Measurable dynamics of meromorphic maps

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Zusammenfassung

Thema dieser Arbeit ist das Verhalten der Iterierten einer meromorphen Funktion auf ihrer Juliamenge. Dabei werden wir Mengen, deren (Lebesgue) Maß Null ist, vernachlässigen und diesen Ansatz mit dem Begriff *messbare Dynamik* bezeichnen. Hauptziel ist die Bestimmung von $\omega(z)$ für fast alle Punkte z der Juliamenge, wobei der ω-*limit set* $\omega(z)$ die Menge aller Häufungs-punkte der Iteriertenfolge $(f^n(z))$ bezeichnet. Dies ist eng verknüpft mit der Frage, ob die betrachtete Funktion rekurrent ist. Da diese Fragestellung nur für Funktionen, deren Juliamenge positives Maß besitzt, sinnvoll ist, interessieren wir uns ebenfalls für das Maß von Juliamengen.

Wir formulieren zunächst hinreichenden Bedingungen dafür, dass eine Funktion nicht rekurrent ist, und wenden diese später für Funktionen mit rationaler Schwarzscher Ableitung und eine verallgemeinerte Sinusfamilie an. Zusätzlich erhalten wir Informationen über das Maß der Juliamenge. Insbesondere können wir Beispiele angeben für Funktionen, deren Fatoumenge positives aber endliches Maß besitzt, und für Funktionen, für die das Maß der Punkte, die unter Iteration nicht gegen unendlich streben, endlich ist.

Unser besonderes Augenmerk liegt jedoch auf ganzen Funktionen, welche nur endlich viele Singularitäten der Umkehrfunktion besitzen, wobei hier gemäß Vielfachheit gezählt wird. Diese Funktionen haben die Form

$$f(z) = \int_0^z P(t)\exp(Q(t))dt + c,$$

wobei P und Q Polynome sind und $c \in \mathbb{C}$. Falls jeder endliche asymptotische Wert ausreichend schnell entkommt, d. h. seine Iterierten ausreichend schnell gegen unendlich streben, so ist eine solche Funktionen nicht rekurrent. Unter zusätzlichen, die kritischen Werte betreffenden Bedingungen können wir darüber hinaus einen *typischen* ω-limit set konkret bestimmen, wobei typisch bedeutet, dass die davon abweichende Menge eine Nullmenge ist. Ist nämlich außerdem jeder kritische Wert enthalten in einer attraktiven Fatou-Komponente, präperiodisch oder entkommt ebenfalls ausreichend schnell, so ist der ω-limit set von fast jedem Punkt der Juliamenge identisch und gleich dem Abschluss des Vorwärtsorbits der Menge der asymptotischen Werte.

Sobald wenigstens ein asymptotischer Wert nicht mehr entkommt, sondern statt dessen präperiodisch ist, ändert sich der typische ω-limit set grundlegend. Ist dies der Fall und ist jede andere endliche Singularität weiterhin präperiodisch, entkommt ausreichend schnell oder liegt in einer attraktiven Komponente der Fatoumenge, so ist der Vorwärtsorbit von fast jedem Punkt der Juliamenge dicht in derselben. Im Falle $J(f) = \mathbb{C}$ ist f dann rekurrent.

Abstract

The subject of this thesis is the behaviour of the iterates of a meromorphic function on the Julia set. We will neglect sets whose (Lebesgue) measure is zero, and refer to this approach with the term *measurable dynamics*. Our main goal is to determine $\omega(z)$ for almost every point z in the Julia set, where the ω-*limit set* $\omega(z)$ denotes the set of accumulation points of the sequence of iterates $(f^n(z))$. We are particularly interested in the question of whether the function is recurrent. Since all this only makes sense if the Julia set has positive measure, we also study the measure of Julia sets.

We give sufficient conditions for a function not to be recurrent and apply these to functions with rational Schwarzian derivative and to a generalised sine family. Moreover we obtain lower estimates for the measure of the Julia set and in particular examples of functions whose measure of the Fatou set is positive but finite and examples of functions, for which all points, except for a set of finite measure, escape - that is, their iterates tend to infinity.

Our main focus, however, is the class of entire transcendental functions with only finitely many singularities of the inverse, counting multiplicity. These functions are of the form

$$f(z) = \int_0^z P(t)\exp(Q(t))dt + c,$$

where P and Q are polynomials and $c \in \mathbb{C}$. Provided that all finite asymptotic values escape sufficiently fast, we obtain that these functions are not recurrent. Under additional assumptions we can even determine a *typical* ω-limit set, where typical is used in the sense that the set of deviating points has measure zero. If additionally every critical value is either contained in an attracting Fatou component, is pre-periodic or also escapes sufficiently fast, then the ω-limit set is the same for almost every point in the Julia set and coincides with the closure of the forward orbit of the set of asymptotic values.

As soon as at least one of the finite asymptotic values does not escape any more but is instead pre-periodic, the typical ω-limit set changes greatly. If this is the case, and any other finite singularities of the inverse remains to be either pre-periodic, be contained in an attractive Fatou component or escape sufficiently fast, the forward orbit of almost every point in the Julia set is dense in the Julia set. If $J(f) = \mathbb{C}$ it follows that $\omega(z) = \hat{\mathbb{C}}$ for almost every $z \in \mathbb{C}$ and that the function f is recurrent.

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1 Introduction

1.1 Context and main results

One of the main ideas in complex dynamics is to divide the plane into the Fatou set of points, where iterates behave stably, i.e. where they form a normal family, and its complement, the Julia set. By definition the dynamics in the Fatou set is the easiest and is understood very well. We are interested in the dynamics of meromorphic functions on their Julia set. Here we roughly describe the context and state the main results. For a more detailed description we refer to §2.

Our starting point is a result of H. Bock [7] (see Theorem 2.24 for the precise statement) which implies that for every meromorphic function f there is a set S of (Lebesgue) measure zero, such that

(i) $\omega(z) = \hat{\mathbb{C}}$ for all $z \in J(f)\backslash S$ or

(ii) $\omega(z) \subset P(f)$ for all $z \in J(f)\backslash S$.

Here the post-singular set $P(f)$ denotes the closure of the union of the forward orbits of all singularities of the inverse function, consisting of critical and asymptotic values. In case (i) the theorem even implies that $J(f) = \mathbb{C}$ and that f is recurrent and ergodic. Here a meromorphic function is called *ergodic* (with respect to the Lebesgue measure), if any completely invariant set has full measure or measure zero. It is called *recurrent* if for every measurable set $A \subset \mathbb{C}$ and almost every point $z \in A$ the forward orbit $O^+(z)$ intersects A infinitely many times. If $P(f) \neq \hat{\mathbb{C}}$, (ii) implies non-recurrence.

It is natural to ask which case holds for a given function. Since a non-empty Fatou set always implies (ii), one can restrict to the cases in which the Julia set consists of the whole complex plane. If the Julia set is not the entire plane, and thus (ii) holds, it would still be interesting to know whether the Julia set has positive measure, since otherwise the statement (ii) would be trivial.

H. Bock also gives sufficient conditions for (i): if f is entire, the set of finite singularities of the inverse function is finite and all of these are pre-periodic but not periodic, then (i) is satisfied. Thus the function $f(z) = 2\pi i \exp(z)$ is an example for (i), in which the post-singular set consists of the asymptotic value, zero, its image $2\pi i$ and the asymptotic value infinity. Other conditions concerning this case are given by L. Keen and J. Kotus [29] (see Theorem 2.27). Conversely it was already shown in 1984, independently by M. Rees [42] and M. Lyubich [32], that the function $f(z) = \exp(z)$ is an example for (ii). Here the post-singular set consists of the forward orbit of the asymptotic value zero, which tends to infinity on the real axis, and the point infinity.

The difference between the dynamics of $\exp(z)$ and $2\pi i \exp(z)$ is caused by the different behaviour of the asymptotic value zero under iteration. One might hope for a classification of the two cases depending on the behaviour of the singularities of the inverse. As a first approach, we restrict to functions with a finite number of singularities of the inverse, all of which have simple orbits. As already mentioned, we can neglect all orbits that would imply the existence of a component of the Fatou set, such as periodic critical points or infinite orbits that converge in $\mathbb{C}$. The simplest orbits that remain are pre-periodic or escaping ones. If one considers meromorphic functions with poles, another interesting case is that of singularities which are mapped eventually onto a pole. This case has been studied by B. Skorulski for the tangent family in [47] and for a larger class of functions in his recent thesis [48].

We are interested in conditions ensuring case (ii). In the third chapter we prove the rather technical Theorem 3.1, which provides a set of sufficient conditions for this case. The proof consists of applying the method developed by M. Rees in order to construct a positive measure set of points, whose iterates show a “spiral” type of behaviour: they are eventually mapped close to some asymptotic value, then follow its orbit for a certain number of iterates, coming close to infinity, until they are mapped again, and even closer than before, to some asymptotic value, etc. These orbits are not dense in $\hat{\mathbb{C}}$, which implies (ii). Since these orbits accumulate at infinity, the only type of components that could possibly intersect this set are wandering and Baker domains. At least for the various families (e.g. functions with a finite number of singularities of the inverse) in which those do not occur, we also obtain that the Julia set has positive measure.

In Chapter 4 we consider functions of the type $f(z) = \int_0^z P(t)\exp(Q(t))dt + c$ with polynomials P and Q and $c \in \mathbb{C}$, such that Q is not constant and P not identically zero. These functions have at most $\deg(Q)$ finite asymptotic values and $\deg(P)$ critical points. In the extremal case that all finite singularities of the inverse are pre-periodic but not periodic, the theorem of H. Bock implies (i). We consider the other extremal case, in which the singularities of the inverse tend to infinity. It turns out that we may neglect the critical values, but have to specify the speed of escape of the asymptotic values. We say that a point z *escapes exponentially* if $|f^n(z)| \geq \exp(|f^{n-1}(z)|^\delta)$ for some $\delta > 0$ and almost all $n \in \mathbb{N}$. Of course, for subsets of $\mathbb{N}$ the term *almost all* means that the complement is finite and we are not referring to the Lebesgue measure. Then Theorem 3.1 yields the following principal result.

1.1 Theorem
Let P and Q be polynomials with P not zero and Q not constant, $c \in \mathbb{C}$ and

$$f(z) := \int_0^z P(t)\exp(Q(t))dt + c.$$

Suppose that all finite asymptotic values escape exponentially. Then the Julia set has positive measure and $\omega(z) \subset P(f)$ for almost all $z \in J(f)$. If $\deg(Q) \geq 3$, then $\operatorname{meas}(F(f)) < \infty$.

Conversely, one may ask whether almost every orbit in the Julia set accumulates at every singularity s of f^{-1}, such that we would get $\omega(z) = P(f)$ for almost every $z \in J(f)$. It is easy to find examples for which this is not the case if s is a critical value. For asymptotic values this question is more interesting and closely related to the well-known open question of whether the Julia set of a polynomial may have positive measure. A positive answer to the latter question would suggest a negative answer to our initial question also for asymptotic values. This will be discussed in §4.4. Under additional assumptions on the critical values, however, the answer on our question is positive. More precisely we get the following.

1.2 Theorem
Let f be as in Theorem 1.1 and again suppose that all finite asymptotic values either escape exponentially. Suppose that every critical point also escapes exponentially, is pre-periodic or is contained in an attractive Fatou-component. Then $\omega(z) = \overline{O^+(A)}$ for almost every point $z \in J(f)$, where A denotes the set of finite asymptotic values.

We define the *multiplicity* of an asymptotic value s as the supremum of the set of all natural numbers n with the following property: there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ the set $f^{-1}(B(s, \epsilon))$ contains at least n unbounded components. Then the functions above have exactly $2\deg(Q)$ asymptotic values, of which $\deg(Q)$ are finite, and $\deg(P)$ critical points, counting multiplicity. Indeed, this family may be characterised as those entire transcendental functions with only finitely many singularities of the inverse, counting multiplicity. This follows from Corollary 2.13, which was proved by G. Elfving in [17]. For an entire transcendental function f with only finitely many singularities of the inverse, all of which are pre-periodic or escape exponentially, the set $P(f)$ does not accumulate in $\mathbb{C}$ and in particular $P(f) \neq \hat{\mathbb{C}}$. Therefore if (ii) is satisfied, the function cannot be recurrent. Thus for this restricted family of functions, the question whether (i) or (ii) is true is equivalent to the question whether f is recurrent or not.

1.3 Theorem
Let f be entire and transcendental with only a finite number of singularities of its inverse, counting multiplicity, such that all these escape exponentially or are pre-periodic, but no critical point is periodic. Then f is not recurrent if and only if all asymptotic values escape exponentially.

In the fifth chapter we discuss applications of Theorem 3.1 for other families, especially transcendental meromorphic functions with rational Schwarzian derivative. These functions may be characterised as those transcendental meromorphic functions with only finitely many singularities of the inverse counting multiplicity. This makes this family a natural candidate with which to continue our studies. Theorem 3.1 applies and again we get sufficient conditions for non-recurrence and estimates for the Julia set.

Finally, we include a chapter on the exponential family. The detailed knowledge of this family allows us to study also the topological structure and the Hausdorff dimension of sets that occur in the preceding chapters. For this family many naturally arising questions can be answered, providing ideas of what to expect for more general families.

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2 Preliminaries

2.1 Notation

Let f^k denote the k-th iterate, and $f^{(k)}$ the k-th derivative of f. Let "meas" denote the Lebesgue measure in the plane, "dist" the Euclidean distance, and "diam" the diameter. For $r > 0$ and $z \in \mathbb{C}$, let $B(z, r)$ denote the open ball of radius r and centre z, $B(M, r) := \bigcup_{z \in M} B(z, r)$ for some $M \subset \mathbb{C}$ and $D(r) := \mathbb{C} \backslash B(0, r)$. For a square S let rS denote the square with the same centre, satisfying $\operatorname{diam}(rS) = r \operatorname{diam}(S)$. For a conformal map $f : D \to \mathbb{C}$ we call $\sup_{z,w \in D} \left| \frac{f'(z)}{f'(w)} \right|$ its *distortion*. A set which is the image of a square by a conformal map ϕ, whose distortion is bounded by $K > 1$, is called a *K-quasi-square*. We note that this is not consistent with the usual definition in the theory of quasi-conformal mappings. However this should not cause confusion, since our quasi-squares are also quasi-squares in that sense.

2.2 Complex analysis

We state the well-known Koebe distortion theorem as it may be found in [40](§1, 1.6, p.21).

2.1 Theorem (Koebe)
Suppose $f : B(0, 1) \to \mathbb{C}$ is conformal with $f(0) = 0$ and $f'(0) = 1$. Then for every $z \in B(0, 1)$ it follows that

$$\frac{1 - |z|}{(1 + |z|)^3} \leq |f'(z)| \leq \frac{1 + |z|}{(1 - |z|)^3}, \tag{1}$$

$$\frac{|z|}{(1 + |z|)^2} \leq |f(z)| \leq \frac{|z|}{(1 - |z|)^2}, \tag{2}$$

$$\frac{1 - |z|}{1 + |z|} \leq \left| z \frac{f'(z)}{f(z)} \right| \leq \frac{1 + |z|}{1 - |z|}. \tag{3}$$

This implies the following fact, which is known as Koebe's $\frac{1}{4}$-Theorem.

2.2 Corollary (Koebe)
Let f be as in Theorem 2.1. Then

$$B\left(0, \frac{1}{4}\right) \subset f\left(B(0, 1)\right). \tag{4}$$

It is much easier to show the following property, which will be sufficient for most of our purposes.

2.3 Lemma
Let $f : B(z_0, r) \to \mathbb{C}$ be holomorphic. Then

$$B\left(f(z_0), \inf_{z\in B(z_0,r)} |f'(z)|r\right) \subset f(B(z_0, r)). \tag{5}$$

Proof. We can assume that f has no critical points. We consider the straight path γ from $f(z_0)$ to the closest boundary point of the image. The pre-image of γ contains a path γ' connecting z_0 with the boundary of $B(z_0, r)$, which is mapped by f one to one onto γ. For the lengths $l(\gamma)$ and $l(\gamma')$ of γ and γ' it follows that

$$r \le l(\gamma') \le \int_\gamma |(f^{-1})'(z)||dz| \le l(\gamma) \sup_{z\in\gamma} |(f^{-1})'(z)| = \frac{l(\gamma)}{\inf_{z\in\gamma} \frac{1}{|(f^{-1})'(z)|}},$$

where f^{-1} denotes the branch of the inverse, which we get by extending that which maps $f(z_0)$ to z_0 along γ. It follows that $l(\gamma) \ge \inf_{z\in B(z_0,r)} |f'(z)|r$. ∎

Rather than disks we will be more interested in the distortion of squares. From Koebe's distortion theorem one can obtain similar estimates for squares. The following lemma will be sufficient for our purpose and follows from Koebe's distortion theorem. However one could also prove this more directly using normal families.

2.4 Lemma
For any $0 < c < 1$ there exists $K_c > 0$ such that for any holomorphic function, which is injective on some square S, the distortion of its restriction to cS is bounded by K_c. Moreover K_c tends to one, if c tends to zero.

The following lemma follows easily from the definition of distortion.

2.5 Lemma
Suppose that the distortion of the conformal map f is bounded by K. Let D and M be measurable subsets of its domain of definition, such that $\operatorname{meas}(D) > 0$. *Then*

$$\frac{\operatorname{meas}(M \cap D)}{\operatorname{meas}(D)} \le \frac{K^2 \operatorname{meas}(f(M) \cap f(D))}{\operatorname{meas}(f(D))}. \tag{6}$$

The term on the left side of (6) is called the *density* of M in D. We state two basic properties of quasi-squares, which we will frequently use. The proofs are straight-forward and may be found in [25].

2.6 Lemma
Let D be a K-quasi-square and $\epsilon > 0$. Then

$$\text{meas}(D) \geq \frac{\text{diam}(D)^2}{2K^2} \text{ and } \text{meas}(D \cap B(\partial D, \epsilon)) \leq 4\epsilon K^2 \,\text{diam}(D). \tag{7}$$

Here ∂S denotes the boundary of a set $S \subset \mathbb{C}$. Finally we state a tool, which we will frequently use to obtain injectivity of a function on certain sets. It is a corollary of the Monodromy Theorem. This may be found in most function theory books, such as [10](§3, p.217).

2.7 Lemma
Let $D' \subset D \subset \mathbb{C}$ be domains and $f : D \to \mathbb{C}$ be holomorphic, such that all singularities of the inverse of f are contained in the unbounded component of $\mathbb{C} \backslash f(D')$. Then f is injective on D'.

To avoid confusion we include a definition of a singularity of f^{-1}.

2.8 Definition
Let $D \subset \hat{\mathbb{C}}$ be a domain, $f : D \to \hat{\mathbb{C}}$ be meromorphic and $s \in \hat{\mathbb{C}}$. Then s is called a singularity of f^{-1} *if there exist*

- *a smooth function $\gamma : [0,1] \to \hat{\mathbb{C}}$ with $\gamma(1) = s$;*
- *a domain $U \subset \hat{\mathbb{C}}$ with $\gamma([0,1)) \subset U$;*
- *a branch ϕ of the inverse of f on U, i.e. $\phi : U \to D$ meromorphic with $f(\phi(z)) = z$,*

such that there is no domain $V \subset \hat{\mathbb{C}}$ with $\gamma([0,1]) \subset V$, and no branch ψ of the inverse of f on V that coincides with ϕ on the component of $U \cap V$ containing $\gamma([0,1))$. We denote the set of singularities of f^{-1} by $\text{sing}(f^{-1})$.

Studying the set $A := \bigcap_{t \in (0,1)} \overline{\phi((t,1))}$, one can classify the singularities of the inverse as follows. Here $\overline{S}$ denotes the closure of a set $S \subset \hat{\mathbb{C}}$.

2.9 Theorem
Let D, f be as above and $s \in \text{sing}(f^{-1})$. Define γ, U and ϕ as in Definition 2.8. Then one of the following cases holds:

- *There exists $z \in D$ with $f(z) = s$ and $\phi(\gamma(t)) \to z$ as $t \to 1$. If neither z nor s coincides with ∞ it follows that $f'(z) = 0$; or*
- $\text{dist}(\phi(\gamma(t)), \partial D) \to 0$ *as $t \to 1$.*

In the first case s is called a critical *value and in the second case an* asymptotic *value.*

In the literature sometimes the closure of our set of singularities of the inverse is denoted by the same name. However if this set is finite, which is the case for all functions we will consider, this makes no difference.

The multiplicity of a critical value is defined to be the sum of all multiplicities of all critical points in its pre-image. As mentioned in the introduction, we also define the multiplicity of an asymptotic value.

2.10 Definition
Let f be a transcendental meromorphic function and a an asymptotic value of f. The multiplicity *of a is defined as the supremum of all natural numbers $n \in \mathbb{N}$ with the property that there is an $\epsilon_0 > 0$ such that such that for all $0 < \epsilon < \epsilon_0$ the set $f^{-1}(B(a, \epsilon))$ contains at least n unbounded connected components.*

It is evident that the pre-image of a neighbourhood of an asymptotic value of an entire function must contain an unbounded component. Thus the multiplicity is always at least one but it can be larger or even infinite.

2.3 Functions with finitely many singularities of the inverse

Here we count the singularities of the inverse according to multiplicity as defined above. Another way of characterising these functions is by means of the *Schwarzian derivative* $S(f) := \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$. This was done by G. Elfving in [17]. He generalised a method introduced by R. Nevanlinna in [38], who considered the case without critical values. This method is summarised in [39](§9, Nr.33, p.345). The following is a restatement of a result from [17](§11, Nr.44, p.57) adapted to out context.

2.11 Theorem (Elfving)
Let f be a meromorphic function with exactly p asymptotic values and r critical values counting multiplicity. Then f is of order $p/2$ and $S(f)$ is a rational function of degree $2p+r-2$, such that every pole a of $S(f)$ has order two and the coefficients $S(f)(z) = \frac{a_0}{(z-a)^2} + \frac{a_1}{z-a} + a_2 + ...$ satisfy the following properties:

(a) there exists $\lambda \in \mathbb{Z}$ such that $a_0 = \frac{1-\lambda^2}{2}$;

(b)

$$\det\begin{pmatrix} a_1 & 2*1(1-\lambda) & 0 & \dots & 0 & 0 \\ a_2 & a_1 & 2*2(2-\lambda) & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ a_{\lambda-1} & a_{\lambda-2} & a_{\lambda-3} & \dots & a_1 & 2(\lambda-1)(-1) \\ a_\lambda & a_{\lambda-1} & a_{\lambda-2} & \dots & a_2 & a_1 \end{pmatrix} = 0$$

Conversely if $R(z)$ is a rational function of degree d, for which infinity is a pole of order $(p-2)$, and every finite pole is of order two and satisfies (a) and (b), then every solution of the differential equation $S(f) = R$ is a meromorphic function of order $p/2$ and has exactly p asymptotic and $d-2p+2$ critical values counting multiplicity.

Elving also studied the asymptotic behaviour of these functions. He showed that there are p so called *critical* directions that divide the plane into p sectors, in each of which f is very close to one of the asymptotic values $a \in \hat{\mathbb{C}}$ in the sense that $\lim_{R\to\infty} f(R\exp(i\phi)) = a$ if ϕ is a non-critical direction in the sector. Two asymptotic values a_1 and a_2 of two adjoining sectors are distinct and every other value $z \in \hat{\mathbb{C}}\backslash\{a_1, a_2\}$ has infinitely many pre-images in the small region in between these two sectors. Thus for functions with only finitely many poles exactly half of the asymptotic values happen to be ∞. For this case Elving obtains the following result [17](§12, Nr. 48, p.60).

2.12 Theorem (Elfving)
Let f be a meromorhic function with $2p$ asymptotic values counting multiplicity, where ∞ is one of multiplicity p. Then there exist polynomials P, Q and R with $\deg(Q) = p$ *and $c \in \mathbb{C}$ such that*

$$f(z) = \int_0^z \frac{P(t)}{R(t)} \exp(Q(t))dt + c$$

Most relevant for us is the following corollary for entire functions.

2.13 Corollary
Let f be entire transcendental, with only finitely many singularities of its inverse counting multiplicity. Then there exist polynomials P, Q and $c \in \mathbb{C}$, such that $f(z) = \int_0^z P(t)\exp(Q(t))dt + c$.

It is easy to see that functions with a rational Schwarzian derivative coincide with those quotients f_1/f_2 of two linearly independent solutions of the differential equation $f_i'' + Af_i = 0$ with $A := S(f)/2$. Studying this equation allows one to obtain stronger growth estimates for f than those described above by G. Elfving. For certain A this equation has been integrated asymptotically by E. Hille [26] and his method has been used by many others afterwards. The following theorem and remark may be found explicitly in the postgraduate notes of J. Langley [30](§4, 4.3.1, p.46), where an excellent summary of this method may be found.

2.14 Theorem (Hille)
Let $c > 0$ and $0 < \epsilon < \pi$. Then there exists a constant $d > 0$, depending only on c and ϵ, with the following properties. Suppose that F is analytic, with $|F(z)| \leq c|z|^{-2}$, in

$$\Omega := \{z : 1 \leq R_0 \leq |z| \leq R_1 < \infty, |\arg z| \leq \pi - \epsilon\}.$$

Then the equation

$$\omega'' + (1 - F(z))\omega \quad = \quad 0$$

has two linearly independent solutions U, V satisfying

$$\begin{aligned} U(z) = \exp(-iz)(1 + \delta_1(z)) \quad &, \quad U'(z) = -i\exp(-iz)(1 + \delta_2(z)), \\ V(z) = \exp(iz)(1 + \delta_3(z)) \quad &, \quad V'(z) = i\exp(iz)(1 + \delta_4(z)), \end{aligned}$$

such that $|\delta_i(z)| \leq d|z|^{-1}$ for $z \in \Omega \backslash \{z : Re(z) < 0, |Im(z)| < R\}$.

2.15 Remark
Ω may be replaced by

$$\Omega' := \{z : 1 \leq R_0 \leq |z| \leq R_1 < \infty, |\arg z - \pi| \leq \pi - \epsilon\}$$

and also by the unbounded region

$$\Omega'' := \{z : 1 \leq R_0 \leq |z| < \infty, |\arg z| \leq \pi - \epsilon\}.$$

To see this, we take a sequence $R_k \to \infty$ and obtain solutions U_k,V_k with uniformly bounded $\delta_{i,k}$ in Ω_k, where Ω_k is Ω with R_1 replaced by R_k. Therefore both form a normal family, and a subsequence of U_k,V_k converges in $\Omega'' = \bigcup_{k \in \mathbb{N}} \Omega_k$.

2.4 Iteration theory

One can probably say that this field was founded independently by G. Julia [27] and P. Fatou [20] around 1918. We begin with the basic notations.

2.16 Definition
Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ *be meromorphic. Then*

$$F(f) := \{z \in \mathbb{C} : (f^n)_{n\in\mathbb{N}} \text{ is defined and normal in } z\}$$

is called the Fatou set *of* f *and*

$$J(f) := \mathbb{C}\backslash F(f)$$

is called the Julia set *of* f.

Sometimes the plane is replaced by the sphere, which is more suitable for rational functions, but for transcendental functions the only difference is that ∞ also belongs to the Julia set. For transcendental entire functions however, which are our main focus, our definition has the advantage that one does not have to worry about the definition of the iterates.

From the definition, it follows immediately that $F(f)$ is open and $J(f)$ is closed. Moreover, both are completely invariant under f. If f is not constant and not a Möbius transformation, which we will always assume from now on, one can also show that $J(f)$ is non-empty, perfect and, if it does not coincide with $\mathbb{C}$, has empty interior. The iteration of Möbius transformations is straight-forward, such we do not loose much generality.

For $z \in \hat{\mathbb{C}}$ the sets $O^{\pm}(z) := \{f^{\pm n}(z) : n \in \mathbb{N} \cup \{0\}\}$, if defined, are called the forward/backward orbits of z. If some $f^n(z)$ is not defined, of course, the forward orbit $O^+(z)$ denotes the set of those iterates, which are defined. Moreover the set $O(z) := O^-(z) \cup O^+(z)$ is called the orbit of z and for $A \subset \hat{\mathbb{C}}$ we define $O^{(\pm)}(A) := \bigcup_{z\in A} O^{(\pm)}(z)$. The simplest forward orbits possible are finite ones, which belong to so called *pre-periodic points* z with the property that $f^p(f^n(z)) = f^n(z)$ for some $n \in \mathbb{N} \cup \{0\}$ and $p \in \mathbb{N}$. If $n = 0$, the point z is called *periodic* and the smallest p with this property its *period*. For a periodic point $z \in \mathbb{C}$ the number $(f^p)'(z)$ is called its *multiplier*. If $z = \infty$ the multiplier is defined by the derivative at the point 0 of the function f^p conjugated with the function $z \mapsto 1/z$. Then it is easy to see that the point z belongs to the Fatou set, if the modulus of the multiplier is smaller than one, in which case z is called *attracting*, and it belongs to the Julia set, if it is larger than one, in which case it is called *repelling*. In the case that the modulus of the multiplier of the periodic point z equals one, in which case z is called *indifferent*, the situation is more complicated. If the multiplier equals $\exp(2\pi i\alpha)$ with $\alpha \in \mathbb{Q}$, in which case z is called *rationally* indifferent it is relatively easy to show that $z \in J(f)$. In the other, the *irrationally indifferent* case, however, this depends on number-theoretical properties of the multiplier. It is possible that $z \in F(f)$, in which case z is called *linearisable*. However, it is also possible that $z \in J(f)$, in which case z is called a *Cremer point*.

As one of the most important theorems in complex dynamics, it has been proved in [2] for meromorphic functions that the set of repelling periodic points is dense in the Julia set. For rational functions this has been shown already by P. Fatou and G. Julia and the latter actually used this set as the starting point of his theory. The first proof for transcendental entire functions is due to N. Baker. There are other dynamically interesting sets, which are dense in the Julia set. For example, if the backward orbit of some point is not finite, it is dense in the Julia set. This follows easily from *Montel's theorem* about normality, which says that if every function of a family of meromorphic mappings omits the same three values, then this family is normal. There are at most two values for which the backward orbit is finite and these are called *exceptional*. The set of exceptional values is denoted by $E(f)$. Moreover, there are always points in the Julia set whose forward orbit is dense in the Julia set. The set of points with this property is called the *transitive* set $Tr(f)$. Trivially, it is also dense in the Julia set. One can obtain points in the transitive set using the so called *self-similarity* property of the Julia set, which says that $J(f)\backslash E(f) \subset O^+(U)$ for any open set U that intersects the Julia set. This self-similarity property follows directly from Montel's theorem. For $K \subset \mathbb{C}\backslash E(f)$ compact and n large enough one even obtains $J(f) \cap K \subset f^n(U)$.

We define a connected component U of the Fatou set to be *periodic* of period $p \in \mathbb{N}$, if $f^p(U) \subset U$ and $\{U_0, U_1, ..., U_{p-1}\}$, where U_j is the component of $F(f)$ containing $f^j(U)$, is called the *cycle* of U. A component is called *pre-periodic* if it is eventually mapped to a periodic one. The dynamics of periodic components of the Fatou set can be classified as follows.

2.17 Theorem (Classification theorem)
Let U be a periodic component of the Fatou set of period p. Then one of the following cases holds:

- *U contains an attracting periodic point z_0 of period p and $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$ and U is called the* immediate attractive basin *of z_0.*

- *∂U contains a periodic point z_0 with multiplier* 1 *and $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$ and U is called* a Leau *or* parabolic *domain.*

- *There exist a conformal map $\phi : U \to B(0,1)$ and $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ such that $\phi(f^p(\phi^{-1}(z))) = \exp(2\pi i\alpha)z$ for $z \in U$. In this case U is called a* Siegel disc.

- *There exist $r > 1$, a conformal map $\phi : U \to B(0,r)\backslash B(0,1)$ and $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ such that $\phi(f^p(\phi^{-1}(z))) = \exp(2\pi i\alpha)z$ for $z \in U$. In this case, U is called a* Herman ring.

- *There exists $z_0 \in \partial U$ such that $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$ but $f^p(z_0)$ is not defined. Then U is called* a Baker domain.

This result is essentially due to H. Cremer [11] and P. Fatou [20] and it describes the possible dynamics of periodic components of the Fatou set, and therefore also of pre-periodic components, very well.

The case of an immediate attractive basin is often further distinguished depending on whether the multiplier of the attracting point z_0 is zero or not. If this is the case, U and z_0 are called *superattractive* and U a *Böttcher domain*. Otherwise, U is called a *Schröder domain*.

Entire functions do not have Herman rings and Baker domains do not occur for rational functions. The absence of Baker domains has also been shown for entire functions, whose set of finite singularities of the inverse is bounded, by A. Eremenko and M. Lyubich in [19]. For meromorphic functions, for which this set is finite, this has been shown by P. J. Rippon and G. M. Stallard in [44]. These results indicate that there is some connection between a Baker domain and the set of singularities of the inverse function. This is also true for all other types of components of the Fatou set. Obviously cycles of Böttcher domains contain a singularity of the inverse, namely the super-attractive critical value. For the remaining periodic components a connection is given by the following theorem.

2.18 Theorem
Let f be meromorphic and $C = \{U_0, ..., U_{p-1}\}$ a periodic cycle of components of $F(f)$.

- *If C is a cycle of Leau or Schröder domains, there exists $j \in \{0, ..., p-1\}$ such that $U_j \cap \operatorname{sing}(f^{-1})$ contains a point, which is not pre-periodic.*

- *If C is a cycle of Siegel discs or Herman rings, then $\partial U_j \subset P(f)$ for all $j \in \{0, ..., p-1\}$.*

As mentioned in the introduction the *post-singular* set $P(f)$ consists of the closure of the forward orbit of all singularities of the inverse. These results were proved by Fatou in [20] for rational maps, but the proofs carry over to the transcendental case.

Components which are not pre-periodic, are called *wandering*. The famous *no wandering domains* theorem of D. Sullivan [50] says that rational functions do not have wandering domains. As in the case of Baker domains, this has been generalised to various other classes of functions including all those which we will consider. For functions of type (36), which we consider in the chapter on entire functions, the absence of wandering domains has been shown by I. N. Baker in

[1]. For entire functions with only finitely many singularities of the inverse, this has been shown by A. Eremenko and M. Lyubich in [19] and by L. R. Goldberg and L. Keen in [24]. For meromorphic functions with the same property this has been shown by I. N. Baker, J. Kotus and Y. Lü in [3]. Again this indicates that a wandering domain somehow requires many singularities.

The above implies that attractive or rationally indifferent periodic points must lie in $P(f)$ and even in $P(f)'$, except for the super-attractive case. Here, for any set S the set S' denotes the set of all accumulation points of S. It is well-known that the same is true for Cremer points.

2.19 Theorem
A Cremer point z_0 of a meromorphic function f is contained in $P(f)'$.

This is proved for rational functions in [5](§9, 9.3.4, p. 195). However the proof carries over to the transcendental case.

2.5 Measurable dynamics

We saw that the set of repelling periodic points and the transitive set are both dense in Julia set. This shows that the dynamics on the Julia set is complicated and highly unstable, which of course was to be expected because of the definition. As mentioned in the abstract, a useful object to describe the dynamics of a point $z \in \mathbb{C}$ is its *ω-limit set* $\omega(z)$, which is the set of accumulation points of the sequence of iterates $(f^n(z))$. With this we can rephrase the above as follows. Every open set in $J(f)$ contains points with a finite ω-limit set as well as points whose ω-limit set coincides with the Julia set itself. In particular, not every point in the Julia set has the same ω-limit set. Our aim is to describe the ω-limit set of all points in the Julia set up to a set of Lebesgue measure zero and refer to this approach with the term *measurable dynamics*. We introduce some related terms from ergodic theory. A meromorphic function is called *ergodic* (with respect to the Lebesgue measure), if any invariant set has full measure or measure zero. It is called *recurrent*, if for every set $A \subset \mathbb{C}$ and almost every point $z \in A$ the cardinality of the set $A \cap O^+(z)$ is infinite.

It is not known whether one can always find a set Z of measure zero, such that all points in $J(f)\backslash Z$ have the same ω-limit set. In other words it is an open question whether there always exists a typical orbit. As indicated in the abstract, we call a point, orbit or ω-limit set *typical*, if the set of points in the Julia set with a distinct ω-limit set has measure zero. This question is related to the well-known open question whether the Julia set of a polynomial may have positive measure. If f is a polynomial itself, this relation is evident. For a class of entire functions, we will discuss this relation at the end of the chapter on entire functions. There are, however, examples for rational functions whose Julia set coincides with the entire plane. In [33] C. McMullen even gives examples for transcendental entire functions with a Julia set which is not the entire plane, but has positive measure. Thus measurable dynamics becomes particularly interesting for transcendental functions and is one of the few subjects which seems to be easier here than in the rational case.

2.20 Theorem (McMullen)
Let $f(z) := \delta \exp(z) + \gamma \exp(-z)$ *with* $\delta, \gamma \in \mathbb{C}^*$. *Then* $\operatorname{meas}(I(f)) > 0$.

Here the set $I(f)$ denotes the *escaping set* $\{z : f^n(z) \to \infty\}$ All functions of the type $\cos(az + b)$ and $\sin(az + b)$ are conjugated by a linear transformation to a function in the above form. Also $I(f) \subset J(f)$, which holds for any entire $f \in \mathcal{B}$ as shown in [19]. Here $\mathcal{B}$ denotes the family of meromorphic functions whose set of finite singularities of the inverse is bounded. Thus the Julia set of the sine-function, which is not the entire plane, has positive measure. Moreover, H. Schubert shows in [46] that the measure of the Fatou set of the sine function in the strip $\{z : 0 \leq \operatorname{Re}(z) \leq 2\pi\}$ is finite, which was conjectured by J. Milnor in [35](§6, Figure 5, p.65).

Of course, there are also examples of transcendental functions for which the Julia set coincides with $\mathbb{C}$. For the exponential function this is the case, which had already been conjectured by P. Fatou and shown by M. Misiurewicz in [36]. For the exponential function, however, $\operatorname{meas}(I(\exp)) = 0$, which follows from the following theorem obtained by M. Lyubich and A. Eremenko in [19](§7, p.1009).

2.21 Theorem (Eremenko, Lyubich)
Let $f \in \mathcal{B}$ *and be entire such that*

$$\liminf_{r\to\infty} \frac{1}{\log r} \int_1^r \theta(t) \frac{dt}{t} > 0, \tag{8}$$

where $R > 0$ *and for every* r *positive* $\theta(r)$ *denotes the linear measure of the set* $\{\theta \in (0, 2\pi) : |f(r \exp(\theta i))| < R\}$. *Then* $\operatorname{meas}(I(f)) = 0$.

2.22 Remark
For all functions that we will consider, (8) is satisfied, since $\theta(r)$ *converges to a positive number (for functions as in Corollary 2.13 this number is* π*), which then equals the* $\liminf$.

Information about the ω-limit set of a typical point for the exponential function is given by M. Lyubich in [32] and M. Rees [42].

2.23 Theorem (Lyubich, Rees)
For almost every $z \in J(\exp)$ it follows that $\omega(z) \subset O^+(0) \cup \{\infty\} = P(\exp)$.

In particular exp is not recurrent. Furthermore Lyubich shows that exp is not ergodic and $\omega(z) = P(\exp)$ for almost every $z \in J(\exp)$. This behaviour is quite typical as it is one of only two cases that occur. This follows from the following classification of H. Bock, which he proved in his thesis [7] and which is a generalisation of similar results for rational functions, obtained by M. Lyubich [31] and C. McMullen [34](§3, 3.9, p.42). For entire functions this result may be found in [6].

2.24 Theorem (Bock)
Let f be meromorphic. Then one of the two following cases holds:

(i) For all $A \subset \mathbb{C}$ of positive measure, all $m \in \mathbb{N}$ and almost all $z \in \mathbb{C}$ there are infinitely many $n \in \mathbb{N}$ with $f^{mn}(z) \in A$;

(ii) $\omega(z) \subset P(f)$ or almost every $z \in J(f)$.

It is easy to see that (i) implies that f is recurrent and ergodic and that $J(f) = \mathbb{C}$. Case (ii) does not rule out either one of these in general. If however $P(f) \neq \hat{\mathbb{C}}$, it implies non-recurrence.

We have seen that exp is an example for (ii). The following theorem from [7] gives sufficient conditions for case (i) and in particular implies that the function $f(z) = 2\pi i \exp(z)$ is an example of this case.

2.25 Theorem (Bock)
Let f be meromorphic with only finitely many finite singularities of its inverse all of which are pre-periodic but not periodic. Then either (i) is satisfied or the set $I(f)$ has full measure.

Theorem 2.25 is easily deduced from Theorem 2.24. We include this simple argument, since we will argue similarly later on, and state the crucial argument as a lemma.

2.26 Lemma
Suppose that z_0 is a repelling periodic point of f and $O^+(z)$ accumulates at z_0 for some $z \in \mathbb{C}$. Then $\omega(z)$ also accumulates at z_0.

Proof. We consider the compact annulus

$$A_r := \{z : r \leq |z - z_0| \leq 2|(f^p)'(z_0)|r\}$$

and assume that $r > 0$ is small enough, such that $|f(z) - z_0| > |z - z_0|$ and $|(f^p)'(z)| < 2|(f^p)'(z_0)|$ for $z \in \overline{B(z_0, r)}$. Since $O^+(z)$ accumulates at z_0, there are infinitely many n such that $f^n(z) \in B(z_0, r)$. The first assumption on r guarantees that the forward orbit must also leave $B(z_0, r)$ infinitely many times. The second assumption implies that this is not possible without falling into A_r. Since A_r is compact, $O^+(z)$ accumulates somewhere in A_r. In other words this means $\omega(z) \cap A_r \neq \emptyset$. The same is true for arbitrarily small r and the result follows. ■

Proof of theorem 2.25. We assume that (ii) is true and that a set S of positive measure accumulates at some point other than ∞. Due to (ii), this point must be an element of $P(f)$. Thus it is mapped by some iterate of f to a periodic point z_0 of f. The conditions of the singularities of the inverse imply $P(f)' = \emptyset$. Together with Theorem 2.18, this assures that there are no Leau or Schröder domains, no Siegel discs and no Herman rings. Therefore z_0 must be repelling or a Cremer point. As a Cremer point it would be contained in $P(f)'$, as shown in Theorem 2.19, which would be a contradiction to $P(f)' = \emptyset$. If it was repelling, Lemma 2.26 above would imply that $\omega(z)$ accumulates at z_0 for every $z \in S$. Since $\omega(z) \subset P(f)$ for almost all $z \in J(f)$ by (ii), this would again contradict $P(f)' = \emptyset$. ■

More sophisticated conditions for ergodicity have been obtained by L. Keen and J. Kotus in [29](§3, 3.3, p.142). First of all they show the following theorem.

2.27 Theorem (Keen, Kotus)
Suppose that $f \in \mathcal{B}$, such that $P(f)'$ is a compact repeller, *i.e. it is compact and $|f'(z)| > k > 1$ for all $z \in P(f)'$. Then* $\text{meas}(\{z : \omega(z) \subset P(f)'\}) = 0$.

In order to get ergodicity one has to assure that the measure of the set $I(f)$ is zero, which is not the case in general. Therefore they generalise Theorem 2.21 to meromorphic functions with poles. However, if there are infinitely many poles, they need additional conditions on these poles. To state these we introduce the following notation. For a pole p of a function $f \in \mathcal{B}$ consider those c_p, m_p and ϕ_p giving the Laurent expansion of f at p the form $\frac{c_p}{(z-p)^{m_p}}(1 + \phi_p(z-p))$ where ϕ_p is analytic and $\phi_p(z-p) = o((z-p))$. Moreover for fixed R with $\text{sing}(f^{-1}) \cup \{2f(0)\} \subset B(0, R)$, let V_p be that component of $f^{-1}(\mathbb{C}\backslash B(0, R))$, which contains a punctured neighbourhood of p.

2.28 Theorem (Keen, Kotus)
Let $f \in \mathcal{B}$ satisfying (8) and $n \in \mathbb{N}$, $b, B, C_1, C_2 > 0$ such that for every pole p the following estimates are satisfied:

$$m_p < n, b < |c_p| < B, |\phi_p(z-p)| < C_1, |\phi_p'(z-p)| < C_1 \text{ and } C_2 < |1 + \phi_p(z-p)|.$$

Then $\text{meas}(I(f)) = 0$.

From these results they deduce that for $f \in \mathcal{B}$ satisfying these conditions and with $P(f)'$ a compact repeller, f is ergodic on the Julia set with respect to the Lebesgue measure.

Finally we give the definition of a density point and include an important property, which we will frequently need.

2.29 Definition
Let $A \subset \mathbb{C}$ be measurable. We call $z \in \mathbb{C}$ a density point *of A if*

$$\frac{\operatorname{meas}(B(z,r) \cap A)}{\operatorname{meas}(B(z,r))} \to 1 \textit{ as } r \to 0.$$

The existence of density points is given by the following result. The proof may be found in [49].

2.30 Theorem
Let $A \subset \mathbb{C}$ be measurable. Then almost every point of A is a density point of A.

3 Non-recurrence

We follow the ideas used by M. Rees [42] for the exponential function. We obtain a set of points with positive measure, whose orbits are not dense in $\mathbb{C}$, and therefore rule out case (i) of Theorem 2.24. Therefore this provides a set of sufficient conditions for case (ii). In order to allow a wide application, and hoping for further generalisations, we state our theorem as generally as possible. It is therefore rather technical.

3.1 Sufficient conditions for case (ii)

3.1 Theorem

Let f be meromorphic, $A \subset \mathbb{C}$ finite and $G \subset \mathbb{C}$, such that

(a) *there exists $\epsilon > 0$, such that the map*

$$\overline{s} : G \to A \cup \{0\}; z \mapsto \begin{cases} s & \text{if } \exists s \in A : |f(z) - s| \leq \exp(-|z|^{\epsilon}) \\ 0 & \text{if } |f(z)| \geq \exp(|z|^{\epsilon}) \end{cases}$$

is well defined and there are $\delta_1, \delta_2 \in \mathbb{R}$, such that for all $z \in G$,

$$|z|^{\delta_1} \leq \left| \frac{f'(z)}{f(z) - \overline{s}(z)} \right| \leq |z|^{\delta_2};$$

(b) *there exist $B > 1$ and $\beta \in (-\infty, 1)$, such that for every measurable set $D \subset \{z : \operatorname{dist}(z, \mathbb{C} \backslash G) \leq 2|z|^{-\delta_1}\}$,*

$$\operatorname{meas}(D) \leq B \operatorname{diam}(D) \sup_{z \in D} |z|^{\beta};$$

(c) *$f^m(s) \overset{m \to \infty}{\to} \infty$ and $B\left(f^m(s), 2\left|f^m(s)\right|^{\tau}\right) \subset G$ for some $\tau > \beta$, almost all $m \in \mathbb{N}$ and all $s \in A$.*

Then the set $T(f) := \{z : \omega(z) \subset \overline{O^+(A)}\}$ has positive measure. Furthermore, there exists $M > 0$ such that for any square $T_0 \subset \left\{z : \operatorname{dist}(z, \mathbb{C} \backslash G) > |z|^{-\delta_1}\right\}$ with $M_0 := \inf_{z \in T_0} |z| > M$ and $\operatorname{diam}(T_0) \geq M_0^{-\delta_2}$,

$$\frac{\operatorname{meas}(T(f) \cap T_0)}{\operatorname{meas}(T_0)} \geq 1 - \exp\left(-\eta M_0^{\epsilon}\right),$$

where $\eta := \frac{\tau - \beta}{\max\{1, 2 - 2\tau\}} > 0$.

3.2 Remark
Since the orbits of all points in $T(f)$ accumulate at infinity, the only components that could possibly intersect $T(f)$ are Baker domains and wandering domains. At least for the various families, for which these do not occur (see preliminaries), we obtain $T(f) \subset J(f)$.

It also makes sense to choose $A = \emptyset$. Then we obtain sufficient conditions for $\operatorname{meas}(I(f)) > 0$, where $I(f)$ denotes the set of escaping points (see 5.1).

Proof of Theorem 3.1. From our conditions (b) and (c) one can deduce that $-\delta_1 < \beta < \tau < 1$. We note that for any $M > 0$ a sufficiently large choice of B allows us to choose G such that $G \cap B(0, M) = \emptyset$. For all $s \in A$ we define

$$m_s := \max(\{m \in \mathbb{N} : f'(f^{m-1}(s)) = 0\} \cup \{0\})$$

and

$$k_s := \min\{k \in \mathbb{N} : (f^{m_s})^{(k)}(s) \neq 0\}.$$

The distortion constant K_c from Lemma 2.4 tends to one as c tends to zero. Thus for $c > 0$ small enough we have $\frac{cK_c}{4} < 1$. Since A is finite one can even find $c > 0$, such that $k_s \arcsin(\frac{cK}{4}) < \pi$ holds for all $s \in A$ and $K := K_c$. Suppose that $\delta > 0$ is small. In fact it turns out that $\delta < \frac{(\tau-\beta)(1-\tau)}{6-5\tau-\beta}$ is sufficient for all requirements needed. Similarly chose $M > 0$ sufficiently large, satisfying many bounds appearing throughout the proof. For now we only require the following two properties. Firstly for any $M_0 > M$ the series defined by

$$M_{k+1} := \exp\left(\min\{1, \frac{1}{2-2\tau}\} M_k^{\epsilon}\right)$$

tends to infinity fast enough, such that

$$\prod_{k\in\mathbb{N}} \left(1 - \frac{1}{4} M_k^{\beta-\tau}\right) \geq 1 - M_1^{\beta-\tau}. \tag{9}$$

Secondly there are no critical points in A_k for all $k \in \mathbb{N} \cup \{0\}$ where

$$\begin{aligned} A_k &:= \left(D\left(\frac{1}{2} M_{k+1}^{\frac{1}{1+2\delta_2-\delta_1+3\delta}}\right) \cap G\right) \\ &\cup \bigcup_{0\leq l\leq m_s} B\left(f^l(s), M_{k+1}^{\delta-1}\right) \setminus \{f^l(s)\} \\ &\cup \bigcup_{l>m_s} B(f^l(s), a_{k,s,l}), \end{aligned}$$

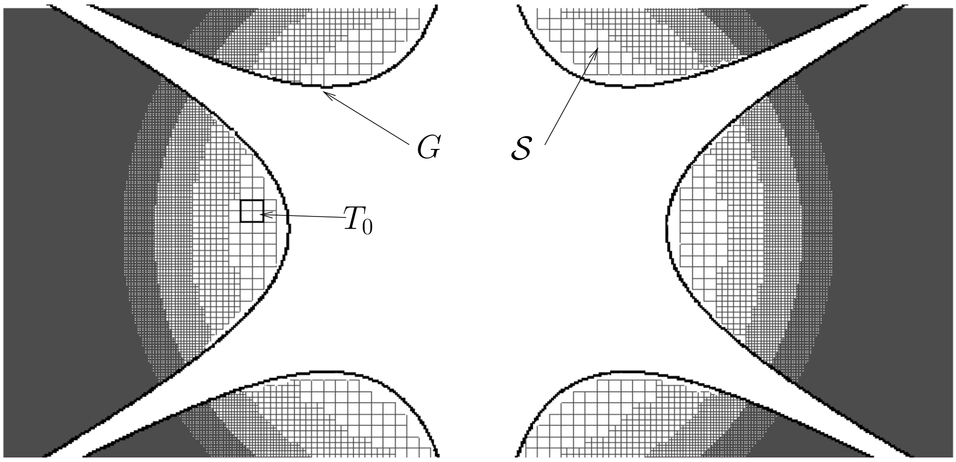

Figure 1: The family $\mathcal{S}$

and $a_{k,s,l} := \sup\left\{|f^j(s)|^{-\delta_2} : j \geq l, |f^{j+1}(s)| \geq M_{k+1}^{\frac{1}{1+2\delta_2-\delta_1+3\delta}}\right\}$. Of course, at this point we only need to study A_0 since the A_k are descending in the sense that $A_{k+1} \subset A_k$ for large k. A_0 does not contain any critical points for M_0 large enough, since those do not accumulate in $\mathbb{C}$ and, with condition (a), G does not contain any critical points. We note that due to (c) every $s \in A$ escapes in G exponentially fast, such that $a_{k,s,l} \leq |f^l(s)|^{-\delta_2}$ for large l. Now let T_0 and M_0 be as in the theorem. We choose a family $\mathcal{S}$ of disjoint open squares $S \subset \{z : \operatorname{dist}(z, \mathbb{C}\backslash G) \geq |z|^{-\delta_1}\}$ satisfying

$$\frac{c}{8}\left(\inf_{z\in\frac{1}{c}S}|z|\right)^{-\delta_2} \leq \operatorname{diam}(S) \leq \frac{c}{2}\left(\sup_{z\in\frac{1}{c}S}|z|\right)^{-\delta_2}, \tag{10}$$

whose union cover $\{z \in G : \operatorname{dist}(z, \mathbb{C}\backslash G) \geq 2|z|^{-\delta_1}\}$ up to measure zero, such that $\overline{T_0} = \bigcup_{S\in X}\overline{S}$ for some finite $X \subset \mathcal{S}$. A picture of this could look like Figure 1. We can get this by covering the whole plane with open squares of a constant diameter, beginning with T_0, cutting these into four until their diameter satisfies the upper bound, and throwing away those intersecting $\left\{z : \operatorname{dist}(z, \mathbb{C}\backslash G) \leq \frac{|z|^{-\delta_1}}{2}\right\}$. For M large enough and $G \cap B(0, M) = \emptyset$ our squares also satisfy the lower bound. We prove the measure estimate in the theorem for all elements of $\mathcal{S}$ including the ones in X. This implies this estimate also for T_0. Thus we proceed with an element of $\mathcal{S}$, which we again call T_0.

With the estimates of condition (a) one can show that if $|z_0|$ is large enough and $B(z_0, |z_0|^{-\delta_1}) \subset G$ then

$$f \text{ is injective on } B\left(z_0, \frac{|z_0|^{-\delta_2}}{4}\right). \tag{11}$$

To see this we use Lemma 2.7 and show first that

$$f\left(B\left(z_0, \frac{|z_0|^{-\delta_2}}{4}\right)\right) \subset B\left(f(z_0), \frac{3|f(z_0)-\overline{s}(z_0)|}{8}\right) \subset f\left(B(z_0, |z_0|^{-\delta_1})\right). \quad (12)$$

If the first inclusion was not true, we would find a point $z \in B\left(z_0, \frac{|z_0|^{-\delta_2}}{4}\right)$ with $|f(z)-f(z_0)| \geq \frac{3|f(z_0)-\overline{s}(z_0)|}{8}$. We choose $|z-z_0|$ minimal with this property, such that for $x \in (z, z_0) := \{(1-t)z + tz_0 : 0 < t < 1\}$ we have

$$|f(x)-\overline{s}(z_0)| \leq |f(x)-f(z_0)| + |f(z_0)-\overline{s}(z_0)| \leq \frac{11}{8}|f(z_0)-\overline{s}(z_0)|.$$

The mean value theorem provides $x \in (z, z_0)$ with

$$|f'(x)| \geq \frac{|f(z)-f(z_0)|}{|z-z_0|} \geq \frac{3}{2}|f(z_0)-\overline{s}(z_0)||z_0|^{\delta_2} \geq \frac{12}{11}|f(x)-\overline{s}(z_0)||z_0|^{\delta_2}.$$

This contradicts (a), since for z_0 large enough $|x-z_0| \leq |z_0|^{-\delta_2}$ is very small, such that $\overline{s}(x) = \overline{s}(z_0)$ and $|x| > (\frac{11}{12})^{-\delta_2}|z_0|$.

The second inclusion from (12) follows from the the fact that there are no critical points in $B(z_0, |z_0|^{-\delta_1}) \subset G$. Thus we may extend the branch of f^{-1}, mapping $f(z_0)$ to z_0 along any path in $B\left(f(z_0), \frac{3|f(z_0)-\overline{s}(z_0)|}{8}\right)$ as long as the image stays in $B(z_0, |z_0|^{-\delta_1})$. As above the mean value theorem together with condition (a) assures this, since for every $x \in B(z_0, |z_0|^{-\delta_1})$ for which $f(x)$ is contained in $B\left(f(z_0), \frac{3|f(z_0)-\overline{s}(z_0)|}{8}\right)$ we know that $|f(x)-\overline{s}(z_0)| \geq \frac{5|f(z_0)-\overline{s}(z_0)|}{8}$, such that

$$|f'(x)| \geq |f(x)-\overline{s}(z_0)||x|^{\delta_1} \geq \frac{1}{2}|f(z_0)-\overline{s}(z_0)||z_0|^{\delta_1}. \quad (13)$$

This implies that the image of any path in $B\left(f(z_0), \frac{3|f(z_0)-\overline{s}(z_0)|}{8}\right)$ in fact stays inside $B(z_0, \frac{3}{4}|z_0|^{-\delta_1})$. From this and (12) it follows that f is injective on $B\left(z_0, \frac{|z_0|^{-\delta_2}}{4}\right)$, as claimed. Together with (10) this implies that the distortion of f on any $S \in \mathcal{S}$ is bounded by K.

Starting with $F_0 := \{T_0\}$ and $n_0(T_0) := 0$, we will define for every $k \in \mathbb{N}$ a family $\mathcal{F}_k$ of disjoint simply connected domains, and functions $n_k : \mathcal{F}_k \to \mathbb{N}$ such that the sets $T_k := \bigcup \mathcal{F}_k = \bigcup_{F \in \mathcal{F}_k} F$ form a decreasing series with the following properties. For every $U \in \mathcal{F}_k$ and the corresponding $V \in \mathcal{F}_{k-1}$ with $U \subset V$:

(i) $D(M_k) \supset f^{n_k(U)}(U) \in \mathcal{S}$ and $\frac{1}{c} f^{n_k(U)}(U) \subset f^{n_k(U)}(V)$;

(ii) $f^j(V) \subset A_k$ for every $n_{k-1}(V) < j < n_k(U)$;

(iii) $\operatorname{meas}(V \cap \bigcup \mathcal{F}_k) \geq \left(1 - \frac{1}{4}M_k^{\beta-\tau}\right) \operatorname{meas}(V)$.

From Condition (ii) it follows that $\omega(z) \subset \bigcap_{k\in\mathbb{N}} A_k = O^+(A) \cup \{\infty\}$ for every $z \in T := \bigcap_{k\in\mathbb{N}} T_k$. Having (iii) for each component of T_k, namely the elements of $\mathcal{F}_k$, implies that meas$(T_k) > \left(1 - \frac{1}{4}M_k^{\beta-\tau}\right)$ meas(T_{k-1}), which, together with the exponential growth of M_k, guarantees that

$$\text{meas}\,(T) \geq \left(\prod_{k=1}^{\infty}\left(1 - \frac{1}{4}M_k^{\beta-\tau}\right)\right)\text{meas}(T_0).$$

Together with (9) this implies the measure estimate from Theorem 3.1.

It remains to construct the sequences. We will do so inductively. We note that the starting step of the induction works the same way as any other step, so we do not consider it separately. We assume the existence of appropriate $\mathcal{F}_k$ and n_k for some $k \in \mathbb{N}$. Let $U \in \mathcal{F}_k$. Then $S := f^{n_k(U)}(U) \in \mathcal{S}$. Due to condition (ii) and the fact that there are no critical points in A_0, one can extend the inverse of $f^{n_k(U)}\big|U$ to $\frac{1}{c}S$ and its distortion on S is bounded by K. Furthermore, $S \subset D(M_k)$ such that we can consider the following cases separately.

Case 1: $f(S) \subset D(M_{k+1})$.

We define $\mathcal{F} := \{R \in \mathcal{S} : \frac{1}{c}R \subset f(S)\}$, $\mathcal{F}_U := \{(f^{n_k(U)+1}|U)^{-1}(R) : R \in \mathcal{F}\}$ and $n_{k+1}(V) := n_k(U) + 1$ for all $W \in \mathcal{F}_U$. See Figure 2. Then (for $\mathcal{F}_U$ in place of $\mathcal{F}_{k+1}$) property (i) holds by definition, while property (ii) is trivial. Since $f|S$ is injective and its distortion is bounded by K, $f(S)$ is a K-quasi-square with

$$\text{diam}(f(S)) \geq \frac{1}{\sqrt{2}}\,\text{diam}(S)\inf_{z\in S}|f'(z)| \geq \sup_{z\in S}|f(z)|^{1-\delta} \tag{14}$$

for M_0 large enough. Here the last inequality holds since the term $|f'(z)|$ is, due to (a), of magnitude $|f(z)| \geq \exp(|z|^\epsilon)$. This is far larger than all other factors that appear, so they may be cancelled by $|f(z)|^\delta$. Also the infimum may be substituted by the supremum, since the distortion is bounded by K.

By definition of $\mathcal{S}$ and $\mathcal{F}$, the set $f(S)\backslash\bigcup\mathcal{F}$ is contained in the union of $\partial\bigcup\mathcal{S}$, which has measure zero, and small neighbourhoods of $\partial f(S)$ and $\mathbb{C}\backslash G$. More precisely, we have

$$\begin{aligned}\text{meas}\left(f(S)\backslash\bigcup\mathcal{F}\right) \;&\leq\; \text{meas}\left(\left\{z \in f(S) : \text{dist}(z, \partial f(S)) \leq |z|^{-\delta_2}\right\}\right)\\ &+\; \text{meas}\left(\left\{z \in f(S) : \text{dist}(z, \mathbb{C}\backslash G) \leq 2|z|^{-\delta_1})\right\}\right).\end{aligned} \tag{15}$$

With condition (b) we can control the second term on the right by

$$\text{meas}\left(\{z \in f(S) : \text{dist}(z, \mathbb{C}\backslash f(S)) \leq 2|z|^{-\delta_1}\}\right) \leq B\,\text{diam}(f(S))\sup_{z\in f(S)}|z|^\beta.$$

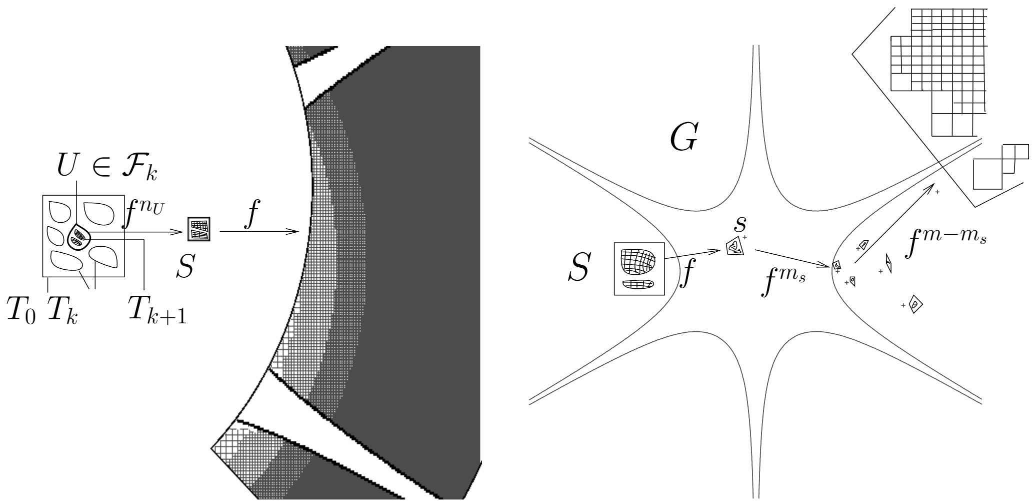

Figure 2: Models for the construction in both cases

$f(S)$ is a K-quasi-square. Therefore we can use (7) and obtain that the measure of an r-neighbourhood of the boundary of $f(S)$ is at most $4rK^2 \operatorname{diam}(f(S))$ and that $\operatorname{meas}(f(S)) \geq \operatorname{diam}(f(S))^2/(2K^2)$. Since the set of the first term in (15) is contained in a $\sup_{z\in f(S)} |z|^{-\delta_2}$-neighbourhood of $\partial f(S)$ and $-\delta_2 < -\delta_1 < \beta$, we obtain, using (14), that

$$\begin{aligned}
\frac{\operatorname{meas}(f(S)\backslash \bigcup \mathcal{F})}{\operatorname{meas}(f(S))} &\leq \frac{2K^2(4K^2 \sup_{z\in f(S)} |z|^{-\delta_2} + B \sup_{z\in f(S)} |z|^{\beta})}{\operatorname{diam}(f(S))} \\
&\leq 8BK^4 \sup_{z\in f(S)} |z|^{\beta+\delta-1} \\
&\leq 8BK^4 M_{k+1}^{\beta+\delta-1} \qquad (16)
\end{aligned}$$

for M large enough. As mentioned, the distortions of $f^{n_k(U)}|U$ and $f|S$ are bounded by K. Therefore the distortion of $f^{n_{k+1}(V)}|U$ is bounded by K^2 and we get

$$\frac{\operatorname{meas}(U\backslash \bigcup \mathcal{F}_U)}{\operatorname{meas}(U)} \leq \frac{K^4 \operatorname{meas}(f(S)\backslash \bigcup \mathcal{F})}{\operatorname{meas}(f(S))},$$

which, together with (16), implies property (iii) for δ small and M large enough.

Case 2: $f(S) \subset B(s, M_{k+1}^{-1})$ for some $s \in A$.

We will study the behaviour of a certain number of iterates on of f on S. See also Figure 2. We begin with the first iterate. Let w be the centre of S. For $z \in S$, (10) implies $|z - w| \leq \frac{c}{4}|w|^{-\delta_2}$ and (a) implies $|f'(w)| \leq |f(w) - s||w|^{\delta_2}$. The mean value theorem provides $x \in [z, w]$ with

$$|f(z) - f(w)| \leq |f'(x)||z - w| \leq K|f'(w)||z - w| \leq \frac{c}{4}K|f(w) - s|.$$

Thus we know that

$$f(S) \subset B\left(s, \left(1+\frac{cK}{4}\right)|f(w)-s|\right) \setminus B\left(s, \left(1-\frac{cK}{4}\right)|f(w)-s|\right). \quad (17)$$

S contains the disc $B\left(w, \frac{c}{16\sqrt{2}|w|^{\delta_2}}\right)$. Thus Lemma 2.3 together with (a) implies that

$$B\left(f(w), \frac{c|f(w)-s|}{16\sqrt{2}K|w|^{\delta_2-\delta_1}}\right) \subset f(S). \quad (18)$$

Next we consider those iterates, in which we cannot avoid critical points. We do this in terms of the power series

$$f^{m_s}(z) = f^{m_s}(s) + (f^{m_s})^{(k_s)}(s)(z-s)^{k_s} + O\left((z-s)^{k_s+1}\right).$$

This provides good estimates for f^{m_s} and its derivative, if $|z-s|$ is very small, which is the case for $z \in f(S)$ since $|f(w)-s| \le \exp(-|w|^{\epsilon})$. The only purpose of our choice of c was to ensure that the diameter of $f(S)$ is small enough to guarantee that $f^{m_s}(s)$ lies in the unbounded component of $\mathbb{C}\backslash f^{m_s+1}(S)$. The reader may prefer to convince himself that this goal is achievable by a sufficiently small choice of c, instead of checking that our concrete choice above is sufficient. Thus Lemma 2.7 implies that f^{m_s} is injective on $f(S)$. The ratio of the outer and inner radii of the annulus in (17) is $C := \left(\frac{1+\frac{cK}{4}}{1-\frac{cK}{4}}\right)$. Thus the image lies in an annulus whose ratio of those radii is very close to C^{k_s} and the distortion is bounded by any constant greater than C^{k_s-1}, say C^{k_s}. Using the factor $1 \pm \frac{cK}{4}$ for the error term of the power series we can deduce from (17) and (18) that

$$f^{m_s+1}(S) \quad \supset \quad B\left(f^{m_s+1}(w), \left|\frac{k_s(f^{m_s})^{(k_s)}(s)(1-\frac{cK}{4})^{k_s}(f(w)-s)^{k_s}}{16\sqrt{2}K|w|^{\delta_2-\delta_1}}\right|\right) \quad (19)$$

and

$$f^{m_s+1}(S) \quad \subset \quad B\left(f^{m_s}(s), \left|(f^{m_s})^{(k_s)}(s)(1+\frac{cK}{4})^{k_s+1}(f(w)-s)^{k_s}\right|\right). \quad (20)$$

Here all but the term $|f(w)-s|$ do not depend on k. One could get similar estimates for $1 \le l \le m_s+1$, that imply $f^l(S) \subset B(f^{l-1}(s), M_{k+1}^{\delta-1})\backslash\{f^{l-1}(s)\}$ for M large enough. This implies that $f^l(S)$ is contained in A_k or, more precisely, in the middle term of its definition.

Next we consider the maximal number of iterates for which we can assure injectivity and bounded distortion. Due to (c) the set $B(f^m(s), 16|f^m(s)|^{-(\delta_2+\delta)})$ is contained in $\{z : \operatorname{dist}(z, \mathbb{C}\backslash G) \ge |z|^{-\delta_1}\} \cap B(f^m(s), |f^m(s)|^{-\delta_2})$ for m large enough. For m large, $|f^m(s)|^\delta > 64$ such that, due to (11), f restricted to this set is injective. By Koebe's 1/4-theorem we get

$$\begin{aligned} f\left(B\left(f^m(s), 8|f^m(s)|^{-(\delta_2+\delta)}\right)\right) &\supset B\left(f^{m+1}(s), \frac{2|f'(f^m)(s)|}{|f^m(s)|^{(\delta_2+\delta)}}\right) \\ &\supset B\left(f^{m+1}(s), \frac{2|f^{m+1}(s)|}{|f^m(s)|^{\delta_2-\delta_1+\delta}}\right). \end{aligned} \tag{21}$$

Here the last inclusion follows from (a) and (c). Due to condition (c) we know that $f^m(s)$ escapes to ∞ in G. For $z \in G$, (a) implies $|f'(z)| \geq |f(z)||z|^{\delta_1}$. Thus $|f'(f^m(s))| \to \infty$ as $m \to \infty$. Thus for m large enough, $m_s \leq l \leq m$ and $r > 0$ small we know that f^{m-l} is expanding on $B(f^l(s), r)$. Consequently the component of $(f^{m-l})^{-1}\left(B\left(f^m(s), 8|f^m(s)|^{-(\delta_2+\delta)}\right)\right)$ containing $f^l(s)$ is contained in $B\left(f^l(s), 8|f^m(s)|^{-(\delta_2+\delta)}\right)$, which does not contain critical points. This allows us to extend the inverse g of f^{m-m_s+1}, mapping $f^{m+1}(s)$ to $f^{m_s}(s)$, to $B\left(f^{m+1}(s), \frac{2|f^{m+1}(s)|}{|f^m(s)|^{\delta_2-\delta_1+\delta}}\right)$. Thus the distortion on half the ball is bounded by some constant $\tilde{K}$. One could use Lemma 2.4 to obtain $\tilde{K} = K_{\frac{1}{\sqrt{2}}}$ or the original distortion theorem 2.1 to obtain $\tilde{K} = 81$. In any case we get

$$\begin{aligned} &\operatorname{dist}\left(f^{m_s}(s), \partial g\left(B\left(f^{m+1}(s), \frac{|f^{m+1}(s)|}{|f^m(s)|^{\delta_2-\delta_1+\delta}}\right)\right)\right) \\ \geq\ & \frac{|f^{m+1}(s)|}{|f^m(s)|^{\delta_2-\delta_1+\delta}} \inf_{z \in B\left(f^{m+1}(s), \frac{|f^{m+1}(s)|}{|f^m(s)|^{\delta_2-\delta_1+\delta}}\right)} |g'(z)| \\ \geq\ & \frac{|f^{m+1}(s)|}{\tilde{K}|f^m(s)|^{\delta_2-\delta_1+\delta}} |g'(f^{m+1}(s))| \\ =\ & \left| \frac{f^{m+1}(s)}{\tilde{K} f^m(s)^{\delta_2-\delta_1+\delta} (f^{m-m_s+1})'(f^{m_s}(s))} \right| \\ =\ & \left| \frac{f^{m+1}(s)}{\tilde{K} f^m(s)^{\delta_2-\delta_1+\delta} \prod_{i=m_s}^{m} f'(f^i(s))} \right| \\ \geq\ & \left| \frac{1}{\tilde{K} f^m(s)^{1+2\delta_2-\delta_1+\delta} f^{m_s}(s)^{\delta_2} \prod_{i=m_s+1}^{m-1} f^i(s)^{1+\delta_2}} \right| \\ \geq\ & |f^m(s)|^{-(1+2\delta_2-\delta_1+2\delta)} \end{aligned}$$

for large m. We define m as the greatest natural number that satisfies

$$|f^{m-1}(s)|^{-(1+2\delta_2-\delta_1+2\delta)} \geq |(f^{m_s})^{(k_s)}(s)| \left(1 + \frac{cK}{4}\right)^{k_s+1} |f(w) - s|^{k_s}.$$

We note that $m \to \infty$ as $|f(w) - s| \to 0$. Thus we can guarantee that m is large by choosing M large. This choice of m guarantees together with (19) that

$f^{m_s+1}(S) \subset g\left(B\left(f^m(s), \frac{|f^m(s)|}{|f^{m-1}(s)|^{\delta_2-\delta_1+\delta}}\right)\right)$. Thus f^{m-m_s} restricted to $f^{m_s+1}(S)$ is injective, its distortion is bounded by $\tilde{K}$, and

$$f^{m+1}(S) \subset B\left(f^m(s), \frac{|f^m(s)|}{|f^{m-1}(s)|^{\delta_2-\delta_1+\delta}}\right) \subset B\left(f^m(s), \frac{|f^m(s)|}{2}\right). \tag{22}$$

The maximal choice of m guarantees that

$$|f^m(s)| \geq |f(w)-s|^{\frac{-k_s}{1+2\delta_2-\delta_1+3\delta}} \tag{23}$$

for M_0 large enough. Together with the discussion below (21), this implies that $f^k(S)$ is contained in A_k for $m_s+1 \leq k \leq m+1$. The exponential growth of $|f^m(s)|$ and our estimates for $|f'|$ imply that

$$|f^m(s)|^{1-\alpha} \leq |(f^{m-k})'(f^k(s))| \leq |f^m(s)|^{1+\alpha} \tag{24}$$

for any $\alpha > 0$ and any natural numbers $k < m$ with m large enough. This allows us to cancel the smaller factors, transferring (19) and (20) by bounded distortion of f^{m-m_s} to

$$B\left(f^{m+1}(w), |f^m(s)|^{1-\delta}|f(w)-s|^{k_s}\right) \subset f^{m+1}(S) \tag{25}$$

and

$$f^{m+1}(S) \subset B\left(f^m(s), |f^m(s)|^{1+\delta}|f(w)-s|^{k_s}\right). \tag{26}$$

Again we distinguish between two cases.

Case 2.1. We have

$$|f^m(s)| \geq |f(w)-s|^{\frac{-k_s}{1-\tau+\delta}}. \tag{27}$$

This is stronger than (23) and with (a) we get

$$|f^m(s)| \geq \exp\left(\frac{k_s}{1-\tau+\delta} M_k^{\epsilon}\right), \tag{28}$$

which together with (22) implies that $f^{m+1}(S) \subset D(M_{k+1})$ for δ sufficiently small. We define $\mathcal{F} := \{R \in \mathcal{S} : \frac{1}{c}R \subset f^{m+1}(S)\}$,

$$\mathcal{F}_U := \{(f^{n_k(U)+m+1}|U)^{-1}(R) : R \in \mathcal{F}\} \text{ and } n_{k+1}(V) := n_k(U)+m+1$$

for all $V \in \mathcal{F}_U$. Again properties (i) and (ii) follow by definition. The distortion of $f|S$ is bounded by K, while the distortion of $f^{m_s}|f(S)$ is bounded by C^{k_s}, and the distortion of $f^{m-m_s}|f^{m_s+1}(S)$ is bounded by $\tilde{K}$. Thus the distortion of $f^{m+1}|S$ is bounded by $\tilde{K}KC^{k_s}$. Therefore $f^{m+1}(S)$ is a $\tilde{K}KC^{k_s}$-quasi-square. For M large enough (27) together with (25) implies that

$$\operatorname{diam}(f^{m+1}(S)) \geq |f^m(s)|^{\tau-2\delta} \geq \sup_{z\in f^{m+1}(S)} |z|^{\tau-3\delta}. \tag{29}$$

As in case 1 we can show that $\operatorname{meas}(f^{m+1}(S)\backslash \bigcup \mathcal{F})$ is bounded above by the measure of the set $\{z : \operatorname{dist}(z, \mathbb{C}\backslash G) \leq 2|z|^{-\delta_1}\}$, which, due to condition (b), does not exceed $B \operatorname{diam}(f^{m+1}(S)) \sup_{z\in f^{m+1}(S)} |z|^{\beta}$ plus the measure of the set $\{z \in f^{m+1}(S) : \operatorname{dist}(z, \partial f^{m+1}(S)) \leq |z|^{-\delta_2}\}$, which is, with (7), bounded above by $4\tilde{K}^2K^2C^{2k_s} \operatorname{diam}(f^{m+1}(S)) \sup_{f^{m+1}(S)} |z|^{-\delta_2}$. Using (7) once more, we obtain that $\operatorname{meas}(f^{m+1}(S)) \geq \operatorname{diam}(f^{m+1}(S))^2/(2\tilde{K}^2K^2C^{2k_s})$. We know that $-\delta_2 < \beta$, such that we can deduce from the above and (29) that

$$\frac{\operatorname{meas}(f^{m+1}(S)\backslash \bigcup \mathcal{F})}{\operatorname{meas}(f^{m+1}(S))} \leq 5B\tilde{K}^2K^2C^{2k_s} \sup_{z\in f^{m+1}(S)} |z|^{\beta+3\delta-\tau} \leq \frac{M_{k+1}^{\beta-\tau}}{4\tilde{K}^2K^4C^{2k_s}}, \tag{30}$$

where the last inequality holds due to (22) and (28) for M large and δ small enough. The distortion $f^{n_k(U)}|U$ is bounded by K. Thus (30) together with Lemma 2.5 imply (iii).

Case 2.2. $|f^m(s)| < |f(w) - s|^{\frac{-k_s}{1-\tau+\delta}}$.

With (26) we get

$$f^{m+1}(S) \subset B(f^m(s), |f^m(s)|^{1+\delta}|f(w) - s|^{k_s}) \subset B(f^m(s), |f^m(s)|^{\tau}), \tag{31}$$

which, because of condition (c), is contained in $\{z : \operatorname{dist}(z, \mathbb{C}\backslash G) \geq |z|^{-\delta_1}\}$. We distinguish between two more cases:

Case 2.2.1. $\operatorname{diam}(f^{m+1}(S)) < \frac{c}{4}|f^m(s)|^{-\delta_2}$.

Then due to (11), $f|f^{m+1}(S)$ is injective and its distortion is bounded by K. We define

$$\mathcal{F}_U := \left\{ (f^{n_k(U)+m+2}|U)^{-1}(T) : T \in \mathcal{S}; \frac{1}{c}T \subset f^{m+2}(S) \right\}$$

and $n_{k+1}(V) := n_k(U) + m + 2$. Then properties (i) and (ii) are again satisfied by definition. The bounds of the distortion of $f|S$,$f^{m_s}|f(S)$ and $f^{m-m_s}|f^{m_s+1}(S)$ are as above. Then $f^{m+2}(S)$ is a $\tilde{K}K^2C^{k_s}$-quasi-square and, with (25), it follows that

$$\operatorname{diam}(f^{m+2}(S)) \geq |f^m(s)|^{1-\delta}|f(w) - s|^{k_s} \inf_{z\in f^{m+1}(S)} |f'(z)| \geq \sup_{z\in f^{m+2}(S)} |z|^{1-\delta}. \tag{32}$$

Here the last inequality follows for M large enough, since for $z \in f^{m+1}(S)$ the magnitude of $|f(z)|$ and, by condition (a), also that of $|f'(z)|$ is $\exp(|f^m(s)|^\epsilon)$. By (23), this is far larger than the other factors. From condition (b) and (7) we get as before

$$\begin{aligned} & \frac{\operatorname{meas}\left(f^{m+2}(S)\setminus\bigcup_{R\in\mathcal{S},\frac{1}{c}R\subset f^{m+2}(S)} R\right)}{\operatorname{meas}(f^{m+2}(S))} \\ \leq\ & 5B\tilde{K}^2K^4C^{2k_s} \sup_{z\in f^{m+2}(S)} |z|^{\beta+\delta-1} \\ \leq\ & \exp\left(\left(\frac{1}{2}\exp\left(\frac{M_k^\epsilon}{1+2\delta_2-\delta_1+3\delta}\right)\right)^\epsilon\right)^{\beta+2\delta-1}. \end{aligned} \tag{33}$$

Here the last inequality follows by (22) and (23). Again the distortion estimates from above and Lemma 2.5 imply that

$$\frac{\operatorname{meas}(U\setminus\mathcal{F}_U)}{\operatorname{meas}(U)} \leq \frac{\tilde{K}^2K^6C^{2k_s}\operatorname{meas}\left(f^{m+2}(S)\setminus\bigcup_{T\in\mathcal{S},\frac{1}{c}T\subset f^{m+2}(S)} T\right)}{\operatorname{meas}(f^{m+2}(S))}.$$

Together with estimate (33) this is far stronger than condition (iii).

Case 2.2.2. $\operatorname{diam}(f^{m+1}(S)) \geq \frac{c}{4}|f^m(s)|^{-\delta_2}$.

We consider a family $\mathcal{F}$ of disjoint open squares $R \subset f^{m+1}(S)$ with diameter $\exp(-|f^m(s)|^{\frac{\epsilon}{2}})$, such that we cover all of $f^{m+1}(S)$ except a set of measure zero and an $\exp(-|f^m(s)|^{\frac{\epsilon}{2}})$-neighbourhood of the boundary. Since $f^{m+1}(S)$ is a $\tilde{K}KC^{k_s}$-quasi-square, (7) implies that

$$\begin{aligned} \operatorname{meas}\left(f^{m+1}(S)\setminus\bigcup\mathcal{F}\right) &\leq 4\tilde{K}^2K^2C^{2k_s}\exp(-|f^m(s)|^{\frac{\epsilon}{2}})\operatorname{diam}(f^{m+1}(S)) \\ &\leq \exp\left(-\left|\frac{f^m(s)}{2}\right|^{\frac{\epsilon}{2}}\right) \end{aligned} \tag{34}$$

for M large enough, since, due to (31), again one factor, namely $\exp(|f^m(s)|^{\frac{\epsilon}{2}})$, dominates all others. We define

$$\mathcal{F}_U := \left\{\left((f^{n_k(U)+m+1}|U)^{-1}\circ(f|R)^{-1}\right)(Q) : R\in\mathcal{F} \text{ and } Q\in\mathcal{S} \text{ with} \frac{1}{c}Q\subset f(R)\right\}$$

and $n_{k+1}(V) := n_k(U)+m+2$ for all $V \in \mathcal{F}_U$. Again properties (i) and (ii) follow directly. The diameter of all $R \in \mathcal{F}$ is small enough such that, due to (11), $f|R$ is injective and its distortion is close to one, say bounded by K. With the mean value theorem we can deduce that

$$\operatorname{diam}(f(R)) \geq \frac{1}{\sqrt{2}}\inf_{z\in R}|f'(z)|\exp(-|f^m(s)|^{\frac{\epsilon}{2}}) \geq \sup_{z\in f(R)}|z|^{1-\delta}$$

for M_0 large enough. Note that with (22) we have $|f(z)| \geq \exp(|\frac{f^m(s)}{2}|^\epsilon)$ for $z \in R$. By the same arguments as above, (b) and (7) imply that

$$\begin{aligned}\frac{\operatorname{meas}\left(f(R)\setminus\bigcup_{Q\in\mathcal{S},\frac{1}{c}Q\subset f(R)}Q\right)}{f(R)} &\leq 5BK^2 \sup_{z\in f(R)} |z|^{\beta+2\delta-1} \\ &\leq 5BK^2 \exp\left(\beta+\delta-1\left|\frac{f^m(s)}{2}\right|^\epsilon\right),\end{aligned}$$

where last inequality may be deduced from (a) and (22). Since the distortion of $f|R$ is bounded by K, we can transfer this with the help of Lemma 2.5 to R losing only a factor of K^2. This estimate for the density in every $R \in \mathcal{F}$ implies the same for their union $\bigcup \mathcal{F}$, which is contained in $f^{m+1}(S)$. More precisely, we know that

$$\begin{aligned}&\frac{\operatorname{meas}\left(\bigcup\mathcal{F}\setminus\bigcup_{R\in\mathcal{F},Q\in S,\frac{1}{c}Q\subset f(R)}(f|R)^{-1}(Q)\right)}{\operatorname{meas}(f^{m+1}(S))} \\ \leq\ & 5BK^4 \exp\left(\beta+\delta-1\left|\frac{f^m(s)}{2}\right|^\epsilon\right).\end{aligned} \tag{35}$$

With the distortion estimates above and with Lemma 2.5 it follows from (34) and (35) that

$$\begin{aligned}&\frac{\operatorname{meas}(U\setminus\bigcup\mathcal{F}_U)}{\operatorname{meas}(U)} \\ \leq\ & \frac{\tilde{K}^2K^4C^{2k_s}\operatorname{meas}\left(f^{m+1}(S)\setminus\bigcup_{R\in\mathcal{F},Q\in S,\frac{1}{c}Q\subset f(R)}(f|R)^{-1}(Q)\right)}{\operatorname{meas}(f^{m+1}(S))} \\ \leq\ & \frac{\tilde{K}^2K^4C^{2k_s}\left(\operatorname{meas}\left(f^{m+1}(S)\setminus\bigcup\mathcal{F}\right)+\operatorname{meas}(\bigcup\mathcal{F}\setminus\bigcup_{R\in\mathcal{F},Q\in S,\frac{1}{c}Q\subset f(R)}(f|R)^{-1}(Q))\right)}{\operatorname{meas}(f^{m+1}(S))} \\ \leq\ & \tilde{K}^2K^4C^{2k_s}\exp\left(-\left|\frac{f^m(s)}{2}\right|^{\frac{\epsilon}{2}}\right)+5K^8\tilde{K}^2C^{2k_s}\exp\left(\beta+\delta-1\left|\frac{f^m(s)}{2}\right|^\epsilon\right).\end{aligned}$$

Together with (23) this is again far stronger than condition (iii) for δ small and M_0 large enough.

The definition $\mathcal{F}_{n+1} := \bigcup_{U\in\mathcal{F}_k}\mathcal{F}_U$ completes the recursive definition, such that all required properties are satisfied. This completes the proof of the theorem. ■

4 Entire functions

This section is about functions of the same type as in Theorem 1.1. First of all we will prove some general properties and introduce some notations which will frequently occur. In the entire chapter let P and Q be polynomials with P not zero and Q not constant, $c \in \mathbb{C}$ and

$$f(z) := \int_0^z P(t)\exp(Q(t))dt + c. \tag{36}$$

For $k \in \{1, .., \deg(Q)\}$ define $\phi_k := \frac{(2k+1)\pi - \arg(q)}{\deg(Q)}$, where q denotes the leading coefficient of $Q(z) = qz^{\deg(Q)} + ...$ For $R \to \infty$ the modulus of $\exp(Q(R\exp(\phi_k i))$ decreases very fast, such that $f(R\exp(\phi_k i))$ converges to the point

$$s_k := \lim_{R\to\infty} \int_0^{R\exp(i\phi_k)} P(t)\exp(Q(t))dt + c, \tag{37}$$

which therefore is an asymptotic value of f. Let A denote the set of finite asymptotic values. For $z \in \mathbb{C}$ choose k such that

$$\phi_k - \frac{\pi}{\deg(Q)} \leq \arg(z) < \phi_k + \frac{\pi}{\deg(Q)}$$

and define $\overline{s}(z) = s_k$.

4.1 Lemma

$$f(z) \;=\; \overline{s}(z) + \frac{P(z)\exp(Q(z))}{Q'(z)} + \mathcal{O}(|z|^{\deg(P)-\deg(Q)})\exp(Q(z)) \text{ as } z \to \infty.$$

Proof. Let $z \in \mathbb{C}$. We define $w := 2|z|\exp(\phi_k i)$ with the k from above. Instead of integrating from 0 to z on a straight path, one might as well go from 0 to infinity in the direction ϕ_k, come back the same way up to w, and finally move forward to z. If z, w are not zeros of Q', one can find a path from w to z avoiding these zeros such that with integration by parts it follows that

$$\begin{aligned} f(z) \;&=\; s_k + \frac{P(z)\exp(Q(z))}{Q'(z)} - \frac{P(w)\exp(Q(w))}{Q'(w)} \\ &+ \int_w^z \left(\frac{P'Q' - PQ''}{(Q')^2}\right)(t)\exp(Q(t))dt \\ &- \int_{2|z|}^{\infty} P(t\exp(\phi_k i))\exp(Q(\exp(\phi_k i)))dt. \end{aligned}$$

It is easy to obtain estimates of the last three terms that imply the claim. ■

4.2 Lemma
Suppose $\delta, \delta' > 0$. Then for M large enough and every element z in

$$G := \{z : |Re(Q(z))| \geq |z|^{\delta}\} \cap D(M),$$

the restriction of f to $B\left(z, \frac{(1-\delta')\pi}{|Q'(z)|}\right)$ is injective.

PROOF. We use Lemma 2.7 and assume the existence of points $z, w \in G$ with $f(z) = f(w)$ and $|z - w| < \frac{(2-2\delta')\pi}{|Q'(z)|}$. Then $f([z, w])$ is a closed curve with a singularity of f^{-1} in a bounded component of its complement. The condition $|f(z) - s| = |f(w) - s|$ together with Lemma 4.1 implies

$$1 - \frac{\delta'}{8} < \frac{|\mathrm{Re}(Q(w))|}{|\mathrm{Re}(Q(z))|} < 1 + \frac{\delta'}{8}$$

for $|w|$ large enough. Then

$$1 - \frac{\delta'}{4} < \frac{|\mathrm{Re}(Q(x))|}{|\mathrm{Re}(Q(z))|} < 1 + \frac{\delta'}{4}$$

for every $x \in [z, w]$. Again by Lemma 4.1 we have

$$1 - \frac{\delta'}{2} < \frac{|f(x)|}{|f(z)|} < 1 + \frac{\delta'}{2}$$

for $|z|$ large enough. Therefore the length of the curve $f([z, w])$ must be at least $\pi(2 - \delta')|f(z)|$. On the other hand, the mean value theorem does not allow this length to exceed $|z - w| \max_{x \in [z,w]} |f'(x)|$, which is bounded above by $\frac{\pi(2-\delta'-\delta'^2) \max_{x\in[z,w]} |Q'(x)|}{|Q'(z)|}|f(z)|$. This is a contradiction to the estimate from above for $|z|$ large enough. Thus the claim follows with Lemma 2.7. ■

4.1 The non-recurrent case

Now we prove the first of the results given in the introduction.

PROOF OF THEOREM 1.1. We verify the properties of Theorem 3.1. Since every $s \in A$ escapes exponentially, there exists $\delta_s > 0$ with $|f^{n+1}(s)| \geq \exp(|f^n(s)|^{\delta_s})$ for almost every $n \in \mathbb{N}$. We chose δ, such that $0 < \delta < \min_{s \in A} \delta_s$. By Lemma 4.1 we have an estimate for $|f|$. For $\epsilon < \delta$ and $\delta_1 < \deg(Q) - 1 < \delta_2$ property (a) follows if we redefine $\overline{s}$ as zero on the part of G where $\mathrm{Re}(Q(z)) > 0$.
Far away from the origin, $\mathbb{C}\backslash G$ consists of neighbourhoods around the pre-images under Q of the imaginary axis, whose widths at a distance R from the origin are of magnitude $R^{-\deg(Q)+1+\delta}$. With the *width* at a distance R from the origin

we mean the diameter of the largest disc that is contained in the set and whose centre has modulus R. For $-\deg(Q)+1+\delta<\beta<1$ and B sufficiently large, (b) follows.

As mentioned above we have $|f^{n+1}(s)| \geq \exp(|f^n(s)|^{\delta_s})$ except for a finite number of $n \in \mathbb{N}$. Thus the real part of $Q(f^n(s))$ is at least of magnitude $|Q(f^n(s))|^{\frac{\delta_s}{\deg(Q)}}$ and the magnitude of the distance of $f^n(s)$ to $\mathbb{C}\backslash G$ is no less than $|f^n(s)|^{-\deg(Q)+1+\delta_s}$. We choose τ, such that $\beta<\tau<-\deg(Q)+1+\min_{s\in A}\delta_s$. Then condition (c) is satisfied. Applying Theorem 3.1, we get $\operatorname{meas}(T(f))>0$. Case (ii) of Theorem 2.24 follows. As explained in Remark 3.2 we know that $T(f) \subset J(f)$ due to a result of I. N. Baker [1].
Assume now $\deg(Q) \geq 3$. Then for δ small enough we have $\deg(Q)-1-\delta>1$. This implies that $\operatorname{meas}(\mathbb{C}\backslash G)<\infty$ and, in the case that $\delta_1>1$, even that $\operatorname{meas}(\{z : \operatorname{dist}(z,\mathbb{C}\backslash G) \leq |z|^{-\delta_1}\})<\infty$. This follows since this set is contained in the set

$$B(0,M) \cup \bigcup_{k=1}^{\deg(Q)} \left\{ z : \left| \arg(z) - \frac{(4k+1)\pi - 2\arg(q)}{2\deg(Q)} \right| \leq 2q|z|^{\delta-\deg(Q)} \right\}.$$

Using polar coordinates one sees that the measure of the last set is bounded by

$$\deg(Q)\int_M^\infty R^{1-\deg(Q)+\delta} d2qR = \frac{2q\deg(Q)M^{2-\deg(Q)+\delta}}{2-\deg(Q)+\delta}$$

We cover the set $\{z : \operatorname{dist}(z,\mathbb{C}\backslash G) \geq 2|z|^{-\delta_1}\}$ up to measure zero with a family $\mathcal{S}$ of disjoint squares $S \subset \{z : \operatorname{dist}(z,\mathbb{C}\backslash G) \geq |z|^{-\delta_1}\}$ whose diameters satisfy $\sup_{z\in S}|z|^{-\delta_2} \leq \operatorname{diam}(S) \leq 4\sup_{z\in S}|z|^{-\delta_2}$. The density of $\mathcal{F}(f)$ in any $S \in \mathcal{S}$ is, due to Theorem 3.1, at most $\exp(-\eta \inf_{z\in S}|z|^{\epsilon})$. Let $R_k := M+k$ for all $k \in \mathbb{N}\cup\{0\}$. Then it follows that

$$\begin{aligned}
\operatorname{meas}(F(f)) &\leq \operatorname{meas}\left(\{z : \operatorname{dist}(z,\mathbb{C}\backslash G) \leq 2|z|^{-\delta_1}\}\right) + \sum_{S\in\mathcal{S}} \operatorname{meas}(\mathcal{F}(f)\cap S) \\
&\leq \pi M^2 + \frac{2q\deg(Q)M^{2-\deg(Q)+\delta}}{2-\deg(Q)+\delta} + \sum_{k\in\mathbb{N}} \sum_{\substack{S\in\mathcal{S} \\ R_k<\inf_{z\in S}|z|\leq R_{k+1}}} \operatorname{meas}(\mathcal{F}(f)\cap S) \\
&\leq \pi M^2 + \frac{2q\deg(Q)M^{2-\deg(Q)+\delta}}{2-\deg(Q)+\delta} + \sum_{k=1}^\infty 2\pi(R_k+1)\exp\left(-\eta R_k^\epsilon\right),
\end{aligned}$$

which is finite. This gives the second part of Theorem 1.1. ■

Theorem 1.1 allows us to construct examples of functions f with the property that $0<\operatorname{meas}(\mathcal{F}(f))<\infty$. For this purpose we arrange the parameters a and b

(e.g. $a = (\frac{27\pi^2}{16})^{1/3}$, $b = \log(\sqrt{a/3})$), such that both critical points of the function

$$f(z) = \exp(z^3 + az + b)$$

are fixed, and the only finite asymptotic value 0 escapes to infinity on the real axis. Thus the Fatou set of f, which consists of those two super attractive basins, has finite measure. In Figure 3, where the $\{z : |\mathrm{Re}(z)| \leq 2, |\mathrm{Im}(z)| \leq 2\}$ is displayed, these two super-attractive basins are coloured black.

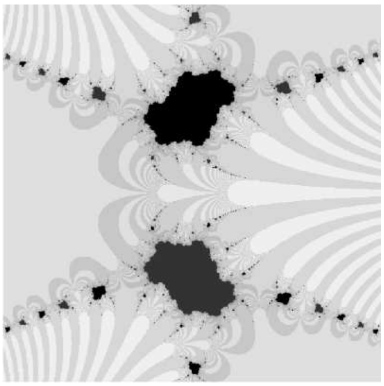

Figure 3: The Fatou set of $f(z) = \exp(z^3 + az + b)$ with $a = (\frac{27\pi^2}{16})^{1/3}$ and $b = \log(\sqrt{a/3})$

We should note that the existence of such examples is not very surprising after the construction of examples with a positive measure Julia set by C. McMullen in [33]. Also the idea of using concrete measure estimates like the one in Theorem 3.1 in order to show finiteness of the measure of subsets of the Fatou set has been used before by H. Schubert, who proved in [46], that the measure of the Fatou set of the sine function in the strip $\{z : 0 \leq \mathrm{Re}(z) \leq 2\pi\}$ is finite, as conjectured by J. Milnor in [35].

4.2 The typical orbit

We just proved that the ω-limit-set of almost every point in the Julia set is contained in the post-singular set, if all finite asymptotic values escape exponentially. Now our aim is to precisely determine the ω-limit-set for almost every point in the Julia set, which we achieve under additional hypothesis.

The key question in this context is the question of whether almost every orbit must accumulate at every asymptotic value. In order not to accumulate at an asymptotic value s, an orbit has to stay out of an entire sector of the plane. In other contexts, sets with this property turn out to have measure zero.

Thus one may expect a positive answer. If, however, the set of points in the Julia set whose orbits are bounded had positive measure, there would be no reason why these orbits should accumulate at a given asymptotic value. It is not known whether this can actually occur, and this is related to the well-known open question of whether the Julia set of a polynomial can have positive measure. A positive answer to the latter question would suggest a negative answer to our initial question, which will be discussed in the final part of this section. In this subsection we will give a positive answer under additional assumptions on the critical values.

We choose $0 < \delta < \min_{a,b\in A} \frac{\mathrm{dist}(a,b)}{2}$, $R > \max_{a\in A} |a|$ and assume that M is sufficiently large. Then for any $z \in G$ there is exactly one $a \in A$ for which $|f(z) - a| < \delta$ or $|f(z)| \geq R$. For $a \in A$ we define G_a to be that part of G for which the first is satisfied and G_∞ the part of G for which the last is satisfied. In order to prove Theorem 1.2 we need the following lemma.

4.3 Lemma
Let $\Gamma := \bigcup_{n\in\mathbb{N}} (f^n)^{-1}(B(a,\epsilon))$ for some $a \in A$ and $\epsilon > 0$. Then there exist positive constants c, C, and a family $\mathcal{F}$ of disjoint domains D, such that

$$\mathrm{diam}(D) \leq \frac{C}{\sup_{z\in D} |Q'(z)|}\ , \quad \frac{\mathrm{meas}(\Gamma \cap D)}{\mathrm{meas}(D)} \geq c \text{ and } \mathrm{meas}\left(G_s \backslash \bigcup \mathcal{F}\right) = 0,$$

if $s = \infty$ or if s is an asymptotic value that escapes exponentially.

Proof. Since a is an asymptotic value, we have $\lim_{R\to\infty} f(R\exp(\phi_a i)) = a$ for some $k \in \{0, 1, .. \deg Q\}$ and $\phi_a = \frac{(2k+1)\pi - q}{\deg Q}$. From Lemma 4.1 it follows that for any $0 < \delta'$ there exists $M > 0$, such that

$$\left\{ z : \phi_a - \frac{(1-\delta')\pi}{2\deg(Q)} < \arg(z) < \phi_a + \frac{(1-\delta')\pi}{2\deg(Q)} \right\} \cap D(M) \subset f^{-1}\left(B(s,\epsilon)\right).$$

Since Lemma 4.1 also gives good estimates for the argument of f in G_∞ it follows that the set

$$\left\{ z : \phi_a - \frac{(1-2\delta')\pi}{2\deg(Q)} < \arg\left(\frac{P(z)}{Q'(z)}\right) + \mathrm{Im}(Q(z)) < \phi_a + \frac{(1-2\delta')\pi}{2\deg(Q)} \right\} \cap G_\infty$$

is contained in the $f^{-2}\left(B(s,\epsilon)\right)$. Every component of this set is an unbounded region, whose width at the distance R from the origin is at least $\frac{1-3\delta'}{\deg(Q)} 2\pi |q|^{-1} R^{1-\deg(Q)}$ for sufficiently large R. We refer to these regions as the *channels*, see the left side of Figure 4, in which these channels are coloured black. In order to be able to display the structure we had to magnify their diameter relative to M. The *gaps* in between these channels have a width of at most $\left(2 - \frac{(1-3\delta')}{\deg(Q)}\right) \pi |q|^{-1} R^{1-\deg(Q)}$,

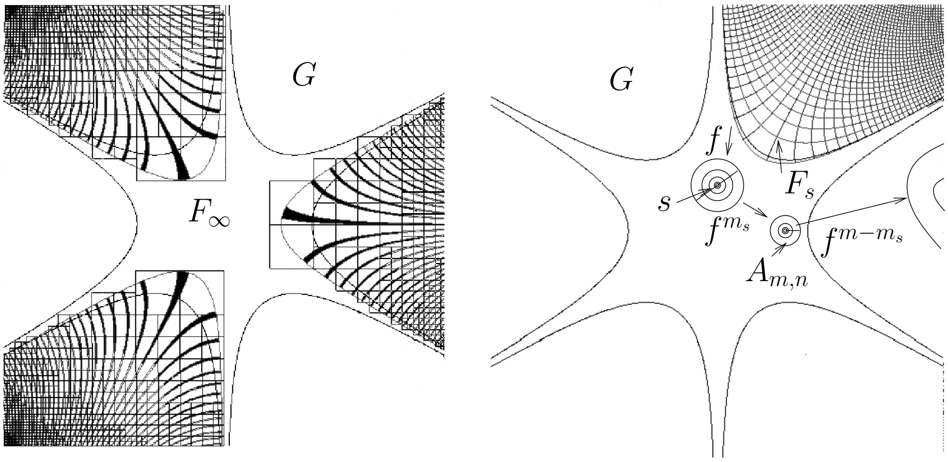

Figure 4: Construction of the family $\mathcal{F}$

still assuming that M is large. The complement of Γ in G_∞ must lie in the gaps between these channels. For a sufficiently small choice of δ' simple geometric arguments give for any constant $C' > \sqrt{2}\pi\left(2 - \frac{1}{\deg(Q)}\right)$ a constant $c' > 0$, such that for any square S intersecting G_∞ with $\operatorname{diam}(S) \geq C'(\inf_{z\in S}|Q(z)|)^{-1}$, the density of Γ in S is bounded below by c'. (For a sufficiently large choice of C' one can choose c' arbitrarily close to $\frac{1}{2\deg(Q)}$). We cover G_∞ up to measure zero by a family $\mathcal{F}_\infty$ of squares S with $C'(\inf_{z\in S}|Q'(z)|)^{-1} < \operatorname{diam}(S) < 4C'(\sup_{z\in S}|Q'(z)|)^{-1}$. We obtain this family in a similar way to the family $\mathcal{S}$ in the proof of Theorem 3.1. We begin with a grid of open squares covering the whole plane with a constant diameter, subdivide these into four parts until they satisfy the upper bound, and finally throw away those not intersecting G_∞. Then our conditions are satisfied with $c = c'$ and $C = 4C'$. The family $\mathcal{F}_\infty$ could look similar to the left side of Figure 4.

Now we need to find a covering $\mathcal{F}_s$ of G_s for every finite asymptotic value s that escapes exponentially. We define $m_s \in \mathbb{N} \cup \{0\}$ as before: minimal such that for $m \geq m_s$ the point $f^m(s)$ is not a critical point of f and choose $m_0 \geq m_s$ such that for $m \geq m_0$ it is none of Q. Then we choose n_m to be the smallest natural number, for which there exists $l_m \leq 4$ with

$$\left|\frac{(f^{m-m_s})'(f^{m_s}(s))Q'(f^m(s))}{(f^{m-m_s-1})'(f^{m_s}(s))Q'(f^{m-1}(s))}\right| = l_m^{n_m}.$$

The exponential escape of s implies that $l_m^{n_m}$ is of magnitude $|f^m(s)|$, such that $l_m \overset{m\to\infty}{\longrightarrow} 4$ and $n_m \overset{m\to\infty}{\longrightarrow} \infty$. For $0 \leq n \leq n_m$ we define

$$R_{m,n} \quad := \quad \left|\frac{(1-\delta')l_m^n\pi}{2(f^{m-m_s})'(f^{m_s}(s))Q'(f^m(s))}\right|$$

and for $n \neq 0$ we consider the slit annulus

$$A_{m,n} := \{z : R_{m,n-1} < |z - f^{m_s}(s)| < R_{m,n},\ z - f^{m_s}(s) \notin \mathbb{R}_{>0}\}.$$

Let $\mathcal{F}_s$ be the family of all connected components of $(f^{m_s+1})^{-1}(A_{m,n})$ intersecting G_s, for $m \geq m_0$ and $n \in \{1, ..., n_m\}$. We try to give an idea of $\mathcal{F}_s$ in Figure 4. For M large enough these components cover G_s up to measure zero. Next we will verify the diameter condition. For large m the annulus $A_{m,k}$ is very close to $f^{m_s}(s)$, such that the power-series of f^{m_s} gives good estimates. If k_s is the multiplicity of f^{m_s} in s, there are k_s pre-images A' of $A_{m,k}$ under f^{m_s} which are contained in $B(s, r) \backslash B(s, r^-)$ with $r^{(-)} := \frac{1+(-)\delta'}{|(f^{m_s})^{(k_s)}(s)|} \left|R_{m,n(-1)}\right|^{\frac{1}{k_s}}$. Since the ratio of the outer and inner radii of this annulus is $\frac{1+\delta'}{1-\delta'} l_m^{\frac{1}{k_s}}$, the distortion of f^{m_s} on these A' is bounded by any constant C_2 which is larger than this to the power of $k_s - 1$ for m_0 large enough.

Every connected component D of $f^{-1}(A')$ intersecting G_s, which is therefore an element of $\mathcal{F}_s$, is a simply connected domain with a diameter of at most $C \inf_{z\in D} \frac{1}{|Q'(z)|}$ for any $C > 4^{\frac{2}{k_s}} + \frac{4^{\frac{1}{k_s}} 2\pi}{k_s}$ and M large enough. This follows since we can connect any two points in A' with a path in A' whose length is at most $(\frac{2\pi}{k_s} + l_m^{\frac{1}{k_s}})r$, and for $z \in D$ we know due to Lemma 4.1 that $|f'(z)|$ is bounded by $|Q'(z)|r^-$. Thus the diameter condition in the claim is satisfied. An analogous upper estimate for $|f'|$ on D implies that distortion of f on such a domain D is bounded by any constant $C_3 > 4^{\frac{1}{k_s}}$ if M is large enough.

It remains to show that the density of Γ in these pre-images is again bounded away from zero by some $c > 0$.

The diameter of A_{m,n_m} is chosen in such a way that for m large enough f^{m-m_s} is injective on this set and its distortion is bounded by $K_{\tilde{c}}$ from Lemma 2.4 with $\tilde{c} < \frac{1}{\sqrt{2} l_m^{n_m - n}}$. This follows since for m large enough, f^{m-m_s-1} is injective on $B(f^{m_s}, R_{m-2,0})$ such that its distortion on $A_{m,n}$ is bounded by any constant larger than one, say $\frac{1-\frac{\delta'}{2}}{1-\delta'}$. Thus we have

$$f^{m-m_s-1}(A_{m,n}) \subset B\left(f^{m-1}(s), \frac{(1-\frac{\delta'}{2})\pi}{2|Q'(f^{m-1}(s))|}\right).$$

By Lemma 4.2 we know that f is injective on the ball with the same centre but twice the radius, such that the distortion estimate for $f^{m-m_s}|A_{m,n}$ follows with Lemma 2.4.

Since s escapes exponentially we may assume that δ is small enough to ensure that $|f^{n+1}(s)| \geq \exp(|f^n(s)|^{2\delta})$ for large n, such that in particular $f^n(s) \in G_\infty$.

To show that the density of Γ in the set $f^{m-m_s}(A_{m,n})$ is bounded below by some constant $c > 0$, we distinguish between large and small n.

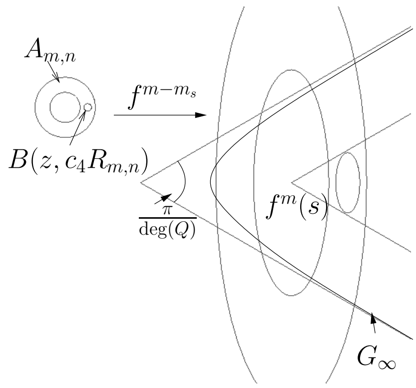

Figure 5: Constructions to obtain estimates for the density of Γ in $A_{m,n}$

If n is small enough, such that $l_m^n \leq |f^m(s)|^\delta$ is satisfied, it follows with the definition of n_m that $n_m - n$ is large. Assuming $\delta < 1/2$ we get $n_m - n \geq n_m/2$ for m large enough. Thus the distortion of f^{m-m_s} on $A_{m,n}$ is bounded by some K_m, which tends to 1 as $m \to \infty$. Thus for large m the set $f^{m-m_s}(A_{m,n})$ is close to the annulus $B\left(f^m(s), \left|\frac{(1-\delta')l_m^n \pi}{2Q'(f^m(s))}\right|\right) \setminus B\left(f^m(s), \left|\frac{(1-\delta')l_m^{n-1}\pi}{2Q'(f^m(s))}\right|\right)$ in the sense that as $m \to \infty$ the measure of density of the complement of $f^{m-m_s}(A_{m,n})$ in this annulus tends to zero. This annulus is contained in G_∞ and its diameter is more than twice as large as the width of the gaps in between the channels of Γ. Thus it has to intersect these channels. More precisely, the diameter assures that the density of Γ in $f^{m-m_s}(A_{m,n})$ is bounded below by a positive constant c_2. Since the distortion of f^{m-m_s} on $A_{m,n}$ is arbitrarily close to one for large m, for $c_3 < c_2$ this carries over by Lemma 2.5 to

$$\frac{\text{meas}(A_{m,n} \cap \Gamma)}{\text{meas}(A_{m,n})} \quad > \quad c_3. \tag{38}$$

For larger n the distortion is still bounded by $K := K_{1/\sqrt{2}}$, such that one could call $f^{m-m_s}(A_{m,n})$ a K-quasi-annulus, whose centre $f^m(s)$ lies in G_∞ and whose diameter is far larger than the gaps in between the channels of Γ. We choose $0 < c_4 < \frac{1}{4}$ smaller than $\sin\left(\frac{\pi}{2\deg(Q)}\right)/K$ and $z \in \partial B\left(f^{m_s}(s), \frac{R_{m,n}+R_{m,n-1}}{2}\right)$ maximising the distance of $f^{m-m_s}(z)$ to $\mathbb{C}\backslash G_\infty$. Then for large m it follows that $f^{m-m_s}(B(z, c_4 R_{m,n})) \subset G_\infty$. To see this, we note that the choice of c_4 guarantees that $f^{m-m_s}(B(z, c_4 R_{m,n}))$ lies in a sector of angle $\pi/\deg(Q)$ and corner $f^m(s)$. The boundary of the component of G_∞ containing $f^m(s)$ is tangent to the boundary of a sector of the same angle and corner 0. This is displayed in Figure 5. Using the family $\mathcal{F}_\infty$ from above as a cover, it is easy to see that the density of Γ in $f^{m-m_s}(B(z, c_4 R_{m,n}))$ is bounded below by any $0 < c_5 < c'$ and m

large enough. This carries over by Lemma 2.5 to $B(z, c_4 \operatorname{diam}(f^{m-m_s}(A_{m,n})))$, in which the density of Γ is at least $\frac{c_5}{K^2}$. We assume $c_3 \leq \frac{c_5 c_4^2}{K^2}$ and get (38) for all n and m if m_0 is large enough. This carries over by Lemma 2.5 to the elements of $\mathcal{F}_s$ and completes the proof for $c = \frac{c_3}{C_2^2 C_3^2}$. ■

4.4 Lemma

Let $B \subset \mathbb{C}$ be finite, such that every $b \in B$ escapes exponentially. Suppose that every singularity of the inverse is either pre-periodic, escapes exponentially or is contained in some attractive basin. Suppose further that $A \not\subset O^+(B)$. Then the set $\{z \in J(f) : \omega(z) \subset O^+(B) \cup \{\infty\}\}$ has zero measure.

Proof. We assume that the set $\{z \in J(f) : \omega(z) \subset O^+(B) \cup \{\infty\}\}$ has positive measure. Then for some $\epsilon > 0$ also the set

$$\begin{aligned} X \;&:=\; \{z \in J(f) : \{\infty\} \neq \omega(z) \subset O^+(B) \cup \{\infty\}\} \\ &\cap\; \{z \in J(f) : \forall a \in A \backslash O^+(B)\ \operatorname{dist}(O^+(z), a) > \epsilon\} \end{aligned}$$

has positive measure. We may assume $\omega(z) \neq \{\infty\}$ since A. Eremenko and M. Lyubich [19] proved that the set of escaping points $I(f)$ has measure zero. The assumption $\operatorname{dist}(O^+(z), a) > \epsilon$ is permissible since X is contained in the countable union of the sets $\{z : \forall a \in A : \operatorname{dist}(a, O^+(z)) > \epsilon_n > 0\}$ where $\epsilon_n \to 0$. Since every $b \in B$ escapes exponentially, there exists some $\delta > 0$ with $|f^{n+1}(b)| \geq \exp(|f^n(b)|^{4\delta})$ for every $b \in B$ and n large enough. Let z_0 be a density point of X. Since the iterates of z_0 do not tend to infinity, there exists a convergent subsequence $f^{\beta(n)}(z_0)$, whose limit must be of the form $f^{n_0}(b)$ with $n_0 \in \mathbb{N} \cup \{0\}$ and $b \in B$. We may assume $n_0 \geq m_b := \max(\{m \in \mathbb{N} : f'(f^{m-1}(b)) = 0\} \cup \{0\})$. For all $n \in \mathbb{N}$ we define $\alpha(n) \geq \beta(n)$ smallest possible with

$$|f^{\alpha(n)}(z_0) - f^{\alpha(n)-\beta(n)+n_0}(b)| \;\;\geq\;\; |f^{\alpha(n)}(z_0)|^{1-\deg(Q)+3\delta}$$

and $B_n := B(f^{\alpha(n)}(z_0), |f^{\alpha(n)}(z_0)|^{1-\deg(Q)+2\delta})$. We will see that for large n the inverse branch g_n of $f^{\alpha(n)}$ mapping $f^{\alpha(n)}(z_0)$ to z_0 may be extended with uniformly bounded distortion to B_n. Furthermore we show that the density of X in B_n does not tend to one. This carries over to $g_n(B_n)$ by Lemma 2.5. Finally we show $\operatorname{diam}(g_n(B_n)) \to 0$, a contradiction to the choice of z_0 as a density point of X.

Since $\operatorname{sing}(f^{-1})$ is bounded, we can extend every branch of f^{-1} to every slit annulus around the origin whose inner radius is larger than the modulus of every singularity of the inverse. For large n the diameter of B_n and the definition of α assure that we can extend the branch of f^{-1} mapping $f^{\alpha(n)}(z_0)$ to $f^{\alpha(n)-1}(z_0)$ to B_n such that the distortion is bounded by a constant, which can be chosen arbitrarily close to one, and for the image B_n^{-1} it follows

$$\frac{\operatorname{diam}(B_n^{-1})}{|f^{\alpha(n)-1}(z_0) - f^{n_0+\alpha(n)-\beta(n)-1}(b)|} \leq |f^{\alpha(n)-1}(z_0)|^{-\frac{\delta}{2}} \tag{39}$$

and

$$B_n^{-1} \subset B\left(f^{n_0+\alpha(n)-\beta(n)-1}(b), 2|f^{n_0+\alpha(n)-\beta(n)-1}(b)|^{1-\deg(Q)+3\delta}\right).$$

Because of the choice of $n_0 \geq m_b$, we can apply the same argument as above $\alpha(n) - \beta(n) - 1$ times, which allows us extend the branch of $(f^{\alpha(n)-\beta(n)-1})^{-1}$, mapping $f^{n_0+\alpha(n)-\beta(n)-1}(b)$ to $f^{n_0}(b)$, to a ball around $f^{n_0+\alpha(n)-\beta(n)-1}(b)$ with a diameter of almost the modulus of its centre. Again this implies that its distortion on $B\left(f^{n_0+\alpha(n)-\beta(n)-1}(b), 2|f^{n_0+\alpha(n)-\beta(n)-1}(b)|^{1-\deg(Q)+3\delta}\right)$ tends to 1 as n tends to infinity. Moreover from (39) it follows that B_n^{-1} is mapped to a small ball $B(f^{\beta(n)}(z_0), r_n)$ with $r_n/|f^{\beta(n)}(z_0) - f^{n_0}(b)| \to 0$ as $n \to \infty$. Since $P(f)$ does not accumulate at $f^{n_0}(b)$, we can extend the branch of the inverse of $f^{\beta(n)}$ mapping $f^{\beta(n)}(z_0)$ to z_0 to $B(f^{\beta(n)}(z_0), |f^{\beta(n)}(z_0) - f^{n_0}(b)|)$. Thus g_n exists and its distortion tends to one as n tends to infinity.

For $s \in A \cup \{\infty\}$ we define G_s as in Lemma 4.3. Due to our choice of $\operatorname{diam}(B_n)$, the density of $\mathbb{C}\backslash G$ in B_n tends to zero as n tends to infinity. For M large enough and $s \in A\backslash O^+(B)$ we have $G_s \cap X = \emptyset$. For $s \in A \cap O^+(B) \cup \{\infty\}$ we can apply Lemma 4.3 to $\Gamma := \mathbb{C}\backslash X$ and obtain a family $\mathcal{F}$ of disjoint domains covering $B_n \cap G_s$ up to measure zero, such that the density of Γ in all of these is bounded below by some positive constant c. The diameter of these domains is much smaller than the diameter of B_n, such that we can neglect the ones intersecting the boundary of B_n. Thus the density of X in B_n does not tend to one.

It remains to show that $\operatorname{diam}(g_n(B_n)) \to 0$, which is equivalent to

$$\left|(f^{\alpha(n)})'(z_0)\right| |f^{\alpha(n)}(z_0)|^{\deg(Q)-1-2\delta} \to \infty.$$

Lemma 4.1 implies the existence of $R > 0$, such that

$$\frac{1}{2}|f'(z)| \leq \frac{|f(z) - \overline{s}(z)|}{|Q'(z)|} \leq 2|f'(z)|$$

for any $z \in \mathbb{C}$ with $|z| \geq R$. We chose $\epsilon' > 0$ small enough such that for $b \in O^+(B) \cap B(0, R)$ and $z \in B(s, \epsilon')$ and

$$m := \min\left\{m \in \mathbb{N} : |f^m(z) - f^m(s)| \geq |f^m(s)|^{1-\deg(Q)+3\delta}\right\}, \tag{40}$$

we have

$$|(f^m)'(z)| \geq \frac{2|f^m(z)|^{1-\deg(Q)+\frac{9\delta}{4}}}{|z - s|}. \tag{41}$$

If the distortion of f^m on $B(s,|z-s|)$ was bounded by a constant, a stronger statement would follow from the mean value theorem. There are however two obstructions for uniformly bounded distortion. There may be critical points on the orbit of s and the distortion of the last iterate $f|B(f^{m-1}(B(s,|z-s|))$ is not necessarily uniformly bounded. Thus in order to obtain the existence of such an ϵ' we need to consider these iterates separately. For ϵ' small enough the power series of f^{m_s} guarantees that the modulus of the derivative of f^{m_s} on $B(s,|z-s|)$ is bounded by $2|(f^{m_s})'(z)|$. Here m_s is again chosen such that $f^n(s)$ is no critical point of f for $n \geq m_s$. The choice of m guarantees that the distortion of $f^{m-m_s-1}|f^{m_s}(B(s,|z-s|))$ is arbitrarily close to one for ϵ' small enough and $f^{m-1}(B(s,|z-s|)) \subset B(f^{m-1}(s), 2|f^{m-1}(s)|^{1-\deg(Q)+3\delta})$. The definition of δ and the diameter of the latter set assures that the magnitude of $\mathrm{Re}(Q(f^{m-1}(x)))$ is the same for all $x \in B(s,|z-s|)$. By Lemma 4.1 this carries over to $|f^m(x)|$ and $|f'(f^{m-1}(x))|$ and gives estimate (41).

Since $\omega(z_0) \subset \overline{O^+(B)}$, there exists $j_0 \in \mathbb{N}$, such that for all $j \geq j_0$ the point $f^j(z_0)$ is not contained in the compact set $\overline{B(0,R)} \setminus \left(\bigcup_{n\in\mathbb{N}, s\in B} B(f^n(s),\epsilon')\right)$. With I we denote the set of $j_0 \leq j \in \mathbb{N}$ with the property that $f^j(z_0) \notin B(0,R)$ and $f^{j+1}(z_0) \in B(0,R)$. Then for every $j_0 \leq j \notin I$ with $|f^j(z_0)| \geq R$ we have $|f'(f^j(z_0))| \geq 1$. This follows since otherwise $|f^{j+1}(z) - s| \leq \frac{2}{|Q'(f^j(z_0))|}$ would imply $j \in I$, if we assume R to be larger than the modulus of every $s \in A$. Now for $j \in I$ we have

$$|f^{j+1}(z_0) - s| \;\leq\; \frac{2|f'(f^j(s))|}{|Q'(f^j(s))|}, \tag{42}$$

which is at most ϵ'. We assume $\epsilon' < \epsilon$, such that $s \in O^+(B)$. We choose m_j as in (40) for $z = f^{j+1}(s)$. Then we have $m_j \leq \alpha(n)$ and (41) together with (42) imply that

$$\begin{aligned} |(f^{m_j})'(f^j(z_0))| &= |f'(f^j(z_0))||(f^{m_j-1})'(f^{j+1}(z_0))| \\ &\geq |Q'(f^j(z_0))||f^{m_j+j}(z_0)|^{1-\deg(Q)+\frac{9\delta}{4}}. \end{aligned} \tag{43}$$

Let $\Delta_j := \min\{i \in I \cup \{\alpha(n)\} : i > j\} - m_j - j$. We may assume $j_0 \in I$ since otherwise we continue with $j_0 \leq j_0' \in I$ smallest possible. By the chain rule we get

$$\begin{aligned} &\left|\frac{(f^{\alpha(n)})'(z_0)}{f^{\alpha(n)}(z_0)^{1-\deg(Q)+2\delta}}\right| \\ =\; &\left|\frac{(f^{j_0})'(z_0)}{f^{\alpha(n)}(z_0)^{1-\deg(Q)+2\delta}}\right| \prod_{j\in I, j_0\leq j<\alpha(n)} |(f^{\Delta_j})'(f^{m_j+j}(z_0))||(f^{m_j})'(f^j(z_0))| \end{aligned}$$

With (43) we obtain a lower estimate of the product above, in which most of the factors in the product cancel each other out. More precisely, for each $j \in I$ except the first and the last in the product, the factor $|f^{m_j+j}(z_0)|$ only remains with a power of $\frac{9}{4}$. If $\Delta_j = 0$ it follows directly from (43) by considering the j-th and the $(j+1)$-st factor of the product together. Of course there the factor $|q|\deg(Q)$ appears as the leading coefficient of $|Q'|$. If $\Delta_j \neq 0$, we have $m_j + j \notin I$, such that

$$|(f^{\Delta_j})'(f^{m_j+j}(z_0))| \geq |f'(f^{m_j+j}(z_0))| \geq \frac{R - \max_{s\in A}|s|}{2}|Q'(f^{m_j+j}(z_0))|.$$

This implies the same as above with a factor, which is larger than $|q|\deg(Q)$ for R large enough.

Finally we know due to the definition of m_j, that for the last j in the product we have $j + m_j = \alpha(n)$ such that this factor cancels out with the denominator in front of the product up to $|f^{\alpha(n)}(z_0)^{\frac{\delta}{4}}|$ and we get

$$\begin{aligned}\left|\frac{(f^{\alpha(n)})'(z_0)}{f^{\alpha(n)}(z_0)^{1-\deg(Q)+2\delta}}\right| &\geq |f^{\alpha(n)}(z_0)|^{\frac{\delta}{4}}|(f^{j_0})'(z_0)||Q'(f^{j_0}(z_0))| \\ &\quad\times \prod_{\alpha(n)-m_j>j\in I} \frac{|q|\deg(Q)}{2}|f^{m_j+j}(z_0)|^{\frac{9\delta}{4}},\end{aligned}$$

which tends to infinity as n does so, and thus completes the proof. ■

Proof of Theorem 1.2. The assumptions on the singular orbits guarantee $P(f)' \cap J(f) = \emptyset$. Due to Theorems 2.18 and 2.19, an indifferent periodic point in the Julia set must be an accumulation point of $P(f)$. Therefore all periodic points in $J(f)$ are repelling. Due to Theorem 3.1, we have $\omega(z) \subset P(f)$ for almost every $z \in J(f)$. If $O^+(z)$ accumulates at a repelling periodic point, $\omega(z)$ also accumulates at this point. This follows from the fact that $O^+(z)$ accumulates at every compact annulus $\{z : r \leq |z - z_0| \leq 2|(f^p)'(z_0)|r\}$ if $r > 0$ small enough and p is the period of the repelling periodic point z_0. Thus for almost every $z \in J(f)$ we have $\omega(z) \subset \overline{O^+(B)}$ if B is the set of singularities, that escape exponentially. Now Lemma 4.4 implies that the set of points that do not accumulate at all asymptotic values has measure zero. This concludes the proof for the inclusion $\omega(z) \supset A$.

Now we assume that there exists some point $s \in B\backslash\overline{O^+(A)}$, such that the set $X^* := \{z \in J(f) : s \in \omega(z) \subset \overline{O^+(B)}\}$ has positive measure. Then the whole proof of Lemma 4.4 works identically, with X^* instead of X: the only difference being that at the point where Lemma 4.3 is used we now use the measure estimate of Theorem 3.1 instead. Since $O^+(T(f))$ is disjoint from $O^+(X^*)$, we get again that X^* contains no density point, contradicting the assumption of positive measure.

More precisely, instead of using the family $\mathcal{F}$ from Lemma 4.3 to see that the density of Γ in $B_n := B(f^{\alpha(n)}(z_0), |f^{\alpha(n)}(z_0)|^{1-\deg(Q)+2\delta})$ is bounded below, we use the family $\mathcal{S}$ from the proof for $\operatorname{meas}(F(f)) < \infty$ of Theorem 1.1 to see that the density of $T(f) := \{z : \omega(z) \subset \overline{O^+(A)}\}$ in B_n is bounded below. We recall that $\mathcal{S}$ was a family of disjoint open squares $S \subset \{z : \operatorname{dist}(z, \mathbb{C}\backslash G) \geq |z|^{1-\deg(Q)+\delta}\}$ with $\sup_{z\in S} |z|^{1-\deg(Q)} \leq \operatorname{diam}(S) \leq 4 \sup_{z\in S} |z|^{1-\deg(Q)}$, such that the measure of $\{z : \operatorname{dist}(z, \mathbb{C}\backslash G) \geq 2|z|^{1-\deg(Q)+\delta}\} \backslash \bigcup \mathcal{S}$ is zero. The density of $T(f)$ in any $B_n \supset S \in \mathcal{S}$ is, due to Theorem 3.1, very close to one for large n. In particular it is bounded below by some positive constant c. The choice of the diameter of B_n implies that the density of $\{z : \operatorname{dist}(z, \mathbb{C}\backslash G) \leq 2|z|^{1-\deg(Q)+\delta}\}$ tends to zero with n. The same is true for the union of those squares in $\mathcal{S}$, that intersect the boundary of B_n. This gives the estimate needed to proceed with the proof of Lemma 4.4. ∎

4.3 The recurrent case

Now we will drop the assumption that all finite asymptotic values escape exponentially and suppose that at least one of them is pre-periodic instead.

4.5 Theorem
Let f be of the form (36). Suppose that at least one finite asymptotic value is pre-periodic, and that every other finite singularity of the inverse is either also pre-periodic, escapes exponentially or is contained in an attractive Fatou component. Then either $\operatorname{meas}(J(f)) = 0$ *or* $J(f) = \mathbb{C}$ *and* $\omega(z) = \hat{\mathbb{C}}$ *for almost every* $z \in J(f)$.

Proof. We assume that the Julia set has positive measure and (ii) holds. Since $P(f)' \cap J(f) = \emptyset$, there are again no indifferent periodic points. From Lemma 4.4 we know that the orbit of almost every $z \in J(f)$ accumulates at least at one point in $P(f)$, which does not escape exponentially and thus has to be pre-periodic. By continuity, $O^+(z)$ accumulates at a repelling periodic point. As above this implies that $\omega(z)$ and therefore also $P(f)$ accumulates at this repelling periodic point. This is a contradiction. ∎

From this we can deduce the last result of the introduction.

Proof of Theorem 1.3. The assumptions on the singularities of the inverse imply that $P(f)' \subset \{\infty\}$ and $J(f) = \mathbb{C}$. If all finite asymptotic values escape exponentially we can apply Theorem 1.1 and obtain a set of positive measure, whose orbits accumulate only at the orbits of the asymptotic values and the point infinity. In particular the function is not recurrent. If the set of pre-periodic asymptotic values is non-empty and case (ii) of Theorem 2.24 holds, we can deduce from Theorem 4.5 that (i) of Theorem 2.24 is satisfied and f is recurrent and ergodic. ■

It is remarkable that for this restricted family the critical points have no influence on the question of whether the function is recurrent or not. This raises the question whether the conditions on the critical values are necessary. We do not claim that our conditions are sharp and indeed expect the same to be true for many other cases. In the next subsection, however, we will see that it will be difficult to dispose of all assumptions on the critical orbits.

4.4 Is there always a typical orbit?

Now we will explain our statement that a positive answer on the question of whether a polynomial may have a Julia set with positive measure would suggest a negative answer on the question of whether the orbit of almost every point in the Julia set of a function the type of (36) must accumulate at every asymptotic value and on the question of whether there always exists a typical orbit.

The crucial difference between transcendental entire functions and polynomials is their behaviour close to infinity. In other words, on bounded sets transcendental functions behave similarly as polynomials. This has been formalised by A. Douady and J. H. Hubbard with the introduction of polynomial-like mappings in [16]. A *polynomial-like* mapping is a proper analytic map $f : D_1 \to D_2$ such that $D_k \subset \mathbb{C}$ are simply connected and the closure of D_1 is contained in D_2. They show that a polynomial-like mapping is quasi-conformally conjugated to a polynomial.

Indeed for some functions f of type (36), it is possible to find domains, on which these are polynomial-like. Now suppose the polynomial P, to which f is conjugated, has a Julia set $J(P)$ of positive measure. Then the image $\phi(J(P))$ of this Julia set under the conjugation function ϕ is a subset of $J(f)$, whose points have bounded orbits. In particular they do not accumulate at an asymptotic value which escapes exponentially.

Concerning the question, whether the Julia set of polynomial may have positive measure, there have been attempts to give positive as well as negative answers. Recently many experts believe that there are polynomials with positive measure. One reason for this and probably the most recent progress on this field has its origin in an idea of A. Douady. One considers polynomials $P_\theta := \exp(i\theta)z + z^2$ in which $\theta \in [0, 2\pi)$. These have an indifferent fixed point at zero. The idea is to construct θ with $J(P_\theta) > 0$ as the limit of the sequence θ_n in which $\theta_n/(2\pi) \in \mathbb{Q}$ for odd n and $\theta_n/(2\pi)$ irrational of bounded type for even n. Thus P_{θ_n} has a Leau domain or a Siegel disc if n is odd or even, respectively. The crucial part is to arrange the θ_n such that $\text{meas}(K(P_{\theta_n})\backslash K(P_{\theta_{n+1}})) \to 0$ fast enough to ensure that the set that remains in the limit has still positive measure. Then this set may be identified as a subset of the Julia set of P_θ, for which zero is a Cremer point. Together with the idea of this construction, A. Douady formulated two conjectures ensuring the success. In [9] A. Chéritat proved one of these and formulated two new conjectures ensuring the other. These new conjectures are undermined by computer experiments, but have not been verified so far. Their construction does not depend on the first terms of the continued fraction expansion of $\theta/(2\pi)$. Thus if it works, it works for a dense set of θ in $[0, 2\pi)$.

One reason of summarising their method was to indicate that the most promising candidates for polynomials with a Julia set of positive measure are polynomials of the type P_θ for a dense set of $\theta \in [0, 2\pi)$. It is possible to show that some of these polynomials indeed occur as conjugated polynomials of polynomial-like mappings of the type (36), for which all finite asymptotic values escape exponentially.

To see this we consider the family of functions

$$f_\theta(z) := \exp\left(\frac{1}{2}\left(\left(\frac{\pi}{2} - \sin(\theta)\right) z^3 - \cos(\theta) z^2 + \left(\frac{3\pi}{2} - \sin(\theta)\right) z - \cos(\theta)\right)\right).$$

Then it is easy to see that the points i and $-i$ are indifferent fixed points, whose multipliers are $\exp(\theta i)$ and $\exp((\pi - \theta)i)$ respectively. Also the point zero is the only asymptotic value, its multiplicity is three and it escapes exponentially. One can see this by showing that for $t \in [0, \infty)$ we have $f_\theta(t) > \exp(t - 1/2)$. For the function $\exp(z + \kappa)$ with $\kappa > -1$ it is well known and easy to show that 0 escapes exponentially on the positive real axis. Hence the same is true for f_θ.

For the concrete value $\theta = \frac{\pi}{2}$ the function f_θ maps the ellipse

$$E := \left\{ z : \left(\frac{5}{6}\text{Re}\left(\exp(\frac{-\pi}{4} i)(z - \frac{7}{5} i)\right)\right)^2 + \left(\text{Im}\left(\exp(-\frac{\pi}{4} i)(z - \frac{7}{5} i)\right)\right)^2 < 1 \right\}$$

two to one to a set, in which its closure is contained. We do not prove this, but believe that Figure 6, where we plotted the boundary of the ellipse and its image, may convince the reader.

By continuity the same is true for f_θ with $\theta \in (\frac{\pi}{2} - \epsilon, \frac{\pi}{2} + \epsilon)$ and ϵ small enough. If there is a dense set of $\theta \in [0, 2\pi)$ with $\text{meas}(J(P_\theta)) > 0$, we can choose such a θ in this interval. From the above it follows that f_θ is polynomial-like as

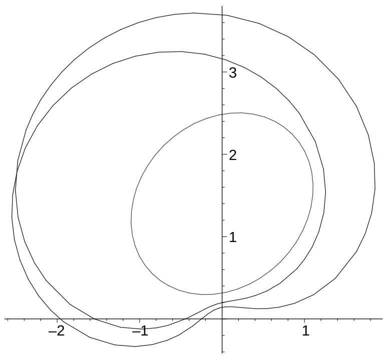

Figure 6: The boundary of the ellipse E and its image

a map from the component $i \in D_1 \subset E$ of $f_\theta^{-1}(E)$ to $D_2 := E$. Thus it is quasi-conformally conjugated to a polynomial P of degree two, i.e. there exists a quasi-conformal map $\phi : D_1 \to \mathbb{C}$ such that $f_\theta = \phi^{-1} \circ P \circ \phi$ on D_1 and $\phi(D_1) \supset J(P)$. The multiplier of an indifferent fixed point is invariant under conjugation by a homeomorphism as shown by V. A. Naishul in [37]. Thus up to conjugation by a Möbius transformation we have $P = P_\theta$. Now the set $S_1 := \phi^{-1}(J(P_\theta))$ is a subset of $J(f)$ with $O^+(z) \subset D_1$ for $z \in S_1$ and in particular $0 \notin \omega(z)$. Since ϕ is quasi-conformal, the assumption $J(P_\theta) > 0$ would imply that $\text{meas}(S_1) > 0$. Thus the answer on the question of whether almost every orbit must accumulate at every asymptotic value, would be negative. Since zero escapes exponentially, Theorem 3.1 implies that the set $S_2 := \{z : \omega(z) \subset \overline{O^+(0)}\}$ also has positive measure. Moreover, since $\omega(z_1) \cap \omega(z_2) = \emptyset$ for $z_2 \in S_1$ and $z_2 \in S_2$, there would not exist a typical orbit.

5 Other applications

5.1 The escaping set

As mentioned in Remark 3.2, one can use the Theorem 3.1 in order to obtain positive measure for the escaping set $I(f)$. For this purpose, we consider the following family containing the sine and cosine family, for which this result was proved by C. McMullen in [33].

5.1 Theorem
Let $f(z) := P(z)\exp(Q(z)) + \tilde{P}(z)\exp(\tilde{Q}(z))$ *for polynomials* $P, \tilde{P} \neq 0$ *and* $Q, \tilde{Q}$, *such that* $n := \deg(\tilde{Q}) = \deg(Q) \geq 0$ *and the arguments of their* $n-th$ *coefficients* $q, \tilde{q}$ *differ by some odd multiple of* $\frac{\pi}{n}$. *Then* $\mathrm{meas}(I(f)) > 0$. *If* $\tilde{Q} = -Q$ *and* $n \geq 3$ *then* $\mathrm{meas}(\mathbb{C}\backslash I(f)) < \infty$.

Sketch of proof. With the same arguments as in Theorem 1.1, one can show that for any $0 < \delta < \beta < 1$, $-1 < \delta_1 < n-1 < \delta_2$, M large enough, $A := \emptyset$ and $G := \{z : |\arg(z) - \frac{(2k+1)\pi - 2\arg(q)}{2n}| \leq |z|^{\delta-1}\}$ conditions (a) and (b) of Theorem 3.1 are satisfied, while condition (c) is trivial. The theorem implies the first part of the claim. The second part follows as in the proof of Theorem 1.1 choosing $1-n < \delta < \beta < -1$. ■

As an example, we consider the function

$$f(z) \;:=\; \exp(z^3) - \exp(-z^3).$$

Its Fatou set is not empty, since it contains a super attractive basin around zero. The theorem above, however, gives $0 < \mathrm{meas}(\mathbb{C}\backslash I(f)) < \infty$. In Figure 6 the Fatou set is black. The picture shows the part of the plane given by $\{z : |\mathrm{Re}(z)| \leq 2, |\mathrm{Im}(z)| \leq 2\}$.

5.2 Functions with rational Schwarzian derivative

The functions discussed in the previous chapter have rational Schwarzian derivative. The asymptotic behaviour of functions with this property is known very well, as described in the preliminaries. It is easy to see that a critical point of f is a pole of $S(f)$. Thus, functions with a rational Schwarzian derivative have only finitely many critical points. If $S(f)(z) = cz^n(1 + o(1))$ as $z \to \infty$ with $c \neq 0$ and $n \geq 0$, there are $n+2$ critical rays defined by $\arg z = \phi$ with $\arg c + (n+2)\phi = 0 (\mathrm{mod}\ 2\pi)$, which divide the complex plane into $n+2$ sectors, in which the asymptotic behaviour of f is known very well. Similarly to the proof of Theorem 1.1, one can show that conditions (a) and (b) of Theorem 3.1

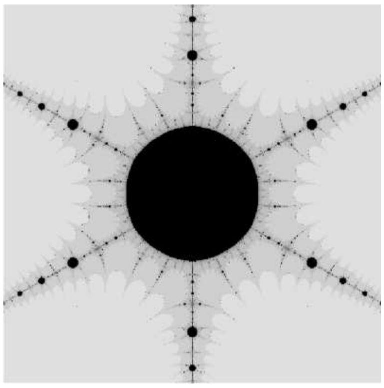

Figure 7: The Fatou set of $f(z) = \exp(z^3) - \exp(-z^3)$

are satisfied. If one of these asymptotic values happens to be ∞, points and also asymptotic values may escape exponentially inside the corresponding sector satisfying condition (c). However, these functions may have infinitely many poles, such that points and in particular asymptotic values can also escape exponentially without satisfying condition (c) of Theorem 3.1. This can only happen if an asymptotic value *a jumps from pole to pole* in the sense that there is a sequence of poles $p_n \to \infty$ and a subsequence f^{k_n} of the iterates with $|f^{k_n}(a) - p_n| \to 0$ very fast. The poles are, however, contained in small neighbourhoods around these critical rays. Thus, we can formulate another more geometric condition in order to guarantee condition (c). More precisely, we get the following.

5.2 Theorem
Let f be a meromorphic function with rational Schwarzian derivative, whose behaviour at infinity is of the form $cz^n(1+\mathrm{o}(1))$ with $c \neq 0$ and $n \geq -1$. Suppose that all finite asymptotic values s tend to ∞ under iteration and there exists some $\epsilon > 0$ such that $\left|\arg(f^m(s)) - \frac{2\pi k + \arg(c)}{n+2}\right| \geq |f^m(s)|^{\epsilon - \frac{n+2}{2}}$ for almost all $m \in \mathbb{N}$ and all $k \in \{0, 1, ..., n+1\}$. Then $\mathrm{meas}(J(f)) > 0$ and $\omega(z) \subset P(f)$ for almost every $z \in J(f)$. If $n \geq 3$, it follows that $\mathrm{meas}(F(f)) < \infty$.

Sketch of proof. The principle is exactly as the proof of Theorem 1.1. First one has to check that the properties of Theorem 3.1 are satisfied. This gives us measure estimates of $T(f)$ that imply case (ii) of Theorem 2.24. We obtain $T(f) \subset J(f)$ again from the absence of Baker and wandering domains, which once more follows from the finiteness of $\mathrm{sing}(f^{-1})$ (see Preliminaries). For meromorphic functions with polynomial Schwarzian derivative this has also been shown by R. L. Devaney and L. Keen in [15].

To check the properties we briefly summarise, how to obtain estimates of the asymptotic behaviour of f. We refer to the post-graduate notes of Jim Langley [30] for more details. As mentioned in the preliminaries, functions with rational Schwarzian derivative $S(f) = 2A$ coincide with those quotients f_1/f_2 of two linearly independent solutions of the differential equation

$$f_i'' + Af_i = 0.$$

For a critical ray with argument ϕ and $R_0 > 0$ large we define

$$Z(z) := \int_{2R_0 e^{i\phi}}^{z} A(t)^{1/2} = \frac{2c^{1/2}}{n+2} z^{(n+2)/2} \left(1 + \mathcal{O}\left(\frac{\ln|z|}{|z|}\right)\right), \quad \text{for } z \to \infty$$

in the set $\{z : R_0 \leq |z|, |\arg z - \phi| \leq \frac{2\pi}{n+2}\}$. Then it is easy to see that for any $\delta' > 0$ there exists R_1 large enough such that Z is univalent in the slightly smaller sector $\{z : |z| \geq R_1, \ |\arg(z) - \phi| < \frac{2\pi}{n+2} - \delta'\}$. With the Liouville transformation $W_i(Z) = A(z)^{1/4} f_i(z)$ and $F_0(Z) := A''(z)/4A(z)^2 - 5A'(z)^2/16A(z)^3$ we get

$$\frac{\partial^2 W_i}{\partial Z^2} + (1 - F_0(Z))W_i \;=\; 0.$$

Since $|F_0(Z)| = \mathcal{O}(|Z|^{-2})$, we can apply Theorem 2.14 for every $j \in \{1, .., n+2\}$ and every critical ray with argument ϕ_j and obtain constants $a_j, b_j, c_j, d_j \in \mathbb{C}$ and V_j, U_j, such that

$$f(z) \;=\; \frac{a_j U_j(Z) + b_j V_j(Z)}{c_j U_j(Z) + d_j V_j(Z)},$$

$$V_j(z) = \exp\left(\frac{2ic^{1/2}}{n+2} z^{\frac{n+2}{2}} \left(1 + \mathcal{O}\left(\frac{\ln|z|}{|z|}\right)\right)\right)(1 + \mathcal{O}(|z|^{\frac{-1}{2}}) \text{ and}$$

$$U_j(z) = \exp\left(\frac{-2ic^{1/2}}{n+2} z^{\frac{n+2}{2}} \left(1 + \mathcal{O}\left(\frac{\ln|z|}{|z|}\right)\right)\right)(1 + \mathcal{O}(|z|^{\frac{-1}{2}})$$

for $z \to \infty$ in $S_j := \{z : 1 \leq R_0 \leq |z|, \ |\arg(z) - \phi_j| < \frac{2\pi}{n+2} - \delta\}$. Thus f tends to the asymptotic value a_j/c_j in $S_j^+ := \{z \in S_j : \arg(z) > \phi_j|\}$ and f tends to b_j/d_j in $S_j^- := \{z \in S_j : \arg(z) < \phi_j\}$. If c_j or d_j happen to be zero, while a_j or b_j are not, we obtain a sector, on which f tends to ∞, such that points may escape exponentially in this sector. Taking the derivative, we get a similar estimate for the derivative. This implies, as in the proof of Theorem 1.1, that f satisfies the conditions of Theorem 3.1 for $0 < \delta < \epsilon$, $\delta - \frac{n}{2} < \beta < 1$, $-1 < \delta_1 < \frac{n}{2} < \delta_2$, M large enough, and $G := \bigcup_{1 \leq j \leq n+2} \{z \in S_j : |\mathrm{Im}(Z_j(z))| \geq |Z_j(z)|^{\frac{2\delta}{n+2}}\}$, where Z_j

is the upper change of coordinates Z for the sector S_j. In the case that $n \geq 3$, we can choose $\delta < \frac{n-2}{2}$. The proof for $\operatorname{meas}(F(f)) < \infty$ also carries over from that of Theorem 1.1. ■

As mentioned, another way of escaping exponentially is to jump from pole to pole. If this is the case for an asymptotic value, the theorem does not apply. The extremal case of an asymptotic value being a pre-pole (i.e. $f^n(z) = \infty$) is the case studied by B. Skorulski in [48]. It seems likely that his methods work for this family, which makes this family a promising candidate to study measurable dynamics under the assumption that a every singularity of the inverse either escapes exponentially, is a pre-pole or pre-periodic. This will be object of further studies.

6 The exponential family

The family of functions $f(z) = \lambda \exp(z)$ with $\lambda \in \mathbb{C}\backslash\{0\}$ is called the *exponential family*. Up to conjugation by an affine function, these are the only transcendental entire functions with exactly one finite singularity of the inverse. In analogy to the family of quadratic polynomials $z^2 + c$, this makes this family a natural and interesting object to study. Thus it is not surprising that it has been the object of much research, including its dynamics. For us this makes it an important source of answers for questions that arise automatically from the chapters before. In particular we are interested in the topological structure of the set $T(f)$ for a given function, especially in the cases in which it has full measure. Another interesting subject is the Hausdorff dimension of the remaining atypical set. One can also ask for the measure, dimension and topological structure of the set of parameter values for P and Q, for which the singularities show a given behaviour. Here one also might consider special sub-families and in particular one-parameter families. We will see that for the exponential family many answers are known providing ideas of what to expect for more general functions.

The dynamics of exponential functions has been studied for a long time. A very complete and recent survey on this topic is the thesis of L. Rempe [43]. The Julia set of $\exp(z)$ is the entire plane. This had already been conjectured by P. Fatou in [21] and was proved by M. Misiurewicz in [36]. For functions with $\lambda \neq 1$, however, this may change greatly. An interesting object to study, whose structure is quite independent of the parameter, is the escaping set $I(f)$. It has an interesting structure from the topological point of view, while its measure is always zero. Since $\partial I(f) = J(f)$, it is also suitable to provide information about the Julia set.

6.1 The escaping set

For a more in-depth discussion of $I(f)$ we refer to the work of R. Devaney, D. Schleicher and many others. E. g. one may look at the articles [8] and [45], where also more references may be found. To ease the study of the literature we give a brief overview, where we introduce the main terms, and afterwards use those that are most convenient, mainly according to personal taste.

The set $I(f)$ consists of an uncountable union of so called (external) rays or hairs plus some of their landing points. These rays or hairs are obtained by introducing symbolic dynamics. The plane is divided into the horizontal strips $S_k := \{z : (2k-1)\pi < \operatorname{Im}(z) - \arg(\lambda) \leq (2k+1)\pi\}$ for all $k \in \mathbb{Z}$. One says the point z has *(external) address* or *itinerary* $s \in \mathbb{Z}^{\mathbb{N}}$ if $f^n(z) \in S_{s_n}$ for all $n \in \mathbb{N}$. Obviously every point has an address, whereas not all sequences in $\mathbb{Z}^{\mathbb{N}}$ occur as addresses of points. Those that are realized are called *allowable* or, since they are characterised by the property that there exists $x > 0$ such that $2\pi|s_k| < \exp^{k-1}(x)$ for all $k \geq 1$, also *exponentially bounded*.

It turns out (see [12]) that for every exponentially bounded address s the set $I(s)$ of points, which escape to infinity with address s, contains the image of some curve, *external ray* or *hair* $g_s : (t_0, \infty) \to \mathbb{C}$ with $\lim_{t\to\infty} \mathrm{Re}(g_s(t)) = \infty$ for some $t_0 \geq 0$. One can obtain g_s as the limit of the functions $g_k := L_{s_1} \circ \ldots \circ L_{s_k} \circ F^k(t)$, where L_n is the branch of $\log(z) - \log(\lambda)$ with image in S_n, $F(t) := \exp(t) - 1$. If σ denotes the shift map with $\sigma(s)_n = s_{n+1}$ for all $n \in \mathbb{N}$, we obtain the functional equation $f(g_s(t)) = g_{\sigma(s)}(F(t))$ for all $t > t_0$ with which we can define g_s on (t_s, ∞) for $t_s := \limsup_{n\to\infty} F^{-(n-1)}(2\pi|s_n|)$ its *minimal potential*. If zero escapes with the address s, we have to allow g_s to take the value ∞ and consider so-called *broken* hairs, which are pre-images of g_s, seperately. This way (see [45]) one actually obtains all of $I(f)$ except some of the *endpoints* or *landing points* and broken hairs. We say a hair *lands* if a continuous extension of g_s to t_s is possible, and call $g_s(t_s) := \lim_{t\to t_s} g(t)$ its endpoint. Moreover, the endpoints that belong to $I(f)$ are exactly the endpoints of so-called *fast* addresses, which are addresses that are not *slow*, which they are if there are infinitely many n such that the shifted addresses $\sigma^n(s)$ are exponentially bounded with the same x. Hairs at fast addresses indeed land, though they are not the only ones.

Thus $I(f)$ consists of a so-called *Cantor Bouquet* of hairs labelled by the address of their points and f permutes these curves according to the shift map.

A well-known but quite surprising feature of the set $I(f)$ is the *dimension paradox*. The set of escaping endpoints has Hausdorff dimension two, while the union of the hairs has Hausdorff dimension one, although every escaping endpoint has a hair attached to it. This has been discovered by B. Karpinska in [28]. The part that the Hausdorff dimension of $I(f)$ is two is due to C. McMullen [33].

6.2 The typical orbit

Now we get back to our main field of interest. The only finite singularity of the inverse is the asymptotic value zero. If it is pre-periodic, it follows from Theorem 2.25 that $\omega(z) = \hat{\mathbb{C}}$ for almost every $z \in \mathbb{C}$. Concerning the escaping case, Theorem 1.2 translates as follows.

6.1 Corollary
Suppose that 0 escapes and $\mathrm{Re}(f^n(0)) \geq |f^n(0)|^\delta$ *for some* $\delta > 0$ *and all but a finite number of* $n \in \mathbb{N}$. *Then* $\omega(z) = P(f)$ *for almost every* $z \in \mathbb{C}$.

This is a generalisation of the result of M. Lyubich and M. Rees for the case $\lambda = 1$. The condition on the singular orbit forces the iterates of the singular value zero to eventually stay out of some parabola around the imaginary axis. This *parabola condition* is weaker than the condition that the asymptotic orbit eventually escapes inside a sector $\{z : |\arg(z)| < \theta\}$ for $\theta < \frac{\pi}{2}$, which we denote by the *sector condition*. Under this sector condition, the same result has been proven in [25]. In this case, M. Urbanski and A. Zdunik [51] even showed that the Hausdorff dimension of the set of atypical points (whose forward orbits accumulate somewhere outside the post-singular set) is strictly smaller than 2.

6.3 The parameter plane

If one considers the set I of escaping parameters, which are those λ for which the asymptotic value 0 escapes, one obtains a set with a similar structure as the set $I(f)$ for a particular parameter. Again it consists of hairs, which are now labelled by the address of the point zero, plus some endpoints. This has been shown by in [22] and [23]. Even the dimension paradox carries over to the parameter plane. That the set of endpoints has Hausdorff dimension two is shown in [41] and that the union of hairs has Hausdorff dimension one may be found in [4].
We are interested in the set of parameters λ, for which our parabola condition is satisfied. From [45] it follows that points on a hair eventually escape inside every parabola around the positive real axis. From the work of M. Förster it follows that the parabola condition is always satisfied if λ lies on a parameter hair. It is also satisfied for some but not all of their endpoints. If λ is the endpoint of the parameter hair with address s, then our condition is equivalent to the condition $|s_n| < F^{n-1}(t_s)^K$ for some $K > 0$ and almost every $n \in \mathbb{N}$. This may be deduced in the same way as the characterisation for the sector condition obtained in [43] (§3, 3.6.4, p. 42). Obviously a positive minimal potential is a necessary condition for our parabola condition.

From the proof in [41] it follows that the set of those λ satisfying the sector condition has Hausdorff dimension two, which is the analogue of the corresponding result in the dynamical plane from [33]. Since the sector condition is stronger than our parabola condition, we know that the set of parameters λ for which 0 escapes exponentially has Hausdorff dimension two.

6.4 The limiting behaviour

We know that $J(f) = \partial I(f)$ which follows easily for any entire function once one knows that $I(f) \neq \emptyset$, which has been shown by A. Eremenko in [18]. Thus the structure of the set $J(f) \backslash I(f)$ can be studied in terms of the accumulation set of the rays or hairs.

The easiest case of the accumulation set of a single hair is the case for which the hair lands. As mentioned above, it is known that hairs with fast addresses land at escaping points and vice-versa. Also hairs with periodic addresses land at periodic points and basically vice-versa, in the sense that every periodic point whose orbit does not intersect S_0 is the landing point of a periodic hair. This carries over to pre-periodicity. Moreover, if an address contains no zeros, the hair lands. All of this may be found in the thesis of L. Rempe [43].

However, forward orbits of points whose addresses do not contain infinitely many zeros cannot accumulate at zero, and thus they belong neither to the set $T(f)$ nor to the transitive set. Unfortunately, there is little known about the limiting behaviour of hairs whose addresses contain infinitely many zeros. The results that are known restrict mainly to the exponential map itself. All results stated from now on refer to the case $\lambda = 1$.

In [12] R. Devaney studied the set $I(s)$ for $s = (0, 0, 0, ...)$ which contains a broken ray and its preimages in S_0. He showed that the accumulation set of $I(s)$ may be compactified to an *indecomposable continuum*, which is a continuum that cannot be written as the union of two proper sub-continua. The ω-limit set of points z in this indecomposable continuum, except those points on the ray and one single repelling periodic point, has exactly the form we are looking for, i.e. $\omega(z) = \overline{O^+(0)}$.

If a bounded address is not constantly zero, but contains infinitely many blocks of zeros whose lengths are increasing sufficiently fast, something similar happens as shown by R. Devaney and X. Jarque in [13]. Here a *block* of length m is a sequence of m integers $t_1, .., t_m$. The length of a block will be indicated as an exponent. With 0^m and 1^m we denote blocks of m zeros and m ones respectively. With the term t^m we denote a block of m non-zero integers.

6.2 Theorem (Jarque, Devaney)
Given an $M > 0$, a sequence (m_k) in $\mathbb{N}$ and a sequence (t^{m_k}) of blocks t^{m_k} of length m_k and $|t_l^{m_k}| < M$ for $1 \leq l \leq m_k$, there is a sequence (n_j) such that the set $\overline{I(s)}$ for the address $s := (t^{m_1}, 0^{n_1}, t^{m_2}, 0^{n_2}, ...)$ is an indecomposable continuum in the Riemann sphere. Moreover, for all $z \in \overline{I(s)} \backslash I(\exp)$, except one single point with a bounded orbit, we have $\omega(z) = \overline{O^+(0)}$.

Thus some subsets of our $T(\exp)$ can be identified as indecomposable continua, occurring as accumulation sets of non-landing hairs of addresses that have blocks of zeros. It seems difficult to show that the whole set $T(\exp)$ is the union of such indecomposable continua. More realistic is the aim to show the same for the subset of $T(\exp)$ which we constructed in the proof of Theorem 3.1, since our construction gives us information on the dynamics on this set, which we denote by $T^*(\exp)$. In particular one can deduce from the proof that the address $s := (t^{m_1}, 0^{n_1}, t^{m_2}, 0^{n_2}, ...)$ of a point in $T^*(\exp)\backslash I(\exp)$ has the property that $n_k \geq (\sum_{j=1}^{k} m_j) + k$. It would therefore be a natural idea to generalise the result of R. Devaney and X. Jarque for every address with this property. There are however two principle obstructions. One needs to quantify the length of the blocks of zeros necessary in the proof and one needs to consider unbounded addresses. While the first obstruction does not require crucial changes in the proof, the second appears to do so.

The result of R. Devaney and X. Jarque raised the question of whether the same happens for every address that contains blocks of zeros of unbounded length. A negative answer was given by L. Rempe in [43], who showed the following.

6.3 Theorem (Rempe)
Suppose that (n_k) is a sequence of positive integers. Then there exists a sequence (m_k) such that the external ray at address $s := (1^{m_1}, 0^{n_1}, 1^{m_2}, ...)$ lands.

Thus the limiting behaviour depends on how fast the length of the blocks of zeros increases. Studying the proof in [43] with respect to the question of how many ones are necessary, one sees that this result may be generalised as follows.

6.4 Theorem
Let $s = (t^{m_1}, 0^{n_1}, t^{m_2}, 0^{n_2}, ...)$ be exponentially bounded and $n_k < (\sum_{j=1}^{k} m_j) - k$ for all $k \in \mathbb{N}$. Then the hair at address s lands.

We give a sketch of the proof that only uses results stated above. Of course idea is same as in the thesis of L. Rempe.

Sketch of proof. We consider the sequence (s^k) of addresses, where s^k arises from s in the following way: from the k-th block of zeros to the right, all zeros are replaced by ones. More precisely, we define

$$s_j^k := \begin{cases} 1 & \text{if} \qquad s_j = 0 \text{ and } l \geq p_k \\ s_j & \text{otherwise} \end{cases}, \tag{44}$$

where $p_k := m_k + 1 + \sum_{j=1}^{k-1}(m_j + n_j)$ is the position of the k-th block of zeros. In other words we have $s^k = (t^{m_1}, 0^{n_1}, t^{m_2}, 0^{n_2}, ..., t^{m_k}, 1^{n_k}, t^{m_{k+1}}, 1^{n_{k+1}}...)$. Since all s^k contain only finitely many zeros, the hairs land, and since they all have the same tail, we know $t_{s^k} = t_s$ for all $k \in \mathbb{N}$. We will show that the condition $n_k < (\sum_{j=1}^{k} m_j) - k$ assures that s^{k+1} is ϵ_k-close to s^k for a sequence (ϵ_k) with $\sum \epsilon_k < \infty$, where two addresses s and s' are called ϵ-close if $|g_s(t) - g_{s'}(t)| < \epsilon$ for all t. This implies the landing of the hair at address s.

More precisely, we show that s^{k+1} is $\frac{3}{\pi^{k-1}}$-close to s^k. For $t_0 > t_s$ and every $0 \leq l \leq \sum_{j=1}^{k} m_j + n_j$ we consider the points $z_l := \exp^{(\sum_{j=1}^{k} m_j + n_j) - l}(g_{s^{k+1}}(t_0))$ and

$w_l := \exp^{(\sum_{j=1}^k m_j+n_j)-l}(g_{s^k}(t_0))$ and show that $|z_{\sum_{j=1}^k m_j+n_j} - w_{\sum_{j=1}^k m_j+n_j}| \leq \frac{3}{\pi^{k-1}}$. The hairs g_{s^k} and $g_{s^{k+1}}$ are pre-images under the map $\exp^{\sum_{j=1}^k m_j+n_j}$ of the hair with address $\sigma^{\sum_{j=1}^k m_j+n_j}(s_k) = (t^{m_{k+1}}, 1^{n_{k+1}}, t^{m_{k+2}}, 1^{n_{k+2}}...)$. However, one has to consider different branches of the logarithm, namely $L^{p_k-1} \circ L_1^{n_k}$ and $L^{p_k-1} \circ L_0^{n_k}$, where $L^{p_k-1} := L_{s_1} \circ ... \circ L_{s_{p_k-1}}$. Thus we have $z_0 = w_0$. As long as $\mathrm{Re}(z_l) \geq 1$ for all $0 \leq l \leq n_k$, the real parts of w_l and z_l are almost the same and the imaginary part of w_l is about 2π larger than that of z_l, e.g. it is easy to check that $\mathrm{Re}(z_l) \leq \mathrm{Re}(w_l) \leq \mathrm{Re}(z_l) + \pi$ and $\mathrm{Im}(w_l) \in (\frac{3\pi}{2}, \frac{5\pi}{2})$, such that we have $|z_l - w_l| \leq 3\pi$. Now any branch of $\log^n$ is a contraction on $\{z : |\mathrm{Im}(z)| > \pi\}$. More precisely, one can show that $|(\exp^n)'(z)| \geq \pi$ for all $n \in \mathbb{N}$ and $z \in \mathbb{C}$ with $|\mathrm{Im}(\exp^n(z)) \geq \pi|$. We divide L^{p_k-1} into the parts from one non-zero entry to the next by considering $l_1 := n_k + 1$ and

$$l_n := \min\{l : l > l_{n-1}, s_{1+(\sum_{j=1}^k m_j+n_j)-l} \neq 0\}$$

for $2 \leq n \leq \sum_{j=1}^k m_j$. Obviously we have $\sum_{j=1}^k m_j \geq k$ such that this argument gives $|z_{\sum_{j=1}^k m_j+n_j} - w_{\sum_{j=1}^k m_j+n_j}| \leq \frac{3}{\pi^{k-1}}$. The only way that $|z_l - w_l|$ becomes large is if $\mathrm{Re}(z_l) < 1$ for some $0 \leq l \leq n_k$, such that $|z_l|$ is very small and z_{l+1} lies far away in the left half-plane. In this case the estimates for the real and the imaginary parts of the points $w_0, ..., w_{l-1}$ from above imply that w_l is lying in the square $\{z : \mathrm{Re}(z) \in (1, 1+\pi), \mathrm{Im}(z) \in (\frac{3\pi}{2}, \frac{5\pi}{2})\}$. Since $z_0 \notin S_0$, we know that $|\mathrm{Im}(z_0)| > \pi$. With the mean value theorem one can deduce from this that $|z_l| \geq \frac{\pi}{|(\exp^l)'(0))|}$, which becomes smallest if $l = n_k$. Then $\mathrm{Re}(z_{n_k+1})$ is negative and $|\mathrm{Re}(z_{n_k+1})| - \mathrm{Re}(w_{n_k+1}) \leq -\log(|z_{n_k}|) \leq |\log((\exp^{n_k})'(0))|$. For the imaginary part we use the trivial estimate $|\mathrm{Im}(z_{n_k+1}) - \mathrm{Im}(w_{n_k+1})| < 2\pi$. As before we consider the non-zero entries l_n for $1 \leq n \leq \sum_{j=1}^k m_j$. For $1 \leq n \leq n_k - 1$ one can show inductively that $||\mathrm{Re}(z_{l_n})| - |\mathrm{Re}(w_{l_n})|| \leq |\log^n((\exp^{n_k})'(0))|$. From this we can deduce that $|z_{l_{n_k}} - w_{l_{n_k}}| \leq 3\pi$, which is not sharp but gives similar estimates as above. With the same contraction argument as above we inductively obtain that $|\mathrm{Re}(w_{l_n}) - \mathrm{Re}(z_{l_n})| \leq \frac{3}{\pi^{n-n_k}}$ for $n_k \leq n \leq \sum_{j=1}^k m_j + n_j$. Then the condition $n_k < (\sum_{j=1}^k m_j + n_j) - k$ from the theorem finally implies that $|z_{\sum_{j=1}^k m_j+n_j} - w_{\sum_{j=1}^k m_j+n_j}| \leq \frac{3}{\pi^{k-1}}$. ∎

6.5 Remark
The condition $n_k < (\sum_{j=1}^k m_j) - k$ *for all* $k \in \mathbb{N}$ *may obviously be replaced by the condition* $n_k < (\sum_{j=1}^k m_j) - k + C$ *for some* $C > 0$ *and all* $k \in \mathbb{N}$ *and also by the condition* $n_k < (\sum_{j=1}^k m_j) - ck + C$ *for some* $c, C > 0$ *and all* $k \in \mathbb{N}$. *This requires only minor changes in the proof.*

We note that the addresses of points in $T^*(\exp)$ fail the condition only slightly. The fact that we were not able to show the landing of the hairs with these addresses is one more reason to expect the limit sets of these hairs to be indecomposable continua. Conversely, the construction of indecomposable continua for these hairs, would make our condition in Theorem 6.4 somehow sharp in the sense that it would squeeze the range in which the limiting behaviour of an address remains unknown to $(\sum_{j=1}^{k} m_j) - k \leq n_k \leq (\sum_{j=1}^{k} m_j) + k$ for all $k \in \mathbb{N}$. However this requires further research. For now I only wish to present all this as a method to study the topological structure $T(\exp)$.

I would also like to mention that indecomposable continua also occur as limit sets of hairs for so-called *Misiurewicz parameters*, for which 0 is pre-periodic. This has been discovered by D. Schleicher and proved in much generality in [43]. A concrete construction for the function $f(z) = 2\pi i \exp(z)$ may be found in [14]. For these, however, it appears to be difficult to determine the dynamics of their points. One can show that the ω-limit set is not contained in $P(f)$. Since the addresses considered in [14] consist only of zeros and ones, it is not possible to identify these points as elements of the transitive set, which in this case has full measure. To obtain indecomposable continua that are contained in $Tr(f)$, one has to generalise the construction for unbounded addresses, which may be difficult.

Nomenclature

$\mathbb{C}$	complex plane
$\mathbb{C}^*$	punctured plane $\mathbb{C}\backslash\{0\}$
$\hat{\mathbb{C}}$	Riemann sphere $\mathbb{C}\cup\{\infty\}$
$\mathbb{N}$	set of natural numbers
$\mathbb{N}_0$	$\mathbb{N}\cup\{0\}$
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{Z}$	set of integers
$\overline{S}$	closure of the set S, page 7
∂S	boundary of the set S, page 7
S'	set of accumulation points of S, page 14
$\mathrm{diam}(M)$	diameter $\sup_{z,w\in M}\lvert z-w\rvert$, page 5
dist	Euclidean distance in the plane, page 5
meas	plane Lebesgue measure, page 5
$B(z,r)$	open ball $\{w : \lvert w-z\rvert < r\}$, page 5
$B(M,r)$	$\bigcup_{z\in M} B(z,r)$, page 5
$D(r)$	$\mathbb{C}\backslash B(0,r)$, page 5
$S(z,r)$	square $\{w\in\mathbb{C} : \mathrm{Re}(z-w) < r \text{ and } \mathrm{Re}(z-w) < r\}$, page 5
rS	square $\{w\in\mathbb{C} : (w-z)/r+z\in S\}$ for square S with centre z, page 5
f^k	k-th iterate of f, page 5
$f^{(k)}$	k-th derivative of f, page 5
$\mathrm{sing}(f^{-1})$	set of singularities of the inverse function, page 7
$F(f)$	Fatou set of f, page 11

$J(f)$	Julia set of f, page 11
$S(f)$	Schwarzian derivative $\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$, page 8
$O^+(z)$	forward orbit $\{f^n(z) : n \in \mathbb{N}_0$ such that $f^n(z)$ is defined$\}$, page 11
$O^-(z)$	backward orbit $\{f^{-n}(z) : n \in \mathbb{N}_0\}$, page 11
$O(z)$	orbit $O^+(z) \cup O^-(z)$, page 11
$O^{(\pm)}(A)$	$\bigcup_{z\in A} O^{(\pm)}(z)$, page 11
$\omega(z)$	set of accumulation points of the sequence $(f^n(z))$, page 14
$P(f)$	post-singular set $\overline{O^+(\operatorname{sing}(f^{-1}))}$, page 13
$T(f)$	$\{z : \omega(z) \subset \overline{O^+(A)}\}$, page 19
$\mathcal{B}$	$\{f : \mathbb{C} \to \hat{\mathbb{C}}$ meromorphic such that $\operatorname{sing}(f^{-1}) \cap \mathbb{C}$ is bounded$\}$, page 15
$E(f)$	set of exceptional values $\{z : O^-(z)$ is finite$\}$, page 12
$I(f)$	escaping set $\{z : f^n(z) \to \infty\}$, page 15
$Tr(f)$	transitive set $\{z : \omega(z) \supset J(f)\}$, page 12
σ	shift map $\sigma(s)_n := s_{n+1}$, page 52
$\theta(r)$	linear measure of set $\{\theta \in (0, 2\pi) : \lvert f(r\exp(\theta i))\rvert < R\}$, page 15
$F(t)$	$\exp(t) - 1$, page 52
K_c	constant given by Lemma 2.4 , page 6
m_s	$\max(\{m \in \mathbb{N} : f'(f^{m-1}(s)) = 0\} \cup \{0\})$, page 20
k_s	$\min\{k \in \mathbb{N} : (f^{m_s})^{(k)}(s) \neq 0\}$, page 20

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Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit, abgesehen von der Beratung durch meinen akademischen Lehrer, in Inhalt und Form selbst angefertigt und keine anderen als die angegeben Hilfsmittel verwendet habe. Einen Artikel mit Resultaten dieser Arbeit habe ich bei der Zeitschrift Fundamenta Matematicae zur Veröffentlichung eingereicht und beim digitalen Archiv arXiv.org veröffentlicht. Diese Arbeit hat weder ganz noch zum Teil an anderer Stelle im Rahmen eines Prüfungsverfahrens vorgelegen. Desweiteren versichere ich, noch keinen Promotionsversuch unternommen zu haben.

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