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BUBBLING SOLUTIONS FOR SUPERCRITICAL PROBLEMS ON MANIFOLDS

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Abstract. Let $(\mathcal{M}, g)$ be a $n-$dimensional compact Riemannian manifold without boundary and Γ be a non degenerate closed geodesic of $(\mathcal{M}, g)$. We prove that the supercritical problem

$$-\Delta_g u + hu = u^{\frac{n+1}{n-3} \pm \epsilon},\ u > 0,\ \text{in } (\mathcal{M}, g)$$

has a solution that concentrates along Γ as ϵ goes to zero, provided the function h and the sectional curvatures along Γ satisfy a suitable condition. A connection with the solution of a class of periodic O.D.E.'s with singularity of attractive or repulsive type is established.

1. Introduction and statement of main results

We deal with the semilinear elliptic equation

$$-\Delta_g u + hu = u^{p-1},\ u > 0,\ \text{in } (\mathcal{M}, g) \tag{1.1}$$

where $(\mathcal{M}, g)$ is a $n-$dimensional compact Riemannian manifold without boundary, h is a C^1-real function on $\mathcal{M}$ such that $-\Delta_g + h$ is coercive and $p > 2$.

For any $p \in (2, 2^*_n)$, where $2^*_n := \frac{2n}{n-2}$ if $n \geqslant 3$ and $2^*_n := +\infty$ if $n = 2$, problem (1.1) has a solution, which can be found by minimization of

$$\mathcal{I}_p(u) = \frac{\int\limits_{\mathcal{M}} \left(|\nabla_g u|^2 + hu^2\right) d\sigma_g}{\left(\int\limits_{\mathcal{M}} |u|^p d\sigma_g\right)^{2/p}}$$

over $H^1_g(\mathcal{M}) \setminus \{0\}$, using the compactness of the embedding $H^1_g(\mathcal{M}) \hookrightarrow L^p_g(\mathcal{M})$.

In the critical case, i.e. $p = 2^*_n$, the situation turns out to be more delicate. In particular, the existence of solutions is related to the position of the potential h with respect to the geometric potential $h_g := \frac{m-2}{4(m-1)} R_g$, where R_g is the scalar curvature of the manifold.

If $h \equiv h_g$, then problem (1.1) is referred to as the Yamabe problem [21] and it has always a solution. After Trudinger [19] discovered a gap in the argument in [21] and gave a proof under some conditions on $(\mathcal{M}, g)$, Aubin [1, 2] showed that whenever $Q(\mathcal{M}, g) < Q(S^n, g_0)$, where (S^n, g_0) is the standard sphere and

$$Q(\mathcal{M}, g) := \inf_{u \in H^1_g(\mathcal{M}) \setminus \{0\}} I_{2^*_n}(u),$$

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there is a solution to the problem, and proved that this holds if $n \geq 6$ and $(\mathcal{M}, g)$ is not locally conformally flat. Finally, Schoen [17] gave a proof in full generality using the Positive Mass Theorem [18].

When $h < h_g$ somewhere in M, existence of a solution is guaranteed by a minimization argument, arguing as in Aubin [1,2]. The situation is extremely delicate when $h \geqslant h_g$ everywhere in $\mathcal{M}$, because blow-up phenomena can occur as pointed out by Druet in [8,9].

The supercritical case $p > 2^*_n$ is even more difficult to deal with. A first result in this direction is a perturbative result due to Micheletti, Pistoia and Vétois [14]. They consider the almost critical problem (1.1) when $p = 2^*_n \pm \epsilon$ with $\epsilon > 0$. If $p = 2^*_n - \epsilon$ the problem (1.1) is slightly subcritical and if $p = 2^*_n + \epsilon$ the problem (1.1) is slightly supercritical. They prove the following results.

Theorem 1.1. *[Micheletti, Pistoia and Vétois [14]] Assume $n \geqslant 6$ and $\xi_0 \in M$ is a non degenerate critical point of $h - \frac{m-2}{4m} R_g$. Then*

(i) *if $h(\xi_0) > \frac{n-2}{4n} R_g(\xi_0)$ then the slightly subcritical problem (1.1) with $p = 2^*_n - 1 - \epsilon$, has a solutions u_ϵ which concentrates at ξ_0 as $\epsilon \to 0$,*

(ii) *if $h(\xi_0) < \frac{n-2}{4n} R_g(\xi_0)$ then the slightly supercritical problem (1.1) with $p = 2^*_n - 1 - \epsilon$, has a solutions u_ϵ which concentrates at ξ_0 as $\epsilon \to 0$.*

Now, for any integer $0 \leqslant k \leqslant n-3$ let $2^*_{n,k} = \frac{2(n-k)}{n-k-2}$ be the $(k+1)-$*st critical exponent.* We remark that $2^*_{n,k} = 2^*_{n-k,0}$ is nothing but the critical exponent for the Sobolev embedding $H^1_h(\mathcal{N}) \hookrightarrow L^q_h(\mathcal{N})$ in a compact $(n-k)-$dimensional Riemannian manifold $(\mathcal{N}, h)$. In particular, $2^*_{n,0} = \frac{2n}{n-2}$ is the usual Sobolev critical exponent.
We can summarize the results proved by Micheletti, Pistoia and Vétois just saying that problem (1.1) when $p \to 2^*_{n,0}$ (i.e. $k = 0$) has positive solutions blowing-up at points. Note that a point is a $0-$dimensional manifold.changed !

A natural question arises:
does problem (1.1) have solutions blowing-up at $k-$dimensional submanifolds when $p \to 2^*_{n,k}$?

In the present paper, we give a positive answer when $k = 1$. More precisely, we prove that if $p \to 2^*_{n,1}$ problem (1.1) has a solution which concentrates along a geodesic Γ of the manifold provided h satisfies a suitable condition. Let us state our main result.

We consider the problem (1.1) with $p = 2^*_{n,1} \pm \epsilon$ and $\epsilon > 0$, i.e.

$$-\Delta_g u + hu = u^{\frac{n+1}{n-3} \pm \epsilon}, \ u > 0 \text{ in } (\mathcal{M}, g) \tag{1.2}$$

We will say that problem (1.2) is slightly $2nd-$*supercritical* if $p = 2^*_{n,1} + \epsilon$ and it is slightly $2nd-$*subcritical* if $p = 2^*_{n,1} - \epsilon$.

In order to state our main result, we need to introduce some geometric notation. Let Γ be a closed nontrivial simple geodesic in $\mathcal{M}$. Given $\xi \in \Gamma$ there is a natural splitting $T_\xi M = T_\xi \Gamma \oplus N_\xi \Gamma$ into the tangent and normal bundle over Γ. It is useful to introduce a local system of coordinates near Γ. Let $\gamma : [0, 2\ell] \to \mathcal{M}$ be an arclenght parametrization of Γ, where 2ℓ is the lenght of Γ. We denote by E_0 a unit tangent vector to Γ. In a neighborhood of a point ξ of Γ we give an orthonormal basis $E_1, \dots, E_N$ of $N_q\Gamma$. We can assume that the E_i's are parallel along Γ, i.e. $\nabla_{E_0} E_i = 0$ for any $i = 1, \dots, N$. The geodesic condition for Γ translates into the condition $\nabla_{E_0} E_0 = 0$. Here ∇ is the connection associated with the metric g. Moreover, the non degeneracy of Γ is equivalent to say that the linear equation

$$\mathcal{J}\phi := \nabla^2_{E_0}\phi + R(\phi, E_0)E_0 = 0 \text{ has only the trivial solution on all of } \Gamma. \tag{1.3}$$

Here $\mathcal{J}$ is the Jacobi operator on Γ corresponding to the second variation of the length functional on curves. For a generic metric g on $\mathcal{M}$ it is well known that all closed geodesics are non degenerate. REFERENCE?
To parametrize a neighborhood of a point of Γ in M we define the *Fermi coordinates*

$$F(x_0, x_1, \dots, x_N) = \exp_{\gamma(x_0)}\left(\sum_{i=1}^{N} x_i E_i(x_0)\right) \tag{1.4}$$

where $\exp_{\gamma(x_0)}$ is the exponential map in $\mathcal{M}$ through the point $\gamma(x_0)$.

Let us introduce the function (see also (4.20))

$$\sigma(x_0) = h(x_0) - \frac{(n-3)}{4(n-2)}\left[R_g(x_0) - (n-1)Ric(\dot\gamma(x_0), \dot\gamma(x_0))\right] \tag{1.5}$$

where R_g is the scalar curvature and Ric denotes the Ricci tensor.erased in normal coordinates

Let $a_n := \frac{2(n-2)}{(n-3)(n+1)}$ and $b_n := \frac{(n-3)^2(n-5)}{4(n+1)}$. erased " (see (4.16) and Remark (4.1))". I don't looking ahead is useful here We introduce the periodic ODE problem

$$\begin{cases} -\ddot\mu + a_n\sigma\mu - \dfrac{b_n}{\mu} = 0 & \text{in } [0, 2\ell] \\ \mu > 0 & \text{in } [0, 2\ell] \\ \mu(0) = \mu(2\ell),\ \dot\mu(0) = \dot\mu(2\ell) \end{cases} \tag{1.6}$$

which has a *singularity of attractive type* at the origin and the periodic ODE problem

$$\begin{cases} -\ddot\mu + a_n\sigma\mu + \dfrac{b_n}{\mu} = 0 & \text{in } [0, 2\ell] \\ \mu > 0 & \text{in } [0, 2\ell] \\ \mu(0) = \mu(2\ell),\ \dot\mu(0) = \dot\mu(2\ell) \end{cases} \tag{1.7}$$

which has a *singularity of repulsive type* at the origin.

Solvability of the slightly $2nd-$*subcritical* problem is strictly related with solvability of (1.6) with *attractive singularity*, while solvability of the slightly $2nd-$*supercritical* problem is strictly related with solvability of (1.7) with *repulsive singularity.* We remark that in the subcritical side the assumption $\sigma(s) > 0$ for any $s \in [0, \ell]$ is enough to find a solution to problem (1.6). In this case, using standard arguments, the solution is just a minimizer of the energy. The supercritical side turns out to be more difficult and the only existence result for problem (1.7) was obtained by del Pino, Manásevich and Montero in [4] when $\sigma(s) < 0$ for any $s \in [0, \ell]$ provided some extra non-resonance conditions are satisfied (see also Proposition 2.1).

As usual in this kind of problem, we also need to assume a gap condition of the form

$$|\epsilon k^2 - \kappa^2| > \nu\sqrt{\epsilon}, \qquad k = 1, 2, \dots\dots \tag{1.8}$$

where $\kappa > 0$ is given explicitly in Lemma 6.2 and ν is positive.

Now we can state our main result.

Theorem 1.2. *Let $n \geq 8$. Let Γ be a simple closed, non degenerate geodesic of $\mathcal{M}$ (see* (1.3)*).*

(i) *Assume the problem* (1.6) *has a non degenerate positive solution* μ_0. *Then, for any* $\nu > 0$ *there exists* $\epsilon_0 > 0$ *such that for any* $\epsilon \in (0, \epsilon_0)$ *which satisfies condition* (1.8)*, the slightly* $2nd-$*subcritical problem* (1.2) *with* $p = 2^*_{n,1} - 1 - \epsilon$*, has a solution* u_ϵ *that concentrates along* Γ *as* $\epsilon \to 0$.

(ii) *Assume the problem* (1.7) *has a non degenerate positive solution* μ_0. *Then, for any* $\nu > 0$ *there exists* $\epsilon_0 > 0$ *such that for any* $\epsilon \in (0, \epsilon_0)$ *which satisfies condition* (1.8)*, the slightly* $2nd-$*supercritical problem* (1.2) *with* $p = 2^*_{n,1} - 1 + \epsilon$*, has a solution* u_ϵ *that concentrates along* Γ *as* $\epsilon \to 0$.

Moreover, the solution u_ϵ can be described in Fermi coordinates as follows:

$$u_\epsilon(x_0, x) = \mu_\epsilon^{-\frac{N-2}{2}} w\left(\mu_\epsilon^{-1}(x - d_\epsilon)\right) + o(1),$$

where

$$\mu_\epsilon(x_0) \sim \sqrt{\epsilon}\mu_0(x_0) \text{ and } d_{\epsilon_k}(x_0) \sim \epsilon d_k(x_0), k = 1, \dots, N,$$

and μ_0 solves either problem (1.6) in the slightly $2nd-$*subcritical* case or problem (1.7) in the slightly $2nd-$*supercritical* case, the d_j's are smooth functions of x_0 and w is the standard bubble

$$w(y) = c_N \frac{1}{(1 + |y|^2)^{\frac{N-2}{2}}}, \qquad y \in \mathbb{R}^N, \qquad c_N = [N(N-2)]^{\frac{N-2}{4}}, \tag{1.9}$$

which is the radial solution of the critical problem $\Delta w + w^p = 0$ in $\mathbb{R}^N$, with $N = n - 1$.

Since the existence of solutions to singular problems (1.6) or (1.7) plays a crucial role in the construction of the solution, in particular in the choice of the concentration parameter μ_ϵ, it is important to point out that existence of solutions to problems (1.6) or (1.7) is strictly linked with the sign of the function σ defined in (1.5), as it is showed in the following Theorem, whose proof is given in Section 2.

Theorem 1.3. *If*

$$\min_{x_0 \in \mathbb{R}} \sigma(x_0) > 0,$$

then problem (1.6) *has a non degenerate solution.*

If $h^* \in C^2(M)$ *is such that*

$$-\left(\frac{(k+1)\pi}{2\ell}\right)^2 < \min_{x_0 \in \mathbb{R}} \sigma_{h^*}(x_0) \leqslant \max_{t \in \mathbb{R}} \sigma_{h^*}(x_0) < -\left(\frac{k\pi}{2\ell}\right)^2 < 0,$$

then for most functions $h \in C^2(M)$ *with* $\|h - h^*\|_{C^0(M)} \leqslant r$*, provided* r *is small enough, the problem* (1.7) *has a non degenerate solution.*

As far as we know, Theorem 1.2 is the first result about existence of solutions to (1.1) which concentrate along geodesic of the manifold M when the exponent p approaches the $2nd-$critical exponent from above. Indeed, in the Euclidean setting, del Pino, Musso and Pacard in [6] built bubbling solutions for a Dirichlet problem when the exponent is close to but less than the second critical exponent. Solutions concentrating in higher dimensional sets and the gap condition have been found in elliptic problems in the Euclidean setting. We mention among, among many results, [10–13] for a Neumann singular perturbation problem and [3] for a Schödinger equation in the plane.

It would be interesting to find a geometric interpretation to problem (1.2). We only observe that the geometric potential

$$\Omega_\Gamma(x_0) := \frac{(n-3)}{4(n-2)}\left[R_g(x_0) - (n-1)Ric(\dot\gamma(x_0), \dot\gamma(x_0))\right]$$

introduced in (1.5) when Γ reduces to a point x_0 is nothing but the usual geometric potential $\frac{(n-2)}{4(n-1)}R_g(x_0)$ which appears in the Yamabe problem. erased "So it seems that when ϵ is zero problem (1.2) is the natural extension to higher critical exponents to the Yamabe problem." I prefer to leave the reader this type of conclusion.

We conjecture that our result can be extended to higher $k-$dimensional minimal submanifolds Γ of $\mathcal{M}$. Indeed, arguments developed by Del Pino, Mamhoudi and Musso in [5] in the Euclidean setting for a Neumann problem could also be applied to equation (1.1). More precisely, we could consider a supercritical problem

$$-\Delta_g u + hu = u^{\frac{m-k+2}{m-k-2}\pm\epsilon},\ u > 0,\ \text{in } (M, g)$$

and we could find conditions on h such that it possesses solutions which concentrate along Γ as ϵ goes to zero. It would interesting to determine the function σ_Γ (the analogue of the function σ introduced in (1.5)) whose sign determines the existence of solutions either to the supercritical case or to the subcritical case.

The proof of our result relies on the infinite-dimensional reduction erased "firstly", others were the first developed by del Pino, Kowalczyk and Wei in [3] and successively adapted by del Pino, Musso and Pacard in [6] to study a problem quite similar to our problem

$$-\Delta u = u^{\frac{m+1}{m-3}-\epsilon} \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega,$$

where Ω is a bounded smooth domain in $\mathbb{R}^m$. We omit many details in several steps of the proof, because they can be carried out, up to some minor modifications, as in [6]. However there is an important difference with respecto to [6] concerning the scaling parameter μ_ϵ, whose choice is crucial for building the solution. The difference is that the extra term $\frac{1}{\mu}$ here is the main order term, see (4.11), and leads to the ODEs (1.6) and (1.7), while in [6] it appears at a higher order.changed the wording

The paper is organized as follows. In Section 2 we study the singular problems (1.6) and (1.7). In Section 3 we build the approximate solution close to the geodesic and in Section 4 we estimate the error. Then, in Section 5 we reduce the problem to a suitable infinite dimensional set of parameters and in Section 6 we study the reduced problem. Section 7 is devoted to the study of a linear problem.

Notation

- For sums we use the standard convention of summing terms where repeated indices appear.
- We will denote by $L^\infty_{2\ell}(\mathbb{R})$, $C^0_{2\ell}(\mathbb{R})$ and $C^2_{2\ell}(\mathbb{R})$ the Banach space of $2\ell-$periodic L^∞, C^0 and C^2 functions, respectively. We will set $\|u\|_\infty := \sup\limits_{\mathbb{R}} |u|$, for any $2\ell-$periodic bounded function u.

2. A periodic ODE with repulsive or attractive singularity

Let us consider the periodic boundary value problem

$$\begin{cases} -\ddot{\mu}+\sigma\mu-\dfrac{c}{\mu}=0 \quad \text{in } [0,2\ell] \\ \mu>0 \quad \text{in } [0,2\ell] \\ \mu(0)=\mu(2\ell),\ \dot{\mu}(0)=\dot{\mu}(2\ell) \end{cases} \tag{2.1}$$

where $c\in\mathbb{R}$ and $\sigma\in C^0_{2\ell}(\mathbb{R})$. The following existence result holds true.

Proposition 2.1. *Assume either*

$$\min_{t\in\mathbb{R}}\sigma(t)>0 \text{ and } c>0 \tag{2.2}$$

or

$$-\left(\frac{(k+1)\pi}{2\ell}\right)^2<\min_{t\in\mathbb{R}}\sigma(t)\leqslant\max_{t\in\mathbb{R}}\sigma(t)<-\left(\frac{k\pi}{2\ell}\right)^2<0 \text{ and } c<0 \tag{2.3}$$

for some integer k. Then problem (2.1) has a periodic solution $\mu_0\in C^2_{2\ell}(\mathbb{R})$.

Proof. If (2.2) holds, the claim follows by standard arguments and if (2.3) holds the claim follows by Theorem 1.1 of [4]. □

Let us consider the linearization of problem (2.1) around μ_0, namely the linear periodic boundary value problem

$$\begin{cases} -\ddot{\mu}+\left(\sigma+\dfrac{c}{\mu_0^2}\right)\mu=0 \quad \text{in } [0,2\ell] \\ \mu(0)=\mu(2\ell),\ \dot{\mu}(0)=\dot{\mu}(2\ell) \end{cases} \tag{2.4}$$

The solution μ_0 is non degenerate if and only if the problem (2.4) has only the trivial solution.

Proposition 2.2. (i) *If (2.2) holds, then the solution μ_0 is non degenerate.*

(ii) *Let $\sigma^*\in C^0_{2\ell}(\mathbb{R})$ and $c\in\mathbb{R}$ as in (2.3). The set*

$$\{\sigma\in B(\sigma^*,r)\ :\ \text{all the positive solutions of (2.1) are nondegenerate}\}$$

is a dense subset of the ball $B(\sigma^*,r):=\left\{\sigma\in C^0_{2\ell}(\mathbb{R})\ :\ \|\sigma-\sigma^*\|_\infty\leqslant r\right\}$ provided the radius r is small enough.

Proof. (i) follows immediately by the maximum principle.
Let us prove (ii). We shall use the following abstract transversality theorem previously used by Quinn [15], Saut and Temam [16] and Uhlenbeck [20].

Theorem 2.3. *Let X,Y,Z be three Banach spaces and $U\subset X$, $V\subset Y$ open subsets. Let $F:U\times V\to Z$ be a $C^\alpha-$map with $\alpha\geqslant 1$. Assume that*

- (ι) *for any $y\in V$, $F(\cdot,y):U\to Z$ is a Fredholm map of index l with $l\leqslant\alpha$;*
- ($\iota\iota$) *0 is a regular value of F, i.e. the operator $F'(x_0,y_0):X\times Y\to Z$ is onto at any point (x_0,y_0) such that $F(x_0,y_0)=0$;*
- ($\iota\iota\iota$) *the map $\pi\circ i:F^{-1}(0)\to Y$ is $\sigma-$proper, i.e. $F^{-1}(0)=\cup_{\eta=1}^{+\infty}C_\eta$ where C_η is a closed set and the restriction $\pi\circ i_{|C_\eta}$ is proper for any η; here $i:F^{-1}(0)\to Y$ is the canonical embedding and $\pi:X\times Y\to Y$ is the projection.*

Then the set $\Theta:=\{y\in V\ :\ 0 \text{ is a regular value of } F(\cdot,y)\}$ is a residual subset of V, i.e. $V\setminus\Theta$ is a countable union of closet subsets without interior points.

In our case the C^2- function F is defined by

$$F : C^2_{2\ell}(\mathbb{R}) \times C^0_{2\ell}(\mathbb{R}) \to C^0_{2\ell}(\mathbb{R}), \quad F(\mu,\sigma) := -\ddot{\mu} + \sigma\mu - \frac{c}{\mu},$$

$X = C^2_{2\ell}(\mathbb{R})$ and $U = \{\mu \in C^2_{2\ell}(\mathbb{R}) \ : \ \min_{\mathbb{R}} \mu > 0\}$, $Y = Z = C^0_{2\ell}(\mathbb{R})$ and $V = B(\sigma^*, r)$ where r is small enough so that condition (2.3) holds for any $\sigma \in V$.
It is not difficult to check that for any $\sigma \in V$ the map $\mu \to F(\mu,\sigma)$ is a Fredholm map of index 0 and then assumption (ι) holds. Let us prove assumption $(\iota\iota)$. We fix $(\mu_0,\sigma_0) \in U \times V$ such that $F(\mu_0,\sigma_0) = 0$. The derivative $D_\sigma F(\mu_0,\sigma_0) : C^0_{2\ell}(\mathbb{R}) \to C^0_{2\ell}(\mathbb{R})$ is the linear map defined by $D_\sigma F(\mu_0,\sigma_0)[\sigma] = \sigma\mu_0$ and it is surjective, because $\mu_0 > 0$.
As far as it concerns assumption $(\iota\iota\iota)$, we have that

$$F^{-1}(0) = \cup_{m=1}^{+\infty} \left\{ (C_m \times B_m) \cap F^{-1}(0) \right\}$$

where

$$C_m = \left\{ \mu \in C^2_{2\ell}(\mathbb{R}) \ : \ \frac{1}{m} \leqslant \min_{\mathbb{R}} \mu \leqslant \max_{\mathbb{R}} \mu \leqslant m \right\} \text{ and } B_m = \overline{B\left(\sigma^*, r - \frac{1}{m}\right)}.$$

We can show that the restriction $\pi \circ i_{|C_m}$ is proper, namely if the sequence $(\sigma_n) \subset B_m$ converges to σ and the sequence $(\mu_n) \subset C_m$ is such that $F(\mu_n,\sigma_n) = 0$ then there exists a subsequence of (μ_n) which converges to $\mu \in C_m$ and $F(\mu,\sigma) = 0$.
That concludes the proof. □

Proof of Theorem 1.3. It follows immediately by Proposition 2.1 and Proposition 2.2. □

3. Construction of the approximate solution close to the geodesic

This section is devoted to the construction of an approximation for a solution to the problem (1.2) in a neighborhood of the geodesic.

3.1. **The problem near to the geodesic.** Let us consider the system of Fermi coordinates (x_0, x) introduced in (1.4). In this language the geodesic Γ is represented by the x_0- axis. We recall that x_0 denotes the arclenght of the curve, 2ℓ represent the total length of the geodesic and $x = (x_1,\ldots,x_N) \in \mathbb{R}^N$. Let us introduce a neighborhood of the geodesic Γ in this system of coordinates

$$D := \left\{ (x_0,x) \in \mathbb{R} \times \mathbb{R}^N \ : \ x_0 \in [-\ell,\ell], \ |x| < \hat{\delta} \right\}, \tag{3.1}$$

where $\hat{\delta} > 0$ is a fixed small number. Then for a function defined in D we write

$$\tilde{u}(x_0,x) = u(F(x_0,x))$$

and we extend $\tilde{u}$ in a satisfying the following periodicity condition

$$\tilde{u}(2\ell,x) = \tilde{u}(0,Ax)$$

where $A = (a_{ij})$ is the invertible matrix defined by the requirement

$$E_i(2\ell) = \sum_{j=1}^N a_{ji} E_j(0). \tag{3.2}$$

Therefore, if u solves equation (1.2) in the neighborhood D of the geodesic, then $\tilde{u}$ solves

$$\begin{cases} \partial_{00}\tilde{u} + \Delta_x \tilde{u} + B(\tilde{u}) - h\tilde{u} + f_\epsilon(\tilde{u}) = 0 \text{ in } D \\ \tilde{u}(x_0 + 2\ell, x) = \tilde{u}(x_0, Ax) \text{ for any } (x_0,x) \in D \end{cases} \tag{3.3}$$

where $f_\epsilon(s) := (s^+)^{p\pm\epsilon}$. For the sake of simplicity, we will refer to $f_\epsilon(s) := (s^+)^{p+\epsilon}$ as the *supercritical case* and to $f_\epsilon(s) := (s^+)^{p-\epsilon}$ as the *subcritical case*.
In (3.3) B is a second order linear operator defined in the following Lemma

Lemma 3.1. *Let u be a smooth function. Then for any $(x_0, x) \in D$ we have*

$$\Delta_g u = \partial_{00}\tilde{u} + \Delta_x \tilde{u} + B(\tilde{u})$$

where B is a second order linear operator defined by

$$\begin{aligned} B(\tilde{u}) :=& A^{00}\partial_{00}\tilde{u} + \sum_j A^{0j}\partial_0\partial_j\tilde{u} + \sum_{i,j}\left(-\frac{1}{3}\sum_{k,l} R_{ikjl}x_k x_l + A^{ij}\right)\partial_i\partial_j\tilde{u} \\ &+ B^0\partial_0\tilde{u} + \sum_j\left(\sum_k\left(\frac{2}{3}R_{ijik} + R_{0j0k}\right)x_k + B^j\right)\partial_j\tilde{u} \end{aligned}$$

where the Riemann tensor R_{ijkl} and the metric g are computed along Γ, depending only on x_0, while the function $A^{\alpha\beta}$ and B^α do depend on (x_0, x) and enjoy the following decomposition:

$$A^{00} = \sum_{k,l} A^{00}_{kl} x_k x_l; \qquad A^{ij} = \sum_{k,l,m} A^{ij}_{kl} x_k x_l x_m; \qquad A^{0j} = \sum_{k,l} A^{0j}_{kl} x_k x_l$$

$$B^0 = \sum_k B^0_k x_k; \qquad B^j = \sum_{k,l} B^j_{kl} x_k x_l$$

where A^{00}_{kl}, A^{ij}_{kl}, A^{0j}_{kl}, B^0_k and B^j_{kl} are smooth functions depending on (x_0, x).

Proof. We argue exactly as in Section 4 of [6] taking into account the following expansion of the metric g in a neighborhood of the geodesic

$$\begin{cases} g_{00}(x) = 1 + \sum_{k,l=1}^N R_{0k0l}x_k x_l + O(|x|^3) \\ g_{0j}(x) = O(|x|^2), \quad j = 1, \ldots, N. \\ g_{ij}(x) = \delta_{ij} + \frac{1}{2}\sum_{k,l} R_{ikjl}x_k x_l + O(|x|^3), \quad i, j = 1, \ldots, N. \end{cases} \tag{3.4}$$

whose proof is postponed in the Appendix. □

3.2. **The scaled problem.** We write an approximated solution of problem (3.3). Let

$$\tilde{u}_\epsilon(x_0, x) = \mu_\epsilon(x_0)^{-\frac{N-2}{2}} w\left(\frac{x - d_\epsilon(x_0)}{\mu_\epsilon(x_0)}\right), \tag{3.5}$$

where the bubble w is defined in (1.9), and d_ϵ satisfies

$$d_\epsilon(0) = A d_\epsilon(2\ell), \text{ with } d_\epsilon(x_0) = (d_{\epsilon 1}(x_0), \ldots, d_{\epsilon N}(x_0)) \tag{3.6}$$

and $A = (a_{ij})$ is the matrix defined by (3.2). In the sequel, $C^2_{2\ell}(\mathbb{R}, \mathbb{R}^N)$ is the space of functions $d : [0, 2\ell] \to \mathbb{R}^N$ which satisfy (3.6).

We will take $d_\epsilon(x_0)$ of the form

$$d_{\epsilon j}(x_0) = \epsilon d_j(x_0) \text{ with } d_j \in C^2_{2\ell}(\mathbb{R}), \ j = 1, \ldots, N \tag{3.7}$$

and the concentration parameter $\mu_\epsilon(x_0)$ is given by

$$\mu_\epsilon(x_0) = \sqrt{\epsilon}\tilde{\mu}_\epsilon(x_0), \ \tilde{\mu}_\epsilon(x_0) = \mu_0(x_0) + (\epsilon \ln \epsilon)\mu_1(x_0) + \epsilon\mu(x_0), \tag{3.8}$$

with $\mu_0, \mu_1, \mu \in C^2_{2\ell}(\mathbb{R})$. We point out that in (3.8) and (3.7) the μ_0, μ_1, μ and d_j, $j = 1, \dots, N$ are unknown functions which will be found in the final step of the infinite-dimensional reduction. In particular, it will turn out that μ_0 is a non degenerate solution to problem (1.6) in the subcritical case or to problem (1.7) in the supercritical case.

Therefore, it is natural to consider the change of variables

$$\tilde{u_\epsilon}(x_0, x) = \mu_\epsilon^{-\frac{N-2}{2}} v\left(\frac{x_0}{\rho}, \frac{x - d_\epsilon}{\mu_\epsilon}\right), \ \rho := \sqrt{\epsilon}, . \tag{3.9}$$

Here $v_\epsilon = v_\epsilon(y_0, y)$ is defined in a region of the form

$$\mathcal{D} = \left\{(y_0, y) \ : \ y_0 \in \left[-\frac{\ell}{\rho}, \frac{\ell}{\rho}\right], \quad |y| < \frac{\eta}{\sqrt{\rho}}\right\}. \tag{3.10}$$

It is clear that if $\tilde{u}_\epsilon(x_0, x)$ solves equation (3.3), then $v_\epsilon = v_\epsilon(y_0, y)$ solves problem

$$\begin{cases} \mathcal{A}(v) - \mu_\epsilon^2 hv + \mu_\epsilon^{\pm\frac{N-2}{2}\epsilon} f_\epsilon(v) = 0 \text{ in } \mathcal{D} \\ v\left(y_0 + \frac{2\ell}{\rho}, y\right) = v(y_0, Ay) \text{ for any } (y_0, y) \in \mathcal{D}. \end{cases} \tag{3.11}$$

We agree that we take $\mu_\epsilon^{+\frac{N-2}{2}\epsilon}$ in the supercritical case, i.e. $f_\epsilon(s) = (s^+)^{p+\epsilon}$ and $\mu_\epsilon^{-\frac{N-2}{2}\epsilon}$ in the subcritical case, i.e. $f_\epsilon(s) = (s^+)^{p-\epsilon}$.
In (3.11) $\mathcal{A}$ is a second order operator of the form defined in the following Lemma, whose proof can be obtained arguing exactly as in Lemma 5.1 of [6].

Lemma 3.2. *After the change of variable* (3.9), *the following holds true:*

$$\mathcal{A}(v) := a_0 \partial_{00} v + \Delta_y v + \tilde{\mathcal{A}}(v),$$

with

$$a_0(\rho y_0) = \rho^{-2} \mu_\epsilon(\rho y_0)^2 = (\mu_0 + \rho\mu)^2 \tag{3.12}$$

and $\tilde{\mathcal{A}}(v) := \sum_{\kappa=0}^2 \mathcal{A}_\kappa(v) + \mathcal{B}(v)$ *where*

$$\begin{aligned} \mathcal{A}_0(v) =& \dot{\mu}_\epsilon^2 \left[D_{yy} v[y]^2 + N D_y v[y] + \frac{N(N-2)}{4} v\right] + \dot{\mu}_\epsilon \left[D_{yy} v[y] + \frac{N-2}{2} D_y v\right] [\dot{d}_\epsilon] \\ &+ D_{yy} v[\dot{d}_\epsilon]^2 - 2\mu_\epsilon \left[\rho^{-1} D_y(\partial_0 v)[\dot{\mu}_\epsilon y + \dot{d}_\epsilon] + \frac{N-2}{2} \dot{\mu}_\epsilon \rho^{-1} \partial_0 v\right] - \mu_\epsilon D_y v[\ddot{d}_\epsilon] \\ &- \mu_\epsilon \ddot{\mu}_\epsilon \left(\frac{N-2}{2} v + D_y v[y]\right) \\ \mathcal{A}_1(v) :=& -\frac{1}{3} \sum R_{ikjl} \left(\mu_\epsilon y_k + d_{\epsilon k}\right) \left(\mu_\epsilon y_l + d_{\epsilon l}\right) \partial_{ij} v \\ \mathcal{A}_2(v) :=& \sum \left(\frac{2}{3} R_{ijik} + R_{0j0k}\right) \left(\mu_\epsilon y_k + d_{\epsilon k}\right) \mu_\epsilon \partial_j v \end{aligned}$$

and the operator $B(v)$ *satisfies*

$$\begin{aligned} \mathcal{B}(v) =& O\left(|\mu_\epsilon y + d_\epsilon|^2\right) \mathcal{A}_0(v) + O\left(|\mu_\epsilon y + d_\epsilon|^3\right) \partial_{ij} v \\ &+ O\left(|\mu_\epsilon y + d_\epsilon|^2\right) \Big[\mu_\epsilon \rho^{-1} \partial_{0j} v + \mu_\epsilon \rho^{-1} \partial_0 v - D_y(\partial_j v)[d_\epsilon] \\ &\qquad - \left(\frac{N-2}{2} \partial_j v + D_y(\partial_j v)[y]\right) \dot{\mu}_\epsilon - D_y v[\dot{d}_\epsilon] \\ &\qquad - \dot{\mu}_\epsilon \left(\frac{N-2}{2} v + D_y v[y]\right) + \mu_\epsilon \partial_j v\Big]. \end{aligned}$$

Our approximation close to the geodesic is

$$\tilde{\boldsymbol{\omega}} = \omega + \omega_1. \tag{3.13}$$

The first order approximation ω is given in (3.15), while the second order approximation ω_1 is given in (3.25). We also set

$$\mathcal{S}_\epsilon(v) := \mathcal{A}(v) - \mu_\epsilon^2 hv + \mu_\epsilon^{\pm\frac{N-2}{2}\epsilon} f_\epsilon(v). \tag{3.14}$$

3.3. **The ansatz: the first order approximation.** We define ω to be

$$\omega := (1+\alpha_\epsilon)w + e_\epsilon(\rho y_0)\chi_\epsilon(y)Z_0(y). \tag{3.15}$$

In the first term of (3.15), w is the bubble defined in (1.9) and $\alpha_\epsilon := \mu_\epsilon^{\frac{(N-2)^2}{8}\epsilon} - 1$ in the subcritical case or $\alpha_\epsilon := \mu_\epsilon^{-\frac{(N-2)^2}{8}\epsilon} - 1$ in the subcritical case. In the second term of (3.15), $\chi_\epsilon(y) := \chi\left(\epsilon^{\frac{1}{2}}|y|\right)$ where χ is a cut-off function such that $\chi(s) = 1$ if $s \leqslant \delta$ and $\chi(s) = 0$ if $s \geqslant 2\delta$ with $\delta > 0$ small but fixed. Moreover, Z_0 denotes the first eigenfunction in $L^2(\mathbb{R}^N)$ of the problem (see Section 7)

$$\Delta Z_0 + pw^{p-1}Z_0 = \lambda_1 Z_0 \text{ in } \mathbb{R}^N, \quad \text{with } \lambda_1 > 0 \text{ and } \int_{\mathbb{R}^N} Z_0^2\, dy = 1. \tag{3.16}$$

Finally, the function $e_\epsilon(x_0)$ is given by

$$e_\epsilon = \epsilon\tilde{e}_\epsilon, \quad \tilde{e}_\epsilon = e_0 + (\epsilon\ln\epsilon)e_1 + \epsilon e, \tag{3.17}$$

with $e_0, e_1, e \in C^2_{2\ell}(\mathbb{R})$. We point out that e_0, e_1 and e are unknown functions which will be chosen in the final step of the infinite-dimensional reduction, together with the functions μ_0, μ and d_j introduced in (3.7) and (3.8).

Let us estimate the error $S_\epsilon(\omega)$ one commits by considering ω a real solution to (3.11), which is itself a function of the parameter functions μ, d, e.
Assume that the functions μ, d, e defined respectively in (3.8), (3.7) and (3.17), satisfy the assumption

$$\|(\mu, d, e)\| := \|\mu\| + \|d\| + \|e\|_\epsilon \leq C \tag{3.18}$$

for some constant $C > 0$, independent of ϵ, where

$$\|\mu\| := \|\ddot{\mu}\|_\infty + \|\dot{\mu}\|_\infty + \|\mu\|_\infty, \ \|d\| := \sum_{j=1}^{N} \|d_j\|_\infty, \tag{3.19}$$

$$\|e\|_\epsilon := \|\epsilon\ddot{e}\|_\infty + \|\epsilon^{\frac{1}{2}}\dot{e}\|_\infty + \|e\|_\infty \tag{3.20}$$

Here and in the rest of the paper, the dot denotes the derivative with respect to x_0.
It is possible to compute the expansion of the error $S_\epsilon(\omega)$ as showed in the following Lemma whose proof is postponed in Section 4.1.

Lemma 3.3. *If $\epsilon > 0$ small enough, then for any $(y_0, y) \in \mathcal{D}$ the following expansion holds*

$$\begin{aligned}
&\mathcal{S}_\epsilon(\omega) = \pm\epsilon w^p \ln w + \epsilon\lambda_1 e_0 Z_0 - \epsilon\mu_0^2 hw + \\
&+ \epsilon \left[\dot\mu_0^2 \left(D_{yy}w[y]^2 + ND_y w[y] + \frac{N(N-2)}{4}w\right) - \mu_0\ddot\mu_0 Z_{N+1} + \right.\\
&\left. +\mu_0^2 \left(-\frac{1}{3}R_{ikjl}y_k y_l \partial_{ij}w + \left(\frac{2}{3}R_{ijik} + R_{0j0k}\right) y_k \partial_j w\right)\right] \\
&+ \epsilon^{\frac{3}{2}} \left[-\mu_0 \partial_j w \ddot d_j - \frac{1}{3}\mu_0 R_{ikjl} y_k y_l \partial_{ij} w + \mu_0 \left(\frac{2}{3}R_{ijik} + R_{0j0k}\right) d_k \partial_j w - 2\dot\mu_0 \partial_j Z_{N+1} \dot d_j\right] \\
&+ \epsilon^2 \left[\left(\rho^2 a_0 \ddot e + \lambda_1 e\right) Z_0 + \left(\sum_{i,j} \dot d_i \dot d_j - \frac{1}{3}R_{ijkl} d_k d_l\right) \partial_{ij} w + \Upsilon_0 + \right.\\
&-2\mu_0 \mu h w + b(\rho y_0, \mu, d, e) w^p + 2\dot\mu_0 \dot\mu \left(D_{yy}w[y]^2 + ND_y w[y] + \frac{N(N-2)}{4}w\right) + \\
&-\mu_0 \ddot\mu Z_{N+1} - \mu\ddot\mu_0 Z_{N+1} + 2\mu_0\mu \left(-\frac{1}{3}R_{ikjl} y_k y_l \partial_{ij} w + \left(\frac{2}{3}R_{ijik} + R_{0j0k}\right) y_k \partial_j w\right) + \\
&-e_0 \ddot\mu_0 \mu_0 Z_{N+1} + \mu_0^2 e_0 \left(-\frac{1}{3}R_{ikjl} y_k y_l \partial_{ij} Z_0 + \left(\frac{2}{3}R_{ijik} + R_{0j0k}\right) y_k \partial_j Z_0\right) + \\
&\left. +\dot\mu_0^2 \left(D_{yy} Z_0[y]^2 + ND_y Z_0[y] + \frac{N(N-2)}{4} Z_0\right) - \mu_0^2 h Z_0\right] \\
&+ \epsilon^{\frac{5}{2}} \left[-\mu \partial_j \ddot d_j - \frac{1}{3}\mu R_{ikjl} y_k d_l \partial_{ij} w - \mu \left(\frac{2}{3}R_{ijik} + R_{0j0k}\right) d_k \mu \partial_j w - 2\dot\mu \partial_j Z_{N+1} \dot d_j \right.\\
&-\mu_0 e_0 \partial_j Z_0 \ddot d_j - \frac{1}{3}\mu_0 e_0 R_{ikjl} y_k d_l \partial_{ij} Z_0 + \mu_0 e_0 \left(\frac{2}{3}R_{ijik} + R_{0j0k}\right) d_k \partial_j Z_0 + \\
&\left. -2\dot\mu_0 e_0 \left(\frac{N-2}{2} D_y Z_0 + D_{yy} Z_0[y]\right) [\dot d]\right] + \epsilon^3 \Theta
\end{aligned} \tag{3.21}$$

where

- Z_0 *is defined in* (3.16) *and* Z_{N+1} *is defined in* (3.23)
- *the first term is "*$-\epsilon w^p \ln w$*" in the subcritical case or "*$+\epsilon w^p \ln w$*" in the supercritical case.*
-
$$\Upsilon_0 = p(p-1)e_0^2 w^{p-2} Z_0^2 + p e_0 w^{p-1} \ln w Z_0 \tag{3.22}$$
- $\Theta = \Theta(y_0, y)$ *is a sum of functions of the form*
$$h_0(\rho y_0)\left[f_1(\mu, d, \dot\mu, \dot d) + o(1) f_2(\mu, d, e, \dot\mu, \dot d, \dot e, \ddot\mu, \ddot d, \ddot e)\right] f_3(y)$$
with
  - h_0 *a smooth function uniformly bounded in* ϵ
  - f_1 *and* f_2 *smooth functions of their arguments, uniformly bounded in* ϵ *when* μ, d *and* e *satisfy* (3.18)
  - f_2 *depending linearly on the argument* $(\ddot\mu, \ddot d, \ddot e)$
  - $o(1) \to 0$ *as* $\epsilon \to 0$ *uniformly when* μ, d *and* e *satisfy* (3.18)
  - $\sup_{y\in\mathbb{R}}(1 + |y|^{N-2})|f_3(y)| < +\infty$

Now, we use formula (3.21) to compute, for each $y_0 \in [-\ell/\rho, +\ell/\rho]$, the $L^2(\mathcal{D}_{y_0})$ the projection of the error $\mathcal{S}_\epsilon(\omega)$ along the elements of the kernel of the linear operator $\mathcal{L}_0 := \Delta_{\mathbb{R}^N} + pw^{p-1}I$

(see Section 7), i.e. the functions

$$Z_k(y) := \partial_k w(y),\ k = 1, \dots, N \text{ and } Z_{N+1}(y) := y \cdot \nabla w(y) + \frac{N-2}{2} w(y). \tag{3.23}$$

Lemma 3.4. *If $\epsilon > 0$ small enough, then for any $x_0 = \rho y_0$ with $y_0 \in [-\ell/\rho, +\ell/\rho]$ the following expansion hold:*

$$\int_{\mathcal{D}_{y_0}} \mathcal{S}_\epsilon(\omega) Z_k \, dy = \epsilon^{\frac{3}{2}} c_1 \mu_0 \left(-\ddot{d}_k + \sum R_{0k0l} d_l \right) + \epsilon^2 \theta, \quad \text{for any } k = 1, \dots, N;$$

moreover, if μ_0 solves either (1.6) *or* (1.7) *there exist $\mu_1, e_0, e_1 \in C^2_{2\ell}(\mathbb{R})$ such that*

$$\int_{\mathcal{D}_{y_0}} \mathcal{S}_\epsilon(\omega) Z_{N+1} \, dy = \epsilon^2 c_2 \mu_0 \left[\alpha_{N+1}(x_0) + c_3 Q(x_0, d) - \ddot{\mu} + \left(a_n \sigma \mp \frac{b_n}{\mu_0^2} \right) \mu \right] + \epsilon^3 |\ln \epsilon| \theta$$

and

$$\int_{\mathcal{D}_{y_0}} \mathcal{S}_\epsilon(\omega) Z_0 \, dy = \epsilon^2 \left[\epsilon a_0 \ddot{e} + \lambda_1 e + \alpha_0(x_0) + c_4 Q(\rho y_0, d) + \beta(x_0) \mu \right] + \epsilon^3 |\ln \epsilon| \theta.$$

Here

- σ *is defined in* (1.5) *and* a_n, b_n *are positive constants depending only on n defined in* (4.16)
- $Q(x_0, d) := \sum \left(\dot{d}_j^2 - \frac{1}{3} R_{ikjl} d_k d_l \right)$
- c_i*'s are constants which depends only on n*
- α_i*'s and β are explicit smooth functions, uniformly bounded in ϵ when μ, d and e satisfy* (3.18)
- $\theta = \theta(x_0)$ *denotes a sum of functions of the form*

  $$h_0(x_0) \left[h_1(\mu, d, e, \dot{\mu}, \dot{e}, \dot{d}) + o(1) h_2(\mu, d, e, \dot{\mu}, \dot{d}, \dot{e}, \ddot{\mu}, \ddot{d}, \ddot{e}) \right],$$

  where
  - h_0 *is a smooth function uniformly bounded in ϵ*
  - h_1 *and h_2 are smooth functions of their arguments, uniformly bounded in ϵ when μ, d and e satisfy* (3.18)
  - h_2 *depends linearly on the argument* $(\ddot{\mu}, \ddot{d}, \ddot{e})$
  - $o(1) \to 0$ *as $\epsilon \to 0$ uniformly when μ, d and e satisfy* (3.18)

The proof is postponed in Section 4.2.

In the sequel we will use the following norms, which are motivated by the linear theory presented in Section 7. For functions ϕ, g defined on a set $\mathcal{D}$ as in (3.10), and for a fixed $2 \le \nu < N$, let

$$\|\phi\|_* := \sup_{\mathcal{D}} (1 + |y|^{\nu-2}) |\phi(y_0, y)| + \sup_{\mathcal{D}} (1 + |x|^{\nu-1}) |D\phi(x_0, x)|,$$
$$\|g\|_{**} := \sup_{\mathcal{D}} (1 + |y|^{\nu}) |g(y_0, y)|.$$

Therefore, from the expansion given in (3.21) we conclude that the error $\mathcal{S}_\epsilon(\omega)$, computed in (3.21), has the properties listed in the following Lemma.

Lemma 3.5. *Let μ_0 and e_0 as in Lemma 3.4 If ϵ is small enough*

$$\mathcal{S}_\epsilon(\omega) = \epsilon S_0 + \epsilon\left[\rho^2 a_0 \ddot{e} + \lambda_1 e\right]\chi_\epsilon Z_0 + N_0 \tag{3.24}$$

where

- *S_0 is a smooth function of ρy_0 uniformly bounded in ϵ*
- *S_0 does not depend on μ, d and e.*
- *$\int_{\mathcal{D}_{y_0}} S_0 Z_j\, dy = 0$ for any $y_0 \in (-\rho^{-1}\ell, \rho^{-1}\ell)$ and for any $j = 0, \dots, N+1$*
- *$\|N_0\|_{**} \le c\epsilon^{\frac{3}{2}}$*

Here c is a positive constant independent of ϵ. All the estimates are uniform with respect to μ, d and e which satisfy (3.18).

3.4. **The ansatz: the second order approximation.** Now we introduce a further correction ω_1 to ω, to get the final approximation $\tilde{\boldsymbol{\omega}} := \omega + \omega_1$. The correction ω_1 is chosen to reduce the size of the error (3.24), killing the term ϵS_0 and it is found in the following Lemma, whose proof can be carried out arguing exactly as in Section 5 of [6].

Lemma 3.6. *If ϵ is small enough there exists a unique solution ω_1 of the problem*

$$\begin{cases} \mathcal{A}(\omega_1) - \mu_\epsilon^2 h\omega_1 + pw^{p-1}\omega_1 = -\epsilon S_0 + \sum_{j=0}^N \sigma_j Z_j & \text{in } \mathcal{D} \\ \\ \int_{\mathcal{D}_{y_0}} \omega_1(y_0, y) Z_j dy = 0 & \text{for any } y_0 \in \left[-\frac{\ell}{\rho}, \frac{\ell}{\rho}\right],\ j = 0, \dots, N+1 \end{cases} \tag{3.25}$$

Moreover, the function ω_1 satisfies

- *$\|\omega_1\|_* \le c\epsilon$ and $\|\partial_0\omega_1\|_* \le c\epsilon^{\frac{3}{2}}$*
- *ω_1 depends smoothly on μ and d and it is independent on e*
- *$\|\omega_1(\mu_1, d_1) - \omega_1(\mu_2, d_2)\|_* \le c\|(\mu_1 - \mu_2, d_1 - d_2)\|$*

and each function σ_j satisfies

- *$\|\sigma_j\|_\infty \le o(1)\epsilon^3$*
- *σ_j depends smoothly on μ and d and it is independent on e*
- *$\|\sigma_j(\mu_1, d_1) - \sigma_j(\mu_2, d_2)\|_\infty \le c\epsilon^2\|(\mu_1 - \mu_2, d_1 - d_2)\|$*

Moreover, it holds true

$$\mathcal{S}_\epsilon(\tilde{\boldsymbol{\omega}}) = \epsilon^{\frac{3}{2}} S_1 + \epsilon\left[\rho^2 a_0 \ddot{e} + \lambda_1 e\right]\chi_\epsilon Z_0 + N_1 + \sum_{j=0}^N \sigma_j Z_j \tag{3.26}$$

where

- *S_1 is a smooth function of ρy_0 uniformly bounded in ϵ*
- *S_1 depends smoothly on μ, d and e.*
- *$\|S_1(\mu_1, d_1, e_1) - S_1(\mu_2, d_2, e_2)\|_{**} \le c\|(\mu_1 - \mu_2, d_1 - d_2, e_1 - e_2)\|$*
- *$\|N_1\|_{**} \le c\epsilon^2$*

Here c is positive constant independent of ϵ. All the estimates are uniform with respect to μ, d and e which satisfy (3.18). *Moreover, the components of $\mathcal{S}_\epsilon(\tilde{\boldsymbol{\omega}})$ along the Z_j's satisfy the estimate in Lemma 3.4.*

4. The error $S_\epsilon(\omega)$

4.1. **The pointwise estimate of the error.** We recall that

$$\mathcal{S}_\epsilon(\omega) = \mathcal{A}(\omega) - \mu_\epsilon^2 h\omega + \mu_\epsilon^{\pm\frac{N-2}{2}\epsilon} f_\epsilon(\omega)$$

where by Lemma 3.2

$$\mathcal{A}(\omega) = a_0\partial_{00}\omega + \Delta_y\omega + \underbrace{\sum_{k=0}^{2}\mathcal{A}_k(\omega) + \mathcal{B}(\omega)}_{\tilde{\mathcal{A}}(\omega)}$$

and

$$\omega(y) = (1+\alpha_\epsilon)w(y) + e_\epsilon(\rho y_0)\chi_\epsilon(y)Z_0(y).$$

Here we recall that

$$\alpha_\epsilon = \mu_\epsilon^{\mp\frac{(N-2)^2}{8}\epsilon} - 1$$

and

$$\Delta\left((1+\alpha_\epsilon)w\right) + \mu_\epsilon^{\pm\frac{N-2}{2}\epsilon} f_0\left((1+\alpha_\epsilon)w\right) = 0 \qquad \text{in } \mathbb{R}^N.$$

Proof of Lemma 3.3. We use Lemma 3.2.
A straightforward computation shows that

$$\begin{aligned}
\mathcal{S}_\epsilon(\omega) = & \underbrace{\sum_{\kappa=0}^{2}\mathcal{A}_\kappa(w) - \mu_\epsilon^2 hw \pm \epsilon w^p \ln w + \left[\rho^2 a_0 \ddot{e}_\epsilon(\rho y_0) + \lambda_1 e_\epsilon(\rho y_0)\right]\chi_\epsilon Z_0}_{J_0} \\
& + \underbrace{\mathcal{B}(w) + a_0 w\partial_{00}\alpha_\epsilon + \tilde{\mathcal{A}}(\alpha_\epsilon w) - \mu_\epsilon^2\alpha_\epsilon hw}_{J_1} \\
& + \underbrace{\mu_\epsilon^{\pm\frac{N-2}{2}\epsilon}\left[f_\epsilon\left((1+\alpha_\epsilon)w\right) - f_0\left((1+\alpha_\epsilon)w\right)\right] \mp \epsilon w^p \ln w}_{J_2} \\
& + \underbrace{\sum_{\kappa=0}^{2}\mathcal{A}_\kappa(e_\epsilon\chi_\epsilon Z_0) - \mu_\epsilon^2 e_\epsilon\chi_\epsilon Z_0 h}_{J_3} \\
& + \underbrace{\mathcal{B}(e_\epsilon\chi_\epsilon Z_0) + e_\epsilon Z_0\Delta\chi_\epsilon + 2e_\epsilon\nabla\chi_\epsilon\nabla Z_0}_{J_4} \\
& + \underbrace{\mu_\epsilon^{\pm\frac{N-2}{2}\epsilon}\left[f_\epsilon(\omega) - f_\epsilon\left((1+\alpha_\epsilon)w\right)\right] - f_0'(w)e_\epsilon\chi_\epsilon Z_0}_{J_5}.
\end{aligned} \tag{4.1}$$

By Lemma 3.2, we get the first term of J_0

$$\begin{aligned}
\sum_{\kappa=0}^{2}\mathcal{A}_\kappa(w) = & \dot{\mu}_\epsilon^2\left[D_{yy}w[y]^2 + ND_yw[y] + \frac{N(N-2)}{4}w\right] \\
& + \dot{\mu}_\epsilon\left[D_{yy}w[y] + \frac{N-2}{2}D_yw\right][\dot{d}_\epsilon] + D_{yy}w[\dot{d}_\epsilon]^2 \\
& - \mu_\epsilon D_yw[\ddot{d}_\epsilon] - \mu_\epsilon\ddot{\mu}_\epsilon\left(\frac{N-2}{2}w + D_yw[y]\right) \\
& - \frac{1}{3}\sum R_{ikjl}\left(\mu_\epsilon y_k + d_{\epsilon k}\right)\left(\mu_\epsilon y_l + d_{\epsilon l}\right)\partial_{ij}w \\
& + \sum\left(\frac{2}{3}R_{ijik} + R_{0j0k}\right)\left(\mu_\epsilon y_k + d_{\epsilon k}\right)\mu_\epsilon\partial_j w + \epsilon^3\Theta \\
= & \epsilon^2\left[\sum\left(\dot{d}_i\dot{d}_j - \frac{1}{3}R_{ikjl}d_kd_l\right)\right]\partial_{ij}w
\end{aligned}$$

$$
\begin{aligned}
&+\rho\epsilon\left[-\tilde\mu D_y w[\ddot d]-\sum\frac{1}{3}\tilde\mu R_{ikjl}y_k d_l\partial_{ij}w+\right.\\
&\qquad\left.+\left(\frac{2}{3}R_{ijik}+R_{0j0k}\right)d_k\tilde\mu\partial_j w-2\dot{\tilde\mu}D_y Z_{N+1}[\dot d]\right]\\
&+\rho^2\left[\dot{\tilde\mu}^2\left[D_{yy}w[y]^2+ND_y w[y]+\frac{N(N-2)}{4}w\right]-\tilde\mu\ddot{\tilde\mu}Z_{N+1}\right.\\
&\qquad\left.+\tilde\mu^2\left(-\frac{1}{3}\sum R_{ikjl}y_k y_l\partial_{ij}w+\left(\frac{2}{3}R_{ijik}+R_{0j0k}\right)y_k\partial_j w\right)\right]+\epsilon^3\Theta, \qquad (4.2)
\end{aligned}
$$

where $\Theta=\Theta(\rho y_0,y)$ has the required properties.

By Lemma 3.2, we deduce that $\mathcal{B}(w)$ is of lower order with respect to $\sum\mathcal{A}_k(w)$. Moreover, by definition of α_ϵ we get that $\alpha_\epsilon=O(\epsilon|\ln\epsilon|)$ as $\epsilon\to 0$. Hence $\alpha_\epsilon\tilde{\mathcal{A}}(w)$ and $\mu_\epsilon\alpha_\epsilon hw$ are terms of lower order with respect to the others. Furthermore $\partial_{00}\alpha_\epsilon=\rho^2O(\alpha_\epsilon)$, so also $a_0\partial_{00}[\alpha_\epsilon w]=O(\epsilon^2|\ln\epsilon|)w$. Therefore,

$$J_1=\epsilon^3\Theta$$

where $\Theta=\Theta(\rho y_0,y)$ is a sum of functions of the form $h_0(\rho y_0)f_1(\mu,d,\dot\mu,\dot d)f_2(y)$, with h_0 a smooth function uniformly bounded in ϵ, f_1 a smooth function of its arguments, homogeneous of degree 3, uniformly bounded in ϵ and $\sup_{y\in\mathbb{R}}(1+|y|^{N-2})|f_2(y)|<+\infty$.

By mean value theorem we deduce that

$$
\begin{aligned}
&J_2=\pm\frac{(n-2)^2}{8}(\epsilon^2\ln\epsilon)w^p(\ln w-1)\pm\epsilon^2 w^p\left(\frac{(n-2)^2}{8}(\ln w-1)\ln\mu+\frac{1}{2}\ln w\right)\\
&+O\left(\epsilon^3|\ln\epsilon|\right). \qquad (4.3)
\end{aligned}
$$

By Lemma 3.2 we also get that

$$
\begin{aligned}
J_3=\epsilon\tilde e&\left\{\epsilon^2\left[\left(\sum\dot d_i\dot d_j-\frac{1}{3}R_{ikjl}d_k d_l\right)\partial_{ij}Z_0\right]\right.\\
&+\rho\epsilon\left[-\tilde\mu D_y Z_0[\ddot d]-\frac{1}{3}\tilde\mu R_{ikjl}y_k d_l\partial_{ij}Z_0+\tilde\mu\left(\frac{2}{3}R_{ijik}+R_{0j0k}\right)d_k\partial_j Z_0\right.\\
&\qquad\left.-2\dot{\tilde\mu}\left(\frac{N-2}{2}D_y Z_0+D_{yy}Z_0[y]\right)[\dot d]\right]\\
&+\rho^2\left[-\ddot{\tilde\mu}\tilde\mu Z_{N+1}+\tilde\mu^2\left(-\frac{1}{3}R_{ikjl}y_k y_l\partial_{ij}Z_0+\left(\frac{2}{3}R_{ijik}+R_{0j0k}\right)y_k\partial_j Z_0\right)+\right.\\
&\qquad\left.\left.+\dot{\tilde\mu}^2\left(D_{yy}Z_0[y]^2+ND_y Z_0[y]+\frac{N(N-2)}{4}Z_0\right)-\tilde\mu^2 hZ_0\right]\right\}\\
&+\rho\epsilon\dot{\tilde e}\left\{\epsilon\left(-2\tilde\mu D_y Z_0[\dot d]\right)+\rho\epsilon\left[-2\tilde\mu\dot{\tilde\mu}D_y Z_0[y]-(N-2)\tilde\mu\dot{\tilde\mu}Z_0\right]\right\}
\end{aligned}
$$

and

$$J_4=\epsilon^3\Theta$$

where $\Theta=\Theta(\rho y_0,y)$ has the required properties.

Finally, standard estimates yield to

$$J_5=\epsilon^2\underbrace{\left[p(p-1)e_0^2w^{p-2}Z_0^2+pe_0w^{p-1}\ln wZ_0\right]}_{\Upsilon_0}+\epsilon^3|\ln\epsilon|\Theta,$$

where $\Theta=\Theta(\rho y_0,y)$ is a sum of functions of the form $h_0(\rho y_0)h_1(\mu,d,e)h_2(y)$ with h_0 a smooth function, uniformly bounded in ϵ, h_1 a smooth function of its arguments and $\sup_{y\in\mathbb{R}}(1+$

$|y|^{N-2})|h_2(y)| < +\infty$.
Collecting all the previous estimates we get the proof. □

4.2. **The components of the error along the Z_j's.**

Proof of Lemma 3.4. The proof consists of two steps. In the first part we compute the expansion in ϵ of the projection assuming that

$$\mu_\epsilon = \rho\tilde{\mu}, \qquad d_{\epsilon j} = \epsilon d_j, \qquad e_\epsilon = \epsilon\tilde{e}.$$

In the second part we will choose the $\epsilon-$order terms μ_0 and e_0 and the $\epsilon\ln\epsilon$-order terms μ_1 and e_1 in the expansion of $\tilde{\mu}$ and $\tilde{e}$.

Arguing as in the proof of Lemma 3.3, we have

$$\mathcal{S}_\epsilon(\omega) \quad = \quad \underbrace{\pm\epsilon w^p\ln w - \rho^2\tilde{\mu}^2 hw}_{I_1} + \underbrace{\sum_{k=0}^{2}\mathcal{A}_k(w)}_{I_2} + \underbrace{\epsilon\left[\rho^2 a_0\ddot{\tilde{e}} + \lambda_1\tilde{e}\right]\chi_\epsilon Z_0}_{I_3} + \underbrace{J_1+\cdots+J_5}_{I_4}.$$

We stress the fact that the first term in I_1 is " $+\,\epsilon w^p\ln w$" in the super-critical case and " $-\,\epsilon w^p\ln w$" in the sub-critical case.

- *The projection of I_1.*

$$\begin{aligned}\int_{\mathcal{D}_{y_0}} I_1 Z_{N+1}\,dy &= \pm\epsilon\int_{\mathcal{D}_{y_0}} w^p\ln w Z_{N+1}\,dy - \rho^2\tilde{\mu}^2\int_{\mathcal{D}_{y_0}} hwZ_{N+1}\,dy\\ &= -\epsilon A_1 + O(\epsilon\rho^N) - \rho^2\tilde{\mu}^2 h(\rho y_0)\int_{\mathbb{R}^N} wZ_{N+1}\,dy + O(\rho^N)\\ &= \epsilon\left[\pm A_1 - \tilde{\mu}^2 h(\rho y_0)A_2\right] + O(\rho^N).\end{aligned}$$

where

$$A_1 = \int_{\mathbb{R}^N} w^p\ln w Z_{N+1}\,dy = \frac{N}{(p+1)^2}\int_{\mathbb{R}^N} w^{p+1}\,dy > 0 \text{ (see Remark 4.1)} \tag{4.4}$$

and

$$A_2 = \int_{\mathbb{R}^N} wZ_{N+1}\,dy < 0 \text{ (see Remark 4.1).} \tag{4.5}$$

$$\begin{aligned}\int_{\mathcal{D}_{y_0}} I_1 Z_k\,dy &= \epsilon\int_{\mathcal{D}_{y_0}} w^p\ln w Z_j\,dy + \rho^2\tilde{\mu}^2\int_{\mathcal{D}_{y_0}} hwZ_j\,dy\\ &= \epsilon\int_{\mathbb{R}^N} w^p\ln w Z_j\,dy + \rho^2\tilde{\mu}^2 h(\rho y_0)\int_{\mathbb{R}^N} wZ_j\,dy + O(\rho^{N+1})\\ &= O(\rho^{N+1}) \text{ for } k = 1,\ldots,N.\end{aligned}$$

$$\begin{aligned}\int_{\mathcal{D}_{y_0}} I_1 Z_0\,dy &= -\epsilon\int_{\mathcal{D}_{y_0}} w^p\ln w Z_0\,dy - \rho^2\tilde{\mu}^2\int_{\mathcal{D}_{y_0}} hwZ_0\,dy\\ &= \epsilon\left[-A_3 - \tilde{\mu}^2 h(\rho y_0)A_4\right] + O(\rho^N),\end{aligned}$$

where

$$A_3 := \int_{\mathbb{R}^N} w^p\ln w Z_0\,dy,\ A_4 := \int_{\mathbb{R}^N} wZ_0\,dy. \tag{4.6}$$

- *The projection of I_2.*
We use estimate (4.2).

$$
\begin{aligned}
\int_{\mathcal{D}_{y_0}} I_2 Z_{N+1}\,dy &= \epsilon^2 \sum \left(\dot d_i \dot d_j - \frac{1}{3} R_{ikjl} d_k d_l\right) \int_{\mathcal{D}_{y_0}} \partial_{ij} w Z_{N+1}\,dy \\
&- \rho\epsilon\tilde\mu \sum \ddot d_j \int_{\mathcal{D}_{y_0}} \partial_j w Z_{N+1}\,dy \\
&- \frac{1}{3}\tilde\mu\rho\epsilon \sum R_{ikjl} d_l \int_{\mathcal{D}_{y_0}} y_k \partial_{ij} w Z_{N+1} \\
&+ \rho\epsilon\tilde\mu \sum \left(\frac{2}{3} R_{ijik} + R_{0j0k}\right) d_k \int_{\mathcal{D}_{y_0}} \partial_j w Z_{N+1}\,dy \\
&- 2\dot{\tilde\mu}\rho\epsilon \sum \dot d_j \int_{\mathcal{D}_{y_0}} \partial_j Z_{N+1} Z_{N+1}\,dy \\
&+ \dot{\tilde\mu}^2\rho^2 \int_{\mathcal{D}_{y_0}} \left[D_{yy} w[y]^2 + N D_y w[y] + \frac{N(N-2)}{4} w\right] Z_{N+1}\,dy \\
&- \tilde\mu\ddot{\tilde\mu}\rho^2 \int_{\mathcal{D}_{y_0}} Z_{N+1}^2\,dy \\
&- \rho^2\tilde\mu^2 \frac{1}{3} \sum R_{ikjl} \int_{\mathcal{D}_{y_0}} y_k y_l \partial_{ij} w Z_{N+1}\,dy \\
&+ \tilde\mu^2\rho^2 \sum \left(\frac{2}{3} R_{ijik} + R_{0j0k}\right) \int_{\mathcal{D}_{y_0}} y_k \partial_j w Z_{N+1}\,dy \\
&= \epsilon^2 \sum \left[\dot d_i^2 - \frac{1}{3} R_{ikil} d_k d_l\right] \int_{\mathbb{R}^N} \partial_{ii} w Z_{N+1}\,dy \\
&+ \tilde\mu^2\rho^2 \sum \left(\frac{2}{3} R_{ijij} + R_{0j0j}\right) \int_{\mathbb{R}^N} y_j \partial_j w Z_{N+1}\,dy + \\
&- \tilde\mu\ddot{\tilde\mu}\rho^2 \int_{\mathcal{D}_{y_0}} Z_{N+1}^2 \\
&- \frac{1}{3}\rho^2\tilde\mu^2 \sum R_{ikjl} \int_{\mathbb{R}^N} y_k y_l \partial_{ij} w Z_{N+1}\,dy \\
&+ \epsilon^3\theta \\
&= \epsilon^2 B_1 \underbrace{\sum \left[\dot d_i^2 - \frac{1}{3} R_{ikil} d_k d_l\right]}_{Q(d,\rho y_0)} \\
&+ \epsilon \left[\tilde\mu^2 \sum \left(\frac{1}{3} R_{ijij} + R_{0j0j}\right) B_2 - \tilde\mu\ddot{\tilde\mu} B_3\right] \\
&+ \epsilon^3\theta
\end{aligned}
$$

where the function $\theta = \theta(\rho y_0)$ has the required properties and

$$
B_1 := \int_{\mathbb{R}^N} \partial_{ii} w Z_{N+1} dy,\ B_2 := \int_{\mathbb{R}^N} y_j \partial_j w Z_{N+1} dy < 0,\ B_3 := \int_{\mathbb{R}^N} Z_{N+1}^2 dy. \tag{4.7}
$$

Here we used the fact that

$$
\sum R_{ikjl} \int_{\mathbb{R}^N} y_k y_l \partial_{ij} w Z_{N+1}\,dy = \sum R_{jiij} \int_{\mathbb{R}^N} y_j \partial_j w Z_{N+1}\,dy,
$$

because R_{ikjl} is antisymmetric (i.e. $R_{ikjl} = -R_{kijl}$),

$$\int_{\mathbb{R}^N} y_k y_l \partial_{ij} w Z_{N+1}\, dy$$
$$= \int_{\mathbb{R}^N} y_k y_l \left(-c_N(N-2)\frac{\delta_{ij}}{(1+|y|^2)^{\frac{N}{2}}} + c_N N(N-2)\frac{y_i y_j}{(1+|y|^2)^{\frac{N+2}{2}}} \right) Z_{N+1}\, dy \tag{4.8}$$

and $\int_{\mathbb{R}^N} \frac{y_k y_l y_i y_j}{(1+|y|^2)^{\frac{N+2}{2}}} Z_{N+1}\, dy$ is symmetric.

$$\begin{aligned}\int_{\mathcal{D}_{y_0}} I_2 Z_k\, dy = \rho\epsilon\tilde{\mu} \Big[-\ddot{d}_k \int_{\mathbb{R}^N} Z_j^2\, dy - \frac{2}{3} R_{iljm} d_l \int_{\mathbb{R}^N} y_m \partial_{ij} w Z_k\, dy \\ + \left(\frac{2}{3} R_{ijil} + R_{0j0l} \right) d_l \int_{\mathbb{R}^N} Z_j^2\, dy \Big] \\ + \rho^2 \epsilon\theta \\ = \epsilon^{\frac{3}{2}} \tilde{\mu} B_4 \left[-\ddot{d}_k + R_{0j0l} d_l \right] + \rho^2 \epsilon\theta,\end{aligned}$$

where

$$B_4 := \int_{\mathbb{R}^N} Z_j^2\, dy,\ j = 1, \ldots, N. \tag{4.9}$$

Here we used the fact that

$$\begin{aligned}&-\frac{2}{3} R_{iljm} \int y_m \partial_{ij} w Z_k\, dy \\ &= -\frac{2}{3} \left[R_{ilik} \int y_k \partial_{ii} w Z_k\, dy + R_{ilki} \int y_l \partial_{ik} w Z_k\, dy + R_{kljj} \int y_j \partial_{kj} w Z_k\, dy \right] \\ &= -\frac{1}{3} B_4 \left[R_{ilik} - R_{ilki} \right] = -\frac{2}{3} B_4 R_{ilik}.\end{aligned}$$

$$\begin{aligned}\int_{\mathcal{D}_{y_0}} I_2 Z_0\, dy = \epsilon^2 \left[\sum \left(\dot{d}_i^2 - \frac{1}{3} R_{ikil} d_k d_l \right) \int_{\mathbb{R}^N} \partial_{ii} w Z_0\, dy \right] \\ + \tilde{\mu}^2 \rho^2 \sum \left(\frac{2}{3} R_{ijij} + R_{0j0j} \right) \int_{\mathbb{R}^N} y_j \partial_j w Z_0\, dy \\ - \rho^2 \tilde{\mu}^2 \frac{1}{3} \sum R_{ikjl} \int_{\mathbb{R}^N} y_k y_l \partial_{ij} w Z_0\, dy + \epsilon^3 r \\ = \epsilon^2 B_5 \underbrace{\sum \left[\dot{d}_i^2 - \frac{1}{3} R_{ikil} d_k d_l \right]}_{Q(d, \rho y_0)} \\ + \epsilon \tilde{\mu}^2 B_6 \sum \left(\frac{1}{3} R_{ijij} + R_{0j0j} \right) \\ + \epsilon^3 \theta,\end{aligned}$$

where

$$B_5 := \int_{\mathbb{R}^N} \partial_{ii} w Z_0\, dy,\ B_6 := \int_{\mathbb{R}^N} y_j \partial_j w Z_0\, dy. \tag{4.10}$$

Here we used (4.8) and we argued as before.

- *The projection of* I_3.

$$\int_{\mathcal{D}_{y_0}} I_3 Z_{N+1}\, dy = o(1)\epsilon^3 \text{ and } \int_{\mathcal{D}_{y_0}} I_3 Z_k\, dy = o(1)\epsilon^3 \text{ for any } k = 1, \dots, N,$$

because of the symmetry and of the orthogonality of Z_0 with Z_{N+1} and Z_j.

$$\int_{\mathcal{D}_{y_0}} I_3 Z_0\, dy = \epsilon \left[\rho^2 a_0 \ddot{\tilde{e}} + \lambda_1 \tilde{e}\right] + o(1)\epsilon^3$$

because $\int_{\mathbb{R}^N} Z_0^2\, dy = 1$.

- *The projection of* I_4.

$$\int_{\mathcal{D}_{y_0}} I_4 Z_{N+1}\, dy = \epsilon^2 \ln \epsilon D_1 + \epsilon^2 b_1(\rho y_0) + \epsilon^3 |\ln \epsilon| \theta$$

$$\int_{\mathcal{D}_{y_0}} I_4 Z_k\, dy = \epsilon^2 \theta \text{ for any } k = 1, \dots, N.$$

$$\int_{\mathcal{D}_{y_0}} I_4 Z_0\, dy = \epsilon^2 \ln \epsilon D_2 + \epsilon^2 b_2(\rho y_0) + \epsilon^3 |\ln \epsilon| \theta$$

where

$$D_1 := \pm \frac{(N-2)^2}{16} A_1,\ \ D_2 := \pm \frac{(N-2)^2}{16} A_3 \text{ (see (4.4) and (4.6))},$$

b_1, b_2 are explicit functions and the function $\theta = \theta(\rho y_0)$ has the required properties .

Hence, summing up the previous calculations we conclude that

$$\begin{aligned}\int_{\mathcal{D}_{y_0}} \mathcal{S}_\epsilon(\omega) Z_{N+1}\, dy = {} & \epsilon \underbrace{\left(\pm A_1 - \mu_0 \ddot{\mu}_0 B_3 + \mu_0^2 g_1\right)}_{\text{the choice of } \mu_0 \Rightarrow =0} \\ & + \epsilon^2 \ln \epsilon \underbrace{\left(-\ddot{\mu}_1 \mu_0 B_3 + \mu_1 \left(-\ddot{\mu}_0 B_3 + 2\mu_0 g_1\right) + D_1\right)}_{\text{the choice of } \mu_1 \Rightarrow =0} \\ & + \epsilon^2 \left(-\ddot{\mu} \mu_0 B_3 + \mu \left(-\ddot{\mu}_0 B_3 + 2\mu_0 g_1\right) + B_1 Q(d, x_0) + b_1(x_0)\right) \\ & + O(\epsilon^3 |\ln \epsilon|) \end{aligned} \tag{4.11}$$

where (see Remark 4.1)

$$g_1(x_0) := -A_2 h(x_0) + \sum \left(\frac{1}{3} R_{ijij} + R_{0j0j}\right) B_2 = -A_2 \sigma(x_0) \tag{4.12}$$

and

$$\begin{aligned}\int_{\mathcal{D}_{y_0}} \mathcal{S}_\epsilon(\omega) Z_0\, dy = {} & \epsilon \underbrace{\left(\lambda_1 e_0 - A_3 + \mu_0^2 g_2\right)}_{\text{the choice of } e_0 \Rightarrow =0} \\ & + \epsilon^2 \ln \epsilon \underbrace{\left(\lambda_1 e_1 + 2\mu_0 \mu_1 + D_2\right)}_{\text{the choice of } e_1 \Rightarrow =0} \\ & + \epsilon^2 \left(\epsilon a_0 \ddot{e} + \lambda_1 e + a_0 \ddot{e}_0 + b_2(x_0) + 2\mu_0 \mu g_2 + B_5 Q(d, x_0)\right) \\ & + O(\epsilon^3 |\ln \epsilon|) \end{aligned} \tag{4.13}$$

where

$$g_2(x_0) := -A_4 h(x_0) + \sum \left(\frac{1}{3} R_{ijij} + R_{0j0j} \right) B_6. \tag{4.14}$$

More precisely, μ_0 solves the periodic O.D.E.

$$-\ddot{\mu}_0 B_3 + g_1 \mu_0 \pm \frac{A_1}{\mu_0} = 0, \ \mu_0 > 0 \text{ in } [0, 2\ell]. \tag{4.15}$$

which is nothing but problem (1.6) or (1.7) where (see Remark 4.1)

$$a_n := -\frac{A_2}{B_3} > 0 \ \text{ and } b_n := \frac{A_1}{B_3} > 0 \text{ (see (4.4), (4.5) and (4.7))}. \tag{4.16}$$

Moreover,

$$e_0 = \frac{A_3 - \mu_0^2 g_2}{\lambda_1}. \tag{4.17}$$

Finally, μ_1 solves the periodic O.D.E.

$$-\ddot{\mu}_1 \mu_0 B_3 + \mu_1 \underbrace{\left(-\ddot{\mu}_0 B_3 + 2\mu_0 g_1\right)}_{= \mu_0 g_1 \mp \frac{A_1}{\mu_0^2}} + D_1 = 0 \text{ in } [0, 2\ell]. \tag{4.18}$$

We point out that μ_1 does exist, because μ_0 is a non degenerate solution of (4.15) (see also Lemma 6.1). Moreover,

$$e_1 = \frac{-2\mu_0\mu_1 - D_2}{\lambda_1}. \tag{4.19}$$

That concludes the proof. □

Remark 4.1. *It holds*

- $g_1(x_0) = -A_2 \sigma(x_0)$ *with* $A_2 < 0$ *(see* (4.5)*)*
- $A_1 > 0$ *(see* (4.4)*)*
- $a_n = -\frac{A_2}{B_3} = \frac{2(N-1)}{(N-2)(N+2)} = \frac{2(n-2)}{(n-3)(n+1)}$ *(see* (4.5) *and* (4.7)*)*
- $b_n = \frac{A_1}{B_3} = \frac{(N-2)^2(N-4)}{4(N+2)} = \frac{(n-3)^2(n-5)}{4(n+1)}$ *(see* (4.4) *and* (4.7)*)*

Proof. It is useful to point out that

$$\frac{B_2}{A_2} = \frac{3(N-2)}{4(N-1)}.$$

Indeed, if we denote by

$$I_p^q := \int_0^{+\infty} \frac{r^q}{(1+r)^p} dr \text{ if } p - q > 1$$

and we use the properties

$$I_{p+1}^q = \frac{p-(q+1)}{p} I_p^q \text{ and } I_{p+1}^{q+1} = \frac{q+1}{p-(q+1)} I_{p+1}^q$$

a straightforward computation shows that

$$A_1 = \frac{N}{(p+1)^2} \int_{\mathbb{R}^N} w^{p+1} \, dy = c_N^2 \frac{(N-2)^4}{8N} \omega_N I_N^{N/2} > 0,$$

$$A_2 = \int_{\mathbb{R}^N} w Z_{N+1} \, dy = -c_N^2 \frac{2(N-1)(N-2)}{N(N-4)} \omega_N I_N^{N/2} < 0,$$

$$B_2 = \int_{\mathbb{R}^N} y_j \partial_j w Z_{N+1} \, dy = -c_N^2 \frac{3(N-2)^2}{2N(N-4)} \omega_N I_N^{N/2} < 0$$

and

$$B_3 = \int_{\mathbb{R}^N} Z_{N+1}^2\, dy = c_N^2 \frac{(N-2)^2(N+2)}{2N(N-4)} \omega_N I_N^{N/2} > 0,$$

where ω_N is the measure of the sphere $\mathbb{S}^{N-1}$. Therefore, we immediately deduce the quantities a_n and b_n, taking into account that $N = n-1$.
Moreover, it is easy to check that

$$\begin{aligned} \frac{1}{3}\sum_{i,j=1}^{N} R_{ijij}(x_0) + \sum_{j=1}^{N} R_{0j0j}(x_0) &= \frac{1}{3}\sum_{i,j=0}^{N} R_{ijij}(x_0) - \frac{1}{3}\sum_{j=1}^{N} R_{0j0j}(x_0) \\ &= \frac{1}{3} R_g(x_0) - \frac{N}{3} Ric\left(\dot\gamma(x_0), \dot\gamma(x_0)\right) \qquad (4.20) \end{aligned}$$

Therefore, the claim follows. □

5. The infinite dimensional reduction

5.1. **The gluing procedure.** Here we perform a gluing procedure that reduces the full problem (1.2) to the scaled problem (3.11) in the neighborhood of the scaled geodesic.
Since the procedure is very similar to that of [6] we briefly sketch it.

We denote by M_ρ the scaled manifold $\frac{1}{\rho}M$, by z the original variable in M_ρ and by $\xi := \rho z$ the corresponding point in M. It is clear that the function $u(x)$ is a solution to (1.2) if and only if the function $v(z) := \rho^{\frac{N-2}{2}} u(\rho z)$ solves the problem

$$\Delta_g v - \rho^2 h v + \rho^{-\frac{N-2}{2}\epsilon} v^{p-\epsilon} = 0 \qquad \text{in } \mathcal{M}_\rho \tag{5.1}$$

The function $\tilde{\boldsymbol{\omega}}(y_0, y)$ constructed in (3.13) defines an approximation to a solution of (1.2) near the geodesic through the natural change of variables (3.9).
It is useful to introduce the following notation. Let $f(z)$ be a function defined in a small neighborhood of the scaled geodesic $\Gamma_\rho := \frac{1}{\rho}\Gamma$. Through the change of variables (3.9) we denote by

$$\tilde f(y_0, y) = \tilde\mu_\epsilon^{-\frac{N-2}{2}}(\rho y_0) f\left(\frac{1}{\rho} F\left(\rho y_0, \mu_\epsilon(\rho y_0) + d_\epsilon(\rho y_0)\right)\right), \tag{5.2}$$

where the point $\rho z = F(\rho y_0, \mu_\epsilon(\rho y_0) + d_\epsilon(\rho y_0)) \in M$ and $\tilde\mu_\epsilon, \mu_\epsilon$ and d_ϵ are defined in (3.8) and (3.7). According this notation, we set $\boldsymbol{\omega} = \boldsymbol{\omega}(z)$ the function corresponding to $\tilde{\boldsymbol{\omega}} = \tilde{\boldsymbol{\omega}}(y_0, y)$.
Let $\delta > 0$ be a fixed number with $4\delta < \hat\delta$, where $\hat\delta$ is given in (3.1). We consider a smooth cut-off function $\zeta_\delta(s)$ such that $\zeta_\delta(s) = 1$ if $0 < s < \delta$ and $\zeta_\delta(s) = 0$ if $s > 2\delta$. Let us consider the cut-off function η_δ^ϵ defined on the manifold M_ρ by

$$\eta_\delta^\epsilon(z) = \zeta_\delta\left(\frac{dist_g(\xi, \Gamma)}{\rho}\right) \qquad \text{for } \rho z = \xi \in M.$$

We remark that with this definition $\eta_\delta^\epsilon(z)$ does not depend on the parameter functions.
We define our global first approximation of the problem (1.2) $\mathbf{w}(z)$ as

$$\mathbf{w}(z) = \eta_\delta^\epsilon(z)\boldsymbol{\omega}(z). \tag{5.3}$$

We look for a solution to problem (5.1) of the form $u = \mathbf{w} + \Phi$, namely

$$\Delta_g \Phi + p\mathbf{w}^{p-1}\Phi + N(\Phi) + E = 0 \text{ in } \mathcal{M}_\rho \tag{5.4}$$

where

$$N(\Phi) = \rho^{-\frac{N-2}{2}\epsilon}(\mathbf{w} + \Phi)^{p-\epsilon} - \mathbf{w}^{p-\epsilon} - p\mathbf{w}^{p-1}\Phi - \rho^2 h(\mathbf{w} + \Phi) \tag{5.5}$$

and

$$E = \Delta_g \mathbf{w} + \mathbf{w}^{p-\epsilon}. \tag{5.6}$$

We look for a solution Φ of (5.4) as $\Phi = \eta_{2\delta}\phi + \psi$ where the function ϕ is such that the corresponding function $\tilde{\phi}$ via the change of variables (5.2) is defined only in $\mathcal{D}$. It is immediate to check that Φ of this form solves (5.4) if the pair (ψ, ϕ) solves the following nonlinear coupled system:

$$\Delta_g\psi + (1-\eta^{\epsilon}_{2\delta})p\mathbf{w}^{p-1}\psi = -2\nabla_g\phi\nabla_g\eta^{\epsilon}_{2\delta} - \phi\Delta_g\eta^{\epsilon}_{2\delta} - (1-\eta^{\epsilon}_{2\delta})N(\eta^{\epsilon}_{2\delta}\phi + \psi) \text{ in } \mathcal{M}_\rho \tag{5.7}$$

and

$$\mathcal{A}(\tilde{\phi}) + p\tilde{\boldsymbol{\omega}}^{p-1}\tilde{\phi} = -\mathcal{N}(\zeta^{\epsilon}_{2\delta}\tilde{\phi} + \tilde{\psi}) - \mathcal{S}_\epsilon(\tilde{\boldsymbol{\omega}}) - p\tilde{\boldsymbol{\omega}}^{p-1}\tilde{\psi} \text{ in } \mathcal{D}, \tag{5.8}$$

where

$$\mathcal{N}(\tilde{\Phi}) = \tilde{\mu}_\epsilon^{-\frac{N-2}{2}\epsilon}(\tilde{\boldsymbol{\omega}} + \tilde{\Phi})^{p-\epsilon} - \mathbf{w}^{p-\epsilon} - p\tilde{\boldsymbol{\omega}}^{p-1}\tilde{\Phi} - \tilde{\mu}_\epsilon^2\tilde{h}\tilde{\Phi},\ \tilde{\Phi} = \zeta^{\epsilon}_{2\delta}\tilde{\phi} + \tilde{\psi}. \tag{5.9}$$

Indeed, problem (5.4) in a scaled neighborhood of the geodesic looks like problem 5.8 and the error E given in (5.6) via the change of variables (5.2) is nothing but the error term $\mathcal{S}_\epsilon(\tilde{\boldsymbol{\omega}})$ defined in (3.26).

Given ϕ such that $\tilde{\phi}$ is defined in $\mathcal{D}$, we first solve problem (5.7) for ψ (see Section 6 of [6]).

Lemma 5.1. *For any $R > 0$ there exists $r > 0$ such that for any function ϕ such that the corresponding function $\tilde{\phi}$ is defined only in $\mathcal{D}$ with $\|\tilde{\phi}\|_* \leqslant r$, there exists a unique solution $\psi = \psi(\phi)$ of* (5.7) *with*

$$\|\psi\|_\infty \le R\epsilon^{\frac{N-4}{2}}\|\tilde{\phi}\|_*.$$

Moreover, the nonlinear operator ψ satisfies a Lipschitz condition of the form

$$\|\psi(\phi_1) - \psi(\phi_2)\|_\infty \le c\epsilon^{\frac{N-4}{2}}\|\phi_1 - \phi_2\|_*, \tag{5.10}$$

for some positive constant c independent on ϵ.

Finally, we substitute $\tilde{\psi} = \tilde{\psi}(\phi)$ (via the change of variables (5.2)) in the equation (5.7) and we reduce the full problem (1.2) to solving the following (nonlocal) problem in $\mathcal{D}$:

$$\mathcal{A}(\tilde{\phi}) + p\tilde{\boldsymbol{\omega}}^{p-1}\tilde{\phi} = -\mathcal{N}(\eta^{\epsilon}_{2\delta}\tilde{\phi} + \tilde{\psi}(\phi)) - \mathcal{S}_\epsilon(\tilde{\boldsymbol{\omega}}) - p\tilde{\boldsymbol{\omega}}^{p-1}\tilde{\psi}(\phi) \text{ in } \mathcal{D}. \tag{5.11}$$

5.2. **The nonlinear projected problem.** We can solve the following projected problem associated to (5.11): *given μ, d and e satisfying* (3.18), *find functions $\tilde{\phi}$ and $c_j(y_0)$ for $j = 0, \ldots, N+1$ such that*

$$\begin{cases} L(\tilde{\phi}) = -S_\epsilon(\tilde{\boldsymbol{\omega}}) + \mathfrak{N}(\tilde{\phi}) + \sum_{j=0}^N c_j Z_j & \text{in } \mathcal{D} \\ \tilde{\phi}\left(y_0 + \frac{2\ell}{\rho}, y\right) = \phi(y_0, Ay) & \text{for any } (y_0, y) \in \mathcal{D}, \\ \int_{\mathcal{D}_{y_0}} \tilde{\phi} Z_j dy = 0 \text{ and for any } y_0 \in \left[-\frac{\ell}{\rho}, \frac{\ell}{\rho}\right],\ j = 0, 1, \ldots, N+1. \end{cases} \tag{5.12}$$

Here $S_\epsilon(\tilde{\boldsymbol{\omega}})$ is given in (3.26) and

$$L(\tilde{\phi}) := \mathcal{A}(\tilde{\phi}) + p\omega^{p-1}\tilde{\phi} \ (\mathcal{A} \text{ is in Lemma 3.2 and } \omega \text{ is in (3.5)}),$$
$$\mathfrak{N}(\tilde{\phi}) := p(\omega^{p-1} - \tilde{\boldsymbol{\omega}}^{p-1})\tilde{\phi} - \mathcal{N}(\zeta^{\epsilon}_{2\delta}\tilde{\phi} + \tilde{\psi}(\phi)) - p\tilde{\boldsymbol{\omega}}^{p-1}\tilde{\psi}(\phi) \ (\mathcal{N} \text{ is in (5.9)}).$$

Proposition 5.2. *There exists $c > 0$ such that for all sufficiently small ϵ and all μ, d and e satisfying* (3.18), *problem* (5.12) *has a unique solution $\tilde{\phi} = \tilde{\phi}(\mu, d, e)$ and $c_j = c_j(\mu, d, e)$ which satisfies*

$$\|\phi\|_* \le c\epsilon^{\frac{3}{2}}. \tag{5.13}$$

Moreover, $\tilde{\phi}$ depends Lipschitz continuously on μ, d and e in the sense

$$\|\tilde{\phi}(\mu_1, d_1, e_1) - \tilde{\phi}(\mu_2, d_2, e_2)\|_* \leqslant \epsilon^{\frac{5}{2}}\|(\mu_1 - \mu_2, d_1 - d_2, e_1 - e_2)\|$$

for some positive constant c independent of ϵ and uniformly with respect to μ, d and e which satisfy (3.18).

Proof. We argue exactly as in Section 7 of [6], using a contraction mapping argument and the linear theory developed in Proposition 7.3. □

6. The reduced problem

6.1. **The reduced system.** We find $N+1$ equations relating μ, d and e to get all the coefficients c_j in (5.12) identically equal to zero. To do this, we multiply equation (5.12) by Z_j, for all $j = 0, \dots, N+1$ and we integrate in y. Thus, the system

$$c_j(\rho y_0) = 0,\ j = 0, 1, \dots, N+1$$

is equivalent to

$$\int_{\mathcal{D}_{y_0}} S_\epsilon(\tilde{\boldsymbol{\omega}}) Z_j\, dy + \int_{\mathcal{D}_{y_0}} \left(L(\tilde{\phi}) - \mathfrak{N}(\tilde{\phi}) \right) Z_j\, dy = 0,\ j = 0, 1, \dots, N+1,$$

for any $y_0 \in \left[-\frac{\ell}{\rho}, \frac{\ell}{\rho} \right]$.
By Proposition 5.2 it follows that

$$\int_{\mathcal{D}_{y_0}} \left(L(\tilde{\phi}) - \mathfrak{N}(\tilde{\phi}) \right) Z_j\, dy = \epsilon^3 \theta,$$

where $\theta = \theta(\rho y_0)$ is as in Lemma 3.4.

Hence the equations $c_j = 0$ are equivalent to the following limit system on $N+2$ nonlinear ordinary differential equations:

$$\begin{cases} L_{N+1}(\mu) := -\ddot{\mu} + \left(a_n \sigma \pm \frac{b_n}{\mu_0^2} \right) \mu = -\alpha_{N+1}(x_0) - c_3 Q(x_0, d) + \epsilon |\ln \epsilon| M_{N+1} \\ \\ L_k(d) := -\ddot{d}_k + \sum_{j=1}^N R_{0j0k} d_j = \sqrt{\epsilon} M_k,\ k = 1, \dots, N \\ \\ L_0(e) := \epsilon a_0 \ddot{e} + \lambda_1 e = -\alpha_0(x_0) - c_4 Q(x_0, d) - \beta(x_0)\mu + \epsilon |\ln \epsilon| M_0 \end{cases} \tag{6.1}$$

where $\mu, d_1, \dots, d_N, e \in C^2_{2\ell}(\mathbb{R})$ and

- the functions α_i and β are explicit functions of x_0, smooth and uniformly bounded in ϵ given in Lemma 3.4
- the operator Q is quadratic in d (see Lemma 3.4) and it is uniformly bounded in $L^\infty_{2\ell}(\mathbb{R})$ for (μ, d, e) satisfying (3.18)
- the operators $M_i = M_i(\mu, d, e)$ can be decomposed as $M_i(\mu, d, e) = A_i(\mu, d, e) + K_i(\mu, d, e)$ where
  - K_i is uniformly bounded in $L^\infty_{2\ell}(\mathbb{R})$ for (μ, d, e) satisfying (3.18) and it is compact
  - A_i depends on (μ, d, e) and their first and second derivatives and it satisfies

    $$\|A_i(\mu_2, d_2, e_2) - A_i(\mu_1, d_1, e_1)\| \leqslant o(1) \|(\mu_2 - \mu_1, d_2 - d_1, e_2 - e_1)\|$$

    uniformly for (μ, d, e) satisfying (3.18)
  - the dependance on $(\ddot{\mu}, \ddot{d}, \ddot{e})$ is linear

Our goal is to solve (6.1) in μ, d and e. To do so, we first analyze the invertibility of the linear operator L_{N+1}.

Lemma 6.1. *For any $f \in L^\infty_{2\ell}(\mathbb{R})$, there exists a unique $\mu \in C^2_{2\ell}(\mathbb{R})$ solution of $L_{N+1}(\mu) = f$. Moreover, there exists c such that*

$$\|\mu\|_\infty + \|\dot{\mu}\|_\infty \le c\|f\|_\infty.$$

Proof. The non degeneracy condition of the solution μ_0 translates into the fact that the periodic O.D.E.

$$-\ddot{\mu} + \left(a_n\sigma \pm \frac{b_n}{\mu_0^2}\right)\mu = 0 \text{ in } [0, 2\ell]$$

has only the trivial solutions. Therefore the claim follows. □

Next, we analyze the invertibility of the linear operator L_0.

Lemma 6.2. *Assume*

$$|\epsilon m^2 - \kappa^2| > \nu\sqrt{\epsilon} \text{ for any } m = 1, 2, \ldots$$

for some ν positive, where

$$\kappa := \frac{\pi}{2}\sqrt{\lambda_1}\int_{-\ell}^{+\ell} \frac{1}{\sqrt{a_0(s)}} ds.$$

For any $f \in C^0_{2\ell}(\mathbb{R}) \cap L^\infty_{2\ell}(\mathbb{R})$, there exists a unique solution $e \in C^2_{2\ell}(\mathbb{R})$ of $L_0(e) = f$. Moreover, there exists c such that

$$\epsilon\|\ddot{e}\|_\infty + \sqrt{\epsilon}\|\dot{e}\|_\infty + \|e\|_\infty \le c\frac{1}{\sqrt{\epsilon}}\|f\|_\infty,$$

Finally, if $f \in C^2_{2\ell}(\mathbb{R})$, then

$$\epsilon\|\ddot{e}\|_\infty + \sqrt{\epsilon}\|\dot{e}\|_\infty + \|e\|_\infty \le c\left[\|\ddot{f}\|_\infty + \|\dot{f}\|_\infty + \|f\|_\infty\right].$$

Proof. We argue as in in Lemma 8.2 of [6]. □

Finally, we consider the invertibility of the linear operator $(L_1, \ldots, L_N)$.

Lemma 6.3. *Assume the geodesic is non degenerate. For any $f = (f_1, \ldots, f_N)$ with $f_k \in L^\infty_{2\ell}(\mathbb{R})$, there exists a $d = (d_1, \ldots, d_N)$ with $d_k \in C^2_{2\ell}(\mathbb{R})$ such that $L_k(d) = f_k$ for any $k = 1, \ldots, N$. Moreover, there exists c such that*

$$\|\ddot{d}\|_\infty + \|\dot{d}\|_\infty + \|d\|_\infty \le c\|f\|_\infty.$$

Proof. It is useful to point out that assumption (1.3) about non degeneracy of Γ in normal coordinates translates exactly into the fact that the linear system of O.D.E.'s

$$-\ddot{d}_k + \sum_{j=1}^{N} R_{0j0k}d_j = 0, \text{ in } [0, 2\ell], \quad k = 1, \ldots, N,$$

has only the trivial solution $d \equiv 0$ satisfying the periodicity condition (3.6). Therefore, the claim follows.

□

6.2. **The choice of parameters: the proof completed!** Now, we are ready to complete the proof, finding parameters which solve the reduced problem (6.1).

First, by Lemma 6.1 we find $\hat\mu_0$ solution of

$$L_{N+1}(\hat\mu_0) = -\alpha_{N+1}(x_0), \quad \text{with } \|\ddot{\hat\mu}_0\|_\infty + \|\dot{\hat\mu}_0\|_\infty + \|\hat\mu_0\|_\infty \le c.$$

Then, by Lemma 6.2 we find $\hat e_0$ solution of

$$L_0(\hat e_0) = -\alpha_0 - \beta\hat\mu_0, , \quad \text{with } \epsilon\|\ddot{\hat e}_0\|_\infty + \sqrt{\epsilon}\|\dot{\hat e}_0\|_\infty + \|\hat e_0\|_\infty \le c.$$

Therefore, $\|(\hat\mu_0, 0, \hat e_0)\| \le c$. Let us define

$$\mu = \hat\mu_0 + \hat\mu_1, \ d = \hat d_1, \ e = \hat e_0 + \hat e_1.$$

The system (6.1) reduces to

$$\begin{cases} L_{N+1}(\hat\mu_1) = -c_3 Q(x_0, \hat d_1) + \epsilon|\ln\epsilon| M_{N+1} \\ \\ L_k(\hat d_1) = \sqrt{\epsilon} M_k, \ k = 1, \dots, N \\ \\ L_0(\hat e_1) = -c_4 Q(x_0, \hat d_1) - \beta(x_0)\hat\mu_1 + \epsilon|\ln\epsilon| M_0 \end{cases} \tag{6.2}$$

Let us observe now that the linear operator

$$\mathcal{L}(\hat\mu_1, \hat d_1, \hat e_1) = (L_{N+1}(\hat\mu_1), L_N(\hat d_1), \dots, L_1(\hat d_1), L_0(\hat e_1))$$

is invertible with bounds for $\mathcal{L}(\hat\mu_1, \hat d_1, \hat e_1) = (f, g, h)$ given by

$$\|(\hat\mu_1, \hat d_1, \hat e_1)\| \le C\left[\|f\|_\infty + \|g\|_\infty + \epsilon^{-1/2}\|h\|_\infty\right].$$

Finally, by the contraction mapping principle it follows that, the problem (6.2) has a unique solution with

$$\|\hat\mu_1\|_\infty < c\epsilon|\ln\epsilon|, \qquad \|\hat d_1\|_\infty < \sqrt{\epsilon}, \qquad \|\hat e_1\|_\infty < \sqrt{\epsilon}|\ln\epsilon|.$$

That concludes the proof.

7. The linear theory

Here we recall a linear theory necessary to solve problem (3.11), which has been developed in Section 3 of [6].

Let us consider the operator $\mathcal{L}_0 := \Delta_{\mathbb{R}^N} + pw^{p-1}$. It is well-known that the L^2- null space of the operator $\mathcal{L}_0$ is $N+1-$ dimensional and spanned by the functions

$$Z_j(y) := \partial_j w(y), \ j = 1, \dots, N \text{ and } Z_{N+1}(y) := y \cdot \nabla w(y) + \frac{N-2}{2} w(y).$$

Moreover it is known that (see [6]) that the operator $\mathcal{L}_0$ has one negative eigenvalue $-\lambda_1 < 0$, whose corresponding eigenfunction Z_0 (normalized to have L^2- norm equal to 1) decays exponentially at infinity with exponential order $O(e^{-\sqrt{\lambda_1}|x|})$.

The following results (see Lemma 3.1 of [6] and also [7]) are useful in order to obtain a priori estimates and a solvability theory for problem (3.11).

Lemma 7.1. *Assume that $\lambda \notin \{0, \pm\sqrt{\lambda_1}\}$. Then for $g \in L^\infty(\mathbb{R}^N)$, there exists a unique bounded solution of*

$$(\mathcal{L}_0 - |\lambda|^2)\psi = g$$

in $\mathbb{R}^N$. Moreover

$$\|\psi\|_{L^\infty} \le c_\lambda \|g\|_{L^\infty}$$

for some constant $c_\lambda > 0$ only depending on λ.

Lemma 7.2. *Let ϕ a bounded solution of*

$$\partial_{00}\phi + \Delta_y \phi + pw^{p-1}\phi = 0 \qquad in \qquad \mathbb{R}^{N+1}.$$

Then $\phi(y_0, y)$ is a linear combination of the functions Z_j, $j = 1, \dots, N+1$, $Z_0(y)\cos(\sqrt{\lambda_1} y_0)$, $Z_0(y)\sin(\sqrt{\lambda_1} y_0)$.

Now, we study a slightly more general problem than (3.11) that involves the essential features needed. For any constant $M > 0$ we consider the domain $\mathcal{D}$ defined as

$$\mathcal{D} := \left\{ (y_0, y) \in \mathbb{R} \times \mathbb{R}^N \quad : \quad |y| < M \right\} \tag{7.1}$$

and given a function ϕ defined on $\mathcal{D}$, an operator of the form

$$L(\phi) := b(y_0)\partial_{00}\phi + \Delta_y \phi + pw^{p-1}\phi + \sum_{i,j} b_{ij}(y_0, y)\partial_{ij}\phi + \sum_i b_i(y_0, y)\partial_i \phi + d(y_0, y)\phi.$$

Then for a given function g we want to solve the following projected problem:

$$\begin{cases} L(\phi) = g + \sum_{j=0}^{N+1} c_j(y_0) Z_j(y) \qquad \text{in } \mathcal{D} \\ \int_{\mathcal{D}_{y_0}} \phi(y_0, y) Z_j(y)\, dy = 0 \qquad \text{for any } y_0 \in \mathbb{R},\ j = 0, \dots, N \end{cases} \tag{7.2}$$

where

$$\mathcal{D}_{y_0} := \left\{ y \in \mathbb{R}^N : (y_0, y) \in \mathcal{D} \right\}.$$

We fix a number $2 \le \nu < N$ and consider the $L^\infty-$ weighted norms

$$\|\phi\|_* := \sup_{\mathcal{D}} (1 + |y|^{\nu-2})|\phi(y_0, y)| + \sup_{\mathcal{D}} (1 + |x|^{\nu-1})|D\phi(x_0, x)|,$$
$$\|g\|_{**} := \sup_{\mathcal{D}} (1 + |y|^{\nu})|g(y_0, y)|.$$

We assume that all functions involved are smooth. The following result (see Proposition 3.2 of [6]) establishes existence and uniform a priori estimates for problem (7.2) in the above norms, provided that appropriate bounds for the coefficients hold.

Proposition 7.3. *Assume that $N \ge 7$ and $N - 2 \le \nu < N$. Assume that there exists $m > 0$ such that*

$$m \le b(y_0) \le m^{-1} \qquad \textit{for any } y_0 \in \mathbb{R}.$$

There exist $\delta > 0$ and $C > 0$ such that if

$$M\|\partial_0 b\|_\infty + \sum_{i,j} (\|b_{ij}\|_\infty + \|Db_{ij}\|_\infty) + \sum_i \|(1 + |y|)b_i\|_\infty + \|(1 + |y|^2)d\|_\infty < \delta \tag{7.3}$$

then for any g with $\|g\|_{**} < \infty$ there exists a unique solution $\phi = T(g)$ of problem (7.2) with $\|\phi\|_* < \infty$ and it holds true that

$$\|\phi\|_* \le C\|g\|_{**}.$$

8. Appendix

8.1. **Proof of** (3.4)**.** Let $E_0, E_1, \dots, E_N$ the coordinate vectors as given in the Introduction. By our choice of coordinates it follows that $\nabla_E E = 0$ on Γ for any vector field E, that is a linear combination (with coefficients depending only on x_0) of the E_j's, $j = 1, \dots, N$.
In particular, for any $i, j = 1, \dots, N$ and for any $t \in \mathbb{R}$, we have $\nabla_{E_i + tE_j}(E_i + tE_j) = 0$ on Γ, which implies $\nabla_{E_i} E_j + \nabla_{E_j} E_i = 0$ for every $i, j = 1, \dots, N$.
Using the fact that E_i's are coordinate vectors for $j = 1, \dots, N$ and in particular $\nabla_{E_a} E_b = \nabla_{E_b} E_a$ for all $a, b = 0, \dots, N$, we obtain that $\nabla E_j E_i = 0$ for every $i, j = 1, \dots, N$. The geodesic coordinate for Γ translates precisely into $\nabla E_0 E_0 = 0$.
These facts immediately yields

$$\partial_m g_{ij} = E_m \langle E_i, E_j \rangle = \langle \nabla_{E_m} E_i, E_j \rangle + \langle E_i, \nabla_{E_m} E_j \rangle = 0 \tag{8.1}$$

on Γ with $i, j, m = 1, \dots, N$.

Moreover, since E_a's are coordinate vectors for $a = 0, \dots, N$, we obtain

$$\begin{aligned} \partial_m g_{0j} &= E_m \langle E_0, E_j \rangle \\ &= \langle \nabla_{E_m} E_0, E_j \rangle + \langle E_0, \nabla_{E_m} E_j \rangle \\ &= \langle \nabla_{E_0} E_m, E_j \rangle + \langle E_0, \nabla_{E_m} E_j \rangle = 0 \end{aligned} \tag{8.2}$$

on Γ with $m, j = 1, \dots, N$.

Here we used the fact that $\nabla_{E_0} E_m = 0$ on Γ, namely that $\nabla_{E_0} E_m$ has zero normal components.
Moreover by (8.1) it follows that

$$\partial_m g_{00} = 0 \qquad \text{on} \quad \Gamma. \tag{8.3}$$

We can also prove that the components R_{0m0j} of the curvature tensor are given by

$$R_{0m0j} = -\frac{1}{2} \partial_{mj} g_{00}. \tag{8.4}$$

Indeed, we have

$$\begin{aligned} -R_{0m0j} &= \langle R(E_0, E_j) E_0, E_m \rangle \\ &= \langle \nabla_{E_0} E_j E_0, E_m \rangle - \langle \nabla_{E_j} \nabla_{E_0} E_0, E_m \rangle \\ &= \langle \nabla_{E_0} \nabla E_j E_0, E_m \rangle - E_j \langle \nabla_{E_0} E_0, E_m \rangle - \langle \nabla_{E_0} E_0, \nabla_{E_j} E_m \rangle \\ &= \langle \nabla_{E_0} \nabla_{E_j} E_0, E_m \rangle - E_j \langle \nabla_{E_0} E_0, E_m \rangle \\ &= \langle \nabla_{E_0} \nabla_{E_j} E_0, E_m \rangle - E_j E_0 \rangle E_0, E_m \rangle + E_j \langle E_0, \nabla_{E_0} E_m \rangle \\ &= \langle \nabla_{E_0} \nabla_{E_j} E_0, E_m \rangle + E_j \langle E_0, \nabla_{E_m} E_0 \rangle \\ &= \frac{1}{2} E_j E_m \langle E_0, E_0 \rangle + E_0 \langle \nabla_{E_j} E_0, E_m \rangle - \langle \nabla_{E_j} E_0, \nabla_{E_0} E_m \rangle \\ &= \frac{1}{2} \partial_{mj} g_{00} \end{aligned}$$

where here we have used the above properties and the fact that

$$\nabla_{E_j} E_0 = \nabla_{E_0} E_j = \frac{1}{2} \partial_j g_{00} E_0 = 0.$$

By (8.2), (8.4), (8.3) and (8.1) the claim follows.

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