Generalized Lagrangian Master Equations

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Abstract

We discuss the geometry of the Lagrangian quantization scheme based on (generalized) Schwinger-Dyson BRST symmetries. When a certain set of ghost fields are integrated out of the path integral, we recover the Batalin-Vilkovisky formalism, now extended to arbitrary functional measures for the classical fields. Keeping the ghosts reveals the crucial role played by a natural connection on the space of fields.

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The Lagrangian quantization scheme of Batalin and Vilkovisky [1] has a direct relation to what we have called "Schwinger-Dyson BRST symmetry", – the BRST symmetry whose Ward identities provide the most general Schwinger-Dyson equations of any given quantum theory [2, 3]. Imposing this Schwinger-Dyson symmetry on the theory leads immediately to a Lagrangian Master Equation [4], which reduces to the Batalin-Vilkovisky Master Equation [1] upon integrating out a certain set of new ghost fields c^A. The "antifields" of the Batalin-Vilkovisky formalism are nothing but the usual antighosts of these new fields c^A [4].

The easiest way to see the need for new ghost fields c^A is to derive the Schwinger-Dyson BRST symmetry from a particular collective field formalism [3]. Since the BRST symmetry in question is related to arbitrary local shifts of all field variables, there is a one-to-one correspondence between all fundamental fields ϕ^A of a given theory and the required collective fields φ^A. The appearance of both the new ghosts c^A and the collective fields φ^A is not fortuitous. For example, if one wishes to quantize a theory in such a manner that it is invariant under BRST and anti-BRST symmetry simultaneously [6], then both of these new fields can simply not be removed from the Master Equation [7]. The new ghosts c^A also play an important rôle when one derives the Lagrangian BRST quantization from the BFV theorem of the Hamiltonian formalism [8].

Gauge field theories can be dealt with at the same level as theories without internal gauge symmetries. The solution to the quantization problem is then entirely given by imposing the Schwinger-Dyson BRST symmetry, and demanding certain boundary conditions on the resulting differential equation. Information about the internal gauge symmetries enters only at the stage where boundary conditions are imposed. These boundary conditions can be chosen to equal those of ref. [1], but more general procedures are also possible [4].

The Schwinger-Dyson BRST symmetry is intimately related to the BRST symmetry of field redefinitions [5]. This is not surprising, because Schwinger-Dyson equations can be viewed as the tool with which to describe the quantized theory independently of a specific path integral representation. In fact, the Schwinger-Dyson BRST symmetry is precisely the gauge-fixed remnant of a hidden local gauge symmetry present in any quantum field theory: The gauge symmetry of local field reparametrizations [3]. Ordinarily one chooses from the outset a basis of field variables with which to describe physics, but the field redefinition theorem ensures – at least under certain mild assumptions about the asymptotic states – that any other choice of variables should describe the same physics. Technically, this can be seen from the invariance of S-matrix elements under field redefinitions. Invariance of the S-matrix under such reparametrizations is precisely a reflection of the local gauge symmetry of field redefinitions [3], in just the same manner as invariance of S-matrix elements under internal gauge transformations reflects the ordinary gauge symmetry of gauge field theories.

Since the Schwinger-Dyson BRST symmetry can be viewed as one particular facet of the general field reparametrization BRST symmetry, one would expect that a more general Lagrangian quantization scheme could be derived from the latter. This should provide a quantization principle independent of the field representation, "covariant" in the space of field variables. Such a generalized quantization procedure should by definition be closely related to the geometric formulation of the Batalin-Vilkovisky formalism, a subject that has recently received considerable attention [9, 10].

The aim of the present paper is to derive this more general covariant Lagrangian quantization prescription starting from the generalized Schwinger-Dyson BRST symmetry, – the

BRST symmetry of field redefinitions. In the process we hope to add some physical insight to the more abstract algebraic considerations of refs. [9, 10]. Our manipulations will throughout be "formal" in the sense that we shall employ standard manipulations in the path integral, assuming the existence of a suitable symmetry-preserving regulator. Some of the subtleties involved in this process, especially at the two-loop level, are discussed in ref. [3].

To set the stage for the generalizations that are to follow, let us first briefly consider the simplest case, that of an action free of internal gauge symmetries. Fields of the classical action S are denoted by ϕ^A; they can be of arbitrary Grassmann parity $\epsilon(\phi^A) \equiv \epsilon_A$.[1] The index A labels collectively all internal quantum numbers and space-time variables. A quantum action S_{ext} that incorporates the correct Schwinger-Dyson BRST symmetry can in this case be taken to be simply [2, 3]: $S_{ext}[\phi, \phi^*, c] = S[\phi] - \phi^*_A c^A$, with a new ghost-antighost pair c^A, ϕ^*_A of Grassmann parities $\epsilon(c^A) = \epsilon(\phi^*_A) = \epsilon_A + 1$. Their ghost number assignments are $gh(c^A) = -gh(\phi^*_A) = 1$. With this set of fields, the Schwinger-Dyson BRST symmetry reads

$$\delta\phi^A = c^A \ , \quad \delta c^A = 0 \ , \quad \delta\phi^*_A = -\frac{\delta^l S}{\delta\phi^A} \ . \tag{1}$$

In this simple case, it is obviously possible to substitute S_{ext} for S in the transformation law for ϕ^*_A, but in general care is required in such a substitution. By correct Schwinger-Dyson equations, we shall always refer to those that formally follow for the classical fields of the classical action, independently of whether the path integral has been given a precise meaning through an appropriate gauge fixing, when needed.

The above choice incorporates the Schwinger-Dyson BRST symmetry (1) in the particular field variables ϕ^A. To find a more covariant formulation, let us perform a field redefinition of all the classical fields ϕ^A. At this stage we restrict ourselves to redefinitions that do not mix in the new ghost fields c^A, ϕ^*_A. We follow to a large extent the formulation presented in ref. [5]. Denote the new field variables by Φ^A, and the transformation by F. Introduce left (L) and right (R) vielbeins $e^A_{(L,R)B}$ and their inverses, $E^A_{(L,R)B}$, through the definition

$$e^A_{(L,R)B}(\Phi) \equiv \frac{\delta^{l,r} F^A(\Phi)}{\delta\Phi^B} \ , \qquad e^A_{(L)B} E^C_{(L)A} = e^C_{(R)A} E^A_{(R)B} = \delta^C_B \ . \tag{2}$$

We next choose to let the ghost-antighost pair transform oppositely under F, *i.e.*, in total:

$$\phi^A = F^A(\Phi) \ , \quad C^A = E^A_{(R)B} c^B \ , \quad \Phi^*_A = \phi^*_B e^B_{(R)A} \ , \tag{3}$$

where C^A and Φ^*_A are the new transformed ghost fields. This has the advantage that the ghost-antighost measure formally, or with a suitable symmetry-respecting regulator, remains invariant under the transformation. Of course, the ϕ^A-measure will in general *not* remain invariant, but acquire a Jacobian factor $\sqrt{g}$, where g is the superdeterminant of the metric

$$g_{AB}(\Phi) = \eta_{CD} e^C_{(L)A}(\Phi) e^D_{(R)B}(\Phi) \ . \tag{4}$$

Consider now the action S_{ext}. Since it must transform as a scalar under F, we immediately have, using (3),

$$S_{ext} = S[F(\Phi)] - \Phi^*_A C^A \ . \tag{5}$$

[1] Our conventions are described in detail in the appendix of ref. [4].

This transformed action is invariant under the transformed Schwinger-Dyson BRST symmetry

$$\delta\Phi^A = C^A \ , \quad \delta C^A = 0 \ , \quad \delta\Phi^*_A = (-1)^{\epsilon_M+1}\Gamma^M_{AK}C^K\Phi^*_M - \frac{\delta^l S}{\delta\Phi^A} \ . \tag{6}$$

It is also straightforward to check that the functional measure is formally invariant. The "connection" Γ^A_{BC} is the (superspace) Christoffel symbol of second kind [11],

$$\Gamma^A_{BC} \equiv \frac{1}{2}(-1)^{\epsilon_A\epsilon_C}\left[(-1)^{\epsilon_C\epsilon_D}\frac{\delta^r g_{BD}(\Phi)}{\delta\Phi^C} + (-1)^{\epsilon_B+\epsilon_C+\epsilon_B\epsilon_C+\epsilon_B\epsilon_D}\frac{\delta^r g_{CD}(\Phi)}{\delta\Phi^B} - \frac{\delta^r g_{BC}(\Phi)}{\delta\Phi^D}\right] g^{DA} \ . \tag{7}$$

In the new set of coordinates, the Schwinger-Dyson equations are Ward identities $0 = \langle\delta[\Phi^*_A G(\Phi)]\rangle$. In detail, with $\Gamma^M_{AM} = (-1)^{\epsilon_M}(\sqrt{g})^{-1}\delta^l(\sqrt{g})/\delta\Phi^A$,

$$0 = \frac{i}{\hbar}(-1)^{\epsilon_G+1}\langle\delta[\Phi^*_A G(\Phi)]\rangle = \left\langle(-1)^{\epsilon_M}\Gamma^M_{AM}(\Phi)G(\Phi) + \left(\frac{i}{\hbar}\right)\frac{\delta^l S}{\delta\Phi^A}G(\Phi) + \frac{\delta^l G}{\delta\Phi^A}\right\rangle \ , \tag{8}$$

where in the last bracket we have integrated out the ghost-antighost pair in order to compare with the conventional formulation of field-covariant Schwinger-Dyson equations. Such equations are normally derived from the invariance of the measure $[d\Phi]$ under arbitrary local shifts, *i.e.*, from

$$0 = Z^{-1}\int[d\Phi]\sqrt{g(\Phi)}\left[(\sqrt{g(\Phi)})^{-1}\frac{\delta^l}{\delta\Phi^A}\left\{e^{-S[\Phi]}\sqrt{g(\Phi)}G(\Phi)\right\}\right] \ . \tag{9}$$

In contrast, in the present formulation these equations are automatically incorporated into the action principle.

The Master Equation for the action S_{ext} in transformed coordinates is derived in as trivial a manner as in the original variables; it is simply the statement that S_{ext} is invariant under the BRST symmetry (6). Thus $0 = \delta S_{ext}$ immediately gives

$$\frac{\delta^r S_{ext}}{\delta\Phi^A}C^A = \frac{\delta^r S_{ext}}{\delta\Phi^*_A}\frac{\delta^l S}{\delta\Phi^A} = \frac{\delta^r S_{ext}}{\delta\Phi^*_A}\frac{\delta^l S_{ext}}{\delta\Phi^A} \ . \tag{10}$$

The extra term in the transformation law for Φ^*_A in eq. (6), which is proportional to the connection Γ^A_{BC}, does not contribute to the Master Equation due to the symmetry properties of Γ^A_{BC} and the ghosts C^A. The Master Equation (10) is of precisely the same form as that of the original S_{ext} [4], except that it is now expressed in the new coordinates.

To extend this construction to field theories in all generality, including those of arbitrarily complicated gauge-symmetry structure, one can proceed by demanding that the above coordinate-covariant Schwinger-Dyson equations for the classical fields are satisfied at the formal level throughout, and even before any gauge fixings. A sufficient, but perhaps not necessary, condition is that the ghosts C^A enter only linearly, and only in the combination $\Phi^*_A C^A$, as in eq. (5). This ensures that the crucial integral over C^A and Φ^*_A is diagonal, and in particular that $\langle C^A\Phi^*_B\rangle = -i\hbar\delta^A_B$, an ingredient needed in eq. (8) to recover the correct Schwinger-Dyson equations. In general, on should not expect to be able to split the extended action S_{ext} into the form $S_{ext}[\Phi, \Phi^*, C] = S[\Phi] - \Phi^*_A C^A$, as in eq. (5). But the above requirement is equivalent to demanding that S_{ext} is of the form $S_{ext}[\Phi, \Phi^*, C] = S^{BV}[\Phi, \Phi^*] - \Phi^*_A C^A$, where S^{BV} is simply everything left over after the term linear in C^A has been taken out.

Correct Schwinger-Dyson equations are obtained even if the extended action S_{ext} is not invariant under the Schwinger-Dyson BRST symmetry, but only transforms in precisely such a manner as to cancel a perhaps non-trivial Jacobian factor from the functional measure. In the *old* coordinates, an arbitrary action $S_{ext}[\phi, \phi^*, c] = S^{BV}[\phi, \phi^*] - \phi^*_A c^A$ which satisfies the full quantum Master Equation [4]

$$\frac{1}{2}(S_{ext}, S_{ext}) = -\frac{\delta^r S_{ext}}{\delta \phi^A} c^A + i\hbar \Delta S_{ext} \ , \tag{11}$$

gives rise to the Batalin-Vilkovisky Master Equation [1] for S^{BV}:

$$\frac{1}{2}(S^{BV}, S^{BV}) = i\hbar \Delta S^{BV} \ . \tag{12}$$

Here $(\cdot,\cdot)$ is the antibracket, and $\Delta \equiv (-1)^{\epsilon_A+1} \frac{\delta^r}{\delta\phi^A}\frac{\delta^r}{\delta\phi^*_A}$ is the correction term from the measure [1]. Both can straightforwardly be derived from the Schwinger-Dyson BRST symmetry [4].

To find the generalized Master Equation in the *new* coordinates, we must be careful when expressing the BRST transformation laws of all fields, ghosts and antighosts in terms of the new variables only. In particular, since Φ^* in general will enter non-trivially apart from the term $\Phi^*_A C^A$, and since these antighosts arise from a Φ-dependent transformation, a new implicit Φ-dependence enters through Φ^*:

$$\begin{aligned} \delta \Phi^A &= C^A \\ \delta C^A &= 0 \\ \delta \Phi^*_A &= (-1)^{\epsilon_M+1} \Gamma^M_{AK} C^K \Phi^*_M + (-1)^{\epsilon_A \epsilon_M + 1} \frac{\delta^r S^{BV}}{\delta \Phi^*_K} \Gamma^M_{KA} \Phi^*_M - \frac{\delta^l S^{BV}}{\delta \Phi^A} \ . \end{aligned} \tag{13}$$

With these transformation rules it is easy to get the following master equation:

$$\frac{\delta^r S^{BV}}{\delta \Phi^*_A} \frac{\delta^l S^{BV}}{\delta \Phi^A} = i\hbar (-1)^{\epsilon_A} \frac{1}{\sqrt{g}} \frac{\delta}{\delta \Phi^A} \left(\sqrt{g} \frac{\delta S^{BV}}{\delta \Phi^*_A} \right) \ . \tag{14}$$

The operator

$$\Delta_\rho \equiv (-1)^{\epsilon_A+1} \frac{1}{\sqrt{g}} \frac{\delta^r}{\delta \Phi^A} \left(\sqrt{g} \frac{\delta^r}{\delta \Phi^*_A} \right) \ , \tag{15}$$

associated with the measure density $\rho = \sqrt{g}$, is the covariant generalization of the Batalin-Vilkovisky operator Δ of eq. (12). Its form can also be inferred from general covariance arguments [9, 10]. Here, it arises straightforwardly from the non-trivial Jacobian factor associated with the BRST transformation (13). Since we have so far restricted ourselves to field transformations among the ϕ's only, the resulting measure density ρ does not depend on Φ^*.

In the case of flat coordinates, there is an interesting direct relation between the Schwinger-Dyson BRST operator (1) and the operator Δ [4]. Namely, if one integrates out the ghosts

c^A but keeps the antighosts ϕ^*_A in the path integral, the operator Δ appears as a "quantum deformation" (proportional to $\hbar$) of the BRST operator δ left over when integrating out the c^A-fields. The quantum deformation of the BRST operator in the conventional Batalin-Vilkovisky formalism has been discussed in ref. [12]. It must be emphasized that the appearance of this quantum deformation in the BRST operator is completely unrelated to the appearance of possible quantum corrections in the Lagrangian Master Equation (12). The quantum correction in the Schwinger-Dyson BRST operator in the unusual form in which the ghosts c^A (but not their antighosts ϕ^*_A) have been integrated out of the functional integral is *always* present. The quantum correction to the Master Equation (12) is non-vanishing only in those particular cases where the functional measure is not invariant under the Schwinger-Dyson BRST symmetry (independently of whether the ghosts c^A have been integrated out or not).

In the covariant case we have seen that Δ in the Master Equation is replaced by the covariant Δ_ρ. Let us now consider integrating out the new ghosts C^A from the path integral, and trace what happens to the Schwinger-Dyson BRST operator in this process. As in the flat case [4], the simple identity

$$\int [dC] F(C^B) \exp\left[-\frac{i}{\hbar}\Phi^*_A C^A\right] = F\left(i\hbar\frac{\delta^l}{\delta\Phi^*_B}\right)\int [dC]\exp\left[-\frac{i}{\hbar}\Phi^*_A C^A\right] \tag{16}$$

is useful here. Consider, inside the path integral, the BRST variation of an arbitrary functional $G[\Phi,\Phi^*]$. Using (16) above, we get

$$\begin{aligned} \delta G[\Phi,\Phi^*] & \\ = \; & \frac{\delta^r G}{\delta\Phi^A}C^A + \frac{\delta^r G}{\delta\Phi^*_A}\left\{(-1)^{\epsilon_M+1}\Gamma^M_{AK}C^K\Phi^*_M + (-1)^{\epsilon_A\epsilon_M+1}\frac{\delta^r S^{BV}}{\delta\Phi^*_K}\Gamma^M_{KA}\Phi^*_M - \frac{\delta^l S^{BV}}{\delta\Phi^A}\right\} \\ \to \; & \frac{\delta^r G}{\delta\Phi^A}\frac{\delta^l S^{BV}}{\delta\Phi^*_A} - \frac{\delta^r G}{\delta\Phi^*_A}\frac{\delta^l S^{BV}}{\delta\Phi^A} + (i\hbar)\left((-1)^{\epsilon_A}\frac{\delta^r}{\delta\Phi^A} + (-1)^{\epsilon_A\epsilon_G+\epsilon_M}\Gamma^M_{AM}\right)\frac{\delta^r}{\delta\Phi^*_A}G\;, \end{aligned} \tag{17}$$

where the arrow indicates that partial integrations are required inside the functional integral. Since

$$\Delta_\rho G = (-1)^{\epsilon_A+1}\frac{1}{\sqrt{g}}\frac{\delta^r}{\delta\Phi^A}\left(\sqrt{g}\frac{\delta^r G}{\delta\Phi^*_A}\right) = \left((-1)^{\epsilon_A+1}\frac{\delta^r}{\delta\Phi^A} + (-1)^{\epsilon_A\epsilon_G+\epsilon_M+1}\Gamma^M_{AM}\right)\frac{\delta^r}{\delta\Phi^*_A}G\;, \tag{18}$$

the equivalent of the Schwinger-Dyson BRST operator after having integrated out the ghosts C^A is indeed, as expected, given by

$$\delta = (\,\cdot\,, S^{BV}) - i\hbar\Delta_\rho \tag{19}$$

in the covariant formulation.[2] This form of the "quantum BRST operator" in the covariant Batalin-Vilkovisky formulation was first considered by Hata and Zwiebach [9]. Here we see that it can be derived straightforwardly from the Schwinger-Dyson BRST operator by integrating out the ghosts C^A. It is only because one chooses such an asymmetric procedure as that of integrating out the ghosts, while keeping the antighosts in the path integral, that

[2]Note that the operator Δ_ρ is nilpotent for any $\sqrt{g}$ that depends only on ϕ^A.

one has to face the unusual situation of having a quantum correction to the BRST operator. The *full* Schwinger-Dyson operator (13), with the ghosts C^A kept, automatically includes both classical and quantum parts, as is customary in quantum field theory.

So far everything has been derived from the flat case using a general coordinate transformation. In effect, all this amounts to is a formulation of the Lagrangian BRST quantization scheme in arbitrary curvilinear coordinates. It is worthwhile to first look at the quantization problem from the point of view of having been given a "space of fields" on which the path integral is to be defined. What is the effect of *curvature* in such a space of fields? To see a possible consequence, we need to go back and determine the Schwinger-Dyson equations on such spaces. As we have seen, once we have the correct Schwinger-Dyson BRST algebra, the quantization prescription follows immediately.

The correct Schwinger-Dyson equations for field theories defined on field spaces with a non-vanishing Riemann tensor (but with zero torsion, see below) can be derived as soon as the functional integral on such spaces is decided upon. Taking it to be of the form of a scalar density function $\rho(\phi) = \sqrt{g(\phi)}$, it is obvious that the Schwinger-Dyson equations (and the Schwinger-Dyson BRST algebra (13) that reproduces them) are of exactly the same kind as in eq. (8). This means that the whole quantization procedure, the Lagrangian Master Equation (14) and the form of the BRST operator (13), carry over directly to this case without modifications.[3] Whereas the case of curvature in the space of fields can thus be treated straightforwardly, a non-trivial aspect enters if we consider field spaces with torsion. We shall return to a discussion of this point elsewhere.

We shall now approach the quantization problem from a different point of view. Suppose we are *given* a measure density $\rho(\phi)$, and the set of transformations that leave the functional measure $d\phi\rho(\phi)$, but not the action $S[\phi]$, invariant. We denote these transformations by

$$\phi^A(x) = g^A(\phi'(x), a(x)) \; , \tag{20}$$

where $a^i(x)$ is a local field parametrizing the transformations. We choose coordinates such that g^A reduces to the identity at $a^i(x) = 0$. Invariance of the functional measure implies a set of identities, generalized Schwinger-Dyson equations:

$$\left\langle \left. \frac{\delta^l g^A}{\delta a^i} \right|_{a=0} \left[\frac{\delta^l F}{\delta \phi^A} + \frac{i}{\hbar} \frac{\delta^l S}{\delta \phi^A} F[\phi] \right] \right\rangle = 0 \; , \tag{21}$$

These Schwinger-Dyson equations are different in form from those obtained by exploring invariance of the measure $d\phi$ under local shifts. But under the conditions stipulated below they have the same content, and can, in fact, be mapped onto one another. To regain the usual Schwinger-Dyson equations from the generalized equations, we must require that $v^A_B \equiv \delta^l g^A/\delta a^B|_{a=0}$ locally has an inverse. When this v^{-1} exists, the generalized Schwinger-Dyson equations are in a one-to-one correspondence with those obtained from exploring invariance of the $d\phi$-measure (without the factor of $\rho(\phi)$) under local shifts. We will assume that the space of fields forms a manifold. If the dimension of the space is N (i.e., $A = 1, \ldots, N$),

[3]The only non-trivial aspect lies in the choice of appropriate boundary conditions for the Master Equation. In contrast to the simple curvilinear case, we may not simply take the standard Batalin-Vilkovisky boundary conditions for Cartesian coordinates and then perform the required field redefinition to obtain the corresponding boundary conditions in new coordinates.

we need precisely those symmetry transformations that locally correspond to shifts. These transformations are parametrized by N fields $a^A(x)$.

The condition that the measure $d\phi\rho(\phi)$ be invariant under the transformation (20) is equivalent to

$$\frac{\delta^r \rho}{\delta \phi^C} - (-1)^{\epsilon_A+\epsilon_C} G^A_{CA}\rho = 0 \ , \tag{22}$$

with

$$G^A_{CA} \equiv -\left(v^{-1}\right)^B_C \frac{\delta^r}{\delta\phi^A}\left(v^A_B\right) \ . \tag{23}$$

How do we now find the modified Schwinger-Dyson BRST symmetry whose Ward identities are the equations (21)? As in ref. [4], we can again follow the collective field approach. We do this by promoting $a^A(x)$ to a genuine field in the path integral, which we integrate over by using a flat measure. The relevant BRST symmetry reads [3]:

$$\begin{aligned}
\delta\phi'^A &= -\left(M^{-1}\right)^A_B \frac{\delta^r g^B}{\delta a^C} c^C \\
\delta a^A &= c^A \\
\delta c^i &= 0 \\
\delta\phi^*_A &= B_A \\
\delta B_A &= 0 \ ,
\end{aligned} \tag{24}$$

where $M^A_B \equiv \delta^r g^A/\delta\phi'^B$. Nilpotency of the transformations (24) is not immediately evident, but can be checked to hold: $\delta^2 = 0$. This is also obvious from its construction in ref. [3]. Next, we choose to gauge-fix on the trivial surface $a^A = 0$. We do this by adding $-\delta[\phi^*_A a^A] = (-1)^{\epsilon_A+1} B_A a^A - \phi^*_A c^A$ to the action S. At this point we can integrate out B_A and a^A, modifying the BRST transformations accordingly. The result is, for the BRST algebra:

$$\begin{aligned}
\delta\phi^A &= -(-1)^{\epsilon_B(\epsilon_A+1)} v^A_B c^B \\
\delta c^A &= 0 \\
\delta\phi^*_A &= (-1)^{\epsilon_B(\epsilon_A+1)} \frac{\delta^l S}{\delta\phi^B} v^B_A \ ,
\end{aligned} \tag{25}$$

where we have used the boundary condition $g^A(\phi', a=0) = \phi'^A$.

One can readily check that the BRST Ward identities $0 = \langle\delta\{\phi^*_A F[\phi]\}\rangle$ precisely coincide with the Schwinger-Dyson equations (21). Equation (25) thus gives us the required Schwinger-Dyson BRST algebra. However, nilpotency of the BRST operator is lost in the process of integrating out B_A and a^A. In contrast to the usual case of $\rho = 1$ [4], nilpotency does not even hold in general on the space of fields ϕ only. This makes this form of the Schwinger-Dyson BRST algebra slightly awkward for the quantization programme. But the version of the collective field formalism we have adhered to until now corresponds to the "Abelianization" of the constraints. As it turns out, the problem of nilpotency of the operator δ is instantly solved if we instead use the non-Abelian formalism (see appendix A of ref. [3]). We shall now describe this in some detail.[4] The non-Abelian Schwinger-Dyson BRST transformations can be chosen in the form

$$\delta\phi'^A = u^A_B(\phi')c^B$$

[4]The notation follows Appendix A of ref. [3].

$$
\begin{aligned}
\delta a^A &= -\nu^A_B(a)c^B \\
\delta c^A &= -\frac{1}{2}(-1)^{\epsilon_B} c^A_{BC} c^C c^B \\
\delta \phi^*_A &= B_A \\
\delta B_A &= 0 \ ,
\end{aligned}
\tag{26}
$$

where

$$
u^A_B(\phi') \equiv \left. \frac{\delta^r g^A(\phi', a)}{\delta a^B} \right|_{a=0} . \tag{27}
$$

The supernumbers c^A_{BC} are the structure coefficients of the supergroup of transformations (20). They satisfy

$$
c^A_{BC} = -(-1)^{\epsilon_B \epsilon_C} c^A_{CB} \ . \tag{28}
$$

A boundary condition is $\nu^A_B(a=0) = \delta^A_B$, and we also have $\lambda^A_B(a)\nu^B_C(a) = \delta^A_C$ [13]. We integrate the collective field over the left- or right-invariant measure of the supergroup of transformations g^A. The full functional measure is then formally (i.e. with a symmetry-preserving regulator) invariant under the BRST transformation (26) if we take, for $[dc]$ and $[d\phi^*]$, the usual (flat) measures, and if we assume that the group of transformations is compact (and in particular $(-1)^{\epsilon_A} c^A_{AB} = 0$). We now gauge-fix the collective field a^A to zero by adding a term $-\delta[\phi^*_A a^A] = (-1)^{\epsilon_A+1} B_A a^A + \phi^*_A \nu^A_B(a) c^B$ to the action S. Integrating over B_A and a_A, we find the modified BRST transformations by substituting for B_A the equation of motion for a^A (at $a^A = 0$). It is important to take into account the contribution from the measure as well. If we define

$$
\bar{\Gamma}^A_{BC} \equiv \left. \frac{\delta^r \nu^A_B}{\delta a^C} \right|_{a=0} , \tag{29}
$$

then the BRST transformations can be written

$$
\begin{aligned}
\delta \phi^A &= u^A_B(\phi) c^B \\
\delta c^A &= -\frac{1}{2}(-1)^{\epsilon_B} c^A_{BC} c^C c^B \\
\delta \phi^*_i &= (-1)^{\epsilon_A} \frac{\delta^l S}{\delta \phi^B} u^B_A(\phi) + i\hbar(-1)^{\epsilon_A+\epsilon_B} \bar{\Gamma}^B_{BA} + (-1)^{\epsilon_A \epsilon_B} \phi^*_M \bar{\Gamma}^M_{BA} c^B \ .
\end{aligned}
\tag{30}
$$

A related BRST construction for field theories with vanishing equations of motion, $\delta S/\delta\phi^A = 0$ has been considered by Okubo [14]. One has $\bar{\Gamma}^G_{KL} - (-1)^{\epsilon_K \epsilon_L} \bar{\Gamma}^G_{LK} = c^G_{KL}$.

Due to the "quantum correction" to the transformation law for ϕ^*_A, the action S itself is not invariant under the transformations (30). However, the measure transforms in just such a manner as to cancel the remaining term. So the combination of action and measure is invariant under (30), as it should be. The last two terms in the transformation law for ϕ^*_A cancel when we consider the Ward identity $0 = \langle \delta[\phi^*_A F(\phi)] \rangle$, leaving us with the correct Schwinger-Dyson equations.

We now perform the change of variables

$$
C^A = u^A_B(\phi) c^B \ , \qquad \Phi^*_A = \phi^*_B \left(u^{-1}\right)^B_A \ . \tag{31}
$$

The result is:

$$
\delta \phi^A = C^A
$$

$$
\begin{aligned}
\delta C^A &= 0 \\
\delta \Phi^*_A &= \frac{\delta^l S}{\delta \phi^A} + (-1)^{\epsilon_M+1}\Gamma^M_{BA} C^B \Phi^*_C + i\hbar(-1)^{\epsilon_A+\epsilon_C}\bar{\Gamma}^C_{CB}\left(u^{-1}\right)^B_A ,
\end{aligned} \tag{32}
$$

where Γ^A_{BC} is defined to be

$$
\Gamma^A_{BC} = G^A_{BC} + (-1)^{\epsilon_A(\epsilon_M+\epsilon_C+1)} u^M_S \bar{\Gamma}^S_{CB}\left(u^{-1}\right)^B_A \left(u^{-1}\right)^C_K , \tag{33}
$$

and where we have introduced the connection[5]

$$
G^D_{AC}(\phi) = (-1)^{\epsilon_A(\epsilon_D+1)} u^D_B \frac{\delta^r \left(u^{-1}\right)^B_A}{\delta \phi^C} . \tag{34}
$$

The action S is again not invariant under the BRST transformation, but the full partition function is, provided that ρ is covariantly conserved with respect to G^A_{BC}:

$$
\frac{\delta \rho}{\delta \phi^A} - (-1)^{\epsilon_A+\epsilon_B}\rho(\phi) G^B_{AB} = 0 . \tag{35}
$$

But this is just the condition (22) that the measure $d\phi\rho(\phi)$ is invariant under the group of transformation g^A. So we again find that the combination of action and measure is invariant under this (now non-Abelian) Schwinger-Dyson BRST transformation.

The advantage of this non-Abelian formulation is that nilpotency of δ when acting on the space of fields ϕ^A is not lost in the process of integrating out the collective field a^A and the Nakanishi-Lautrup field B_A. This means that the BRST operator δ can be used to gauge-fix internal gauge symmetries as well, and it is therefore meaningful to formulate the quantization prescription in terms of a Lagrangian Master Equation. This equation follows again from the simple requirement that the combination of action and measure remain invariant under the Schwinger-Dyson BRST symmetry. Let us write $S_{ext} = S^{BV}[\phi, \Phi^*] + \Phi^*_A C^A$. Since S^{BV} now depends on Φ^*, we find again that the transformation law for Φ^*_A has to be modified slightly. The resulting transformation is

$$
\delta \Phi^*_A = \frac{\delta^l S^{BV}}{\delta \phi^A} + (-1)^{\epsilon_M+1}\Gamma^M_{AK} C^K \Phi^*_M + (-1)^{\epsilon_A\epsilon_M}\frac{\delta^r S^{BV}}{\delta \Phi^*_B}\Gamma^M_{BA}\Phi^*_M + i\hbar(-1)^{\epsilon_A+\epsilon_C}\bar{\Gamma}^C_{CB}\left(u^{-1}\right)^B_A , \tag{36}
$$

with the transformations for ϕ and C left untouched. Note that only S^{BV} enters in the transformation law for Φ^*. The condition that the path integral remains Schwinger-Dyson BRST-invariant leads precisely to the standard Master Equation for S^{BV}:

$$
\frac{\delta^r S^{BV}}{\delta \Phi^*_A}\frac{\delta^l S^{BV}}{\delta \phi^A} = -i\hbar \Delta_\rho S^{BV} . \tag{37}
$$

We wish to emphasize that in the present formulation this is a highly non-trivial result of delicate cancellations between action and measure, as well as of the continuity equation (35).

Boundary conditions need to be imposed on S^{BV}. A first requirement is that $S_{cl}[\phi] = S^{BV}[\phi, \Phi^* = 0]$, where S_{cl} is the classical action. This is needed to ensure that Schwinger-Dyson equations for $S_{ext} = S^{BV}[\phi, \Phi^*] + \Phi^*_A C^A$ formally agree with those of S_{cl} before any

[5]This definition is consistent with the one given in eq. (23).

of the possible internal gauge symmetries have been fixed.[6] One further boundary condition is needed to ensure regularity of S^{BV}, i.e., invertibility of the propagator matrix.

We see that knowing the group of transformations that leave the measure $d\phi\rho(\phi)$ invariant naturally leads to an object (G^A_{BC}) that transforms as a connection on the space of fields. This connection itself has only indirect physical significance, since just the traced-over object $(-1)^{\epsilon_A}G^A_{CA}$ appears in the Schwinger-Dyson equations. Note that the Schwinger-Dyson BRST transformations (32) are ambiguous as far as the connection is concerned. We can replace any suitable connection G^A_{BC} with $G^A_{BC}+\tilde{G}^A_{BC}$ as long as $\tilde{G}^A_{BC}$ has the correct symmetry properties under exchange of the lower indices, and as long as $(-1)^{\epsilon_A}\tilde{G}^A_{BA}=0$. Such a replacement is void of physical content. It is conceivable that the redundancy in the choice of connection is a reflection of the large group of symmetries of the covariant Master Equation. If so, this could permit a geometric interpretation of the group of invariances directly on the space of fields.

We end this paper with some general comments. We have throughout restricted ourselves to either transformations of the fields ϕ^A that do not depend on the ghosts c^A or antighosts ("antifields" in the language of Batalin and Vilkovisky) ϕ^*_A, or, in the last part, on symmetries of functional measures of the fields ϕ^A only. We have done this on the assumption that eventually only symmetry properties related to the original classical fields (part of ϕ^A) are of physical importance. This means that we have really only been interested in the subset of transformations involving ϕ^A that refer to the classical fields, and not to the usual ghosts, antighosts, auxiliary fields, ghosts-for-ghosts, etc., which may be required to complete the quantization programme, and which form another part of ϕ^A. Such a point of view may be too restrictive, and there is indeed nothing preventing a more general setting in which all fields ϕ^A are mixed with each other and with ghosts c^A and antighosts ϕ^*_A. These more general transformations must of course obey the quite restrictive condition of preserving Grassmann parities and ghost numbers. The discussion in refs. [9, 10] goes along such lines (for the case where the ghosts c^A have been integrated out, and where the remaining fields ϕ^A and antighosts ϕ^*_A thus are canonical variables under the antibracket). One may in that case phrase the canonical framework in terms of a supersymplectic formalism that resembles the usual symplectic formulation of classical Hamiltonian mechanics. The different ghost number and Grassmann parity assignments between "coordinates" (ϕ^A) and "momenta" (ϕ^*_A) does, however, make the analogy with classical mechanics somewhat limited. It is difficult and rather tedious to formulate correct boundary conditions to be imposed on the Master Equations in any other frame than that of (the analogue of) Darboux coordinates on the supersymplectic manifold.

As we have shown in this letter, the analogue of Batalin-Vilkovisky quantization on spaces with non-trivial measure densities can be derived straightforwardly from the underlying Schwinger-Dyson BRST algebra. It is not coincidental that upon integrating out the ghosts c^A, the Master Equation for theories with non-trivial $\rho(\phi)$-measures formally matches the one of Schwarz [10], although we have not made use of the fact that a Darboux frame exists in which $\rho = 1$. This is because the existence of such a frame is a sufficient but not necessary condition for having nilpotency of the operator Δ_ρ. As we have seen, the existence

[6] For the special case of no internal gauge symmetries, $S^{BV}[\phi,\Phi^*] = S_{cl}[\phi]$, and it is then straightforward to see that the Ward identities of the symmetry (32) and (36) yield the correct Schwinger-Dyson equations for $S_{cl}[\phi]$.

of a coordinate frame with, in the language of ref. [10], $\rho(\phi, \phi^*) = \rho(\phi)$, also ensures that $\Delta_\rho^2 = 0$. It is only when leaving this density $\rho(\phi)$ in the measure (instead of exponentiating it into a "one-loop correction" of the extended action) that the full geometric picture discussed in this paper emerges.

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