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Mixed finite element solution for the navier-stokes equations*

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SUMMARY: A mixed finite element method for the solution of the Navier-Stokes equations is presented. The scheme uses discontinuous finite elements that allow the utilization of upwind derivatives. This makes the scheme more stable and accelerates the convergence. Moreover the boundary layer is better considered for high Reynolds numbers. The existence of a solution and the convergence of the scheme are established. Results inside a square cavity for $Re \leqq 10000$ for the recirculating flow, the stream function and their second derivatives, viscous stress tensor and velocity vector are obtained. Between $Re=1000$ and $Re=2000$ a third secondary vortex appears near the upper right corner.

1. INTRODUCCION

The main difficulty of the numerical solution of the Navier-Stokes equations for an incompressible viscous fluid is to simulate the behaviour when viscosity becomes small, or equivalently, when the Reynolds numbers becomes large. In this case near the walls zones will appear where the gradient of the velocity is large and the viscous effects are important.

One of the models proposed to describe separated flow corresponding to small viscosity is that due to G.K. Batchelor (1956). The model postulates at high Reynolds numbers the existence of a central core of constant vorticity sorrounded by a thin viscous layer. In studies done by Burgraff (1966) and recently by Nallasamy & Krishna (1978) provide comparisons of the uniform vorticity model with numerical solutions of the Navier-Stokes equations.

The square cavity with a steady sliding top wall is a geometrically simple case and for which the recirculating flow has no analytical solution, is a prototype of separated flow for which Batchelor's model would be applicable. From a computational point of view the cavity flow is an easy model problem which allows comparisons among numerical methods.

The method that will be presented was introduced by Hermann (1) in 1967 to solve the bending plate problem. Other formulations of the method were given by Hellan (2) in 1967 and Visser (3) in 1969.

In 1973, Johnson (4) justified the method proving the existence of a solution and obtaining an estimate of the error.

In 1976, Brezzi and Raviart (5) proved that the method is the best in the sense that the order of convergence is $O(h^{k+1})$ with k the degree of polynomials employed.

The stationary Navier-Stokes equations in a bounded domain $\Omega \subset \mathbb{R}^n$ are:

$$(1.1)\quad \begin{cases} -\nu\Delta \vec{u} + \vec{u}\cdot\nabla\vec{u} + \nabla p = \vec{f} \text{ in } \Omega \\ \operatorname{div}\vec{u} = 0 \text{ in } \Omega \\ \vec{u} = 0 \text{ on } \partial\Omega \end{cases}$$

where ν is the kinematic viscosity; p the pressure; $\vec{u} = (u_i)_{1\leq i\leq n}$ the velocity vector; $\vec{f} = (f_i)_{1\leq i\leq n}$ the body external forces.

A classical variational formulation of the problem is (Cf.(6)):

$$(1.2)\quad \begin{cases} \text{Find } \Omega \text{ a pair } (\vec{u}, p) \text{ in } V \times L^2(\Omega)/\mathbb{R} \text{ satisfying:} \\ \nu\sum_{i,j=1}^{n} \int_\Omega \partial_j u_i\, \partial_j v_i\, dx + (\vec{u}\cdot\nabla\vec{u}, \vec{v}) - (p, \operatorname{div}\vec{v}) \\ = (\vec{f}, \vec{v});\ \forall\, \vec{v} \in (H_0^1(\Omega))^n \end{cases}$$

where $V = \left\{ \vec{v} \in (H_0^1(\Omega))^n\colon \operatorname{div}\vec{v} = 0 \right\}$.

There is a solution $(\vec{u}, p)$ in $V \times L^2(\Omega)/\mathbb{R}$ of (1.2) (equivently of (1.1)) if Ω is a bounded domain of $\mathbb{R}^n$, $n \leq 4$, with a Lipschitz continous boundary (Cf.(7)).

The solution is unique if the viscosity is relatively large with respect the norm of the body external forces.

In two dimensions the Navier-Stokes equations can be stated in terms of the stream function as follows:

$$(1.3)\quad \begin{cases} \text{Find } \psi \text{ in } H_0^2(\Omega) \text{ such that:} \\ \nu\int_\Omega \Delta\psi\,\Delta\varphi\, dx + \int_\Omega \Delta\psi(\partial_2\psi\partial_1\varphi - \partial_1\partial_2\varphi)dx \\ = \int_\Omega \vec{f}\,\overrightarrow{\operatorname{curl}}\,\varphi\, dx;\ \forall\varphi\in H_0^2(\Omega) \end{cases}$$

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The problems (1.2) and (1.1) are equivalent. That is:

— if $(\vec{u}, p)$ is the solution of (1.2), the only function $\Psi \in H_0^2(\Omega)$ such that $\vec{u} = \overrightarrow{\mathrm{curl}}\Psi = (\partial_2\Psi, -\partial_1\Psi)$ is the solution of (1.3);

— if Ψ is the solution of (1.3) there exists p in $L^2(\Omega)/\mathbb{R}$ such that the pair $(\vec{u}, p) = (\overrightarrow{\mathrm{curl}}\Psi, p)$ is the solution of (1.2).

To interpret the problem (1.3), the function Ψ satisfies

$$(1.4) \quad \begin{cases} \nu\Delta^2\Psi - \partial_1(\Delta\Psi\partial_2\Psi) + \partial_2(\Delta\Psi\partial_1\Psi) = \mathrm{curl}\,\vec{f} \text{ in } \Omega \\ \Psi = \dfrac{\partial\Psi}{\partial n} = 0 \text{ on } \partial\Omega \end{cases}$$

2. FORMULATION OF THE METHOD FOR THE STOKES PROBLEM

The Stokes problem has the following formulation:

$$(2.1) \quad \begin{cases} \text{Find } \lambda \text{ in } H_0^2(\Omega) \text{ satisfying} \\ \nu\,\Delta^2\lambda = \mathrm{curl}\,\vec{f} \quad \text{in } \Omega \end{cases}$$

Let λ in $H_0^2(\Omega)$ be a solution of (2.1). If we set $\sigma_{ij} = \partial_{ij}^2\lambda$, $1 \leq i, j \leq 2$ then $\sigma = (\sigma_{ij})_{1\leq i,\, j\leq 2}$ and λ satisfy the equations

$$\sum_{i,j=1}^{2} \int_\Omega (\sigma_{ij} - \partial_{ij}^2\lambda)\,\tau_{ij}\,dx = 0 \quad ;$$

$$\forall \tau = (\tau_{ij})_{1\leq i,j\leq 2} \text{ in } (L^2(\Omega))^4,\ \tau_{12} = \tau_{21}$$

$$\sum_{i,j=1}^{2} \int_\Omega \sigma_{ij}\,\partial_{ij}^2\mu\,dx - \int_\Omega \overrightarrow{\mathrm{curl}\,f}\,dx = 0 \quad ; \quad \forall \mu \in H_0^2(\Omega)$$

We assume that Ω is a convex polygon. Let ζ_h be a triangulation of Ω by triangles K with sides of length less than h.

Let $\tau = (\tau_{ij})_{1\leq i,j\leq 2}$ in $(L^2(\Omega))^4$, $\tau_{12} = \tau_{21}$ so that $\tau_{ij}/K \in H^1(K)$, $\forall$ K $\in \zeta_h$, then

$$\sum_{i,j=1}^{2} \int_\Omega \tau_{ij}\,\partial_{ij}^2\lambda\,dx = -\sum_{K\in\zeta h}\sum_{i,j=1}^{2} \int_K \partial_j\,\tau_{ij}\,\partial_i\lambda\,dx + $$

$$+ \sum_{K\in\zeta h}\sum_{i,j=1}^{2} \int_{\partial K} \frac{\partial\lambda}{\partial t}\,\tau_{ij}\,t_i\,n_j\,ds$$

$$+ \sum_{K\in\zeta h}\sum_{i,j=1}^{2} \int_{\partial K} \frac{\partial\lambda}{\partial n}\,\tau_{ij}\,n_i\,n_j\,ds$$

where $\vec{n} = (n_1, n_2)$ is the unit normal vector and $\vec{t} = (n_2, -n_1)$ is the unit tangent along the boundary ∂K.

If we impose the condition denoted (MN):

$$\sum_{i,j=1}^{2} \tau_{ij}/K_1\,n_i^{K_1}\,n_j^{K_1} = \sum_{i,j=1}^{2} \tau_{ij}/K_2\,n_i^{K_2}\,n_j^{K_2}$$

$$\text{over } K_1 \cap K_2 \neq \emptyset$$

then $\sigma = (\sigma_{ij})_{1\leq i,j\leq 2}$ and λ satisfy the system

$$(2.2) \quad \begin{cases} a(\sigma, \tau) + b(\tau, \lambda) = 0 \quad ; \quad \forall \tau \in \underset{\sim}{\widetilde{V}} \\ \nu\, b(\sigma, \mu) + \displaystyle\int_\Omega \mu\,\overrightarrow{\mathrm{curl}\,f}\,dx = 0 \quad ; \\ \forall \mu \in H_0^s(\Omega) \text{ with } 1 < s < 3/2 \end{cases}$$

where the space

$$\underset{\sim}{\widetilde{V}} = \left\{ \begin{array}{l} \tau = (\tau_{ij})_{1\leq i,j\leq 2} \in (L^2(\Omega))^4, \\ \tau_{12} = \tau_{21}\ ,\ \tau_{ij}/K \in H^1(K)\ ,\ \forall K \in \zeta_h, \\ ,\ \tau \text{ satisfy the condition (MN)} \end{array} \right\}$$

$$a(\sigma, \tau) = \sum_{i,j=1}^{2} \int_\Omega \sigma_{ij}\,\tau_{ij}\,dx$$

$$b(\sigma, \mu) = \sum_{K\in\zeta_h}\sum_{i,j=1}^{2} \Big(\int_K \partial_j\,\tau_j\,\tau_{ij}\,\partial_i\mu\,dx - \int_{\partial K} \frac{\partial\mu}{\partial t}\,\tau_{ij}\,n_i\,t_j\,ds \Big)$$

Brezzi and Raviart (5) proved that the method (2.2) is optimal.

3. FORMULATION OF THE METHOD FOR THE NAVIER-STOKES EQUATIONS

We shall consider the formulation (1.2). It is easy obtain "centered" formulation of the method by introducing the term

$$\int_\Omega (\sigma_{11} + \sigma_{22})\,(\partial_2\lambda\,\partial_1\mu - \partial_1\lambda\,\partial_2\mu)\,dx$$

in the second equation of (2.2).

But the problem is to simulate efficiently the behaviour of the fluid when the viscosity becomes small. For that Fortin (8) proposed to consider formulations called "decentered". The method is a generalization of that introduced by Lesaint (9) for the approximation of hyperbolic equations of first degree by means of discontinuous finite elements.

The difficulty is the discretization of the nonlinear term $\int_\Omega (\vec{u} \cdot \nabla\vec{u})\vec{v}\,dx$ where $\vec{u} = \overrightarrow{\mathrm{curl}}\lambda$, $\vec{v} = \overrightarrow{\mathrm{curl}}\,\mu$ since the functions λ, $\mu \in H_0^s(\Psi)$, $1<s<3/2$ are not sufficiently smooth.

For that we introduce the space M_h defined as

$M_h = H_0^s(\Omega) \cap \{\mu \in C^0(\Omega): \mu/K \in P_k \ , \ \forall K \in \zeta_h\}$, with $k \in \mathbb{N}$.

If λ, $\mu \in M_h$ and $\vec{u} = \overrightarrow{\text{curl}}\,\lambda$ then $\vec{u} \cdot \vec{n} = -\frac{\partial \lambda}{\partial t}$, that is $\vec{u} \cdot \vec{n}$ is continuous across the boundary ∂K, i.e. $\vec{u} \cdot \vec{n}_{K_1} = -\vec{u} \cdot \vec{n}_{K_2}$ on $K_1 \cap K_2 \neq \emptyset$. This allows us to define a partition of the boundary $\partial K = \partial K_+ \cap \partial K_-$ with ∂K_+ the zone where the fluid exits, that is $\vec{u} \cdot \vec{n} > 0$ and $\partial K_- = \partial K - \partial K_+$. By definition

$$\vec{u}^+ = \begin{cases} \vec{u}/K & \text{on } \partial K_+ \\ \vec{u}^e = \vec{u}/K' & \text{on } \partial K_- \cap \partial K' \ , \ K' \text{ next to } K \end{cases}$$

We consider the next discretization of the nonlinear convective term by using upwind derivatives:

$$- \sum_{K \in \zeta h} \sum_{i,j=1}^{2} \int_K u_i u_j (\partial_j v_i) \, dx$$

$$+ \sum_{K \in \zeta h} \int_{\partial K} (\vec{u}^+ \cdot \vec{v}) (\vec{u} \cdot \vec{n}) \, ds$$

denoted $d(\lambda, \lambda, \mu)$ and introduced in the second equations of (2.2).

We obtain the mixed formulation

$$(3.1) \quad \begin{cases} \text{Find a pair } (\sigma, \lambda) \text{ in } V_h \times M_h \text{ such that} \\ a(\sigma, \tau) + b(\tau, \lambda) = 0 \ ; \ \forall \tau \in V_h \\ \nu\, b(\sigma, \mu) + d(\lambda, \lambda, \mu) \\ \qquad + \int_\Omega \mu \ \text{curl}\, \vec{f} \, dx = 0 \ ; \ \forall \mu \in M_h \end{cases}$$

where

$$V_h = \{\tau = (\tau_{ij})_{1 \leq i,j \leq 2} \in \underset{\sim}{\tilde{V}} \ : \ \tau_{ij}/K \in P_{k-1} \ , \ \forall K \in \zeta_h\}$$

Concerning the existence of a solution for the system (3.1) we have the next result (Cf. (10), (11)).

Denoting $F_h: M_h \rightarrow V_h \ ; \ \mu_h \rightarrow \sigma_h$ where σ_h is the solution of the problem $a(\sigma_h, \tau_h) = -b(\tau_h, \mu_h)$, $\forall \tau_h \in V_h$ we have

Theorem 3.1

If

i) $d(\mu_h, \mu_h, \mu_h) \leq 0 \ ; \ \forall \mu_h \in M_h$ (3.2)

ii) $\| \lambda_h \|_{0,\Omega} \leq C \ \| F_h(\lambda_h) \|_{0,\Omega} \ ; \ \forall \lambda_h \in M_h$ (3.3)

there exists at least one solution (σ_h, λ_h) in $V_h \times M_h$ for the system (3.1).

Concerning the convergence of the scheme we have the following result (Cf. (10), (11)):

Theorem 3.2

Suppose that (3.2) and (3.3) hold for all $h>0$. Then there exist a pair (σ, λ) in $(L^2(\Omega))^3 \times H^s(\Omega)$ and a subsequence of the sequence (σ_h, λ_h) denoted (σ_h, λ_h) where (σ_h, λ_h) is a solution of (3.1) so that

$\sigma_h \rightarrow \sigma$ weakly in $(L^2(\Omega))^3$

$\lambda_h \rightarrow \lambda$ strongly in $H^1(\Omega)$.

Theorem 3.3

The function λ belongs to $H_0^2(\Omega)$ and is the solution of the Navier-Stokes equations in terms of the stream function.

Remark

The error estimate for this type of scheme is an open problem.

4. THE METHOD OF SOLUTION

If the boundary $\partial\Omega$ is moving with a relative velocity g in $H^{1/2}(\partial\Omega)$ then the conditions on the boundary are $\vec{u} \cdot \vec{n} = 0$ and $\vec{u} \cdot \vec{t} = g$. That is $\frac{\partial \lambda}{\partial t} = 0$, $\frac{\partial \lambda}{\partial n} = g$. Let λ_0 in $H^2(\Omega)$ be such that $\lambda_0 = 0$ and $\frac{\partial \lambda_0}{\partial n} = g$ over $\partial\Omega$.

Then we write the problem

Find $\lambda \in H^2(\Omega)$, $\lambda - \lambda_0$ in $H_0^2(\Omega)$ such that

$$-\nu \int_\Omega \Delta\lambda \Delta\varphi \, dx + \int_\Omega \Delta\lambda (\partial_1 \lambda \partial_2 \varphi - \partial_2 \lambda \partial_1 \varphi) \, dx +$$

$$\int_\Omega \varphi \ \text{curl}\, \vec{f} \, dx = 0; \ \forall \varphi \in H^2(\Omega)$$

The boundary condition $\frac{\partial \lambda}{\partial n} = g$ is introduced in the first equation of (3.1):

$$a(\sigma, \tau) + b(\tau, \lambda) = \sum_{i,j=1}^{2} \int_{\partial\Omega} \tau_{ij} n_i n_j \, g \, ds$$

We shall use the iterative method

1) $(\sigma_h^0, \lambda_h^0)$ is the initial value in $V_h \times M_h$

2) (θ_h, ψ_h) in $V_h \times M_h$ is the solution of

$$(4.1) \quad \begin{cases} a(\theta_h, \tau_h) + b(\tau_h, \psi_h) \\ \quad = \sum_{i,j=1}^{2} \int_{\partial\Omega} (\tau_h)_{ij} n_i n_j \, g \, ds; \ \forall \tau_h \in V_h \\ \nu b(\theta_h, \mu_h) + d(\lambda_h^n, \psi_h, \mu_h) \\ \quad + \int_\Omega \mu_h \ \text{curl}\, f \, dx = 0; \ \forall \mu_h \in M_h \end{cases}$$

3) $(\sigma_h^{n+1}, \lambda_h^{n+1}) = (\sigma_h^n, \lambda_h^n) + \gamma\,(\theta_h - \sigma_h^n, \psi_h - \lambda_h^n)$ with $0 < \gamma < 2$.

Remarks concerning the iterative method

The existence theorem in the paragraph 3 may be adapted if we consider $d(\bar{\lambda}, \lambda, \mu)$ with $\bar{\lambda}$ fixed and the hypothese

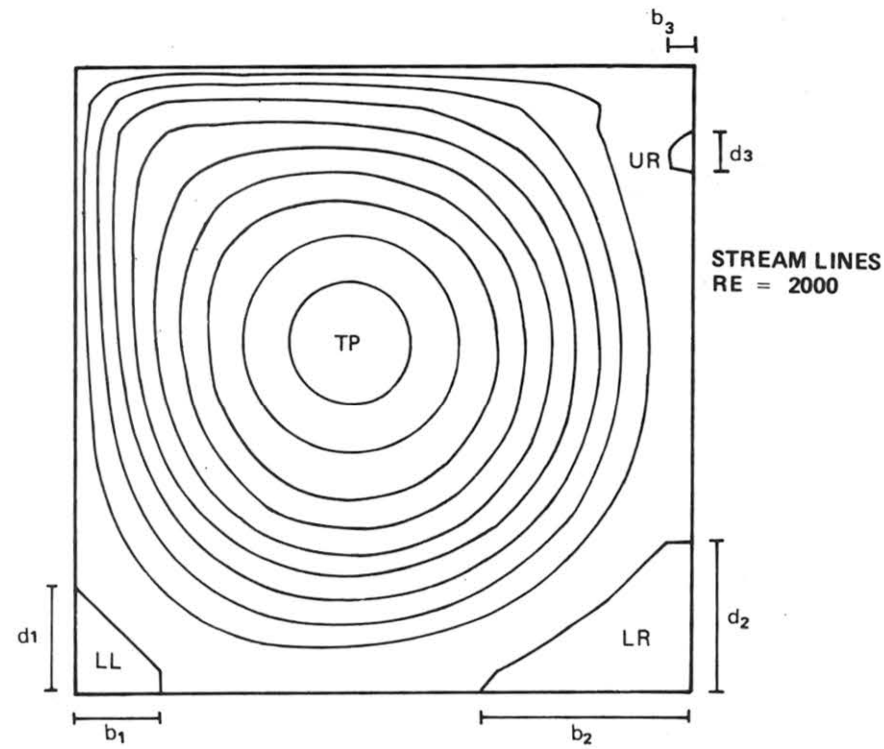


FIG. 1 DEFINITION SKETCH

TP PRIMARY VORTEX
LL LOWER LEFT VORTEX
LR LOWER RIGHT VORTEX
UR UPPER RIGHT VORTEX
h GRID SIZE

Fig. 1. Definition sketch.

$d(\lambda_h, \psi_h, \psi_h) \leq 0, \forall \psi_h$ in M_h . Then there exist a solution for the linealized system (4.1). The unicity is easy to see.

Fortin and Thomasset (12) used $d(\lambda_h^n, \lambda_h^n, \mu_h)$. The matrix is the same each iteration.

The system (4.1) is solved by using the frontal method (Cf. (13)). The difficulty of the linearization (4.1) is the increase in the band with and that the matrix is different with each iteration. However the convergence is rapid.

5. PRACTICAL IMPLEMENTATION OF THE METHOD

It was demonstrated (5) that the degrees of freedom by element and for the space V_h are

$\frac{3k(k+1)}{2}$. Therefore, the total degrees of freedom by element is

$$DL(k) = \frac{3k(k+1)}{2} + \frac{(k+1)\,(k+2)}{2}$$

The result on the degrees of freedom of a tensor-valued function $\tau \in V_h$:

"Let K be in τ_h and let $\tau = (\tau_{ij})_{1\leq\ i,\ j\leq 2}$ be a tensor-valued function such that k $\tau_{ij} \in P_{K-1}$ and $\tau_{12} = \tau_{21}$. Then τ is uniquely determined by the values of

$$\int_{K_i'} (\sum_{i,\ j=1}^{2} \tau_{ij} n_i n_j)\ q\ ds\ ;\ q \in P_{k-1},\ 1 \leq i \leq 3$$

$$\int \tau_{ij}\ q\ dx\ ;\ q \in P_{k-2},\ 1 \leq i, j \leq 2$$

where K_i' , i = 1, 2, 3 denotes the sides of K'' was also established in (5).

We refer to (11) for technical details.

6. NUMERICAL RESULTS FOR THE RECIRCULATING FLOW INSIDE A SQUARE CAVITY

Let Ω the unit square]0,1[x]0,1[. We assume that the wall y = 1 is moving with a relative velocity g = −1.

We shall apply the method for k=2. We use regular meshes that allow easy comparison with finite-difference methods.

We approximate λ_h by continuous functions which are pieced together from second degree polynomials defined over the triangles and approximate σ_{ij}^h by linear functions over triangles.

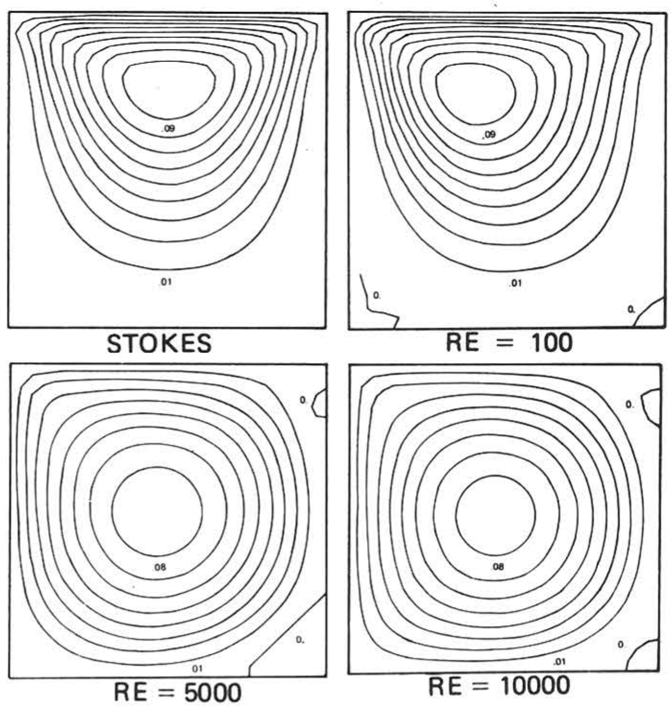


Fig. 2. Streamlines in the square cavity.

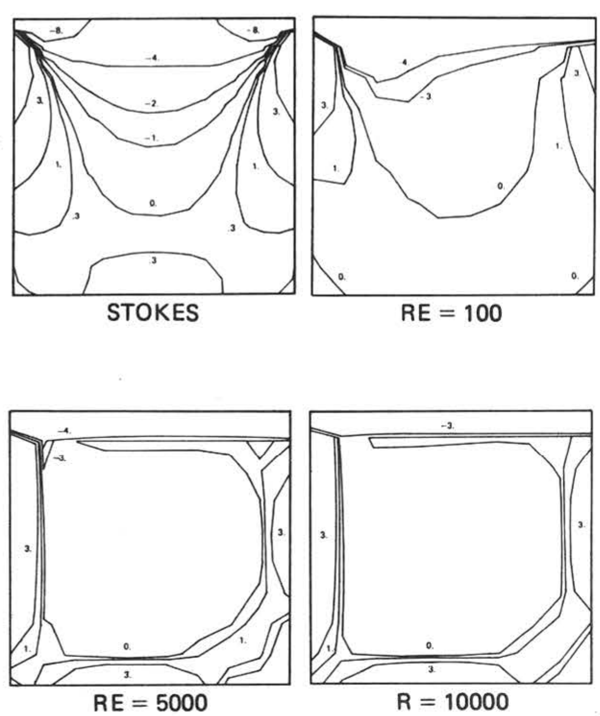


Fig. 3. Lines of constant vorticity in the square cavity.

We have fifteen degrees of freedom, six corresponding to λ_h and nine to σ_h.

We calculate an approximation $(\bar{\lambda}, \bar{\sigma})$ of (λ_h, σ_h). We recall that λ_h is an approximate value of λ the stream function; σ_{ij}^h approximatly equal to $\partial_{ij}^2 \lambda$, $1 \leq i, j \leq 2$. With those we may approximate $\triangle\lambda_h$ the vorticity; $\vec{u} = \vec{curl}\, \lambda = (\partial_2\lambda, -\partial_1\lambda)$ the velocity vector; D_{ij}, $1 \leq i, j \leq 2$ the viscous stress tensor where $D_{11} = 2\,\sigma_{12}^h$, $D_{22} = -2\,\sigma_{12}^h$, $D_{12} = D_{21} = \sigma_{22}^h - \sigma_{11}^h$.

Referent the numerical performance of the method, for a grid of 200 elements, h = 1/10 and 1681 unknowns, the region needed was 650 K, the time 31 seconds, 7 iterations and 1050 output-input. The computations were carried out on a IBM 370/158.

Remarks about the results

Let λ denote the stream function and $\sigma_{ij} = \partial_{ij}^2 \lambda$, $1 \leq i, j \leq 2$. The velocity vector $\vec{u} = (u_1, u_2)$ is equal to $\vec{curl}\, \lambda$. Then $u_1 = \partial_2\lambda$ and $u_2 = -\partial_1\lambda$. For that σ_{22} (respectively $-\sigma_{11}$) may be interpreted as the fluid acceleration in the horizontal direction (vertical), that is σ_{22} $(-\sigma_{11})$ is the variation of the horizontal component (vertical) of the velocity vector. Velocities on the centerlines for Reynolds numbers up 10000 are known (Cf. (14), (12)). For instance, for Re=0 the acceleration is positive on x=.5 for $0 \leq y \leq .5$, negative at all other points. The graph of σ_{22} for Re=0 confirme this result (Fig. 5). Similarly the acceleration is zero on y=.5 at x=.21 and x=.79, positive over

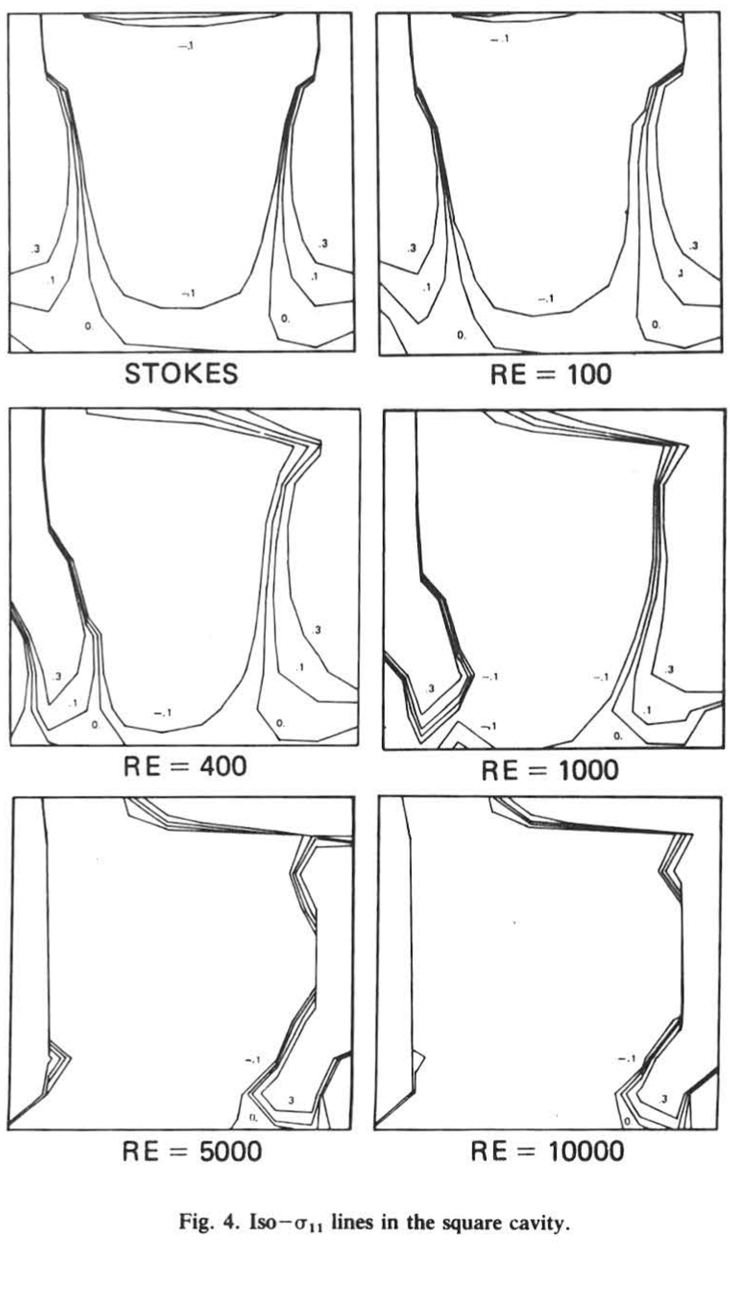


Fig. 4. Iso$-\sigma_{11}$ lines in the square cavity.

(0,.21), (.79,1) negative on the remaining interval and symmetric. The graph of σ_{11} (Fig. 4) for Re=0 confirms this results.

On the other hand, $\partial_1\lambda$ is zero on x=.5 then $\partial_{21}(.5,y)=0$ for all y in (0,1), that is σ_{12} is zero on x=.5 a fact confirmed by the graph (Fig. 6). But the graph is not too exact at the centre of primary wortex for Re=0 (where the velocity is zero) because it is hard to approximate; for Re=400 the non intersection of lines may be interpreted by the viscous nature of the primary vortex.

The viscous stress tensor

The tensor can be stated in terms of the stream function as

$$D_{11} = 2\partial^2_{12}\lambda \ , \ D_{22} = -2\partial^2_{12}\lambda \ , \ D_{12} = \partial^2_{22}\lambda - \partial^2_{11}\lambda$$

Figure 6 shows the iso$-$ σ_{12} lines in the square cavity and Figure 7 shows the iso$-$ $(\sigma_{22}-\sigma_{11})$ lines again in the square cavity.

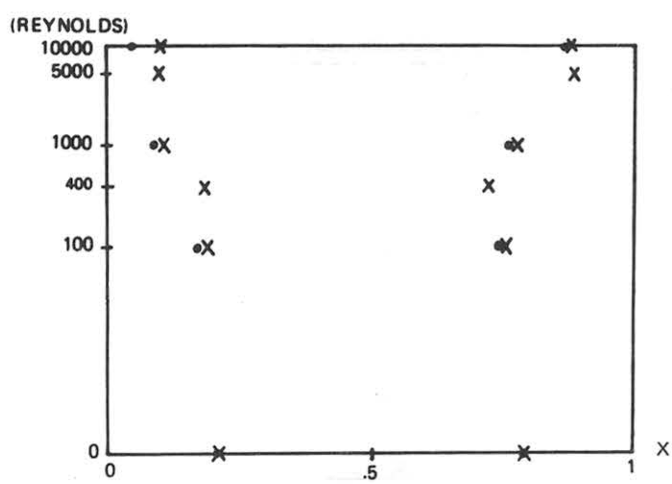
The upper right vortex

Between Re=1000 and Re=2000 (exactly Re=1500 according to (15)) there appears a third secondary vortex near the upper upstream corner. It size increases with the Reynolds numbers. The dimensions are given in Table 1.

Table 1

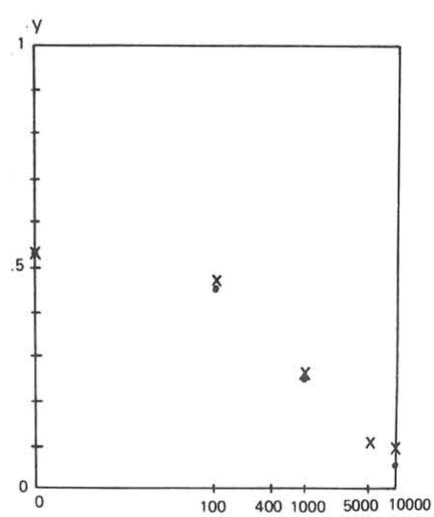
TP CENTRE AND DIMENSIONS OF SECONDARY VORTICES

h	Reynolds Numbers	TP Centre		Vortex LL b_1	Vortex LL d_1	Vortex LR b_2	Vortex LR d_2	Vortex UR b_3	Vortex UR d_3
1/10	0	.50	.75	—	—	—	—	—	—
1/10	100	.40	.75	.14	.18	.10	.10	—	—
1/10	400	.41	.65	.30	.35	.14	.10	—	—
1/10	1000	.43	.57	.27	.29	.23	.13	—	—
1/10	2000	.45	.55	.15	.17	.33	.24	.04	.07
1/10	5000	.46	.53	—	—	.28	.27	.04	.10
1/12	10000	.47	.51	—	—	.10	.11	.06	.13



RE	NALLASAMY		THIS STUDY	
0	.21	.79	.21	.79
100	.17	.75	.19	.76
400	--	--	.19	.73
1000	.09	.76	.11	.78
5000	--	--	.10	.89
10000	.05	.88	.10	.89

TABLE 2 POINTS OF VANISHING VERTICAL ACCELERATION



RE	NALLASAMY	THIS STUDY
0	.53	.53
100	.45	.47
400	--	.35
1000	.25	.26
5000	--	.11
10000	.05	.10

TABLE 3 POINTS OF VANISHING HORIZONTAL ACCELERATION

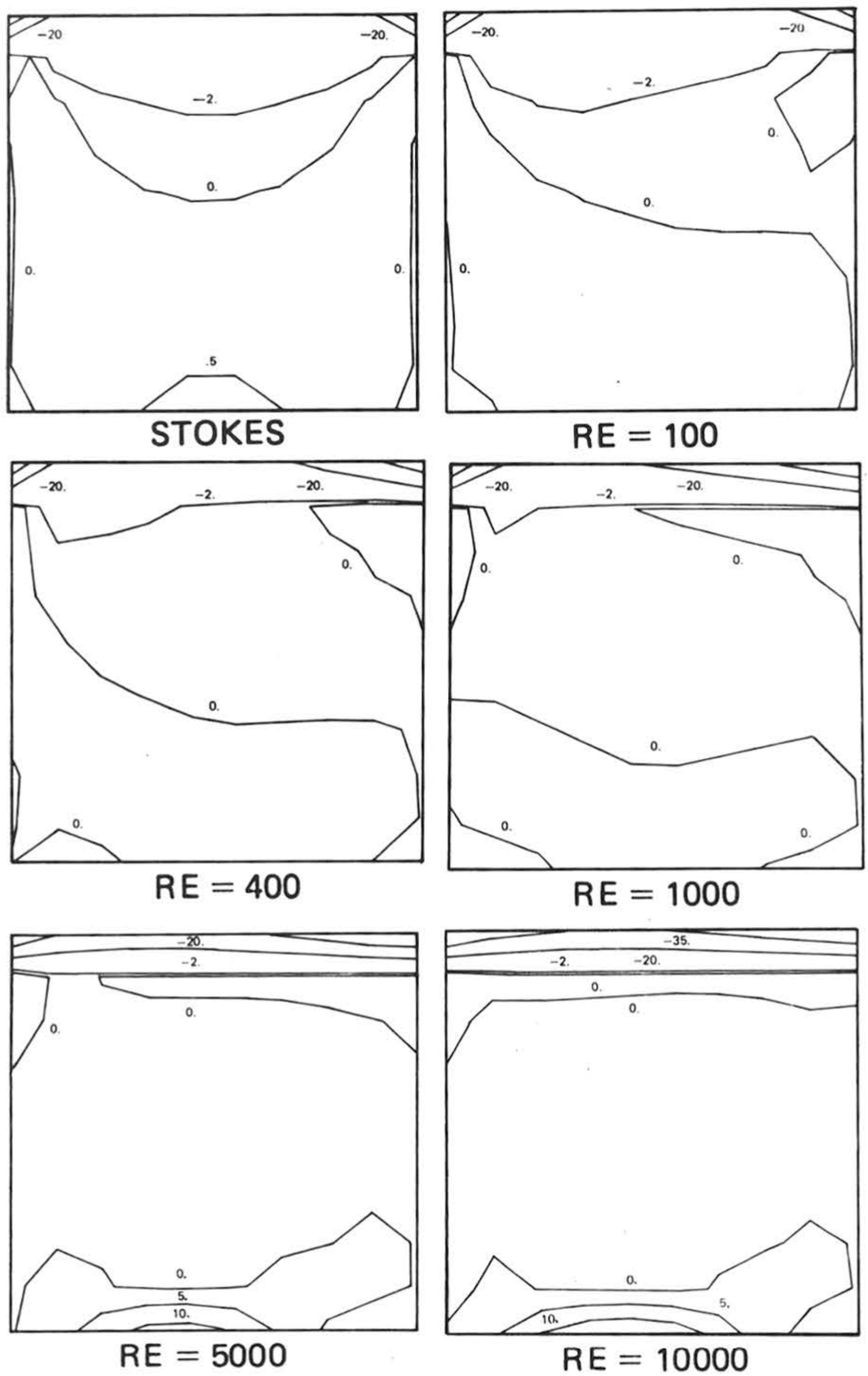


Fig. 5. Iso$-\sigma_{22}$ lines in the square cavity.

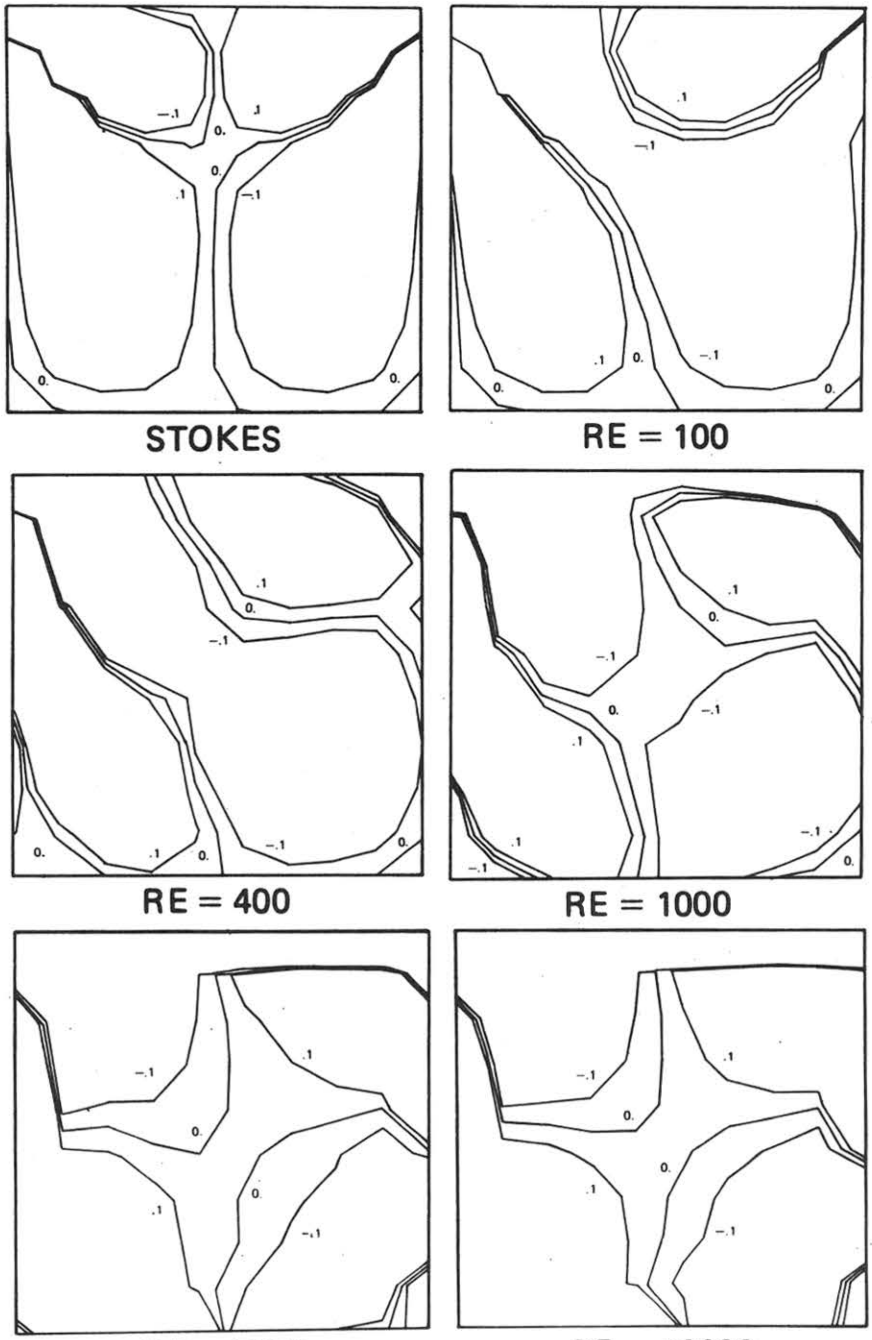


Fig. 6. Iso $-\sigma_{12}$ lines in the square cavity.

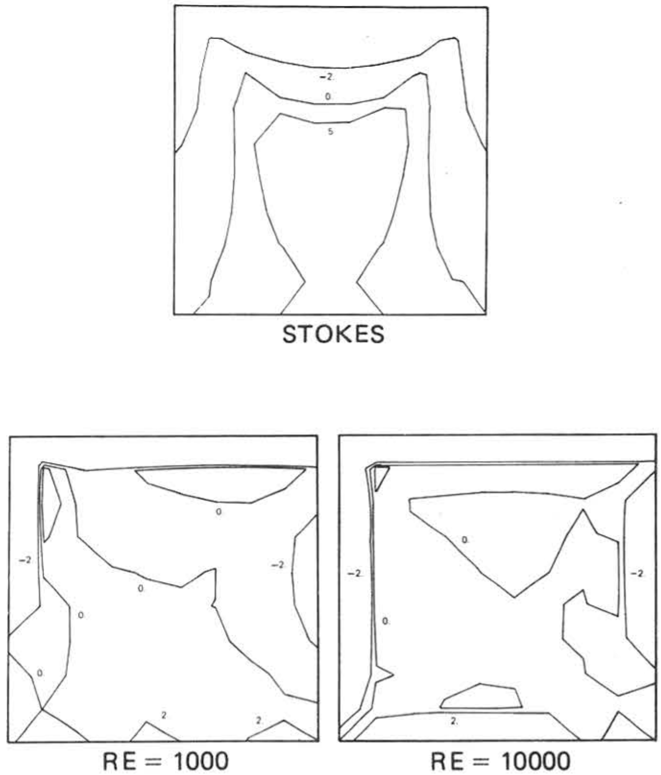


Fig. 7. Iso$-D_{12}$ lines in the square cavity.

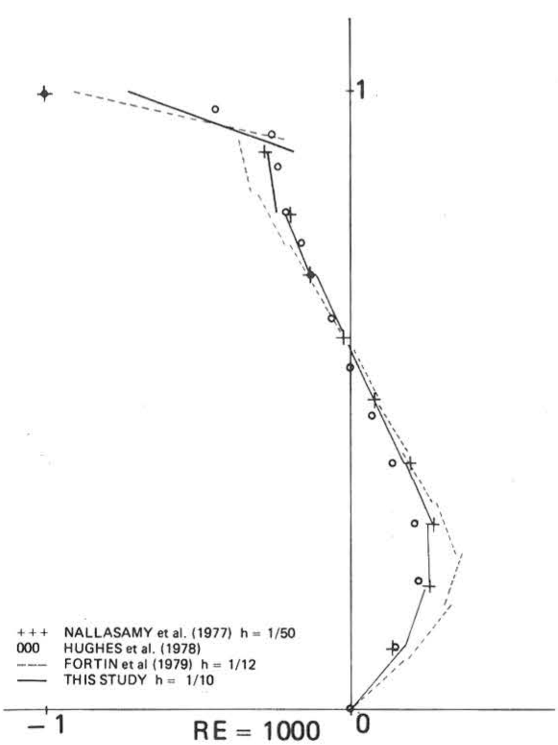


Fig. 8. Velocity profiles on the centreline of the cavity.

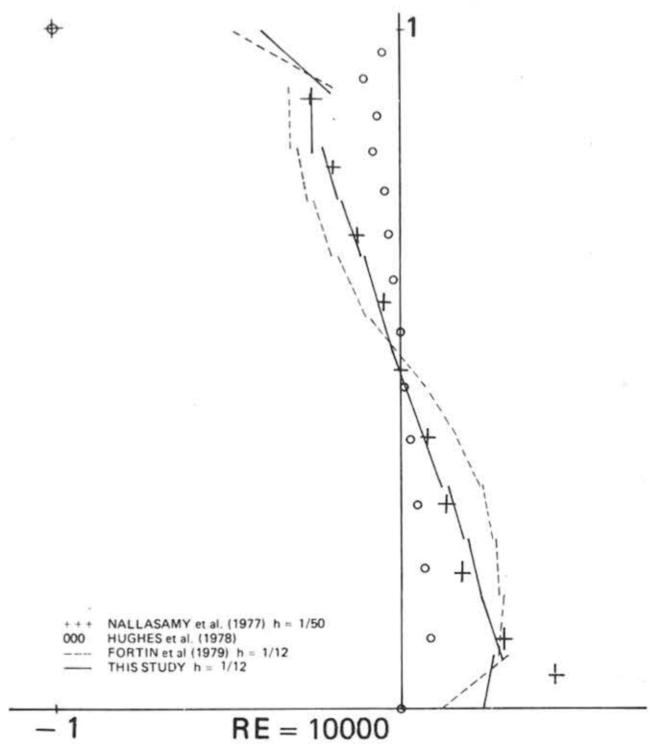


Fig. 9. Velocity profiles on the centreline of the cavity.

7. CONCLUDING REMARKS

The results for the stream function and the vorticity agree with others obtained by different methods. It should be noted that the vorticity is not an unknown.

The utilization of upwind derivatives is effective and the frontal method for the linear system solution is very good.

The results for the viscous stress tensor seems realistic. The method is convenient in time, precision, size of central memory. Four unknowns are calculated directly. Three additional ones may be approximated.

REFERENCES

(1) L. HERMANN. *Finite element bending analysis for plates*, J. Eng. Mech. Div. ASCE. EMS. 93 (1967).

(2) K. HELLAN. *Analysis of elastic plates in fluxure by a simplified finite element method*, Acta Polytechnica Scandinavica, Ci 46, Trondheim (1967).

(3) W. VISSER. *A refined mixed type plate bending element*, A.I.A.A. Journal 7 (1969).

(4) C. JOHNSON. *On the convergence of a mixed finite element method for plate bending problems*, Numer. Math. 21 (1973).

(5) F. BREZZI and P.A. RAVIART. *Mixed finite element methods for 4th order elliptic equations, in: J. Miller (ed.), Topic in Numerical Analysis III (Academic Press, 1977), pp. 33-56.*

(6) J.L. LIONS *Quelques méthodes de résolution des problemes aux limites non linéaires*, (Dunod, Paris, 1968).

(7) V. GIRAULT and P.A. RAVIART. *Finite Element Approximation of the Navier-Stokes Equations*, (Lectures Notes in Mathematics 749, Springer-Verlag, Berlin, 1979).

(8) M. FORTIN. *Résolution numérique des équations de Navier-Stokes par des éléments finis de type mixte*, IRIA-LABORIA Tech. Report 184 (1976).

(9) P. LESAINT. *Sur la resolution des systèmes hyperboliques du premier ordre par des méthodes d'éléments finis*, Thèse, Université de Paris VI, (1975).

(10) C. JOHNSON. *A mixed finite element method for the Navier-Stokes equations*, RAIRO 12 (1978).

(11) J. FIGUEROA. *La méthode d'éléments finis de Herrmann-Johnson por les équations de Navier-Stokes stationnaires*, Thèse de 3ème cycle, Université de Paris VI (1979).

(12) M. FORTIN and F. THOMASSET. *Mixed finite element methods for incompressible flow problems, J. of Computational Physic 31 (1979).*

(13) P. JOLY. *La méthode frontale*, Universite Paris VI Tech. Report 77/008 LA 189 (1977).

(14) M. NALLASAMY and K. KRISHNA PRASAD. *On cavity flow at high Reynolds numbers*, J. Fluid Mechanics 79 N° 2 (1977), pp. 391-414.

(15) M.D. OLSON and SHIH-YU TUANN. *New finite element results for the square cavity*, Computers and Fluids, Vol. 7 (1978), pp. 123-135.