Contribuciones Científicas y Tecnológicas Nº 66 - Área Matemática VI, Mayo 1984

Extensión de espacios duales de Frechet*

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RESUMEN: Se extiende la definición de Grothendieck de DF-espacio al caso de cuerpos valuados no arquimedeanos y se prueban algunas de las principales propiedades. Se caracterizan los espacios topológicos ultrarregulares para los cuales el espacio de funciones continuas con valores en un cuerpo resulta un DF-espacio provisto de la topología compacto abierta.

Finalmente se dan condiciones necesarias y suficientes para que el espacio de funciones con valores vectoriales resulte con DF-espacio.

SUMMARY: Grothendieck's definition of DF-space is extended to the case where the field is nonarchimedean valued and some of the principal properties are proved.

The ultraregular topological spaces for which the space of continuous functions with scalar values results a DF-space with the compact-open topology are characterized.

Finally, necessary and sufficient conditions are given for the space of continuous functions with vector values to be a DF-space.

Throughout this paper (F, | |) denotes a nonarchimedean (n.a.) valued field and we consider only non trivial valuations. We recall that (F, | |) is a spherically complete field if every family of balls pairwise non-disjoint has a non-empty intersection or equivalenty if every totally ordered collection of closed balls has non-empty intersection.

We use Van Tiel's results [5], specially on duality theory, and recall that it needs the Hahn-Banach theorem, or equivalently that the field spherically complete (Ingleton [2]). For other results about Functional Analysis over nonarchimedean valued fields we refer to Prolla [4].

A filter is F-convex if it has a filterbase of translations of F-convex sets. A topological vector space is locally F-convex if the zero-neighborhood filter is F-convex. A locally F-convex space is Frechet n.a. if it is metrizable n.a. and complete; if it is normed n.a. and complete it is a Banach n.a. space.

We consider only Hausdorff locally F-convex spaces usually denoted by (E, τ) or E.

1. N.A. DF-SPACES

1.1. Definition:

A family of bounded set $\{B_i\}_{i\in I}$ in E is fundamental if every bounded subset of E is contained in some B_i.

1.2. Definition:

(E, τ) is a n.a. DF-space if:

i) There exists a countable fundamental family of bounded sets in E.

ii) Every bornivorous set in E which is the intersection of a countable family of F-convex O-neighborhoods is a O-neighborhood.

If (F, | |) is a spherically complete field then we can use ii)' instead of condition ii);

ii)' Every strongly bounded set of E' which is the union of a countable family of equicontinuous sets is equicontinuous.

1.3. Observation:

An absorbent set that is the intersection of a countable family of F-convex zero-neighborhoods is called a *d - F - barrel*. A locally F-convex space where every bornivorous d - F - barrel is a zero-neighborhood is called a d - F evaluable space. Now we can say that E is a *n.a. DF-space* if is a d - F - evaluable space and it verifies condition i). Consequently every Banach n.a. space and more generally, every normed n.a. space is a n.a. DF-space.

1.4. Notation:

In a space of functions, we denote by M(A; S), the subset of all functions f which verify: f(A) $\subseteq$ S.

1.5. Proposition:

Let (F, | |) be a spherically complete field. If (E, τ_E) is a metrizable locally F-convex space then E'_β is a DF-space.

* Manuscrito revisado y aprobado en forma definitiva en marzo de 1984.

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Proof:
Let $\{U_n\}_{n\in\mathbb{N}}$ be a countable fundamental system of zero-neighborhoods in (E, τ_E). We consider $\{U_n^o\}_{n\in\mathbb{N}}$ the family of the polar sets of U_n in E'. As each U_n^o is equicontinuous, $\{U_n^o\}_{n\in\mathbb{N}}$ is a countable family of strongly bounded subsets of E'. Now let L be a strongly bounded set in E'; then L^o is a zero neighborhood in E, hence there is $n \in \mathbb{N}$ such that $U_n \subseteq L$, and therefore $L^o \subseteq U_n^o$; that is $\{U_n^o\}_{n\in\mathbb{N}}$ is a fundamental family of bounded sets in E'_β.

Now, let $V = \bigcap_{n\in\mathbb{N}} V_n$ be a bornivorous set, where each V_n is a F-convex zero-neighborhood in E'_β; then V is a F-convex closed set. By the nonarchimedean bipolar theorem [5], V^{oo} is contained in λV for every $\lambda \in F$ with $|\lambda| > 1$. As U^o is a bounded set in E' if U is a zero neighborhood in E, there exists $\alpha \in F$ such that U^o is contained in αV then, $V^o \subseteq \alpha U^{oo} \subseteq \alpha\gamma U$ where $\gamma \in F$ and $|\gamma| > 1$; so V^o is bounded in (E, τ_E). Finally V^{oo} is a zero neighborhood in E' and consequently V is a zero-neighborhood in E'.

1.6. Proposition:

Let (E, τ_E) be a DF-space and $(F, |\ |)$ a spherically complete field. Then the dual space E'_β is a Frechet space.

Proof:
Let $\{B_n\}$ be a countable fundamental family of bounded sets in E and let $\{U_m\}_{m\in\mathbb{N}}$ be a countable basis of zero neighborhoods in $(F, |\ |)$, then the family $\{M(B_n, U_m)\}_{n,m\in\mathbb{N}}$ is a countable basis of zero-neighborhoods in E'_β.

Now we consider a Cauchy sequence $(f_i)_{i\in\mathbb{N}}$ in E'_β, as E is a DF-space and (f_i) is a strongly bounded set, we have that $(f_i)_{i\in\mathbb{N}}$ is equicontinuous. On the other hand, we define $f(x) = \lim f_i(x)$; then f is contained in the clousure of (f_i) in F^E with the topology of simple convergence. Finally we can see that f is in E'_β.

1.7. Proposition:

Let $(F, |\ |)$ be a spherically complete field and E be a DF-space. If M is a closed subspace of E then E/M is a DF-space.

Proof:
If ν is the quotient map from E to E/M, we define $\psi : M^o \to (E/M)'$ by $\psi(f) = f \circ \nu^{-1}$, ψ is a topological vector space isomorphism of $(M^o, \beta(E'; E))$ onto $((E/M)', \beta((E/M))$. It is clear that ψ is a linear isomorphism. To prove continuity, as M^o is a metrizable space with the induced topology by the strong dual of the DF-space E, it is sufficient to prove that ψ maps null sequences into bounded sets in $(E/M)'$: Let $(f_i)_{i\in\mathbb{N}}$ be a null sequence in M^o; $\{f_i\}$) being a strongly bounded set in E', it is equicontinuous en E'.

Thus $\{\psi(f_i)\} = \{f_i \circ \nu^{-1}\}$ is equicontinuous in $(E/M)'$ hence $\{\psi(f_i)\}$ is bounded in (E/M).

To prove ψ is an open mapping, we consider the 0-neighborhood.

$$U = M^o \cap A^o$$

where A is a bounded set in E. It is easy to show that $(\nu(A))^o$ is a 0-neighborhood in $(E/M)'$ contained in $\psi(U)$. Thus ψ is a topological linear isomorphism of M^o over $(E/M)'$ with the corresponding strong topology. Let L be a bounded set in E/M; as L^o is a 0-neighborhood in $(E/M)'$ there is a bounded set $A \subseteq E$, that we may assume to be Γ-closed by ([5]) such that

$$(*) \qquad \psi(A^o \cap M^o) \subseteq L^o$$

so $\nu^{-1}(L) \subseteq (A^o \cap M^o)^o$, and $L \subseteq \nu(A^{oo} \cup M^{oo})$. But A and M are Γ-closed $L \subseteq \nu(A)$. It follows that there exists a countable fundamental system of bounded sets $\{\nu(B_n)\}$ in E/M, with $\{B_n\}$ a countable fundamental system of bounded sets of E.

Now we consider a sequence of O-neighborhoods $\{V_n\}$ in E/M such that the intersection $V = \cap V_n$ is a bornivorous set in E/M. It is clear that $\nu^{-1}(V)$ is a bornivorous set which is the intersection of a countable family of F-convex O-neighborhoods in the DF-space E; hence V is a O-neighborhood in E/M.

1.8. **Lemma:**

Let E be a topological vector space with a countable fundamental system of O-neighborhoods. If $\{H_n\}_n$ is a sequence of bounded sets, there exists a bounded sequence $\{\mu_n\}$ in $(F, |\ |)$ such that

$$\bigcup_{n\in\mathbb{N}} \mu_n H_n$$

is a bounded set.

Proof:
Let $\{V_n\}$ be a decreasing fundamental sequence of O-neighborhoods in E. For each pair (V_n, H_n), there exists $\delta_n > O$ such that for all $\lambda \in F$, with $|\lambda| \geqslant \delta_n$, $H_n \subseteq \lambda V_n$.

We consider $\delta'_1 = \delta_1$ for (V_1, H_1) and $\delta'_s = \max\{\delta'_{s-1}, \delta_s\}$, $s \geqslant 2$; in this way, we obtain an increasing sequence of real numbers.

If $\lambda_i \in F$ and $|\lambda_i| > \delta'_i$, then the union $\bigcup_{i\in\mathbb{N}} \lambda_i^{-1} H_i$ is a bounded set; indeed, let V be a O-neighborhood in E which we can assume, without loss of generality, that is circled, there is, n_o in $\mathbb{N}$ such that $V_m \subseteq V$ for $m \geqslant n_o$. It is clear that $\lambda_m^{-1} H_m$ is in V for $m > n_o$. Thus the union $\bigcup_{s=n_o}^{\infty} H_s$ is contained in V.

On the other hand, if $A = \bigcup_{s=1}^{n_o-1} \lambda_s^{-1} H_s$ there exists $\delta > 1$ such that for $\alpha \in F$, $|\alpha| > \delta$, $A \subseteq \alpha V$. Hence if $\mu_i = \lambda_i^{-1}$, for the bounded sequence $\{\mu_i\}$ we verify that $\bigcup_{i\in\mathbb{N}} \mu_i H_i$ is a bounded set.

1.9.**Proposition:**

Let E be a DF-space over $(F, |\ |)$ an spherically complete field. If $\{V_n\}_{n\in\mathbb{N}}$ is a sequence of O-neighborhoods of E, there exists a sequence $\{\lambda_n\}$ in F such that $V = \bigcap_{n\in\mathbb{N}} \lambda_n V_n$ is a O-neighborhood in E.

Proof:
Without loss of generality we may suposse that V_n is F-convex for each n in $\mathbb{N}$. We consider H_n the polar of V_n;

$H_n = V_n^o$ is equicontinuous, hence strongly bounded in E'. Since E is a DF-space, E'_β is a metrizable space; by Lemma 1.8 there exists $\{\mu_n\}$ in F such that

$$H = \bigcup_{n \in \mathbb{N}} \mu_n H_n$$

is a bounded set in E'_β. But H is a strongly bounded countable union of equicontinuous sets, hence H is equicontinuous and H^o is a O-neighborhood in E.

On the other hand, if $\alpha_n \in F$ and $|\alpha_n| > 1$ for $n \in \mathbb{N}$, using bipolar properties [5] we obtain

$$H^o \subseteq \cap \mu_n^{-1} \alpha_n V_n$$

let $\lambda_n = \mu_n^{-1} \alpha_n$; then $V = \bigcap_{n \in \mathbb{N}} \lambda_n V_n$ is a O-neighbourhood in E.

2. SPACES OF CONTINUOUS FUNCTIONS

Here we consider only ultraregular topological spaces, i.e. the Hausdorff spaces with a fundamental system of clopen sets at each point.

We call *W-compact* the topological spaces where every countable union of compact sets is relatively compact (those spaces appear in Warner's paper [6]). There are spaces with a weaker condition, the *strongly countable compact* spaces: every countable set is relatively compact.

We also consider the space of continuous functions C(X; E), where X is ultraregular and E is a locally F-convex space. We deal only with the compact open topology, where a fundamental system of zero-neighborhoods is formed by the sets $M(K; U) = \{f \in C(X; E) / f(K) \subseteq U\}$ where $K \subseteq X$ are compacts set and U are O-neighborhoods of E. If $E = \mathbb{F}$ we write C(X) instead of C(X; E).

Let X be an ultraregular space, (E, τ_E) a locally F-convex Hausdorff space and $(F, |\ |)$ a nonarchimedean nontrivially valued field.

2.1. Proposition:

Let $M_1, \ldots M_k$ be F-convex sets in E, U a F-convex O-neighborhood in E and K a compact subset of X; then

$$M(K; U + \sum_{m=1}^{k} M_m) \subseteq M(K; U) + \sum_{m=1}^{k} M(X; M_m)$$

Proof:
If f is in $M(K; U + \sum_{m=1}^{k} M_m)$ we consider the compact set

$$S = \{X \in K / P_U (f(x)) \geq 1\}$$

where P_U is the Minkowski seminorm corresponding to the F-convex set U.
We consider also the clopen set
$V_j = \{X \in X / P_U (F(x) - f(x_j)) < 1/2\}$
where $x_j \in S$. As S is a compact set, there exists $t \in \mathbb{N}$ such that

$$S \subseteq \bigcup_{j=1}^{t} V_j$$

Without loss of generality we may suppose that this is a disjoint union.

On the other hand for each x_j, $j = 1, \ldots, t$, there exists $\sum_{m=1}^{k} h_j^m \in \sum_{m=1}^{k} M_m$ such that

$$f(x_j) + \sum_{m=1}^{k} h_j^m \in U$$

Finally we can show that

$$f = f + \sum_{m=1}^{k} \sum_{j=1}^{k} \chi_{V_j} \otimes h_j^m - \sum_{m=1}^{k} \sum_{j=1}^{t} \chi_{V_j} \otimes h_j^m$$

belongs to

$$M(K; U) + \sum_{m=1}^{k} M(X; M_n)$$

this is easy to see, since:

a) $f + \sum_{n=1}^{k} \sum_{j=1}^{t} \chi_{V_j} \otimes h_j^m =$

$$= f - \sum_{j=1}^{t} \chi_{V_j} \otimes f(x_j) + \sum_{j=1}^{t} \chi_{V_j} \otimes f(x_j)$$

$$+ \sum_{m=1}^{k} \sum_{j=1}^{t} \chi_{V_j} \otimes h_j^m$$

b) $(f - \sum_{j=1}^{t} \chi_{V_j} \otimes f(x_j)) (K) \subseteq U$

c) $(\sum_{j=1}^{t} \chi_{V_j} \otimes f(x_j) + \sum_{m=1}^{k} \sum_{j=1}^{t} \chi_{V_j} \otimes h_j^m) (K) \subseteq U$
and U is F-convex

2.2. Proposition:

Let X be a strongly countably compact topological space, and let $\{B_n\}_{n \in \mathbb{N}}$ be a fundamental sequence of bounded sets in E, then $\{M(X, B_n)\}_{n \in \mathbb{N}}$ is too a fundamental sequence of bounded sets in C(X, E) with the compact open topology.

Proof:
We suppose that there exists a bounded set L in C(X; E) such that $L \cap M(X; B_n)$ is non empty for each natural n. Let f_n in $L \cap M(X; B_n)$ and $x_n \in X$ be such that

$$f_n(x_n) \notin B_n$$

We consider $A = \{x_n / n \in \mathbb{N}\}$, $\bar{A}$ is compact in X.
We claim that $\bigcup_{n \in \mathbb{N}} f_n(\bar{A})$ is bounded. Indeed, let V be a O-neighborhood in E. A_s L is bounded, there exist $\delta > 0$ such that if $\lambda \in F$ and $|\lambda| > \delta$,

$$L \subseteq \lambda \cdot M(\bar{A}, V)$$

hence $\{f_n(a) / n \in \mathbb{N}, a \in \bar{A}\}$ is contained in $\lambda \cdot V$ and therefore $\bigcup_{n \in \mathbb{N}} f_n(\bar{A})$ is bounded.
As $\{B_n\}$ is fundamental, there exists $j \in \mathbb{N}$ such that

$$\bigcup_{n \in \mathbb{N}} f_n(\bar{A}) \subseteq B_j$$

in particular we obtain that $f_j(x_j) \in B_j$ and this contradiction proves our proposition.

2.3. Theorem:

Let X be a W-compact topological space and let $(F, |\ |)$ be a spherically complete field. If E is a DF-space then C(X ; E) is a DF-space.

Proof:
Let $\{V_n\}$ be a sequence of F-convex 0-neighborhoods of $C(X ; E)$. We set $V = \bigcap_{n\in\mathbb{N}} V_n$, we must prove that if V is a bornivorous set then V is a O-neighborhood in $C(X ; E)$. We will build sequences $\{\lambda_i\}$ of scalares in F , $\{U_i\}$ of O-neighborhoods in E and $\{K_i\}$ of compact sets in X with the properties:

1) $M(X ; \lambda_i B_i) \subseteq V$ for all i in $\mathbb{N}$ ·

2) $M(K_i ; U_i) \subseteq V_i$ for all i in $\mathbb{N}$ ·

3) $\lambda_j B_j \subseteq U_i$ for all i, j in $\mathbb{N}$ ·

where $\{B_i\}_{i\in\mathbb{N}}$ is a fundamental sequence of bounded sets in E, that we may suppose F-convex and closed.

If there exist λ_i, U_i , K_i for i= 1, 2,..., n with the properties 1, 2 and 3 then there exists λ_{n+1} in F such that:

a) $\lambda_{n+1} B_{n+1} \subseteq \bigcap_{i=1}^{n} U_i$

b) $M(X ; \lambda_{n+1} B_{n+1}) \subseteq V$,

also there exists a compact subset K_{n+1} of X and a O-neighborhood W_{n+1} on E such that

c) $M(K_{n+1} , W_{n+1}) \subseteq V_{n+1}$

Now we construct the set.

$$U_{n+1} = W_{n+1} + \sum_{i=1}^{n+1} \lambda_i B_i ;$$

this set is a F-convex O-neighborhood that satisfies

$\lambda_j B_j \subseteq U_{n+1}$

for all j = 1, 2, ... , n+1. By a) we obtain

$\lambda_{n+1} B_{n+1} \subseteq U_i$

for all i = 1, ... , n.

Furthermore $\lambda_j B_j \subseteq U_i$ for every i=1, ..., n+1 and j = 1, ... , n+1. Then we have found λ_{n+1}· U_{n+1} , K_{n+1} that verify properties 1) and 3).

We still have to check that $M(K_{n+1} , U_{n+1})$ is contained in V_{n+1}. It is true that

d) $\sum_{i=1}^{n+1} M(X ; \lambda_i B_i) \subseteq V_{n+1}$

since the intersection $\bigcap_{n\in\mathbb{N}} V_i$ is in the F-convex set V_{n+1} and 1) is verified for i = 1, 2, ..., n+1. Using c) and d)

$$M(K_{n+1} ; W_{n+1}) + \sum_{i=1}^{n+1} M(X; \lambda_i B_i) \subseteq V_{n+1} .$$

By proposition it is posible to set

$$M(K_{n+1} ; W_{n+1} + \sum_{i=1}^{n+1} \lambda_i B_i) \subseteq V_{n+1}$$

Hence $M(K_{n+1} , U_{n+1}) \subseteq V_{n+1}$ ·

We call U the intersection of the sequence $\{U_n\}_{n\in\mathbb{N}}$; by 3), U is a O-neighborhood in E. Finally as X is W-compact, there exists a compact set K that contains the union of the sequence $\{K_n\}_{n\in\mathbb{N}}$ · Then

$$M(K;U) \subseteq \bigcap_{n\in\mathbb{N}} M(K; U_n) \subseteq \bigcap_{n\in\mathbb{N}} M(K_n , U_n) \subseteq \bigcap_{n\in\mathbb{N}} V_n = V$$

hence, the second property required to be a DF-space is verified.

The existence of a fundamental sequence of bounded sets is guaranteed by proposition 2.2 and the trivial fact that every W-compact space is a strongly countable compact space.

2.4. Theorem:

Let (F, | |) be a spherically complete field. Then $C(X ; F)$ is a DF-space if and only if X is a W-compact space.

Proof:
As (F, | |) is a DF-space Theorem 2.3 implies that $C(X ; F)$is a DF- space, whenever X is W-compact.

Conversely, if $\{K_n\}_{n\in\mathbb{N}}$ is a sequence of compact subsets of X, we consider the zero neighborhood $M(K_n , B_1)$ (where $B_\delta = : \{\lambda \in F / \mid \lambda \mid < \delta \})$; by proposition 1.8 there exist a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ in F such that

$$\bigcap_{n\in\mathbb{N}} \lambda_n M(K_n , B_1)$$

is a zero neighborhood in $C(X ; F)$.

Therefore there exists a compact subset K of X and $\delta \in \mathbb{R}$ such that

$$M(K; B_\delta) \subseteq \bigcap_{n\in\mathbb{N}} \lambda_n M(K_n ; B_1)$$

We claim that K_n is contained in K for every $n \in \mathbb{N}$.

If not, there exists $n_o \in \mathbb{N}$ such that $K_{n_o} \not\subseteq K$; then there is x in K_{n_o} which is outside K and there is $f \in C(X ; F)$ such that

$f(x) = 1$ and $f(K) = \{0\}$

we consider $\gamma \in F$ with $\mid \gamma \mid > \mid \lambda_{n_o} \mid$ then $\gamma f \in M(K ; B_\delta)$ but $\gamma f \notin \lambda_{n_o} M(K_{n_o} , B_1)$. This contradiction proves our claim.

In 1.6 we proved that the property of being a DF-space is preserved under the formation of quotient topologies. In [3] we showed that $C(X; F)$ and E can be considered as complementary closed subspaces of $C(X ; E)$. Now we can formulate of following:

2.5. Corollary:
Let X be a topological space and (F ; | |) be a spherically complete field. Then $C(X ; F)$ and E are DF-spaces if and only if $C(X ; E)$ is a DF-space.

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