On the computation and inversion of the cumulative noncentral beta distribution function

Amparo Gil

Departamento de Matemática Aplicada y CC. de la Computación.
ETSI Caminos. Universidad de Cantabria. 39005-Santander, Spain.

Javier Segura

Departamento de Matemáticas, Estadística y Computación,
Univ. de Cantabria, 39005 Santander, Spain.

Nico M. Temme

IAA, 1825 BD 25, Alkmaar, The Netherlands.*

e-mail: amparo.gil@unican.es, javier.segura@unican.es, nico.temme@cwi.nl

Abstract

The computation and inversion of the noncentral beta distribution $B_{p,q}(x,y)$ (or the noncentral F-distribution, a particular case of $B_{p,q}(x,y)$) play an important role in different applications. In this paper we study the stability of recursions satisfied by $B_{p,q}(x,y)$ and its complementary function and describe asymptotic expansions useful for computing the function when the parameters are large. We also consider the inversion problem of finding x or y when a value of $B_{p,q}(x,y)$ is given. We provide approximations to x and y which can be used as starting values of methods for solving nonlinear equations (such as Newton) if higher accuracy is needed.

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*Former address: Centrum Wiskunde & Informatica (CWI), Science Park 123, 1098 XG Amsterdam, The Netherlands

1   Introduction

The cumulative noncentral beta distribution can be defined as[1]

$$B_{p,q}(x,y) = e^{-x/2} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{x}{2}\right)^j I_y(p+j,q), \tag{1.1}$$

where $I_y(p,q)$ is the central beta distribution

$$I_y(p,q) = \frac{1}{B(p,q)} \int_0^y t^{p-1}(1-t)^{q-1} dt, \quad B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{1.2}$$

These functions satisfy the recurrence relation

$$(p+j)f_{j+1} = (p+j+(p+j+q-1)y) f_j - (p+j+q-1)y f_{j-1}, \tag{1.3}$$

where $f_j = I_y(p+j,q)$. The recursion is not stable in the forward direction, but we can use a backward recursion scheme (see [5, §4.6]).

The complementary function is defined by

$$\bar{B}_{p,q}(x,y) = 1 - B_{p,q}(x,y), \tag{1.4}$$

and using $I_y(p,q) = 1 - I_{1-y}(q,p)$ we can write the series

$$\bar{B}_{p,q}(x,y) = e^{-x/2} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{x}{2}\right)^j I_{1-y}(q,p+j). \tag{1.5}$$

This is related with the so-called beta distribution of type II, $J_x(p,q,\lambda)$ [4], by $J_y(p,q,x) = \bar{B}_{q,p}(1-y,x)$.

Using (1.2) in (1.1) we get

$$B_{p,q}(x,y) = \frac{e^{-x/2}}{B(p,q)} \int_0^y t^{p-1}(1-t)^{q-1} M\left(p+q, p, \tfrac{1}{2}xt\right) dt, \tag{1.6}$$

where $M(a,b,z)$ is the confluent hypergeometric function or Kummer function; see [9, Ch. 13]. Similarly for $\bar{B}_{p,q}(x,y)$ but with the integration going from y to 1.

Therefore the probability density function can be written

$$\frac{\partial B_{p,q}(x,y)}{\partial y} = \frac{e^{-x/2}}{B(p,q)} y^{p-1}(1-y)^{q-1} M\left(p+q, p, \tfrac{1}{2}xy\right) > 0. \tag{1.7}$$

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[1]Different notations are used in the literature; for this we refer to §2.

The derivative with respect to x can be computed using (1.1). We get

$$\frac{\partial B_{p,q}(x,y)}{\partial x} = \tfrac{1}{2}\left(B_{p+1,q}(x,y) - B_{p,q}(x,y)\right), \tag{1.8}$$

which, using the recurrence relation (3.1) that we later obtain, can be written

$$\frac{\partial B_{p,q}(x,y)}{\partial x} = -\frac{e^{-x/2}}{2pB(p,q)}y^p(1-y)^q M\left(p+q,p+1,\tfrac{1}{2}xy\right) < 0. \tag{1.9}$$

By using the recurrence relations for the Kummer functions [9, §13.3(i)] it is possible to check, that, as expected, taking the derivative of (1.7) with respect to x gives the same value as the derivative of (1.9) with respect to y.

Particular values are (some are evident and the rest become evident in a later analysis)

$$
\begin{aligned}
B_{p,q}(x,0) &= 0, & B_{p,q}(x,1) &= 1, \\
B_{p,q}(0,y) &= I_y(p,q), & B_{p,q}(+\infty,y) &= 0, \\
B_{+\infty,q}(x,y) &= 0, & B_{p,+\infty}(x,y) &= 1,
\end{aligned} \tag{1.10}
$$

and, as corresponds to a cumulative distribution function, we have $0 \leq B_{p,q}(x,y) \leq 1$ for $p,q > 0$ and $x,y \geq 0$.

In this paper, we give useful expressions for computing $B_{p,q}(x,y)$ and its complementary function: inhomogeneous and three and four-term homogeneous recurrence relations; asymptotic expansions. For the three and four-term recurrence relations, their stability is also analyzed. Numerical examples illustrate the accuracy of the expressions. We also consider the inversion problem of finding x or y when a value of $B_{p,q}(x,y)$ is given. We provide approximations to x and y which can be used as starting values of methods for solving nonlinear equations (such as Newton) if higher accuracy is needed.

2   Other notations in the literature

We have chosen for the definition and notation in (1.1), knowing that several other definitions and notations exist. We like to separate the parameters p,q from x,y. The first set can be used for recursions, the second set for differentiation.

First of all, the noncentral beta distribution and the noncentral-F distribution are the same, up to notation of the parameters. The cumulative

distribution function $F(w, \nu_1, \nu_2, \lambda)$ of the noncentral F-distribution, with ν_1, ν_2 degrees of freedom and noncentrality parameter λ and the noncentral beta distribution are related as follows:

$$F(w, \nu_1, \nu_2, \lambda) = B_{p,q}(\lambda, x), \tag{2.1}$$

where

$$\nu_1 = 2p, \quad \nu_2 = 2q, \quad x = \frac{\nu_1 w}{\nu_1 w + \nu_2}. \tag{2.2}$$

In [11] and [7] a function $q(\lambda, \omega; 2a, 2b)$ is defined by

$$1 - q(\lambda, \omega; 2a, 2b) = e^{-\lambda} x^a (1-x)^b \sum_{n=0}^{\infty} \frac{\Gamma(a+b+n)}{\Gamma(b)\Gamma(a+1+n)} x^n \sum_{j=0}^{n} \frac{\lambda^j}{j!}, \tag{2.3}$$

where $x = \omega/(\omega + 1)$. In our notation we have

$$1 - q(\lambda, \omega; 2a, 2b) = B_{a,b}(2\lambda, x). \tag{2.4}$$

For a proof we refer to the Appendix. It follows that (see (1.5))

$$q(\lambda, \omega; 2a, 2b) = \bar{B}_{a,b}(2\lambda, x). \tag{2.5}$$

In [4], [1] and [2] the definition of the noncentral beta function is

$$I_x(a, b; \lambda) = e^{-\frac{1}{2}\lambda} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{x}{2}\right)^j I_x(a+j, b), \tag{2.6}$$

and the relation with our definition is

$$I_x(a, b; \lambda) = B_{a,b}(\lambda, x). \tag{2.7}$$

3   Recurrence relations

We obtain two-term inhomogeneous recurrence relations for the beta distributions, as well as three and four-term homogeneous recurrence relations, and we discuss the stability of these recursive processes.

3.1 First order recurrence relations

Using the recurrence relations for the central distribution given in [10, 8.17(iv)] together with the definition (1.1) it is easy to get analogous relations for the noncentral case.

Using [10, 8.17.20] we obtain

$$B_{p,q}(x,y) = B_{p+1,q}(x,y) + \frac{e^{-x/2}}{pB(p,q)} y^p (1-y)^q M\left(p+q, p+1, \tfrac{1}{2}xy\right), \quad (3.1)$$

from [10, 8.17.21]

$$B_{p,q}(x,y) = B_{p,q+1}(x,y) - \frac{e^{-x/2}}{qB(p,q)} y^p (1-y)^q M\left(p+q, p, \tfrac{1}{2}xy\right), \quad (3.2)$$

and from [10, 8.17.19]

$$B_{p,q}(x,y) = B_{p-1,q+1}(x,y) - \frac{e^{-x/2}}{qB(p,q)} y^{p-1} (1-y)^q M\left(p+q, p, \tfrac{1}{2}xy\right), \quad (3.3)$$

For the complementary function, $\bar{B}_{p,q}(x,y) = 1 - B_{p,q}(x,y)$ the same recursions hold but with the sign of the inhomogeneous term reversed, for instance, the first recursion reads

$$\bar{B}_{p,q}(x,y) = \bar{B}_{p+1,q}(x,y) - \frac{e^{-x/2}}{pB(p,q)} y^p (1-y)^q M\left(p+q, p+1, \tfrac{1}{2}xy\right). \quad (3.4)$$

In view of the recursions, it appears that $B_{p,q}(x,y)$ $(\bar{B}_{p,q}(x,y))$ can be computed in the direction of decreasing (increasing) p and in the direction of increasing (decreasing) q. In turn, this means that $B_{p,q}(x,y)$ is minimal with respect to p and $\bar{B}_{p,q}(x,y)$ is minimal with respect to q.

3.2 Homogeneous three-term recurrence relations

Next we provide equivalent three-term recurrence relations which display similar stability as the two-term relations but which have some advantages, as we next discuss.

For instance, using the recurrence relation (3.1) we can write $(B_{p+1,q} - B_{p,q})/(B_{p,q} - B_{p-1,q})$ in terms of a ratio of Kummer functions, and this leads to

$$y_{p+1} - (1 + c_{p,q})y_p + c_{p,q}y_{p-1} = 0,$$

$$c_{p,q} = \frac{p+q-1}{p} y \frac{M(p+q, p+1, xy/2)}{M(p+q-1, p, xy/2)}, \quad (3.5)$$

where $y_{p+j} = B_{p+j,q}$. The three-term recurrence is also satisfied by $y_{p+j} = \bar{B}(p+j,q)$.

The coefficient $c_{p,q}$ can be computed in terms of a continued fraction because the Kummer function is minimal for the recurrence relation in the $(+,+)$ direction (see [12]); in addition, using the behavior predicted by the Perron-Kreuser theorem (see [8] or [3] for a more recent account of this useful theorem) for this recurrence relation, we obtain that $c_{p,q}(x,y) \to y$ as $p \to +\infty$. The application of the Perron-Kreuser theorem to (3.5), gives us information on the behavior as $p \to +\infty$ and we conclude that as $p \to +\infty$,

$$\frac{B_{p+1,q}}{B_{p,q}} \sim c_{p,q} \sim y, \quad \frac{\bar{B}_{p+1,q}}{\bar{B}_{p,q}} \sim 1 \tag{3.6}$$

and, as announced, $B_{p,q}$ is minimal while $\bar{B}_{p,q}$ is dominant. Because $B_{p,q}$ is minimal, Pincherle's theorem gives the following continued fraction representation

$$\frac{B_{p+1,q}}{B_{p,q}} = \frac{c_{p,q}}{1 + c_{p,q}-} \frac{c_{p+1,q}}{1 + c_{p+1,q}-} \cdots,$$

where, as commented before, the coefficients $c_{p,q}$ can also be computed using a continued fraction representation.

Proceeding similarly with the recurrence relation (3.2) we arrive to the three-term recursion

$$y_{q+1} - (1 + \bar{c}_{p,q})y_q + \bar{c}_{p,q}y_{q-1} = 0,$$

$$\bar{c}_{p,q} = \frac{p+q-1}{q}(1-y)\frac{M(p+q,p,xy/2)}{M(p+q-1,p,xy/2)} \tag{3.7}$$

where $y_{q+j} = B_{p,q+j}$ or $y_{q+j} = \bar{B}_{p,q+j}$.

We conclude that as $q \to +\infty$,

$$\frac{\bar{B}_{p,q+1}}{\bar{B}_{p,q}} \sim \bar{c}_{p,q} \sim 1 - y, \quad \frac{B_{p,q+1}}{B_{p,q}} \sim 1 \tag{3.8}$$

and, as announced, $\bar{B}_{p,q}$ is minimal while $B_{p,q}$ is dominant. As before, there is a continued fraction representation for $\bar{B}_{p,q+1}/\bar{B}_{p,q}$.

These recurrence relations can be useful for computing both the beta and the complementary beta distributions, provided the recurrences are applied in their numerically stable directions: decreasing p and increasing q for $B_{p,q}$ and the opposite directions for $\bar{B}_{p,q}$. There is, however, the added difficulty that the coefficients $c_{p,q}$ or $\bar{c}_{p,q}$ are not simple and should be evaluated by some means (for instance, with the associated continued fraction). These

coefficients are, on the other hand, easier to compute and less prone to possible overflows/underflows than the inhomogeneous terms in (3.1) and (3.2).

3.3 Homogeneous four-term recurrence relations

Next, we provide four-term recurrence relations for $B_{p,q}$ and $\bar{B}_{p,q}$. These recurrences have some advantages, but also some disadvantages. The main advantage is that the coefficients are simple. The disadvantages are, on one hand that three function values are needed to start the recurrence, and on the other hand that their applicability is more restricted for reasons of stability; as we next discuss, apart from $B_{p,q}$ and $\bar{B}_{p,q}$, these recurrences introduce an independent third independent solution which is spurious and that in some cases compromises the stability.

For obtaining these recurrences we note that from (3.1) (similarly (3.2)), we can write the Kummer function on the right as the difference of $B_{p+1,q}$ and $B_{p,q}$. Similarly, we can do the same for two additional consecutive Kummer functions by shifting p. Then, relating these three consecutive Kummer functions by its corresponding three-term recurrence relation [9, §13.3(i)], we can find a recurrence for four consecutive values $B_{p+j,q}$, $j = 0, 1, 2, 3$. We get:

$$c_3 y_{p+3} + c_2 y_{p+2} + c_1 y_{p+1} + c_0 y_p = 0, \tag{3.9}$$

where

$$
\begin{aligned}
c_0 &= (p + q)y^2, \\
c_1 &= -y\left(p + 1 - \tfrac{1}{2}xy + y(p + q)\right), \\
c_2 &= y\left(p + 1 - \tfrac{1}{2}xy\right) - \tfrac{1}{2}xy, \\
c_3 &= \tfrac{1}{2}xy,
\end{aligned}
\tag{3.10}
$$

and $y_{p+j} = B_{p+j,q}(x, y)$ (or $y_{p+j} = \bar{B}_{p+j,q}(x, y)$).

Considering now the Perron-Kreuser theorem, we observe that the dominant solution is such that

$$\lim_{p \to \infty} \frac{1}{p} \frac{y_{p+1}}{y_p} = -\frac{2}{x}. \tag{3.11}$$

This solution is spurious because it is not any of the solutions we are interested in. Therefore, this recurrence should never be used in the direction of increasing p, neither for computing $B_{p,q}(x, y)$ or $\bar{B}_{p,q}(x, y)$. Then, we have a second subspace of solutions, of dimension two, with all the solutions such

that $\limsup |y_n|^{1/n}$ is finite. The characteristic equation for this subspace of solutions is $y\lambda^2 - y(1 + y)\lambda + y^2 = 0$, with solutions $\lambda = 1$ and $\lambda = y$. This corresponds to $y_p = B_{p,q}(x, y)$ for $\lambda = y$ and to $y_p = \bar{B}_{p,q}(x, y)$ (or $y_p = 1$) for $\lambda = 1$ (see (3.6)). Because $0 < y < 1$, $B_{p,q}(x, y)$ is minimal as p increases, while $\bar{B}_{p,q}(x, y)$ is dominant over $B_{p,q}(x, y)$, but it is not a dominant solution of the recurrence, which increases much faster according to (3.11). We therefore conclude that this recurrence can be only used for computing $B_{p,q}(x, y)$ and only in the backward direction (decreasing p); the recurrence should not be used for $\bar{B}_{p,q}(x, y)$ neither in the forward nor in the backward direction.

As for the recurrence over q, we have

$$d_3 y_{q+3} + d_2 y_{q+2} + d_1 y_{q+1} + d_0 y_q = 0, \qquad (3.12)$$

where

$$
\begin{aligned}
d_0 &= -(p + q)(1 - y)^2, \\
d_1 &= (1 - y)\left((p + q)(1 - y) + p + 2q + 2 + \tfrac{1}{2}xy\right), \\
d_2 &= -(1 - y)\left(p + 2q + 2 + \tfrac{1}{2}xy\right) - q - 2, \\
d_3 &= q + 2,
\end{aligned}
\qquad (3.13)
$$

where we can have $y_{q+j} = B_{p,q+j}(x, y)$ or $y_{q+j} = \bar{B}_{p,q+j}(x, y)$.

The coefficients are all linear in q and this implies that the characteristic equation is of third degree, namely: $(\lambda - 1)(\lambda - (1 - y))^2 = 0$. We have the simple root $\lambda = 1$ which corresponds to $B_{p,q}$ because of (3.8); therefore this is a dominant solution and it can be computed by forward recursion (increasing q). With respect to the second (double) characteristic root $\lambda = 1 - y$, the Perron-Kreuser is not so conclusive, and the information we have is that there is a subspace of solutions of dimension two associated to this root, and that all the solutions in this subset satisfy

$$\limsup_{n \to \infty} |y_q|^{1/q} = |1 - y|,$$

but we cannot conclude from here that $\bar{B}_{p,q}$ is minimal (and, in fact, it does not appear to be). Therefore, we conclude that the recurrence can only be safely used for $B_{p,q}$ and only in the forward direction (increasing q).

3.4   Computational examples

Examples of the numerical performance of the homogeneous three-term recurrence relations (3.5) and (3.7), are shown Table 3.4. The initial function

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TTRR	Function/Direction	IPVs/Accuracy	FPVs/Accuracy
(3.5)	$B_{p,q}(50, 0.4)$ Backward	$p = 300, q = 200$ $\sim 5 \, 10^{-16}$	$p = 50, q = 200$ $1.9 \, 10^{-14}$
(3.5)	$\bar{B}_{p,q}(50, 0.4)$ Forward	$p = 30, q = 200$ $\sim 5 \, 10^{-16}$	$p = 280, q = 200$ $7 \, 10^{-15}$
(3.7)	$B_{p,q}(50, 0.4)$ Forward	$p = 30, q = 20$ $\sim 5 \, 10^{-16}$	$p = 30, q = 270$ $4.6 \, 10^{-14}$
(3.7)	$\bar{B}_{p,q}(50, 0.4)$ Backward	$p = 30, q = 300$ $\sim 5 \, 10^{-16}$	$p = 30, q = 50$ $1.5 \, 10^{-14}$

Table 1. Examples of computations using the homogeneous three-term recurrence relations (3.5) and (3.7). In the second column, the function ($B_{p,q}(x, y)$ or $\bar{B}_{p,q}(x, y)$) and the direction of application of the TTRR (forward or backward) are specified. The third column shows the initial values (IPVs) of the parameters p and q and the accuracy of the function values. The fourth column shows the final parameter values (FPVs) after applying the TTRR and the accuracy of the function values.

values needed for starting the recurrences and the final function values used for testing, have been computed with Maple with a large number of digits. The recurrences have been implemented with $Digits := 16$ (double precision). As can be seen, the recursions are stable when applied in the correct direction. We have checked that similar results are obtained when using the first order recurrences discussed in Section 3.1.

As an example of the performance of the four-term recurrence relation for computing $B_{p,q}(x, y)$, we consider (3.9) for $p = 1000$ and $q = 1200$, $x = 10$, $y = 0.2$. As we will later show in the paper, using very few terms in the asymptotic expansion (6.17) we can compute $B_{p,q}(x, y)$ with an accuracy close to single precision $\sim 10^{-8}$ for these large values of the parameters p and q. For using the recurrence relation (3.9) three starting values are needed: $B_{p,1200}(10, 0.2)$ for $p = 1003, 1002, 1001$. We evaluate these functions with an accuracy of $3.4 \, 10^{-8}$ for the three values of p using just 3 terms in the asymptotic expansion (6.17). Then, we use (6.17) backwards down to $p = 200$. When checking the accuracy of the last function value computed ($B_{200,1200}(10, 0.2)$) we find that this is exactly $3.4 \, 10^{-8}$, i.e. there isn't any loss of accuracy when using (3.9). This shows the numerical stability of the recursion. Furthermore, because the solution is minimal, the ratio of consecutive functions tends to converge to the exact solution; in our example, which we compute with 16 digits, all digits of $B_{200,1200}(10, 0.2)/B_{199,1200}(10, 0.2)$ are correct.

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As mentioned before, for computing the function $\bar{B}_{p,q}(x,y)$ the use of the four-term recurrence relations should always be avoided. As an example, let us consider (3.9) for $p = 300$, $q = 200$, $x = 10$, $y = 0.2$. The three starting values needed to apply the recursion forward ($\bar{B}_{300,200}(10, 0.2)$, $\bar{B}_{301,200}(10, 0.2)$, $\bar{B}_{302,200}(10, 0.2)$) have been computed with Maple using a large number of digits. When implemented in double precision, we obtain a loss of accuracy of four significant digits in the function value when the recurrence relation (3.9) is used to compute $\bar{B}_{307,200}(10, 0.2)$ (just five steps of the recursive process has been applied.)

4 Expansions in terms of Kummer functions

The first-order recursions, when iterated, lead to series in terms of Kummer functions that can be useful for computing the beta distributions, particularly for small y.

Observe that the recursion in (3.1) has the form $B_{p+1,q}(x,y) = B_{p,q}(x,y) + h_p$, and that, hence, $B_{p,q}(x,y) = B_{p+N+1,q}(x,y) + h_p + h_{p+1} + \ldots + h_{p+N+1}$, $N = 0, 1, 2, \ldots$. This gives

$$B_{p,q}(x,y) = B_{p+N+1,q}(x,y) +$$

$$\frac{e^{-x/2}y^p(1-y)^q}{pB(p,q)} \sum_{j=0}^{N} y^j \frac{(p+q)_j}{(p+1)_j} M\left(p+q+j, p+1+j, \tfrac{1}{2}xy\right). \tag{4.1}$$

Taking into account (3.6) it is clear that $B_{p+N+1,q} \to 0$ as $N \to +\infty$. Therefore

$$B_{p,q}(x,y) = \frac{e^{-\frac{1}{2}x}y^p(1-y)^q}{pB(p,q)} \sum_{j=0}^{\infty} y^j \frac{(p+q)_j}{(p+1)_j} M\left(p+q+j, p+1+j, \tfrac{1}{2}xy\right).$$

$$\tag{4.2}$$

Using $M(a,b,z) = e^z M(b-a,b,-z)$, we obtain

$$B_{p,q}(x,y) = \frac{e^{\frac{1}{2}x(y-1)}y^p(1-y)^q}{pB(p,q)} \sum_{j=0}^{\infty} y^j \frac{(p+q)_j}{(p+1)_j} M\left(1-q, p+1+j, -\tfrac{1}{2}xy\right).$$

$$\tag{4.3}$$

Observing that $M(a, b + \lambda, z) = \mathcal{O}(1)$ as $\lambda \to +\infty$, we conclude again that the series is convergent if $0 \le y < 1$.

By using (see [9, §13.3(ii)])

$$\frac{d^j M(a+b, a+1, z)}{dz^j} = \frac{(a+b)_j}{(a+1)_j} M\left(a+b+j, a+1+j, z\right), \tag{4.4}$$

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it follows that we can write

$$B_{p,q}(x,y) = \frac{e^{-x/2}y^p(1-y)^q}{pB(p,q)} \sum_{j=0}^{\infty} y^j \frac{d^j M(p+q,p+1,z)}{dz^j}, \quad z = \tfrac{1}{2}xy. \quad (4.5)$$

For the complementary function we obtain the series expansion

$$\bar{B}_{p,q}(x,y) = \frac{e^{-x/2}y^p(1-y)^q}{qB(p,q)} \sum_{j=0}^{\infty} (1-y)^j \frac{(p+q)_j}{(q)_j} M\left(p+q+j,p,\tfrac{1}{2}xy\right).$$
$$(4.6)$$

5 Contour integral representation

For deriving asymptotic expansions it is convenient to have integral representations in terms of elementary functions. We use (see [9, §13.4(ii)]) the contour integral

$$M(a,b,z) = \frac{\Gamma(b)\Gamma(1+a-b)}{\Gamma(a)} \frac{1}{2\pi i} \int_0^{(1+)} e^{zt}t^{a-1}(t-1)^{b-a-1}\,dt, \quad (5.1)$$

where $a > 0$, $b - a \neq 1, 2, 3, \ldots$, and the contour starts at the origin, encircles the point $t = 1$ anti-clockwise, and returns to the origin. The multivalued factors t^{a-1} and $(t-1)^{b-a-1}$ assume their principal values.

When we use the expansion in (4.5) and the representation in (5.1) we obtain, after interchanging the order of summation and integration, the contour integral

$$B_{p,q}(x,y) = \frac{e^{-x/2}y^p(1-y)^q}{2\pi i} \int_0^{(1+)} e^{\frac{1}{2}xyt}t^{p+q-1}(t-1)^{-q} \frac{dt}{1-yt}, \quad (5.2)$$

where we have assumed that $|yt| < 1$. It follows that the point $t = 1/y$ should be outside the contour, that is, the contour cuts the positive real axis between 1 and $1/y$.

When we extend the contour in order to include the point $t = 1/y$ inside the contour, calculate the residue (which equals 1), we obtain

$$B_{p,q}(x,y) = 1 + \frac{e^{-x/2}y^p(1-y)^q}{2\pi i} \int_0^{(1+,1/y+)} e^{\frac{1}{2}xyt}t^{p+q-1}(t-1)^{-q} \frac{dt}{1-yt},$$
$$(5.3)$$

where the contour cuts the real axis on the right of the point $t = 1/y$. It follows that we have for the complementary function the representation

$$\bar{B}_{p,q}(x,y) = \frac{e^{-x/2}y^p(1-y)^q}{2\pi i} \int_0^{(1+,1/y+)} e^{\frac{1}{2}xyt}t^{p+q-1}(t-1)^{-q} \frac{dt}{yt-1}. \quad (5.4)$$

6 Asymptotic expansions

We introduce the notations

$$p = r\cos^2\theta, \quad q = r\sin^2\theta, \quad r = p+q, \quad 0 \le \theta \le \tfrac{1}{2}\pi, \quad z = \tfrac{1}{2}xy = r\xi. \tag{6.1}$$

We derive an asymptotic expansion for large values of z with p and q bounded, and one with large values of r with ξ bounded.

6.1 Asymptotic expansion for large values of z

For large values of z we use the integral representation in (5.2) and apply Watson's lemma for contour integrals, see [14, §2.2]. In the lemma an integral is considered of the form

$$G_\lambda(z) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} s^{\lambda-1} q(s) e^{zs}\, ds, \tag{6.2}$$

where λ is a real or complex constant and the path runs around the branch cut of the multivalued function $s^{\lambda-1}$ from $-\infty$ to 0. Along the lower side of the negative real axis we have $\mathrm{ph}\, s = -\pi$, along the upper side $\mathrm{ph}\, s = \pi$. It is assumed that $q(s)$ is analytic at the origin and inside the contour. The expansion $q(s) = \sum_{n=0}^{\infty} a_n s^n$ gives the asymptotic expansion

$$G(z) \sim \sum_{n=0}^{\infty} \frac{1}{\Gamma(1-\lambda-n)} \frac{a_n}{z^{n+\lambda}}, \quad z \to \infty. \tag{6.3}$$

In this result all fractional powers have their principal values.

The integral in (5.2) is not of the form in (6.2), we have a finite contour starting at $t = 0$, but it is clear that the dominant exponential contributions e^{zt} are coming from $t > 1$. We expand

$$t^{p+q-1} = \sum_{n=0}^{\infty} a_n (t-1)^n, \quad a_n = (-1)^n \frac{(1-p-q)_n}{n!}, \tag{6.4}$$

and

$$\frac{1}{1-yt} = \frac{1}{1-y} \sum_{n=0}^{\infty} b_n (t-1)^n, \quad b_n = \left(\frac{y}{1-y}\right)^n. \tag{6.5}$$

This gives

$$\frac{t^{p+q-1}}{1-yt} = \frac{1}{1-y} \sum_{n=0}^{\infty} c_n(t-1)^n, \quad c_n = \sum_{m=0}^{n} a_m b_{n-m}. \tag{6.6}$$

When we use this expansion for the integral in (5.2), we obtain

$$B_{p,q}(x,y) \sim \frac{e^{-(1-y)x/2}y^p(1-y)^{q-1}z^{q-1}}{\Gamma(q)} \sum_{n=0}^{\infty}(-1)^n(1-q)_n\frac{c_n}{z^n}, \quad z \to \infty, \tag{6.7}$$

where z is defined in (6.1). This expansion can be used for p and q of order $\mathcal{O}(1)$ and for $0 < y \leq 1 - \delta$, where δ is a fixed positive small number.

Remark 1. The expansion in (6.7) becomes finite and exact when q is a positive integer. In that case it corresponds with the finite expansion given in [1, Eq. (11)]. We have verified and confirmed this relation, although the representation of the two results is quite different, not only because of a different notation.

6.2 Asymptotic expansion for large values of r

We write the integral in (5.2) in the form

$$B_{p,q}(x,y) = \frac{e^{-x/2}y^p(1-y)^q}{2\pi i} \int_0^{(1+)} e^{r\phi(t)} \frac{dt}{t(1-yt)}, \tag{6.8}$$

where

$$\phi(t) = \ln t - \sin^2\theta \ln(t-1) + \xi t, \quad \phi'(t) = \frac{\xi t^2 + \left(\cos^2\theta - \xi\right)t - 1}{t(t-1)}. \tag{6.9}$$

There are two saddle points, one is negative, the other one is given by

$$t_0 = \frac{\xi - \cos^2\theta + \sqrt{\left(\cos^2\theta - \xi\right)^2 + 4\xi}}{2\xi}, \tag{6.10}$$

which satisfies $t_0 > 1$. Limiting values with respect to ξ are

$$\lim_{\xi \to 0} t_0 = \frac{1}{\cos^2\theta}, \quad \lim_{\xi \to \infty} t_0 = 1. \tag{6.11}$$

In the following analysis we assume that, next to large values of r, the parameter θ is well inside $(0, \frac{1}{2}\pi)$, that is, $0 < \delta \leq \theta \leq \frac{1}{2}\pi - \delta$, where δ is small and not depending on r. Also, we assume that $t_0 \geq 1 + \delta$.

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The integrand of the integral in (6.8) has a pole at $t = t_p = 1/y$, and we assume that $t_0 < t_p$. Later we will relax this assumption. Equality $t_0 = t_p$ happens when $y = y_0$, where

$$y_0 = \frac{x + 2p}{x + 2p + 2q}.$$ (6.12)

Hence, we assume that $y < y_0$. This value y_0 is a transition value with respect to y: when $y \leq y_0$ we have (roughly) $B_{p,q}(x,y) \leq \bar{B}_{p,q}(x,y)$, and $B_{p,q}(x,y)$ is the smaller (primary) function for numerical computations; when $y_0 \leq y \leq 1$, it is better to compute $\bar{B}_{p,q}(x,y)$ as the primary function, and use $B_{p,q}(x,y) = 1 - \bar{B}_{p,q}(x,y)$.

In terms of x the transition value is

$$x_0 = 2\frac{(p+q)y - p}{1 - y}.$$ (6.13)

$B_{p,q}(x,y)$ is the primary function for $x \geq x_0$. For example, when we take $y = 0.6$, $p = 4.5$, $q = 5.5$, we have $x_0 = 7.5$, and with $x = x_0$ we compute $B_{p,q}(x,y) \doteq 0.4975$.

We use the substitution

$$\phi(t) - \phi(t_0) = \tfrac{1}{2}w^2,$$ (6.14)

where we assume that for $t > 1$ and $w \in \mathbb{R}$: $\mathrm{sign}(t - t_0) = \mathrm{sign}(w)$. This transforms (6.8) into

$$B_{p,q}(x,y) = \frac{e^{-x/2 + r\phi(t_0)}y^p(1-y)^q}{2\pi i} \int_{-i\infty}^{i\infty} e^{\frac{1}{2}rw^2} f(w)\, dw,$$ (6.15)

where

$$f(w) = \frac{1}{t(1-yt)}\frac{dt}{dw}, \quad \frac{dt}{dw} = \frac{w}{\phi'(t)}.$$ (6.16)

Substituting the expansion $f(w) = \sum_{k=0}^{\infty} f_k w^k$, we obtain

$$B_{p,q}(x,y) \sim \frac{e^{-x/2 + r\phi(t_0)}y^p(1-y)^q}{\sqrt{2\pi r}} \sum_{k=0}^{\infty} (-1)^k f_{2k}\frac{2^k \left(\frac{1}{2}\right)_k}{r^k}, \quad r \to \infty.$$ (6.17)

This expansion can be used, uniformly with respect to $y \in [0, y_0 - \delta]$, where δ is a small fixed positive number. When the values of y, p, q are given, the expansion can be used when $x > x_0$ (see (6.13)).

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The coefficients follow from the inversion of the transformation in (6.14). First we find the coefficients of the expansion

$$t = t_0 + t_1 w + t_2 w^2 + \ldots, \tag{6.18}$$

of which t_1 is given by

$$t_1 = \frac{dt}{dw}\bigg|_{w=0} = \frac{1}{\sqrt{\phi''(t_0)}} = \frac{t_0(t_0-1)}{\sqrt{\sin^2\theta\, t_0^2 - (t_0-1)^2}}. \tag{6.19}$$

The next ones are

$$t_2 = -\frac{\phi_3}{6\phi_2^2}, \quad t_3 = \frac{5\phi_3^2 - 3\phi_2\phi_4}{72\phi_2^{\frac{7}{2}}}, \quad t_4 = \frac{45\phi_4\phi_3\phi_2 - 40\phi_3^3 - 9\phi_5\phi_2^2}{1080\phi_2^5}, \tag{6.20}$$

where ϕ_k is the kth derivative of $\phi(t)$ at t_0.

The first coefficient f_0 of the expansion in (6.17) is given by

$$f_0 = \frac{t_1}{t_0(1-yt_0)} = \frac{t_0-1}{(1-yt_0)\sqrt{\sin^2\theta\, t_0^2 - (t_0-1)^2}}, \tag{6.21}$$

and the next one is

$$\begin{aligned}
f_2 = \frac{1}{24\phi_2^{\frac{7}{2}}t_0^3(1-yt_0)^3}\Big(&24\phi_2^2 + 12\phi_2(\phi_3 - 6y\phi_2)t_0 + \\
&(72\phi_2^2 y^2 + 5\phi_3^2 - 36\phi_3 y\phi_2 - 3\phi_4\phi_2)t_0^2 + \\
&2y(12\phi_3 y\phi_2 + 3\phi_4\phi_2 - 5\phi_3^2)t_0^3 + y^2(5\phi_3^2 - 3\phi_4\phi_2)t_0^4\Big).
\end{aligned} \tag{6.22}$$

6.2.1 The influence of the pole: error function approximation

When in (6.8) the pole at $t = t_p = 1/y$ is close to the saddle point t_0 the asymptotic approach has to be modified. For example, the shown coefficients f_0 and f_2 have poles when $t_0 = t_p$.

When we derived the expansion in (6.17) we have assumed that $t_0 < t_p$. This point t_p has a corresponding point $w = \zeta$ that follows from the transformation in (6.14). When $t_p > t_0$, we have $\zeta > 0$, when $t_p < t_0$ we should take $\zeta < 0$. Hence,

$$\tfrac{1}{2}\zeta^2 = \phi(t_p) - \phi(t_0), \quad \mathrm{sign}(\zeta) = \mathrm{sign}(t_p - t_0). \tag{6.23}$$

We write the function $f(w)$ defined in (6.16) in the form

$$f(w) = \frac{A}{w-\zeta} + g(w), \tag{6.24}$$

with the condition on A that $g(w)$ is analytic at $w = \zeta$. We find that $A = 1$ and because

$$e^{-r\phi(t_p)} = y^p(1-y)^q e^{-\frac{1}{2}x} \quad \implies \quad y^p(1-y)^q e^{-\frac{1}{2}x} e^{r\phi(t_0)} = e^{-\frac{1}{2}r\zeta^2}, \quad (6.25)$$

(6.15) becomes

$$B_{p,q}(x,y) = \frac{e^{-\frac{1}{2}r\zeta^2}}{2\pi i} \int_{-i\infty}^{i\infty} e^{\frac{1}{2}rw^2} \frac{dw}{w-\zeta} + \frac{e^{-\frac{1}{2}r\zeta^2}}{2\pi i} \int_{-i\infty}^{i\infty} e^{\frac{1}{2}rw^2} g(w)\, dw. \quad (6.26)$$

Using [13, 7.2.3, 7.7.2], we obtain

$$B_{p,q}(x,y) = \tfrac{1}{2}\mathrm{erfc}\left(\zeta\sqrt{r/2}\right) + \frac{e^{-\frac{1}{2}r\zeta^2}}{2\pi i} \int_{-i\infty}^{i\infty} e^{\frac{1}{2}rw^2} g(w)\, dw, \quad (6.27)$$

where erfc z is the complementary error function defined by

$$\mathrm{erfc}\, z = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\, dt. \quad (6.28)$$

For the complementary function introduced in (1.4) we find

$$\bar{B}_{p,q}(x,y) = \tfrac{1}{2}\mathrm{erfc}\left(-\zeta\sqrt{r/2}\right) - \frac{e^{-\frac{1}{2}r\zeta^2}}{2\pi i} \int_{-i\infty}^{i\infty} e^{\frac{1}{2}rw^2} g(w)\, dw, \quad (6.29)$$

where we have used $1 - \tfrac{1}{2}\mathrm{erfc}\, z = \tfrac{1}{2}\mathrm{erfc}(-z)$.

An expansion of the integral now follows from substituting the expansion $g(w) = \sum_{k=0}^{\infty} g_k w^k$ to obtain

$$B_{p,q}(x,y) = \tfrac{1}{2}\mathrm{erfc}\left(\zeta\sqrt{r/2}\right) + \frac{e^{-\frac{1}{2}r\zeta^2}}{\sqrt{2\pi r}} R_r(\zeta),$$

$$R_r(\zeta) \sim \sum_{k=0}^{\infty} (-1)^k g_{2k} \frac{2^k \left(\frac{1}{2}\right)_k}{r^k}, \quad r \to \infty. \quad (6.30)$$

We have derived this expansion under the condition $t_0 < t_p$, that is, for $y < 1/t_0$, but by taking $\zeta < 0$ when $t_0 > t_p$, we have (see (6.23)) $\mathrm{sign}(\zeta) = \mathrm{sign}(t_p - t_0)$, and we can use (6.30) for all $y \in (0,1)$.

For the asymptotic expansion in (6.30) we assume that (see (6.1))

$$y \in [\delta_1, 1 - \delta_1], \quad \theta \in \left[\delta_2, \tfrac{1}{2}\pi - \delta_2\right], \quad \xi \geq 0, \quad (6.31)$$

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where δ_j are fixed small positive numbers.

When the argument of the complementary error function is large, we can use the asymptotic expansion

$$\tfrac{1}{2}\operatorname{erfc}\left(\zeta\sqrt{r/2}\right) \sim \frac{e^{-\frac{1}{2}r\zeta^2}}{\sqrt{2\pi r}} \sum_{k=0}^{\infty} (-1)^k \frac{2^k \left(\tfrac{1}{2}\right)_k}{r^k} \frac{1}{\zeta^{2k+1}}, \tag{6.32}$$

and we recover the expansion in (6.17). This means that the coefficients g_{2k} follow from f_{2k} by the relation

$$g_{2k} = f_{2k} - \frac{1}{\zeta^{2k+1}}, \quad k = 0, 1, 2, \dots . \tag{6.33}$$

When $\zeta \to 0$ this relation remains valid, because the pole in f_{2k} is properly cancelled analytically. The numerical evaluation of g_{2k} for small values of $|\zeta|$ (that is, small values of $|t_p - t_0|$) needs some extra care.

Remark 2. The variable ζ introduced in (6.23) is a function of p, q, x and y. Clearly, $\zeta = 0$ when saddle point and pole coincide: $t_p = t_0$. This happens when y and x assume the transition values y_0 and x_0, see (6.12)–(6.13). By using the simple relation

$$\frac{\partial \zeta}{\partial x} = \frac{y(t_p - t_0)}{2r\zeta}, \tag{6.34}$$

it is easy to derive the expansion

$$x = \sum_{k=0}^{\infty} x_k \zeta^k, \tag{6.35}$$

where x_0 is the transition value $x_0 = 2(ry - p)/(1 - y)$ (see (6.13)). The next two coefficients are

$$x_1 = \frac{2\sqrt{r(q - r(1 - y)^2)}}{1 - y}, \quad x_2 = \frac{2(r(1 - y)^3 - q)}{3(r(1 - y)^2 - q)(1 - y)}. \tag{6.36}$$

The coefficients of the expansion

$$y = \sum_{k=0}^{\infty} y_k \zeta^k, \tag{6.37}$$

can be obtained from

$$\frac{\partial \zeta}{\partial y} = \frac{xy(1 - y)(t_p - t_0) + (2r + x)(y - y_0)}{2yr(1 - y)\zeta}. \tag{6.38}$$

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Here, y_0 is the transition value $y_0 = (x + 2p)/(x + 2r)$ (see (6.12)) and the next coefficients are

$$y_1 = -\frac{2\sqrt{q}\,\sqrt{4pr + 4rx + x^2}}{\sqrt{r}\,(2r + x)^2},$$

$$y_2 = \frac{2(16pr^2(p - q) + 8rx(8rp^2 - p^2 - 3r^2) + 12rx^2(p + r) + 8rx^3 + x^4)}{3r(2r + x)^3(4pr + 4rx + x^2)}.$$

$$(6.39)$$

6.3 Computational examples

As an example of the computational performance of the expansions, Table 6.3 shows relative errors in the computation of $B_{p,q}(x, y)$ using the asymptotic expansions for the parameter values considered in Table 2 of [1] and few additional values. When q is a positive integer, we have in (6.7) a finite sum. Then, as can be seen in the Table, the result provided by the expansion is exact for $q = 5$ by taking $n = 4$. In [1] also a finite sum is given of a different form when their b is a positive integer.

It is interesting to note that for few of the parameter values tested, the asymptotic expansion (6.30) provide an accuracy much better than single precision (10^{-8}) just using three coefficients g_{2k} (6.33).

7 Inversion of the noncentral beta distribution

We consider the inversion problem of finding x or y when a value of $B_{p,q}(x, y)$ is given.

As mentioned in §2, x is the noncentrality parameter λ of the noncentral F-distribution $F(w, \nu_1, \nu_2, \lambda)$. For $x = 0$ we have $B_{p,q}(0, y) = I_y(p, q)$, the incomplete beta function. This follows from (1.1). Also, from (1.9), $B_{p,q}(x, y)$ is a decreasing function of x. Therefore when we consider the inversion for nonnegative values of x with a given value z, we should write

$$B_{p,q}(x, y) = z, \quad 0 \le z \le I_y(p, q), \quad p, q, y, z \text{ given.} \quad (7.1)$$

For the inversion with respect to y, we have the equation

$$B_{p,q}(x, y) = z, \quad 0 \le z \le 1, \quad p, q, x, z \text{ given.} \quad (7.2)$$

This corresponds with finding the quantiles of the distribution.

We try to find an approximation of x or y by using the representation in (6.30) and by inverting first the complementary error function, that is by

AE	p	q	x	y	$B_{p,q}^{AE}(x,y)$	Rel. error
(6.7), $n=5$	2.3	3.5	54	0.8640	0.2760082728547706	$5.6\,10^{-4}$
(6.7), $n=5$	2.3	3.5	140	0.9000	0.03608547984275312	$2\,10^{-5}$
(6.7), $n=5$	2.3	3.5	250	0.9000	0.0005034732632828640	$8.9\,10^{-7}$
(6.7), $n=4$	5	5	54	0.8640	0.4563026193369792	$0^{(*)}$
(6.7), $n=4$	5	5	140	0.9000	0.1041334930397555	$0^{(*)}$
(6.7), $n=4$	5	5	170	0.9560	0.6022421650011662	$0^{(*)}$
(6.30), $k=2$	10	10	54	0.8686	0.9187790583189610	$5.7\,10^{-8}$
(6.30), $k=2$	10	10	140	0.9000	0.6008070986289955	$1.4\,10^{-8}$
(6.30), $k=2$	10	10	250	0.9000	0.09028986850391792	$5.3\,10^{-7}$
(6.30), $k=2$	20	20	54	0.8787	0.9998676573798253	$9.0\,10^{-12}$
(6.30), $k=2$	20	20	140	0.9000	0.9925975041637949	$5.2\,10^{-10}$
(6.30), $k=2$	20	20	250	0.9220	0.9641190712607291	$1.7\,10^{-9}$
(6.17), $k=2$	30	30	100	0.1	$5.341313347397197\,10^{-33}$	$3.7\,10^{-6}$
(6.17), $k=2$	30	30	150	0.1	$5.175358340461182\,10^{-42}$	$1.9\,10^{-6}$
(6.17), $k=2$	30	30	250	0.1	$3.252685735589340\,10^{-60}$	$7.8\,10^{-7}$

Table 2. Relative errors in the computation of $B_{p,q}(x,y)$ using the asymptotic expansions (6.7), (6.17) and (6.30) for different values of the parameters. The number of terms used in each of the expansions is shown in the first column. $^{(*)}$ The asymptotic expansion is exact in this case.

finding ζ_0 from the equation

$$\tfrac{1}{2}\mathrm{erfc}\left(\zeta_0\sqrt{r/2}\right) = z. \tag{7.3}$$

A simple and efficient algorithm for computing the inverse of the complementary error function is included, for example, in the package described in [6]. Hence, we consider the numerical inversion of (7.3) as a known problem and concentrate on finding from the computed value ζ_0 the requested value of x or y.

To obtain x or y when ζ_0 is small, we can use the expansions given in (6.35) or (6.37) with $\zeta = \zeta_0$, and for the general inversion problem we use (6.23), written in the form

$$\ln\left(\frac{(2x)^{p/r}(S-2xy)^{q/r}}{(1-y)^{q/r}S}\right) + \frac{2x-S}{4r} - \tfrac{1}{2}\zeta_0^2 = 0, \tag{7.4}$$

where S is defined by (see also (6.10))

$$t_0 = \frac{S}{2xy}, \quad S = xy - 2p + \sqrt{x^2y^2 - 4pxy + 8rxy + 4p^2}. \tag{7.5}$$

After inverting equation (7.3), we obtain an approximation to the value of x or y.

A better approximation of ζ follows from obtaining ζ_1 in the expansion $\zeta \sim \zeta_0 + \zeta_1/r$. From [15, Eq. D.21] we take[2]

$$\zeta_1 = \frac{1}{\zeta_0}\ln\left(1 + \zeta_0 g_0\right), \tag{7.6}$$

where g_0 follows from (6.33) and f_0 is given in (6.21).

7.1   Examples for the inversion with respect to x

We take $p = 10$, $q = 15$, $y = 0.45$. For these values the inversion problem makes sense if we take $z \le I_y(p,q) \doteq 0.7009$ (see (7.1)). So, we can take $z = 0.5$, which gives $\zeta_0 = 0$. The first approximation of x that follows from the expansion in (6.35) is $x \doteq x_0 = \frac{50}{11} \doteq 4.54545$. With these values we have $B_{10,15}(x_0, 0.45) \doteq 0.50952$.

Figure 1 shows $f(x)$, the left-hand side of Eq. (7.4), for two values of z in Eq. (7.3) ($z^{(1)} = 0.4$, which gives $\zeta_0^{(1)} \doteq 0.05067$, and $z^{(2)} = 0.6$, which

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[2]In the cited formula we should take $q = 0$ for the present case.
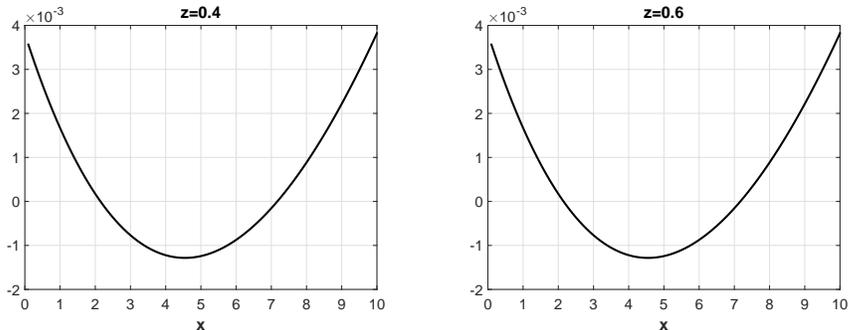
Figure 1: Plot of Eq. (7.4) written in the form $f(x) = 0$, for $p = 10$, $q = 15$, $y = 0.45$ and $z = 0.1, 0.3$ (z as in (7.1) and (7.3)).

gives $\zeta_0^{(2)} \doteq -0.05067$). The shown graphs in Figure 1 attain their minimum value at the transition value $x_0 = \frac{50}{11}$ (see (6.13) and Remark 2).

The function $f(x)$ shown in Figure 1 has two zeros for both choices of $z^{(j)}$. For $z^{(1)} = 0.4$, giving a positive ζ_0, the zero that is larger than x_0 is an approximation of x, because when $B_{p,q}(x,y)$ is smaller than 0.5, x is larger than x_0 (see (6.13)). For $z^{(2)} = 0.6$ the corresponding x is smaller than x_0.

The x-values resulting in the inversion of (7.3) for these z-values are $x^{(1)} \doteq 7.1704$, $x^{(2)} \doteq 2.1475$, respectively. In order to check the accuracy of the approximations, we can compute the values of the noncentral beta function for $x^{(1)}$ and $x^{(2)}$: $B_{10,15}(x^{(1)}, 0.45) \doteq 0.40900$, $B_{10,15}(x^{(2)}, 0.45) \doteq 0.60915$.

The values $x^{(j)}$ for the parameters considered in the plots of Figure 1 are obtained by using the expansion of x given in (6.35). We have used terms up to $x_5\zeta_0^5$, and this term is smaller than $0.77e-5$. This shows the performance of the series. As can be seen, in both cases it is possible to avoid the inversion of the nonlinear equation (7.4) and obtain a good approximation to the x-values using (6.35). In general, these approximations could also be used as starting values of methods for solving nonlinear equations in case higher accuracy is required.

The approximations of x obtained in these examples have a relative error of about 0.03. A better result can be obtained in examples with larger parameter p and q; in the present case $r = p + q = 25$, which is not a very large asymptotic parameter. Also, in the examples only the complementary error function in representation (6.30) is used. A first influence of the R_r-term is contained in ζ_1 given in (7.6). When we take this term into account,

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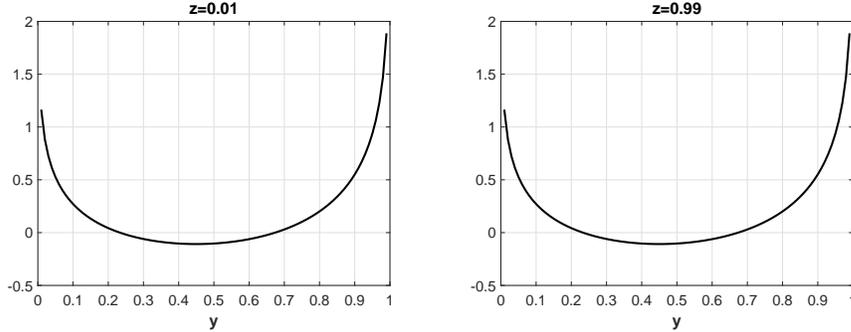
Figure 2: Plot of Eq. (7.4) written in the form $g(y) = 0$, for $p = 10$, $q = 15$, $x = 4.5$ and $z = 0.01$, 0.99 (z as in (7.1) and (7.3)).

we find for the example with $z^{(1)} = 0.4$, $\zeta^{(1)} \doteq 0.05523$, giving $x^{(1)} \doteq 7.4176$ and $B_{10,15}(x^{(1)}, 0.45) \doteq 0.40014$, which is a better result.

7.2 Examples for the inversion with respect to y

We take $p = 10$, $q = 15$, $x = 4.5$. For $z = 0.5$, the solution of (7.3) is $\zeta_0 = 0$. The first approximation of y that follows from the expansion in (6.37) is $y \doteq y_0 = 49/109 \doteq 0.44954$. With these values we have $B_{10,15}(4.5, y_0) \doteq 0.5095$.

To compute y from Eq. (7.4) when x, p, q, z are given, we call the left-hand side of this equation $g(y)$. In Figure 2 we show the graphs of $g(y)$ for values of ζ_0 that follow from the inversion of Eq. (7.3) for $z^{(1)} = 0.01$ and $z^{(2)} = 0.99$.

The values of ζ_0 are $\zeta_0^{(1)} \doteq 0.4653$, with $y^{(1)} \doteq 0.2330$ (we take the zero left of $y_0 = \frac{49}{109} \doteq 0.4495$ because, when $B_{p,q}(x, y)$ is smaller than 0.5, y is smaller than y_0 (see (6.12)). For $z^{(2)} = 0.99$ the corresponding y is larger than y_0. We have $\zeta_0^{(2)} \doteq -0.4652$, with $y^{(2)} \doteq 0.6739$. The computed values of the noncentral beta distribution are $B_{10,15}(4.5, y^{(1)}) \doteq 0.011360$ and $B_{10,15}(4.5, y^{(2)}) \doteq 0.98999$.

We see a better result for the second y-value. This can be explained by observing that, when the argument of the complementary error function in (6.30) is negative and large in absolute value, this terms approaches 1 very fast, while the other term is exponentially small. In the present example the argument of the complementary error function is $\zeta_0^{(2)} \sqrt{r/2} \doteq -1.6447$.

8 Appendix

We prove the relation in (2.4) by using interchanging summation as in

$$\sum_{n=0}^{\infty} A_n \sum_{j=0}^{n} B_j = \sum_{j=0}^{\infty} B_j \sum_{n=j}^{\infty} A_n, \tag{8.1}$$

where

$$A_n = \frac{\Gamma(a+b+n)}{\Gamma(b)\Gamma(a+1+n)} x^n, \quad B_j = \frac{\lambda^j}{j!}. \tag{8.2}$$

We have

$$\begin{aligned}
\sum_{n=j}^{\infty} A_n &= \sum_{n=0}^{\infty} A_{n+j} = x^j \sum_{n=0}^{\infty} \frac{\Gamma(a+b+n+j)}{\Gamma(b)\Gamma(a+1+n+j)} x^n \\
&= x^j \frac{\Gamma(a+b+j)}{\Gamma(b)\Gamma(a+1+j)} \sum_{n=0}^{\infty} \frac{(a+b+j)_n}{(a+1+j)_n} x^n \\
&= \frac{x^j}{(a+j)B(a+j,b)} {}_2F_1 \left(\begin{matrix} a+b+j, 1 \\ a+1+j \end{matrix} ; x \right) \\
&= x^{-a}(1-x)^{-b} I_x(a+j,b),
\end{aligned} \tag{8.3}$$

where we have used [10, 8.17.8]. From this result, the relation in (2.4) easily follows.

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