WEB LINE RESONANT FREQUENCIES

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ABSTRACT

The goal of this paper is to investigate the resonant frequencies of a system of idle rollers and web spans. Of particular interest is the determination of the minimum resonant frequency as a function of the number of idle rollers, web span lengths between idle rollers, web properties such as the modulus of elasticity and area of cross-section, and inertia and radius of the idle rollers. Any conclusions from the analysis that can provide insights into maximizing the minimum resonant frequency by choosing particular web paths or span lengths are of benefit to the web machine designer as well as the control system engineer. Knowledge of the minimum resonant frequency and the factors that influence it can assist the machine designer in reconfiguring web paths and/or number of idle rollers between two driven rollers such that the machine induced vibrations do not excite resonances. Further, the knowledge of the minimum resonant frequency will assist the control engineer to better select the bandwidth of the closed-loop system as well as the crossover frequency of the controller.

A simple linear model that describes the dynamics of tension in a web span is considered. It is shown with experimentation on a web platform that the simple model is able to predict resonant frequencies in an idler roller system. Therefore, the simple model is used to investigate the behavior of the minimum resonant frequency as a function of the number of idle rollers and span lengths. Analytical solutions for computing the minimum resonant frequency for one idler (one idle roller and two spans) and two idler (two idle rollers and three spans) systems are given. Numerical analysis and discussions on how to maximize the minimum resonant frequency for two, three, and four idler systems are also given. An analytical approximation of the minimum resonant frequency for equal span lengths and any number of idle rollers is derived. This approximation tends to converge to the actual minimum resonant frequency with increase in the number of idle rollers. This is useful in getting a quick and reliable estimate of the minimum resonant frequency in festoons/accumulators that contain many idle rollers and equal span lengths.

INTRODUCTION

This paper will investigate a system of idle rollers and spans between two driven rollers and suggest ways to maximize the minimum resonant frequency based on the selection of one or more of the following: (i) span lengths, (ii) number of idle rollers, and (iii) web path. The approach consists of constructing a simplified idler-system model that can predict resonant frequencies. This system model is based on a simplified model for web tension between two idle rollers and web velocity dynamics on an idle roller. To verify that such a simplified idler-system model can predict resonant frequencies, experiments were conducted on a web platform for different web paths and number of idle rollers. A representative sample of the results will be shown and discussed. Analytical and numerical analysis of the behavior of the minimum resonant frequency will be discussed together with recommendations on how to choose span lengths and web paths within two driven rollers to maximize the minimum resonant frequency. Further, an analytical estimate of the minimum resonant frequency is derived for the case of an idler-system containing equal span lengths and any number of idle rollers (an accumulator is an example of such a system). This estimate is a lower bound for the minimum resonant frequency and its accuracy improves with the increase in the number of idle rollers.

NOMENCLATURE

A	Area of cross-section of web
A_n	$(2n + 1) \times (2n + 1)$ tridiagonal system matrix
D	Matrix determinant
E	Modulus of elasticity of web material
J	Roller inertia
L	Length of i-th span
R	Roller radius
T	Web tension
V	Web speed on roller
C	Coefficients of the characteristic polynomial of A_n
P	Scaled differential operator, s/ν
f	Resonant frequency (Hz)
f_1	Minimum resonant frequency (Hz)
$\hat{f}_1$	Estimate of minimum resonant frequency (Hz)
k	Equivalent stiffness, $k = EA/L$
m	Equivalent mass, $m = J/R^2$
s	Laplace variable
x	State vector
ν	Frequency (Hz)
ω	Angular velocity (rad/sec)

Subscripts

i	Roller or span index, $i = 0, 1, 2, \ldots$
min	Minimum value
n	Number of idle rollers, $n = 1, 2, \ldots$
p	Related to pull roll
r	Reference value
w	Related to winder or rewind

IDLER SYSTEM DYNAMIC MODEL

The angular velocity dynamics of an idle roller with the web under tension on either side of the roller can be obtained by applying Newton's second law as

$$J_i \dot{\omega}_i = R_i (T_i - T_{i-1}) \qquad \{1\}$$

Assuming that the web is not slipping on the roller, the web velocity on the *i*-th roller is given by

$$V_i = R_i \omega_i \qquad \{2\}$$

Substitution of {2} into {1} and simplification gives

$$\frac{J_i}{R_i^2} \dot{V}_i = T_i - T_{i-1} \qquad \{3\}$$

The web tension dynamics is obtained by applying the principle of mass conservation to a control volume between two consecutive rollers:

$$L_i \dot{T}_i = EA(V_{i+1} - V_i) + (T_{i-1} V_i - T_i V_{i+1}) \qquad \{4\}$$

The nonlinear web span tension dynamics can be linearized around given reference velocity and tension values by defining the perturbations around the reference value: $\Delta V_i = V_i - V_r$ and $\Delta T_i = T_i - T_r$. Substitution of these into the nonlinear web tension dynamics and using $\dot{T}_r = \dot{V}_r = 0$ gives

$$L_i \Delta \dot{T}_i = EA(\Delta V_{i+1} - \Delta V_i) + T_r(\Delta V_i - \Delta V_{i+1}) + V_r(\Delta T_{i-1} - \Delta T_i) + (\Delta T_{i-1} \Delta V_i - \Delta T_i \Delta V_{i+1}) \qquad \{5\}$$

Ignoring the second-order perturbation term (last term)

$$L_i \Delta \dot{T}_i = (EA - T_r)(\Delta V_{i+1} - \Delta V_i) + V_r(\Delta T_{i-1} - \Delta T_i) \qquad \{6\}$$

Since $\Delta \dot{T}_i = \dot{T}_i$, $\Delta V_{i+1} - \Delta V_i = V_{i+1} - V_i$ and $\Delta T_{i+1} - \Delta T_i = T_{i+1} - T_i$, the linearized dynamics can be written as

$$L_i \dot{T}_i = (EA - T_r)(V_{i+1} - V_i) + V_r(T_{i-1} - T_i) \qquad \{7\}$$

Since the quantity EA is typically much larger than the reference tension T_r and reference velocity V_r, the following simplified model of web tension is considered for the prediction of resonant frequencies for a system of idle rollers:

$$L_i \dot{T}_i = EA(V_{i+1} - V_i) \qquad \{8\}$$

Note that this model does not consider strain transport from upstream to downstream spans and hence is not suitable for control system design; one has to consider the nonlinear model {4} or the full linearized model {7} for control design purposes. But the simplified model given by {8} is capable of predicting resonant frequencies (data from a series of experiments which are shown later support this) and allows for the ease of computation and analysis of the minimum resonant frequency.

Therefore, the simplified model for the span tension and roller dynamics given by the following equations is considered in this study:

$$\dot{T}_i = \frac{EA}{L_i}\left(V_{i+1} - V_i\right) \tag{9}$$

$$\dot{V}_i = \frac{1}{m_i}\left(T_i - T_{i-1}\right) \tag{10}$$

Governing equations for a system consisting of any number of idlers and spans can be obtained by recursively using the above equations. For example, the dynamic equation for a system consisting of one idle roller and two web spans is given by

$$\underbrace{\begin{bmatrix} \dot{T}_0 \\ \dot{V}_1 \\ \dot{T}_1 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} 0 & \frac{EA}{L_0} & 0 \\ -\frac{1}{m_1} & 0 & \frac{1}{m_1} \\ 0 & -\frac{EA}{L_1} & 0 \end{bmatrix}}_{\mathbf{A}_1} \underbrace{\begin{bmatrix} T_0 \\ V_1 \\ T_1 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} -\frac{EA}{L_0} & 0 \\ 0 & 0 \\ 0 & \frac{EA}{L_1} \end{bmatrix}}_{\mathbf{B}_1} \begin{bmatrix} V_p \\ V_w \end{bmatrix} \tag{11}$$

In the above system, the state vector is x = $[T_0, V_1, T_1]^\top$, the pull roll velocity, V_p, and the rewind velocity, V_w, are controlled variables and thus are treated as input variables. For a two-idler system with state vector x = $[T_0, V_1, T_1, V_2, T_2]^\top$, the system matrix is given by

$$\mathbf{A}_2 = \begin{bmatrix} 0 & \frac{EA}{L_0} & 0 & 0 & 0 \\ -\frac{1}{m_1} & 0 & \frac{1}{m_1} & 0 & 0 \\ 0 & -\frac{EA}{L_1} & 0 & \frac{EA}{L_1} & 0 \\ 0 & 0 & -\frac{1}{m_2} & 0 & \frac{1}{m_2} \\ 0 & 0 & 0 & -\frac{EA}{L_2} & 0 \end{bmatrix} = \begin{bmatrix} \multicolumn{3}{c}{\multirow{3}{*}{$\mathbf{A}_1$}} & 0 & 0 \\ & & & 0 & 0 \\ & & & \frac{EA}{L_1} & 0 \\ 0 & 0 & -\frac{1}{m_2} & 0 & \frac{1}{m_2} \\ 0 & 0 & 0 & -\frac{EA}{L_2} & 0 \end{bmatrix}$$

Similarly, for a three-idler system with state vector x = $[T_0, V_1, T_1, V_2, T_2, V_3, T_3]^\top$, the system matrix is

$$\mathbf{A}_3 = \begin{bmatrix} 0 & \frac{EA}{L_0} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{m_1} & 0 & \frac{1}{m_1} & 0 & 0 & 0 & 0 \\ 0 & -\frac{EA}{L_1} & 0 & \frac{EA}{L_1} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{m_2} & 0 & \frac{1}{m_2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{EA}{L_2} & 0 & \frac{EA}{L_2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{m_3} & 0 & \frac{1}{m_3} \\ 0 & 0 & 0 & 0 & 0 & -\frac{EA}{L_3} & 0 \end{bmatrix} = \begin{bmatrix} \multicolumn{5}{c}{\multirow{5}{*}{$\mathbf{A}_2$}} & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & \frac{EA}{L_2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{m_3} & 0 & \frac{1}{m_3} \\ 0 & 0 & 0 & 0 & 0 & -\frac{EA}{L_3} & 0 \end{bmatrix}$$

One can build up the system matrix from lower-order system matrices as more idle rollers are added. In the system matrix of an n-idler system, A_n, the system matrix of $(n-1)$-idler system appears in block-diagonal form as shown above. Therefore, one can recursively write down the system matrix for any number of idle rollers and spans. In general, for a system with n idlers, consisting of $n + 1$ web spans, the size of the system matrix is $(2n + 1) \times (2n + 1)$. The system matrix is a tridiagonal matrix with all its eigenvalues (complex conjugate pairs) on the imaginary axis of the complex plane with one of those at the origin. The zero eigenvalue corresponds to the "rigid body" mode of the system. The resonant frequencies of the idler system are equal to the modulus of the non-zero imaginary-axis eigenvalues of the system matrix. Therefore, to study the resonant frequencies, one needs to analytically obtain the eigenvalues of the system matrix. Closed-form analytical solutions can be obtained for one- and two-idler systems in a form suitable for analysis. Although closed-form solutions for systems consisting of a large number of idlers can be obtained using symbolic computational tools, the expressions for resonant frequencies can be quite complicated. Thus, it is difficult to ascertain from the symbolic expressions which one corresponds to the minimum resonant frequency. Even when it is possible to distinguish the minimum resonant frequency, it is difficult to obtain meaningful conclusions on its dependence on physical parameters such as web modulus, span lengths, and roller inertias. For systems consisting of more than two idle rollers, one typically resorts to numerical analysis.

Validity of the Simple Model to Predict Resonant Frequencies

To verify the simple system model given by equations {9} and {10}, frequency response experiments were conducted on the Euclid Web Line (EWL) for the following systems: one path with 5 idlers between the S-wrap roll and pull roll and two paths with 5 and 7 idlers, respectively, between the pull roll and rewind roll. The pull roll velocity reference V_{pr} is treated as the input variable and web tension T is the output variable. The web tension in each section is measured by a load cell. Web tension data were obtained by sweeping a sinusoidal input containing various frequencies.

The pull roll was run in the velocity regulation mode and the following reference velocity V_{pr} was considered:

$$V_{pr} = \bar{V}_r + M(\sin 2\pi\nu_1 t + \sin 2\pi\nu_2 t + \cdots + \sin 2\pi\nu_n t) \qquad \{12\}$$

where $\bar{V}_r$ denotes a constant base speed, M is the amplitude of the disturbance, and ν_1, ν_2, . . . , ν_n are the disturbance input frequencies. The velocity PI controller for the pull roll was chosen as:

$$C_{pv}(s) = \frac{k_{ps}(s + \omega_{ld})}{s} \qquad \{13\}$$

where $k_{ps} = J_p k_{rf}$, $k_{rf} = 40$ and J_p is the total inertia reflected to the motor side. Also, $\omega_{ld} = k_{rf}/4\zeta^2$, where $\zeta = 1.1$ is the damping ratio of the system.

The control strategy for the rewind roll is a cascaded tension-velocity controller. The inner velocity loop controller is

$$C_{rv}(s) = \frac{k_{ps}(s + \omega_{ld})}{s} \qquad \{14\}$$

where $k_{ps} = J_r k_{rf}$ where $k_{rf} = 15$ and J_r is the rewind roll total inertia reflected to the motor side. Also, $\omega_{ld} = k_{rf}/4\zeta^2$, where $\zeta = 1.1$ is the damping ratio of the system.

For the outer tension control loop, a de-tuned PI controller with very low gains was chosen to avoid web breakage. The tension controller can be written as

$$C_{rt}(s) = 2\left(\frac{s+3}{s}\right) \qquad \{15\}$$

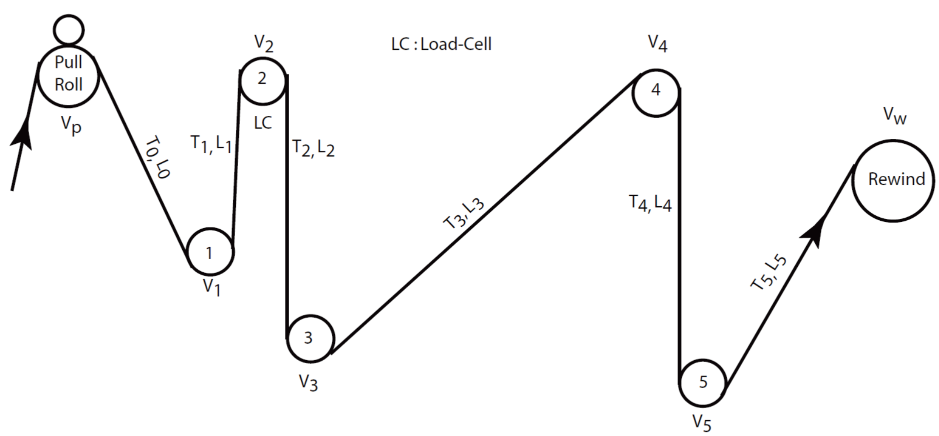
Three web paths were considered as shown in Figures 1, 2, 3. The corresponding span lengths for the three paths are shown in tables below the figures. The inertia J_i and radius R_i of the idle rollers are equal to 0.069874 lb-ft^2 and 0.125 ft, respectively. The web material used in the experimentation is Tyvek with the value of EA equal to 2000 lbf. The pull roll base speed $\overline{V}_r$ was fixed at 150 FPM, the disturbance amplitude M was 5 FPM, and the disturbance frequencies ni were chosen from 1 Hz to 19 Hz in steps of 1 Hz. Additional non- integer disturbance frequency components were added to get tension data close to a resonant frequency.

At the beginning of the experiment, the web line was running at the steady base speed $\overline{V}_r$, then a disturbance was injected into the system and web tension was measured by a load cell in each section. Computer simulations with the state space system models for the same web paths were conducted in MATLAB. To compare the experimental data with simulation data, FFT of each data was obtained.

Comparison of the tension data from experiments and model simulations for the three cases are shown in Figures 4, 5, and 6. In each case it is evident that the model simulation is able to capture the minimum resonant frequency observed in experiments.

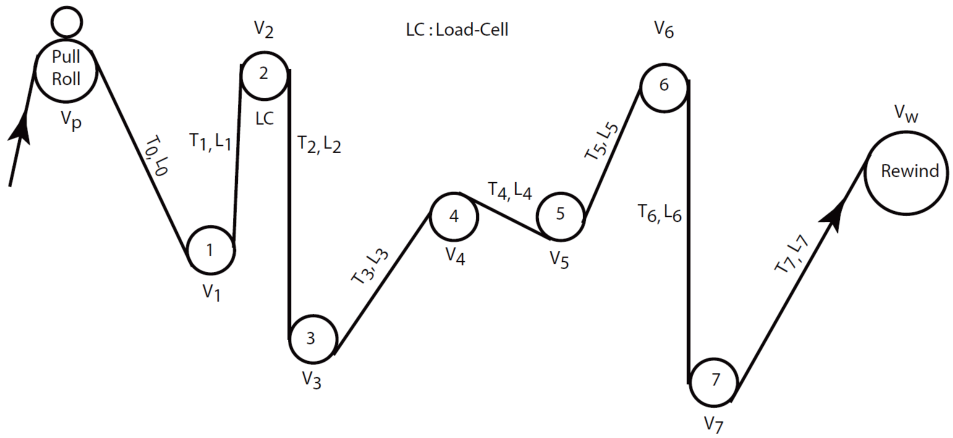
ANALYTICAL SOLUTIONS FOR ONE & TWO IDLER SYSTEMS

Due to the complexity of solving analytically for the system resonant frequencies (modulus of the complex-conjugate eigenvalues of the system matrices), one-and two-idler systems will be discussed here to highlight the process.



L_0	L_1	L_2	L_3	L_4	L_5
32.5”	19.8”	31.2”	54.5”	34.8”	32.6”

Figure 1 – Rewind Section: Five-Idler System Configuration

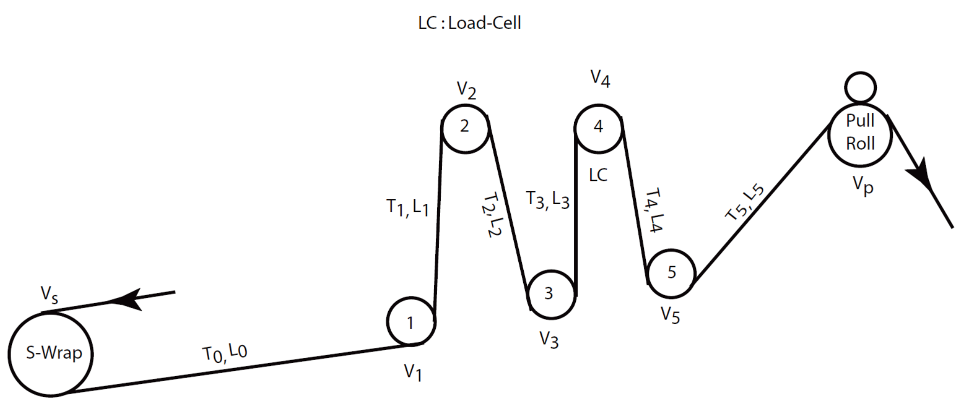


L_0	L_1	L_2	L_3	L_4	L_5	L_6	L_7
32.5"	19.8"	31.2"	26.8"	5.3"	23.3"	34.8"	32.6"

Figure 2 – Rewind Section: Seven-Idler System Configuration

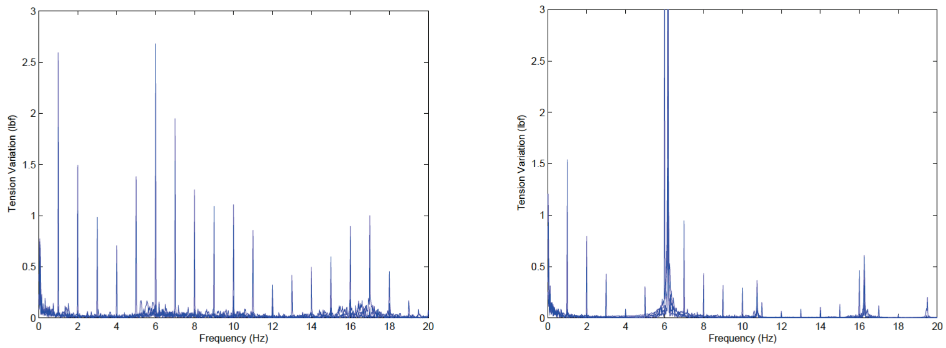
For the one-idler system shown in Figure 7, the resonant frequencies are:

$$f_0 = 0$$
$$f_1 = \frac{1}{2\pi}\sqrt{\frac{EA}{m}} \cdot \sqrt{\frac{1}{L_0} + \frac{1}{L_1}} \tag{16}$$



L_0	L_1	L_2	L_3	L_4	L_5
65.3"	34.2"	13.0"	14.3"	16.0"	35.0"

Figure 3 – Pull Roll Section: Five-Idler System Configuration

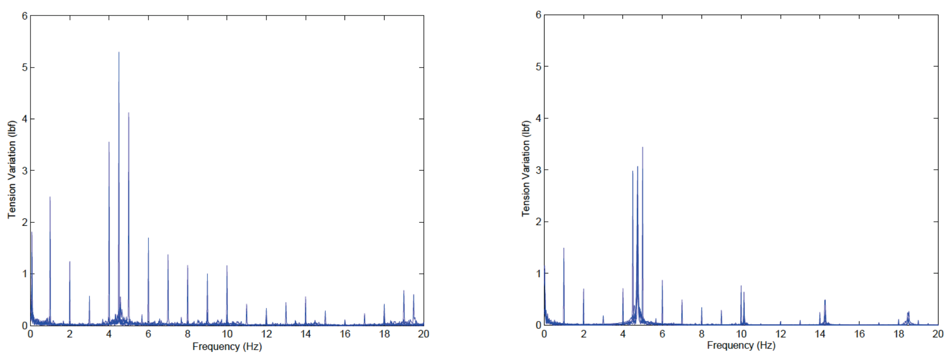


(a) Experiment (b) Model Simulation

Figure 4 – Rewind Section: Five-Idler System Frequency Response Comparison

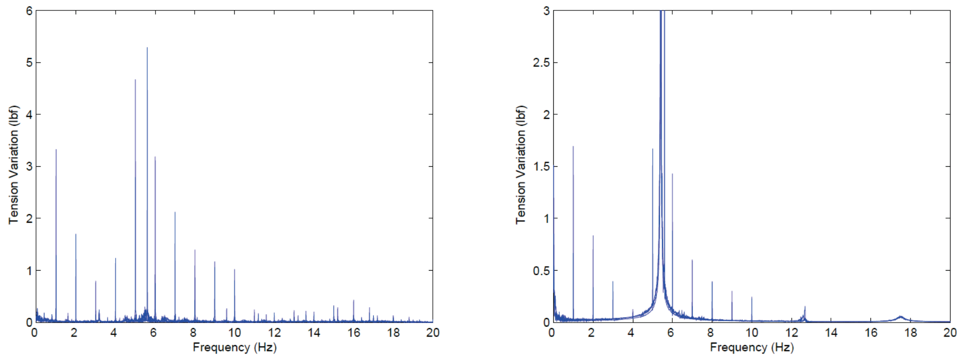
where f_0 corresponds to the rigid-body mode of the system and f_1 is the minimum resonant frequency. Observe that an increase in L_0 and/or L_1 results in a decrease in the minimum resonant frequency $f1$. Assuming the total span length $L_0 + L_1 = 2L$, with a constant L, among all combinations of L_0 and L_1 satisfying this assumption, the minimum resonant frequency is of the smallest value when $L_0 = L_1 = L$, and it is given by $f_{1\min} = (1/2\,\pi)\sqrt{2EA/mL}$.

For the two-idler system shown in Figure 8, the system resonant frequencies



(a) Experiment (b) Model Simulation

Figure 5 – Rewind Section: Seven-Idler System Frequency Response Comparison



(a) Collected Data (b) Simulation Result

Figure 6 – Pull Roll Section: Five-Idler System Frequency Response Comparison

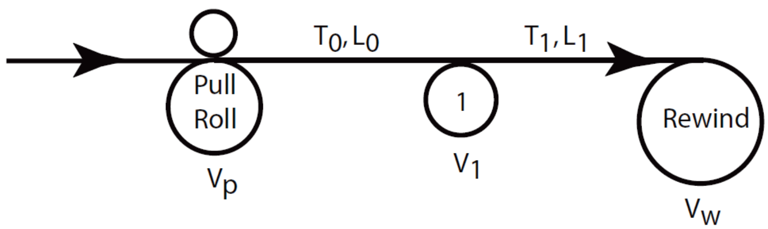


Figure 7 – One-Idler System

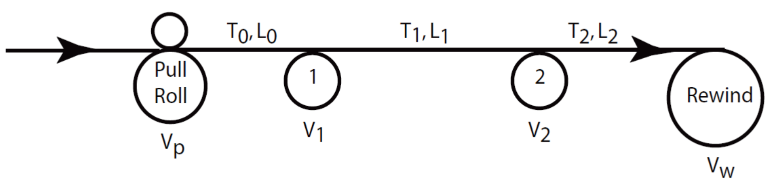


Figure 8 – Two-Idler System

are:

$$f_1 = \frac{1}{2\pi}\sqrt{\frac{EA}{m}} \cdot \sqrt{\frac{-\sqrt{L_1^2(L_2-L_0)^2+4(L_0L_2)^2}+L_0L_1+2L_0L_2+L_1L2}{2L_0L_1L_2}} \quad \{17\}$$

$$f_2 = \frac{1}{2\pi}\sqrt{\frac{EA}{m}} \cdot \sqrt{\frac{\sqrt{L_1^2(L_2-L_0)^2+4(L_0L_2)^2}+L_0L_1+2L_0L_2+L_1L2}{2L_0L_1L_2}} \quad \{18\}$$

The minimum resonant frequency is f_1 for all L_0, L_1, and L_2. Also, if the end span lengths are the same, that is, $L_0 = L_2 = L$, the minimum resonant frequency is independent of L_1 and is given by

$$f_1 = \frac{1}{2\pi}\sqrt{\frac{EA}{mL}} \quad \{19\}$$

Therefore, if the end span lengths are the same in a three-span system, the middle span length can be arbitrarily increased without affecting the minimum resonant frequency. Also, from the above expressions for resonant frequencies, it is clear that interchanging the values of $L0$ and $L2$ does not change the resonant frequencies.

NUMERICAL ANALYSIS

It is generally difficult to obtain useful expressions for the resonant frequencies if the number of idlers is more than two. Even if the expressions for resonant frequencies can be found, it may not be possible to identify the minimum one. Numerical analysis of the resonant frequencies, by varying dependent physical quantities such as span lengths, roller inertias, and modulus of elasticity, can provide some insights into the behavior of the minimum resonant frequency as a function of these physical quantities. In the following, the minimum resonant frequencies of the two-, three-, and four-idler systems are numerically analyzed as a function of the span lengths and modulus of elasticity. For simplicity, all the idlers are treated as identical with inertia $J = 0.069874$ lbm-ft^2 (= 0.00217 lbf- ft-s^2) and radius $R = 0.125$ ft, which results in $m = J/R^2 = 4.4719$ lbm (= 0.1389 lbf-ft^{-1}-s^2). Further, except for the last numerical simulation, EA is taken to be 2000 lbf.

Two-Idler System

Numerical simulations of the model for the two-idler system are performed using the following procedure:

1. Choose L_0 and L_2 such that $L_0+L_2 = 10$ ft.
2. Choose the middle span length as $L_1 = 50$ ft.
3. Increase L_0 from 0.1 ft to 9.9 ft in steps of 0.1 ft and correspondingly decrease L_2 from 9.9 ft to 0.1 ft to comply with step 1.
4. For each pair of L_0 and L_2, solve the system matrix A_2 for the minimal resonant frequency f_1, and plot the trajectory of f_1 in 3D space as a function of L_0 and L_2.
5. Repeat the above steps for two different L_1 values, 5 ft and 10 ft.

The minimum resonant frequency as a function of L_1 and L_2 is shown in Figure 9. Figure 9 verifies the fact that when $L_0 = L_2$, the minimum resonant frequency takes the same value irrespective of the length L_1; in the figure $L_0 = L_2 = 5$ ft is a single point for all L_1. Figure 9 also reveals some other guidelines on choosing L_0 and L_1 if the middle span length L_1 is fixed:

1. If $L_0 + L_2 = 2L < L_1$, choosing $L_0 = L_2 = L$ will maximize the minimum resonant frequency.
2. If $L_0+L_2 =L_1$, any combination of L_0 and L_1 will result in the same minimum resonant frequency.
3. If $L_0 +L_2 = 2L > L_1$, the minimum resonant frequency can be increased by increasing $|L_0 - L_2|$. The minimum value of f_1 will be attained for $L_0 =L_2 =L$.

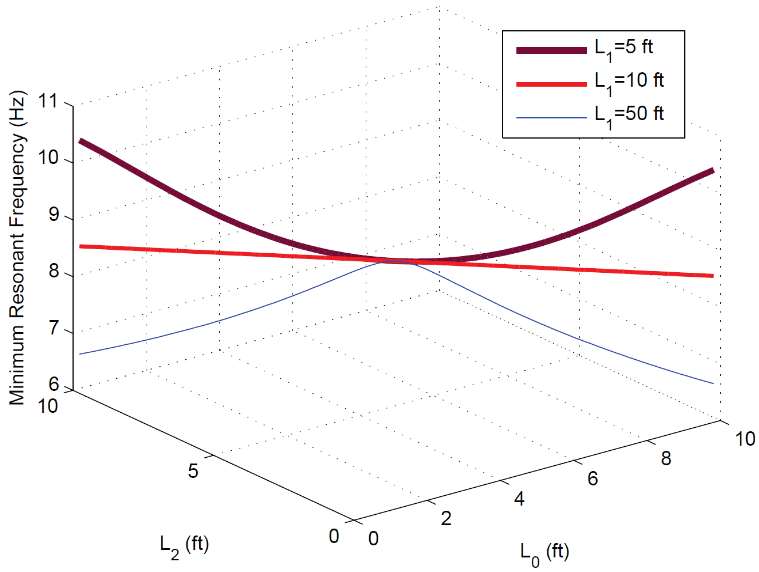


Figure 9 Two-Idler System Minimum Resonant Frequency: $L_0 + L_2 = 10$ ft

Three-Idler System

Numerical simulations for the three-idler system with a procedure similar to the two-idler case give some useful insights into the behavior of the minimum resonant frequency. The first observation is related to the symmetric property of the span length distribution which is similar to the one discussed for the two-idler case. If the minimum resonant frequency f_1 is written as a function of all span lengths, that is $f_1(L_0, L_1, L_2, L_3)$, then the following relationship can be obtained:

$$f_1(L_0, L_1, L_2, L_3) = f_1(L_3, L_2, L_1, L_0) \neq f_1(L_0, L_2, L_1, L_3) = f_1(L_3, L_1, L_2, L_0).$$

The procedure followed for the three-idler system: Choose $L_1 = L_2 = 40$ ft, vary the total span length of the first and last span, $L_0 + L_3 = 2L$ from 20 ft to 60 ft in steps of 5 ft, and for each total span length value the same strategy as in the two-idler case is employed. The minimum resonant frequency as a function of L_0 and L_3 is shown in Figure 10.

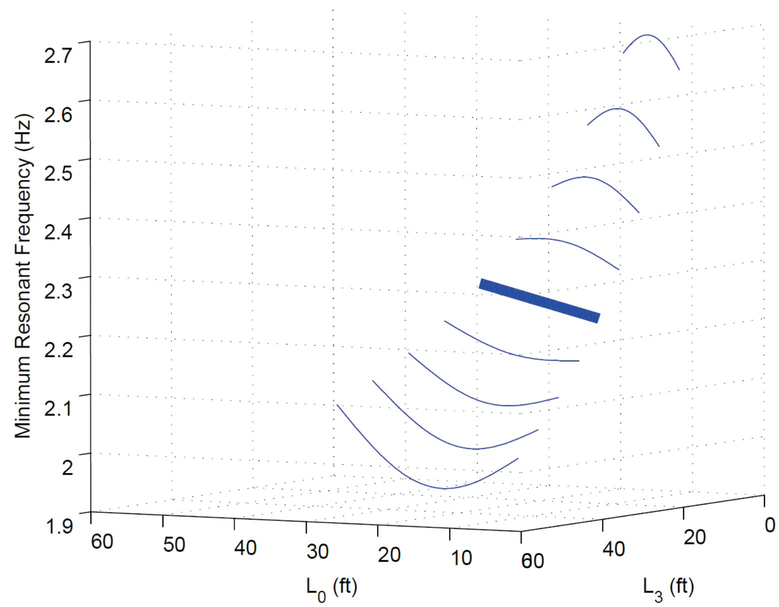


Figure 10 – Three-Idler System Minimum Resonant Frequency: $L_1 = L_2 = 40$ ft, $L_0 + L_3 = 2L$, $2L = 20$ to 60 ft

In Figure 10, the middle thick straight line corresponds to the case when $L_0 + L_3 = L_1 = L_2 = 40$ ft, and the following observations can be drawn:

1. If $L_0 + L_3 = 2L < L_1$, choosing $L_0 = L_3 = L$ will maximize the minimum resonant frequency.
2. If $L_0 + L_3 = L_1 = L_2$, any combination of L_0 and L_3 will result in the same minimum resonant frequency.
3. If $L_0 + L_3 = 2L > L_1$, f_1 is maximized by increasing $|L_0 - L_3|$. The minimum value of f_1 will be attained when $L_0 = L_3 = L$.

Four-Idler System

The following observations about the minimum resonant frequency can be drawn from the numerical simulations for the four-idler system:

1. If $L_0 = L_4$ and $L_1 = L_3$, $f_1(L_0,L_1,L_2,L_3,L_4) = f_1(L_0,L_1,\alpha L_2,L_3,L_4)$ where $\alpha > 0$, that is, the minimum resonant frequency, f_1, is independent of L_2.
2. The symmetry property also applied to the four-idler case, that is, $f_1(L_0,L_1,L_2,L_3,L_4) = f_{min}(L_4,L_3,L_2,L_1,L_0)$.

The minimum resonant frequency as a function of L_0 and L_4 is shown in Figure 11 with the following numbers: $L_1 = L_3 = 30$ ft, $L_2 = 40$ ft, and $L_0 + L_4 = 2L$ varies from 20 ft to 60 ft in steps of 5 ft. In the figure the middle thick line corresponds to the case $L_0 + L_4 = L_2 = 40$ ft.

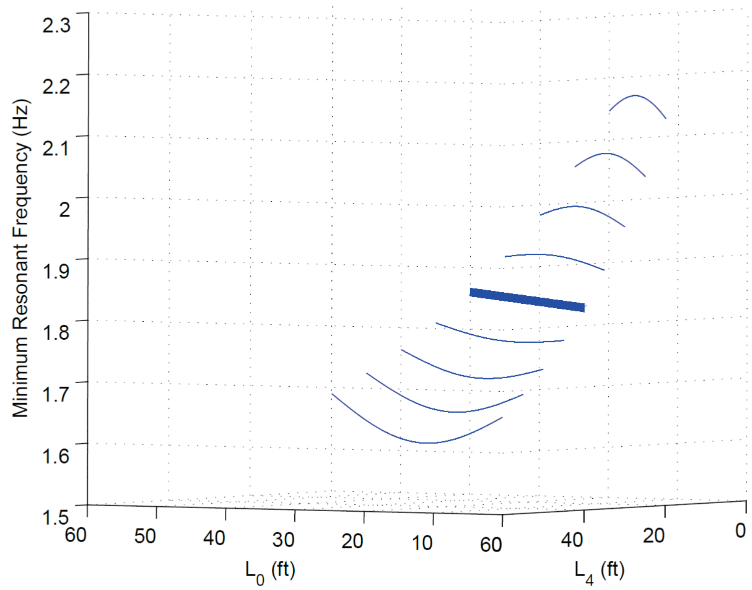


Figure 11 – Four-Idler System Minimum Resonant Frequency: $L_1 = L_3 = 30$ ft, $L_2 = 40$ ft, $L_0 + L_4 = 2L$, $2L = 20$ to 60 ft

The following additional observations can be made based on the numerical simulations:

1. If $L_0 + L_4 = 2L < L_2$, choosing $L_0 = L_4 = L$ will maximize the minimum resonant frequency.
2. If $L_0 + L_4 = L_2$, any combination of L_0 and L_4 will result in the same minimum resonant frequency.
3. If $L_0 + L_4 = 2L > L_2$, f_1 can be increased by increasing the quantity $|L_0 - L_4|$. The minimum value for $f1$ will be attained when $L_0 = L_4 = L$.

In addition, a simulation result similar to Figure 9 for the two-idler case is generated for the four-idler case as shown in Figure 12.

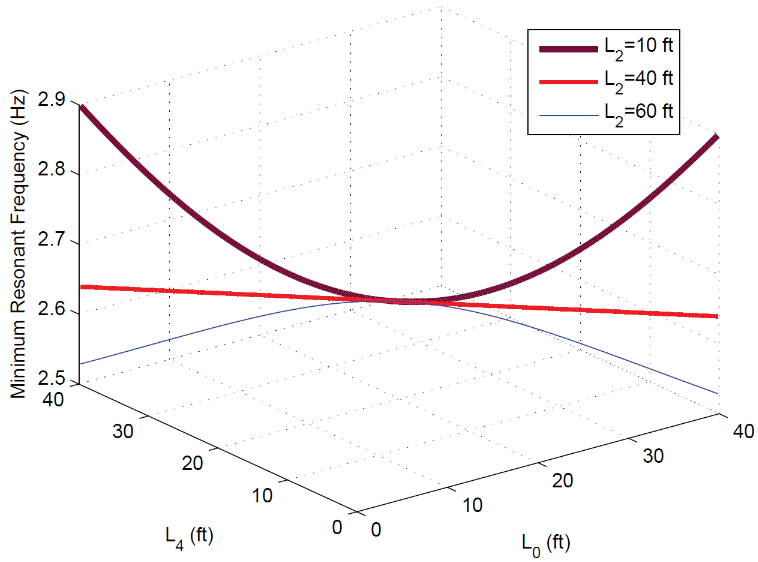


Figure 12 – Four-Idler System Minimum Resonant Frequency: $L_0 + L_4 = 40$ ft, $L_1 = L_3 =$ 20 ft and varying L_2

Effect of *EA/m*

Since the minimum resonant frequency is a function of the quantity $\sqrt{EA/m}$ (as is evident from the expressions for the one- and two-idler cases), it is reasonable to assume that it affects the minimum resonant frequency in a similar fashion. Numerical simulation of the model has been done to verify this by fixing the total length of all the spans, varying number of idlers, and considering three values for the quantity *EA* – 1000 lbf, 2000 lbf, and 4000 lbf. Figure 13 shows the results of the simulation. It is clear that increasing *EA* increases the minimum resonant frequency while increasing the number of idle rollers, for the same total length of spans, decreases the minimum resonant frequency.

MASS-SPRING SYSTEM ANALOGY

Based on the considered model for the web span tension dynamics and web velocity on the roller, one can give a mass-spring system analogy for the system of idle rollers and spans. For example, mass-spring systems analogous to the two-idler and four-idler cases are shown in Figures 14 and 15, respectively. The velocities V_p and V_w are the input web velocities (driven roller velocities are maintained by a control system). Note that the rigid-body mode corresponds to all the involved velocities being equal.

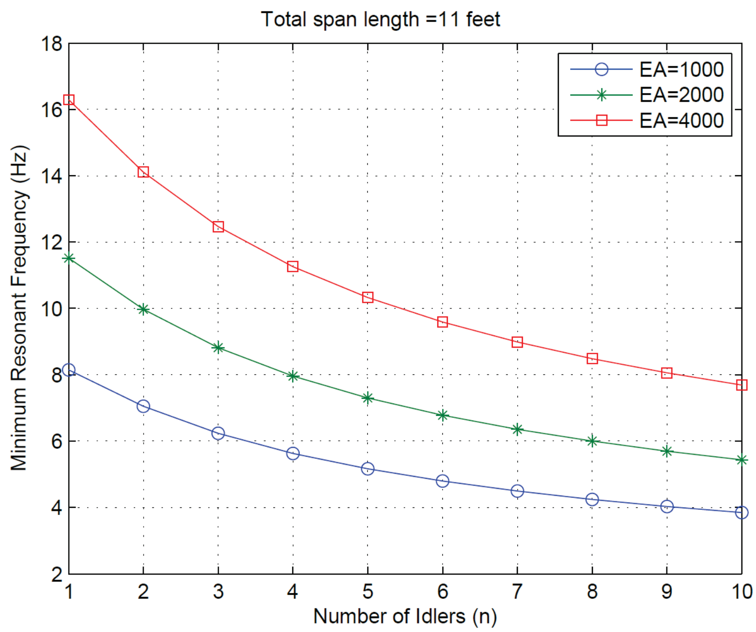


Figure 13 – Effect of *EA* on the Minimum Resonant Frequency

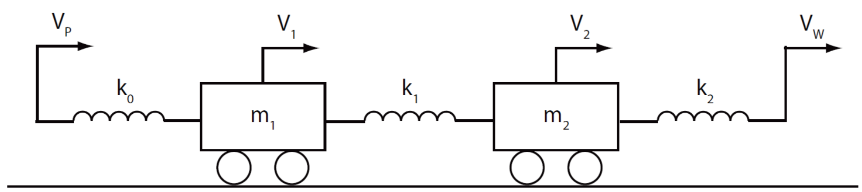


Figure 14 – Two-Idler System Analogy

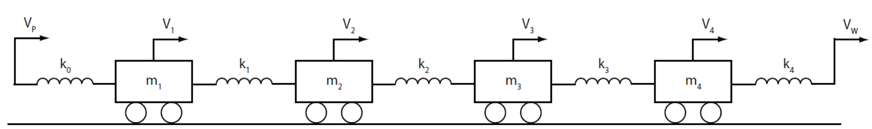


Figure 15 – Four-Idler System Analogy

The mass spring system has five independent energy storage elements – 3 springs and 2 masses. Therefore, the motion of the system is characterized by five state variables, these are the forces in the three springs (F_{k0}, F_{k1}, F_{k2}) and the velocities of the two masses (V_1, V_2). The dynamic equations of motion are given by

$$\begin{aligned} m_1\dot{V}_1 &= -F_{k0} + F_{k1} \\ m_2\dot{V}_1 &= -F_{k1} + F_{k2} \\ \dot{F}_{k0} &= k_0(V_1 - V_p) \\ \dot{F}_{k1} &= k_1(V_2 - V_1) \\ \dot{F}_{k2} &= k_2(V_w - V_2) \end{aligned}$$

In matrix form,

$$\begin{bmatrix} \dot{F}_{k0} \\ \dot{V}_1 \\ \dot{F}_{k1} \\ \dot{V}_2 \\ \dot{F}_{k2} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & k_0 & 0 & 0 & 0 \\ -\frac{1}{m_1} & 0 & \frac{1}{m_1} & 0 & 0 \\ 0 & -k_1 & 0 & k_1 & 0 \\ 0 & 0 & -\frac{1}{m_2} & 0 & \frac{1}{m_2} \\ 0 & 0 & 0 & -k_2 & 0 \end{bmatrix}}_{\mathbf{A}_2} \begin{bmatrix} F_{k0} \\ V_1 \\ F_{k1} \\ V_2 \\ F_{k2} \end{bmatrix} + \begin{bmatrix} 0 & -k_0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ k_2 & 0 \end{bmatrix} \begin{bmatrix} V_p \\ V_w \end{bmatrix} \quad \{20\}$$

The equations of motion for the mass-spring system shown in Figure 14 are the same as the two-idler system. Therefore, the forces in the springs are analogous to tensions in the web spans and the velocities of the masses are analogous to web velocities on the rollers.

ANALYTICAL APPROXIMATION OF THE MINIMUM RESONANT FREQUENCY FOR EQUAL SPAN LENGTHS AND ANY NUMBER OF IDLE ROLLERS

In this section, an analytical estimate of the minimum resonant frequency of a system with many spans with equal lengths is derived. One example of such a system is the accumulator system where the span lengths are all equal. The system matrix for an n-idler system is given by

$$\mathbf{A}_n = \begin{bmatrix} 0 & k & & & & \\ -\frac{1}{m} & 0 & \frac{1}{m} & & 0 & \\ & -k & 0 & k & & \\ & & -\frac{1}{m} & 0 & \ddots & \\ & 0 & & \ddots & \ddots & \frac{1}{m} \\ & & & & -k & 0 \end{bmatrix}_{(2n+1)\times(2n+1)} \quad \{21\}$$

where $m = J/R^2$, $k = EA/L$, and L is the web span length.

The eigenvalues of the system matrix are the roots of the characteristic equation:

$$\det(sI-\mathbf{A}_n)=\begin{vmatrix} s & -k & & & & \\ \frac{1}{m} & s & -\frac{1}{m} & & 0 & \\ & k & s & -k & & \\ & & \frac{1}{m} & s & \ddots & \\ & 0 & & \ddots & \ddots & -\frac{1}{m} \\ & & & & k & s \end{vmatrix}_{(2n+1)\times(2n+1)}=0 \qquad \{22\}$$

To simplify equation {22}, one matrix determinant property that will be used is the following: If a row or a column of a determinant is multiplied by a scalar k, then the determinant is multiplied by k. Note that the dimension of the system matrix is always odd, $2n+1$. With these properties in mind, the characteristic equation can be obtained by the following procedure:

1. Define $s = p\omega$ where $\omega = \sqrt{k/m} = \sqrt{EA/mL}$. Substitute this into equation {22} to get $\det(p\omega I - A_n)$.
2. Divide all odd rows and even columns of $\det(p\omega I - A_n)$ by $\sqrt{k}$.
3. Divide all even rows and odd columns of $\det(p\omega I - A_n)$ by $\sqrt{1/m}$.
4. Define a special determinant:

$$D_i=\begin{vmatrix} p & -1 & & & & \\ 1 & p & -1 & & 0 & \\ & 1 & p & -1 & & \\ & & 1 & p & \ddots & \\ & 0 & & \ddots & \ddots & -1 \\ & & & & 1 & p \end{vmatrix}_{i\times i}$$

Now

$$\det(sI-A_n)=\omega^{2n+1}D_{2n+1}=\omega^{2n+1}\begin{vmatrix} p & -1 & & & & \\ 1 & p & -1 & & 0 & \\ & 1 & p & -1 & & \\ & & 1 & p & \ddots & \\ & 0 & & \ddots & \ddots & -1 \\ & & & & 1 & p \end{vmatrix}_{(2n+1)\times(2n+1)}=0 \qquad \{23\}$$

For D_i, the following recursive equation is valid:

$$D_i = pD_{i-1} + D_{i-2} \qquad (i = 3, 4, \cdots) \qquad \{24\}$$

One can easily compute $D_1 = p$ and $D_2 = p^2 + 1$; based on these two as initial conditions, D_{2n+1} can be obtained by using equation {24} recursively. The roots of $D_{2n+1} = 0$ are obtained first, followed by the resonant frequencies. If p is a root of $D_{2n+1} = 0$, the corresponding resonant frequency is $\left|p(1/2\pi)\sqrt{EA/mL}\right|$ Hz.

It is possible to obtain a lower bound of the minimum resonant frequency by considering the characteristic polynomial $D_{2n+1} = 0$ which is given by

$$p(p^{2n} + c_{2(n-1)}p^{2(n-1)} + \cdots c_2 p^2 + c_0) = 0 \tag{25}$$

The values of c_0 and c_2 for increasing number of idle rollers are given in Table 2. By induction, the coefficients c_0 and c_2 for the general case of an n-idler system

n	c_0	c_2
1	$2 = 1+1$	1
2	$3 = 2+1$	4
3	$4 = 3+1$	$10 = 3^2 + 1^2$
4	$5 = 4+1$	$20 = 4^2 + 2^2$
5	$6 = 5+1$	$35 = 5^2 + 3^2 + 1^2$
6	$7 = 6+1$	$56 = 6^2 + 4^2 + 2^2$
7	$8 = 7+1$	$84 = 7^2 + 5^2 + 3^2 + 1^2$
8	$9 = 8+1$	$120 = 8^2 + 6^2 + 4^2 + 2^2$

Table 2: Values of c_0 and c_1 as a function of n

are given by

$$\begin{aligned} c_0 &= n+1 \\ c_2 &= \sum_{i=0}^{floor(n/2)} (n-2i)^2 \end{aligned} \tag{26}$$

where, for odd n, the function $floor(n/2)$ gives the largest integer number smaller than $n/2$.

Since $p = s/\omega$ and $\omega = \sqrt{EA/mL}$, if $\sqrt{EA/mL} > 1$, then higher powers of p would be much smaller. Hence, the characteristic equation $D_{2n+1} = 0$ can be approximated by

$$p(c_2 p^2 + c_0) = 0 \tag{27}$$

Since the complex conjugate roots of the above equation are $\pm j\sqrt{(c_0/c_2)}$, an estimate of the minimum resonant frequency f_1 (in Hz) is given by

$$\hat{f}_1 = \frac{\omega}{2\pi}\sqrt{\frac{c_0}{c_2}} \tag{28}$$

In summary, for an n-idler system with identical idle rollers and equal span lengths, L, an estimate of the minimum resonant frequency is

$$\hat{f}_1 = \frac{\omega}{2\pi}\sqrt{\frac{n+1}{\sum_{i=0}^{(floor(n/2))}(n-2i)^2}} \qquad \text{if } \omega = \sqrt{\frac{EA}{mL}} > 1 \tag{29}$$

The requirement of $\omega > 1$ is satisfied for many materials, since the quantity EA is typically much larger than mL. Further, this estimate is a lower bound on the minimum resonant frequency, and so it can be safely used. Table 3 gives the values of the estimate and the actual frequency for values of n ranging from 1 to 10. It can be observed that as the number of idlers is increased, the estimate tends to converge to the actual frequency.

# of Idlers n	Estimated $\hat{f}_1/(1/2\pi)\sqrt{EA/mL}$	Actual $f_1/(1/2\pi)\sqrt{EA/mL}$
1	1.4142	1.4142
2	0.8660	1.0000
3	0.6325	0.7654
4	0.5000	0.6180
5	0.4140	0.5176
6	0.3536	0.4450
7	0.3086	0.3902
8	0.2739	0.3473
9	0.2462	0.3129
10	0.2236	0.2846

Table 3 – Estimated and actual minimal resonant frequency

SUMMARY AND CONCLUSIONS

A simple dynamic model for web tension dynamics within a span is considered for predicting the resonant frequencies in a subsection of a web line between two driven rollers containing idle rollers and spans. Based on this model one can write the system dynamics for any number of idle rollers and spans. With this model, the system matrix is a tridiagonal matrix which is suitable for analytical and numerical analysis. The ability of the model to predict resonant frequencies (at least the minimum resonant frequency) was verified by conducting experiments on a web line, and comparing the experimental data with the data from model simulations.

Analytical expressions of the resonant frequencies for the one- and two-idler systems are given and discussed. The focus of this is on the effect of span lengths on the resonant frequency; it was assumed that the idle rollers are identical for both analytical and numerical analysis. In the one-idler case, the minimum value of the resonant frequency occurs when the span lengths adjacent to the idle roller are equal. For the two-idler case, two useful observations can be made: (1) If the length of the two end spans are equal, then the minimum resonant frequency does not depend on the middle span length. (2) The two end span lengths can be interchanged without affecting the resonant frequencies. Numerical analysis was also done for the three- and four-idler cases. It was found that behavior of the minimum resonant frequency is symmetric in terms of span lengths similar to that of the two-idler case.

For systems containing identical rollers with equal span lengths (such as an accumulator), an estimate of the minimum resonant frequency was derived in terms of the number of idle rollers. The estimate is a lower bound on the minimum resonant frequency, and approaches the actual value with increase in the number of idle rollers.

This study is expected to assist the web machine designer as well as the control systems engineer. For the web machine designer, it helps in the selection of span lengths and placement of idle rollers to maximize the minimum resonant frequency, and in designing the machine structure to avoid web and idle roller induced resonances. For the control system engineer, this leads to limitations on the bandwidth of the overall closed-loop system.

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Web Line Resonant Frequencies

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Oklahoma State University,
USA

Name & Affiliation	**Question**
	In Figure 13, it appears that as the number of idlers increases, the idler frequency decreases. Why is that?
Name & Affiliation	**Answer**
P. R. Pagilla, Oklahoma State University	The length of the total span was kept the same for the idler system. For instance, I might have one idler with a total span of 10 meters or I could have two idlers with two spans 5 meters long. In the case with two idlers I have also doubled the inertia.
Name & Affiliation	**Question**
Bob Lucas, Winder Science	In the process of your modeling, have you considered the length the web runs upstream and downstream of a given idler relative to that idler's circumference in case that idler has some eccentricity to it? This can be a forcing function that introduces a resonance.
Name & Affiliation	**Answer**
P. R. Pagilla, Oklahoma State University	We did not. Carlo did those experiments in the idler paper, but not here.
Name & Affiliation	**Question**
Ron Lynch, Procter & Gamble	You have shown us a spring mass system where you have kept the length of the spring constant but the distance or the stretch length would be changing with span length. Is that equivalent to an idler case where the span length is constant but the amount of material in between is changing? In your car on wheels, the length, the amount of the spring remains unchanged. In the idler case, the amount of the spring between the two idlers is changing. The distance between the idlers doesn't change. Are they equivalent systems?
Name & Affiliation	**Answer**
P. R. Pagilla, Oklahoma State University	Yes.
Name & Affiliation	**Question**
J. J. Shelton, Oklahoma State University	There is really very little in the literature regarding rollers as masses and webs as springs, which is what you are studying.
Name & Affiliation	**Answer**
P. R. Pagilla, Oklahoma State University	This work was not dominated by viscous forces. The resonant frequencies don't change with friction. The friction is hard to validate. This study was only for stiff webs. We used Tyvek in our tests.