can directly be obtained from the beam parameters. It is referred to as the 'dynamical friction coefficient' of the Fokker-Planck equation. Explicitly, $\beta_f$ is proportional to the particle density and the fourth power of the charge state.

For *unbunched* beams propagating through *linear* structures, we derived a simple formula that provides us with a characteristic $e$-folding time $\tau_{\mathrm{ef}}$ for the transverse beam emittance $\varepsilon_\perp$:

$$\tau_{\mathrm{ef}} = 4\left[\beta_f\, I_e\right]^{-1} .$$

The dimensionless quantity $I_e$ represents the 'ellipticity' of the matched beam along the focusing period, hence is a feature of the specific lattice only.

The effect of emittance growth due to the relaxation of temperature differences has important consequences for systems, where a temperature imbalance is restored periodically by the focusing forces. Namely, in systems containing quadrupoles, a temperature equilibrium can never be reached. In other words, strong focusing devices are always associated with a positive 'ellipticity' $I_e > 0$, leading to a continuous, non-saturating growth of the transverse beam emittance.

Our formalism is easily generalized to three dimensions, which is necessary to study collision effects within *bunched* beams and *coasting* beams in storage rings. Intra-beam scattering acts to relax the temperature imbalances within the beam. This process is always accompanied by an increase of the total beam emittance. In principle, intra-beam scattering would not cause any emittance growth as long as the temperatures within the beam are balanced, i.e. if the beam is in the state of thermodynamical equilibrium. On the other hand, real focusing lattices do not enable the beam to take on an equilibrium temperature due to the non-continuous external forces acting on the particles. Depending on the specific beam optics, the scattering effect thus always 'generates' a positive growth rate of the total beam emittance. For the $e$-folding time $\tau_{\mathrm{ef}}$ of the total beam emittance $\varepsilon_{\mathrm{tot}}$, we found:

$$\tau_{\mathrm{ef}} = 9\left[\beta_f\,(I_{xy} + I_{xz} + I_{yz})\right]^{-1} ,$$

wherein $I_{xy}$, $I_{xz}$, and $I_{yz}$ denote the three possible 'ellipticities', which are related to the course of the beam temperature ratios along the focusing period in the $x$, $y$-, $x$, $z$-, and $y$, $z$-planes, respectively.

If we want to reduce the scattering induced emittance growth rate in a storage ring, we must design the beam optics in a way that the 'ellipticities' of the envelopes along the ring circumference are minimized, which means that the matched envelope functions should be as smooth as possible.

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