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WELL-POSEDNESS, A SHORT SURVEY

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Abstract

In this paper we analyze the property of Tykhonov well-posedness in relation to other well-posedness properties which are ordinal and, as stated in the title, we give a survey on some important results on well-posedness in scalar optimization and in scalar inequalities.

1 Introduction

The notion of well-posedness is significant for several mathematical problems and, in particular, it plays an important role in establishing convergence of algorithms for solving scalar optimization problems.

Two notions of well-posedness, extensively investigated in many papers [4],[10],[16] are known in scalar optimization: one is based on the behaviour of a prescribed class of sequences of approximate solutions, the other on the continuous dependence of the solution on the data of the problem. These notions are named, respectively, Tykhonov and Hadamard well-posedness.

The notion of well-posedness was generalized also to other contexts: variational inequalities, Nash equilibria, saddle point problems and vector optimization problems, all special cases of an equilibrium problem. For instance, Lignola and Morgan [7] investigated well-posedness for optimization problems with constraints defined by variational inequalities; Huang [6] discussed well-posedness for vector optimization problems while Patrone and Margiocco [8], [12], [14] Tykhonov well-posedness for Nash equilibria. Cavazzuti and Morgan [2], at least, given a definition of well-posed saddle point problems and related results. In all these cases the idea is an extension of the concept of minimizing sequences seen as approximate solutions.

The aim of this survey is to collect the two notions of well-posedness, Tykhonov well-posedness and Hadamard well-posedness, and to show some links between these two

different notions. It is organized as follows. In section 2, after notations and definitions needed in addressing our study, we consider the two different definitions of well-posedness while subsequently in section 3 we present some strengthened versions of well-posedness (for instance well-posedness in the sense of Levitin and Polyak) and some notions generalized in case in which there is not uniqueness of solutions. Section 4 contains the relationships between Tykhonov and Hadamard well-posedness while the last section of the paper contains some basics on the notion of well-posedness of a scalar variational inequality. The notion of well-posedness in vector case will be considered separately.

2. Notations, definitions and preliminaries

In scalar optimization the different notions of well-posedness are based either on the behaviour of “appropriate” minimizing sequences (one requires the convergence of such sequences to a solution of the problem) or on the dependence of the optimal solutions on the data of optimization problem. This section is devoted exactly to a study of these notions.

The first notion of well-posedness of an optimization problem was introduced in 1966 by Tykhonov and later took his name. Given a real-valued function $f : R^n \rightarrow R$ and a closed, nonempty and convex subset K of R^n , we consider the scalar optimization problem:

$$P(f, K) \quad \min_{x \in K} f(x)$$

which we denote in the sequel by $P(f, K)$ and which consists in finding $x_0 \in K$ such that

$$f(x_0) = \inf \{f(x) : x \in K\} = \inf_K f(x)$$

The set, possibly empty, of the solutions of the minimization problem $P(f, K)$ is denoted by $\operatorname{argmin} P(f, K)$.

It is well known that usually every numerical method solving the optimization problem $P(f, K)$ produces a sequence.

A sequence $\{x_n\}_{n=1}^{\infty} \subset K$ with the property:

$$f(x_n) \rightarrow \inf_K f(x) = f(x_0) \quad \text{as } n \rightarrow +\infty$$

is said *minimizing sequence* for the problem $P(f, K)$. Such sequence is also called *sequence of approximate solutions* for the problem $P(f, K)$. The following classical definition states the notion of well-posedness of a given optimization problem $P(f, K)$ and it implies unique solvability of $P(f, K)$ together with convergence of the numerical methods of minimization for problem $P(f, K)$.

Definition 2.1[4]: *The minimization problem $P(f, K)$ is called Tykhonov well-posed if it has unique solution $x_0 \in X$ and, moreover, every minimizing sequence for $P(f, K)$ converges to x_0 .*

More precisely, $P(f, K)$ is Tykhonov well-posed if exists exactly one $x_0 \in K$ such that $f(x_0) \leq f(x)$ for all $x \in K$ and $x_n \rightarrow x_0$ whenever $f(x_n) \rightarrow \inf_K f(x) = f(x_0)$ and i.e. the Tykhonov well-posedness of the minimization problem $P(f, K)$ requires existence and uniqueness of minimum point x_0 towards which every approximate solution of $P(f, K)$ converges. In other words, to consider well-posedness of Tykhonov type, one introduces the notion of “approximating sequence” for the solutions of optimization problems and requires convergence of such sequences to a solution of the problem. For more details see [4] and [8].

Example: Let $K = R$. If $f(x) = x^2 e^{-x}$, $P(f, K)$ has a unique minimum $x_0 = 0$ but it is not Tykhonov well-posed, since the sequences $\{x_n\} = \{n\}$ is minimizing but it does not converges to $x_0 = 0$. If $f(x) = x^2$ then $P(f, K)$ is Tykhonov well-posed.

For convex functions in finite dimensions the uniqueness of the solution is enough to guarantee its Tykhonov well-posedness while this is no longer valid in infinite dimensions [15]. In analogous way the uniqueness of the solution of a minimization problem $P(f, K)$ is enough to guarantee its well-posedness in the case when K is compact. Instead, the uniqueness of the solution is not enough to guarantee Tykhonov well-posedness for continuous functions. A simple example of a problem with a unique solution but which is not Tykhonov well-posed is the following:

$$f(x) = x^2 / (x^4 + 1) \quad \forall x \in X$$

Obviously $P(f, K)$ has a unique solution at zero, i.e. the argmin $P(f, K) = \{0\}$, while $x_n = n$, $n = 1, 2, \dots$ provides a minimizing sequence for $P(f, K)$ which does not converge to this unique solution. Hence $P(f, K)$ is not Tykhonov well-posed. Therefore for continuous functions Tykhonov well-posedness of $P(f, K)$ simply means that every minimizing sequence of $P(f, K)$ is convergent.

The following results are known [4].

Proposition 2.1: *Let $f : K \subseteq R^n \rightarrow R$ be a convex function and let K be convex set. If $P(f, K)$ has a unique solution, then $P(f, K)$ is Tykhonov well-posed.*

Proposition 2.2: *If K is a nonempty convex and closed subset of R^n and if f is lower hemicontinuous bounded from below and uniformly quasi convex on K , then $P(f, K)$ is Tykhonov well-posed.*

The next fundamental theorem [4] gives an alternative characterization of Tykhonov well-posed problems; it states that Tykhonov well-posedness of $P(f, K)$ can be characterized by behaviour of $\text{diam}[\varepsilon - \arg \min(f, K)]$ as $\varepsilon \rightarrow 0$.

Theorem 2.1: *If $P(f, K)$ is Tykhonov well-posed, then*

$$\text{diam}[\varepsilon - \arg \min(f, K)] \rightarrow 0 \text{ as } \varepsilon \rightarrow 0$$

where $\varepsilon - \arg \min(f, k) = \{x \in K : f(x) \leq \varepsilon + \inf_K f(x)\}$ is the set of ε -minimizers of f over K .

Conversely, if f is lower semicontinuous and bounded from below on K , $\text{diam}[\varepsilon - \arg \min(f, K)] \rightarrow 0$ as $\varepsilon \rightarrow 0$ implies Tykhonov well-posedness of $P(f, K)$.

The second notion of well-posedness is inspired by the classical idea of J. Hadamard to the beginning of previous century: it requires existence and uniqueness of solution of the optimization problem together with continuous dependence of solution on the problem's data.

Definition 2.2: *The minimization problem $P(f, K)$ is said to be Hadamard well-posed if it has unique solution $x_0 \in K$ and x_0 depends continuously on the data of the problem.*

This is the well-known condition of well-posedness considered in the study of differential equations, translated for minimum problems. The essence of this notion is that a "small" change of the data of the problem yields a "small" change of the solution. In fact, very often the mathematical

model of a phenomenon is so complicated that it is necessary to simplify it and replace it by other model which is “near” the original and, at the same time, one wants to be sure that the new problem will have a solution which is “near” the original one. The well-known variational principle of Ekeland [5] asserts just that a particular optimization problem can be replaced by other which is near the original and has a unique solution.

3. Some generalizations

The idea of the behaviour of the minimizing sequences was used by different authors to extend this concept to two directions: first for strengthened notions and second for the case in which the optimal solution is not unique. These notions are not suitable for numerical methods, where the function f is approximated by a family or a sequence of functions. For this reason new notions of well-posedness have been introduced and studied. Before, however, we consider two generalizations of the notion of minimizing sequence. The first one was introduced and studied by Levitin and Polyak.

Definition 3.1: Let K be a nonempty subset of R^n . The sequences $\{x_n\}_{n=1}^{\infty} \subset K$ is a Levitin-Polyak minimizing sequences for the minimization problem $P(f, K)$ if $\forall x_n \in K$:

$$f(x_n) \rightarrow \inf_K f(x) \quad \text{and} \quad d(x_n, K) \rightarrow 0$$

where $d(x, K) = \inf \{\|x - y\| : y \in K\}$ is the distance function from the point x to the set K while $\|\cdot\|$ is the Euclidean norm.

In other words, a sequences $\{x_n\}_{n=1}^{\infty}$ is a Levitin-Polyak minimizing sequences for $P(f, K)$ if not only $\{f(x_n)\}_{n=1}^{\infty}$

approaches the greatest lower bound of f over K but also the sequence $\{x_n\}_{n=1}^{\infty}$ tends (with respect to the norm) to K .

Then, the well-posedness concept can be strengthened as follows.

Definition 3.2: *The minimization problem $P(f, K)$ is called Levitin-Polyak well-posed if it has unique solution $x_0 \in K$ and moreover, every Levitin-Polyak minimizing sequence for $P(f, K)$ converges to x_0 .*

Of course, this definition is stronger than the Tykhonov one since one wants each sequence from a larger set of minimizing sequences to be convergent to the unique solution, i.e. Levitin-Polyak well-posedness implies Tykhonov well-posedness. The converse is true provided that f is uniformly continuous but not necessarily true if f is only continuous. It is enough to consider $K = R \times \{0\} \subset R^2$ $f(x, y) = x^2 - y^2(x + x^4)$ and generalized minimizing sequences $\{1/n\}$.

As Tykhonov well-posedness can be characterized by the behaviour of $diam[\varepsilon - \arg \min(f, K)]$, as Levitin Polyak well-posedness can be characterized by the behaviour of the set:

$$L(\varepsilon) = \{x \in K : dist(x, K) \leq \varepsilon \text{ and } f(x) \leq \inf f(K) + \varepsilon\}$$

defined for $\varepsilon > 0$ and for f bounded from below on K . In analogy with the theorem 2.1 the following result gives:

Theorem 3.1 [4]: *If K is closed and f is lower semicontinuous and bounded from below on K , then $diam L(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ implies Levitin-Polyak well-posedness of $P(f, K)$*

A second generalization of the usual notion of minimizing sequences is the following:

Definition 3.3: A sequence $\{x_n\}_{n=1}^{\infty} \subset K$ is said to be a generalized minimizing sequence for the minimization problem $P(f, K)$ if are fulfilled both:

$$d(x_n, K) \rightarrow 0 \quad \text{and} \quad \limsup f(x_n) \leq \inf_K f(x)$$

Consequently another strengthened version of the well-posedness is:

Definition 3.4: The minimization problem $P(f, K)$ is said strongly well-posed if it has unique solution $x_0 \in K$ and, moreover, every generalized minimizing sequences for $P(f, K)$ converges to x_0 .

Obviously, in general strong well-posedness of the problem $P(f, K)$ implies Levitin-Polyak one, which in its turn implies Tykhonov well-posedness. It is important underline that, unlike def. 2.2, each of the previous definitions (def 2.1, def 3.2, def 3.4), widely studied in many papers [8],[9],[10], [15], is based on the behaviour of a certain set of minimizing sequences.

While the requirement of existence in the previous definitions is crucial, the uniqueness condition is more debatable. Many problems in linear programming or multicriteria optimization problems are in fact usually considered as well-posed problems, although they do not satisfy this uniqueness requirement. More precisely the different notions of well-posedness admit generalizations which do not require uniqueness of the solution. In particular, the concept of Tykhonov well posedness can be extend to minimum problems without uniqueness of the optimal solutions. This new definition requires existence, but not uniqueness, of

solution of $P(f, K)$, and, for every minimizing sequences, the convergence of some subsequence towards some optimal solution.

Definition 3.5: *The problem $P(f, K)$ is called Tykhonov well-posed in the generalized sense if every its minimizing sequence has some subsequence converging to a solution of a $P(f, K)$, i.e. to an element of $\arg \min(f, K)$.*

More precisely the problem $P(f, K)$ is called Tykhonov well-posed in the generalized sense if $\arg \min(f, K) \neq \emptyset$ and every sequence $x_n \in K$ such that $f(x_n) \rightarrow \inf f(K)$ has some subsequence $y_n \rightarrow y$ with $y \in \arg \min(f, K)$.

Obviously, if the problem $P(f, K)$ is Tykhonov well-posed in the generalized sense then it has a non-empty compact set of solutions, i.e. $\arg \min(f, K)$ is nonempty and compact.

When $\arg \min(f, K)$ is a singleton, the previous definition reduces to the classical notion of Tykhonov well-posedness or rather the problem $P(f, K)$ is Tykhonov well-posedness if it is Tykhonov well-posed in the generalized Tykhonov sense and $\arg \min(f, K)$ is a singleton; thus generalized well-posedness is really a generalization of Tykhonov well-posedness.

The corresponding generalization of Levitin-Polyak well-posedness is:

Definition 3.7: *The minimization problem $P(f, K)$ is called generalized Levitin-Polyak well-posed if every Levitin-Polyak minimizing sequence $\{x_n\} \subset K$ for $P(f, K)$ has a subsequence converging to a solution of $P(f, K)$.*

Any of the notions of generalized well-posedness together with the uniqueness of the solution is equivalent obviously to corresponding non generalized notion.

4. Relations between Hadamard and Tykhonov well-posedness

Almost all the literature deals with different notions of well-posedness, even if especially with Tykhonov types well-posedness. Some researchers have investigated the relations between these notions of well-posedness but there is no general research to such relations. At first sight the two notions seem to be independent but, at least in the convex case, there are some works showing a connection between the two properties: for instance [8],[19]. The two notions (Tykhonov and Hadamard well-posedness) are equivalent at least for continuous objective functions. The links between Hadamard and Tykhonov well-posedness have been studied in [8], [10], and in [15], [16]. There, besides uniqueness, additional structures are involved: in [8], for example, basic ingredient is convexity.

The object of this section is to describe generally the relations between Hadamard and Tykhonov well-posedness: a central role is provided by the well-known Hausdorff convergence.

Definition 4.1: Let $\{A_n\}$ be a sequence of subsets of R^n . We say that A_n converges according to Hausdorff to $A \subset R^n$

when
$$e(A_n, A) \rightarrow 0$$

where
$$e(A_n, A) = \sup_{a \in A_n} d(a, A) \quad \text{with} \quad d(a, A) = \inf_{b \in A} \|a - b\|$$

The following theorems show the relations between Tykhonov and Hadamard well-posedness.

Theorem 4.1 [8]: *Let K be a closed convex subset of R^n and let $f : K \rightarrow R$ be a convex continuous function with a unique minimum point on every closed and convex subset of K . If $P(f, K)$ is Hadamard well-posed, with respect to the Hausdorff convergence, then $P(f, K)$ is Tykhonov well-posed on every closed and convex subset of K .*

Theorem 4.2 [8]: *Let $f : R^n \rightarrow R$ be a convex function uniformly continuous on every bounded set. If $P(f, K)$ is Tykhonov well-posed on every closed and convex set, then $P(f, K)$ is Hadamard well-posed, with respect to the Hausdorff convergence.*

Tykhonov well-posedness does not, in general, imply Hadamard well-posedness if the objective function is only continuous.

5. Well-posed variational inequalities

In this section, some notions of well-posedness for scalar variational inequalities and their connections with optimization problems are presented.

In recent years variational inequality theory, which was introduced in the sixties of previous century, has emerged as a powerful tool for handling optimization problems and its importance is not only from the theoretical point of view but also provides an unified and efficient framework for a wide spectrum of applied problems. In other words the theory of variational inequalities provides a convenient mathematical apparatus for obtain result relating to a large number of problems with a wide range of applications in economics, social, pure and applied sciences. In fact, it is well known that many equilibrium problems arising in finance, economics, transportation science and contact problems in

elasticity can be formulated in terms of the variational inequalities [13].

Given an operator $F : K \rightarrow R^n$ with K a nonempty, closed and convex subset of R^n , we consider the problem which consist in finding a vector $x^* \in K$ such that:

$$SVI(F, K) \quad \langle F(x^*), y - x^* \rangle \geq 0 \quad \forall y \in K$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in R^n .

The previous problem is known as the *Stampacchia variational inequality* (for short $SVI(F, K)$) and was introduced and studied mainly by Stampacchia in 1964 [17].

Definition 5.1: *The variational inequality $SVI(F, K)$ is well-posed when:*

- i) $G(\varepsilon) \neq \emptyset \quad \forall \varepsilon > 0$
- ii) $\text{diam } G(\varepsilon) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$

where $G(\varepsilon) = \{x \in K : \langle F(x), x - y \rangle \geq \varepsilon \|y - x\| \quad \forall y \in K\}$.

Some results concerning the existence and uniqueness of a solution of variational inequality are investigated.

Under very weak hypotheses, if the variational inequality is well-posed, the solution of variational inequality exist and is unique. Before, we recall the following well Known concepts that will be useful later on:

Definition 5.2: *The operator F is said to be*

- i) *monotone on K if: $\forall x, y \in K \quad \langle F(x) - F(y), x - y \rangle \geq 0$*
- ii) *pseudomonotone on K if: $\forall x, y \in K$
 $\langle F(x), x - y \rangle \geq 0 \Rightarrow \langle F(y), x - y \rangle \geq 0$*

- iii) *hemicontinuous on K if $x_n \rightarrow x$ implies $F(x_n) \rightarrow F(x)$*

It is well known that monotonicity implies pseudomonotonicity but the converse is not true.

Proposition 5.1 [4],[9]: *Let F be an hemicontinuous and monotone function. If the variational inequality $SVI(F, K)$ is well-posed, then $SVI(F, K)$ has an unique solution.*

Proof: The set of solution of $SVI(F, K)$ is exactly $\bigcap_{\varepsilon \rightarrow 0} T(\varepsilon)$. If the variational inequality is well-posed, then $\bigcap_{\varepsilon \rightarrow 0} T(\varepsilon)$ is nonempty and shrinks to a single point.

In [4] can be found the converse of previous proposition:

Proposition 5.2: *Let F be an hemicontinuous and monotone function. If $SVI(F, K)$ has exactly one solution, then $SVI(F, K)$ is well-posed.*

Now, let us turn briefly our attention to scalar variational inequalities of differential type (i.e. in which the operator F involved is the gradient of a given function f). Assume that $f : R^n \rightarrow R$ is differentiable on an open set containing K , and denote by f' the gradient of f . We recall that:

- a point $x^* \in K$ is a solution of a Stampacchia variational inequality of differential type when:

$$SVI(f', K) \quad \langle f'(x^*), y - x^* \rangle \geq 0 \quad \forall y \in K$$

- an optimization problem can be expressed by means of a variational inequality involving the gradient of f .

The links between variational inequalities of differential type and optimization problems have been deeply studied [9], [11],[18]. More precisely, in [4], the following characterizations for Tykhonov well-posed problem are given:

Proposition 5.3: *Let K be a closed convex subset of R^n and let be $Z(\varepsilon) = \{x \in K : f(x) \leq f(y) + \varepsilon \|x - y\| \quad \forall y \in K\}$.*

If $Z(\varepsilon) \neq \emptyset \quad \forall \varepsilon > 0$ and $\text{diam} Z(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, then $P(f, K)$ is Tykhonov well-posed. The converse holds provided f is convex on K .

Proposition 5.4: Let f be differentiable and bounded from below on K . Then, $\forall \varepsilon > 0$:

$$T(\varepsilon) = \{x \in K : \langle f'(x), x - y \rangle \leq \varepsilon \|y - x\| \quad \forall y \in K\} \neq \emptyset$$

If $\text{diam} T(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, then $P(f, K)$ is Tykhonov well-posed. The converse holds if moreover f is convex on K .

Since it is well known [9] that the minimization problem is equivalent to a variational inequality involving the operator f' , the following definition that gives the notion of well-posedness for a variational inequality of differential type follows.

Definition 5.3 [4]: Let f be a function convex, lower semi-continuous, differentiable and bounded from below on a closed convex set K . The variational inequality $SVI(f', K)$ is well-posed when:

- i) $T(\varepsilon) \neq \emptyset \quad \forall \varepsilon > 0$
- ii) $\text{diam} T(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$

An important extension call "well-posed" those variational inequalities (governed by an operator that is not necessarily a derivative) such that the diameter of the level set of the operator decreases to zero.

The following theorem gives the link between Tykhonov well-posedness of $P(f, K)$ and well-posedness of $SVI(f', K)$:

Theorem 5.1 [4], [9]: Let f be bounded from below and differentiable on an open set containing K . If $SVI(f', K)$ is

well-posed, then problem $P(f, K)$ is Tykhonov well-posed. The converse is true if f is convex.

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