INVARIANT OPERATOR RANGES AND SIMILARITY DOMINANCE IN BANACH AND VON NEUMANN ALGEBRAS

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This thesis dedicated to someone.

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ABSTRACT

Invariant Operator Ranges and Similarity Dominance in Banach and von Neumann Algebras

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Suppose $\mathcal{M}$ is a von Neumann algebra. An **operator range in** $\mathcal{M}$ is the range of an operator in $\mathcal{M}$. When $\mathcal{M} = B(H)$, the algebra of operators on a Hilbert space H, R. Douglas and C. Foiaş proved that if $S, T \in B(H)$, and T is not algebraic, and if S leaves invariant every T-invariant operator range, then $S = f(T)$ for some entire function f.

In the first part of this thesis, we prove versions of this result when $B(H)$ is replaced with a factor von Neumann algebra $\mathcal{M}$ and T is normal. Then using the direct integral theory, we extend our result to an arbitrary von Neumann algebra.

In the second part of the thesis, we investigate the notion of **similarity dominance.** Suppose $\mathcal{A}$ is a unital Banach algebra and $S, T \in \mathcal{A}$. We say that T sim-dominates S provided, for every $R > 0$,

$$\sup\left(\left\{\left\|A^{-1}SA\right\| : A \in \mathcal{A},\ A \text{ invertible},\ \left\|A^{-1}TA\right\| \leq R\right\}\right) < \infty.$$

When $\mathcal{A}$ is the algebra $B(H)$, J. B. Conway and D. Hadwin proved that T sim-dominates S implies $S = \varphi(T)$ for some entire function φ. We prove this for a large class of operators in a type III factor von Neumann algebra.

We also prove, for any unital Banach algebra $\mathcal{A}$, if T sim-dominates S, then S is in the approximate double commutant of T in $\mathcal{A}$.

Moreover, we prove that sim-domination is preserved under approximate similarity.

CHAPTER 1

INTRODUCTION

1.1 Invariant Operator Ranges

Suppose H is a Hilbert space and $B(H)$ is the set of (bounded, linear) operators on H. By an **operator range** we mean a linear subspace in H which is the range of some operator in $B(H)$.

In 1971 P.A. Fillmore and J. P. Williams [1] surveyed a number of foundational results on operator ranges. They began by various characterizations of operator ranges, proved a number of results around the notion of similar and unitarily equivalent operator ranges and discussed the consequences in the original context of similar and equivalent operators.

In 1972 C. Foiaş [2] studied operator ranges invariant under given algebras of operators. In particular, he proved a version of Burnside's theorem: If $\mathcal{S}$ is a strongly (or weakly) closed unital subalgebra of $B(H)$, and $\{0\}, H$ are the only operator ranges invariant under $\mathcal{S}$, then $\mathcal{S} = B(H)$.

In 1979 E. Nordgren, M. Radjabalipour, H. Radjavi and P. Rosenthal [3] considered two general questions regarding operator ranges: (1) Given a lattice of operator ranges, what can be said about the operators leaving them invariant? (2) Given an algebra of operators, what can be said about its lattice of invariant operator ranges? They initiated the study of these problems by considering singly generated lattices and algebras and proved two amazing theorems. Suppose P is any operator in $B(H)$, and let $\mathcal{A}(P)$ be the algebra of operators leaving the range of P invariant. The first result is a structure theorem for the algebra $\mathcal{A}(P)$. It can be written as the sum of a certain algebra of upper triangular matrices and an algebra of lower triangular matrices relative to a decomposition of the space corresponding to certain spectral subspaces of P. Regarding the second question, they proved that every operator has an uncountable set of invariant operator ranges, any pair of which intersect only in $\{0\}$.

In 1976 R. G. Douglas and C. Foiaş [4] proved

Theorem 1.1.1. *Suppose $S, T \in B(H)$, T is not algebraic (i.e., $p(T) \neq 0$ for every nonzero polynomial p), and S leaves invariant every T-invariant vector subspace of H. Then there is a polynomial p such that $S = p(T)$.*

The second Douglas-Foiaş theorem is more surprising;

Theorem 1.1.2 (Douglas-Foiaş)**.** *If $S, T \in B(H)$, T is not algebraic, and S leaves invariant every T-invariant operator range, then there is an entire function $\varphi : \mathbb{C} \to \mathbb{C}$ such that $T = \varphi(S)$.*

If $T \in B(H)$, we let Lat(T) denote the set of all T-invariant (closed linear) subspaces of H. We let $\mathrm{Lat}_0(T)$ denote the set of all T-invariant vector subspaces of H, and we let $\mathrm{Lat}_{1/2}(T)$ denote the set of all T-invariant operator ranges. If $\mathcal{L}$ is a collection of vector subspaces of H, we define Alg$(\mathcal{L})$ to be the set of all operators in $B(H)$ that leave all the elements of $\mathcal{L}$ invariant. Thus the two Douglas-Foiaş theorems say that if $T \in B(H)$ and T is not algebraic, then

1. $\mathrm{AlgLat}_0(T) = \{p(T) : p \text{ is a polynomial}\}$, and

2. $\mathrm{AlgLat}_{1/2}(T) = \{\varphi(T) : \varphi \text{ is an entire function}\}$.

In [5] D. Hadwin gave a nearly linear-algebraic proof of the first Douglas-Foiaş theorem that holds in an arbitrary Banach space. Later, D. Hadwin and S.-C. Ong [6] used a result in [5] to give a generalization of the second Douglas-Foiaş theorem that was also almost purely algebraic. In both of these generalizations the assumption that $S \in B(H)$ was replaced with $S : H \to H$ is linear (although $S \in B(H)$ follows from the conclusions).

From this point onward, we will refer to the second Douglas-Foiaş theorem simply as the Douglas-Foiaş theorem.

In [11] J. B. Conway and D. Hadwin improved the Douglas-Foiaş theorem in terms of ranges of compact operators that intertwine positive multiples of the unilateral shift operator. They also proved a version of the Douglas-Foiaş theorem for type I von Neumann algebras. If $\mathcal{M}$ is a von Neumann algebra and $T \in \mathcal{M}$ we define

$$\mathrm{Lat}_{1/2}\left(T, \mathcal{M}\right)$$

to be the set of all ranges of operators **in** $\mathcal{M}$ that are T-invariant, and we define

$$\mathrm{AlgLat}_{1/2}(T, \mathcal{M}) = \left\{S \in \mathcal{M} : \mathrm{Lat}_{1/2}\left(T, \mathcal{M}\right) \subset \mathrm{Lat}_{1/2}\left(S, \mathcal{M}\right)\right\}.$$

We denote the **center** of $\mathcal{M}$ by $\mathcal{Z}(\mathcal{M})$, i.e., the elements of $\mathcal{M}$ that commute with every element of $\mathcal{M}$. We say that an element T of $\mathcal{M}$ is **algebraic over the center of** $\mathcal{M}$, if there is a positive integer n and elements $c_0, \ldots, c_n \in \mathcal{Z}\left(\mathcal{M}\right)$ with $c_n \neq 0$, such that

$$c_0 + c_1 T + \cdots + c_n T^n = 0,$$

i.e., there is a nonzero polynomial p in $\mathcal{Z}\left(\mathcal{M}\right)[t]$ such that $p\left(T\right) = 0$. If $\mathcal{M}$ is a factor von Neumann algebra (e.g., $\mathcal{M} = B\left(H\right)$), then $\mathcal{Z}\left(\mathcal{M}\right) = \mathbb{C}1$ and , therefore the notions of algebraic over the center and algebraic are identical in this case.

If $\varphi : \mathbb{C} \to \mathcal{Z}\left(\mathcal{M}\right)$ is an entire function, we can write

$$\varphi\left(z\right) = \sum_{n=0}^{\infty} c_n z^n ,$$

with $c_0, c_1, \ldots \in \mathcal{Z}\left(\mathcal{M}\right)$. The radius of convergence R of such a power series is given by

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{\|c_n\|}.$$

If $T \in \mathcal{M}$ we can evaluate φ at T by

$$\varphi\left(T\right) = \sum_{n=0}^{\infty} c_n T^n \in \mathcal{M}.$$

J. B. Conway and D. Hadwin [11] proved the following result.

Theorem 1.1.3. *Suppose $\mathcal{M}$ is a type I von Neumann algebra acting on a separable Hilbert space, $T \in \mathcal{M}$ and T is not algebraic over $\mathcal{Z}(\mathcal{M})$. Then $\mathrm{AlgLat}_{1/2}(T, \mathcal{M})$ is the set of all $\varphi(T)$ with $\varphi : \mathbb{C} \to \mathcal{Z}(\mathcal{M})$ entire.*

In the first part of this thesis we explore $\mathrm{AlgLat}_{1/2}(T, \mathcal{M})$ for von Neumann algebras that are not necessarily type I. We prove a version of Theorem 1.1.2 for a normal operator T in all factor von Neumann algebras. Then, using the direct integral theory, we extend our result to a general von Neumann algebra.

1.2 Similarity Dominance

In [11] Conway and Hadwin introduced the notion of similarity domination. Suppose $\mathcal{A}$ is a unital Banach algebra and $S, T \in \mathcal{A}$. We say that T **sim-dominates** S in $\mathcal{A}$, provided, for every $R > 0$,

$$\sup\left(\left\{\left\|A^{-1}SA\right\| : A \in \mathcal{A},\ A \text{ invertible},\ \left\|A^{-1}TA\right\| \leq R\right\}\right) < \infty.$$

They proved

Theorem 1.2.1. *If $S, T \in B(H)$ and T similarity-dominates S, then there is an entire function $\varphi : \mathbb{C} \to \mathbb{C}$ such that $S = \varphi(T)$.*

Here there is no assumption that T is algebraic. In the second part of this thesis, we explore similarity domination in arbitrary Banach algebras and prove a version of the above theorem for a large class of operators in a type III factor von Neumann algebra.

CHAPTER 2

$\mathrm{ALGLAT}_{1/2}(T, \mathcal{M})$

2.1 Preliminaries

The following theorem of Douglas will be used quite often in later sections.

Theorem 2.1.1 (R. G. Douglas [1][7]). *Suppose $A, B \in B(H)$. The following conditions are equivalent:*

1. $ran(A) \subset ran(B)$
2. $AA^* \leq \lambda^2 BB^*$ *for some constant* $\lambda > 0$.
3. $A = BC$ *for some* $C \in B(H)$

Remark. This theorem holds more generally in a von Neumann algebra. The operator C in part 3 can be chosen in $W^*(A, B)$ such that $\|C\| \leq \lambda$.

Remark. We can state part 3 of the Douglas's theorem differently. Define

$$B^{-1} = (B|_{ker(B)^\perp})^{-1} : ran(B) \to ker(B)^\perp.$$

Then $A = BC$ is the same as saying $B^{-1}A \in B(H)$.

Corollary 2.1.2. *Suppose $\mathcal{M} \subset B(H)$ is a von Neumann algebra and $T, D \in \mathcal{M}$. The following are equivalent*

$$\begin{aligned} T(ran(D)) \subset ran(D) &\Leftrightarrow ran(TD) \subset ran(D) \\ &\Leftrightarrow \exists \lambda > 0, \text{ such that } TDD^*T^* \leq \lambda DD* \\ &\Leftrightarrow \exists C \in \mathcal{M} \text{ such that } TD = DC \\ &\Leftrightarrow \left\| D^{-1}TD \right\| < \infty \end{aligned}$$

Proof. This follows immediately from the remarks and the theorem of Douglas. □

Remark. If X is a Banach space, $T, D \in B(X)$ and $TD = DT$ then

$$T(ran(D)) = ran(TD) = ran(DT) \subset ran(D).$$

2.2 Measurable cross-sections

Suppose (Y, d) is a separable metric space and μ is a σ-finite measure on the sigma-algebra Bor(Y) of Borel subsets of Y. A subset $E \subset Y$ is **μ-measurable** if and only if there are Borel sets $A, B \subset Y$ such that $A \subset E$, $E \backslash A \subset B$ and $\mu(B) = 0$. A subset $E \subset Y$ is **absolutely measurable** if, for every σ-finite measure μ on Bor(Y) it follows that E is μ-measurable. The collection of absolutely measurable sets is a σ-algebra (usually properly) containing Bor(Y). Suppose (W, ρ) is another separable metric space and $f : Y \to W$. We say that f is **absolutely measurable** if and only if, for every absolutely measurable subset $A \subset W$, it follows that$f^{-1}(A)$ is absolutely measurable in Y. It is not hard to show that f is absolutely measurable if and only if, for every $A \in$ Bor(W), $f^{-1}(A)$ is absolutely measurable. In the context of a complete σ-finite measure around, absolute measurability is the same as measurability. The following lemma is elementary.

Lemma 2.2.1. *Suppose (Y, d), (W, ρ), (Z, δ) are separable metric spaces, $B \subset Y$ is absolutely measurable and $f : Y \to W$ and $g : W \to Z$ are absolutely measurable. Suppose (Ω, Σ, μ) is* *a complete (i.e., $E \in \Sigma$, $F \subset E$, $\mu(E) = 0$ implies $F \in \Sigma$). Suppose $\varphi : \Omega \to Y$ is Σ-Bor(Y) measurable. Then*

1. *$\varphi^{-1}(B) \in \Sigma$, and*

2. *$f \circ \varphi : \Omega \to W$ is Σ-Bor(W) measurable.*

3. *$g \circ f : Y \to Z$ is absolutely measurable.*

The most significant theorem on this subject can be found in [20], chapters 3 and 4.

Theorem 2.2.2. *Suppose X is a Borel subset of a complete metric space, (Y, d) is a separable metric space and $f : X \to Y$ is continuous. Then*

1. *$f(X)$ is an absolutely measurable subset of Y, and*

2. *There is an absolutely measurable function $\gamma : f(X) \to X$ such that, for every $y \in f(X)$,*

$$f(\gamma(y)) = y.$$

As an application we prove a modified version of a result [15] by C. Pearcy.

Theorem 2.2.3. *Suppose $n \in \mathbb{N}$ and (Ω, Σ, μ) is a complete σ-finite measure space and $\varphi : \Omega \to \mathbb{M}_n(\mathbb{C})$ is a measurable map such that, for every $\omega \in \Omega$, $\varphi(\omega)$ is a normal matrix. Then there is a measurable map $u : \Omega \to \mathbb{M}_n(\mathbb{C})$ and a measurable function $d : \Omega \to \{1, 2, \ldots, n\}$ such that*

1. *$u(\omega)$ is unitary for every $\omega \in \Omega$,*

2. $u(\omega)^* \varphi(\omega) u(\omega) = \begin{pmatrix} t_1(\omega) & 0 & \cdots & 0 \\ 0 & t_2(\omega) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_n(\omega) \end{pmatrix} = diag(t_1(\omega), \ldots, t_n(\omega))$

3. $Card\left(\left\{t_1(\omega), \ldots, t_{d(\omega)}(\omega)\right\}\right) = d(\omega) = Card(\{t_1(\omega), \ldots, t_n(\omega)\})$.

Proof. Let $\mathcal{N}$ be the set of normal $n \times n$ matrices, and let $\mathcal{U}$ be the set of unitary $n \times n$ matrices. Let

$$V = \mathcal{N} \times \mathcal{U} \times \prod_{k=1}^{n} \mathbb{C} \times \{1, \ldots, n\} \times \prod_{1 \leq i < j \leq n} \mathbb{C} \backslash \{0\} \times \{1, \ldots, n\}^{\{1,\ldots,n\}}$$

with the product topology. Then V is a complete separable metric space (with a different metric on $\mathbb{C} \backslash \{0\}$).

Let X be the set of all $(T, U, (t_1, \ldots, t_n), d, \{c_{ij} : 1 \leq i < j \leq n\}, h)$ in V such that

a. $U^* T U = diag\,(t_1, \ldots, t_n)$

b. If $1 \leq i < j \leq d$, then $t_i - t_j = c_{ij}$,

c. For all $k \in \{1, \ldots, n\}$, $h(k) \leq d$

d. For all $k \in \{1, \ldots, n\}$, $t_k = t_{h(k)}$.

It is easily shown that X is a closed subset of V, so X is a complete separable metric space.

Define $f : X \to \mathcal{N}$ as the projection onto the first coordinate. Since every normal matrix is unitarily equivalent to a diagonal matrix, and since any permutation of the diagonal entries preserves unitary equivalence, we see that $f(X) = \mathcal{N}$. We know from Theorem 2.2.2 there is an absolutely measurable function $\gamma : \mathcal{N} \to X$ such that, for every $T \in \mathcal{N}$, $f(\gamma(T)) = T$. Since $\varphi : \Omega \to \mathcal{N}$ is measurable, $\gamma \circ \varphi$ is measurable. We can write

$$(\gamma \circ \varphi)(\omega) =$$

$$(\varphi(\omega), U(\omega), (t_1(\omega), \ldots, t_n(\omega)), d(\omega), \{c_{ij}(\omega) : 1 \leq i < j \leq n\}, h_\omega).$$

Thus $d : \Omega \to \{1, \ldots, n\}$, $t_k : \Omega \to \mathbb{C}$ $(1 \leq k \leq n)$ are measurable, and from the definition of X, we see that statements (1)-(3) are true. $\square$

2.3 Measurable Families

A family $\{\mathcal{M}_\omega : \omega \in \Omega\}$ is a **measurable family** of von Neumann algebras if, there are sequences of SOT measurable functions f_n and g_n from Ω into the unit ball of $B(H)$ so that $\mathcal{M}_\omega$ is the von Neumann algebra generated by the set $\{f_n(\omega) : n \in \mathbb{N}\}$, $\mathcal{M}_\omega^{'}$ is the von Neumann algebra generated by the set $\{g_n(\omega) : n \in \mathbb{N}\}$, and each of those sets is SOT dense in the unit ball of the von Neumann algebra it generates.

2.4 Direct Integrals

Suppose (Ω, Σ, μ) is a complete finite measure space, and suppose H is a separable Hilbert space. Suppose X is a Banach space and $f : \Omega \to X$ is a function. We define $|f| : \Omega \to [0, \infty)$ by

$$|f(\omega)| = \|f(\omega)\|.$$

We define $L^2(\mu, H) = \{f | f : \Omega \to H \text{ is measurable and } |f| \in L^2(\mu)\}$ and we define

$$L^\infty(\mu, B(H)) = \{\varphi : \Omega \to B(H) \text{ is SOT measurable, } |\varphi| \in L^\infty(\mu)\}.$$

As usual, in both cases, we identify two functions that are equal almost everywhere. note that $L^2(\mu, H)$ is a Hilbert space with $\|f\|_2 = \||f|\|_2$ and inner product

$$\langle f, g\rangle = \int_\Omega \langle f(\omega), g(\omega)\rangle \, d\mu(\omega).$$

If $\varphi \in L^\infty(\mu, B(H))$ we can identify φ with an operator on $L^2(\mu, H)$ that sends f to φf defined by

$$(\varphi f)(\omega) = \varphi(\omega) f(\omega).$$

We define the **direct integral** of the measurable family $\{\mathcal{M}_\omega : \omega \in \Omega\}$ of von Neumann algebras as

$$\int_\Omega^\oplus \mathcal{M}_\omega \, d\mu(\omega) = \{\varphi \in L^\infty(\mu, B(H)) : \varphi(\omega) \in \mathcal{M}_\omega \text{ a.e. } (\mu)\}.$$

Another notation we use for the operator identified with $\varphi \in L^{\infty}(\mu, B(H))$ is

$$\int_{\Omega}^{\oplus} \varphi(\omega)\, d\mu(\omega).$$

We also use the notation, if $T \in \int_{\Omega}^{\oplus} \mathcal{M}_{\omega}\, d\mu(\omega)$ we write

$$T = \int_{\Omega}^{\oplus} T_{\omega}\, d\mu(\omega) = \int_{\Omega}^{\oplus} T(\omega)\, d\mu(\omega)\ .$$

We also sometimes write

$$L^{2}(\mu, H) = \int_{\Omega}^{\oplus} H\, d\mu(\omega)$$

and denote a vector $f \in L^{2}(\mu, H)$ as

$$f = \int_{\Omega}^{\oplus} f(\omega)\, d\mu(\omega).$$

In this notation we have

$$T(f) = \int_{\Omega}^{\oplus} T_{\omega}(f(\omega))\, d\mu(\omega) \in \int_{\Omega}^{\oplus} H\, d\mu(\omega)\ .$$

We also sometimes write

$$L^{\infty}(\mu, B(H)) = \int_{\Omega}^{\oplus} B(H)\, d\mu(\omega).$$

Theorem 2.4.1. *Suppose $\mathcal{M} = \int_{\Omega}^{\oplus} \mathcal{M}_{\omega}\, d\mu(\omega)$ is a direct integral decomposition of a measurable family of von Neumann algebras on a separable Hilbert space H with (Ω, Σ, μ) a complete finite measure space, so $\mathcal{M} \subset B(L^{2}(\mu, H))$. Suppose $T = \int_{\Omega}^{\oplus} T_{\omega}\, d\mu(\omega)$ and $S = \int_{\Omega}^{\oplus} S_{\omega}\, d\mu(\omega)$ are in $\mathcal{M}$. If $S \in \mathrm{AlgLat}_{1/2}(T, \mathcal{M})$, then*

$$S_{\omega} \in \mathrm{AlgLat}_{1/2}(T_{\omega}, \mathcal{M}_{\omega}) \text{ a.e. } (\mu).$$

Proof. Since $\{\mathcal{M}_\omega : \omega \in \Omega\}$ is a measurable family, there is a sequence $\{\psi_1, \psi_2, \ldots\}$ of $*$-SOT measurable functions from Ω into $\mathcal{B}$ such that, for every $\omega \in \Omega$,

$$\{\psi_1(\omega), \psi_2(\omega), \ldots\}^{-*\text{-SOT}} = \{A \in \mathcal{M}'_\omega : \|A\| \le 1\}.$$

Thus an $A = \int_\Omega^\oplus A_\omega \, d\mu$ in $L^\infty(\mu, B(H))$ is in $\mathcal{M}$ if and only if

$$A_\omega \psi_n(\omega) = \psi_n(\omega) A_\omega \text{ a.e. } (\mu)$$

for all $n \in \mathbb{N}$.

Let $\mathcal{B} = \{A \in B(H) : \|A\| \le 1\}$ with the $*$-SOT, and let $\mathcal{B}_o = \{T \in \mathcal{B} : T = T^* \text{ and } T \neq 0\}$. We know, since H is separable, $\mathcal{B}$ is a complete separable metric space with a metric d. Also, since $\mathcal{B}^{sa} = \{T \in \mathcal{B} : T = T^*\}$ is $*$-SOT closed, it is also a complete separable metric space. Since $\mathcal{B}_o$ is relatively open in $\mathcal{B}^{sa}$, we know that $\mathcal{B}_o$ is a complete separable metric space with an equivalent metric d_o.

We then have

$$\mathcal{X} = \mathcal{B} \times \mathcal{B} \times \mathcal{B} \times \prod_{n=1}^{\infty} \mathcal{B} \times \prod_{n=1}^{\infty} \mathcal{B}_o$$

with the product $*$-SOT topology.

For each positive integer m, let $\mathcal{V}_m$ be the set of all $(A, B, D, \{F_n\}, \{G_n\})$ in $\mathcal{X}$ such that

1. $mDD^* - ADD^*A^* \ge 0$
2. $DF_n - F_nD = 0$ for all $n \in \mathbb{N}$,
3. $[nDD^* - BDD^*B^*]^- = G_n$ for all $n \in \mathbb{N}$.

Clearly, $\mathcal{V}_m$ is a closed subset of $\mathcal{X}$, which means that $\mathcal{V}_m$ is a complete separable metric space. Define the continuous maps $\pi : \mathcal{X} \to \mathcal{B} \times \mathcal{B} \times \prod_{n=1}^{\infty} \mathcal{B} = \mathcal{Y}$ by

$$\pi\left(\left(A, B, D, \{F_n\}, \{G_n\}\right)\right) = \left(A, B, \{F_n\}\right)$$

and $\rho : \mathcal{X} \to \mathcal{B}$ by

$$\rho\left(\left(A, B, D, \{F_n\}, \{G_n\}\right)\right) = D.$$

Then, by Theorem 2.2.2, $\pi\left(\mathcal{V}_m\right)$ is an absolutely measurable subset of $\mathcal{Y}$ and there is an absolutely measurable cross-section $\eta_m : \pi\left(\mathcal{V}_m\right) \to \mathcal{V}_m$ with

$$\left(\pi \circ \eta_m\right)(y) = y$$

for every $y \in \pi\left(\mathcal{V}_m\right)$. Also $\rho \circ \eta_m : \pi_m\left(\mathcal{V}_m\right) \to \mathcal{B}$ is absolutely measurable.

It is clear that $\pi\left(\mathcal{V}_m\right)$ is the set of all $\left(A, B, \{F_n\}\right)$ for which there exists $D \in \{F_1, F_2, \ldots\}' \cap \mathcal{B}$ such that $A\left(D\left(H\right)\right) \subset D\left(H\right)$ and $B\left(D\left(H\right)\right) \not\subset D\left(H\right)$.

Clearly, there is no harm in assuming $\|S\|, \|T\| \leq 1$, so that $\|S_\omega\|, \|T_\omega\| \leq 1$ for every $\omega \in \Omega$. Thus the map

$$\left(S, T, \{\psi_n\}\right) : \Omega \to \mathcal{Y}$$

defined by

$$\left(S, T, \{\psi_n\}\right)(\omega) = \left(S_\omega, T_\omega, \{\psi_n(\omega)\}\right)$$

is measurable and

$$\Omega_m = \left(S, T, \{\psi_n\}\right)^{-1}\left(\pi\left(\mathcal{V}_m\right)\right)$$

is the set of all $\omega \in \Omega$ for which there exists $D \in \mathcal{M}_\omega$ such that

$$mDD^* - T_\omega DD^* T_\omega^* \geq 0$$

and S_ω does not leave the range of D_ω invariant.

Define $D_m : \Omega \to \mathcal{B}$ by

$$D_m(\omega) = \begin{cases} 1 & \text{if } \omega \notin \Omega_m \\ \rho_m\left(\left(S, T, \{\psi_n\}\right)(\omega)\right) & \text{if } \omega \in \Omega_m \end{cases}$$

Then

$$mDD^* - TDD^*T^* \geq 0.$$

Thus there exists a positive integer N such that

$$ND_\omega D_\omega^* - S_\omega D_\omega D_\omega^* S^* \geq 0 \text{ a.e. } (\mu).$$

It follows that $\mu(\Omega_m) = 0$ for each $m \in \mathbb{N}$. Since $\cup_{m=1}^{\infty}\Omega_m$ has measure 0 and is the set of all $\omega \in \Omega$ such that there exists $D \in \mathcal{M}_\omega$ whose range is invariant for T_ω but not for S_ω, we see that

$$S_\omega \in \text{AlgLat}_{1/2}(T_\omega, \mathcal{M}_\omega) \text{ a.e. } (\mu).$$

$\square$

Lemma 2.4.2. *Suppose $T = T_1 \oplus T_2 \oplus \cdots$ and $S = S_1 \oplus S_2 \oplus \cdots$ are elements of the von Neumann algebra $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots$ and $S \in \text{AlgLat}_{1/2}(T, \mathcal{M})$. Then*

1. *$S_n \in \text{AlgLat}_{1/2}(T_n, \mathcal{M}_n)$ for each $n \geq 1$, and*

2. *If $T_n \to A$ and $S_n \to B$ in the $*$-SOT, then*

$$AB = BA.$$

Proof. Let $\mathbb{C}_{\mathbb{Q}} = \mathbb{Q} + i\mathbb{Q}$ be the set of complex numbers whose real and imaginary parts are both rational. We can write

$$\mathbb{C}_{\mathbb{Q}} = \{z_1, z_2, \ldots\}.$$

Let $D = \left(e^{z_1 T_1} / \left\|e^{z_1 T_1}\right\|\right) \oplus \left(e^{z_2 T_2} / \left\|e^{z_2 T_2}\right\|\right) \oplus \cdots$. Then $TD = DT$. Thus there exists $W = W_1 \oplus W_2 \oplus \cdots \in \mathcal{M}$ such that $SD = DW$. Thus, for every $n \in \mathbb{N}$,

$$S_n D_n = D_n W_n,$$

so

$$\left\|e^{-z_n T_n} S_n e^{z_n T_n}\right\| = \left\|D_n^{-1} S_n D_n\right\| = \|W_n\| \leq \|W\|.$$

Now suppose $\lambda \in \mathbb{C}$. Then there is a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that

$$\lim_{k \to \infty} z_{n_k} = \lambda.$$

Thus $S_{n_k} \to A$ and $T_{n_k} \to B$ in the $*$-SOT, so $e^{z_{n_k} T_{n_k}} \to e^{\lambda A}$ and $e^{-z_{n_k} T_{n_k}} \to e^{-\lambda A}$ in the $*$-SOT. Hence $e^{-z_{n_k} T_{n_k}} S_{n_k} e^{z_{n_k} T_{n_k}} \to e^{-\lambda A} B e^{\lambda A}$ in the $*$-SOT. Thus

$$\left\|e^{-\lambda A} B e^{\lambda A}\right\| \leq \sup_k \left\|e^{-z_{n_k} T_{n_k}} S_{n_k} e^{z_{n_k} T_{n_k}}\right\| \leq \|W\|.$$

Thus the function $F : \mathbb{C} \to \mathcal{M}$ defined by

$$F(\lambda) = e^{-\lambda A} B e^{\lambda A}$$

is a bounded entire function. Thus, by Liouville's theorem, F is constant. Hence

$$0 = F'(0) = -AB + BA,$$

which implies $AB = BA$. □

Definition 2.4.3 (Hadwin-Hoover[16])**.** Suppose (Ω, Σ, μ) is a measure space, Y is a separable metric space and $\varphi : \Omega \to Y$ is measurable. Then the **essential range** of φ, denoted by ess-ran(φ) is

$$Y \backslash \cup \left\{U \subset Y : U \text{ is open}, \mu\left(\varphi^{-1}(U)\right) = 0\right\}.$$

Lemma 2.4.4. *Suppose (Ω, Σ, μ) is a measure space, Y is a separable metric space and $\varphi : \Omega \to Y$ is measurable. Then*

1. *$\varphi(\omega) \in$ ess-ran(φ) a.e. (μ)*

2. *If $y \in$ ess-ran(φ) and $y \in U$ and $U \subset Y$ is open, then $\mu\left(\varphi^{-1}(U)\right) > 0$.*

Lemma 2.4.5. *Suppose (Ω, Σ, μ) is a measure space with the following property.*

for every $E \in \Sigma$, with $\mu(E) > 0$, and for every $0 < \varepsilon < \mu(E)$, there exists $F \in \Sigma$, $F \subset E$, such that $0 < \mu(E) < \varepsilon$.

Suppose $\{E_n\}_{n=1}^{\infty}$ is a sequence in Σ with $\mu(E_n) > 0$ for every $n \in \mathbb{N}$. Then there exists a ***mutually disjoint*** *sequence $\{F_n\}_{n=1}^{\infty}$ in Σ, such that $F_n \subset E_n$, and $\mu(F_n) > 0$, for all $n \in \mathbb{N}$.*

Proof. Consider the sequence of projections $\{\chi_{E_n}\}$ in $L^\infty(\Omega, \mu)$. By Theorem 2.6.5, there exists an orthogonal sequence of nonzero projections $\{\chi_{F_n}\}$, with $\chi_{F_n} \leq \chi_{E_n}$ for all $n \in \mathbb{N}$. Since $\{\chi_{F_n}\}$ is an orthogonal family, it follows that $F_n \cap F_m = \varnothing$ for all $m, n \in \mathbb{N}$. Since $\chi_{F_n} \neq 0$, it follows that $\mu(F_n) > 0$ for all $n \in \mathbb{N}$. □

Theorem 2.4.6. *Suppose $\mathcal{M} = \int_\Omega^\oplus \mathcal{M}_\omega \, d\mu(\omega)$ is a direct integral decomposition of a measurable family of von Neumann algebras on a separable Hilbert space H with (Ω, Σ, μ) a complete finite measure space, so $\mathcal{M} \subset B\left(L^2(\mu, H)\right)$. Suppose $T = \int_\Omega^\oplus T_\omega \, d\mu(\omega)$ and $S = \int_\Omega^\oplus S_\omega \, d\mu(\omega)$ are in $\mathcal{M}$. If $S \in \mathrm{AlgLat}_{1/2}(T, \mathcal{M})$ and μ is nonatomic, then*

$$ST = TS.$$

Proof. We can assume that $\|T\| \leq 1$ and $\|S\| \leq 1$ and we can therefore assume (by redefining on a set of measure 0) $\|T_\omega\| \leq 1$ and $\|S_\omega\| \leq 1$ for every $\omega \in \Omega$. Similarly, by Theorem 2.4.1, we can assume that $S_\omega \in \mathrm{AlgLat}_{1/2}(T_\omega, \mathcal{M}_\omega)$ for every $\omega \in \Omega$. Let $\mathcal{B} = \{A \in B(H) : \|A\| \leq 1\}$ and let d be a metric on $\mathcal{B}$ that gives the $*$-SOT and makes $\mathcal{B}$ a complete separable metric space. Such a metric d exists because H is separable. Define $\varphi : \Omega \to \mathcal{B} \times \mathcal{B}$ by

$$\varphi(\omega) = (T_\omega, S_\omega).$$

Suppose $(A, B) \in \text{ess-ran}(\varphi)$. For each positive integer n, let

$$U_n = \{(C, D) \in \mathcal{B} \times \mathcal{B} : d(A, C) + d(B, D) < 1/n\}.$$

Then U_n is open in $\mathcal{B} \times \mathcal{B}$ and $(A, B) \in U$. Thus $\mu(\varphi^{-1}(U_n)) > 0$. We know from lemma 2.4.5 that we can find mutually disjoint subsets $E_n \subset \varphi^{-1}(U_n)$, such that for all $n \in \mathbb{N}, \mu(E_n) > 0$. Define

$$U(\omega) = \begin{cases} 1 & \text{if } \omega \notin \bigcup_{n=1}^{\infty} E_n \\ \frac{e^{z_n T_\omega}}{\|e^{z_n T_\omega}\|} & \text{if } \omega \in E_n \end{cases}.$$

Then $\|U\| \leq 1$ and for every $\omega \in \Omega$,

$$T(\omega)U(\omega) = U(\omega)T(\omega).$$

Thus there exists a bounded operator $C = \int_\Omega^\oplus C_\omega \, d\mu(\omega) \in \mathcal{M}$ such that $SU = UC$. Since $S_\omega \in \mathrm{AlgLat}_{1/2}(T_\omega, \mathcal{M}_\omega)$, we have for every $\omega \in \Omega$

$$S(\omega)U(\omega) = U(\omega)C(\omega).$$

Thus

$$\left\| e^{-z_n T(\omega)} S(\omega) e^{z_n T(\omega)} \right\| \leq \|C\|.$$

Suppose $z \in \mathbb{C}$. There exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$such that

$$\lim_{k \to \infty} z_{n_k} = z.$$

Choose $\omega_{n_k} \in E_{n_k}$. It follows from the definition of E_{n_k} that, $S(\omega_{n_k}) \to B$, and $T(\omega_{n_k}) \to A$ in the *-SOT. Hence $e^{-z_{n_k} T(\omega_{n_k})} S(\omega_{n_k}) e^{z_{n_k} T(\omega_{n_k})} \to e^{-zA} B e^{zA}$ in the *-SOT. Thus

$$\left\| e^{-zA} B e^{zA} \right\| \leq \sup_{k \in \mathbb{N}} \left\| e^{-z_{n_k} T(\omega_{n_k})} S(\omega_{n_k}) e^{z_{n_k} T(\omega_{n_k})} \right\| \leq \|C\| .$$

Proceeding as in lemma 2.4.2, we see that $AB = BA$ for every $A, B \in$ ess-ran(φ). However,

$$\varphi(\omega) = (T_\omega, S_\omega) \in \text{ess-ran}(\varphi),\ a.e.(\mu)$$

Thus $ST = TS$. □

2.5 The Central Decomposition

Suppose $1 \leq n \leq \infty = \aleph_0$. We define ℓ_n^2 be the space of square summable sequences with the inner product $\langle x, y \rangle = \sum_{i=1}^{n} x_i \bar{y}_i$. Here is the statement of the Central Decomposition Theorem [19].

Theorem 2.5.1. *Suppose $\mathcal{M}$ is a von Neumann algebra on a separable Hilbert space H. Then there is a family $(\Omega_n, \Sigma_n, \mu_n)$ of finite measure spaces and measurable families $\{\mathcal{M}_{n,\omega} : \omega \in \Omega_n\}$ of von Neumann algebras on $B\left(\ell_n^2\right)$ ($1 \leq n \leq \infty$) such that*

1. $\mathcal{M} = \sum_{1 \leq n \leq \infty}^{\oplus} \int_{\Omega_n}^{\oplus} \mathcal{M}_{n,\omega} \, d\mu_n(\omega)$
2. $\mathcal{M}_{n,\omega}$ *is a factor von Neumann algebra for every* n *and* ω
3. $\mathcal{Z}(\mathcal{M}) = \sum_{1 \leq n \leq \infty}^{\oplus} \int_{\Omega_n}^{\oplus} \mathbb{C} \cdot 1 \, d\mu_n(\omega)$, *which is isomorphic to* $\sum_{1 \leq n \leq \infty}^{\oplus} L^\infty(\mu_n)$.

This is called the **central decomposition** of $\mathcal{M}$.

We can prove the following.

Theorem 2.5.2. *Suppose $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra with $\mathcal{H}$ separable such that the center $\mathcal{Z}(\mathcal{M})$ has no minimal projections. If $S, T \in \mathcal{M}$ and $S \in \mathrm{AlgLat}_{1/2}(T, \mathcal{M})$, then $ST = TS$.*

Proof. Relative to the above central decomposition of $\mathcal{M}$, the fact that $\mathcal{Z}(\mathcal{M})$ has no minimal projections says that each μ_n is nonatomic. We can write

$$T = \sum_{1 \le n \le \infty}^{\oplus} T_n = \sum_{1 \le n \le \infty}^{\oplus} \int_{\Omega_n}^{\oplus} T_n(\omega)\ d\mu_n(\omega)$$

and

$$S = \sum_{1 \le n \le \infty}^{\oplus} S_n = \sum_{1 \le n \le \infty}^{\oplus} \int_{\Omega_n}^{\oplus} S_n(\omega)\ d\mu_n(\omega).$$

It follows from Lemma 2.4.2, that, for each n

$$S_n \in \text{AlgLat}_{1/2}\left(T_n, \int_{\Omega_n}^{\oplus} \mathcal{M}_{n,\omega}\, d\mu_n(\omega)\right).$$

It follows from Theorem 2.4.6 that, for every n, $S_nT_n = T_nS_n$. Thus $ST = TS$. □

2.6 Normal Operators in a Factor

This first result holds for an arbitrary von Neumann algebra.

Theorem 2.6.1. *Suppose $\mathcal{M} \subset B(H)$ is a von Neumann algebra. Suppose $S, T \in \mathcal{M}$ and $S \in \mathrm{AlgLat}_{1/2}(T, \mathcal{M})$, and T is normal. Then S is normal and $ST = TS$.*

Proof. Let $\mathcal{A}$ be a masa in $\mathcal{M}$ containing T. Suppose $P \in \mathcal{A}$ is a projection. Then $PT = TP$. Hence $ran(P)$ and $ran(1-p)$ are T-invariant, and hence S-invariant. Thus $SP = PS$ for every projection in $\mathcal{A}$. Suppose $W \in \mathcal{A}$. Since $\mathcal{A}$ is weakly closed, $\mathcal{A}$ is a von Neumann algebra and hence contains all spectral projections of W. Thus $SW = WS$ for every $W \in \mathcal{A}$. But $\mathcal{A}$ is a masa, so $S, S^* \in \mathcal{A}$. Thus $ST = TS$ and $SS^* = S^*S$. □

Corollary 2.6.2. *Assume the hypotheses of Theorem 2.6.1. If $E \subset \sigma(T)$ is a Borel set, the spectral projection $\chi_E(T)$ commutes with S.*

Proof. We know from Theorem 2.6.1 that S commutes with T. Since T is normal, S commutes with all spectral projections of T by Fuglede's Theorem. □

The main theorem in this section is the following. This is a far cry from the Douglas-Foiaş Theorem (1.1.2), in which the function is entire.

Theorem 2.6.3. *Suppose $\mathcal{M} \subset B(H)$ is a factor von Neumann algebra on a separable Hilbert $S, T \in \mathcal{M}$ and $S \in \mathrm{AlgLat}_{1/2}(T, \mathcal{M})$, and T is normal. Then there is a continuous function φ on $\sigma(T)$ such that*

$$S = \varphi(T).$$

The proof will be done in a series of lemmas. If we first consider a type I_n factor with $1 \leq n < \infty$, then $\mathcal{M}$ is isomorphic to $\mathbb{M}_n(\mathbb{C}) = B(\mathbb{C}^n)$ and the result is well-known. If $\mathcal{M}$ is a type I_∞ factor, then $\mathcal{M} = B(\ell^2)$ and an even stronger result follows theorem 1.1.2.

The remaining types of factors are type II_1, II_∞ and III.

If $\mathcal{M}$ is a type II_1 factor, then $\mathcal{M}$ has a faithful normal tracial state τ and two projections p and q in $\mathcal{M}$ are unitarily equivalent in $\mathcal{M}$ if and only if $\tau(p) = \tau(q)$. Moreover, if $\mathcal{C}$ is a maximal chain of projections in $\mathcal{M}$, we can write

$$\mathcal{C} = \{p_t : 0 \leq t \leq 1\}$$

with $\tau(p_t) = t$ for every $t \in [0,1]$. Thus if $0 < s < \tau(p) \leq 1$, there is a projection $q \in \mathcal{M}$ such that $q \leq p$ and $\tau(q) = s$.

If $\mathcal{M}$ is a type II_∞ factor, there is a type II_1 factor $\mathcal{R}$ such that

$$\mathcal{M} = \left\{ A = (A_{ij}) \in B(H) : A_{ij} \in \mathcal{R}, \text{for } 1 \leq i, j < \infty \right\}.$$

We can define a faithful normal weight ρ with domain the set of positive elements in $\mathcal{M}$ by

$$\rho(A) = \sum_{i=1}^{\infty} \tau(A_{i,i}) \in [0, \infty].$$

It is known that two projections $p, q \in \mathcal{M}$ are Murray von Neumann equivalent if and only if $\rho(p) = \rho(q)$. They are unitarily equivalent in $\mathcal{M}$ if and only if we also have $\rho(1-p) = \rho(1-q)$. Also, if $p \neq 0$ and $0 \leq s < \rho(p)$, then there is a projection $q \leq p$ such that $\rho(q) = s$.

In a type III factor $\mathcal{M}$ all nonzero projections are Murray von Neumann equivalent and if p, q are projections with $0 \neq p \neq 1$ and $0 \neq q \neq 1$, then p and q are unitarily equivalent in $\mathcal{M}$. Also if φ is a state on $\mathcal{M}$ and $0 < s < \varphi(p)$, then there is a projection $q < p$ such that $\varphi(q) = s$.

One property of an arbitrary von Neumann algebra $\mathcal{M}$ is the following. If $\{p_i : i \in I\}$ and $\{q_i : i \in I\}$ are orthogonal families of projections whose sum is 1, and if each p_i is Murray von Neumann equivalent to q_i, then there is a unitary operator $U \in \mathcal{M}$ such that, for every $i \in I$,

$$U^* p_i U = q_i.$$

Lemma 2.6.4. *Suppose $\mathcal{M} \subset B(H)$ is a von Neumann algebra with no minimal projections and a faithful normal state φ. Suppose P_1 and P_2 are nonzero projections in $\mathcal{M}$ and $0 < \varepsilon < \varphi(P_1)$. Then there are mutually orthogonal nonzero subprojections $Q_1 \leq P_1$ and $Q_2 \leq P_2$ such that $\varphi(P_1 - Q_1) < \varepsilon$.*

Proof. According to Halmos' standard form [9], we can write

$$H = H_1 \oplus H_2 \oplus H_3 \oplus H_4 \oplus H_5 \oplus H_6 \text{ (with } H_5 = H_6)$$

so that

$$P_1 = 1 \oplus 1 \oplus 0 \oplus 0 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$P_2 = 1 \oplus 0 \oplus 1 \oplus 0 \oplus \begin{pmatrix} x & \sqrt{x - x^2} \\ \sqrt{x - x^2} & 1 - x \end{pmatrix}$$

with $x \in P_{H_5} \mathcal{M} P_{H_5}$ and $0 < x < 1$.

For each $k \in \{1, 2, 3, 4\}$ we can choose a masa $\mathcal{D}_k \subset P_{H_k} \mathcal{M} P_{H_k}$ and we can choose a masa $D_5 \subset P_5 \mathcal{M} P_5$ that contains x. Thus $W^* (P_1, P_2)' \cap \mathcal{M}$ contains

$$\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{D}_3 \oplus \mathcal{D}_4 \oplus \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : A \in \mathcal{D}_5 \right\}.$$

Clearly $\mathcal{D}$ has no minimal projections, so we can choose a projection $E \in \mathcal{D}$ with $E \leq P_{H_1} + P_{H_3} + P_{H_5} + P_{H_6}$ such that $0 < \varphi(E) < \varepsilon$. If we let $Q_1 = P_1 (1 - E)$ and $Q_2 = EP_2$, the proof is complete. □

Theorem 2.6.5. *Suppose $\mathcal{M} \subset B(H)$ is a von Neumann algebra with no minimal projections and a faithful normal state φ. Suppose $P_1, P_2, \ldots$ are nonzero projections in $\mathcal{M}$. Then there are nonzero subprojections $Q_n \leq P_n$ so that $\{Q_1, Q_2, \ldots\}$ is orthogonal. Moreover, if $\mathcal{M}$ is a factor, we can also have that Q_{2n-1} and Q_{2n} are Murray von Neumann equivalent for all $n \in \mathbb{N}$.*

Proof. We can use mathematical induction (constructing $P_{1,n}, \ldots, P_{n,n}$ at the nth stage) and Lemma 2.6.4 to construct projections

$$\{P_{n,k} : n \leq k \leq \infty\}$$

such that

1. $P_{n,n} \leq P_{n,n+1} \leq \cdots \leq P_n$ for all $n \in \mathbb{N}$,

2. $\varphi(P_{n,k}) < \left(\frac{1}{3}\right)^k \varphi(P_n)$ for $1 \leq n \leq k < \infty$,

3. $P_{n,n} \perp (P_m - P_{m,m})$ for $1 \leq m < n < \infty$.

We then let $Q_n = P_{n,n} - \sum_{k=n+1}^{\infty} P_{n,k}$.

If $\mathcal{M}$ is a factor, then one of Q_{2n-1} and Q_{2n} is Murray von Neumann equivalent to a subprojection of the other for each $n \in \mathbb{N}$. □

Suppose T is an operator on a Hilbert space H. We let $\mathfrak{R}(T)$ denote the projection onto the range of T. Then

$$\mathfrak{R}(T) = \lim_{n\to\infty} (TT^*)^{1/n},$$

where the convergence is in the strong operator topology.

Note that if $\mathcal{M}$ is a von Neumann algebra with a faithful normal state φ, and if P is a nonzero projection and $0 < t < \varphi(P)$, there is a projection $P_1 \leq P$ in $\mathcal{M}$ such that $\varphi(P) = t$.

Theorem 2.6.6. *Suppose $\mathcal{M}$ is a von Neumann algebra with no minimal projections and a faithful normal state φ. Suppose $P_1, P_1', P_2, P_2', \ldots$ are nonzero projections in $\mathcal{M}$. Then $\mathcal{M}$ contains nonzero projections $Q_1, Q_1', Q_2, Q_2', \ldots$ such that*

1. *$\{Q_1, Q_1', Q_2, Q_2', \ldots\}$ is orthogonal*

2. *For every $n \in \mathbb{N}$, $Q_n \leq P_n$ and $Q_n' \leq P_n'$.*

3. *If $\mathcal{M}$ is a factor, then, for every $n \in \mathbb{N}$, P_n and P_n' are Murray von Neumann equivalent.*

Proof. First suppose P_1 and P_2 are projections. We can write $P_1P_2 = (P_1P_2P_1)^{1/2} V$ as a polar decomposition where the partial isometry has an initial space V^*V is the projection onto $[\ker(P_1P_2)]^{\perp}$, so $V^*V \leq P_2$ and $VV^* = \mathfrak{R}(P_1P_2) \leq P_1$. Thus $V : \mathfrak{R}(VV^*) \to \mathfrak{R}(P_1P_2), V = P_1(P_2(H))^{-}$ is unitary. If $P_1P_2 = 0$, then $P_1(H) \perp P_2(H)$.

More generally, suppose $P_1P_2 \neq 0$ and $\varepsilon > 0$. Then $\varphi(VV^*) > 0$. We can choose a projection $E \leq VV^*$ in $\mathcal{M}$ so that $0 < \varphi(E) < \varepsilon$. Then $V^*(E(H))$ is a closed subspace of $V^*(H)$. Thus $F = V^*EV$ is a projection and $F = V^*EV \leq V^*V \leq P_2$. Now we can find a subprojection F_2 of F such that $0 < \varphi(F_2) < \varepsilon$. Let $E_2 = \mathfrak{R}(P_1F_2) = VF_2V^*$. Then

$F_2 \leq P_2$ and $F_2 \perp P_1 - E_1$. □

From this point onward, we assume $\mathcal{M}$ is a factor of type II_1, II_∞ or III and $T \in \mathcal{M}$ is normal and $S \in$AlgLat$_{1/2}(T, \mathcal{M})$. Therefore S is normal and S commutes with every spectral projection of T.

Definition 2.6.7. Suppose $E \subset \sigma(T)$ is a Borel set. We define S_E to be the restriction of S to the range of the spectral projection $\chi_E(T) \in \mathcal{M}$.

Remark. It follows from the Corollary 2.6.2 that $S_E \in B(\chi_E(T)(H))$. We see that S_E is normal because $S = S_E \oplus S_{sp(T)\setminus E}$.

Lemma 2.6.8. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $E \subset sp(T)$ is a Borel set,*

$$diam(E) < \delta \implies diam(sp(S_E)) < \varepsilon.$$

Proof. By way of contradiction there exists a sequence $\{E_n\}$ of Borel subsets of $\sigma(T)$ and an $\varepsilon > 0$ such that, for every $n \in \mathbb{N}$,

1. diam$(E_n) < 1/2^n$, and

2. diam$(\sigma(S_{E_n})) \geq \varepsilon$.

For each $n \in \mathbb{N}$ we can choose $z_n \in \sigma\left(T|_{\chi_{E_n}(T)(H)}\right)$. Thus

$$\|(T - z_n)\chi_{E_n}\| \leq \text{diam}(E_n) \leq \frac{1}{2^n}.$$

We can choose $\alpha_n, \beta_n \in \sigma(S_{E_n}) \subset \sigma(S)$ such that, for each $n \in \mathbb{N}$,

$$|\alpha_n - \beta_n| \geq \varepsilon.$$

Let $r_n = 1/2^n$ for each $n \in \mathbb{N}$. Thus $\chi_{D(\alpha_n, r_n)}(S_{E_n})$ and $\chi_{D(\beta_n, r_n)}(S_{E_n})$ are nonzero subprojections of $\chi_{E_n}(T)$. It follows from Theorem 2.6.6 and the fact that $\mathcal{M}$ is a factor of type II_1, II_∞ or

III, that there is an orthogonal family $\{P_{n_k} : n \in \mathbb{N}, k \in \{1,2\}\}$ of nonzero projections such that $P_{n_1} \leq \chi_{D(\alpha_n, r_n)}(S_{E_n})$ and $P_{n_2} \leq \chi_{D(\beta_n, r_n)}(S_{E_n})$ for all $n \in \mathbb{N}$ and $k = 1, 2$. Since $\mathcal{M}$ is a factor, one of P_{n_1} and P_{n_2} is Murray-von Neumann equivalent to a subprojection of the other. Hence we can assume that P_{n_1} and P_{n_2} are Murray-von Neumann equivalent. Thus there is a partial isometry $V_n \in \mathcal{M}$ such that $V_n^* V_n = P_{n_1}$ and $V_n V_n^* = P_{n_2}$. Since the map π_n such that

$$\pi_n\left(\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\right) = aP_{n_1} + bV_n + cV_n^* + dP_{n_2}$$

is a $*$-homomorphism on $\mathbb{M}_2(\mathbb{C})$, we see that

$$\|aP_{n_1} + bV_n + cV_n^* + dP_{n_2}\| = \left\|\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\right\|.$$

For each positive integer n, we define $A_n = \frac{1}{n}I_2 + \frac{1}{2}\left(\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right)$ and let $D_n = \pi(A_n)$. Clearly

$$\|D_n\| = \|A_n\| = 1 + \frac{1}{n} \leq 2.$$

If we view D_n acting on $(P_{n_1} + P_{n_2})(H) = H_n$ we can view D_n as being invertible and $D_n^{-1} = \pi_n(A_n^{-1})$. However, we have $D_n^{-1} D_n = \pi_n(1) = P_{n_1} + P_{n_2}$. We will use the notation D_n^{-1} for $\pi(A_n^{-1})$ even though it is not the inverse in $\mathcal{M}$ of D_n.

Since $\frac{1}{2}\left(\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right)$ is a rank-one projection, we have $\|A_n^{-1}\| = n$ for $n \in \mathbb{N}$.

A simple computation shows that, for each $n \in \mathbb{N}$,

$$\left\|D_n^{-1}(\alpha_n P_{n_1} + \beta_n P_{n_2}) D_n\right\| = \left\|A_n^{-1}\left(\begin{array}{cc} \alpha_n & 0 \\ 0 & \beta_n \end{array}\right) A_n\right\| \geq n|\alpha_n - \beta_n| \geq n\varepsilon.$$

For each $n \in \mathbb{N}$, let $Q_n = P_{n_1} + P_{n_2}$ and let

$$D_\infty = Q_\infty = 1 - \sum_{n=1}^{\infty} Q_n.$$

We define

$$D = Q_\infty + \sum_{n=1}^{\infty} D_n \in \mathcal{M}.$$

It is clear that $\|D\| \leq 2$ and that $\ker(D) = \{0\}$. Thus the (unbounded) inverse D^{-1} of D is

$$D^{-1} = Q_\infty + \sum_{n=1}^{\infty} D_n^{-1}.$$

We have $Q_n D = D Q_n = D_n$ and $Q_n D^{-1} = D^{-1} Q_n = D_n^{-1}$ for each $n \in \mathbb{N}$. We now want to show that

$$\left\| D^{-1} T D \right\| < \infty,$$

i.e., the range of D is T-invariant. We have

$$\left\| D^{-1} T D \right\| \leq$$

$$\leq \sup_{1 \leq n \leq \infty} \left\| D_n^{-1} T D_n \right\| + \left\| D^{-1} T D - \sum_{n=1}^{\infty} D_n^{-1} T D_n \right\|$$

$$\leq \sup_{1 \leq n \leq \infty} \left\| D_n^{-1} T D_n \right\| + \sum_{1 \leq n \leq \infty} \left\| Q_n \left(D^{-1} T D - \sum_{n=1}^{\infty} D_n^{-1} T D_n \right) \right\|$$

$$= \sup_{1 \leq n \leq \infty} \left\| D_n^{-1} T D_n \right\| + \sum_{1 \leq n \leq \infty} \left\| Q_n \left(D_n^{-1} T D - D_n^{-1} T Q_n D \right) \right\|$$

$$\leq \sup_{1 \leq n \leq \infty} \left\| D_n^{-1} T D_n \right\| + \sum_{1 \leq n \leq \infty} \left\| D_n^{-1} \right\| \left\| Q_n T - T Q_n \right\| \left\| D \right\|.$$

However,

$$\left\| D_n^{-1} T D_n \right\| = \left\| D_n^{-1} (T - z_n) D_n \right\| + |z_n| \left\| D_n^{-1} D_n \right\|$$

$$= \left\| D_n^{-1} (T - z_n) \chi_{E_n}(T) D_n \right\| + |z_n| \leq \left\| D_n^{-1} \right\| \text{diam}(E_n) \left\| D \right\| + \left\| T \right\|$$

$$\leq \frac{n}{2^n} 2 + \|T\| \leq 1 + \|T\|.$$

Thus $\sup_{1 \leq n \leq \infty} \|D_n^{-1} T D_n\| \leq 1 + \|T\|$.

Next, for $1 \leq n < \infty$,

$$\left\|D_n^{-1}\right\| \|Q_n T - T Q_n\| \|D\| = \left\|D_n^{-1}\right\| \|Q_n (T - z_n) - (T - z_n) Q_n\| \|D\|$$

$$= \left\|D_n^{-1}\right\| \|Q_n \chi_{E_n}(T)(T - z_n) - (T - z_n) \chi_{E_n}(T) Q_n\| \|D\|$$
$$\leq n \frac{2}{2^n} \|D\| \leq n/2^{n-2}.$$

Thus

$$\sum_{n=1}^{\infty} \left\|D_n^{-1}\right\| \|Q_n T - T Q_n\| \|D\| \leq \sum_{n=1}^{\infty} n/2^{n-2} < \infty.$$

Also

$$\left\|Q_\infty D^{-1} T D\right\| = \|Q_\infty T D\| \leq 2 \|T\|.$$

Hence $\|D^{-1}TD\| < \infty$.

We now want to show that $D^{-1}SD$ is not bounded.

We know

$$\left\|D^{-1}SD\right\| \geq \sup_{1 \leq n < \infty} \left\|Q_n D^{-1} S D Q_n\right\| = \sup_{1 \leq n < \infty} \left\|D_n^{-1} Q_n S Q_n D_n\right\|$$

$$\sup_{1 \leq n < \infty} \left\|D_n^{-1} (P_{n_1} + P_{n_2}) S (P_{n_1} + P_{n_2}) D_n\right\|.$$

However,

$$\|P_{n_1}(S - \alpha_n)\| = \|P_{n_1}(S_{E_n} - \alpha_n)\| \leq \frac{1}{2^n}.$$

Similarly,

$$\|(S - \alpha_n) P_{n_1}\| \leq \frac{1}{2^n}, \|P_{n_2}(S - \beta_n)\| \leq \frac{1}{2^n}, \ \|(S - \beta_n) P_{n_2}\| \leq \frac{1}{2^n}.$$

Thus,

$$\left\|D_n^{-1} (P_{n_1} + P_{n_2}) S (P_{n_1} + P_{n_2}) D_n\right\| \geq$$
$$\left\|D_n^{-1} (\alpha_n P_{n_1} + \beta_n P_{n_2}) D_n\right\| - \left\|D_n^{-1}\right\| \frac{4}{2^n}$$
$$\geq n\varepsilon - 4 \|D\| / 2^n.$$

Thus $\|D^{-1}DS\| \not< \infty$. This contradiction proves our lemma.

□

Lemma 2.6.9. *For every $\lambda \in \sigma(T)$ there exists $\alpha \in \mathbb{C}$ such that*

$$\bigcap_{0<r<\infty} \sigma(S_{D(\lambda,r)}) = \{\alpha\}.$$

Proof. By Lemma 2.6.8 $diam(\sigma\left(S_D(\lambda, 1/n)\right) \ \rightarrow \ n$ as $n \ \rightarrow \ \infty$. Since for every $n \in \mathbb{N}$, $D(\lambda, 1/(n+1)) \subset D(\lambda, 1/n)$, we have $\{\sigma\left(S_D(\lambda, 1/n)\right)\}$ is a decreasing chain (with respect to $\subset$) of compact (closed) subsets of $\mathbb{C}$. Hence the lemma follows from Cantor's intersection theorem for complete metric spaces. □

Lemma 2.6.10. *Define*

$$f : \sigma(T) \to \mathbb{C}$$
$$\{f(\lambda)\} = \bigcap_{0<r<\infty} \sigma(S_{D(\lambda,r)}).$$

Then f is uniformly continuous on $\sigma(T)$, and $S = f\left(T\right)$.

Proof. The function f is a well defined by Lemma 2.6.9. Given $\varepsilon > 0$, Lemma 2.6.8 provides $\delta > 0$ such that for every $z, a \in \sigma(T)$, if $|z - a| < \delta/2$, then $diam(S_D((z + a)/2, \delta/2)) < \varepsilon$. If $|z-a| < \delta/2$, then $z, a \in D((z+a)/2, \delta/2)$. Moreover, $D(z, \delta/2-|z-a|)$ and $D(a, \delta/2-|z-a|)$ are both subsets of $D((z + a)/2, \delta/2)$. From the definition of f,

$$f(z) \in \sigma(S_{D(z,\delta/2-|z-a|)}) \subset \sigma(S_{D((z+a)/2,\delta/2)}),$$

and

$$f(a) \in \sigma(S_{D(a,\delta/2-|z-a|)}) \subset \sigma(S_{D((z+a)/2,\delta/2)}).$$

Therefore

$$|f(z) - f(a)| \leq diam(\sigma(S_{D((z+a)/2,\delta/2)}) < \varepsilon.$$

Thus f is uniformly continuous on $\sigma(T)$.

Next we show that $S = f(T)$. Let $\varepsilon > 0$. It follows from Lemma 2.6.8, and the fact that f is uniformly continuous that there exists $\delta > 0$ such that for every Borel subset $E \subset \sigma(T)$, and for every $z, \lambda \in \sigma(T)$,

1. $diam(E) < \delta \Rightarrow diam(\sigma(S_E)) < \varepsilon$,

2. $|z - \lambda| \leq \delta \Rightarrow |f(z) - f(\lambda)| \leq \varepsilon$.

Suppose $\lambda \in E \subset \sigma(T)$ and $diam(E) < \delta$. It follows from definition of f that $f(\lambda) \in \sigma(S_{D(\lambda,\delta)})$. It follows from (1) that $diam(S_D(\lambda, \delta)) < \varepsilon$. If $z \in \sigma(S_{D(\lambda,\delta)})$, then

$$|z - f(\lambda)| \leq diam(\sigma(S_{D(\lambda,\delta)})) < \varepsilon.$$

Define a continuous mapping $h : \sigma(S_{D(\lambda,\delta)}) \to \mathbb{C}$, by $h(z) = z - f(\lambda)$. Then

$$\begin{aligned}\left\|S_{D(\lambda,\delta)} - f(\lambda)\chi_{D(\lambda,\delta)}(T)\right\| &= \left\|h\left(S_{D(\lambda,\delta)}\right)\right\| \\ &= \sup_{z\in\sigma(S_D(\lambda,\delta))} |h(z)| \\ &< \varepsilon.\end{aligned}$$

Since $diam(E) < \delta$ it is clear that $E \subset D(\lambda, \delta)$. Let $F = D(\lambda, \delta) \setminus E$. Then

$$\chi_E(T) \perp \chi_F(T)$$

$$\chi_{D(\lambda,\delta)}(T) = \chi_E(T) \oplus \chi_F(T)$$

$$S_{D(\lambda,\delta)} = S_E \oplus S_F.$$

Thus

$$S_{D(\lambda,\delta)} - f(\lambda)\chi_{D(\lambda,\delta)}(T) = (S_E - f(\lambda)X_E(T)) \oplus (S_F - f(\lambda)X_F(T)),$$

It follows that

$$\begin{aligned}\|S_E - f(\lambda)\chi_E(T)\| &\leq \max\{\|S_E - f(\lambda)\chi_E(T)\|, \|S_F - f(\lambda)\chi_F(T)\|\} \\ &= \left\|S_{D(\lambda,\delta)} - f(\lambda)\chi_{D(\lambda,\delta)}(T)\right\| \\ &< \varepsilon\end{aligned}$$

Thus far we have shown that if E is a Borel subset of $\sigma(T)$, and $\lambda \in E$, then

$$diam(E) < \delta \Rightarrow \|S_E - f(\lambda)X_E(T)\| < \varepsilon. \tag{2.1}$$

Now consider a partition of $\sigma(T)$ into disjoint, nonempty subsets $\{E_1, E_2, \ldots, E_n\}$ such that $diam(E_k) < \delta$, for every $1 \leq k \leq n$. Let $T_{E_k} = T\chi_{E_k}(T)$. Write

$$I = \bigoplus_{k=1}^{\infty} \chi_{E_k}(T), \;\; S = \bigoplus_{k=1}^{n} S_{E_k}, \;\; T = \bigoplus_{k=1}^{n} T_{E_k}.$$

Choose $\lambda_1 \in E_1, \lambda_2 \in E_2, \ldots, \lambda_n \in E_n$. $diam(\sigma(T_{E_k})) \leq \delta$ since $\sigma(T_{E_k}) \subset \overline{E_k}$. If $z \in \sigma(T_{E_k})$, then $z \in \overline{E_k}$. Hence $|z - \lambda_k| \leq \delta$. Thus $|f(z) - f(\lambda_k)| \leq \varepsilon$. The mapping $g : \sigma(T_{E_K}) \to \mathbb{C}$, by $g(z) = f(\lambda_k) - f(z)$ is continuous. Thus for every $1 \leq k \leq n$,

$$\begin{aligned}\|f(\lambda_k)\chi_{E_k}(T) - f(T_{E_k})\| = \|g(T_{E_K})\| &= \sup_{z\in\sigma(T_{E_k})} |g(z)| \\ &= \sup_{z\in\sigma(T_{E_k})} |f(\lambda_k) - f(z)| \\ &\leq \varepsilon.\end{aligned} \tag{2.2}$$

Let

$$D = \bigoplus_{k=1}^{n} f(\lambda_k)\chi_{E_k}(T).$$

Then

$$
\begin{aligned}
\|S - f(T)\| &\le \|S - D\| + \|D - f(T)\| \\
&= \left\| \bigoplus_{k=1}^{n} S_{E_k} - \bigoplus_{k=1}^{n} f(\lambda_k)\chi_{E_k}(T) \right\| + \left\| f\left(\bigoplus_{k=1}^{n} T_{E_k} \right) - \bigoplus_{k=1}^{n} f(\lambda_k)\chi_{E_k}(T) \right\| \\
&= \left\| \bigoplus_{k=1}^{n} S_{E_k} - f(\lambda_k)\chi_{E_k}(T) \right\| + \left\| \bigoplus_{k=1}^{n} f\left(T_{E_k}\right) - f(\lambda_k)\chi_{E_k}(T) \right\| \\
&= \max_{1\le k\le n} \|S_{E_k} - f(\lambda_k)\chi_{E_k}(T)\| + \max_{1\le k\le n} \|f\left(T_{E_k}\right) - f(\lambda_k)\chi_{E_k}(T)\| \\
&\le \varepsilon + \varepsilon = 2\varepsilon
\end{aligned}
$$

where the last inequality follows from equations (2.1) and (2.2) above. Thus $S = f(T)$. $\square$

2.7 Normal Operators in a type I_n von Neumann algebra

Suppose $\mathcal{M}$ is a type I_n von Neumann algebra. By a result in [19], there is a family of probability spaces $\{(\Omega_i, \Sigma_i, \mu_i) : i \in I\}$, such that $\mathcal{M}$ is isomorphic (not unitarily equivalent to)

$$
\sum_{i\in I}^{\oplus} \mathbb{M}_n\left(L^\infty\left(\mu_i\right)\right).
$$

We need a simple result about Vandermonde matrices. Suppose $t_1, \ldots, t_d$ are d distinct complex numbers. The Vandermonde matrix $V\left(t_1, \ldots, t_d\right)$ is defined as

$$
V\left(t_1, \ldots, t_d\right) = \begin{pmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{d-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_d & t_d^2 & \cdots & t_d^{d-1} \end{pmatrix}.
$$

It is well known that $V\left(t_1, \ldots, t_d\right)$ is invertible. Here are some additional facts.

Lemma 2.7.1. *Suppose $t_1, \ldots, t_d$ are d distinct complex numbers, $s_1, \ldots, s_d \in \mathbb{C}$, and $p(z) = c_0 + c_1 z + \cdots + c_{d-1} z^{d-1}$ is the (unique) polynomial with degree less than d such that $p(t_k) = s_k$ for $1 \leq k \leq d$. Then*

1. $V(t_1, \ldots, t_d) \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{d-1} \end{pmatrix} = \begin{pmatrix} p(t_1) \\ p(t_2) \\ \vdots \\ p(t_d) \end{pmatrix}$

2. *The first column of* $V(t_1, \ldots, t_d)^{-1}$*diag*$(s_1, \ldots, s_d) V(t_1, \ldots, t_d) = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{d-1} \end{pmatrix}$.

3. $V(t_1, \ldots, t_d)^{-1} \operatorname{diag}(t_1, \ldots, t_d) V(t_1, \ldots, t_d) = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & \cdots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_{d-1} \end{pmatrix}$, *where*

$$a_0 + a_1 z + \cdots + a_{d-1} z^{d-1} = z^d - (z - t_1) \cdots (z - t_d).$$

4. $\left\| V(t_1, \ldots, t_d)^{-1} \operatorname{diag}(t_1, \ldots, t_d) V(t_1, \ldots, t_d) \right\| \leq (1 + R)^d$, *where* $R = \max_{1 \leq k \leq d} |t_k|$.

Proof. (1). This is trivial.

(2). The first column of $V(t_1, \ldots, t_d)^{-1}$diag$(s_1, \ldots, s_d) V(t_1, \ldots, t_d)$ is

$$V(t_1, \ldots, t_d)^{-1} \operatorname{diag}(s_1, \ldots, s_d) V(t_1, \ldots, t_d) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= V(t_1, \dots, t_d)^{-1} \operatorname{diag}(s_1, \dots, s_d) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = V(t_1, \dots, t_d)^{-1} \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_d \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{d-1} \end{pmatrix},$$

where the last equality follows from part (1).

(3). Suppose $V(t_1, \dots, t_d)^{-1}\operatorname{diag}(s_1, \dots, s_d)V(t_1, \dots, t_d) = B$. Then

$$\operatorname{diag}(s_1, \dots, s_d)V(t_1, \dots, t_d) = \begin{pmatrix} t_1 & t_1^2 & \cdots & t_1^d \\ t_2 & t_2^2 & \cdots & t_2^d \\ \vdots & \vdots & \vdots & \ddots \\ t_d & t_d^2 & \cdots & t_d^d \end{pmatrix} = V(t_1, \dots, t_d)B.$$

If B_j denotes the j^{th} column of the matrix B, then

$$V(t_1, \dots, t_d)B_j = \begin{pmatrix} t_1^j \\ t_2^j \\ \vdots \\ t_d^j \end{pmatrix} = (j+1)^{th}\text{column of } V(t_1, \dots, t_d).$$

Therefore B_j is the column vector with a 1 at the $(j+1)^{th}$ component and 0 everywhere else, for $1 \le j < d$. Suppose the last column of B is

$$B_d = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{d-1} \end{pmatrix},$$

and let $P(z) = a_0 + a_1 z + \dots a_{d-1} z^{d-1}$ be the unique polynomial of degree $< d$ such that $p(t_k) = t_k^d$ for $1 \le k \le d$. It is clear that $p(z) = z^d - (z - t_1)(z - t_2) \dots (z - t_d)$.

(4) The columns of the matrix B in part (3) are all unit vectors with a simple form except for the last column. The elements a_k of the last column of B are coefficients of $p(z) = z^d - (z - t_1)(z - t_2) \dots (z - t_d)$. Thus to estimate $\|B\|$, we expand the polynomial and estimate the magnitude of the coefficients. Let $R = \max_{1 \le k \le d} |t_k|$. Then $|a_k| \le C(d, d-k) R^{d-k}$. Thus

$$\sum_{k=0}^{d} |a_k| \le \sum_{k=0}^{d} C(d, d-k) R^{d-k} \le (1+R)^d.$$

Hence $\|B\| \le (1+R)^d$. □

Theorem 2.7.2. *Suppose $n \in \mathbb{N}$ and $\mathcal{M}$ is a type I_n von Neumann algebra. Suppose $S, T \in \mathcal{M}$, T is normal and $S \in \mathrm{AlgLat}_{1/2}(T, \mathcal{M})$. Then there are elements $c_0, c_1, \dots, c_{n-1} \in \mathcal{Z}(\mathcal{M})$ such that*

$$S = c_0 + c_1 T + \cdots + c_{n-1} T^{n-1}.$$

Proof. Write

$$\mathcal{M} = \sum_{i \in I}^{\oplus} \mathbb{M}_n \left(L^\infty \left(\mu_i \right) \right),$$

$$T = \sum_{i \in I}^{\oplus} T_i,$$

and

$$S = \sum_{i \in I}^{\oplus} S_i \, .$$

Theorem 2.2.3 provides a unitary $U = \sum_{i \in I}^{\oplus} U_i \in \mathcal{M}$ such that, for every $i \in I$, there are measurable functions $d_i : \Omega_i \to \{1, \dots, n\}$ and $t_{i,1}, \dots, t_{i,n} : \Omega_i \to \mathbb{C}$ such that

$$U_i^* (\omega) \, T_i (\omega) \, U_i (\omega) = \mathrm{diag} \left(t_{i,1} (\omega), \dots, t_{i,n} (\omega) \right)$$

and

$$Card \left(\left\{ t_{i,1} (\omega), \dots, t_{i,d_i(\omega)} (\omega) \right\} \right) = d_i (\omega) = Card \left(\left\{ t_{i,1} (\omega), \dots, t_{i,n} (\omega) \right\} \right).$$

Since the theorem is unchanged if we replace T with U^*TU and S with U^*SU, we can assume that

$$T_i(\omega) = \text{diag}(t_{i,1}(\omega), \ldots, t_{i,n}(\omega))$$

holds.

For each $i \in I$ and $\omega \in \Omega_i$ define

$$W_{i,\omega} = \begin{pmatrix} V\left(t_{i,1}(\omega), \ldots, t_{i,d_i(\omega)}(\omega)\right) & 0 \\ 0 & I_{n-d_i(\omega)} \end{pmatrix}.$$

Then $W = \sum_{i\in I}^{\oplus} W_i \in \mathcal{M}$ and has norm at most $(1 + \|T\|)^n$. Also if for each $i \in I$ and each $\omega \in \Omega_i$ we define

$$D_{i,\omega} = W_{i,\omega}^{-1} T_i(\omega) W_{i,\omega}$$

and define

$$D = \sum_{i\in I}^{\oplus} D_i \in \mathcal{M},$$

then

$$TW = WD$$

and

$$\|D\| \le (1 + \|T\|)^n.$$

Since $S \in \text{AlgLat}_{1/2}(T, \mathcal{M})$ we have, for each $i \in I$, $S_i \in \text{AlgLat}_{1/2}(T_i, \mathbb{M}_n(L^\infty(\mu_i)))$. It follows from Theorem 2.4.1 that we can assume, for every $\omega \in \Omega_i$

$$S_i(\omega) \in \text{AlgLat}_{1/2}(T_i(\omega), \mathbb{M}_n(\mathbb{C})).$$

Since every normal matrix is reflexive, it follows that there is a polynomial

$$p_{i,\omega}(z) = c_{i,0}(\omega) + c_{i,1}(\omega) z + \cdots + c_{i,n-1}(\omega) z^{n-1}$$

with $c_{i,k}(\omega) = 0$ when $d_i(\omega) \le k \le n-1$, i.e, the degree of $p_{i,\omega}$ is less than $d_i(\omega)$.

Thus, for each $i \in I$ and each $\omega \in \Omega_i$,

$$S_i(\omega) = \text{diag}\left(p_{i,\omega}\left(t_{i,1}(\omega)\right), \ldots, p_{i,\omega}\left(t_{i,n}(\omega)\right)\right).$$

Since $S \in \text{AlgLat}_{1/2}(T, \mathcal{M})$ and $TW = WD$, there exists $B = \sum_{i \in I}^{\oplus} B_i \in \mathcal{M}$ such that $SW = WB$. Hence, we can assume for every $i \in I$ and each $\omega \in \Omega_i$ that

$$W_{i,\omega}^{-1} S_{i,}(\omega) W_{i,\omega} = B_{i,\omega},$$

and therefore

$$\left\| W_{i,\omega}^{-1} S_{i,}(\omega) W_{i,\omega} \right\| \leq \|B\|.$$

By Lemma 2.7.1, the first column of $W_{i,\omega}^{-1} S_{i,}(\omega) W_{i,\omega}$ is

$$\begin{pmatrix} c_{i,0}(\omega) \\ c_{i,1}(\omega) \\ \vdots \\ c_{d-1}(\omega) \end{pmatrix}.$$

It follows that each $c_{i,k}$ is measurable and

$$\sup\left\{ |c_{i,k}(\omega)| : i \in I, \omega \in \Omega_i \right\} \leq \|B\|.$$

We can define $C_0, \ldots C_{n-1} \in \mathcal{Z}(\mathcal{M})$ where

$$C_k = \sum_{i \in I}^{\oplus} C_{k,i}$$

and

$$C_{k,i}(\omega) = c_{i,k}(\omega) I.$$

We clearly have $S = C_0 + C_1 T + \cdots + C_{n-1} T^{n-1}$. $\square$

2.8 Normal Operators in an Arbitrary von Neumann Algebra on a Separable Hilbert Space

Theorem 2.8.1. *Suppose $\mathcal{M}$ is a von Neumann algebra acting on a separable Hilbert space. If $S, T \in \mathcal{M}$, T is normal, and $S \in \mathrm{AlgLat}_{1/2}(T, \mathcal{M})$, then*

$$S \in C^*\left(\{T\} \cup \mathcal{Z}\left(\mathcal{M}\right)\right).$$

Proof. Case 1: First assume $\mathcal{M} = \sum_{n \in E}^{\oplus} \mathcal{M}_n$ with $E \subset \mathbb{N}$ and each $\mathcal{M}_n$ is a factor. Write $T = \sum_{n \in E}^{\oplus} T_n$ and $S = \sum_{n \in E}^{\oplus} T_n$. Since $S \in \mathrm{AlgLat}_{1/2}\left(T, \mathcal{M}\right)$, for each $n \in E$, $S_n \in \mathrm{AlgLat}_{1/2}\left(T_n, \mathcal{M}_n\right)$. Thus, for each $n \in \mathbb{N}$ there is a continuous function $f_n : \mathbb{C} \to \mathbb{C}$ such that $S_n = f_n\left(T_n\right)$. If E is finite, we are done. Thus we can assume $E = \mathbb{N}$. We know from [10] that there is a sequence $\{P_m\}$ of projections in $\mathcal{M}$ such that

$$\lim_{m \to \infty} \|P_m T - T P_m\| = 0$$

and

$$\lim_{m \to \infty} \|P_m S - S P_m\| = \mathrm{dist}\left(S, C^*\left(\{T\} \cup \mathcal{Z}\left(\mathcal{M}\right)\right)\right) = 2\varepsilon.$$

Assume via contradiction that $\varepsilon > 0$. We can assume that for every $m \in \mathbb{N}$,

$$\|P_m S - S P_m\| > \varepsilon\ .$$

For each $m \in \mathbb{N}$ we can write $P_m = \sum_{k \in \mathbb{N}}^{\oplus} P_{m,k}$. Since

$$\lim_{m \to \infty} \left[\sup_{k \in \mathbb{N}} \|P_{m,k} T_k - T_k P_{m,k}\|\right] = 0$$

and $S_k = f_k\left(T_k\right)$, we have, for each $k \in \mathbb{N}$

$$\lim_{m \to \infty} \|P_{m,k} S_k - S_k P_{m,k}\| = 0.$$

It follows that there are integers $1 \leq k_1 < k_2 < \cdots$ and projections $Q_1, Q_2, \ldots$ such that, for each $s \in \mathbb{N}$,

$$\|Q_s T_{k_s} - T_{k_s} Q_s\| < 1/2^s \text{ and } \|Q_s S_{k_s} - S_{k_s} Q_s\| > \varepsilon.$$

Since for every operator A and every projection Q

$$\|QA - AQ\| = \max\left(\|(1-Q)\, AQ\|, \left\|\left(1 - Q^{\perp}\right) AQ^{\perp}\right\|\right),$$

by replacing Q_s with $1 - Q_s = Q_s^{\perp}$ if necessary, we can assume that for every $s \in \mathbb{N}$

$$\|(1 - Q_s)\, S_{k_s} Q_s\| > \varepsilon.$$

We define $A = \sum_{n\in\mathbb{N}}^{\oplus} A_n$ and $B = \sum_{n\in\mathbb{N}}^{\oplus} B_n$ in $\mathcal{M}$ by

$$A_n = \begin{cases} Q_s + \frac{1}{n} Q_s^{\perp} & \text{if } n = k_s \text{ for some } s \in \mathbb{N} \\ I & \text{otherwise} \end{cases}$$

$$B_n = \begin{cases} \left(Q_s + \frac{1}{n} Q_s^{\perp}\right)^{-1} T_{k_s} \left(Q_s + \frac{1}{n} Q_s^{\perp}\right) & \text{if } n = k_s \text{ for some } s \in \mathbb{N} \\ I & \text{otherwise} \end{cases}$$

Then, for every $n \in \mathbb{N}$,

$$T_n A_n = A_n B_n, \text{ and } \|B_n\| \leq \|T_n\| + \left(n + \frac{1}{n}\right)/2^{n.}$$

Thus $TA = AB$, so there exists $C = \sum_{n\in\mathbb{N}}^{\oplus} C_n$ in $\mathcal{M}$ such that

$$SA = AC.$$

This means

$$A_{k_s}^{-1} S_{k_s} A_{k_s} = C_{k_s}.$$

This contradicts the fact that, for every $s \in \mathbb{N}$,

$$\|C\| \geq \|C_{k_s}\| = \left\|A_{k_s}^{-1} S_{k_s} A_{k_s}\right\| \geq \left\|(1 - Q_{k_s})\, A_{k_s}^{-1} S_{k_s} A_{k_s} Q_{k_s}\right\|$$

$$\geq \left\|(1 - Q_{k_s})\, A_{k_s}^{-1} S_{k_s} A_{k_s} Q_{k_s}\right\|$$

$$\geq s \left\|(1 - Q_{k_s})\, S_{k_s} Q_{k_s}\right\| \geq s\varepsilon.$$

Case 2. There is a nonatomic σ-finite measure space (Ω, Σ, μ) and a measurable family $\{\mathcal{M}_\omega : \omega \in \Omega\}$ of factor von Neumann algebras such that $\mathcal{M} = \int_\Omega^\oplus \mathcal{M}_\omega \, d\mu(\omega)$. We can write $T = \int_\Omega^\oplus T_\omega \, d\mu(\omega)$ and $S = \int_\Omega^\oplus S_\omega \, d\mu(\omega)$. Assume, via contradiction that

$$\operatorname{dist}(S, C^*(\{T\} \cup \mathcal{Z}(\mathcal{M}))) = 2\varepsilon > 0.$$

Arguing as in Case 1, there is a sequence $\{P_n\}$ of projections in $\mathcal{M}$ such that, for every $n \in \mathbb{N}$,

$$\|P_n T - T P_n\| < 1/2^n \text{ and } \|P_n S - S P_n\| > \varepsilon.$$

Then there is a sequence $\{E_n\}$ of measurable sets with positive measure such that, for every $n \in \mathbb{N}$ and every $\omega \in E_n$,

$$\|P_n(\omega) T_\omega - T_\omega P_n(\omega)\| < 1/2^n \text{ and } \|P_n(\omega) S_\omega - S_\omega P_n(\varepsilon)\| > \varepsilon.$$

Since μ is nonatomic we can replace each E_n with a subset with positive measure, so that the sets E_n are pairwise disjoint. Since E_n is the union of the

$$\{\omega \in E_n : \|(1 - P_n(\omega)) S_\omega P_n(\omega)\| > \varepsilon\} \cup \left\{\omega \in E_n : \left\|\left(1 - P_n^\perp(\omega)\right) S_\omega P_n^\perp(\omega)\right\| > \varepsilon\right\},$$

we can assume that, for every $\omega \in E_n$

$$\|(1 - P_n(\omega)) S_\omega P_n(\omega)\| > \varepsilon.$$

Following the proof of Case 1, we define $A, B \in \mathcal{M}$ by

$$A(\omega) = \begin{cases} P_n(\omega) + \frac{1}{n} P_n^{\perp}(\omega) & \text{if } \omega \in E_n \text{ for some } n \in \mathbb{N} \\ I & \text{otherwise} \end{cases}$$

and

$$B(\omega) = \begin{cases} A(\omega)^{-1} T_\omega\, A(\omega) & \text{if } \omega \in E_n \text{ for some } n \in \mathbb{N} \\ I & \text{otherwise} \end{cases}.$$

Then as in Case 1, $A, B \in \mathcal{M}$ and $TA = AB$. Since $S \in \mathrm{AlgLat}_{1/2}(T, \mathcal{M})$, there exists $C = \int_\Omega^\oplus C(\omega)\ d\mu(\omega)$ such that $SA = AC$. Thus, since $\mu(E_n) > 0$, we know for each positive integer n,

$$\varepsilon n \leq \left\| A(\omega)^{-1} S(\omega) A(\omega) \chi_{E_n}(\omega) \right\|_\infty = \|C(\omega)\|_\infty = \|C\| < \infty.$$

This contradiction proves Case 2.

General Case. Using the central decomposition for $\mathcal{M}$ [19], we can write

$$\mathcal{M} = \mathcal{N} \oplus \mathcal{R}$$

where $\mathcal{N}$ satisfies the condition of Case 1 and $\mathcal{R}$ satisfies the condition of Case 2. It easily follows from Cases 1 and 2 that the general case is true. □

2.9 Some General Lemmas

Lemma 2.9.1. *Suppose $\mathcal{B}$ is a von Neumann algebra, $\mathcal{A} \subset \mathcal{B}$ is von Neumann subalgebra, and $S, T \in \mathcal{A}$. If $S \in \mathrm{AlgLat}_{1/2}(T, \mathcal{B})$, then $S \in \mathrm{AlgLat}_{1/2}(T, \mathcal{A})$.*

Proof. Suppose $D \in \mathcal{A}$ and $T(Ran(D)) \subset Ran(D)$. Then, $D \in \mathcal{B}$, so $S(Ran(D)) \subset Ran(D)$ and therefore $S \in \mathrm{AlgLat}_{1/2}(T, \mathcal{B})$. □

Corollary 2.9.2. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras and $\pi : \mathcal{A} \to \mathcal{B}$ is an isometric unital $*$-homomorphism such that $\pi(\mathcal{A})$ is a von Neumann algebra. Suppose $S, T \in \mathcal{A}$. If $\pi(S) \in \mathrm{AlgLat}_{1/2}(\pi(T), \mathcal{B})$, then $S \in \mathrm{AlgLat}_{1/2}(T, \mathcal{A})$.*

Proof. Suppose $D \in \mathcal{A}$ and $T(Ran(D)) \subset Ran(D)$. Then there exists a bounded $C \in \mathcal{A}$ such that $TD = DC$. Therefore $\pi(TD) = \pi(T)\pi(D) = \pi(D)\pi(C)$. It follows from the previous lemma that $\pi(S) \in \mathrm{AlgLat}_{1/2}(\pi(T), \pi(\mathcal{A}))$. Thus there exists $C_1 \in \pi(\mathcal{A})$, such that $\pi(S)\pi(D) = \pi(D)C_1$. But $\pi : \mathcal{A} \to \pi(\mathcal{A})$ is a $*$-isomorphism. Hence $C_1 = \pi(C)$, for some $C \in \mathcal{A}$. Thus $SD = DC$, and $S \in \mathrm{AlgLat}_{1/2}(T, \mathcal{A})$. □

Theorem 2.9.3. *Suppose $\mathcal{M}$ is a von Neumann algebra, $\pi : B(\ell^2) \to \mathcal{M}$ is a unital isometric $*$-homomorphism, $S, T \in \pi(B(\ell^2))$, T is not algebraic and $S \in \mathrm{AlgLat}_{1/2}(T, \mathcal{M})$. Then there exists an entire function φ such that*

$$S = \varphi(T).$$

Proof. There exist $S_1, T_1 \in B(\ell^2)$ such that $S = \pi(S_1)$ and $T = \pi(T_1)$. Since $\pi(S_1) \in \mathrm{AlgLat}_{1/2}(\pi(T_1), \mathcal{M})$, it follows from the previous lemma that $S_1 \in \mathrm{AlgLat}_{1/2}(T_1, B(\ell^2))$. Therefore Douglas-Foiaş theorem implies that there exists an entire function $\varphi(z) = \sum_{n=0}^{\infty} c_n z^n$ such that $S_1 = \varphi(T_1)$. Thus

$$\begin{aligned} S = \pi(S_1) = \pi(\varphi(T_1)) = \pi(\lim_{N\to\infty} \sum_{n=0}^{N} c_n T_1^n) &= \lim_{n\to\infty} \pi(\sum_{n=0}^{N} c_n T_1^n) \\ &= \lim_{n\to\infty} \sum_{n=0}^{N} c_n \pi(T_1)^n = \sum_{n=0}^{\infty} c_n \pi(T_1)^n = \varphi(\pi(T_1)) \\ &= \varphi(T). \end{aligned}$$

□

Corollary 2.9.4. *Suppose $\mathcal{M}$ and ρ are as in Theorem 3.2.14, $X, Y, W \in \mathcal{M}$, W is invertible, $X_1 = W^{-1}XW, Y_1 = W^{-1}YW \in \rho(B(\ell^2)), Y \in \mathrm{AlgLat}_{1/2}(X, \mathcal{M})$, and X is not algebraic. Then there is an entire function $\varphi : \mathbb{C} \to \mathbb{C}$ such that $Y = \varphi(X)$.*

Proof. By Theorem 2.9.3, $Y_1 = \varphi(X_1)$. Thus $W^{-1}YW = \varphi(W^{-1}XW) = W^{-1}\varphi(X)W$. Thus $Y = \varphi(X)$.

□

CHAPTER 3

SIMILARITY DOMINANCE

3.1 Preliminaries

In [11], J. B. Conway and D. Hadwin introduced the notion of Similarity Dominance.

Definition 3.1.1 (Similarity Dominance)**.** Suppose $\mathcal{A}$ is a unital Banach algebra and $S, T \in \mathcal{A}$. We say that T sim-dominates S provided, for every $R > 0$,

$$\sup\left(\left\{\left\|A^{-1}SA\right\| : A \in \mathcal{A},\ A \text{ invertible},\ \left\|A^{-1}TA\right\| \leq R\right\}\right) < \infty.$$

Theorem 3.1.2 (J. B. Conway, D. Hadwin)**.** *Suppose H is a separable Hilbert space, and $S, T \in B(H)$. If T sim-dominates S in $B(H)$, then $S = \varphi(T)$ for an entire function φ.*

One of our goals in this chapter is to prove a version of Theorem 3.1.2 for a large class of operators in a type III factor von Neumann algebra.

Another goal of this chapter is to explore the interplay between Sim-Domination, Approximate Double Commutants and Approximate Similarity.

In [8] D. Hadwin introduced the notion of a Double Commutant of a subset of operators in $B(H)$. He proved an asymptotic version of the von Neumann's Double Commutant Theorem, in which C^* algebras play the role of von Neumann algebras. He then used this theorem to to investigate asymptotic versions of similarity, reflexivity and reductivity.

We begin by a review of the basic concepts initially studied in [8], and outline the main results of section 3.2.

Definition 3.1.3 (Approximate Double Commutant [8]). Suppose $\mathcal{S} \subset B(H)$. The Approximate Double Commutant of S, denoted by $\mathrm{Appr}(\mathcal{S})''$ is

$$\mathrm{Appr}(\mathcal{S})'' = \{T \in B(H) : \|A_n T - T A_n\| \to 0\}$$

for every bounded net $\{A_n\}$ in $B(H)$ for which $\|A_n S - S A_n\| \to 0$, for every $S \in \mathcal{S}$.

More generally we can define the Relative Approximate Double Commutant of a set of operators in a unital Banach algebra.

Definition 3.1.4 (Relative Approximate Double Commutant [10]). Suppose $\mathcal{B}$ is a unital Banach Algebra, and $\mathcal{S} \subset \mathcal{B}$. We define the approximate double commutant of $\mathcal{S}$ in $\mathcal{B}$, denoted by $\mathrm{Appr}(\mathcal{S}, \mathcal{B})''$ as in definition 3.1.3, with the additional requirement that T's and A_n's belong to $\mathcal{B}$.

Suppose $\mathcal{A}$ is a unital Banach Algebra and $S, T \in \mathcal{A}$. One of our results in the next section states that if T sim-dominates S in $\mathcal{A}$, then $S \in \mathrm{Appr}(T, \mathcal{A})''$. That is, if $\{A_n\}$ is a bounded sequence in $\mathcal{A}$ such that $\lim_{n\to\infty} \|A_n T - T A_n\| = 0$, then $\lim_{n\to\infty} \|A_n S - S A_n\| = 0$.

Definition 3.1.5 (Invertibly Bounded Sequence [8]). A sequence $\{W_n\}$ in a Banach algebra $\mathcal{B}$ is *invertibly bounded* if each W_n is invertible and $\sup_{n\in\mathbb{N}} \max\left(\|W_n\|, \|W_n^{-1}\|\right) < \infty$.

Definition 3.1.6 (Approximate Similarity [8]). Suppose $\mathcal{B}$ is a unital Banach algebra. Two operators $S, T \in \mathcal{B}$ are *approximately similar* if there is a sequence $\{W_n\}$ of invertibly bounded operators in $\mathcal{B}$ such that $\|W_n^{-1} T W_n - S\| \to 0$.

Definition 3.1.7 (Approximately Similar Pair). A pair (S, T) in a Banach algebra $\mathcal{B}$ is *approximately similar* to a pair (S_1, T_1) if and only if there is an invertibly bounded sequence $\{W_n\}$ in $\mathcal{B}$ such that

$$\lim_{n\to\infty} \left\|W_n^{-1} T W_n - T_1\right\| + \left\|W_n^{-1} S W_n - S_1\right\| = 0.$$

We will prove that sim-domination is preserved under approximate similarity, i.e., if $\{A_n\}$ is an invertibly bounded sequence in $\mathcal{A}$ with $\|A_n^{-1} T A_n - T'\| \to 0$, then there is an $S' \in \mathcal{A}$ such that $\|A_n^{-1} S A_n - S'\| \to 0$, and, T' sim-dominates S'.

Definition 3.1.8. We say that elements S, T in a unital C*-algebra $\mathcal{A}$ are *approximately equivalent* in $\mathcal{A}$ if and only if there is a sequence $\{U_n\}$ of unitary operators such that

$$\lim_{n\to\infty} \|U_n^* T U_n - S\| = 0.$$

Here we list several results of J. B. Conway and D. Hadwin [11], to be used in section 3.2.

Lemma 3.1.9. *Suppose* $A, T \in B(H), A \geq 0$ *and* $T(ran(A)) \subset ran(A)$*. Then, for every* $\varepsilon > 0$

$$\left\|(A+\varepsilon)^{-1}T(A+\varepsilon)\right\| \leq \|T\| + \left\|A^{-1}TA\right\|.$$

Lemma 3.1.10. *Suppose* $A, S \in B(H), A \geq 0$ *and*

$$\sup_{\varepsilon>0} \left\|(A+\varepsilon)^{-1}S(A+\varepsilon)\right\| < \infty.$$

Then, $S(ran(A)) \subset ran(A)$.

Lemma 3.1.11. *Suppose* $T \in B(H)$*,* M *is a Hilbert space,* $W : M \to H$*, and* $T(ran(W)) \subset ran(W)$*. Then* $T\left(ran(WW^*)^{1/2}\right) \subset ran(WW^*)^{1/2}$ *and*

$$\left\|W^{-1}TW\right\| = \left\|\left((WW^*)^{1/2}\right)^{-1} T(WW^*)^{1/2}\right\|.$$

The following lemma is motivated by an argument in Theorem 7 in [11]. We mention the proof here for convenience.

Lemma 3.1.12. *Suppose* $\mathcal{A}$ *is a unital Banach algebra,* $S, T \in \mathcal{A}$ *and* T *sim-dominates* S *in* $\mathcal{A}$*. Then* $S \in \{T\}''$.

Proof. Suppose $A \in \mathcal{A}$ and $TA = AT$. Then $e^{\lambda A}T = Te^{\lambda A}$, and

$$\sup_{\lambda \in \mathbb{C}} \left\| e^{-\lambda A} T e^{\lambda A} \right\| = \|T\| .$$

Since T sim-dominates S,

$$\sup_{\lambda \in \mathbb{C}} \left\| e^{-\lambda A} S e^{\lambda A} \right\| < \infty.$$

Thus the map $\varphi(\lambda) = e^{-\lambda A} S e^{\lambda A}$ is a bounded entire function, which, by generalized Liouville theorem, is constant. Computing

$$0 = \varphi'(0) = -AS + SA,$$

we see that $AS = SA$. Thus $S \in \{T\}''$. □

3.2 Main Results

Lemma 3.2.1. *Suppose $\mathcal{A}$ is a Banach algebra, $A, S, T \in \mathcal{A}$ and A is invertible. Then T sim-dominates S in $\mathcal{A}$ if and only if $A^{-1}TA$ sim-dominates $A^{-1}SA$ in $\mathcal{A}$.*

Proof. This follows from the fact that

$$\left\{ \left\| W^{-1} A^{-1} S A W \right\| : W \in \mathcal{A} \text{ is invertible, } \left\| W^{-1} A^{-1} T A W \right\| < R \right\} =$$
$$\left\{ \left\| (AW)^{-1} S (AW) \right\| : W \in \mathcal{A} \text{ is invertible, } \left\| (AW)^{-1} T (AW) \right\| < R \right\} =$$
$$\left\{ \left\| W^{-1} S W \right\| : W \in \mathcal{A} \text{ is invertible, } \left\| W^{-1} T W \right\| < R \right\}$$

□

Lemma 3.2.2. *Suppose $\mathcal{A}$ is a unital Banach algebra and $T \in \mathcal{A}$. Then*

$$\{S \in \mathcal{A} : T \text{ sim-dominates } S \text{ in } \mathcal{A}\}$$

is an algebra containing T and the center $\mathcal{Z}(\mathcal{A})$ of $\mathcal{A}$.

Proof. It is clear that T sim-dominates T. If $R, S \in \mathcal{A}$, and $\alpha \in \mathbb{C}$, then

$$\left\|W^{-1}(RS)W\right\| = \left\|W^{-1}RWW^{-1}SW\right\| \leq \left\|W^{-1}RW\right\| \left\|W^{-1}SW\right\|,$$

and

$$\left\|W^{-1}(\alpha R + S)W\right\| \leq |\alpha| \left\|W^{-1}RW\right\| + \left\|W^{-1}SW\right\|.$$

If $R \in \mathcal{Z}(\mathcal{M})$, then

$$\left\|W^{-1}RW\right\| = \|R\|.$$

□

Theorem 3.2.3. *Suppose $\mathcal{A}$ is a unital Banach algebra, $S, T \in \mathcal{A}$ and T sim-dominates S in $\mathcal{A}$. Then $S \in Appr(T, \mathcal{A})''$.*

Proof. Consider the mappings

$$\mathcal{A} \xrightarrow{\rho} \ell^\infty(\mathcal{A}) \xrightarrow{\eta} \ell^\infty(\mathcal{A})/C_0(\mathcal{A}),$$

where $\rho(T) = (T, T, T, \dots)$, and η is the quotient map. Define $\pi : \mathcal{A} \to \ell^\infty(\mathcal{A})/C_0(\mathcal{A})$ by $\pi = \eta \circ \rho$. We first show that $\pi(T)$ sim-dominates $\pi(S)$ in $\ell^\infty(\mathcal{A})/C_0(\mathcal{A})$.

Suppose $W \in \ell^\infty(\mathcal{A})/C_0(\mathcal{A})$ is such that $\|W^{-1}\pi(T)W\| < R$. We need to show that there is a constant β_R depending on R, such that $\|W^{-1}\pi(S)W\| < \beta_R$. If W is invertible, then there exists a $V \in \ell^\infty(\mathcal{A})/C_0(\mathcal{A})$ such that $WV = VW = 1$. Let $W = \eta((w_n)), V = \eta((v_n))$. Thus $\eta((w_n v_n - 1)) = \eta((v_n w_n - 1)) = 0$. Therefore $(w_n v_n - 1)$ and $(v_n w_n - 1) \in C_0(\mathcal{A})$. This means that $\lim_{n\to\infty} \|w_n v_n - 1\| = 0$, and that $(w_n v_n)$ is eventually invertible. We have the same conclusion for $(v_n w_n)$. Thus $(v_n w_n)(v_n w_n)^{-1} = 1$ and $(w_n v_n)^{-1}(w_n v_n) = 1$ eventually. Hence (v_n) has a left and a right inverse eventually, and is therefore eventually invertible. We have the same conclusion for (w_n). We may replace the finitely many initial terms of these sequences (those which may not be invertible) by 1, and thus we may assume that they are invertible.

Since $(w_nv_n)^{-1}, (v_n)$ and (w_n) are bounded, $(w_n^{-1}), (v_n^{-1})$ are also bounded sequences. Thus

$$\begin{aligned}\left\|W^{-1}\pi(T)W\right\| &= \left\|\eta(w_n^{-1})\eta(T,T,\ldots,T)\eta(w_n)\right\| \\ &= \left\|\eta(w_n^{-1}Tw_n)\right\| \\ &= \limsup\left\|w_n^{-1}Tw_n\right\| < R.\end{aligned}$$

Therefore there exists $N_w \in \mathbb{N}$ such that

$$\sup_{n>N_w}\left\|w_n^{-1}Tw_n\right\| < R.$$

Since T sim-dominates $S \in \mathcal{A}$, there exists a constant β_R, such that

$$\sup_{n>N_w}\left\|w_n^{-1}Sw_n\right\| < \beta_R.$$

Therefore

$$\begin{aligned}\limsup\left\|w_n^{-1}Sw_n\right\| &= \left\|\eta(w_n^{-1}Sw_n)\right\| \\ &= \left\|\eta(w_n^{-1})\eta(S,S,\ldots,S)\eta(w_n)\right\| \\ &= \left\|W^{-1}\pi(S)W\right\| < \beta_R.\end{aligned}$$

This shows that $\pi(T)$ sim-dominates $\pi(S)$. Thus $\pi(S) \in \{\pi(T)\}''$ by Lemma 3.1.12, which is the same as saying $S \in \mathit{Appr}(T)''$. □

Theorem 3.2.4. *Suppose $\mathcal{A}$ is a unital centrally prime C*-algebra with center $\mathcal{Z}(\mathcal{A})$. Suppose $S, T \in \mathcal{A}$,T is normal and T sim-dominates S in $\mathcal{A}$. Then $S \in C^*(\{T\} \cup \mathcal{Z}(\mathcal{A}))$. Also if S is in the algebra generated by $\{T\} \cup \mathcal{Z}(\mathcal{A})$, then T sim-dominates S in $\mathcal{A}$.*

Proof. Since T sim-dominates S, Theorem 3.2.3 yields $S \in App(T, \mathcal{A})''$. From [12]

$$App\left(C^*\left(\{T\} \cup \mathcal{Z}(\mathcal{M})\right), \mathcal{A}\right)'' = C^*\left(\{T\} \cup \mathcal{Z}(\mathcal{A})\right).$$

Thus

$$S \in App(T, \mathcal{A})'' \subset App\left(C^*\left(\{T\} \cup \mathcal{Z}(\mathcal{M})\right), \mathcal{A}\right)'' = C^*\left(\{T\} \cup \mathcal{Z}(\mathcal{A})\right).$$

□

Theorem 3.2.5. *Suppose $\mathcal{A}$ is a unital Banach algebra, $S, T \in \mathcal{A}$ and T sim-dominates S in $\mathcal{A}$. Suppose $\{W_n\}$ is an invertibly bounded sequence in $\mathcal{A}$ with $\sup_{n\in\mathbb{N}} \max\left(\|W_n\|, \|W_n^{-1}\|\right) = M$. Suppose $T' \in \mathcal{A}$ and $\|W_n^{-1} T W_n - T'\| \to 0$. Then*

1. *There exists $S' \in \mathcal{A}$ such that $\|W_n^{-1} S W_n - S'\| \to 0$.*
2. *T' sim-dominates S' in $\mathcal{A}$.*
3. *If $\varphi : \mathbb{C} \to \mathbb{C}$ is an entire function, then $S = \varphi(T) \Leftrightarrow S_1 = \varphi(T_1)$.*

Proof. Define

$$\mathcal{S} = \Big\{A \in \mathcal{B} : \{W_n^{-1} A W_n\} \text{is convergent, whenever } \{W_n\}$$
$$\text{is an invertibly bounded sequence such that} \{W_n^{-1} T W_n\} \text{ is norm convergent.}\Big\}$$

(1) Theorem 3.4 in [8] yields $\mathcal{S} = \mathrm{Appr}\,(T)''$. By Theorem 3.2.3, $S \in \mathrm{Appr}\,(T)''$. Hence $S \in \mathcal{S}$. Thus there exists $S_1 \in \mathcal{B}$ such that $\lim_{n\to\infty} \|W_n^{-1} S W_n - S_1\| = 0$.

(2) Assume by way of contradiction that T_1 does not sim-dominate S_1. Thus There exists a sequence $\{B_n\}$ in $\mathcal{A}$ and $R > 0$, such that for every $n \in \mathbb{N}$, $\|B_n^{-1}T_1B_n\| < R$, but $\|B_n^{-1}S_1B_n\| > 2^n$. For every $n, k \in \mathbb{N}$, define

$$C_{n,k} = W_k^{-1}B_nW_k.$$

Thus,

$$\left\|C_{n,k}^{-1}TC_{n,k}\right\| \leq \left\|C_{n,k}^{-1}\left(T - W_kT_1W_k^{-1}\right)C_{n,k}\right\| + \left\|C_{n,k}^{-1}W_kT_1W_k^{-1}C_{n,k}\right\|. \tag{3.1}$$

Since $W_kT_1W_k^{-1} \to T$, for every $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ such that

$$\left\|T - W_kT_1W_k^{-1}\right\| \leq \frac{1/n}{M^4\left\|B_n\right\|\left\|B_n^{-1}\right\|}, \tag{3.2}$$

for all $k \geq k_n$. Therefore

$$\begin{aligned}\left\|C_{n,k}^{-1}\left(T - W_kT_1W_k^{-1}\right)C_{n,k}\right\| &\leq \left\|C_{n,k}^{-1}\right\|\left\|T - W_kT_1W_k^{-1}\right\|\left\|C_{n,k}\right\| \qquad (3.3)\\ &= \left\|W_kB_n^{-1}W_k^{-1}\right\|\left\|T - W_kT_1W_k^{-1}\right\|\left\|W_k^{-1}B_nW_k\right\|\\ &\leq M^4\left\|B_n^{-1}\right\|\left\|B_n\right\|\left\|T - W_kT_1W_k^{-1}\right\|\\ &\leq \frac{1}{n}\frac{M^4\left\|B_n\right\|\left\|B_n^{-1}\right\|}{M^4\left\|B_n\right\|\left\|B_n^{-1}\right\|}\\ &\leq 1.\end{aligned}$$

From part (1), $W_kS_1W_k^{-1} \to S$. Thus for every $n \in \mathbb{N}$, there exists $k'_n \in \mathbb{N}$ such that $k \geq k'_n$ implies that relations (3.2) and (3.3) remain true when T and T_1 are replaced by S and S_1 in that order. Hence if $t_n = \max\{k_n, k'_n\}$, (3.2) and (3.3) remain true for pairs S, S_1 and T, T_1. Thus

$$\sup_{n\in\mathbb{N}}\left\|C_{n,t_n}^{-1}\left(S - W_{t_n}S_1W_{t_n}'^{-1}\right)C_{n,t_n}\right\| \leq 1, \tag{3.4}$$

$$\sup_{n\in\mathbb{N}}\left\|C_{n,t_n}^{-1}\left(S - W_{t_n}T_1W_{t_n}'^{-1}\right)C_{n,t_n}\right\| \leq 1. \tag{3.5}$$

Also, for every $n, k \in \mathbb{N}$,

$$\left\|C_{n,k}^{-1}W_kT_1W_k^{-1}C_{n,k}\right\| = \left\|W_kB_n^{-1}T_1B_nW_k^{-1}\right\| \leq M^2R. \tag{3.6}$$

Putting (3.1), (3.5), and (3.6) together yields

$$\sup_{n\in\mathbb{N}} \left\| C_{n,t_n}^{-1} T C_{n,t_n} \right\| \le M^2 R + 1.$$

Since T sim-dominates S,

$$\sup_{n\in\mathbb{N}} \left\| C_{n,t_n}^{-1} S C_{n,t_n} \right\| < \infty. \tag{3.7}$$

From (3.4) and (3.1)

$$\begin{aligned}
\left\| C_{n,t_n}^{-1} S C_{n,t_n} \right\| &\ge \left\| C_{n,t_n}^{-1} W_{t_n} S_1 W_{t_n}^{-1} C_{n,t_n} \right\| - \left\| C_{n,t_n}^{-1} \left(S - W_{t_n} S_1 W_{t_n}^{-1} \right) C_{n,k_n} \right\| \\
&\ge \left\| W_{t_n} B_n^{-1} S_1 B_n W_{t_n}^{-1} \right\| - 1 \\
&\ge 1/M^2 \left\| B_n^{-1} S_1 B_n \right\| - 1 \\
&\ge 2^n / M^2 - 1,
\end{aligned}$$

for every $n \in \mathbb{N}$. This contradicts (3.7) and completes the proof of part (2).

(3) Define a mapping $\pi : \text{Appr}\,(T)'' \to \text{Appr}\,(\pi(T))''$ by $\pi(A) = \lim_{n\to\infty} W_n^{-1} A W_n$. Then by Theorem 3.4 in [8], π is a bounded unital algebra isomorphism. Thus

$$S_1 = \pi(S), \text{ and} \tag{3.8}$$

$$T_1 = \pi(T). \tag{3.9}$$

Suppose $\varphi : \mathbb{C} \to \mathbb{C}$ is an entire function represented by $\phi(z) = \sum_{n=0}^{\infty} a_k z^k$. Then we have

$$\pi(\varphi(T)) = \pi(\sum_{k=0}^{\infty} a_k T^k) = \pi(\lim_{N\to\infty} \sum_{k=0}^{N} a_k T^k) = \lim_{N\to\infty} \pi(\sum_{k=0}^{N} a_k T^k) \tag{3.10}$$

$$= \lim_{N\to\infty} \sum_{k=0}^{N} a_k \pi(T)^k = \sum_{k=0}^{\infty} a_k \pi(T)^k = \varphi(\pi(T)). \tag{3.11}$$

Thus

$$\begin{aligned} S = \varphi(T) &\iff \pi(S) = \pi(\varphi(T)) && \pi \text{ is bijective} \\ &\iff \pi(S) = \varphi(\pi(T)) && \text{equations 3.10 and 3.11} \\ &\iff S_1 = \varphi(T_1). && \text{equations 3.8 and 3.9} \end{aligned}$$

□

Lemma 3.2.6. *Suppose $\mathcal{B}$ is a unital Banach algebra, $1 \in \mathcal{A} \subset \mathcal{B}$ is a closed subalgebra, and $S, T \in \mathcal{A}$. If T sim-dominates S in $\mathcal{B}$, then T sim-dominates S in $\mathcal{A}$.*

Proof. Suppose $\sup_{n\in\mathbb{N}} \|W_n^{-1}TW_n\| < R$ for an invertible sequence $\{W_n\}$ in $\mathcal{A}$ and for some $R > 0$. Since $\{W_n\}$ is an invertible sequence in $\mathcal{B}$, and T sim-dominates S in $\mathcal{B}$, $\sup_{n\in\mathbb{N}} \|W_n^{-1}SW_n\| < \infty$. It follows that T sim-dominates S in $\mathcal{A}$. □

Corollary 3.2.7. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are unital Banach algebras and $\pi : \mathcal{A} \to \mathcal{B}$ is a bounded injective unital homomorphism such that $\pi(\mathcal{A})$ is closed in $\mathcal{B}$. Suppose $S, T \in \mathcal{A}$. If $\pi(T)$ sim-dominates $\pi(S)$ in $\mathcal{B}$, then T sim-dominates S in $\mathcal{A}$.*

Proof. $\pi(\mathcal{A})$ is a unital Banach subalgebra of $\mathcal{B}$. By Lemma 3.2.6, $\pi(T)$ sim-dominates $\pi(S)$ in $\pi(\mathcal{A})$. The mapping $\pi : \mathcal{A} \to \pi(\mathcal{A})$ is an isomorphism. Thus T sim-dominates S in $\mathcal{A}$. The details are as follows. Suppose

$$\sup_{n\in\mathbb{N}} \left\|W_n^{-1}TW_n\right\| < R.$$

Then for every $n \in \mathbb{N}$,

$$\left\|\pi(W_n)^{-1}\pi(T)\pi(W_n)\right\| = \left\|\pi\left(W_n^{-1}TW_n\right)\right\| \leq \|\pi\| \left\|W_n^{-1}TW_n\right\| \leq R\,\|\pi\|\,.$$

Since $\pi(T)$ sim-dominates $\pi(S)$ in $\pi(\mathcal{A})$,

$$\sup_{n\in\mathbb{N}} \left\|\pi(W_n)^{-1}\pi(S)\pi(W_n)\right\| < \infty.$$

Thus

$$\begin{aligned}\sup_{n\in\mathbb{N}}\left\|W_n^{-1}SW_n\right\| &= \sup_{n\in\mathbb{N}}\left\|\pi^{-1}\left(\pi\left(W_n^{-1}SW_n\right)\right)\right\| \\ &\leq \sup_{n\in\mathbb{N}}\left\|\pi^{-1}\right\|\left\|\pi(W_n)^{-1}\pi(S)\pi(W_n)\right\| \\ &< \infty.\end{aligned}$$

It follows that T sim-dominates S in $\mathcal{A}$. □

Theorem 3.2.8. *Suppose $\mathcal{M}$ is a von Neumann algebra and $T \in \mathcal{M}$. Then*

$$\{S \in \mathcal{M} : T \textit{ sim-dominates } S \textit{ in } \mathcal{M}\} \subset \mathrm{AlgLat}_{1/2}(T, \mathcal{M}).$$

Proof. Suppose T sim-dominates S in $\mathcal{M}$ and $T(ran(D)) \subset ran(D)$, for some $D \in \mathcal{M}$. By lemma 3.1.11, we may assume without loss of generality that D is a positive operator. Since $T(ran(D)) \subset ran(D)$, and $D > 0$, by lemma 3.1.9

$$\sup_{\epsilon>0}\left\|(D+\epsilon)^{-1}T(D+\epsilon)\right\| \leq \|T\| + \left\|D^{-1}TD\right\| < \infty.$$

Since T sim-dominates S, and $D+\epsilon$ is invertible for any $\epsilon > 0$,

$$\sup_{\epsilon>0}\left\|(D+\epsilon)^{-1}S(D+\epsilon)\right\| < \infty.$$

It follows from Lemma 3.1.10 that $S(ran(D)) \subset ran(D)$. □

Lemma 3.2.9. *Suppose $\mathcal{A}$ is a unital Banach algebra, $S, T \in \mathcal{A}$ and T sim-dominates S in $\mathcal{A}$. Suppose $\{P_n\}$ is a bounded sequence of idempotents in $\mathcal{A}$ such that*

$$\|(1-P_n)\,TP_n\| \to 0.$$

Then

$$\|(1-P_n)\,SP_n\| \to 0.$$

Proof. Let $d_n = \|(1 - P_n)TP_n\| + 1/n$ and define

$$D_n = \frac{1}{d_n} P_n + (1 - P_n).$$

D_n is invertible in $\mathcal{A}$ for every $n \in \mathbb{N}$ and the inverse is $D_n^{-1} = d_n P_n + (1 - P_n)$. Moreover, $\{D_n^{-1} T D_n\}$ is a bounded sequence as the following computation shows.

$$\begin{aligned} \left\|D_n^{-1} T D_n\right\| &= \left\|(d_n P_n + (1 - P_n))\, T \left(\frac{1}{d_n} P_n + (1 - P_n)\right)\right\| \\ &\leq \|P_n\|^2 \|T\| + \|P_n\| \|1 - P_n\| \|T\| + \|1 - P_n\|^2 \|T\| \qquad (3.12) \\ &+ \frac{1}{d_n} \|(1 - P_n)TP_n\| \qquad (3.13) \end{aligned}$$

Since $\{P_n\}$ is a bounded sequence, all three terms in (3.12) are bounded, and the last term (3.13) is less than 1 by definition of d_n. Thus there exists $R > 0$, such that $\sup_{n\in\mathbb{N}} \|D_n^{-1} T D_n\| < R$. Since T sim-dominates S, there exists $\beta_R > 0$ such that $\sup_{n\in\mathbb{N}} \|D_n^{-1} S D_n\| < \beta_R$. Therefore

$$\begin{aligned} \beta_R \geq \left\|D_n^{-1} S D_n\right\| &= \left\|(d_n P_n + (1 - P_n))\, S \left(\frac{1}{d_n} P_n + (1 - P_n)\right)\right\| \\ &\geq \frac{1}{dn} \|(1 - P_n)SP_n\| - \|P_n S P_n + d_n P_n S(1 - P_n) + (1 - P_n)S(1 - P_n)\| . \end{aligned}$$

Thus

$$\begin{aligned} \frac{1}{dn} \|(1 - P_n)SP_n\| &\leq \beta_R + \|P_n S P_n\| + d_n \|P_n S(1 - P_n)\| + \|(1 - P_n)S(1 - P_n)\| \\ &< \infty \end{aligned}$$

$\lim_{n\to\infty} 1/d_n = \infty$, thus, $\lim_{n\to\infty} \|(1 - P_n)SP_n\| = 0$ in order for the left hand side to be bounded. $\square$

Corollary 3.2.10. *Suppose H is a Hilbert space and $S, T \in B(H)$. If T sim-dominates S in $B(H)$, then $S \in \mathrm{ApprAlgLat}\,(T)$.*

Proof. Let $\{P_n\}$ be a sequence of projections in $B(H)$ such that $\|(1-P_n)TP_n\| \to 0$. Since $\|P_n\| \leq 1$ for all $n \in \mathbb{N}$, it follows from Lemma 3.2.9 that $\lim_{n\to\infty} \|(1-P_n)SP_n\| = 0$. Thus $S \in \mathrm{ApprAlgLat}(T)$. □

Lemma 3.2.11. *Suppose X is a Banach space and $S, T \in B(X)$. If T sim-dominates S and $P \in B(X)$ is an idempotent such that $(1-P)TP = 0$, then $(1-P)SP = 0$.*

Proof. This follows from Lemma 3.2.9, with $P_n = P$, for all $n \in \mathbb{N}$. □

Theorem 3.2.12. *Suppose X is a Banach space, $S, T \in B(X)$ and T is algebraic. If T sim-dominates S in $B(X)$, then there exists a polynomial $p \in \mathbb{C}[x]$ such that $S = p(T)$.*

Proof. First we show that $S \in$ AlgLat(T). Suppose $T(M) \subset M$, where M is a closed subspace of X. Since T is algebraic, there exists a polynomial $0 \neq m \in \mathbb{C}[x]$ such that $m(T) = 0$. Thus $q(T) = m(T)u(T) + r(T) = r(T)$ with $deg(r) < deg(m)$, for any $q \in \mathbb{C}[x]$. Thus $\{p(T) : p \in \mathbb{C}[x]\}$ is a finite dimensional subspace of $B(X)$. For any $x \in X$, define

$$M_x = \{p(T)x : p \in \mathbb{C}[x]\}.$$

Note that M_x is a finite dimensional invariant subspace for T, and

$$M = \bigcup_{x\in M} M_x.$$

By a lemma, we may assume M_x is the range of some bounded idempotent $P_x \in B(X)$. Thus $(1-P_x)TP_x = 0$. By Lemma 3.2.11 $(1-P_x)SP_x = 0$. Thus M_x is T-invariant and

$$S(M) = S\left(\bigcup_{x\in M} M_x\right) \subset \bigcup_{x\in M} S(M_x) \subset \bigcup_{x\in M} M_x \subset M.$$

This shows that M is S-invariant and $S \in \mathrm{AlgLat}(T)$. Moreover, $S \in \{T\}''$ by lemma 3.1.12, and therefore $ST = TS$. A theorem of Hadwin and Nordgren [17] now implies that $S = p(T)$ for some polynomial $p \in \mathbb{C}[x]$.

□

Theorem 3.2.13. *Suppose $\mathcal{A}$ is a finite-dimensional semisimple unital Banach algebra and $T \in \mathcal{A}$. Then*

1. *If $S \in \mathcal{A}$ and T sim-dominates S in $\mathcal{A}$, there is a polynomial $p(z) = c_0 + c_1 z + \cdots + c_n z^n$ with $c_0, \ldots, c_n \in \mathcal{Z}(\mathcal{A})$ such that $S = p(T)$.*
2. *$\{S \in \mathcal{A} : T$ sim-dominates S in $\mathcal{A}\}$ is the algebra generated by $\{T\} \cup \mathcal{Z}(\mathcal{A})$.*

Proof. 1) Artin's theorem implies that

$$\mathcal{A} = \bigoplus_{k=1}^{N} \mathbb{M}_{n_k}(D_k),$$

where D_k are finite dimensional division algebras, and $n_k, N \in \mathbb{N}$. Since D_k is finite dimensional, it is a Banach algebra, and a Banach algebra that is a division ring is isomorphic to $\mathbb{C}$. Thus we may write

$$\mathcal{A} = \bigoplus_{k=1}^{N} \mathbb{M}_{n_k}(\mathbb{C}).$$

Let

$$T = \bigoplus_{i=k}^{N} T_k, \quad S = \bigoplus_{k=1}^{N} S_k.$$

It follows from Theorem 3.2.12 that $S_k = P_k(T_k)$ for a polynomial $P_k \in \mathbb{C}[x]$ for $k \in \mathbb{N}, 1 \leq k \leq N$. Let $m = \max_{1 \leq k \leq N} deg(P_k)$. We can write

$$S_k = P_k(T_k) = a_{k,0} + a_{k,1} T_k + \cdots + a_{k,m} T_k^m,$$

where $a_{k,j} = 0$ for $j > deg(P_k)$.

Let $I_{n_k} \in \mathbb{M}_{n_k}(\mathbb{C})$ be the identity matrix For every $j \in \mathbb{N}$, $1 \leq j \leq m$, and define

$$A_j = \bigoplus_{k=1}^{N} a_{k,j} I_k.$$

Since $\mathcal{Z}\left(\mathcal{M}_{n_k}(\mathbb{C})\right) = \mathbb{C} I_k$, it follows that $A_j \in \mathcal{Z}(\mathcal{M})$ for every $j \in \mathbb{N}, 1 \leq j \leq m$. Thus $P(x) = \sum_{j=0}^{m} A_j x^j$ is a polynomial over the center of $\mathcal{M}$ and

$$S = \bigoplus_{k=1}^{N} P_k(T_k) = \bigoplus_{k=1}^{N} \sum_{j=1}^{m} a_{k,j} T_k^j = \sum_{j=1}^{m} \bigoplus_{k=1}^{N} a_{k,j} T_k^j = \sum_{j=1}^{m} \bigoplus_{k=1}^{N} a_{k,j} I_k T^j = \sum_{j=1}^{m} A_j T^j = P(T).$$

(2) This follows easily from part 1 and Lemma 3.2.2. □

Theorem 3.2.14 (Kadison[19]). *If $\mathcal{M}$ is a type II_∞ or type III factor von Neumann algebra acting on a separable Hilbert space, then there exists a unital isometric $*$-homomorphism $\rho : B\left(\ell^2\right) \to \mathcal{M}$. Moreover, if $\mathcal{M}$ is a II_∞ factor with faithful normal tracial weight τ, we can choose ρ so that for every $A \in B\left(\ell^2\right)$,*

$$\rho\left(\tau\left(A^* A\right)\right) = \infty.$$

Corollary 3.2.15. *Suppose $\mathcal{M}$ and ρ are as in Theorem 3.2.14, $X, Y \in \mathcal{M}$, X sim-dominates Y in $\mathcal{M}$, and X is not algebraic. If $X_1, Y_1 \in \rho\left(B\left(\ell^2\right)\right)$ and (X, Y) is approximately similar to (X_1, Y_1), then there exists an entire function $\varphi : \mathbb{C} \to \mathbb{C}$ such that $Y = \varphi(X)$.*

Proof. Since (X, Y) is approximately similar to (X_1, Y_1) and X sim-dominates $Y \in \mathcal{M}$, Theorem 3.2.5 implies that X_1 sim-dominates Y_1 in $\mathcal{M}$. There exist $S, T \in B(\ell^2)$ such that $X_1 = \rho(T)$ and $Y_1 = \rho(S)$. Since sim-domination is preserved under isomorphism, T sim-dominates S in $B(\ell^2)$. It follows from Theorem 3.1.2 that there exists an entire function $\varphi : \mathbb{C} \to \mathbb{C}$ such that $S = \phi(T)$. Thus

$$Y_1 = \rho(S) = \rho(\phi(T)) = \phi(\rho(T)) = \phi(X_1).$$

Thus by Theorem 3.2.5, $Y = \phi(X)$. □

Theorem 3.2.16 (Shen, Hadwin [10])**.** *Suppose $\mathcal{M}$ is a factor von Neumann algebra and $\mathcal{A}$ is a countably generated unital AH C*-subalgebra of $\mathcal{M}$. Then $\mathcal{A} = Appr\,(\mathcal{A}, \mathcal{M})''$.*

Corollary 3.2.17. *Suppose $\mathcal{M}$ is a factor von Neumann algebra, $T \in \mathcal{M}$ and $C^*(\mathcal{T})$ is AH. Then*

$$\{S \in \mathcal{M} : T \textit{ sim-dominates } S \textit{ in } \mathcal{M}\} \subset C^*(T).$$

Proof. $C^*(T) = \mathrm{Appr}\,(C^*(T), \mathcal{M})''$ by Theorem 3.2.16. If T sim-dominates S in $\mathcal{M}$, then by Theorem 3.2.3, $S \in \mathrm{Appr}\,(T, \mathcal{M})''$. But $\mathrm{Appr}\,(T, \mathcal{M})'' \subset \mathrm{Appr}\,(C^*(T), \mathcal{M})''$. Thus $S \in C^*(T)$. □

Theorem 3.2.18 (Ding, Hadwin [13])**.** *Suppose $\mathcal{M}$ is a type III factor von Neumann algebra acting on a separable Hilbert space, $\mathcal{A}$ is a separable unital AH C*-subalgebra of $\mathcal{M}$, and $\pi : \mathcal{A} \to \mathcal{M}$ is an injective unital $*$-homomorphism. Then there exists a sequence $\{U_n\}$ of unitary operators in $\mathcal{M}$ such that for every $A \in \mathcal{A}$, $\lim_{n\to\infty} \|U_n^* A U_n - \pi(A)\| = 0$.*

Theorem 3.2.19 (Li, Shen, Shi [14])**.** *Suppose $\mathcal{M}$ is a type II_∞ factor von Neumann algebra with a normal tracial weight τ acting on a separable Hilbert space, $\mathcal{A}$ is a separable nuclear unital C*-subalgebra of $\mathcal{M}$, and $\pi : \mathcal{A} \to \mathcal{M}$ is a unital $*$-homomorphism. Suppose for every $0 \neq A \in \mathcal{A}, \tau(A^*A) = \tau(\pi(A^*A)) = \infty$. Then there exists a sequence $\{U_n\}$ of unitary operators in $\mathcal{M}$ such that for every $0 \neq A \in \mathcal{A}$, $\lim_{n\to\infty} \|U_n^* A U_n - \pi(A)\| = 0$.*

Theorem 3.2.20. *Suppose $\mathcal{M}$ is a type III factor von Neumann algebra, $S, T \in \mathcal{M}$, T sim-dominates S in $\mathcal{M}$ and T is not algebraic. Suppose $T_1 \in \mathcal{M}$ such that*

1. *T_1 is approximately similar to T and*
2. *$C^*(T_1)$ is AH.*

Then there exists an entire function $\varphi : \mathbb{C} \to \mathbb{C}$ such that $S = \varphi(T)$.

Proof. By Theorem 3.2.5, there exists $S_1 \in \mathcal{M}$ such that (S,T) is approximately similar to (S_1, T_1), and T_1 sim-dominates S_1 in $\mathcal{M}$. By Corollary 3.2.17, $S_1 \in C^*(T_1)$. There exists an injective unital *-homomorphism $\gamma : C^*(T) \to B(\ell^2)$ by the the GNS construction. Let $\rho : B(\ell^2) \to \mathcal{M}$ be as in Theorem 3.2.14. It follows that the composition map $\pi = \rho \circ \gamma : C^*(T_1) \to \mathcal{M}$ is an injective unital *-homomorphism. Hence by Theorem 3.2.18, there exists a sequence $\{U_n\}$ of unitaries in $C^*(T_1)$, such that

$$\lim_{n\to\infty} \|U_n^* T_1 U_n - \pi(T_1)\| = 0, \text{ and,}$$
$$\lim_{n\to\infty} \|U_n^* S_1 U_n - \pi(S_1)\| = 0.$$

Thus (T_1, S_1) is approximately similar to $(\pi(T_1), \pi(S_1))$. Since $\pi(T_1), \pi(S_1) \in \rho(B(\ell^2))$, Corollary 3.2.15 implies that $S_1 = \varphi(T_1)$ for some entire function $\varphi : \mathbb{C} \to \mathbb{C}$. Thus $S = \varphi(T)$, By Theorem 3.2.5.

□

Theorem 3.2.21. *Suppose $\mathcal{M}$ is a II_∞ factor von Neumann algebra with a faithful normal tracial weight τ. Suppose $S, T \in \mathcal{M}$, T sim-dominates S in $\mathcal{M}$, T is not algebraic, (S,T) is approximately similar to (S_1, T_1) in $\mathcal{M}$ and*

1. *Either*
   - *(a) $C^*(S_1, T_1)$ is nuclear, or*
   - *(b) $C^*(T_1)$ is AH*
2. *For every $0 \neq A \in C^*(S_1, T_1)$, $\tau(A^*A) = \infty$.*

Then there exists an entire function $\varphi : \mathbb{C} \to \mathbb{C}$ such that $S = \varphi(T)$.

Proof. By Theorem 3.2.5, T_1 sim-dominates S_1. If $C^*(T_1)$ is AH, $S_1 \in C^*(T_1)$ by Corollary 3.2.17. Since every AH C*-algebra is nuclear, if $1\,(b)$ holds, then $1\,(a)$ holds. By Theorem 3.2.14, there is a unital isometric $*$-homomorphism $\rho : B(\ell^2) \to \mathcal{M}$ such that, for every $0 \neq D \in B(\ell^2)$, $\tau(\rho(D)^* \rho(D)) = \infty$. By GNS construction, there exists a unital isometric $*$-homomorphism $\gamma : C^*(S_1, T_1) \to B(\ell^2)$. Thus, applying Theorem 3.2.19 to $\pi = \rho \circ \gamma$, provides a sequence $\{U_n\}$ of unitary operators in $\mathcal{M}$ such that

$$\|U_n^* T_1 U_n - (\rho \circ \gamma)(T_1)\| \to 0$$

and

$$\|U_n^* S_1 U_n - (\rho \circ \gamma)(S_1)\| \to 0.$$

Following the proof of Theorem 3.2.20, we see that there exists an entire function $\varphi : \mathbb{C} \to \mathbb{C}$ such that $S = \varphi(T)$. $\square$

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