Discrepancy of Arithmetic Structures

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Preface

Discrepancy theory is divided into two main parts—the geometric and the combinatorial discrepancy theory. The aim of the first is to distribute n points in some space as balanced as possible with respect to a special set of (mostly geometrically defined) subsets. The discrepancy measures how far an optimal (that means best possible) distribution deviates from a perfectly uniform distribution.

In this thesis we investigate problems that are located in the second part—the combinatorial discrepancy. For a given finite *hypergraph* $\mathcal{H} = (V, \mathcal{E})$ (the elements of the finite set V are called *vertices* and those of $\mathcal{E} \subseteq 2^V$ *hyperedges*) one likes to find a partition of V into two sets such that this partition divides every hyperedge into two preferably equal parts. We can express the partition through a color-function, which assigns to every vertex the color *red* or *green*. Then the goal is to color V in such a way that every hyperedge contains the same number of red and green vertices. Here the discrepancy measures the distance of an optimal coloring from a total equipartition in every hyperedge.

A common way to quantify the discrepancy of a hypergraph is to represent the colors *red* and *green* by the integers -1 and 1. Then for a given coloring $\chi : V \to \{-1, 1\}$ and every hyperedge $E \in \mathcal{E}$ the imbalance of E can be expressed as $|\chi(E)|$, where $\chi(E) = \sum_{x \in E} \chi(x)$. This value is the excess of 1's or -1's in E. The discrepancy of $\mathcal{H}$ with respect to a coloring χ is the maximal imbalance of all hyperedges. If we minimize this over all possible colorings χ of $\mathcal{H}$, we get the discrepancy of $\mathcal{H}$.

A natural way to extend the combinatorial discrepancy theory was to use colorings with an arbitrary but fixed number c of colors. Once again one likes to color the vertices of $\mathcal{H}$ in such a way that in every hyperedge E every color appears in the same amount. The quantification of the discrepancy of a hypergraph is more difficult than for two colors. Doerr and Srivastav [DS03] overcome this problem by taking a special set of c c–dimensional vectors as set of colors. Then the imbalance of a hyperedge E with respect to a given coloring χ can be expressed as $\| \sum_{x \in E} \chi(x) \|_\infty$. The discrepancy of $\mathcal{H}$ is defined in the same way as for two colors using the imbalances of the hyperedges. We give an exact definition in Chapter 1.

The hypergraph of arithmetic progressions in $[N] := \{1, 2, \ldots N\}$ is one of the classical hypergraphs in combinatorial discrepancy theory. In Chapter 2 we first give an overview of the history of determing its discrepancy of order $\Theta(N^{1/4})$. Afterwards we give a brief introduction to Fourier analysis on locally compact Abelian groups. From this we derive a method for determing lower bounds for the discrepancy of hypergraphs that are related to the classical hypergraph of arithmetic progressions in $[N]$. In the rest of this chapter we look for the discrepancy of several such hypergraphs. For determing lower bounds we use this Fourier analytic method. For the upper bound proofs in this chapter we will use the probabilistic method, the partial coloring method, and the recursive coloring method.

The first hypergraph that we investigate in Chapter 2 is the hypergraph of sums of arithmetic progressions in $[N]$. The hyperedges of this hypergraph are unlike in the classical hypergraph not the arithmetic progressions but sums of k arithmetic progressions. We give a lower bound for the discrepancy of this hypergraph of order $\Omega(N^{k/(2k+2)})$. Doerr, Srivastav, and Wehr [DSW04] generalized the classical hypergraph of arithmetic progressions to higher dimensions. That means they investigated the hypergraph of Cartesian products of d arithmetic progressions in $[N]^d$ and determined the order of its discrepancy $\Theta(N^{d/4})$. Motivated by this result, we look for the discrepancy of the subhypergraph of all Cartesian products of d arithmetic progressions with one common difference. We establish a lower bound of order $\Omega(N^{d/(2d+2)})$ for the discrepancy of this hypergraph and prove that this bound is almost tight.

Afterwards we study the hypergraph of arithmetic progressions in $\mathbb{Z}_p$ (p prime) and the hypergraph of Bohr neighborhoods in $\mathbb{Z}_p$. These hypergraphs are examples of a class of hypergraphs in $\mathbb{Z}_p$ that we will call *$\mathbb{Z}_p$–invariant hypergraphs*. We give a general lower bound for the discrepancy of $\mathbb{Z}_p$–invariant hypergraphs. Furthermore, we investigate the discrepancy of the hypergraph of all arithmetic progressions in $\mathbb{Z}_p$ that are centered in $0 \in \mathbb{Z}_p$.

In Chapter 3 we look for the connection between the discrepancy of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ and the discrepancy of its d–fold symmetric product $\Delta^d\mathcal{H} := (V^d, \{E^d \mid E \in \mathcal{E}\})$. In the 2–color-case one can show that the discrepancy of $\Delta^d\mathcal{H}$ is bounded by the discrepancy of $\mathcal{H}$. But we will see that things are not that easy, if the number of colors c is larger than 2. We give certain conditions for the values of the dimension d and the number of colors c under which the estimate for the 2–color-case stays valid. For all other values of c and d that do not fulfill these conditions there are hypergraphs with arbitrary large discrepancy such that $\operatorname{disc}(\Delta^d\mathcal{H}, c)$ is much larger than $\operatorname{disc}(\mathcal{H}, c)$. We also show an upper bound of the kind $\operatorname{disc}(\Delta^d\mathcal{H}, c) \leq \operatorname{disc}(\Delta^{d'}\mathcal{H}, c)$ for some $d' < d$ under certain conditions.

The declustering problem is the subject of Chapter 4. Our aim is to allocate data that is given in a d–dimensional grid on M, $M \geq 3$, parallel working storage devices such that typical requests find their data evenly distributed on the devices. The idea is to reduce the time needed for the retrieval of the data. We will see that the declustering problem is indeed a discrepancy problem. In this situation the discrepancy measures for a given assignment (called *declustering scheme*) the largest deviation between the amount of data of a request that is stored on a single device and its average value (the $1/M$–fraction of the whole request). But for the fast retrieval of data it is no problem if for a request some storage devices have a workload less than the average workload. Only positive deviations are of interest. Thus, we have to modify the discrepancy function in this case. We call this modified discrepancy the *positive discrepancy*. Instead of all deviations, we maximize here only over all positive deviations. The exact definition of the positive discrepancy can be found in Chapter 1.

We are investigating the case of rectangular requests to a d–dimensional grid. Using deep results from geometric discrepancy theory, we get a declustering scheme that has a worst case additive error of order $O_d(\log^{d-1} M)$. There is a declustering scheme known before with the same upper bound that works only if M is a power of a prime p and the dimension d is less or equal p. In contrast to this our scheme can be applied for all M and all dimensions $d \leq q_1 + 1$, where q_1 is the smallest prime power in the canonical factorization of M into prime powers. In particular, our scheme can be applied for arbitrary M in dimension $d = 2$ and $d = 3$, and if M is a power of 2, only $d \leq M + 1$ is needed. These are very interesting cases for applications.

Additionally, we give a lower bound for the worst case additive error of declustering schemes. We prove the lower bound of $\Omega_d(\log^{(d-1)/2} M)$. This bound was already stated before in an article of Sinha, Bhatia, and Chen [SBC03]. But unfortunately, their proof contains an error. We show where this error occurs and give a correct proof of the same bound. At the end of this chapter we give an alternative proof for the lower bound.

In the last Chapter we are interested in the discrepancy of the following hypergraph $\mathcal{H} = (\mathbb{F}_q^r, \mathcal{E})$. The set of vertices is the r–dimensional vector space $\mathbb{F}_q^r$ over the finite field $\mathbb{F}_q$, where q is a power of a prime p. The set of hyperedges is the set of all linear hyperplanes of $\mathbb{F}_q^r$, i.e., the set of all subspaces of codimension 1. Put $n := |\mathbb{F}_q^r| = q^r$ and $z := \frac{(q-1) \mod c}{c}$. Using an extension of the eigenvalue bound to the c–color discrepancy, we state the lower bound of order $\Omega_q(\sqrt{nz(1-z)})$. In the case $c|(q-1)$ this lower bound becomes trivial. But this is not surprising, because in this situation, one can easily calculate that the c–color discrepancy is exactly $\frac{c-1}{c}$. We show that the discrepancy of $\mathcal{H}$ is of order $\Theta_q(\sqrt{nz(1-z)})$. For the proof of the upper bound we use the bounded VC-dimension of $\mathcal{H}$.

Having determined the order of the c–color discrepancy of $\mathcal{H}$, we investigate the positive c–color discrepancy of $\mathcal{H}$. Trivially the upper bound of the c–color discrepancy is also an upper bound for the positive c–color discrepancy. And an easy argument shows that an $\frac{1}{c-1}$–fraction of the lower discrepancy bound is a lower bound for the positive c–color discrepancy. But this results in a gap of order c between the lower and the upper bound. Using Fourier analysis on the additive group $\mathbb{F}_q^r$, we shorten this gap to a factor of $\sqrt{c}$. That means, we proof a lower bound for the positive c–color discrepancy of $\mathcal{H}$ of order $\Omega_q(\sqrt{\frac{nz(1-z)}{c}})$. For a large number of colors, precisely if $c \geq qn^{1/3}$, we close the gap, proving that the order of the positive c–color discrepancy in this situation is $\Theta_q(\sqrt{\frac{n}{c}})$.

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Chapter 1

Introduction to Discrepancy Theory

In this chapter we give a short introduction to the field of discrepancy theory. As mentioned in the preface, this field is divided into to main parts, the geometric and the combinatorial discrepancy theory. Although we are only interested in combinatorial discrepancy problems in this thesis, we will also use results from geometric discrepancy theory in a combinatorial setting that is geometrically motivated. This will happen for the declustering problem in Chapter 4. Therefore, we start here with the geometric discrepancy.

1.1 Geometric Discrepancy Theory

The basic question in geometric discrepancy theory is "How well can we distribute n points in a given space with respect to a set of (geometrically defined) subsets of this space?" Many interesting problems arise from this question, but we restrict ourselves here to the following problem:

> Distribute n points in the d–dimensional unit-cube $[0,1]^d$ such that for all axis-parallel rectangles $R = [x_1, y_1] \times [x_2, y_2] \times \ldots \times [x_d, y_d]$ the number of points in R deviates not to much from the wanted fraction $n \cdot \mathrm{vol}(R)$ of all points, where $\mathrm{vol}(R) = \prod_{i=1}^{d}(y_i - x_i)$ is the volume of R.

Let us make this more precise. For a set $\mathcal{P} \subseteq [0,1]^d$ and a rectangle R we define

$$\mathrm{D}(\mathcal{P}, R) := \left| |\mathcal{P} \cap R| - n \cdot \mathrm{vol}(R) \right|.$$

Let $\mathcal{R}_d$ denote the set of all axis-parallel rectangles in the d–dimensional unit-cube $[0,1]^d$. Let us also fix the following set $\mathcal{C}_d := \{C_x \mid x \in [0,1]^d\}$ of so called *corners*, where

$C_x := [0, x_1] \times [0, x_2] \times \ldots \times [0, x_d]$. We define the *discrepancy of $\mathcal{P}$ for axis-parallel rectangles* by

$$\mathrm{D}(\mathcal{P}, \mathcal{R}_d) := \sup_{R \in \mathcal{R}_d} \mathrm{D}(\mathcal{P}, R).$$

Finally the *discrepancy for n–point sets* in the unit-cube $[0,1]^d$ with respect to axis-parallel rectangles is defined as

$$\mathrm{D}(n, \mathcal{R}_d) := \inf_{\substack{\mathcal{P} \subseteq [0,1]^d \\ |\mathcal{P}| = n}} \mathrm{D}(\mathcal{P}, \mathcal{R}_d).$$

This discrepancy is also called the *L_∞–discrepancy.* The *L_p–discrepancy* $(1 \leq p < \infty)$ for a set $\mathcal{P} \subseteq [0,1]^d$ for example for corners is defined as $\mathrm{D}_p(\mathcal{P}, \mathcal{C}_d) := \left(\int_{[0,1]^d} \mathrm{D}(\mathcal{P}, C_x)^p \, \mathrm{dx} \right)^{1/p}$.

It is easy to see that the one-dimensional case is trivial, since the set $\{\frac{1}{n+1}, \frac{2}{n+1}, \ldots, \frac{n}{n+1}\}$ is the n–point set with the best-possible discrepancy. We get $\mathrm{D}(n, \mathcal{R}_1) = 1$. But for dimension $d \geq 2$ things become much more complicated. In the year 1954 Roth [Rot54] proved a lower bound for the L_2–discrepancy with respect to corners. A direct consequence is the following theorem.

Theorem 1.1 (Roth)**.** *Let $d \geq 2$. The discrepancy for n–point sets with respect to axis-parallel rectangles fulfills*

$$\mathrm{D}(n, \mathcal{R}_d) = \Omega_d(\log^{(d-1)/2} n).$$

For dimension $d = 2$ Roth's Theorem gives a lower bound of order $\log^{1/2} n$. Schmidt [Sch72] improved this bound.

Theorem 1.2 (Schmidt)**.** *The discrepancy for n–point sets with respect to axis-parallel rectangles in $[0,1]^d$ fulfills*

$$\mathrm{D}(n, \mathcal{R}_2) = \Omega(\log n).$$

There are several ways to define low-discrepancy point sets. For dimension $d = 2$ we have for instance the well-known Van der Corput set. The Halton-Hammersley set is its extension to arbitrary dimensions $d \geq 2$. It is defined in the following way:

> Let $p_1, p_2, \ldots, p_{d-1}$ be the first $d-1$ primes. Let n be a positive integer. Then we define the n–point set
>
> $$\mathcal{P} := \left\{ \left(\frac{i}{n}, r_{p_1}(i), r_{p_2}(i), \ldots, r_{p_{d-1}}(i) \right) \middle| \, 0 \leq i \leq n-1 \right\},$$
>
> where $r_{p_k}(i)$ is defined like this: let $i = a_0 + p_k a_1 + p_k^2 a_2 + p_k^3 a_3 + \ldots$ be the p_k–ary notion for i, then $r_{p_k}(i) := \frac{a_0}{p} + \frac{a_1}{p^2} + \frac{a_2}{p^3} + \frac{a_3}{p^4} + \ldots$.

The next theorem gives an upper bound for the discrepancy of the Halton-Hammersley point set and therefore also an upper bound for $\mathrm{D}(n, \mathcal{R}_d)$.

Theorem 1.3 (Halton-Hammersley set). *Let $d \geq 2$. The discrepancy of the Halton-Hammersley set is bounded from above by $O_d(\log^{d-1} n)$.*

Thus, the geometric discrepancy for n–point sets in the unit square $[0,1]^2$ is determined up to constant factors. For dimensions $d \geq 3$ there remains a gap of $\log^{(d-1)/2} n$ between the lower and the upper bound. Closing this gap was called "the great open problem" in [BC87]. Up to slight improvements Roth's lower bound [Rot54] is still the best known lower bound for the L_∞–discrepancy in dimensions $d \geq 3$.

Before the problems above were investigated, a related geometric discrepancy theory problem was considered. Not the discrepancy of n–point sets for a fixed positive integer n but the discrepancy of a sequence $x : \mathbb{N} \to [0,1]^d$ was analyzed. We define the discrepancy function of x by

$$\mathrm{D}(x, \mathcal{R}_d, n) := \mathrm{D}(\{x_1, x_2, \ldots, x_n\}, \mathcal{R}_d)$$

for all $n \in \mathbb{N}$. The same can be done for the set $\mathcal{C}_d$.

There is a direct connection between low-discrepancy sets in $[0,1]^{d+1}$ and low-discrepancy sequences in $[0,1]^d$ for all $d \geq 1$. There are constants $c_1 > 0$ and $c_2 > 0$ such that

- for every set $\mathcal{P} \subseteq [0,1]^d$ $(n := |\mathcal{P}|)$ there is a sequence $x : \mathbb{N} \to [0,1]^{d-1}$ such that $\mathrm{D}(x, \mathcal{C}_{d-1}, k) \leq c_1 \, \mathrm{D}(\mathcal{P}, \mathcal{C}_d)$ for every $1 \leq k \leq n$ and
- for every sequence $x : \mathbb{N} \to [0,1]^d$ and every $n \in \mathbb{N}$ there is an n–point set $\mathcal{P} \subseteq [0,1]^{d+1}$ such that $\mathrm{D}(\mathcal{P}, \mathcal{C}_{d+1}) \leq c_2 \max_{k \in [n]} \mathrm{D}(x, \mathcal{C}_d, k)$.

In particular, the discrepancy problem for sequences is non-trivial even in the interval $[0,1]$.

One of the most interesting applications for low-discrepancy sets respectively low-discrepancy sequences stems from the field of numerical integration. Let $f : [0,1]^d \to \mathbb{R}$ be an integrable function. One likes to approximate the integral $\int_{[0,1]^d} f(x)\,\mathrm{dx}$ by finitely many evaluation of the function, i.e., by the value $\frac{1}{|\mathcal{P}|}\sum_{x \in \mathcal{P}} f(x)$ for some finite set $\mathcal{P} \subseteq [0,1]^d$. Instead of a finite set we can also take a sequence $x : \mathbb{N} \to [0,1]^d$ and approximate the integral by the value $\frac{1}{n}\sum_{i=1}^{n} f(x)$, $n \in \mathbb{N}$. But because of the direct connection between both problems, we consider here only the case of approximation by a finite set of sampling points. There arises the question, what properties a suitable set for the approximation should have. In other words: "Which sets minimize the approximation error?" It is clear that we have to claim a "good behavior" of the function. This is obvious, because the integral $\int_{[0,1]^d} f(x)\,\mathrm{dx}$ does not change, if we modify the value of f in finitely many points. Thus, the approximation error can get arbitrary large, if we do not ask for a "good behavior" of the function for which we like to approximate the integral. Let $V(f)$ be the so-called variation of f in the sense of Hardy and Krause. Then the Koksma-Hlawka inequality [Kok43, Hla61] states

$$\left| \int_{[0,1]^d} f(x)\,\mathrm{dx} - \frac{1}{|\mathcal{P}|} \sum_{x \in \mathcal{P}} f(x) \right| \leq \frac{1}{|\mathcal{P}|} \mathrm{D}(\mathcal{P}, \mathcal{C}_d) V(f).$$

1.2 Combinatorial Discrepancy Theory

At first we give here a short overview of the classical combinatorial discrepancy theory, i.e., the discrepancy in two colors. After this we describe the extension of the combinatorial discrepancy theory to an arbitrary (but fixed) number of colors. And in the last subsection we will introduce a related discrepancy notion, the positive multi-color discrepancy.

1.2.1 Two-Color Discrepancy

Let V be an n–element set and $\mathcal{E}$ a system of m subsets of V. We call the tuple $\mathcal{H} := (V, \mathcal{E})$ a *finite hypergraph*. The elements of V are called *vertices* and those of $\mathcal{E}$ *hyperedges*. Our aim is to color the vertices of $\mathcal{H}$ with the colors *red* and *green* in such a way that every hyperedge $E \in \mathcal{E}$ contains roughly the same number of red vertices as of green vertices. It is easy to see that it is not possible to color every finite hypergraph in a perfectly balanced manner. If we take for example the whole power set of V as set of hyperedges $\mathcal{E}$, there are monochromatic hyperedges of size at least $\frac{n}{2}$ for every coloring. That means the deviation from a perfectly balanced partition of all hyperedges can be very large.

Let us now quantify this deviation. We define the discrepancy of a finite hypergraph, which is a measure that tells us, how balanced a hypergraph can be colored with two colors. Instead of the two colors *red* and *green* we use here the "colors" -1 and 1 for the sake of calculation. Let $\chi : V \to \{-1, 1\}$ be a coloring of the hypergraph $\mathcal{H} = (V, \mathcal{E})$. For every hyperedge $E \in \mathcal{E}$ we denote by $\chi(E) := \sum_{x \in E} \chi(x)$ the coloring of E. Clearly, $|\chi(E)|$ is the excess of 1s compared with -1s, if $\chi(E) \geq 0$, and the other way around, if $\chi(E) < 0$. The discrepancy of $\mathcal{H}$ with respect to the coloring χ is defined by

$$\mathrm{disc}(\mathcal{H}, \chi) := \max_{E \in \mathcal{E}} |\chi(E)|$$

and the *discrepancy* of $\mathcal{H}$ by

$$\mathrm{disc}(\mathcal{H}) := \min_{\chi : V \to \{-1,1\}} \mathrm{disc}(\mathcal{H}, \chi),$$

where the minimum is taken over all 2^n possible colorings $\chi : V \to \{-1, 1\}$.

It is in general not easy to estimate the discrepancy of a hypergraph. Clearly the discrepancy is depending on the way the hyperedges are overlapping. The discrepancy is clearly maximal if $\mathcal{E} = 2^V$, whereas for a hypergraph, whose hyperedges do not overlap, the discrepancy is always bounded by 1. One could think that the number of hyperedges is a good indicator for the discrepancy of a hypergraph, but an easy example demonstrates that this is not the case. Let V be an n-element set (n even) and $V = V_1 \dot\cup V_2$ with $|V_1| = |V_2| = \frac{n}{2}$. As hyperedges we take all subsets E of V with $|E \cap V_1| = |E \cap V_2|$. By coloring all elements of V_1 with -1 and those of V_2 with 1 we see that the discrepancy of this hypergraph is 0. But the number of hyperedges is very large, more precisely it is at least of order $\Omega(\frac{1}{n}2^n)$. On the other hand there are hypergraphs having at most n hyperedges with discrepancy of order $\Theta(\sqrt{n})$. One example for this is the hypergraph of linear hyperplanes in the finite field $\mathbb{F}_2^q$ in Chapter 5.

Thus, we have seen that we cannot say anything about the discrepancy of a hypergraph by just knowing the number of its hyperedges. Another surprising example is the following. As we have mentioned before, the discrepancy of a hypergraph is trivial if for any two distinct hyperedges E_1 and E_2 there holds $E_1 \cap E_2 = \emptyset$. If we claim slightly less than this, namely that two distinct hyperedges intersect in at most one vertex, things are changing completely. We consider the extreme example in this situation: a finite projective plane with $n = k^2 + k + 1$ points and n lines for some $k \in \mathbb{N}$. Clearly in this situation the set of vertices is the set of points and the set of hyperedges of the regarded hypergraph is the set of lines of the projective plane. We know that any two distinct lines intersect in exactly one point. In [BS95] one can find that the discrepancy of this hypergraph is at least of order $\Omega(\sqrt[4]{n})$. Matoušek [Mat95] showed that this bound is tight up to constant factors. Thus, we see that the discrepancy of a hypergraph can be extremely large, even if the hyperedges intersect in at most one vertex.

Having mentioned this surprising examples, we now give some upper bounds for the discrepancy of general hypergraphs.

Theorem 1.4. *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph with n vertices and m hyperedges. Set $s := \max\{|E| \mid E \in \mathcal{E}\}$. It holds $\mathrm{disc}(\mathcal{H}) \leq \sqrt{2s\ln(4m)}$.*

This theorem is derived by a random coloring. If we give each vertex of $\mathcal{H}$ with equal probability the color -1 or 1, one can show that the probability that this coloring yields a discrepancy of at most $\sqrt{2s\ln(4m)}$ is at least $\frac{1}{2}$. Thus, there is a coloring with at most this discrepancy.

If we do not know anything about the size of the largest hyperedge, we get still an upper bound of order $O(\sqrt{n \log m})$. Spencer [Spe85] improved this bound.

Theorem 1.5 (Spencer)**.** *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph with n vertices and $m \geq n$ hyperedges. Then*

$$\mathrm{disc}(\mathcal{H}) = O(\sqrt{n \log(\tfrac{m}{n})}).$$

In particular, if m=O(n), then $\mathrm{disc}(\mathcal{H}) = O(\sqrt{n})$.

In the case $m = n$ it holds for n large enough $\operatorname{disc}(\mathcal{H}) \leq 6\sqrt{n}$. This is the reason why this result is know as "Six Standard Deviations Suffice". This bound is tight apart from constant factors, which can be seen in the following example. Let H be an $n \times n$–Hadamard matrix, i.e., a $\{-1, 1\}$–matrix whose column vectors are mutually orthogonal and have all 1s in the first row. Let $\mathcal{H}$ be the hypergraph whose incidence matrix is obtained from H by replacing all -1s with 0s. Then one can show $\operatorname{disc}(\mathcal{H}) > \frac{1}{2}\sqrt{n}$.

The last theorem bounded the discrepancy in terms of the number of vertices and hyperedges. Another possibility is to bound the discrepancy of a hypergraph by its maximum degree, i.e., the maximal number of hyperedges in which a vertex is contained.

Theorem 1.6 (Beck, Fiala)**.** *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph and $t \in \mathbb{N}$ such that every vertex of $\mathcal{H}$ is contained in at most t hyperedges. Then*

$$\operatorname{disc}(\mathcal{H}) \leq 2t - 1.$$

In their paper Beck and Fiala [BF81] conjectured that $\operatorname{disc}(\mathcal{H}) = O(\sqrt{t})$, but the only tiny improvement for an upper bound solely depending on the maximum degree of $\mathcal{H}$ since then was made by Bednarchak and Helm [BH97]. They replaced the term $2t - 1$ by $2t - 3$.

A consequence[1] of Beck's paper [Bec81] is that $\operatorname{disc}(\mathcal{H}) = O(\sqrt{t} \log m \log n)$, where n is the number of vertices and m the number of hyperedges of $\mathcal{H}$. The paper is dealing with the discrepancy of the hypergraph of arithmetic progressions in the first N natural numbers. We will mention this problem in the next chapter. Proving that $\Omega(N^{1/4})$ lower bound of Roth is nearly sharp, Beck invented the partial coloring method. Let us denote by $\deg(\mathcal{H})$ the maximal degree of a hypergraph $\mathcal{H}$ and by $\mathcal{H}_k$ the hypergraph $(V, \{E \in \mathcal{E} \mid |E| \geq k\})$. Beck proved the following theorem[2].

Theorem 1.7 (Beck)**.** *Let $\mathcal{H} = (V, \mathcal{E})$ be a finite hypergraph, $n := |V|$ and $m := |\mathcal{E}|$. Let M and K be natural numbers such that*

$$\deg(\mathcal{H}_M) \leq K.$$

Then

$$\operatorname{disc}(\mathcal{H}) \leq c(M + K \log K)^{1/2} \log^{1/2} m \log n,$$

where $c > 0$ is an absolute constant.

We get the following Corollary.

[1] Apply Corollary 1.8 with $t := \deg(\mathcal{H})$.

[2] This version of the theorem can be found in [BS95]. The original paper states a slightly stronger upper bound.

Corollary 1.8. *If there exists a constant t such that*

$$\deg(\mathcal{H}_t) \leq t,$$

then there is an absolute constant $c > 0$ such that

$$\operatorname{disc}(\mathcal{H}) \leq c\sqrt{t} \log m \log n.$$

We shortly mention here two other discrepancy concepts: the hereditary and the linear discrepancy. Both are needed in the next subsection to state upper bounds for the multi-color discrepancy.

Hereditary Discrepancy

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph and $V_0 \subseteq V$ a subset of the vertices. We call $\mathcal{H}_{|V_0} := (V_0, \{E \cap V_0 \mid E \in \mathcal{E}\})$ the induced subhypergraph of $\mathcal{H}$ on V_0. By just knowing the discrepancy of a hypergraph, we do not have any information about the discrepancy of its induced subhypergraphs. It is even possible that a hypergraph on n vertices has discrepancy 0 and there is an induced subhypergraph with discrepancy a constant fraction of n. This can be checked in the hypergraph with set of vertices $V = V_1 \dot{\cup} V_2$ below the discrepancy definition, just looking at the induced subhypergraph on V_1. On the other hand every hypergraph is the induced subhypergraph of a hypergraph with discrepancy 0, obtained by "doubling" all its vertices. Thus, it is interesting to ask for the maximum discrepancy of any induced subhypergraph of a hypergraph. The *hereditary discrepancy* of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is defined by

$$\operatorname{herdisc}(\mathcal{H}) := \max_{V_0 \subseteq V} \operatorname{disc}(\mathcal{H}_{|V_0}).$$

All upper bounds given in this section are also valid for the hereditary discrepancy, because the parameters n, m and t are non-increasing for all induced subhypergraphs.

Linear Discrepancy

Let us now define the linear discrepancy, which like the hereditary discrepancy plays a big role in the recursive coloring approach for multi-color discrepancies. The concept is the following. So far every vertex should in average belong with equal probability to one of the two color-classes. And the discrepancy function $\chi(E)$ of a hyperedge E can be written as

$$\chi(E) = \left(|E \cap \chi^{-1}(1)| - \tfrac{1}{2}|E|\right) + \left(\tfrac{1}{2}|E| - |E \cap \chi^{-1}(-1)|\right),$$

which is the differences between the number of 1s in E and the wanted number of 1s in E (namely $\frac{1}{2}|E|$) plus the differences between the wanted number of -1s in E and the number of -1s in E. Now we assign to every vertex v a weight $p_{-1}(v)$ describing the ratio it should in average belong to the color-class $\chi^{-1}(-1)$. A natural generalization of the discrepancy function in the weighted case is

$$\left(||E \cap \chi^{-1}(1)| - \sum_{x \in E} (1 - p_{-1}(x)) \right) + \left(\sum_{x \in E} p_{-1}(x) - |E \cap \chi^{-1}(-1)| \right).$$

Thus, the weighted discrepancy function can be written as $\sum_{x\in E}(\chi - (1 - 2p_{-1}))(x)$. As $p_{-1}(v)$ passes through the whole interval $[0, 1]$, the term $p(v) := 1-2p_{-1}(v)$ passes through the whole interval $[-1, 1]$. The *linear discrepancy* is the maximal discrepancy for any possible weight-function $p : V \to [-1, 1]$. Hence, it is defined as

$$\operatorname{lindisc}(\mathcal{H}) := \max_{p:V\to[-1,1]} \min_{\chi:V\to\{-1,1\}} \max_{E\in\mathcal{E}} |(p - \chi)(E)|.$$

There are two trivial interrelations between the different discrepancy notions. We have $\operatorname{disc}(\mathcal{H}) \le \operatorname{herdisc}(\mathcal{H})$ and $\operatorname{disc}(\mathcal{H}) \le \operatorname{lindisc}(\mathcal{H})$. A result due to Beck and Spencer [BS84] and Lovász, Spencer and Vesztergombi [LSV86] shows the following.

Theorem 1.9. *For any finite hypergraph $\mathcal{H} = (V, \mathcal{E})$ it holds* $\operatorname{lindisc}(\mathcal{H}) \le 2 \operatorname{herdisc}(\mathcal{H})$.

This was of course only a brief introduction to the combinatorial discrepancy theory (in two colors). For a more detailed overview of this field we refer to the chapter "*Discrepancy Theory*" of Beck and Sós [BS95] and the fourth chapter of Matoušek's book [Mat99].

1.2.2 Multi-Color Discrepancy

So far, we have only investigated 2–partitions of the set of vertices of a hypergraph and how balanced the induced partition on each hyperedge is. But it is also interesting to ask how balanced a partition in other numbers of partition classes can be with respect to the partitions induced on all hyperedges. In other words, we like to extend the discrepancy notion in this section from two colors to an arbitrary number of c colors. This was done by Doerr [Doe00] and Doerr and Srivastav [DS03]. We follow their approach for a very brief introduction to multi-color discrepancies. A special case of this c–partitioning problem occurred, before Doerr and Srivastav investigated the discrepancy theory in more colors, in a paper concerning communication complexity [BHK01] and two other results were known before not using the explicit notion of multi-color discrepancy. The first can be found in [BS95] and states in the notion of multi-color discrepancy (that we will introduce here shortly) that every hypergraph with a totally unimodular incidence matrix has discrepancy less than one in any number of colors $c \ge 2$. The second can be found in the paper of Beck and Fiala [BF81] mentioned before. It bounds the discrepancy of hypergraphs with m hyperedges in any number of colors by $O(\sqrt{m \log m})$.

Let us now quantify the c–color discrepancy that means the discrepancy of a partition of the set of vertices of a hypergraph into $c \ge 2$ partition classes. It is a natural way to realize the c–partitioning by a c–coloring, i.e., a function $\chi : V \to [c]$ from the set of vertices into the set of the first c positive integers. In some applications, e.g. in [BHK01], it is useful to take other c–element sets as set of colors, and we will consider a set of c–dimensional vectors later on that has advantages for the calculation of the discrepancy. We define the discrepancy of a hyperedge $E \in \mathcal{E}$ in color $i \in [c]$ with respect to the c–coloring χ by

$$\operatorname{disc}_{\chi,i}(E) := \left| |\chi^{-1}(i) \cap E| - \tfrac{1}{c}|E| \right|,$$

the *discrepancy of* $\mathcal{H}$ *with respect to* χ by

$$\operatorname{disc}(\mathcal{H}, c, \chi) := \max_{E \in \mathcal{E}, i \in [c]} \operatorname{disc}_{\chi,i}(E)$$

and the *discrepancy of* $\mathcal{H}$ *in* c *colors* by

$$\operatorname{disc}(\mathcal{H}, c) := \min_{\chi: V \to [c]} \operatorname{disc}(\mathcal{H}, c, \chi).$$

It is obvious that $\operatorname{disc}(\mathcal{H}, 2) = \frac{1}{2} \operatorname{disc}(\mathcal{H})$. The reason for this is that $\operatorname{disc}(\mathcal{H}, 2)$ measures for every hyperedge the deviation of the actual number of vertices in one color from the wanted value of this, which is exactly half of the deviation between the number of vertices in the two color-classes induced on the hyperedges measured by $\operatorname{disc}(\mathcal{H})$.

The calculation of the discrepancy of a hyperedge $E \in \mathcal{E}$ with respect to a coloring $\chi : V \to \{-1, 1\}$ is very easy. Using the abbreviation $\chi(E) = \sum_{x \in E} \chi(x)$ it is just $|\chi(E)|$. Maximizing this term over all $E \in \mathcal{E}$ we get the discrepancy $\operatorname{disc}(\mathcal{H}, \chi)$ of $\mathcal{H}$ with respect to the coloring χ. For the c–color discrepancy things are not that easy. Doerr and Srivastav [DS03] solved this problem by taking the following set of c–dimensional real-valued vectors as set of colors. For every $i \in [c]$ define $m^{(i)} \in \mathbb{R}^c$ by

$$m_j^{(i)} := \begin{cases} \frac{c-1}{c}, & \text{if } i = j, \\ -\frac{1}{c}, & \text{otherwise} \end{cases}$$

and set $M_c := \{m^{(i)} \mid i \in [c]\}$. One can straightforward check that

$$\operatorname{disc}(\mathcal{H}, c) = \min_{\chi: V \to M_c} \max_{E \in \mathcal{E}} \| \sum_{x \in E} \chi(x) \|_\infty.$$

It is an interesting question whether there are relations between the discrepancy in different numbers of colors. Doerr [Doe00] showed that $\operatorname{disc}(\mathcal{H}, c_2) \le \frac{c_1}{c_2} \operatorname{disc}(\mathcal{H}, c_1)$ if c_2 divides c_1. But we cannot hope for many other estimations like this. Quite the contrary, for every number of colors $c \ge 2$ Doerr presents a hypergraph $\mathcal{H}$ with discrepancy $\operatorname{disc}(\mathcal{H}, c) = 0$ and a discrepancy of constant fraction of the number of vertices n in almost all other numbers of colors.

Nevertheless, the upper bounds from Subsection 1.2.1 can be extended to the multi-color discrepancy, because they are hereditary, i.e., they hold also for the hereditary discrepancy. We get this directly from the following theorem which is a consequence of the recursive coloring approach of Doerr.

Theorem 1.10. *Let* $\mathcal{H} = (V, \mathcal{E})$ *be a finite hypergraph, then*

$$\operatorname{disc}(\mathcal{H}, c) \le 2.0005 \operatorname{herdisc}(\mathcal{H})$$

in any number of colors c.

The idea of this recursive coloring approach is to achieve a c–coloring of the hypergraph $\mathcal{H}$ with low discrepancy by iterated 2–colorings. That means, the first step is to split the set of vertices in two color-classes at a certain ratio. Then this splitting process is continued on the hypergraphs induced on the color-classes. This is also possible for numbers of colors differerent from powers of two by choosing a suitable ratio for each splitting process. For this the linear discrepancy and the fact that $\operatorname{lindisc}(\mathcal{H}) \leq 2 \operatorname{herdisc}(\mathcal{H})$ is needed. How good this iteration works depends highly on the number of colors. The factor 2.0005 in Theorem 1.10 stems from the worst case.

For some hypergraphs it is possible to show that for induced subhypergraphs on a smaller set of vertices the discrepancy falls off. In this case of "decreasing discrepancy" Doerr gives a refinement of the recursive coloring method. The following theorem is a consequence of a more general theorem stated by Doerr.

Theorem 1.11. *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. Let $c \geq 2$, $\alpha \in]0,1[$ and $C > 0$. Assume that for all $V_0 \subseteq V$ with $|V_0| \geq \frac{1}{c}|V|$ and all $q \in [0,1]$ with $q|V_0| \in \mathbb{Z}$ there is a 2–coloring $\chi : V_0 \to \{-1, 1\}$ such that $|\chi^{-1}(1)| = q|V_0|$ and $||E \cap V_0 \cap \chi^{-1}(1)| - q|E \cap V_0|| \leq C|V_0|^\alpha$ holds for all $E \in \mathcal{E}$.*

Then there is a constant $c_\alpha > 0$ only depending on α such that the c–color discrepancy of $\mathcal{H}$ is bounded by

$$\operatorname{disc}(\mathcal{H}, c) \leq C c_\alpha \left(\frac{n}{c}\right)^\alpha + 1.$$

Using this recursive approach Doerr generalizes the upper bound of Spencer for general hypergraphs $\mathcal{H} = (V, \mathcal{E})$ with n vertices and $m \geq n$ hyperedges and states for the c–color discrepancy

$$\operatorname{disc}(\mathcal{H}, c) = O\left(\sqrt{\frac{n}{c} \log\left(\frac{mc}{n}\right)}\right)$$

and in the case $m = n$

$$\operatorname{disc}(\mathcal{H}, c) = O\left(\sqrt{\frac{n}{c} \log c}\right).$$

The bound of Beck-Fiala is extended to the c–color discrepancy in the following way. For any hypergraph $\mathcal{H} = (V, \mathcal{E})$ with maximum degree t holds

$$\operatorname{disc}(\mathcal{H}, c) \leq 2.0005\, t.$$

1.2.3 Positive Multi-Color Discrepancy

In this subsection we define another discrepancy notion. We will use it in the Chapter 4 and Chapter 5. The discrepancy function, respectively c–color discrepancy function, gives us only information about the absolute difference between the actual number and the wanted number of vertices in one color. But we do not know if this difference is caused by an excess or a lack of vertices in one color. Especially the declustering problem from Chapter 4 is a good motivation to ask for the positive discrepancy of a hypergraph, i.e., for the maximal excess of vertices in one color and hyperedge over the wanted value.

A declustering scheme distributes data given in a higher-dimensional grid on a number of parallel-working storage devices in such a way that typical requests, e.g. subgrids, find their data evenly distributed over all devices. The idea of declustering is to allow fast data retrieval. In this situation the discrepancy function measures the absolute difference between the actual and the average workload of the devices for a request. But the processing time of a request is determined only by the device with the biggest workload. Thus, only the positive deviations from the average workload are of interest. This is measured by the positive discrepancy that we introduce now. Let $\mathcal{H} = (V, \mathcal{E})$ be a finite hypergraph, and let $\chi : V \to \{-1, 1\}$ be a c–coloring of $\mathcal{H}$. We define for every $E \in \mathcal{E}$ and every color $i \in [c]$

$$\operatorname{disc}^+_{\chi,i}(E) := |\chi^{-1}(i) \cap E| - \frac{1}{c}|E|.$$

This quantity may be negative, but for fixed $E \in \mathcal{E}$ there is at least one color $i \in [c]$ for which it is not. We define the *positive discrepancy of* $\mathcal{H}$ *with respect to* χ by

$$\operatorname{disc}^+(\mathcal{H}, c, \chi) := \max_{E\in\mathcal{E}, i\in[c]} \operatorname{disc}^+_{\chi,i}(E)$$

and the *positive discrepancy of* $\mathcal{H}$ by

$$\operatorname{disc}^+(\mathcal{H}, c) := \min_{\chi:V\to[c]} \operatorname{disc}^+(\mathcal{H}, c, \chi).$$

It is obvious that the positive discrepancy of a hypergraph is bounded by its discrepancy. But we can also give a lower bound for the positive discrepancy in terms of the discrepancy. Let $\chi : V \to [c]$ be a c–coloring of $\mathcal{H} = (V, \mathcal{E})$ with $\operatorname{disc}^+(\mathcal{H}, c, \chi) = \operatorname{disc}^+(\mathcal{H}, c)$. Furthermore choose $E \in \mathcal{E}$ and $i_0 \in [c]$ such that $\operatorname{disc}(\mathcal{H}, c, \chi) = ||\chi^{-1}(i_0) \cap E| - \frac{1}{c}|E||$. If $|\chi^{-1}(i_0) \cap E| - \frac{1}{c}|E| \geq 0$ then $\operatorname{disc}^+(\mathcal{H}, c) = \operatorname{disc}(\mathcal{H}, c)$. Thus, we can assume $|\chi^{-1}(i_0) \cap E| - \frac{1}{c}|E| < 0$. Using $\sum_{i\in[c]}(|\chi^{-1}(i) \cap E| - \frac{1}{c}|E|) = 0$, we get the existence of an $i_1 \in [c] \setminus \{i_0\}$ with

$$|\chi^{-1}(i_1) \cap E| - \tfrac{1}{c}|E| \geq \frac{1}{c-1}||\chi^{-1}(i_0) \cap E| - \tfrac{1}{c}|E|| = \frac{1}{c-1}\operatorname{disc}(\mathcal{H}, c, \chi) \geq \frac{1}{c-1}\operatorname{disc}(\mathcal{H}, c).$$

Summarizing the above discussion, we get

$$\frac{1}{c-1}\operatorname{disc}(\mathcal{H}, c) \leq \operatorname{disc}^+(\mathcal{H}, c) \leq \operatorname{disc}(\mathcal{H}, c). \tag{1.1}$$

That means, the positive discrepancy is strictly related to the discrepancy of a hypergraph. But in some situation this $(c-1)$–gap in (1.1) is much to large. In Chapter 5 we shorten this gap to a factor of order $\sqrt{c}$ and close the gap completely in a special case with the help of Fourier analysis. In Chapter 4 the discrepancy will be of polylogarithmic order in the number of colors. Thus, estimate 1.1 alone would give a trivial lower bound for $\operatorname{disc}^+(\mathcal{H}, c)$.

Chapter 2

Arithmetic Progressions

One of the classical discrepancy problems was to determine the discrepancy of the hypergraph $\mathcal{H}_N$ of all arithmetic progressions in the positive integers $[N] = \{1, 2, \ldots, N\}$ for an $N \in \mathbb{N}$, i.e., all sets of the form

$$A_{a,d,l} := \{a + id \mid 0 \leq i \leq l - 1\} \subseteq [N]$$

with starting point a, difference d and length l. In section 2.1 *The Classical Discrepancy Problem* we sketch the progress on this problem.

From this classical discrepancy problem several new discrepancy problems arise. In order to be in the position to prove a discrepancy bound for related hypergraphs in the sections at the end of this chapter, we have to make some preparatory work first. In section 2.2 *The Fourier Analytic Method* we present a method for determing lower bounds for the discrepancy of those hypergraphs. To attain upper discrepancy bounds we use the probabilistic method and the partial coloring method that we have already mentioned in Chapter 1.

Using this methods we investigate the discrepancy of hypergraphs related to the hypergraph of arithmetic progressions.

2.1 The Classical Discrepancy Problem

The hypergraph $\mathcal{H}_N = ([N], \mathcal{E}_N)$ of all arithmetic progressions in the first N positive integers is one of the most investigated hypergraphs in combinatorial discrepancy theory. Asking for its discrepancy was motivated by a famous result of van der Waerden [vdW27]. He showed that for every coloring of the integers with two colors, there exists a monochromatic arithmetic progression of arbitrary length. In the context of discrepancy theory this means, for every $k \in \mathbb{N}$ there exists an $N \in \mathbb{N}$ only depending on k such that $\operatorname{disc}(\mathcal{H}_N) \geq k$.

In the year 1964 Roth [Rot64] presented for the hypergraph $\mathcal{H}_N$ the famous lower bound of order $\Omega(N^{1/4})$ not given in the explicit notion of discrepancy. His proof uses harmonic analysis. The probabilistic method immediately gives an upper bound of order $O(\sqrt{N \log N})$. It was a long-standing open problem to close this big gap between lower and upper bound. The first significant improvement for the upper bound was done by Sárközy [Sár74]. Using the upper bound of $O(\sqrt{p \log p})$ for arithmetic progressions in $\mathbb{Z}_p$ (ring of integers modulo a prime p) he stated an upper bound for $\mathcal{H}_N$ of order $O((N \log N)^{1/3})$. It is also interesting that with a slight weakening of the result (little increasing of the power of $\log N$) this proof can be made constructive by using the Legendre symbol.

It was Beck [Bec81] who showed that Roth's lower bound is nearly sharp. Inventing the partial coloring method, which we have mention shortly in Chapter 1, he proved that $\operatorname{disc}(\mathcal{H}_N) = O(N^{1/4} \log^{5/2} N)$. About 30 years after Roth stated the lower bound, Matoušek and Spencer [MS96] used a refinement of Beck's partial coloring method namely the entropy method[1] to prove the asymptotically tight upper bound of order $O(N^{1/4})$ for $\mathcal{H}_N$. Thus, they showed that Roth's lower bound is optimal up to a constant factor.

Therefore this classical discrepancy problem was solved. But there were many other problems arising from this. For example one could ask for the discrepancy of $\mathcal{H}_N$ in an arbitrary number of colors. Doerr and Srivastav [DS03] proved the lower bound $\operatorname{disc}(\mathcal{H}_N, c) = \Omega(c^{-0.5} N^{0.25})$ and the upper bound $\operatorname{disc}(\mathcal{H}_N, c) = O(c^{-0.16} N^{0.25})$. Another interesting way of extending the original problem is to take the grid $[N]^d$ as set of vertices. There are at least two ways for this generalization to higher dimensions. One of these is due to Valkó [Val02]. He investigated the following hypergraph. The set of hyperedges is the set of all one-dimensional arithmetic progressions in $[N]^d$, i.e., all sets

$$A_{a,b,L} := \{a + kb \mid k \in \{0, 1, \ldots, L-1\}\}$$

with $a \in [N]^d$, $b \in \mathbb{Z}^d \setminus \{0\}$ and $L \in [N]$ that are subsets of $[N]^d$. For this hypergraph Valkó determined a lower bound of order $\Omega_d(N^{d/(2d+2)})$ using the Fourier analytic method that we present in the next section. The upper bound of order $O_d(N^{d/(2d+2)} \log^{5/2} N)$ for this hypergraph is derived by Beck's partial coloring method.

A second approach of generalizing the classical hypergraph $\mathcal{H}_N$ to higher dimensions was done by Doerr, Srivastav and Wehr [DSW04]. Instead of looking for one-dimensional arithmetic progressions in the d–dimensional grid $[N]^d$ they considered Cartesian products of d arithmetic progressions. That means the set of hyperedges is

$$\mathcal{E} := \{A_1 \times A_2 \times \ldots \times A_d \mid A_i \text{ arithmetic progression in } [N]\}.$$

They prove that the discrepancy of this hypergraph is of order $\Theta(N^{d/4})$. The lower bound proof uses Fourier analysis on the additive group $\mathbb{Z}^d$, while the upper bound is derived by a product coloring using the one-dimensional upper bound of Matoušek and Spencer.

[1] Spencer [Spe85] developed this technique for the proof of the upper bound of $O(\sqrt{n})$ for the discrepancy of hypergraphs with n vertices and hyperedges.

In this chapter we investigate the discrepancy of several hypergraphs that are also related to the classical hypergraph of arithmetic progressions in $[N]$. In Section 2.3 we are interested in the discrepancy of the hypergraph of sums of k arithmetic progressions in $[N]$. The hypergraph of higher-dimensional arithmetic progressions with common difference in $[N]^d$ in Section 2.4 is a special subhypergraph of the hypergraph studied by Doerr, Srivastav and Wehr [DSW04]. Where they considered all Cartesian products of d arithmetic progressions in $[N]$, we take only Cartesian products of d arithmetic progressions with the same difference δ into account.

In the last sections of this chapter we investigate the hypergraph of arithmetic progressions in $\mathbb{Z}_p$ and related hypergraphs. Unlike in the classical hypergraph, arithmetic progressions in $\mathbb{Z}_p$ can be wrapped around (several times). We will see in Section 2.5 that the hypergraph of arithmetic progressions in $\mathbb{Z}_p$ is a special example of a class of hypergraphs in $\mathbb{Z}_p$ that we will call *$\mathbb{Z}_p$–invariant*. Another $\mathbb{Z}_p$–invariant hypergraph is the hypergraph of Bohr neighborhoods that we will discuss in Section 2.8.

All hypergraphs in this chapter but one are invariant under shifting, i.e., for all hyperedges E and all vertices x the set $(x + E) \cap V$ is also a hyperedge. The only exception is the hypergraph of centered arithmetic progressions in $\mathbb{Z}_p$ that we treat in Section 2.7. This is the hypergraph of all arithmetic progressions in $\mathbb{Z}_p$ that are symmetric to $0 \in \mathbb{Z}_p$. The invariance under shifting is essential for the Fourier analytic method from Section 2.2. We use a special color-function to overcome this obstacle.

2.2 The Fourier Analytic Method

In this section we give first a short introduction to Fourier analysis on locally compact Abelian groups. Afterwards we prove a theorem and a corollary of this theorem that we will use later on to state lower bounds for hypergraphs related to the hypergraph of arithmetic progressions. We start with some basic definitions.

Definition 2.1. An **Abelian group** $(G, +)$ is a pair consisting of a set G and a binary operation $+ : G \times G \to G$ such that

(i) $x + y = y + x$ for all $x, y \in G$,

(ii) $(x + y) + z = x + (y + z)$ for all $x, y, z \in G$,

(iii) there exists an element $0 \in G$ with $x + 0 = x$ for all $x \in G$,

(iv) for every $x \in G$ there exists an element $-x \in G$ with $x + (-x) = 0$.

Instead of $x + (-y)$ we use the abbreviation $x - y$.

Definition 2.2. A **topological space** $(X, \mathcal{T})$ is a pair consisting of a set X and a set $\mathcal{T} \subseteq \mathcal{P}(X)$ of subsets of X such that

(i) $\emptyset, X \in \mathcal{T}$,

(ii) The union of any collection of sets in $\mathcal{T}$ is also in $\mathcal{T}$.

(iii) The intersection of finitely many sets in $\mathcal{T}$ is also in $\mathcal{T}$.

The elements of $\mathcal{T}$ are called **open sets** of X. A subset $C \subseteq X$ is called **closed set**, if the complement $X \setminus C$ is an open set.

Definition 2.3. Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be two topological spaces.

(i) A map $f : X_1 \to X_2$ is called **continuous**, if for every $U \in \mathcal{T}_2$ the set $f^{-1}(U) := \{x \in X_1 \mid f(x) \in U\}$ is in $\mathcal{T}_1$.

(ii) Let $X := X_1 \times X_2$. The **product topology** on X is defined to be the coarsest topology (i.e., the topology with the fewest open sets) on X for which the mappings $\pi_i : X \to X_i, (x_1, x_2) \mapsto x_i$ $(i = 1, 2)$ are continuous.

Definition 2.4. A **topological Abelian group** $(G, +, \mathcal{T})$ is a triple, where $(G, +)$ is an Abelian group and $(G, \mathcal{T})$ is a topological space such that $f : G \times G \to G, (x, y) \mapsto x - y$ is a continuous mapping with respect to the product topology on $G \times G$.

Definition 2.5. Let $(X, \mathcal{T})$ be a topological space.

(i) A set $Y \subseteq X$ is called **compact**, if for every collection $(U_i)_{i \in I}$ in $\mathcal{T}$ with $Y \subseteq \bigcup_{i \in I} U_i$ there exists a finite collection $(U_i)_{i \in J}$ (i.e. $J \subseteq I$ and $|J| < \infty$) with $Y \subseteq \bigcup_{i \in J} U_i$.

(ii) A set $B \subseteq X$ is called a **neighborhood** of a point $x \in X$, if there is a $U \in \mathcal{T}$ with $x \in U \subseteq B$.

(iii) $(X, \mathcal{T})$ is called **locally compact**, if every $x \in X$ has a compact neighborhood.

Definition 2.6. We call a topological Abelian group $(G, +, \mathcal{T})$ **locally compact Abelian group**, if $(X, \mathcal{T})$ is a locally compact topological space. We will use the abbreviation G instead of $(G, +, \mathcal{T})$.

Now we give a very short overview over the basics of Fourier analysis on locally compact Abelian groups that we will need for our discrepancy theoretical considerations later on. For an extensive introduction in this field we refer to the book "*Fourier Analysis on Groups*" by W. Rudin [Rud62].

Definition 2.7. Let $(G, +, \mathcal{T})$ be a locally compact Abelian group. A **character** on G is a function $\gamma : G \to \mathbb{C}$ with

(i) $|\gamma(x)| = 1$ for all $x \in G$,

(ii) $\gamma(x + y) = \gamma(x)\gamma(y)$ for all $x, y \in G$.

Let $\widehat{G}$ denote the set of all characters on G.

Remark 2.8. *$\widehat{G}$ is a subset of the set $\mathbb{C}^G$ of all complex-valued functions on G. By $f + g : G \to \mathbb{C}, x \mapsto (f + g)(x) := f(x)g(x)$ we have given an addition on $\mathbb{C}^G$.*

Theorem 2.9. *Let G be a locally compact Abelian group. Then*

(i) $\widehat{G}$ is closed under the addition $+$ in $\mathbb{C}^G$.

(ii) There exists a topology $\mathcal{T}_{\widehat{G}}$ on $\widehat{G}$ such that $\widehat{G}$ is a locally compact Abelian group.

$\widehat{G}$ is called the **dual group** *of G.*

For a locally compact Abelian group G and all $1 \leq p < \infty$ we denote by $L^p(G)$ the subset of all Borel functions f with $\|f\|_p := (\int_G |f(x)|^p dx)^{1/p} < \infty$, where the used measure is the up to a positive constant unique Haar measure. Now we are able to define the Fourier transform for functions in $L^1(G)$ and the convolution of two functions in $L^1(G)$.

Definition 2.10 (Fourier transform)**.** Let G be a locally compact Abelian group and $f \in L^1(G)$. The **Fourier transform** $\widehat{f} : \widehat{G} \to \mathbb{C}$ is defined by

$$\widehat{f}(\gamma) := \int_G f(x)\gamma(-x)dx, \qquad \gamma \in \widehat{G}.$$

Definition 2.11 (Convolution)**.** Let G be a locally compact Abelian group and $f, g \in L^1(G)$. The **convolution** $f * g : G \to \mathbb{C}$ is defined by

$$(f * g)(y) := \int_G f(x)g(y - x)dx, \qquad y \in G.$$

Remark 2.12. *For instance in Rudin [Rud62] one can find the proofs that the Fourier transform and the convolution are well defined. For $f, g \in L^1(G)$ the convolution $f * g$ is also in $L^1(G)$ and it holds $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.*

The following two theorems are the key for the use of Fourier analysis in discrepancy theory. The first shows that the Fourier transform is multiplicative on the Banach algebra $L^1(G)$, where the multiplication on $L^1(G)$ is the convolution. The second is the well-known Plancherel Theorem for locally compact Abelian groups. Both proofs can be found in Rudin [Rud62].

Theorem 2.13. *Let G be a locally compact Abelian group. For all $f, g \in L^1(G)$ we have $\widehat{f * g} = \widehat{f}\widehat{g}$.*

Theorem 2.14. *The Haar measure on $\widetilde{G}$ can be normalized such that the Fourier transform is an isometry, i.e.*

$$\|\widehat{f}\|_2 = \|f\|_2, \quad \text{for all } f \in L^1(G) \cap L^2(G).$$

We have to mention here that, if G is discrete, the integral $\int_G \cdot dx$ is nothing but the sum $\sum_{x \in G}$. And the same holds for $\widehat{G}$. In our discrepancy theoretical problems we are only concerned with discrete locally compact Abelian groups G. For our applications we state the following corollary.

Corollary 2.15. *Let G be a discrete locally compact Abelian group and let $\alpha := \int_{\widehat{G}} 1 d\gamma$. Let $f \in L^1(G) \cap L^2(G)$. Then*

$$\|\widehat{f}\|_2^2 = \alpha \|f\|_2^2.$$

Proof. Let $f : G \to \mathbb{C}$ be defined by $f(0) := 1$ and $f(x) := 0$ for all $x \in G \setminus \{0\}$. Then $f \in L^1(G) \cap L^2(G)$ and $\|f\|_2^2 = 1$. We have

$$\|\widehat{f}\|_2^2 = \int\limits_{\widehat{G}} |\widehat{f}(\gamma)|^2 d\gamma = \int\limits_{\widehat{G}} |\gamma(0)|^2 d\gamma = \int\limits_{\widehat{G}} 1 d\gamma = \alpha \|f\|_2^2.$$

Now the assertion follows from Theorem 2.14. □

After this very short introduction to Fourier analysis on locally compact Abelian groups, we are now able to prove the next theorem. It will be a useful tool for the determination of lower discrepancy bounds of hypergraphs. We will apply it to hypergraphs that are related to the hypergraph of arithmetic progressions. The benefit of this theorem is the separation of two kinds of influence to the discrepancy function. The first is the coloring of the hypergraph, which is represented by a color-function f in this theorem. The second arises from the set of hyperedges. The set $\mathcal{E}_0$ in the theorem is so to speak a basic set of hyperedges. In this chapter all hypergraphs with one exception (the hypergraph of centered arithmetic progressions in $\mathbb{Z}_p$, for which a special color-function will help to overcome this problem) are invariant under all possible translations. That means for all hyperedges $E \in \mathcal{E}$ and all vertices $x \in V$ the intersection of $x + E := \{x + y \mid y \in E\}$ with the set of vertices V is also an element of $\mathcal{E}$. For all $A \subseteq G$ we define $f(A) := \sum_{a \in A} f(a)$. We will use this definition for a suitable color-function for a given coloring of the hypergraph. Then, for every hyperedge E, $|f(E)|$ will be strictly related to the discrepancy of E with respect to this coloring. Thus, the left hand side of the equation in the following theorem is the sum over the squared "discrepancies" of all the translates of sets in $\mathcal{E}_0$.

Theorem 2.16. *Let G be a locally compact Abelian group, $f \in L^1(G) \cap L^2(G)$ and let $\mathcal{E}_0 \subseteq \mathcal{P}(G)$. Furthermore, let $\alpha := \int_{\widehat{G}} 1 d\gamma$. Then*

$$\sum_{E \in \mathcal{E}_0} \sum_{x \in G} |f(x+E)|^2 = \frac{1}{\alpha} \int\limits_{\widehat{G}} |\widehat{f}(r)|^2 \sum_{E \in \mathcal{E}_0} |\mathbb{1}_{-E}(r)|^2 dr.$$

Before we proof this theorem, we show the benefit of this separation is the following. If we are able to state a lower bound $\gamma > 0$ for the expression $\sum\limits_{E \in \mathcal{E}_0} |\widehat{\mathbb{1}}_{-E}(r)|^2$ for almost all $r \in \widehat{G}$, i.e., for all $r \in \widehat{G}$ except for a null set, the next corollary gives a lower bound for $\sum\limits_{E \in \mathcal{E}_0} \sum\limits_{x \in G} |f(x+E)|^2$.

Corollary 2.17. *Let $\gamma > 0$ be chosen such that $\sum\limits_{E \in \mathcal{E}_0} |\widehat{\mathbb{1}}_{-E}(r)|^2 \geq \gamma$ for almost all $r \in \widehat{G}$. Let $f \in L^1(G) \cap L^2(G)$. Then*

$$\sum_{E \in \mathcal{E}_0} \sum_{x \in G} |f(x+E)|^2 \geq \gamma \|f\|_2^2.$$

Proof. Using Theorem 2.16 and the Corollary 2.15 we have

$$\begin{aligned} \sum_{E \in \mathcal{E}_0} \sum_{x \in G} |f(x+E)|^2 &\overset{\textit{Theorem } 2.16}{=} \frac{1}{\alpha} \int\limits_{\widehat{G}} |\widehat{f}(r)|^2 \sum_{E \in \mathcal{E}_0} |\mathbb{1}_{-E}(r)|^2 dr \\ &\geq \frac{\gamma}{\alpha} \int\limits_{\widehat{G}} |\widehat{f}(r)|^2 dr \\ &\overset{\textit{Corollary } 2.15}{=} \gamma \|f\|_2^2. \end{aligned}$$

□

Remark 2.18. *The counterpart of Theorem 2.16 in the context of geometric discrepancies can be found for instance in the book "Geometric Discrepancy" [Mat99]. The corresponding theorem is called "point component/shape component separation" by Matoušek.*

Proof of Theorem 2.16. Let $E \in \mathcal{E}_0$ and $x \in G$. We have

$$f(x+E) = \sum_{y \in x+E} f(y) = \sum_{y \in G} f(y) \mathbb{1}_{x+E}(y) = \sum_{y \in G} f(y) \mathbb{1}_{-E}(x-y) = (f * \mathbb{1}_{-E})(x).$$

Thus,

$$\sum_{E\in\mathcal{E}_0}\sum_{x\in G}|f(x+E)|^2 = \sum_{E\in\mathcal{E}_0}\sum_{x\in G}|(f*\mathbb{1}_{-E})(x)|^2$$
$$\overset{Corollary\ 2.15}{=} \frac{1}{\alpha}\sum_{E\in\mathcal{E}_0}\int_{\widehat{G}}|(\widehat{f*\mathbb{1}_{-E}})(r)|^2dr$$
$$\overset{Theorem\ 2.13}{=} \frac{1}{\alpha}\int_{\widehat{G}}|\widehat{f}(r)|^2\sum_{E\in\mathcal{E}_0}|\widehat{\mathbb{1}}_{-E}(r)|^2dr.$$

□

2.3 Discrepancy of Sums of Arithmetic Progressions

In this section we want to investigate the discrepancy of a hypergraph that is related to the classical hypergraph of arithmetic progressions in the first N integers. But instead of all arithmetic progressions in $[N]$ we take all sums of k arithmetic progressions in $[N]$ as set of hyperedges, where k is a fixed positive integer. Let $\mathcal{A}$ be the set of all arithmetic progressions in $\mathbb{Z}$. We are considering the hypergraph $\mathcal{H}_{N,k} = ([N], \mathcal{E}_{N,k})$, where

$$\mathcal{E}_{N,k} := \{(A_1 + A_2 + \dots A_k) \cap [N] \mid A_i \in \mathcal{A}\}.$$

We prove the lower bound $\mathrm{disc}(\mathcal{H}_{N,k}, c) = \Omega_k(\frac{N^{k/(2k+2)}}{\sqrt{c}})$. Note that our result gives back Roth's lower bound respectively Doerr and Srivastav's multi-color version of it in the case $k = 1$. The results of this section can be found in [Heb05].

Theorem 2.19. *For all positive integers k we have* $\mathrm{disc}(\mathcal{H}_{N,k}, c) = \Omega_k\left(\frac{N^{k/(2k+2)}}{\sqrt{c}}\right)$.

The structure of the hyperedges of $\mathcal{H}_{N,k}$ is not as regular as the structure of arithmetic progressions. For instance in the sum of two or more arithmetic progressions some elements can have several possibilities to be expressed as sum of elements of the arithmetic progressions. This causes problems for the calculus. And we cannot apply Fourier analysis in a direct way. Instead of this we look for hyperedges of $\mathcal{H}_{N,k}$, which do not have those ambiguities. We are calculating a lower bound for the discrepancy of the subhypergraph containing only this special hyperedges, which is of course also a lower bound for $\mathrm{disc}(\mathcal{H}_{N,k})$.

For convenience, we assume that $2^{k-1}|N^{1/(k+1)}$. Bertrand's postulate (also called Chebyshev's theorem) states the existence of prime numbers p_i for all $i \in \{1, 2, \dots k-1\}$ with $2^{i-k+1}N^{1/(k+1)} < p_i < 2^{i-k+2}N^{1/(k+1)}$. Every sum of k arithmetic progressions is characterized by a starting point, a k–tuple $\delta = (\delta_1, \delta_2, \dots, \delta_k)$ of differences and a k–tuple

$L = (L_1, L_2, \ldots, L_k)$ which fixes the length of the k arithmetic progressions. Let us introduce here the special set of hyperedges for which we will determine a lower discrepancy bound. All of these hyperedges have the same k–tuple $L = (L_1, L_2, \ldots, L_k)$ fixing the length of the k arithmetic progressions that are summed up. We define the length of the i–th arithmetic progression ($i \in \{1, 2, \ldots k\}$) by

$$L_i := 2^{i-k-1} N^{\frac{1}{k+1}}.$$

Let $\widetilde{\Delta} := \prod_{i=1}^{k} \{1, 2, \ldots, 2L_i\}$. We define a set Δ of k–tuples of differences by

$$\Delta := \{(\delta_1, \delta_2, \ldots, \delta_k) \mid \delta_i = \prod_{j=1}^{i} \widetilde{\delta}_j \prod_{j=i}^{k-1} p_j, 1 \le i \le k, (\widetilde{\delta}_1, \widetilde{\delta}_2, \ldots, \widetilde{\delta}_k) \in \widetilde{\Delta}\}.$$

For all $j \in \mathbb{Z}$ and all $\delta = (\delta_1, \delta_2, \ldots, \delta_k) \in \Delta$ we set

$$A_{j,\delta} := \left\{ j + \sum_{i=1}^{k} a_i \delta_i \mid a_i \in \{0, 1, 2, \ldots, L_i - 1\}, 1 \le i \le k \right\} \cap [N].$$

The next lemma shows that we can get $A_{j,\delta} \neq \emptyset$ only for a small set of $j \in \mathbb{Z}$.

Lemma 2.20. *Let $j \in \mathbb{Z}$ and $\delta \in \Delta$ with $A_{j,\delta} \neq \emptyset$. Then $j \in \{-\frac{N}{2} + 1, -\frac{N}{2} + 2, \ldots, N - 1, N\}$.*

Proof. It is obvious that $A_{j,\delta} = \emptyset$ for all $j \ge N + 1$ and all $\delta \in \Delta$. Let $j \le -\frac{N}{2}$ and $\delta \in \Delta$. Then

$$\begin{aligned}
&\max \left\{ j + \sum_{i=1}^{k} a_i \delta_i \mid a_i \in \{0, 1, 2, \ldots, L_i - 1\}, 1 \le i \le k \right\} \\
&< \; -\frac{N}{2} + \sum_{i=1}^{k} L_i \delta_i \\
&\le \; -\frac{N}{2} + \sum_{i=1}^{k} L_i \prod_{j=1}^{i} (2L_j) \prod_{j=i}^{k-1} (2^{j-k+2} N^{1/(k+1)}) \\
&= \; -\frac{N}{2} + \sum_{i=1}^{k} 2^{i-k-1} N^{1/(k+1)} \prod_{j=1}^{i} (2^{j-k} N^{1/(k+1)}) \prod_{j=i}^{k-1} (2^{j-k+2} N^{1/(k+1)}) \\
&= \; N \left(-\frac{1}{2} + \sum_{i=1}^{k} 2^{-1} \prod_{j=1}^{k} 2^{j-k} \right) \\
&= \; N \left(-\frac{1}{2} + k 2^{-(k^2-k+2)/2} \right) \\
&\le \; 0.
\end{aligned}$$

Thus, $A_{j,\delta} = \emptyset$. This proves the lemma. ☐

The non-trivial hyperedges $A_{j,\delta}$ with $j \in \mathbb{Z}$ and $\delta \in \Delta$ are building the subhypergraph mentioned above. We set $E_\delta := A_{0,\delta}$ for all $\delta \in \Delta$.

For the proof of Theorem 2.19 we will use the Fourier analytic method from Section 2.2. Thus, we need to introduce the Fourier transform in $\mathbb{Z}$. Let $\mathbb{T}$ denote the one-dimensional torus. $\mathbb{T}$ is isomorphic to the group $\mathbb{R}/\mathbb{Z}$. The assertion of the following proposition is well known.

Proposition 2.21.

(i) $\widehat{\mathbb{Z}} \cong \mathbb{T}$.

(ii) The Fourier transform of a function $f \in L^1(\mathbb{Z})$ *can be written as*

$$\widehat{f} : [0,1] \to \mathbb{C}, \quad \alpha \mapsto \sum_{z \in \mathbb{Z}} f(z) e^{-2\pi i z \alpha}.$$

The next lemma states that for every c-coloring of $\mathcal{H}_{N,k}$ the discrepancy is of the order stated in Theorem 2.19 or there is a color-class that consists of at least $\frac{N}{c}$ vertices but also not much more.

Lemma 2.22. *Let* $\chi : V \to [c]$ *be a* c*–coloring of* $\mathcal{H}_{N,k}$ *and* $\alpha > 0$*, then it holds* $\operatorname{disc}(\mathcal{H}_{N,k}, c, \chi) > \alpha \frac{N^{k/(2k+2)}}{\sqrt{c}}$ *or there is a color* $i \in [c]$ *such that it holds for* $A := \chi^{-1}(i)$ *and* $\delta_A := \frac{1}{N}|A|$

$$0 \le \delta_A - \frac{1}{c} \le \alpha c^{-1/2} N^{-(k+2)/(2k+2)}.$$

Proof. There is at least one color $i \in [x]$ with $|\chi^{-1}(i)| \ge \frac{N}{c}$. If there is a color $i \in [c]$ with $|\chi^{-1}(i)| - \frac{N}{c} > \alpha \frac{N^{k/(2k+2)}}{\sqrt{c}}$, then $\operatorname{disc}(\mathcal{H}_{N,k}, c, \chi) > \alpha \frac{N^{k/(2k+2)}}{\sqrt{c}}$, because $[N]$ itself can be expressed as sum of k arithmetic progressions. Thus, we can assume that there is no color $i \in [c]$ with $|\chi^{-1}(i)| - \frac{N}{c} > \alpha \frac{N^{k/(2k+2)}}{\sqrt{c}}$. In particular, this yields the existence of a color $i \in [c]$ such that it holds for $A := \chi^{-1}(i)$

$$0 \le |A| - \frac{N}{c} \le \alpha \frac{N^{k/(2k+2)}}{\sqrt{c}}.$$

Set $\delta_A := \frac{1}{N}|A|$. Then

$$0 \le \delta_A - \frac{1}{c} \le \alpha c^{-1/2} N^{-(k+2)/(2k+2)}.$$

□

Lemma 2.22 helps us in the proof of Theorem 2.19 in the following way. Since we want to prove a discrepancy of order at least $\Omega_k(\frac{N^{k/(2k+2)}}{\sqrt{c}})$, we can assume the latter case. Using this color-class A and the Fourier analytic method, we will show the existence of a hyperedge E with $||E \cap A| - \delta_A|E|| \geq \beta\frac{N^{k/(2k+2)}}{\sqrt{c}}$ for a constant $\beta > 0$. Since δ_A is roughly $\frac{1}{c}$, the triangle-inequality shows the lower bound.

For all $\delta \in \Delta$ and all $i \in \{1, 2, \dots k\}$ let $\eta_{\delta_i} : \mathbb{Z} \to \{0, 1\}$ be defined by

$$\eta_{\delta_i}(j) := \begin{cases} 1\,, & \text{if } -j \in \delta_i\{0, 1, 2, \dots, L_i - 1\}, \\ 0\,, & \text{otherwise.} \end{cases}$$

The function $\eta_\delta := \eta_{\delta_1} * \eta_{\delta_2} * \dots * \eta_{\delta_k}$ is an indicator function for the set $-E_\delta$ as the following lemma states.

Lemma 2.23. *Let $\delta \in \Delta$. Then $\eta_\delta(x) = \mathbb{1}_{-E_\delta}(x)$ for all $x \in \mathbb{Z}$.*

Proof. For every $x \in \mathbb{Z}$ we have

$$\eta_\delta(x) = \sum_{x_1, x_2, \dots, x_{k-1} \in \mathbb{Z}} \eta_{\delta_1}(x_1)\eta_{\delta_2}(x_2) \dots \eta_{\delta_{k-1}}(x_{k-1})\eta_{\delta_k}(x - x_1 - x_2 - \dots - x_{k-1}).$$

Thus, for every $x \in \mathbb{Z}$ we have $\eta_\delta(x) > 0$ if and only if there exist $x_1, x_2, \dots, x_k \in \mathbb{Z}$ with $x = \sum_{i=1}^k x_i$ and $\eta_{\delta_i}(x_i) = 1$ for all $i \in \{1, \dots, k\}$. But this again holds if and only if $-x \in E_\delta$. Let $x \in \mathbb{Z}$ with $\eta_\delta(x) \geq 1$ and $e, e' \in \mathbb{Z}^k$ with

(i) $e_i, e'_i \in \{0, 1, 2, \dots, L_i - 1\}$, for all $i \in \{1, 2, \dots, k\}$,

(ii) $-x = \sum_{i=1}^k e_i\delta_i = \sum_{i=1}^k e'_i\delta_i$.

We show $e = e'$ by descending induction over the components of e respectively e'. One can easily see that it is sufficient to prove $e_i = e'_i$ for all $i \in \{2, 3, \dots, k\}$. We have

$$p_{k-1} \quad \Big| \quad \sum_{i=1}^{k-1}(e'_i - e_i)\delta_i = (e_k - e'_k)\delta_k = (e_k - e'_k)\prod_{i=1}^{k}\widetilde{\delta}_i.$$

Since $p_{k-1} > N^{\frac{1}{k+1}} \geq \widetilde{\delta}_i$ for all $i \in \{1, 2, \dots k\}$, $p_{k-1} > N^{\frac{1}{k+1}} > |e_k - e'_k|$ and p_{k-1} is a prime number we have $e_k = e'_k$. In the case $k = 2$ we get $e = e'$. Thus we can assume $k \geq 3$. Let $i \in \{3, 4, \dots k\}$ such that we already know $e_j = e'_j$ for all $j \in \{i, i+1, \dots, k\}$. Thus, we have

$$\prod_{j=i-2}^{k-1} p_j \quad \Big| \quad \sum_{j=1}^{i-2}(e'_j - e_j)\delta_j = (e_{i-1} - e'_{i-1})\delta_{i-1} = (e_{i-1} - e'_{i-1})\prod_{j=1}^{i-1}\widetilde{\delta}_j \prod_{j=i-1}^{k-1} p_j.$$

Using $p_{i-2} > 2^{i-k-1}N^{\frac{1}{k+1}} \geq \widetilde{\delta}_j$ for all $j \in \{1, 2, \dots i-1\}$, $p_{i-2} > 2^{i-k-1}N^{\frac{1}{k+1}} > |e_{i-1} - e'_{i-1}|$ and that p_{i-2} is a prime number, we get $e_{i-1} = e'_{i-1}$, which concludes the proof. □

The next lemma gives an estimation for exponential sums that we use later on for the proof of Theorem 2.19.

Lemma 2.24. *Let $\alpha \in \mathbb{R}$, $L \in \mathbb{N}$. There exists an integer $\delta \in \{1, 2, \dots, 2L\}$ such that*

$$\left| \sum_{j=0}^{L-1} e^{2\pi i \delta j \alpha} \right|^2 \geq \left(\frac{2}{\pi} L \right)^2.$$

Proof. If $\delta\alpha \in \mathbb{Z}$ for a $\delta \in \{1, 2, \dots, 2L\}$ the assertion is trivially fulfilled. Thus, we can assume that this is not the case. Using $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$ and hence $1 - \cos(2\alpha) = 2\sin^2(\alpha)$ we get for all $\beta \in \mathbb{R} \setminus \mathbb{Z}$

$$\begin{aligned} \left| \sum_{j=0}^{L-1} e^{2\pi i j \beta} \right|^2 &= \left| \frac{1 - e^{2\pi i \beta L}}{1 - e^{2\pi i \beta}} \right|^2 \\ &= \frac{\left(1 - e^{2\pi i \beta L}\right)\left(1 - e^{-2\pi i \beta L}\right)}{\left(1 - e^{2\pi i \beta}\right)\left(1 - e^{-2\pi i \beta}\right)} \\ &= \frac{2 - 2\,\mathrm{Real}(e^{2\pi i \beta L})}{2 - 2\,\mathrm{Real}(e^{2\pi i \beta})} \\ &= \frac{1 - \cos 2\pi\beta L}{1 - \cos 2\pi\beta} \\ &= \frac{\sin^2(\pi\beta L)}{\sin^2(\pi\beta)}. \end{aligned} \tag{2.1}$$

We prove

$$\left| \frac{\sin(\pi\beta L)}{\sin(\pi\beta)} \right| \geq \frac{2}{\pi} L, \text{ if } |\beta| \leq \frac{1}{2L} \text{ and } \beta \neq 0. \tag{2.2}$$

Since $\left|\frac{\sin(x)}{x}\right| \leq 1$, we have $\left|\frac{\pi\beta}{\sin(\pi\beta)}\right| \geq 1$. For the function $f(x) := \frac{\sin(x)}{x}$ defined on $\mathbb{R} \setminus \{0\}$ we get $\lim_{x \to 0} f(x) = 1$ and $f(\pm\frac{\pi}{2}) = \frac{2}{\pi}$. Now the derivation of the function f is $f'(x) = \frac{x\cos(x) - \sin(x)}{x^2}$. Thus, if $x \leq \tan(x)$ then $f'(x) \leq 0$. Therefore f is monotone decreasing in $(0, \frac{\pi}{2}]$ and monotone increasing in $[-\frac{\pi}{2}, 0)$ because of the symmetry of f. Hence $\frac{\sin(\pi\beta L)}{\pi\beta L} \geq \frac{2}{\pi}$ holds using $|\pi\beta L| \leq \frac{\pi}{2}$. Thus, we have $\left|\frac{\sin(\pi\beta L)}{\pi\beta L}\right| \left|\frac{\pi\beta}{\sin(\pi\beta)}\right| \geq \frac{\pi}{2}$. Our next aim is to show the existence of a $\delta \in \{1, 2, \dots, 2L\}$ and a $z \in \mathbb{Z}$ with

$$|\delta\alpha - z| \leq \tfrac{1}{2L}. \tag{2.3}$$

For every $i \in \{1, 2, \dots, 2L\}$ we define $x_i := i\alpha - \lfloor i\alpha \rfloor$, where $\lfloor x \rfloor$ is the largest integer less or equal than x for each $x \in \mathbb{R}$.
Now there is either an $i \in \{1, 2, \dots, 2L\}$ with $x_i \in [0, \frac{1}{2L}]$ or a $j \in \{1, 2, \dots, 2L - 1\}$ with

$$\left| \left\{ i \in \{1, 2, \dots, 2L\} \,\middle|\, x_i \in \left[\tfrac{j}{2L}, \tfrac{j+1}{2L}\right] \right\} \right| \geq 2.$$

In the former case we are done. In the latter case let $i_1, i_2 \in \{1, 2, \ldots, 2L\}$ with $i_1 < i_2$ and $x_1, x_2 \in [\frac{j}{2L}, \frac{j+1}{2L}]$. Set $\delta := i_2 - i_1 \in \{1, 2, \ldots, 2L\}$. It holds

$$\begin{aligned} |\delta\alpha - (\lfloor i_2\alpha \rfloor - \lfloor i_2\alpha \rfloor)| &= |i_2\alpha - \lfloor i_2\alpha \rfloor - (i_1\alpha - \lfloor i_1\alpha \rfloor)| \\ &= |x_{i_2} - x_{i_1}| \\ &\leq \tfrac{1}{2L}. \end{aligned}$$

Thus, there are a $\delta \in \{1, 2, \ldots 2L\}$, a $z \in \mathbb{Z}$ and a $\beta \in \mathbb{R}$ such that

(i) $\delta\alpha = z + \beta$,

(ii) $|\beta| \leq \frac{1}{2L}$ and $\beta \neq 0$.

Hence,

$$\begin{aligned} \left| \sum_{j=0}^{L-1} e^{2\pi i \delta j \alpha} \right|^2 &\overset{(2.1)}{=} \frac{\sin^2(\pi\delta\alpha L)}{\sin^2(\pi\delta\alpha)} \\ &= \left| \frac{\sin(\pi\beta L + \pi z L)}{\sin(\pi\beta + \pi z)} \right|^2 \\ &= \left| \frac{\sin(\pi\beta L)}{\sin(\pi\beta z)} \right|^2 \\ &\overset{(2.2)}{\geq} \left(\frac{2}{\pi} L \right)^2. \end{aligned}$$

□

The following lemma gives us the needed estimation for the proof of Theorem 2.19.

Lemma 2.25. *Let $\alpha \in [0, 1]$. There exists a constant $c_1 > 0$ only depending on k such that*

$$\sum_{\delta \in \Delta} |\hat{\mathbb{1}}_{-E_\delta}(\alpha)|^2 \geq c_1 N^{\frac{2k}{k+1}}.$$

Proof. First we observe that for all $\delta \in \Delta$ and all $t \in \{1, 2, \ldots, k\}$

$$\begin{aligned} \hat{\eta}_{\delta_t}(\alpha) &= \sum_{x \in \mathbb{Z}} \eta_{\delta_t}(x) e^{-2\pi i x \alpha} \\ &= \sum_{j=0}^{L_t - 1} e^{2\pi i j \delta_t \alpha}. \end{aligned}$$

Thus, Theorem 2.13 yields

$$\begin{aligned}\sum_{\delta\in\Delta}|\hat{\mathbb{1}}_{-E_\delta}(\alpha)|^2=\sum_{\delta\in\Delta}|\hat{\eta}_\delta(\alpha)|^2 &= \sum_{\delta\in\Delta}\prod_{t=1}^{k}|\hat{\eta}_{\delta_t}(\alpha)|^2\\ &= \sum_{\widetilde{\delta_1}=1}^{2L_1}\cdots\sum_{\widetilde{\delta_k}=1}^{2L_k}\prod_{t=1}^{k}\left|\sum_{j=0}^{L_t-1}e^{2\pi ij\alpha\prod_{s=1}^{t}\widetilde{\delta_s}\prod_{s=t}^{k-1}p_s}\right|^2 \end{aligned}\tag{2.4}$$

Using Lemma 2.24 we can find a $\bar{\delta}_1\in\{1,2,\dots,2L_1\}$ with

$$\left|\sum_{j=0}^{L_1-1}e^{2\pi ij\bar{\delta}_1\left(\alpha\prod_{s=1}^{k-1}p_s\right)}\right|^2\geq\left(\frac{2}{\pi}L_1\right)^2.$$

In the same way we get $\bar{\delta}_2\in\{1,2,\dots,2L_2\}$, $\bar{\delta}_3\in\{1,2,\dots,2L_3\}$ up to $\bar{\delta}_k\in\{1,2,\dots,2L_k\}$ one by one such that for all $t\in\{1,2,\dots,k\}$ we have

$$\left|\sum_{j=0}^{L_t-1}e^{2\pi ij\bar{\delta}_t\left(\alpha\prod_{s=1}^{t-1}\bar{\delta}_s\prod_{s=t}^{k-1}p_s\right)}\right|^2\geq\left(\frac{2}{\pi}L_t\right)^2.\tag{2.5}$$

Thus, using (2.4) and (2.5) we get for an appropriate constant $c_1>0$ only depending on k:

$$\begin{aligned}\sum_{\delta\in\Delta}|\hat{\mathbb{1}}_{-E_\delta}(\alpha)|^2 &\overset{(2.4)}{=} \prod_{t=1}^{k}\left|\sum_{j=0}^{L_t-1}e^{2\pi ij\alpha\prod_{s=1}^{t}\bar{\delta}_s\prod_{s=t}^{k-1}p_s}\right|^2\\ &\overset{(2.5)}{\geq}\prod_{t=1}^{k}\left(\frac{2}{\pi}L_t\right)^2\\ &\geq c_1N^{\frac{2k}{k+1}}.\end{aligned}$$

□

Proof of Theorem 2.19: Let $\chi:V\to[c]$ be a c–coloring of $\mathcal{H}_{N,k}$. Using Lemma 2.22 we can assume that there is a color $i\in[c]$ such that we get for $A:=\chi^{-1}(i)$ and $\delta_A:=\frac{1}{N}|A|$

$$0\leq\delta_A-\frac{1}{c}\leq\alpha c^{-1/2}N^{-(k+2)/(2k+2)}\tag{2.6}$$

for a constant $0<\alpha\leq\frac{1}{2}$ that we fix later on in the proof. Otherwise Lemma 2.22 yields $\operatorname{disc}(\mathcal{H}_{N,k},c,\chi)>\alpha\frac{N^{k/(2k+2)}}{\sqrt{c}}$. Let $f_A:\mathbb{Z}\to\mathbb{C}$ be defined by

$$f_A(x):=\begin{cases}1-\delta_A & : \ x\in A,\\ -\delta_A & : \ x\in[N]\setminus A,\\ 0 & : \ x\in\mathbb{Z}\setminus[N],\end{cases}$$

for every $x \in \mathbb{Z}$. For every subset $X \subseteq [N]$ we have

$$f_A(X) := \sum_{x \in X} f_A(x) = \sum_{x \in X \cap A} (1 - \delta_A) + \sum_{x \in X \setminus A} (-\delta_A) = |X \cap A| - \delta_A |X|.$$

The estimation $\sum_{\delta \in \Delta} |\hat{\mathbb{1}}_{-E_\delta}(\alpha)|^2 \geq c_1 N^{\frac{2k}{k+1}}$ from Lemma 2.25 allows us to apply Corollary 2.17. We get

$$\begin{aligned}
\sum_{\delta \in \Delta} \sum_{j \in \mathbb{Z}} |f_A(A_{j,\delta})|^2 &= \sum_{\delta \in \Delta} \sum_{j \in \mathbb{Z}} |f_A(j + E_\delta)|^2 \\
&\geq c_1 N^{\frac{2k}{k+1}} \|f_A\|_2^2 \\
&= c_1 N^{\frac{2k}{k+1}} \left(\delta_A N (1 - \delta_A)^2 + (1 - \delta_A) N (-\delta_A)^2\right) \\
&= c_1 \delta_A (1 - \delta_A) N^{\frac{3k+1}{k+1}}
\end{aligned}$$

It holds $|\Delta| = |\widetilde{\Delta}| = \prod_{i=1}^{k} (2L_i) = O(N^{\frac{k}{k+1}})$. Hence there exists a constant $c_2 > 0$ and a $\delta_0 \in \Delta$ such that

$$\sum_{j \in \mathbb{Z}} |f_A(A_{j,\delta_0})|^2 \geq \frac{c_1}{|\Delta|} \delta_A (1 - \delta_A) N^{\frac{3k+1}{k+1}} \geq c_2 \delta_A (1 - \delta_A) N^{\frac{2k+1}{k+1}}.$$

Lemma 2.20 yields that we have $A_{j,\delta_0} = \emptyset$ for all $j \in \mathbb{Z} \setminus \{-\frac{N}{2} + 1, -\frac{N}{2} + 1, \dots, N - 1, N\}$. Therefore we can find a $j_0 \in \{-\frac{N}{2} + 1, -\frac{N}{2} + 1, \dots, N - 1, N\}$ such that

$$|f_A(A_{j_0,\delta_0})| \geq \sqrt{\frac{c_2}{2}} \sqrt{\delta_A (1 - \delta_A)} N^{\frac{k}{2k+2}}.$$

Set $x := \delta_A - \frac{1}{c}$. It holds $0 \leq x \leq \alpha c^{-1/2} N^{-(k+2)/(2k+2)} \leq \frac{1}{2\sqrt{c}}$. For $c_3 := \sqrt{\frac{c_2}{8}}$ we get

$$\begin{aligned}
|f_A(A_{j_0,\delta_0})| &\geq \sqrt{\frac{c_2}{2}} \sqrt{\delta_A (1 - \delta_A)} N^{\frac{k}{2k+2}} \\
&= \sqrt{\frac{c_2}{2}} \sqrt{\left(\frac{1}{c} + x\right) \left(\frac{c-1}{c} - x\right)} N^{\frac{k}{2k+2}} \\
&= \sqrt{\frac{c_2}{2}} \sqrt{\frac{c-1}{c^2} + \frac{c-2}{c} x - x^2 N^{\frac{k}{2k+2}}} \\
&= \sqrt{\frac{c_2}{2}} \sqrt{\frac{1}{2c} - \frac{1}{4c}} N^{\frac{k}{2k+2}} \\
&= c_3 \frac{N^{\frac{k}{2k+2}}}{\sqrt{c}}.
\end{aligned}$$

Now we fix the constant α in (2.6). W.l.o.g. we can assume $c_3 \leq 1$ and set $\alpha := \frac{c_3}{2}$. Then

$$\begin{aligned}
\mathrm{disc}(\mathcal{H}_{N,k}, c, \chi) &\geq \left| |A_{j_0,\delta_0}| - \frac{1}{c}|A_{j_0,\delta_0}| \right| \\
&= \left| |A_{j_0,\delta_0}| - \delta_A |A_{j_0,\delta_0}| + \left(\delta_A - \frac{1}{c}\right) |A_{j_0,\delta_0}| \right| \\
&\geq ||A_{j_0,\delta_0}| - \delta_A |A_{j_0,\delta_0}|| - \left|\delta_A - \frac{1}{c}\right| |A_{j_0,\delta_0}| \\
&\geq c_3 \frac{N^{\frac{k}{2k+2}}}{\sqrt{c}} - \frac{c_3}{2} c^{-1/2} N^{-(k+2)/(2k+2)} N \\
&= \frac{c_3}{2} \frac{N^{\frac{k}{2k+2}}}{\sqrt{c}}.
\end{aligned}$$

Thus, we have shown $\mathrm{disc}(\mathcal{H}_{N,k}, c) \geq \alpha \frac{N^{\frac{k}{2k+2}}}{\sqrt{c}}$, where the constant $\alpha > 0$ depends only on k. □

We have studied the discrepancy of the hypergraph of sums of k arithmetic progressions. Our main result is the lower bound of $\Omega(N^{k/(2k+2)})$ for this hypergraph. Here $k = 1$ gives back Roth's lower bound for the hypergraph of arithmetic progressions. For the proof of this lower bound we used Fourier analysis on $\mathbb{Z}$ and a special set of hyperedges. There is a large gap between this lower bound and the upper bound of $O(N^{1/2} \log^{1/2}(N))$, which is easily derived by the probabilistic method [AS00]. Our believe is that—as for the hypergraph of arithmetic progressions—the probabilistic method tells not the whole truth and a better upper bound can be found.

2.4 Discrepancy of d–dimensional Arithmetic Progressions with Common Difference

In the Section 2.1 we mentioned the generalization of the hypergraph $\mathcal{H}_N = ([N], \mathcal{E}_N)$ of arithmetic progressions in $[N]$ to higher dimensions by Doerr, Srivastav and Wehr [DSW04]. They investigated the order of discrepancy for the hypergraph $\mathcal{H}_{N,d} = ([N]^d, \mathcal{E}_{N,d})$, where the set of hyperedges is $\mathcal{E}_{N,d} := \{\mathop{\mathsf{X}}_{i=1}^{d} A_{a_i,\delta_i,L_i} \mid A_{a_i,\delta_i,L_i} \in \mathcal{E}_N\}$ and proved that it holds $\mathrm{disc}(\mathcal{H}_{N,d}) = \Theta_d(N^{\frac{d}{4}})$. But what is the effect for the discrepancy, if we take only Cartesian products of arithmetic progressions into account that have a common difference? In this section we determine a lower and an upper bound for the c–color discrepancy of the hypergraph $\mathcal{H}'_{N,d} = ([N]^d, \mathcal{E}'_{N,d})$ with

$$\mathcal{E}'_{N,d} := \left\{ \bigtimes_{i=1}^{d} A_{a_i,\delta,L_i} \mid A_{a_i,\delta,L_i} \in \mathcal{E}_N \right\}.$$

We prove the following.

Theorem 2.26. *Let $d, N \in \mathbb{N}$. It holds*

(i) $\operatorname{disc}(\mathcal{H}'_{N,d}, c) = \Omega_d\left(\frac{1}{\sqrt{c}} N^{\frac{d}{2d+2}}\right)$.

(ii) $\operatorname{disc}(\mathcal{H}'_{N,d}, c) = O_d\left(N^{\frac{d}{2d+2}} \log^{\frac{3}{2}d+2} N\right)$.

2.4.1 The Lower Bound

For the proof of the lower bound we take only a special set of hyperedges into account. More precisely we look for all hyperedges

$$A_{a,\delta} := \{a + \delta b \mid b \in \{0, 1, \ldots, L-1\}^d\} \cap [N]^d$$

for $L := \frac{1}{2} N^{\frac{1}{d+1}}$, all $\delta \in \Delta := \left[N^{\frac{d}{d+1}}\right]$ and all $a \in \mathbb{Z}^d$. For convenience let us assume that $\frac{1}{2} N^{\frac{1}{d+1}} \in \mathbb{N}$. It is easy to see that there are at most $\left(\frac{3}{2} N\right)^d$ elements $a \in \mathbb{Z}^d$ such that $A_{a,\delta} \neq \emptyset$ for some $\delta \in \Delta$. Thus, we look for the discrepancy of a subhypergraph of $\mathcal{H}'_{N,d}$, which consists of at most $\left(\frac{3}{2}\right)^d N^{d+\frac{d}{d+1}}$ hyperedges. A lower discrepancy bound for this subhypergraph is trivially also a lower bound for $\operatorname{disc}(\mathcal{H}'_{N,d}, c)$. We want to apply the Fourier analytic method from Section 2.2. For this, we have to introduce the Fourier transform in $\mathbb{Z}^d$. By $\mathbb{T}^d$ we denote the d–dimensional torus. $\mathbb{T}^d$ is isomorphic to the group $(\mathbb{R}/\mathbb{Z})^d$. The following is well known.

Proposition 2.27.

(i) $\widehat{\mathbb{Z}^d} \cong \mathbb{T}^d$.

(ii) The Fourier transform of a function $f \in L^1(\mathbb{Z}^d)$ can be written as

$$\widehat{f} : [0,1]^d \to \mathbb{C}, \quad \alpha \mapsto \sum_{z \in \mathbb{Z}^d} f(z) e^{-2\pi i \langle z, \alpha \rangle}.$$

By $\langle z, \alpha \rangle$ we mean the common inner product $\langle z, \alpha \rangle = \sum_{j=1}^{d} z_j \alpha_j$.

Using the next lemma, we can assume for every c–coloring $\chi : [N]^d \to [c]$ of $\mathcal{H}'_{N,d}$ in the proof of the lower bound that there is a color $i \in [c]$ such that at least but also not much more than $\frac{N^d}{c}$ elements of $[N]^d$ are colored in the color i.

Lemma 2.28. *Let $\chi : [N]^d \to [c]$ be a c–coloring of $\mathcal{H}'_{N,d}$ and $\alpha > 0$. Then it holds $\operatorname{disc}(\mathcal{H}'_{N,d}, c, \chi) > \alpha\frac{N^{d/(2d+2)}}{\sqrt{c}}$ or there exists a color $i \in [c]$ such that*

$$0 \le \delta_A - \frac{1}{c} \le \alpha c^{-1/2} N^{-d+d/(2d+2)}$$

for $A := \chi^{-1}(i)$ and $\delta_A := \frac{1}{N^d}|A|$.

Proof. $\frac{N^d}{c}$ is the average size of a color-class of χ. Therefore, there exists at least one color $i \in [c]$ with $|\chi^{-1}(i)| \ge \frac{N^d}{c}$. If there is a color $i \in [c]$ such that $|\chi^{-1}(i)| - \frac{N^d}{c} > \alpha\frac{N^{d/(2d+2)}}{\sqrt{c}}$, then we get with $[N]^d$ itself as a d–dimensional arithmetic progression that has common difference 1

$$\begin{aligned} \operatorname{disc}(\mathcal{H}'_{N,d}, c, \chi) &\ge ||\chi^{-1}(i) \cap [N]^d| - \frac{|[N]^d|}{c}| \\ &= ||\chi^{-1}(i)| - \frac{N^d|}{c}| \\ &> \alpha\frac{N^{d/(2d+2)}}{\sqrt{c}}. \end{aligned}$$

Thus, we can assume that there is no such color. This yields the existence of a color $i \in [c]$ with $0 \le |\chi^{-1}(i)| - \frac{N^d}{c} \le \alpha\frac{N^{d/(2d+2)}}{\sqrt{c}}$. We set $A := \chi^{-1}(i)$ and $\delta_A := \frac{1}{N^d}|A|$ and get

$$0 \le \delta_A - \frac{1}{c} \le \alpha c^{-1/2} N^{-d+d/(2d+2)}.$$

□

For the lower bound proof, we have to estimate Fourier coefficients of the indicator functions of the special hyperedges mentioned above. For this estimation we will use the following lemma.

Lemma 2.29. *Let $\alpha \in [0,1]^d$ and $J := \{0, 1, \dots, L-1\}^d$. There exists a $\delta \in \Delta$ with*

$$\left|\sum_{j \in J} e^{2\pi i \delta \langle j, \alpha\rangle}\right|^2 \ge \left(\frac{2}{\pi} L\right)^{2d} = \left(\frac{1}{\pi}\right)^{2d} N^{\frac{2d}{d+1}}.$$

Proof. It holds

$$\begin{aligned} \left|\sum_{j \in J} e^{2\pi i \delta \langle j, \alpha\rangle}\right|^2 &= \left|\sum_{j_1=0}^{L-1}\sum_{j_2=0}^{L-1}\cdots\sum_{j_d=0}^{L-1}\prod_{k=1}^{d} e^{2\pi i \delta j_k \alpha_k}\right|^2 \\ &= \prod_{k=1}^{d}\left|\sum_{j_k=0}^{L-1} e^{2\pi i \delta j_k \alpha_k}\right|^2. \end{aligned}$$

In the proof of Lemma 2.24 we have shown the following two facts:

(i) $\left|\sum_{j=0}^{L-1} e^{2\pi i j\beta}\right|^2 = \frac{\sin^2(\pi\beta L)}{\sin^2(\pi\beta)}$ for all $\beta \in \mathbb{R}\setminus\mathbb{Z}$ and

(ii) $\left|\frac{\sin(\pi\beta L)}{\sin(\pi\beta)}\right| \geq \frac{2}{\pi}L$, if $|\beta| \leq \frac{1}{2L}$ and $\beta \neq 0$.

Thus, we have to show the existence of a $\delta \in \Delta = [(2L)^d]$ such that for every $k \in [d]$ there are $z_k \in \mathbb{Z}$ and $\beta_k \in \mathbb{R}$ with

(a) $\delta\alpha_k = z_k + \beta_k$ and

(b) $|\beta_k| \leq \frac{1}{2L}$.

For every $j \in [(2L)^d]$ and every $k \in [d]$ we set $x_{j,k} := j\alpha_k - \lfloor j\alpha_k \rfloor$, where $\lfloor x \rfloor$ is the largest integer less or equal x for every $x \in \mathbb{R}$. There is a $j \in [(2L)^d]$ such that $x_{j,k} \in [0, \frac{1}{2L}]$ for all $k \in [d]$ or using the pigeon hole-principle there exists an $h \in \{0, 1, \ldots, 2L-1\}^d \setminus \{(0, 0, \ldots, 0)\}$ such that the set

$$M_h := \left\{ j \in [(2L)^d] \,\middle|\, x_{j,k} \in \left[\frac{h_k}{2L}, \frac{h_k+1}{2L}\right], k \in [d] \right\}$$

contains at least two elements. In the first case we are done. Thus, we can assume the second case and choose $j_1, j_2 \in M_h$ with $j_1 < j_2$. Set $\delta := j_2 - j_1$. Then $\delta \in [(2L)^d]$ and we get for all $k \in [d]$:

$$\begin{aligned} |\delta\alpha_k - (\lfloor j_2\alpha_k \rfloor - \lfloor j_1\alpha_k \rfloor)| &= |j_2\alpha_k - \lfloor j_2\alpha_k \rfloor - (j_1\alpha_k - \lfloor j_1\alpha_k \rfloor)| \\ &= |x_{j_2,k} - x_{j_1,k}| \\ &\leq \frac{1}{2L}. \end{aligned}$$

This proves the existence of a $\delta_0 \in [(2L)^d]$ and of $z_k \in \mathbb{Z}$ and $\beta_k \in \mathbb{R}$ for all $k \in [d]$ such that it holds $\delta_0\alpha_k = z_k + \beta_k$ and $|\beta_k| \leq \frac{1}{2L}$ for all $k \in [d]$. Let $k \in [d]$. If $\beta_k = 0$, we have

$$\left|\sum_{j_k=0}^{L-1} e^{2\pi i \delta_0 j_k \alpha_k}\right|^2 = \left|\sum_{j_k=0}^{L-1} e^{2\pi i j_k z_k}\right|^2 = L^2.$$

If $\beta_k \neq 0$, we get

$$\left|\sum_{j_k=0}^{L-1} e^{2\pi i \delta_0 j_k \alpha_k}\right|^2 = \left|\frac{\sin(\pi\beta_k L + \pi z_k L)}{\sin(\pi\beta_k + \pi z_k)}\right|^2 = \left|\frac{\sin(\pi\beta_k L)}{\sin(\pi\beta_k)}\right|^2 \geq \left(\frac{2}{\pi}L\right)^2.$$

Thus,

$$\left|\sum_{j\in J} e^{2\pi i\delta\langle j,\alpha\rangle}\right|^2 = \prod_{k=1}^{d}\left|\sum_{j_k=0}^{L-1} e^{2\pi i\delta_0 j_k\alpha_k}\right|^2 \geq \left(\frac{2}{\pi}L\right)^{2d} = \left(\frac{1}{\pi}\right)^{2d} N^{\frac{2d}{d+1}}.$$

□

Proof of Theorem 2.26 (i). Let $\chi : [N]^d \to [c]$ be a c–coloring of $\mathcal{H}'_{N,d}$. Using Lemma 2.28 we can assume the existence of a color $i \in [c]$ such that we get for $A := \chi^{-1}(i)$ and $\delta_A := \frac{1}{N^d}|A|$

$$0 \leq \delta_A - \frac{1}{c} \leq \alpha c^{-1/2} N^{-d+d/(2d+2)} \tag{2.7}$$

for a constant $0 < \alpha \leq \frac{1}{2}$ that we fix later on in this proof. Otherwise Lemma 2.28 yields $\operatorname{disc}(\mathcal{H}'_{N,d}, c, \chi) > \alpha \frac{N^{d/(2d+2)}}{\sqrt{c}}$. We define the function $f_A : \mathbb{Z}^d \to \mathbb{C}$ by

$$f_A(x) := \begin{cases} 1-\delta_A & : \; x \in A, \\ -\delta_A & : \; x \in [N]^d \setminus A, \\ 0 & : \; x \in \mathbb{Z}^d \setminus [N]^d, \end{cases}$$

for all $x \in \mathbb{Z}$. For every subset $X \subseteq [N]$ we have

$$f_A(X) := \sum_{x\in X} f_A(x) = \sum_{x\in X\cap A}(1-\delta_A) + \sum_{x\in X\setminus A}(-\delta_A) = |X\cap A| - \delta_A|X|.$$

Since δ_A is about $\frac{1}{c}$, for every hyperedge $E \in \mathcal{E}'_{N,d}$ the discrepancy of E in the color i is approximately $|f_A(E)|$.

For every $\delta \in \Delta$ we set $E_\delta := A_{0,\delta}$ and get the following equation for the Fourier coefficients of the indicator function $\mathbb{1}_{-E_\delta}$. Let $\alpha \in [0,1]^d$.

$$\begin{aligned} \widehat{\mathbb{1}}_{-E_\delta}(\alpha) &= \sum_{z\in\mathbb{Z}^d} \mathbb{1}_{-E_\delta}(z) e^{-2\pi i\langle z,\alpha\rangle} \\ &= \sum_{z\in E_\delta} e^{2\pi i\langle z,\alpha\rangle} \\ &= \sum_{j\in J} e^{2\pi i\delta\langle j,\alpha\rangle}, \end{aligned}$$

with $J = \{0,1,\ldots,L-1\}^d$. Thus, Lemma 2.29 yields the existence of a $\delta \in \Delta$ with

$$\left|\widehat{\mathbb{1}}_{-E_\delta}(\alpha)\right|^2 = \left|\sum_{j\in J} e^{2\pi i\delta\langle j,\alpha\rangle}\right|^2 \geq \left(\frac{1}{\pi}\right)^{2d} N^{\frac{2d}{d+1}}.$$

Hence, it holds for every $\alpha \in [0,1]^d$:

$$\sum_{\delta\in\Delta} \left|\widehat{\mathbb{1}}_{-E_\delta}(\alpha)\right|^2 \geq \left(\frac{1}{\pi}\right)^{2d} N^{\frac{2d}{d+1}}. \tag{2.8}$$

Using this estimation, we can apply Corollary 2.17 and get

$$\begin{aligned}\sum_{\delta\in\Delta}\sum_{j\in\mathbb{Z}^d} |f_A(A_{j,\delta})|^2 &= \sum_{\delta\in\Delta}\sum_{j\in\mathbb{Z}^d} |f_A(j+E_\delta)|^2 \\ &\geq \left(\frac{1}{\pi}\right)^{2d} N^{\frac{2d}{d+1}} \|f_A\|_2^2 \\ &= \left(\frac{1}{\pi}\right)^{2d} N^{\frac{2d}{d+1}} \left(\delta_A N^d (1-\delta_A)^2 + (1-\delta_A)N^d(-\delta_A)^2\right) \\ &= \left(\frac{1}{\pi}\right)^{2d} \delta_A(1-\delta_A) N^{\frac{2d}{d+1}+d}.\end{aligned}$$

It holds $|\Delta| = |\widetilde{\Delta}| = N^{\frac{d}{d+1}}$. Hence there exists a $\delta_0 \in \Delta$ such that

$$\sum_{j\in\mathbb{Z}} |f_A(A_{j,\delta_0})|^2 \geq \left(\frac{1}{\pi}\right)^{2d} \delta_A(1-\delta_A) N^{\frac{d}{d+1}+d}.$$

As we have mentioned before, there are at most $\left(\frac{3}{2}N\right)^d$ elements $j \in \mathbb{Z}^d$ such that $f_A(A_{j,\delta_0}) \neq 0$. Therefore we can find a $j_0 \in \mathbb{Z}^d$ such that

$$|f_A(A_{j_0,\delta_0})| \geq \frac{1}{\pi^d}\left(\frac{2}{3}\right)^{d/2} \sqrt{\delta_A(1-\delta_A)} N^{\frac{d}{2d+2}} \geq \frac{1}{4^d}\sqrt{\delta_A(1-\delta_A)} N^{\frac{d}{2d+2}}.$$

Set $x := \delta_A - \frac{1}{c}$. It holds $0 \leq x \leq \alpha c^{-1/2} N^{-d+d/(2d+2)} \leq \frac{1}{2\sqrt{c}}$. Thus, we get

$$\begin{aligned}|f_A(A_{j_0,\delta_0})| &\geq \frac{1}{4^d}\sqrt{\delta_A(1-\delta_A)} N^{\frac{d}{2d+2}} \\ &= \frac{1}{4^d}\sqrt{\left(\frac{1}{c}+x\right)\left(\frac{c-1}{c}-x\right)} N^{\frac{d}{2d+2}} \\ &= \frac{1}{4^d}\sqrt{\frac{c-1}{c^2}+\frac{c-2}{c}x-x^2} N^{\frac{d}{2d+2}} \\ &= \frac{1}{4^d}\sqrt{\frac{1}{2c}-\frac{1}{4c}} N^{\frac{d}{2d+2}} \\ &= \frac{1}{2^{2d+1}}\frac{N^{\frac{d}{2d+2}}}{\sqrt{c}}.\end{aligned}$$

We fix the constant α in (2.7). Set $\alpha := \frac{1}{2^{2d+2}}$. Then

$$\begin{aligned}
\mathrm{disc}(\mathcal{H}'_{N,d}, c, \chi) &\geq \left| |A_{j_0,\delta_0}| - \frac{1}{c}|A_{j_0,\delta_0}| \right| \\
&= \left| |A_{j_0,\delta_0}| - \delta_A |A_{j_0,\delta_0}| + \left(\delta_A - \frac{1}{c}\right) |A_{j_0,\delta_0}| \right| \\
&\geq ||A_{j_0,\delta_0}| - \delta_A|A_{j_0,\delta_0}|| - \left|\delta_A - \frac{1}{c}\right| |A_{j_0,\delta_0}| \\
&\geq \frac{1}{2^{2d+1}} \frac{N^{\frac{d}{2d+2}}}{\sqrt{c}} - \frac{1}{2^{2d+2}} c^{-1/2} N^{-d+d/(2d+2)} N^d \\
&= \frac{1}{2^{2d+2}} \frac{N^{\frac{d}{2d+2}}}{\sqrt{c}}.
\end{aligned}$$

Thus, we have shown $\mathrm{disc}(\mathcal{H}'_{N,d}, c) \geq \frac{1}{2^{2d+2}} \frac{N^{\frac{d}{2d+2}}}{\sqrt{c}}$. This proves the lower bound in Theorem 2.26. □

2.4.2 The Upper Bound

We follow the approach of Beck [Bec81] giving an upper bound for the hereditary discrepancy of the hypergraph $\mathcal{H}'_{N,d}$. Then, applying Theorem 1.10, we prove Theorem 2.26 (ii). Let us define the subhypergraph of $\mathcal{H}'_{N,d}$ of all *elementary* d–dimensional arithmetic progressions with common difference. For every $\delta \in \mathbb{N}$, every $a \in [\delta]^d$, every $s \in \mathbb{N}_0^d$ and every $f \in \mathbb{N}_0^d$ we set

$$AP(a, \delta, s, f) := \mathop{\times}_{i=1}^{d} \{a_i + j\delta \mid f_i 2^{s_i} \leq j \leq (f_i + 1)2^{s_i}\}.$$

We define the hypergraph $\mathcal{H}_{el} := ([N]^d, \mathcal{E}_{el})$, where

$$\mathcal{E}_{el} := \{AP(a, \delta, s, f) \subseteq [N]^d \mid \delta \in \mathbb{N}, a \in [\delta]^d, s \in \mathbb{N}_0^d, f \in \mathbb{N}_0^d\}.$$

Lemma 2.30. *There exists a constant $c > 0$ such that*

$$\mathrm{herdisc}(\mathcal{H}'_{N,d}) \leq c \log^d N \,\mathrm{herdisc}(\mathcal{H}_{el}).$$

Proof. Every hyperedge of $\mathcal{H}'_{N,d}$ can be decomposed into at most $c \log^d N$ hyperedges of $\mathcal{H}_{el}$ for an appropriate constant $c > 0$. This decomposition can be found in [Weh97]. Also in every induced subhypergraph of $\mathcal{H}'_{N,d}$ this decomposition can be applied. This proves the assertion. □

Thus, we can look for the hereditary discrepancy of $\mathcal{H}_{el}$ and apply Lemma 2.30 afterwards.

Proof of Theorem 2.26 (ii). To use Beck's Corollary 1.8 (Partial Coloring Method), we have to determine a threshold t such that the maximal degree of the hypergraph $\mathcal{H}_{el,t} := ([N]^d, \mathcal{E}_{el,t})$, where $\mathcal{E}_{el,t} := \{E \in \mathcal{E}_{el} \mid |E| \geq t\}$, is bounded by t. Let

$$S(a, \delta, t) := \{s \in S \mid 2^{\sum_{i=1}^d s_i} \geq t, a_i + (2^{s_i} - 1)\delta \leq N (i \in [d])\}$$

for all $a \in [N]^d$, $\delta \in \mathbb{N}$ and all $t > 0$. Using that for every $m \in [N]^d$ there is only one vector $a \in [\delta]^d$ with $a_i \equiv m_i \pmod{\delta}$,

$$\begin{aligned}\deg(\mathcal{H}_{el,t}) &= \max_{m \in [N]^d} |\{AP(a, \delta, s, f) \in \mathcal{E}_{el,t} \mid m \in AP(a, \delta, s, f)\}| \\ &\leq \sum_{\delta=1}^{\lfloor N/\sqrt[d]{t} \rfloor} |S(a, \delta, t)|.\end{aligned}$$

Here δ cannot be larger than $\lfloor N/\sqrt[d]{t} \rfloor$, since for every $E = \mathop{\mathsf{X}}_{i=1}^{d} A_i \in \mathcal{E}_{el}$ with $|E| \geq t$ there is at least one $i \in [d]$ with $|A_i| \geq \sqrt[d]{t}$. We have $|S(a, \delta, t)| \leq \log^d N$ for all $\delta \geq 2$. Thus, there is a constant $c_1 > 0$ such that

$$\deg(\mathcal{H}_{el,t}) \leq c_1 \frac{N}{\sqrt[d]{t}} \log^d N.$$

We set $t := c_1^{\frac{d}{d+1}} N^{\frac{d}{d+1}} \log^{\frac{d^2}{d+1}} N$ and get

$$\deg(\mathcal{H}_{el,t}) \leq t.$$

This estimation holds obviously also for all induced subhypergraphs of $\mathcal{H}_{el}$. Thus, Corollary 1.8 yields for a constant $c_2 > 0$ only depending on the dimension d

$$\operatorname{herdisc}(\mathcal{H}_{el}) \leq c_2 N^{\frac{d}{2d+2}} \log^{\frac{d^2}{2d+2}} N \log^2 N = c_2 N^{\frac{d}{2d+2}} \log^{\frac{d^2+4d+4}{2d+2}} N.$$

Now Lemma 2.30 yields for a constant $c_3 > 0$ only depending on d

$$\operatorname{herdisc}(\mathcal{H}'_{N,d}) \leq c_3 N^{\frac{d}{2d+2}} \log^{\frac{3d^2+6d+4}{2d+2}} N \leq c_3 N^{\frac{d}{2d+2}} \log^{\frac{3}{2}d+2} N.$$

We apply Theorem 1.10 and get for a constant $c_0 > 0$ that is only depending on the dimension d

$$\operatorname{disc}(\mathcal{H}'_{N,d}, c) \leq c_0 N^{\frac{d}{2d+2}} \log^{\frac{3}{2}d+2} N.$$

□

Remark 2.31. *We have used the recursive coloring approach but not the refinement of this. The consequence of this is that there is a gap of $\sqrt{c}$ (additionally to the polylogarithmic terms in N) between the lower and the upper bound in Theorem 2.26. We think that one can apply also the refinement looking at the hereditary discrepancy in a more precise way. This should improve the upper bound by a factor of $c^{\frac{d}{2d+2}}$ which is almost $\sqrt{c}$ (slightly increasing the power of the* log*–term probably).*

2.5 Hypergraphs in $\mathbb{Z}_p$

We want to determine a lower bound for the c–color discrepancy of hypergraphs $\mathcal{H} = (\mathbb{Z}_p, \mathcal{E})$ that are somehow adapted to the structure of $\mathbb{Z}_p$. This means that the hypergraphs are invariant under the multiplication with non-trivial elements of $\mathbb{Z}_p$ and also invariant under the addition (shift) with elements of $\mathbb{Z}_p$. Actually we only need to know that there is a hyperedge $E \in \mathcal{E}$ such that for all $a \in \mathbb{Z}_p \setminus \{0\}$ and all $b \in \mathbb{Z}_p$ it holds $b+aE := \{b+ax \mid x \in E\} \in \mathcal{E}$. In the next two sections we will use the results of this section to determine lower bounds for the hypergraph of arithmetic progressions in $\mathbb{Z}_p$, the hypergraph of centered arithmetic progressions in $\mathbb{Z}_p$ and the hypergraph of Bohr neighborhoods in $\mathbb{Z}_p$.

The Field $\mathbb{Z}_p$

Before looking for the discrepancy lower bound we introduce the field $\mathbb{Z}_p$. Let p be a prime number. For every $n, a \in \mathbb{N}$ with $n \geq 1$ we define $n\mathbb{Z} := \{nz \mid z \in \mathbb{Z}\}$ and $a + n\mathbb{Z} := \{a + nz \mid z \in \mathbb{Z}\}$. We denote the set of all residue classes of $\mathbb{Z}$ modulo p by

$$\mathbb{Z}_p := \{a + p\mathbb{Z} \mid a \in \mathbb{Z}\}.$$

Clearly we have $|\mathbb{Z}_p| = p$. Together with the addition $+ : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p, \quad (a+p\mathbb{Z}, b+p\mathbb{Z}) \mapsto (a+b) + p\mathbb{Z}$ and the multiplication $\cdot : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p, \quad (a+p\mathbb{Z}, b+p\mathbb{Z}) \mapsto ab + p\mathbb{Z}$ the triple $(\mathbb{Z}_p, +, \cdot)$ is a field. It is the up to isomorphism unique field with p elements. We should mention that for all $x \in a + p\mathbb{Z}$ and $y \in b + p\mathbb{Z}$ it holds $x + y \in (a + b) + p\mathbb{Z}$ and $xy \in ab + p\mathbb{Z}$. Therefore we can take arbitrary representatives $x \in a + p\mathbb{Z}$ for our calculations. And we will often use the representative x instead of $a + p\mathbb{Z}$.

Fourier Analysis in $\mathbb{Z}_p$

We want to use the Fourier analysis for locally compact Abelian groups from Section 2.2 to determine a lower bound for the discrepancy of hypergraphs in $\mathbb{Z}_p$. Thus, we need to introduce the Fourier transform in $\mathbb{Z}_p$.

Proposition 2.32.

(i) $\widehat{\mathbb{Z}}_p \cong \mathbb{Z}_p$.

(ii) The Fourier transform of a function $f \in L^1(\mathbb{Z}_p)$ *can be written as*

$$\widehat{f} : \widehat{\mathbb{Z}}_p \to \mathbb{C}, \quad r \mapsto \sum_{x \in \mathbb{Z}_p} f(x) e^{-\frac{2\pi i}{p} xr}.$$

Proof.

(i) Let $\gamma \in \widehat{\mathbb{Z}}_p$. Then $\gamma(0) = \gamma(0+0) = \gamma(0)\gamma(0)$ and hence $\gamma(0) = 1$. Furthermore we have

$$(\gamma(1))^p = \gamma(\sum_{j=1}^{p} 1) = \gamma(0) = 1.$$

Thus, $\gamma(1)$ is a p–th unit root. The value $\gamma(1)$ clearly determines the function γ, because $\mathbb{Z}_p$ is generated by 1. And on the other hand for every p–th unit root $\xi_j := e^{\frac{2\pi i}{p}j}$ the function

$$\gamma_j : \mathbb{Z}_p \to \mathbb{C}, \quad x \mapsto \xi_j^x = e^{\frac{2\pi i}{p}jx}$$

is a character on $\mathbb{Z}_p$. Thus, $\widehat{\mathbb{Z}}_p = \{\gamma_j \mid j \in \mathbb{Z}_p\}$ and one can check that $\mu : \mathbb{Z}_p \to \widehat{\mathbb{Z}}_p, \quad j \mapsto \gamma_j$ is an isomorphism from $(\mathbb{Z}_p, +)$ onto $(\widehat{\mathbb{Z}}_p, \cdot)$.

(ii) If we identify $(\widehat{\mathbb{Z}}_p, \cdot)$ with $(\mathbb{Z}_p, +)$ via μ, the Fourier transform is of the stated form.

□

The Lower Bound

Let $\mathcal{H} = (\mathbb{Z}_p, \mathcal{E})$ be a hypergraph with the property that there is an $E \in \mathcal{E}$ such that for all $a \in \mathbb{Z}_p \setminus \{0\}$ and $b \in \mathbb{Z}_p$ it holds $b + aE \in \mathcal{E}$. We call such hypergraphs $\mathbb{Z}_p$*–invariant* with respect to the hyperedge E. A canonical example for hypergraphs with this property is the hypergraph of arithmetic progressions in $\mathbb{Z}_p$. We can take e.g. $E := \{0, 1, \ldots, \frac{p-1}{2}\}$. Then $b + aE$ is the arithmetic progression with starting point b, difference a and length $\frac{p+1}{2}$.

We will use the next lemma to derive a large discrepancy for hypergraphs that are $\mathbb{Z}_p$–invariant. The function f in the lemma will play the role of a color-function.

Lemma 2.33. *Let $\mathcal{H} = (\mathbb{Z}_p, \mathcal{E})$ be a hypergraph that is $\mathbb{Z}_p$–invariant with respect to $E \in \mathcal{E}$. Let $\delta_E := \frac{1}{p}|E|$ be the density of E in $\mathbb{Z}_p$ and $f : \mathbb{Z}_p \to \mathbb{C}$ be a function. There exist $a \in \mathbb{Z}_p \setminus \{0\}$ and $b \in \mathbb{Z}_p$ with*

$$|f(b + aE)| \geq \sqrt{\delta_E(1-\delta_E)}\|f\|_2.$$

Proof. We want to apply Corollary 2.17. Thus, we have to calculate an appropriate $\gamma > 0$ such that $\sum_{a \in \mathbb{Z}_p \setminus \{0\}} |\widehat{\mathbb{1}}_{-aE}(r)|^2 \geq \gamma$ for all $r \in \mathbb{Z}_p$. First of all we have

$$\sum_{a \in \mathbb{Z}_p \setminus \{0\}} |\widehat{\mathbb{1}}_{-aE}(0)|^2 = \sum_{a \in \mathbb{Z}_p \setminus \{0\}} |(-aE)|^2 = (p-1)|E|^2.$$

Let $r \in \mathbb{Z}_p \setminus \{0\}$. It holds for all $a \in \mathbb{Z}_p \setminus \{0\}$

$$\widehat{\mathbb{1}}_{-aE}(r) = \sum_{x \in (-aE)} e^{-\frac{2\pi i}{p} xr} = \sum_{x \in (-E)} e^{-\frac{2\pi i}{p} xar} = \widehat{\mathbb{1}}_{-E}(ar).$$

The multiplication with r is a bijection on $\mathbb{Z}_p \setminus \{0\}$. Hence

$$\begin{aligned}
\sum_{a \in \mathbb{Z}_p \setminus \{0\}} |\widehat{\mathbb{1}}_{-aE}(r)|^2 = \sum_{a \in \mathbb{Z}_p \setminus \{0\}} |\widehat{\mathbb{1}}_{-E}(ar)|^2 &= \sum_{a \in \mathbb{Z}_p \setminus \{0\}} |\widehat{\mathbb{1}}_{-E}(a)|^2 \\
&= \left(\sum_{a \in \mathbb{Z}_p} |\widehat{\mathbb{1}}_{-E}(a)|^2 \right) - |\widehat{\mathbb{1}}_{-E}(0)|^2 \\
&\overset{Cor.\ 2.15}{=} p \sum_{a \in \mathbb{Z}_p} |\mathbb{1}_{-E}(a)|^2 - |E|^2 \\
&= |E|(p - |E|).
\end{aligned}$$

Therefore it holds $\sum_{a \in \mathbb{Z}_p \setminus \{0\}} |\widehat{\mathbb{1}}_{-aE}(r)|^2 \geq |E|(p - |E|)$ for all $z \in \mathbb{Z}_p$. Now we can apply Corollary 2.17 and get

$$\sum_{a \in \mathbb{Z}_p \setminus \{0\}} \sum_{b \in \mathbb{Z}_p} |f(b + aE)|^2 \geq |E|(p - |E|)\|f\|_2^2 = p^2 \delta_E (1 - \delta_E) \|f\|_2^2.$$

This is a sum of $p(p-1)$ terms. Thus, there are $a \in \mathbb{Z}_p \setminus \{0\}$ and all $b \in \mathbb{Z}_p$ with

$$|f(b + aE)| \geq \sqrt{\frac{p^2}{p(p-1)} \delta_E (1 - \delta_E) \|f\|_2^2} \geq \sqrt{\delta_E (1 - \delta_E)} \|f\|_2.$$

□

Our strategy to determine a lower bound for the discrepancy of a $\mathbb{Z}_p$–invariant hypergraph $\mathcal{H} = (\mathbb{Z}_p, \mathcal{E})$ is the following. For every c–coloring χ of $\mathcal{H}$ there exists a color $i \in \{1, 2, \ldots, c\}$ with $|\chi^{-1}(i)| \geq \frac{p}{c}$. Now for a constant $\alpha > 0$ that we choose to optimize the lower bound, there exists a color $i \in \{1, 2, \ldots, c\}$ with $|\chi^{-1}(i)| > \frac{p}{c} + \alpha\sqrt{\frac{p}{c}}$ or there exists no such color. In the first case an average argument yields that one of the translations $b + E$ ($b \in \mathbb{Z}_p$) is a hyperedge with discrepancy of order $\Omega(\sqrt{\frac{p}{c}})$. In the latter case we will derive a discrepancy lower bound of the same order using the next lemma. In this case there exists a color $i \in \{1, 2, \ldots, c\}$ such that for $A := \chi^{-1}(i)$ it holds $\frac{p}{c} \leq |A| \leq \frac{p}{c} + \alpha\sqrt{\frac{p}{c}}$. Let $\delta_A := \frac{1}{p}|A|$ be the density of A in $\mathbb{Z}_p$. We define the color-function $f_A : \mathbb{Z}_p \to \mathbb{C}$ by

$$f_A(x) := \begin{cases} 1 - \delta_A & : \quad x \in A, \\ -\delta_A & : \quad x \in \mathbb{Z}_p \setminus A. \end{cases}$$

So we have $f_A(x) = \mathbb{1}_A(x) - \delta_A$ for all $x \in \mathbb{Z}_p$. The concept of this function is the following. Recall the definition $f_A(X) = \sum_{x \in X} f_A(x)$ for all $X \subseteq \mathbb{Z}_p$. It holds for every $E \in \mathcal{E}$

$$f_A(E) = \sum_{x \in E} f_A(x) = \sum_{x \in E \cap A} (1 - \delta_A) + \sum_{x \in E \setminus A} (-\delta_A) = |E \cap A| - \delta_A |E|.$$

Therefore $|f_A(E)|$ is a kind of discrepancy of the hyperedge E with respect to the color-class A. The next lemma will provide a large value $|f_A(E)|$ for an $E \in \mathcal{E}$. Using the triangle-inequality we get a discrepancy of order $\Omega(\sqrt{\frac{p}{c}})$.

Lemma 2.34. *Let $A, E \subseteq \mathbb{Z}_p$, $\delta_A := \frac{1}{p}|A|$ and $\delta_E := \frac{1}{p}|E|$. Furthermore let $f_A : \mathbb{Z}_p \to \mathbb{C}$, $x \mapsto \mathbb{1}_A - \delta_A$. There exist $a \in \mathbb{Z}_p \setminus \{0\}$ and $b \in \mathbb{Z}_p$ such that*

$$|f_A(b + aE)| \geq \sqrt{\delta_A(1 - \delta_A)\delta_E(1 - \delta_E)p}.$$

Proof. Lemma 2.33 yields the existence of $a \in \mathbb{Z}_p \setminus \{0\}$ and $b \in \mathbb{Z}_p$ such that

$$\begin{aligned} |f_A(b + aE)| &\geq \sqrt{\delta_E(1 - \delta_E)}\|f\|_2 \\ &= \sqrt{\delta_E(1 - \delta_E)}\sqrt{\sum_{x \in A}(1 - \delta_A)^2 + \sum_{x \in \mathbb{Z}_p \setminus A}(-\delta_A)^2} \\ &= \sqrt{\delta_E(1 - \delta_E)}\sqrt{p\delta_A(1 - \delta_A)^2 + p(1 - \delta_A)\delta_A^2} \\ &= \sqrt{\delta_A(1 - \delta_A)\delta_E(1 - \delta_E)p}. \end{aligned}$$

□

In the next theorem we state a lower bound for the discrepancy of a hypergraph $\mathcal{H} = (\mathbb{Z}_p, \mathcal{E})$ that is $\mathbb{Z}_p$–invariant with respect to an $E \in \mathcal{E}$.

Theorem 2.35. *Let $\mathcal{H} = (\mathbb{Z}_p, \mathcal{E})$ be a hypergraph that is $\mathbb{Z}_p$–invariant with respect to an $E \in \mathcal{E}$. Let $\delta_E := \frac{1}{p}|E|$. We have*

$$\operatorname{disc}(\mathcal{H}, c) \geq \sqrt{\frac{\delta_E(1 - \delta_E)}{10}}\sqrt{\frac{p}{c}}.$$

Using this theorem we can derive lower bounds for the hypergraph of (centered) arithmetic progressions in $\mathbb{Z}_p$ and for the hypergraph of Bohr neighborhoods in $\mathbb{Z}_p$. But we will use Lemma 2.34 to achieve better constants for the discrepancy lower bounds of this hypergraphs in the next two sections.

If $|\mathcal{E}|$ is bounded by a polynomial in p the probabilistic method provides an upper bound $\operatorname{disc}(\mathcal{H}, c) = O(\sqrt{\frac{p}{c}\log p})$. Thus, the lower bound in Theorem 2.35 is tight up to a logarithmic factor.

Proof of Theorem 2.35. There exists at least one color $i \in \{1,2,\dots,c\}$ with $|\chi^{-1}(i)| \geq \frac{p}{c}$. If there exists a color $i \in \{1,2,\dots,c\}$ such that $|\chi^{-1}(i)| > \frac{p}{c} + \sqrt{\frac{(1-\delta_E)p}{10\delta_E c}}$, we have

$$\begin{aligned}
\sum_{b\in\mathbb{Z}_p}\left(\big|(b+E)\cap\chi^{-1}(i)\big| - \frac{1}{c}|b+E|\right) &= \left(\sum_{b\in\mathbb{Z}_p}\sum_{a\in\chi^{-1}(i)} \mathbb{1}_{b+E}(a)\right) - \frac{p}{c}|E| \\
&= \left(\sum_{a\in\chi^{-1}(i)}\sum_{x\in E}\sum_{b\in\mathbb{Z}_p}\delta_{a,b+x}\right) - \frac{p}{c}|E| \\
&= \left(\sum_{a\in\chi^{-1}(i)}\sum_{x\in E} 1\right) - \frac{p}{c}|E| \\
&= |E|\left(|A| - \frac{p}{c}\right) \\
&> p\delta_E\sqrt{\frac{(1-\delta_E)p}{10\delta_E c}} = p\sqrt{\frac{\delta_E(1-\delta_E)}{10}}\sqrt{\frac{p}{c}}.
\end{aligned}$$

Let $b \in \mathbb{Z}_p$ such that $|(b+E)\cap\chi^{-1}(i)| - \frac{1}{c}|b+E|$ is maximal. It holds

$$\operatorname{disc}(\mathcal{H},c,\chi) \geq \left|\big|(b+E)\cap\chi^{-1}(i)\big| - \frac{1}{c}|b+E|\right| \geq \sqrt{\frac{\delta_E(1-\delta_E)}{10}}\sqrt{\frac{p}{c}}.$$

Thus, we can assume that there is no color $i \in \{1,2,\dots,c\}$ such that $|\chi^{-1}(i)| > \frac{p}{c} + \sqrt{\frac{(1-\delta_E)p}{10\delta_E c}}$. But then there must be at least one color $i \in \{1,2,\dots,c\}$ with

$$0 \leq |\chi^{-1}(i)| - \frac{p}{c} \leq \sqrt{\frac{(1-\delta_E)p}{10\delta_E c}}.$$

We set $A := \chi^{-1}(i)$ and $\delta_A := \frac{1}{p}|A|$. For the density δ_A of A in $\mathbb{Z}_p$ it holds

$$0 \leq \delta_A - \frac{1}{c} \leq \sqrt{\frac{1-\delta_E}{10\delta_E pc}}.$$

Now Lemma 2.34 yields for the function $f_A : \mathbb{Z}_p \to \mathbb{C}$, $x \mapsto \mathbb{1}_A(x) - \delta_A$ the existence of an $a \in \mathbb{Z}_p \setminus \{0\}$ and a $b \in \mathbb{Z}_p$ such that

$$|f_A(b+aE)| \geq \sqrt{\delta_A(1-\delta_A)\delta_E(1-\delta_E)p}. \tag{2.9}$$

Now we give an estimation for the term $\delta_A(1-\delta_A)$ in (2.9). We set $x := \delta_A - \frac{1}{c} \geq 0$ and get

$$\begin{aligned}\delta_A(1-\delta_A) = \left(\frac{1}{c}+x\right)\left(\frac{c-1}{c}-x\right) &= \frac{c-1}{c^2}+\frac{c-2}{c}x-x^2\\ &\geq \frac{c-1}{c^2}-x^2\\ &\geq \frac{c-1}{c^2}-\frac{1-\delta_E}{10\delta_E pc}.\end{aligned}$$

Since $|E| \geq 1$ we have $\frac{1-\delta_E}{\delta_E p} \leq \frac{p-1}{p} \leq 1$. Using $\frac{c-1}{c^2} \geq \frac{1}{2c}$, we get

$$\delta_A(1-\delta_A) \geq \frac{1}{c}\left(\frac{1}{2}-\frac{1}{10}\right) = \frac{2}{5c}. \tag{2.10}$$

Thus (2.9) and (2.10) yield the existence of an $a \in \mathbb{Z}_p \setminus \{0\}$ and a $b \in \mathbb{Z}_p$ such that

$$|f_A(b+aE)| \geq \sqrt{\frac{2\delta_E(1-\delta_E)}{5}}\sqrt{\frac{p}{c}} = 2\sqrt{\frac{\delta_E(1-\delta_E)}{10}}\sqrt{\frac{p}{c}}.$$

Using the triangle-inequality we get

$$\begin{aligned}\operatorname{disc}(\mathcal{H},c,\chi) &\geq \left||(b+aE)\cap A| - \frac{1}{c}|b+aE|\right|\\ &= \left||(b+aE)\cap A| - \delta_A|b+aE| + (\delta_A-\frac{1}{c})|b+aE|\right|\\ &= \left|f_A(b+aE) + (\delta_A-\frac{1}{c})\delta_E p\right|\\ &\geq |f_A(b+aE)| - \left|(\delta_A-\frac{1}{c})\delta_E p\right|\\ &\geq 2\sqrt{\frac{\delta_E(1-\delta_E)}{10}}\sqrt{\frac{p}{c}} - \sqrt{\frac{1-\delta_E}{10\delta_E pc}}\delta_E p\\ &\geq \sqrt{\frac{\delta_E(1-\delta_E)}{10}}\sqrt{\frac{p}{c}}.\end{aligned}$$

□

2.6 Discrepancy of Arithmetic Progressions in $\mathbb{Z}_p$

In this section we want to determine lower and upper bounds for the discrepancy of the hypergraph $\mathcal{H}_{\mathbb{Z}_p}$ of arithmetic progressions in $\mathbb{Z}_p$. Unlike in the hyper of arithmetic progressions in $[N]$, an arithmetic progression in $\mathbb{Z}_p$ can be wrapped around (several times).

Thus, it is clear that the discrepancy of $\mathcal{H}_{\mathbb{Z}_p}$ is at least of order $\Omega(p^{1/4})$. But we will see that the discrepancy of $\mathcal{H}_{\mathbb{Z}_p}$ is much larger.

The Hypergraph of Arithmetic Progressions in $\mathbb{Z}_p$

We define the hypergraph $\mathcal{H}_{\mathbb{Z}_p} := (\mathbb{Z}_p, \mathcal{E}_{\mathbb{Z}_p})$ of arithmetic progressions in $\mathbb{Z}_p$, where

$$\mathcal{E}_{\mathbb{Z}_p} := \{A_{a,\delta,L} \mid 0 \leq a \leq p-1, \quad 1 \leq \delta \leq p-1, \quad 1 \leq L \leq p\}$$

and $A_{a,\delta,L} := \{(a + j\delta) + p\mathbb{Z} \mid 0 \leq j \leq L-1\}$. We should mention here that $|A_{a,\delta,L}| = L$. This fact, which is trivial for arithmetic progressions in $[N]$, is not that trivial in $\mathbb{Z}_p$. It is the consequence of $(\delta, p) = 1$. It is also mentionable that for every arithmetic progression $A_{a,\delta,L}$ the complement in $\mathbb{Z}_p$ is also an arithmetic progression, namely $A_{a+L\delta,\delta,p-L}$. And for every arithmetic progression $A_{a,\delta,L}$ there exists exactly one alternative representation, namely

$$A_{a,\delta,L} = A_{a+(L-1)\delta,p-\delta,L}.$$

This is just the same arithmetic progression but passed through in the opposite direction.

Discrepancy of $\mathcal{H}_{\mathbb{Z}_p}$

Now we give a lower and an upper bound for the c–color discrepancy of $\mathcal{H}_{\mathbb{Z}_p}$.

Theorem 2.36. *There exists a constant $\alpha > 0$ such that it holds for the hypergraph of the arithmetic progressions in $\mathbb{Z}_p$*

$$\frac{1}{3}\sqrt{\frac{p}{c}} \leq \operatorname{disc}(\mathcal{H}_{\mathbb{Z}_p}, c) \leq \alpha\sqrt{\frac{p}{c}\ln p} + 1.$$

Proof. In the case $p = 2$ it is easy to check that $\operatorname{disc}(\mathcal{H}_{\mathbb{Z}_p}, c) = \frac{c-1}{c}$. Thus, we can assume that $p \geq 3$. In particular p is odd. The proof is divided into two parts. We first derive the lower bound and afterwards cite a theorem for the upper bound.

Let $\chi : \mathbb{Z}_p \to \{1, 2, \ldots, c\}$ be a c–coloring. There exists a color $i \in \{1, 2, \ldots, c\}$ with $|\chi^{-1}(i)| \geq \frac{p}{c}$. If there is a color $i \in \{1, \ldots, c\}$ with

$$|\chi^{-1}(i)| - \frac{p}{c} > \frac{1}{3}\sqrt{\frac{p}{c}},$$

then $\operatorname{disc}(\mathcal{H}_{\mathbb{Z}_p}, c, \chi) > \frac{1}{3}\sqrt{\frac{p}{c}}$, because $\mathbb{Z}_p$ itself is an arithmetic progression in $\mathbb{Z}_p$. Thus, we can assume that there is a color $i \in \{1, 2, \ldots, c\}$ such that for $A := \chi^{-1}(i)$ it holds

$$0 \leq |A| - \frac{p}{c} \leq \frac{1}{3}\sqrt{\frac{p}{c}}.$$

Set $\delta_A := \frac{|A|}{p}$. We define the function $f_A : \mathbb{Z}_p \to \mathbb{C}$ by

$$f_A(x) := \begin{cases} 1 - \delta_A & : \quad x \in A, \\ -\delta_A & : \quad \textit{otherwise}. \end{cases}$$

Recall that we have for every subset $X \subseteq \mathbb{Z}_p$:

$$f_A(X) = \sum_{x \in X} f_A(x) = \sum_{x \in A \cap X} (1 - \delta_A) + \sum_{x \in X \setminus A} (-\delta_A) = |A \cap X| - \delta_A |X|.$$

Now our aim is to find an arithmetic progression P in $\mathbb{Z}_p$ such that $|f_A(P)| \geq \frac{1}{3}\sqrt{\frac{p}{c}}$. Afterwards we will see that the discrepancy of P or the complement of P in $\mathbb{Z}_p$ with respect to χ is at least $|f_A(P)|$. Since the complement of an arithmetic progression in $\mathbb{Z}_p$ is also an arithmetic progression in $\mathbb{Z}_p$ this will prove the lower bound.

We set $E := A_{0,1,\frac{p+1}{2}}$. For every $a \in \mathbb{Z}_p \setminus \{0\}$ and every $b \in \mathbb{Z}_p$

$$b + aE = A_{b,a,\frac{p+1}{2}}$$

is the arithmetic progression with starting point b, difference a and length $\frac{p+1}{2}$ in $\mathbb{Z}_p$. Using Lemma 2.34 we get the existence of an $a \in \mathbb{Z}_p \setminus \{0\}$ and a $b \in \mathbb{Z}_p$ such that

$$\begin{aligned} |f_A(b + aE)| &\geq \sqrt{\delta_A(1 - \delta_A)\delta_E(1 - \delta_E)p} \\ &= \sqrt{\delta_A(1 - \delta_A)\frac{p+1}{2p}\frac{p-1}{2p}p} \end{aligned}$$

It is easy to see that the assertion of the theorem is valid for $p \in \{2, 3\}$. Thus, we can assume $p \geq 5$ and get

$$|f_A(b + aE)| \geq \sqrt{\tfrac{p^2-1}{4p^2}}\sqrt{\delta_A(1 - \delta_A)p} \geq \sqrt{\tfrac{24}{100}}\sqrt{\delta_A(1 - \delta_A)p} \geq \tfrac{\sqrt{6}}{5}\sqrt{\delta_A(1 - \delta_A)p}.$$

Let $x := \delta_A - \frac{1}{c}$ then $0 \leq x \leq \frac{1}{3\sqrt{pc}}$. Thus, we have

$$\delta_A(1 - \delta_A) = \left(\frac{1}{c} + x\right)\left(\frac{c-1}{c} - x\right) = \frac{c-1}{c^2} + \frac{c-2}{c}x - x^2 \leq \frac{c-1}{c^2} - \frac{1}{9pc}.$$

Hence using $p \geq 5$ we get

$$\begin{aligned} |f_A(b + aE)| &\geq \frac{\sqrt{6}}{5}\sqrt{\frac{c-1}{c} - \frac{1}{9p}}\sqrt{\frac{p}{c}} \\ &\geq \frac{\sqrt{6}}{5}\sqrt{\frac{1}{2} - \frac{1}{45}}\sqrt{\frac{p}{c}} \\ &> \frac{1}{3}\sqrt{\frac{p}{c}}. \end{aligned}$$

If $f_A(b + aE) > 0$ we define $Q := b + aE$. In the case $f_A(b + aE) < 0$ we set $Q := \mathbb{Z}_p \setminus (b + aE)$. Q is an arithmetic progression in $\mathbb{Z}_p$ that fulfills $f_A(Q) \geq \frac{1}{3}\sqrt{\frac{p}{c}}$, because

$f_A(\mathbb{Z}_p \setminus (b+aE)) = f_A(\mathbb{Z}_p) - f_A(b+aE) = -f_A(b+aE)$. Thus, we get

$$\begin{aligned}\operatorname{disc}(\mathcal{H}_{\mathbb{Z}_p}, c, \chi) &\geq |A \cap Q| - \tfrac{1}{c}|Q| \\ &= |A \cap Q| - \delta_A|Q| + (\delta_A - \tfrac{1}{c})|Q| \\ &\geq f_A(Q) \\ &> \frac{1}{3}\sqrt{\frac{p}{c}}.\end{aligned}$$

Now we prove the upper bound. The number of hyperedges in $\mathcal{H}_{\mathbb{Z}_p}$ is bounded by p^3 because every tuple (a, δ, L) with $0 \leq a \leq p-1$, $1 \leq \delta \leq p-1$ and $1 \leq L \leq p$ there is only one arithmetic progression P in $\mathbb{Z}_p$ with $P = A_{a,\delta,L}$ and every hyperedge of $\mathcal{H}_{\mathbb{Z}_p}$ is of this form. Applying Theorem 3.14 from [DS03] we can find an $\alpha > 0$ such that

$$\operatorname{disc}(\mathcal{H}_{\mathbb{Z}_p}, c) \leq \alpha\sqrt{\frac{p}{c}\ln p} + 1.$$

□

At the end of this section we want to note that the lower and the upper bound from Theorem 2.36 hold also for the positive discrepancy $\operatorname{disc}^+(\mathcal{H}_{\mathbb{Z}_p}, c)$, where only positive deviations of the number of vertices in each color from the expected value are taken into account for every hyperedge. For the exact definition of the positive discrepancy of a hypergraph, we refer to the Chapter 1.

Every upper bound for the discrepancy of a hypergraph is also an upper bound for the positive discrepancy, because $\operatorname{disc}^+(\mathcal{H}_{\mathbb{Z}_p}, c) \leq \operatorname{disc}(\mathcal{H}_{\mathbb{Z}_p}, c)$. Thus, only the lower bound proof of Theorem 2.36 has to be checked. But in fact we have not mentioned just a large deviation in this proof, but a large positive deviation. Therefore the same proof holds also for the following corollary.

Corollary 2.37. *There exists a constant $\alpha > 0$ such that it holds for the hypergraph of arithmetic progressions in $\mathbb{Z}_p$*

$$\frac{1}{3}\sqrt{\frac{p}{c}} \leq \operatorname{disc}^+(\mathcal{H}_{\mathbb{Z}_p}, c) \leq \alpha\sqrt{\frac{p}{c}\ln p} + 1.$$

2.7 Discrepancy of Centered Arithmetic Progressions in $\mathbb{Z}_p$

Until now all discrepancy lower bounds in this chapter used the fact that for all hyperedges E and all $x \in \mathbb{Z}_p$ (respectively $\mathbb{Z}$) the intersection of $x+E$ with the set of vertices was also a hyperedge. In this section we are interested in the discrepancy of a hypergraph that has not this property. It is the hypergraph $\mathcal{H}_{\mathbb{Z}_p,c} = (V, \mathcal{E}_{\mathbb{Z}_p,c})$ of centered arithmetic progressions in $\mathbb{Z}_p$. That means the set of vertices is once again $\mathbb{Z}_p$, but the set of hyperedges is unlike in Section 2.6 the set of all arithmetic progressions in $\mathbb{Z}_p$. $\mathcal{E}_{\mathbb{Z}_p,c}$ is the set of all arithmetic progressions in $\mathbb{Z}_p$ that are centered in $0 \in \mathbb{Z}_p$, i.e., the set of all

$$C_{\delta,L} := \{j\delta \mid -L \leq j \leq L\}$$

with $\delta \in \mathbb{Z}_p \setminus \{0\}$ and $0 \leq L \leq \frac{p-1}{2}$. We want to use Lemma 2.33, but this is only possible for $\mathbb{Z}_p$–invariant hypergraphs. We will solve this problem by treating the hypergraph of arithmetic progressions in $\mathbb{Z}_p$ with a modified color-function. Using this new color-function we will derive a large discrepancy for hypergraph $\mathcal{H}_{\mathbb{Z}_p,c}$.

Discrepancy of $\mathcal{H}_{\mathbb{Z}_p,c}$

In the next theorem we give a lower and an upper bound for the c–color discrepancy ($c \geq 3$) of the hypergraph of centered arithmetic progressions in $\mathbb{Z}_p$.

Theorem 2.38. *Let $c \geq 3$. For the hypergraph of centered arithmetic progressions in $\mathbb{Z}_p$ there exists a constant $\alpha > 0$ such that*

$$\frac{1}{31}\sqrt{\frac{p}{c}} \leq \operatorname{disc}(\mathcal{H}_{\mathbb{Z}_p,c}, c) \leq \alpha\sqrt{\frac{p}{c}\ln p} + 1.$$

Remark 2.39. *We state bounds for the c–color discrepancy of $\mathcal{H}_{\mathbb{Z}_p,c}$ only for $c \geq 3$, because the 2–color discrepancy of $\mathcal{H}_{\mathbb{Z}_p,c}$ is trivial. It is easy to check that*

$$\operatorname{disc}(\mathcal{H}_{\mathbb{Z}_p,c}) = 1.$$

This discrepancy can be realized with the coloring

$$\chi : \mathbb{Z}_p \to \{-1, 1\}, \quad x \mapsto \begin{cases} 1 & : \quad x \in \{0, \ldots, \frac{p-1}{2}\}, \\ -1 & : \quad x \in \{\frac{p+1}{2}, \ldots, p-1\}. \end{cases}$$

using the symmetry of the hyperedges.

For the proof of the Theorem 2.38 we need two lemmas. First we define for a given subset $A \subseteq \mathbb{Z}_p$ with the density $\delta_A := \frac{1}{p}|A| < \frac{1}{2}$ the function

$$g_A : \mathbb{Z}_p \to \mathbb{C}, \quad x \mapsto \begin{cases} 2 - 2\delta_A & : \quad \{x, -x\} \subseteq A, \\ 1 - 2\delta_A & : \quad \{x, -x\} \neq \{x, -x\} \cap A \neq \emptyset, \\ -2\delta_A & : \quad \{x, -x\} \cap A = \emptyset. \end{cases} \tag{2.11}$$

Let $f_A : \mathbb{Z}_p \to \mathbb{C}$ be defined as in Section 2.6, i.e., $f_A(x) = 1 - \delta_A$ if $x \in A$ and $f(x) = -\delta_A$ otherwise. The function g_A is a symmetrisation of f_A, i.e.,

$$g_A(x) = f_A(x) + f_A(-x), \quad \text{for all } x \in \mathbb{Z}_p.$$

This symmetrisation is the key to the proof of Theorem 2.38. The next lemma gives an estimation for $\|g_A\|_2^2$.

Lemma 2.40. *Let A be a subset of $\mathbb{Z}_p$ with density $\delta_A = \frac{1}{p}|A| < \frac{1}{2}$, then*

$$\|g_A\|_2^2 \geq 4p\delta_A(\tfrac{1}{2} - \delta_A).$$

Proof. We set $\mu := \frac{1}{p}|\{x \in \mathbb{Z}_p \mid \{x, -x\} \neq \{x, -x\} \cap A \neq \emptyset\}|$. μ is the density of the subset of $\mathbb{Z}_p$ of all x with $\mathbb{1}_A(x) \neq \mathbb{1}_A(-x)$. For this density we get $\mu \leq 2\delta_A$, because for every $y \in \{x \in \mathbb{Z}_p \mid \{x, -x\} \neq \{x, -x\} \cap A \neq \emptyset\}$ either $y \in A$ or $-y \in A$. Since there are $(\delta_A - \frac{1}{2}\mu)p$ elements $x \in \mathbb{Z}_p$ with $g_A(x) = 2 - 2\delta_A$, μp elements $x \in \mathbb{Z}_p$ with $g_A(x) = 1 - 2\delta_A$ and $(1 - \delta_A - \frac{1}{2}\mu)p$ elements $x \in \mathbb{Z}_p$ with $g_A(x) = -2\delta_A$, we have

$$\begin{aligned}
\|g_A\|_2^2 &= \sum_{x \in \mathbb{Z}_p} |g_A(x)|^2 \\
&= (\delta_A - \tfrac{1}{2}\mu)p(2 - 2\delta_A)^2 + \mu p(1 - 2\delta_A)^2 + (1 - \delta_A - \tfrac{1}{2}\mu)p(-2\delta_A)^2 \\
&= p[\mu(-2 + 4\delta_A - 2\delta_A^2 + 1 - 4\delta_A + 4\delta_A^2 - 2\delta_A^2) + 4\delta_A(1 - \delta_A)] \\
&= 4p[\delta_A(1 - \delta_A) - \tfrac{1}{4}\mu]
\end{aligned}$$

Using $\mu \leq 2\delta_A$ we get $\|g_A\|_2^2 \geq 4p\delta_A(\frac{1}{2} - \delta_A)$. □

With the function g_A multi-sets come into play. Let $E \subseteq \mathbb{Z}_p$ and let M be the multi-set $E \cup (-E)$, i.e., every element of $E \cap (-E)$ is twice in M. For every multi-set K let us denote by $K(x)$ the frequency of occurrence of x in K. Thus, we have

$$M(x) = \mathbb{1}_E(x) = \mathbb{1}_E(-x)$$

for all $x \in \mathbb{Z}_p$. For a multi-set K and a function $f : \mathbb{Z}_p \to \mathbb{C}$ we extend the definition $f(X) = \sum_{x \in X} f(x)$ for subsets $X \subseteq \mathbb{Z}_p$ to

$$f(K) := \sum_{x \in \mathbb{Z}_p} K(x) f(x).$$

With this definition we have

$$g_A(E) = \sum_{x \in E} g_A(x) = \sum_{x \in E} f_A(x) + f_A(-x) = \sum_{x \in \mathbb{Z}_p} M(x) f_A(x) = f_A(M). \tag{2.12}$$

The proof of Theorem 2.38 will use this equation in the following way. As in Section 2.6 f_A is the color-function for a color-class A. $|f_A(M)|$ is a kind of discrepancy of the multi-set $M = E \cap (-E)$. Via (2.12) we can calculate a lower bound for $|f_A(M)|$ using the modified color-function g_A. If E is an arithmetic progression with starting point 0 in $\mathbb{Z}_p$, the multi-set $M = E \cap (-E)$ can be separated into at most three sets that are either a centered arithmetic progression or the complement of a centered arithmetic progression in $\mathbb{Z}_p$, as the following lemma states. The third set not mentioned in the lemma is the set $M_0 := \{x \in \mathbb{Z}_p \mid M(x) \geq 1\}$, which is obviously a centered arithmetic progression.

Lemma 2.41. *Let P be an arithmetic progression in $\mathbb{Z}_p$. The set $P \cap (-P)$ is the union of at most two sets that are either a centered arithmetic progression or the complement of a centered arithmetic progression in $\mathbb{Z}_p$.*

Proof. Let $\alpha \in \mathbb{Z}_p$, $\beta \in \mathbb{Z}_p \setminus \{0\}$ and $1 \le L \le p-1$ such that $P = \{\alpha + i\beta \mid 0 \le i \le L-1\}$. We can assume that $L < p$, because otherwise $P \cap (-P) = P = \mathbb{Z}_p$ and there is nothing left to prove. Let $x \in P \cap (-P)$. There exist $i, j \in \{0, 1, \dots, L-1\}$ such that

$$x \equiv \alpha + i\beta \equiv -\alpha - j\beta \mod p \tag{2.13}$$

Thus, we have $(i + j)\beta \equiv -2\alpha \mod p$. Let $k_1 \in \{0, 1, \dots, p-1\}$ be the unique element with $k_1\beta \equiv -2\alpha \mod p$. Set $k_2 := k_1 + p$. $\{k_1, k_2\}$ is the solution set of the congruence $k\beta \equiv -2\alpha$ in the set $\{0, 1, \dots, 2p-1\}$. We make a case differentiation.

Case 1: $k_1 > 2L - 2$

Then (2.13) has no solution $(i, j) \in \{0, 1, \dots, L-1\}^2$. Hence $P \cap (-P) = \emptyset$.

Case 2: $L - 1 < k_1 \le 2L - 2$

We have $k_2 = k_1 + p > L - 1 + p > 2L - 2$. Therefore the solutions $(i, j) \in \{0, 1, \dots, L-1\}^2$ of the congruence (2.13) are exactly the solutions of the equation $i + j = k_1$. The possible values for i respectively j are the elements of the set $Y := \{k_1 - L + 1, k_1 - L + 2, \dots, L-1\}$. Thus, one can check that $P \cap (-P) = \{\alpha + i\beta \mid i \in Y\}$ is a centered arithmetic progression, if $|P \cap (-P)|$ is odd, and the complement of a centered arithmetic progression in $\mathbb{Z}_p$ if $|P \cap (-P)|$ is even.

Case 3: $k_1 \le L - 1$ and $k_2 > 2L - 2$

This case is analog to the second case, but the set Y becomes the set $\{0, 1, \dots, k_1\}$.

Case 4: $k_1 \le L - 1$ and $k_2 \le 2L - 2$

The solutions $(i, j) \in \{0, 1, \dots, L-1\}^2$ of the congruence (2.13) are either the solutions of the equation $i + j = k_1$ or the solutions of the equation $i + j = k_2$. The corresponding solution sets are $Y_1 := \{0, 1, \dots, k_1\}$ and $Y_2 := \{k_2 - L + 1, k_2 - L + 2, \dots, L-1\}$. It holds

$$P \cap (-P) = \{\alpha + i\beta \mid i \in Y_1\} \dot\cup \{\alpha + i\beta \mid i \in Y_2\}$$

and $\{\alpha + i\beta \mid i \in Y_k\}$ $(k = 1, 2)$ is a centered arithmetic progression, if $|Y_k|$ is odd, and the complement of a centered arithmetic progression in $\mathbb{Z}_p$ if $|Y_k|$ is even.

□

Proof of Theorem 2.38. The upper bound for the discrepancy of the hypergraph $\mathcal{H}_{\mathbb{Z}_p}$ of all arithmetic progression in $\mathbb{Z}_p$ from Theorem 2.36 is also an upper bound for the discrepancy of $\mathcal{H}_{\mathbb{Z}_p,c}$, since $\mathcal{E}_{\mathbb{Z}_p,c} \subseteq \mathcal{E}_{\mathbb{Z}_p}$. Thus, only the lower bound is left to prove. Fix a c–coloring $\chi : \mathbb{Z}_p \to \{1, 2, \ldots, c\}$ of $\mathcal{H}_{\mathbb{Z}_p,c}$. The assertion is trivial for $p \in \{2, 3\}$. Hence, we can assume $p \geq 5$. We investigate the case $c \geq 4$ first. After this we will check the lower bound also for the case $c = 3$. In the case $c \geq 4$, which we treat now, we show $\operatorname{disc}(\mathcal{H}_{\mathbb{Z}_p,c}, c) \geq \frac{1}{25}\sqrt{\frac{p}{c}}\operatorname{disc}(\mathcal{H}_{\mathbb{Z}_p,c}, c)$, which implies the stated lower bound.

There exists at least one color $i \in \{1, 2, \ldots, c\}$ with $|\chi^{-1}(i)| \geq \frac{p}{c}$. If there is a color $i \in \{1, 2, \ldots, c\}$ such that $|\chi^{-1}(i)| > \frac{p}{c} + \frac{1}{25}\sqrt{\frac{p}{c}}$, then

$$\operatorname{disc}(\mathcal{H}_{\mathbb{Z}_p,c}, c, \chi) \geq \left| |\mathbb{Z}_p \cap \chi^{-1}(i)| - \frac{1}{c}|\mathbb{Z}_p| \right| > \frac{1}{25}\sqrt{\frac{p}{c}},$$

because $\mathbb{Z}_p$ itself is a centered arithmetic progression in $\mathbb{Z}_p$. Thus, we can assume that this is not the case and get the existence of a color $i \in \{1, 2, \ldots, c\}$ such that it holds for $A := \chi^{-1}(i)$

$$\frac{p}{c} \leq |A| \leq \frac{p}{c} + \frac{1}{25}\sqrt{\frac{p}{c}}.$$

For the density $\delta_A := \frac{1}{p}|A|$ of A in $\mathbb{Z}_p$ we have

$$0 \leq \delta_A - \frac{1}{c} \leq \frac{1}{25}\sqrt{\frac{1}{pc}}.$$

Set $E := \{0, 1, \ldots, \frac{p-1}{2}\}$ and $\delta_E := |E| = \frac{p+1}{2}$. For the function g_A defined in (2.11) we get by Lemma 2.33 the existence of an $a \in \mathbb{Z}_p \setminus \{0\}$ and a $b \in \mathbb{Z}_p$ such that

$$|g_A(b + aE)| \geq \sqrt{\delta_E(1 - \delta_E)}\|g_A\|_2 = \frac{1}{2}\sqrt{\frac{p^2 - 1}{p^2}}\|g_A\|_2.$$

Using Lemma 2.40 we have

$$|g_A(b + aE)| \geq \sqrt{\frac{p^2 - 1}{p^2}}\sqrt{\delta_A\left(\frac{1}{2} - \delta_A\right)p}.$$

$b + aE$ is an arithmetic progression with starting point b, difference a and length $\frac{p+1}{2}$. Setting $x := \delta_A - \frac{1}{c}$ we get $0 \leq x \leq \frac{1}{25}\sqrt{\frac{1}{pc}}$. Therefore,

$$\delta_A\left(\frac{1}{2} - \delta_A\right) = \left(\frac{1}{c} + x\right)\left(\frac{1}{2} - \frac{1}{c} - x\right) = \frac{c-2}{2c^2} + \frac{c-4}{2c}x - x^2 \geq \frac{c-2}{2c^2} - \frac{1}{625pc}$$

and hence (using $p \geq 5$ and $c \geq 4$)

$$\begin{aligned} |g_A(b+aE)| &\geq \sqrt{\frac{p^2-1}{p^2}}\sqrt{\left(\frac{c-2}{2c^2}-\frac{1}{625pc}\right)p} \\ &\geq \sqrt{\frac{24}{25}\left(\frac{1}{4}-\frac{1}{3125}\right)}\sqrt{\frac{p}{c}} \\ &> \frac{12}{25}\sqrt{\frac{p}{c}}. \end{aligned}$$

Since $|g_A(\mathbb{Z}_p \setminus (b+aE))| = |g_A(b+aE)|$, we can assume that $0 \notin (b+aE)$. Every arithmetic progression in $\mathbb{Z}_p$ can be supplemented step by step until it is the whole set $\mathbb{Z}_p$, because p is a prime. Thus, we can supplement $b+aE$ to an arithmetic progression P_1 with starting point 0. Using the triangle-inequality P_1 or $P_1 \setminus (b+aE)$ is an arithmetic progression P with starting point 0 and

$$|g_A(P)| \geq \frac{1}{2}|g_A(b+aE)| > \frac{6}{25}\sqrt{\frac{p}{c}}.$$

We define the multi-set $M := P \cap (-P)$ and get as mentioned in the discussion after Lemma 2.40

$$|f_A(M)| = |g_A(P)|.$$

Lemma 2.41 states that $P \cap (-P)$ is the union of at most two sets that are either a centered arithmetic progression or the complement of a centered arithmetic progression in $\mathbb{Z}_p$. Since the set $M_0 := \{x \in \mathbb{Z}_p \mid M(x) \geq 1\}$ is a centered arithmetic progression, the multi-set M is the union of at most three sets that are either a centered arithmetic progression or the complement of a centered arithmetic progression in $\mathbb{Z}_p$. Once again the triangle-inequality yields that it holds for at least one of this sets that we denote by P_0

$$|f_A(P_0)| \geq \frac{1}{3}|f_A(M)| = \frac{1}{3}|g_A(P)| > \frac{2}{25}\sqrt{\frac{p}{c}}.$$

It holds $|f_A(\mathbb{Z}_p \setminus P_0)| = |f_A(P_0)| > \frac{2}{25}\sqrt{\frac{p}{c}}$. Therefore we can assume that P_0 is a centered arithmetic progression in $\mathbb{Z}_p$. The triangle-inequality yields

$$\begin{aligned}
\operatorname{disc}(\mathcal{H}_{\mathbb{Z}_p,c}, c, \chi) &\geq \left| |P_0 \cap A| - \frac{1}{c}|P_0| \right| \\
&= \left| |P_0 \cap A| - \delta_A |P_0| + \left(\delta_A - \frac{1}{c}\right) |P_0| \right| \\
&\geq ||P_0 \cap A| - \delta_A |P_0|| - \left| \left(\delta_A - \frac{1}{c}\right) p \right| \\
&= |f_A(P_0)| - \left| |A| - \frac{p}{c} \right| \\
&> \frac{2}{25}\sqrt{\frac{p}{c}} - \frac{1}{25}\sqrt{\frac{p}{c}} \\
&= \frac{1}{25}\sqrt{\frac{p}{c}}.
\end{aligned}$$

It remains the case $c = 3$. There exists a color $i \in \{1, 2, 3\}$ such that $|\chi^{-1}(i)| \leq \frac{p}{c}$. If there is also a color $i \in \{1, 2, 3\}$ with $|\chi^{-1}(i)| < \frac{p}{c} - \frac{1}{31}\sqrt{\frac{p}{c}}$, then

$$\operatorname{disc}(\mathcal{H}_{\mathbb{Z}_p,c}, c, \chi) \geq \left| |\mathbb{Z}_p \cap \chi^{-1}(i)| - \frac{1}{c}|\mathbb{Z}_p| \right| > \frac{1}{31}\sqrt{\frac{p}{c}},$$

because $\mathbb{Z}_p$ itself is a centered arithmetic progression in $\mathbb{Z}_p$. Thus, we can assume that this is not the case. Then there exists a color $i \in \{1, 2, 3\}$ such that we get for $A := \chi^{-1}(i)$

$$\frac{p}{c} \geq |A| \geq \frac{p}{c} - \frac{1}{31}\sqrt{\frac{p}{c}}.$$

The density $\delta_A := \frac{1}{p}|A|$ of A in $\mathbb{Z}_p$ satisfies

$$0 \leq \frac{1}{c} - \delta_A \leq \frac{1}{31}\sqrt{\frac{1}{pc}}.$$

Now analogously to the case $c \geq 4$ we get for $E := \{0, 1, \ldots, \frac{p-1}{2}\}$ the existence of an $a \in \mathbb{Z}_p \setminus \{0\}$ and a $b \in \mathbb{Z}_p$ such that it holds for the arithmetic progression $b + aE$

$$|g_A(b + aE)| \geq \sqrt{\frac{p^2 - 1}{p^2}} \sqrt{\delta_A \left(\frac{1}{2} - \delta_A\right) p}.$$

But in this case, setting $x := \frac{1}{c} - \delta_A$, we get $0 \leq x \leq \frac{1}{31}\sqrt{\frac{1}{pc}}$ and hence

$$\delta_A \left(\frac{1}{2} - \delta_A\right) = \left(\frac{1}{c} - x\right)\left(\frac{1}{2} - \frac{1}{c} + x\right) = \frac{c-2}{2c^2} + \frac{4-c}{2c}x - x^2 \geq \frac{c-2}{2c^2} - \frac{1}{961pc}.$$

Using $p \geq 5$ and $c = 3$ we have

$$\begin{aligned} |g_A(b+aE)| &\geq \sqrt{\frac{p^2-1}{p^2}}\sqrt{\left(\frac{c-2}{2c^2}-\frac{1}{961pc}\right)p} \\ &\geq \sqrt{\frac{24}{25}\left(\frac{1}{6}-\frac{1}{4805}\right)}\sqrt{\frac{p}{c}} \\ &> \frac{12}{31}\sqrt{\frac{p}{c}} \end{aligned}$$

The rest of the proof is analog to the case $p \geq 4$ just replacing the constants $\frac{x}{25}$ by $\frac{x}{31}$. □

2.8 Bohr Neighborhoods

In this section we shortly point out the relevance of Bohr neighborhoods[2] in the field of additive number theory. We also give bounds for the size of Bohr neighborhoods and, using the fact that the set of all translates of d–dimensional Bohr neighborhoods is $\mathbb{Z}_p$–invariant, we give bounds for the discrepancy of the corresponding hypergraph.

Let p be a prime number. For all residue classes $z \in \mathbb{Z}_p$ let us denote by $|z|$ the distance of z to $0 \in \mathbb{Z}_p$. To make it more precise:

$$|z| = \min\{|x| \mid x \in z\}.$$

Let $R = \{r_1, r_2, \ldots, r_d\}$ be a d–element subset of $\mathbb{Z}_p \setminus \{0\}$ and $0 < \varepsilon \leq \frac{1}{2}$. We define the *$d$–dimensional Bohr neighborhood* $B(R, \varepsilon)$ by

$$B(R,\varepsilon) := \{z \in \mathbb{Z}_p \mid \ |zr| \leq \varepsilon p \text{ for all } r \in R\}.$$

It is obvious that a d–dimensional Bohr neighborhood is the intersection of d arithmetic progressions that are centered in $0 \in \mathbb{Z}_p$. Let $0 < \varepsilon \leq \frac{1}{2}$, $L := \lfloor \varepsilon p \rfloor$ and $R = \{r_1, r_2, \ldots, r_d\} \subseteq \mathbb{Z}_p \setminus \{0\}$, then

$$B(R,\varepsilon) = \bigcap_{i=1}^{d} \{jr_i^{-1} \mid j \in \{-L, -L+1, \ldots, 0, \ldots, L-1, L\}\}.$$

The hypergraph of centered arithmetic progressions in $\mathbb{Z}_p$ investigated in Section 2.7 is indeed also the hypergraph of all one-dimensional Bohr neighborhoods in $\mathbb{Z}_p$.

[2]In honor of the mathematician Harald Bohr, who also played in the national soccer team of Denmark and won the silver medal in the Olympic games 1908 in London.

Bohr Neighborhoods in Additive Number Theory

Bohr neighborhoods are a useful structure in some parts of additive number theory. We consider the question whether there is a 3–term arithmetic progression in a set A or not. Roth [Rot53] proved the following theorem giving a lower bound for the density of the set A such that the existence of a 3–term arithmetic progression is guaranteed in A.

Theorem 2.42 (Roth)**.** *There exists a constant $c > 0$ such that for every $N \in \mathbb{N}$ and every $A \subseteq [N]$ with*

$$\delta_A = \frac{|A|}{N} > c\frac{1}{\log \log N}$$

there is a non-trivial 3–term arithmetic progression[3] *in A.*

As a consequence of this theorem one gets that every subset of the integers of positive density contains infinitely many progressions of length 3. Szemerédi [Sze90] and Heath-Brown [HB87] improved Roth's bound to $(\log N)^{-c}$ for an absolute constant $c > 0$. Szemerédi produced an explicit constant $c = 1/20$. The so far best known bound for the density of a set that suffices to guarantee a 3–term arithmetic progression was given by Bourgain [Bou99].

Theorem 2.43 (Bourgain)**.** *There exists a constant $c > 0$ such that for every $N \in \mathbb{N}$ and every $A \subseteq [N]$ with*

$$\delta_A = \frac{|A|}{N} > c\left(\frac{\log \log N}{\log N}\right)^{1/2}$$

there is a non-trivial 3–term arithmetic progression in A.

The reason why Bourgain obtains a much stronger bound than Roth can be found in the article "Some unsolved problems in additive/combinatorial number theory" by W.T. Gowers [Gow]. He says: "The main source of inefficiency in Roth's argument is the fact that one passes many times to a subprogression of size the square root of what one had before. This means that the iteration argument is very costly. Moreover, at each stage of iteration, one obtains increased density on a mod–N arithmetic progression of *linear* size and simply discards almost all of this information in the process of restricting to a 'genuine' arithmetic progression. Bourgain does not throw away information in this way. Instead, he tries to find increased density not on arithmetic progressions but on translates of *Bohr neighborhoods*, which are sets of the form $\{x \in \mathbb{Z}_N : r_i x \in [-\delta_i N, \delta_i N]\}$. Note that these sets are just intersections of a few mod–N arithmetic progressions. Roughly speaking, if a set A is not evenly distributed inside a Bohr neighborhood B, then, using a large Fourier coefficient of $A \cap B$, one can pick out a new mod–N arithmetic progression P such that the density of A inside of $B \cap P$, which still is a Bohr neighborhood, is larger. The reason this approach can be expected to work is that Bohr neighborhoods have a great

[3] A non-trivial 3–term arithmetic progression is a set of distinct elements a, b and c with $a + c = 2b$.

deal of arithmetic structure: indeed, they are rather similar to multidimensional arithmetic progressions[4]."

Bohr neighborhoods were also used in other famous articles. We mention here two examples. The first is Green's paper [Gre05] in which he shows that any set containing a positive proportion of the primes contains a 3–term arithmetic progression. In the second paper [GT] Green and Tao prove that the primes contain arbitrarily long arithmetic progressions.

Some Properties of Bohr Neighborhoods

As we have seen before, every d–dimensional Bohr neighborhood is the intersection of d arithmetic progressions. It is hopeless to give non-trivial bounds to the size of the intersection of arithmetic progressions in the integers, but using the special structure of $\mathbb{Z}_p$ we can prove lower and upper bounds for the size of d–dimensional Bohr neighborhoods that are only depending on the dimension d, the parameter ε and the size p of $\mathbb{Z}_p$.

Proposition 2.44. *Let R be a d–element subset of $\mathbb{Z}_p \setminus \{0\}$, $0 < \varepsilon \leq \frac{1}{2}$ and let $\varepsilon_0 := 1/\lceil \frac{1}{\varepsilon} \rceil$. The size of the Bohr neighborhood $B(R, \varepsilon)$ fulfills*[5]

(i) $|B(R,\varepsilon)| \geq \lfloor \varepsilon_0^d p \rfloor$.

(ii) $|B(R,\varepsilon)| \leq 17\frac{\varepsilon p}{d}$, *if* $\varepsilon \leq \frac{1}{12}$.

For the proof of the upper bound we need the following theorem that was proved by Lev [Lev01]. We define for all $\varepsilon \in [0, \frac{1}{4}]$ and all $A \subseteq \mathbb{Z}_p$

$$T_A(\varepsilon) := \left| \left\{ z \in \mathbb{Z}_p \setminus \{0\} : \left| \sum_{a \in A} e^{2\pi i \frac{az}{p}} \right| > |A| \cos 2\pi\varepsilon \right\} \right|.$$

Theorem 2.45. *For any set $A \subset \mathbb{Z}_p$ with $n := |A| \geq 4$ and any $\varepsilon \in [0, \frac{1}{12}]$ it holds*

$$T_A(\varepsilon) \leq 4\sqrt{3}\frac{p}{n}\varepsilon(1 + n^{-2})(1 + 2(2\pi\varepsilon)^{2/3}).$$

Proof of Proposition 2.44.

(i) For every $x \in \mathbb{Z}_p$ let $\bar{x}$ be the representative of the residue class x in the set $\{0, 1, \ldots, p-1\}$ or in other words $\bar{x} := \min(x \cap \mathbb{N}_0)$. Let $k := \lceil \frac{1}{\varepsilon} \rceil$, and for all $i \in [k]$

$$A_i := \{x \in \mathbb{Z}_p \mid \bar{x} \in [(i-1)\lceil \varepsilon p \rceil, i\lceil \varepsilon p \rceil)\}.$$

[4]The multidimensional arithmetic progressions are those sets that we called sums of k arithmetic progressions in Section 2.3

[5]Note that the upper bound is more precisely $\max\{17\frac{\varepsilon p}{d}, 4\}$.

Then $\mathbb{Z}_p = \bigcup_{i=1}^k A_i$ and we get for every $i \in [k]$ and every $x, y \in A_i$:

$$|x - y| \leq |\bar{x} - \bar{y}| \leq \varepsilon p.$$

We define for each $v \in [k]^d$ the set

$$M_v := \{x \in \mathbb{Z}_p \mid r_i x \in A_{v_i}, i \in [d]\}.$$

It is obvious that $M_{(1,1,\ldots,1)} \cup M_{(k,k,\ldots,k)} \subseteq B(R, \varepsilon)$. Thus, we can assume that $|M_{(1,1,\ldots,1)} \cup M_{(k,k,\ldots,k)}| < \lfloor \varepsilon_0^d p \rfloor$. But then, by the pigeon hole-principle, there is a $v \in [k]^d \setminus \{(1, 1, \ldots, 1), (k, k, \ldots, k)\}$ such that

$$|M_v| \geq \frac{p}{k^d - 1} > \varepsilon_0^d p.$$

Hence, $|M_v| \geq \lfloor \varepsilon_0^d p \rfloor + 1$. Let $x_0 \in M_v$. The set $\{x - x_0 \mid x \in M_v \setminus \{x_0\}\}$ is a subset of $B(R, \varepsilon)$ containing at least $\lfloor \varepsilon_0^d p \rfloor$ elements.

(ii) Let $\varepsilon \leq \frac{1}{12}$. For every $r \in R$ and every $z \in B(R, \varepsilon)$ it holds $|\frac{rz}{p}| \leq \varepsilon$. Thus,

$$\begin{aligned}
|\widehat{\mathbb{1}}_{-B(R,\varepsilon)}(r)| &= \left| \sum_{z \in \mathbb{Z}_p} \mathbb{1}_{-B(R,\varepsilon)}(z) e^{-2\pi i \frac{rz}{p}} \right| \\
&= \left| \sum_{z \in B(R,\varepsilon)} e^{2\pi i \frac{rz}{p}} \right| \\
&\geq \mathcal{R}\mathrm{e} \left(\sum_{z \in B(R,\varepsilon)} e^{2\pi i \frac{rz}{p}} \right) \\
&= \sum_{z \in B(R,\varepsilon)} \cos(2\pi \tfrac{rz}{p}) \\
&> |B(R, \varepsilon)| \cos(2\pi\varepsilon).
\end{aligned}$$

Hence, we have $T_{B(R,\varepsilon)}(\varepsilon) \geq d$. On the other hand, if $n := |B(R, \varepsilon)| \geq 4$, Theorem 2.45 yields

$$T_{B(R,\varepsilon)}(\varepsilon) \leq 4\sqrt{3}\frac{p}{n}\varepsilon(1 + n^{-2})(1 + 2(2\pi\varepsilon)^{2/3}) \leq 4\sqrt{3}\frac{17}{16}(1 + 2(\tfrac{\pi}{6})^{2/3})\frac{p\varepsilon}{n} < 17\frac{p\varepsilon}{n}.$$

Therefore we have $|B(R, \varepsilon)| \leq 17\frac{p\varepsilon}{d}$.

□

Having given bounds on the size of a Bohr neighborhood, we now look for its dimension. The structure of a Bohr neighborhood is highly depending on the set R. Even though the set R may contain a number of elements, it could happen that the Bohr neighborhood $B(R, \varepsilon)$ is so to speak in effect only a one-dimensional Bohr neighborhood as in the following example.

Example 2.46. *Let p be a prime number, $d \in \mathbb{N}$ and $0 < \varepsilon < \frac{1}{2d}$. Set $R := \{1, 2, \ldots, d\}$. Then*

$$B(R, \varepsilon) = B(\{1\}, \varepsilon/d)$$

Proof. $B(\{1\}, \varepsilon/d) \subseteq B(R, \varepsilon)$ is obvious. But every element $x \in B(R, \varepsilon)$ fulfills in particular $|xd| \leq \varepsilon$ and $|x| \leq \varepsilon$. Using the fact that $d\varepsilon p < \frac{1}{2}p$, we get $|x| \leq \varepsilon/d$. □

The reason for this extreme situation is that there is a "high dependence" in the set $R = \{1, 2, \ldots, d\}$. Of course every subset of $\mathbb{Z}_p$ of size at least two is linearly dependent in $\mathbb{Z}_p$ as a one-dimensional vector space over the field $(\mathbb{Z}_p, +, \cdot)$. But the phrase "high dependence" is meant in the following sense. We call a set $X \subseteq \mathbb{Z}_p$ *k–independent*, if and only if for every non-trivial function $f : X \to \mathbb{Z}_p$ with

$$\sum_{x \in X} f(x)x = 0$$

there is a $x \in X$ with $|f(x)| \geq k$. We call the largest $k \in \mathbb{N}$ for which $X \subseteq \mathbb{Z}_p$ is k–independent the *independence number* $\mathrm{ind}(X)$ of X.

Proposition 2.47. *The independence number is invariant under multiplications with non-trivial elements, i.e., for every $X \subseteq \mathbb{Z}_p$ and every $z \in \mathbb{Z}_p \setminus \{0\}$ it holds*

$$\mathrm{ind}(zX) = \mathrm{ind}(X).$$

Proof. Since $X = z^{-1}zX$, it is sufficient to show that zX is $\mathrm{ind}(X)$–independent. Let $f : zX \to \mathbb{Z}_p$ be a non-trivial function with $\sum_{v \in zX} f(v)v = 0$. Then

$$\sum_{x \in X} f(zx)x = z^{-1} \sum_{x \in X} f(zx)zx = 0.$$

Thus, there is an $x \in X$ such that $|f(zx)| \geq \mathrm{ind}(X)$, since X is $\mathrm{ind}(X)$–independent. This proves that zX is also $\mathrm{ind}(X)$–independent. □

The next proposition gives an answer to the question how large the independence number of a d–element set in $\mathbb{Z}_p$ can be.

Proposition 2.48.

(i) For every d–element set $X \subseteq \mathbb{Z}_p$ it holds $\mathrm{ind}(X) \leq \lfloor \sqrt[d]{p} \rfloor$.

(ii) Let $x := \lfloor \sqrt[d]{p} \rfloor$ and $X := \{1, x, x^2, \ldots, x^{d-1}\}$. Then $\mathrm{ind}(X) = \lfloor \sqrt[d]{p} \rfloor$.

Proof.

(i) Let $X \subseteq \mathbb{Z}_p$ with $|X| = d$. Set $k := \mathrm{ind}(X)$. Assume that $k \geq \lfloor \sqrt[d]{p} \rfloor + 1$. We regard a subset of $(\mathbb{Z}_p)^X$, namely the set of all functions $f : X \to \{0, 1, \ldots, k-1\}$. This set of functions contains $k^d > p$ elements. Thus, there are two distinct functions f_1 and f_2 in this set of functions such that

$$\sum_{x \in X} f_1(x)x = \sum_{x \in X} f_2(x)x.$$

The function $f_1 - f_2$ has the following properties. It is non-trivial, for every $x \in X$ we have $|(f_1 - f_2)(x)| \leq k-1$ and $\sum_{x \in X}(f_1 - f_2)(x)x = 0$. Therefore $\mathrm{ind}(X) \leq k-1$. This contradiction proves $\mathrm{ind}(X) \leq \lfloor \sqrt[d]{p} \rfloor$.

(ii) Let $\Lambda := \{-x+1, \ldots, 0, \ldots, x-1\}$ and $\lambda \in \Lambda^d$ such that $\sum_{i=1}^{d} \lambda_i x^{i-1} = 0$. We prove $\lambda_i = 0$ for all $i \in [d]$. We consider the problem in $\mathbb{Z}$ instead of $\mathbb{Z}_p$ by using the representatives of λ_i ($i \in [d]$) and x in $\{0, 1, 2, \ldots, p-1\}$. Therefore we know that p divides $\sum_{i=1}^{d} \lambda_i x^{i-1}$. It holds

$$\left| \sum_{i=1}^{d} \lambda_i x^{i-1} \right| \leq \sum_{i=1}^{d} (x-1)x^{i-1} = x^d - 1 \leq p - 1.$$

Thus, $\sum_{i=1}^{d} \lambda_i x^{i-1} = 0$. We show $\lambda_i = 0$ for all $i \in [d]$ by induction. We assume that there is an $i \in [d]$ such that $\lambda_i \neq 0$. Let $j := \min\{i \in [d] \mid \lambda_i \neq 0\}$. It is obvious that $j < d$, since otherwise $\sum_{i=1}^{d} \lambda_i x^{i-1} = \lambda_d x^{d-1} \neq 0$. Modulo x^j we have

$$\lambda_j x^{j-1} \equiv \sum_{i=j}^{d} \lambda_i x^{i-1} \equiv \sum_{i=1}^{d} \lambda_i x^{i-1} \equiv 0 \pmod{x^j}.$$

Using $|\lambda_j x^{j-1}| \leq (x-1)x^{j-1} < x^j$, we get $\lambda_j = 0$, which is a contradiction. Thus, $\mathrm{ind}(X) \geq x = \lfloor \sqrt[d]{p} \rfloor$. Applying (i) we get $\mathrm{ind}(X) = \lfloor \sqrt[d]{p} \rfloor$.

□

It is easy to see that for $d \geq 2$ and every $2 \leq k \leq \sqrt[d]{p}k$ there exists a d–element subset X of $\mathbb{Z}_p \setminus \{0\}$ with $\mathrm{ind}(X) = k$.

Discrepancy of Bohr Neighborhoods

Here we introduce the hypergraph $\mathcal{H}_{k,d,\varepsilon} = (\mathbb{Z}_p, \mathcal{E}_{k,d,\varepsilon})$ of all translates of Bohr Neighborhoods $B(R, \varepsilon)$ such that R is a d–element subset of $\mathbb{Z}_p^* := \mathbb{Z}_p \setminus \{0\}$ with $\mathrm{ind}(R) = k$. To make it more precise, let $2 \leq d \leq p-1$, $2 \leq k \leq \lfloor \sqrt[d]{p} \rfloor$ and $\mathcal{R}_{k,d} := \{R \in \binom{\mathbb{Z}_p^*}{d} \mid \mathrm{ind}(R) = k\}$, where $\binom{\mathbb{Z}_p^*}{d}$ is the set of all d–element subsets of $\mathbb{Z}_p^*$. We define

$$\mathcal{E}_{k,d,\varepsilon} := \{a + B(R, \varepsilon) \mid a \in \mathbb{Z}_p, \quad R \in \mathcal{R}_{k,d}\}.$$

Lemma 2.49. *The hypergraph $\mathcal{H}_{k,d,\varepsilon}$ is $\mathbb{Z}_p$–invariant with respect to every hyperedge.*

Proof. We have to prove that for every hyperedge $E \in \mathcal{E}_{k,d,\varepsilon}$, every $a \in \mathbb{Z}_p$ and every $b \in \mathbb{Z}_p^*$ the set $a + bE$ is also in $\mathcal{E}_{k,d,\varepsilon}$. Let $a_0 + B(R, \varepsilon)$ be an element of $\mathcal{E}_{k,d,\varepsilon}$, $a \in \mathbb{Z}_p$ and $b \in \mathbb{Z}_p^*$. It holds

$$\begin{aligned} bB(R,\varepsilon) &= b\{z \in \mathbb{Z}_p : |rz| \leq \varepsilon p, r \in R\} \\ &= \{z \in \mathbb{Z}_p : |rb^{-1}z| \leq \varepsilon p, r \in R\} \\ &= \{z \in \mathbb{Z}_p : |rz| \leq \varepsilon p, r \in b^{-1}R\} \\ &= B(b^{-1}R, \varepsilon). \end{aligned}$$

Thus,

$$a + b(a_0 + B(R,\varepsilon)) = (a + ba_0) + bB(R,\varepsilon) = (a + ba_0) + B(b^{-1}R, \varepsilon).$$

By Proposition 2.47 we have $\operatorname{ind}(b^{-1}R) = \operatorname{ind}(R)$. Hence, $a + b(a_0 + B(R, \varepsilon)) \in \mathcal{E}_{k,d,\varepsilon}$ and the assertion follows. □

Let $\delta_{k,d,\varepsilon} := \frac{1}{2} - \min\{|\frac{1}{p}|E| - \frac{1}{2}| \mid E \in \mathcal{E}_{k,d,\varepsilon}\}$. Then either $\delta_{k,d,\varepsilon}$ or $1 - \delta_{k,d,\varepsilon}$ is the density of a hyperedge that is nearest possible to $\frac{1}{2}$. The following theorem gives estimates to the discrepancy of $\mathcal{H}_{k,d,\varepsilon}$

Theorem 2.50. *Let $p > 2$ be a prime number, $2 \leq d \leq p-1$, $2 \leq k \leq \lfloor \sqrt[d]{p} \rfloor$ and $0 < \varepsilon \leq \frac{1}{2}$. Then there is a constant $\alpha > 0$ (only depending on d) such that*

$$\sqrt{\frac{\delta_{k,d,\varepsilon}(1-\delta_{k,d,\varepsilon})}{10}}\sqrt{p}c \operatorname{disc} \mathcal{H}_{k,d,\varepsilon}, c \leq \alpha\sqrt{\frac{p}{c}\ln p} + 1.$$

Proof. $\mathcal{H}_{k,d,\varepsilon}$ is a hypergraph that is $\mathbb{Z}_p$–invariant with respect to a hyperedge that has density $\delta_{k,d,\varepsilon}$ or $1 - \delta_{k,d,\varepsilon}$. Thus, we can apply Theorem 2.35 and get the stated lower bound.

The number of hyperedges in $\mathcal{H}_{k,d,\varepsilon}$ is bounded by p^{d+1}. Using Theorem 3.14 from [DS03] we can find an $\alpha > 0$ (only depending on d) such that

$$\operatorname{disc}(\mathcal{H}_{k,d,\varepsilon}, c) \leq \alpha\sqrt{\frac{p}{c}\ln p} + 1.$$

□

Remark 2.51. *Although the Theorem 2.50 is only stated for $d \geq 2$, it is also valid for the hypergraph of all translates of one-dimensional Bohr neighborhoods. But the independence number makes no sense for one-dimensional Bohr neighborhoods.*

Chapter 3

Discrepancy of Products of Hypergraphs

For a hypergraph $\mathcal{H} = (V, \mathcal{E})$, its d–fold symmetric product is $\Delta^d\mathcal{H} = (V^d, \{E^d | E \in \mathcal{E}\})$. We give several upper and lower bounds for the c–color discrepancy of such products. In particular, we show that the bound $\operatorname{disc}(\Delta^d\mathcal{H}, 2) \leq \operatorname{disc}(\mathcal{H}, 2)$ proven for all d by Doerr, Srivastav, and Wehr [DSW04] cannot be extended to more than $c = 2$ colors. In fact, for any c and d such that c does not divide $d!$, there are hypergraphs having arbitrary large discrepancy and $\operatorname{disc}(\Delta^d\mathcal{H}, c) = \Omega_d(\operatorname{disc}(\mathcal{H}, c)^d)$. Apart from constant factors (depending on c and d), in these cases the symmetric product behaves no better than the general direct product $\mathcal{H}^d$, which satisfies $\operatorname{disc}(\mathcal{H}^d, c) = O_{c,d}(\operatorname{disc}(\mathcal{H}, c)^d)$. The results of this chapter can be found in [DGH05].

3.1 Introduction

We investigate the discrepancy of certain products of hypergraphs. In [DSW04] Doerr, Srivastav and Wehr noted the following. For a hypergraph $\mathcal{H} = (V, \mathcal{E})$ define the d–fold direct product by

$$\mathcal{H}^d \;:=\; (V^d, \{E_1 \times \cdots \times E_d \,|\, E_i \in \mathcal{E}\})$$

and the d–fold symmetric product by

$$\Delta^d\mathcal{H} \;:=\; (V^d, \{E^d \,|\, E \in \mathcal{E}\}).$$

Then for the (two-color) discrepancy we have

$$\begin{aligned} \operatorname{disc}(\mathcal{H}^d) &\leq \operatorname{disc}(\mathcal{H})^d, \\ \operatorname{disc}(\Delta^d\mathcal{H}) &\leq \operatorname{disc}(\mathcal{H}). \end{aligned}$$

In this chapter, we show that the situation is more complicated for discrepancies in more than two colors. In particular, it depends highly on the dimension d and the number of colors, whether the discrepancy of symmetric products is more like the discrepancy of the original hypergraph or the d–th power thereof. In this more general setting, the product bound proven in [DSW04] is

$$\operatorname{disc}(\mathcal{H}^d, c) \le c^{d-1} \operatorname{disc}(\mathcal{H}, c)^d. \tag{3.1}$$

However, as we show in this chapter the relation $\operatorname{disc}(\Delta^d\mathcal{H}, c) = O(\operatorname{disc}(\mathcal{H}, c))$ does not hold in general. In Section 3.2, we give a characterization of those values of c and d, for which it is fulfilled for every hypergraph $\mathcal{H}$. In particular, we present for all c, d, k such that c does not divide $d!$ a hypergraph $\mathcal{H}$ having $\operatorname{disc}(\mathcal{H}, c) \ge k$ and $\operatorname{disc}(\Delta^d\mathcal{H}, c) = \Omega_{c,d}(k^d)$. In the light of (3.1), this is largest possible apart from factors depending on c and d only.

On the other hand, there are further situations where this worst case does not occur. We prove some in Section 3.3, but the complete picture seems to be complicated.

3.2 Symmetric Direct Products Having Large Discrepancy

A set $\{x_1, \dots, x_k\}$ of integers with $x_1 < \dots < x_k$ is denoted by $\{x_1, \dots, x_k\}_<$. For a set S we put

$$\binom{S}{k} = \{T \subseteq S \mid |T| = k\}.$$

Furthermore, let S_k be the symmetric group on $[k]$. For $l, d \in \mathbb{N}$ with $l \le d$ let $P_l(d)$ be the set of all partitions of $[d]$ into l non-empty subsets. Let $e_1 = (1, 0, \dots, 0)$, …, $e_d = (0, \dots, 0, 1)$ be the standard basis of $\mathbb{R}^d$. For $c \in \mathbb{N}$ and $\lambda \in \mathbb{N}_0$ we write $c \mid \lambda$ if there exists an $m \in \mathbb{N}_0$ with $mc = \lambda$.

Definition 3.1. Let $d \in \mathbb{N}$, $l \in [d]$ and $T \subseteq \mathbb{N}$ be a finite set. For $J = \{J_1, \dots, J_l\} \in P_l(d)$ with $\min J_1 < \dots < \min J_l$ put $f_i = f_i(J) = \sum_{j \in J_i} e_j$, $i = 1, \dots, l$. Let $\sigma \in S_l$. We call

$$S_J^\sigma(T) := \Big\{ \sum_{i=1}^{l} \alpha_{\sigma(i)} f_i \,\Big|\, \{\alpha_1, \dots, \alpha_l\}_< \subseteq T \Big\}$$

an *l–dimensional simplex in T^d*. If $l = d$, we simply write $S^\sigma(T)$ instead of $S_J^\sigma(T)$ (as $|P_d(d)| = 1$).

Remark 3.2. *If $S(d, l)$, $d, l \in \mathbb{N}$, denote the* Stirling numbers of the second kind, *then $|P_l(d)| = S(d, l)$ (see, e.g. [Rio58]). We have*

$$S(d, l) = \sum_{j=0}^{l} \frac{(-1)^j (l-j)^d}{j!\,(l-j)!}\,. \tag{3.2}$$

Let $T \subseteq \mathbb{N}$ be finite. If $|T| \geq l$, we have $S_I^\sigma(T) \neq S_J^\tau(T)$ as long as $I \neq J$ or $\sigma \neq \tau$. Thus the number of l–dimensional simplices in T^d is $l!\, S(d,l)$. If $|T| < l$, then there exists obviously no non-empty l–dimensional simplex in T^d.

Theorem 3.3. *Let $c, d \in \mathbb{N}$.*

If $c \,|\, k!\, S(d,k)$ for all $k \in \{2, \ldots, d\}$, then every hypergraph $\mathcal{H}$ satisfies

$$\operatorname{disc}(\Delta^d \mathcal{H}, c) \leq \operatorname{disc}(\mathcal{H}, c)\,. \tag{3.3}$$

If $c \not|\, k!\, S(d,k)$ for some $k \in \{2, \ldots, d\}$, then there exists a hypergraph $\mathcal{K}$ such that

$$\operatorname{disc}(\Delta^d \mathcal{K}, c) \geq \frac{1}{3\,k!} \operatorname{disc}(\mathcal{K}, c)^k\,, \tag{3.4}$$

and $\mathcal{K}$ can be chosen to have arbitrary large discrepancy $\operatorname{disc}(\mathcal{K}, c)$.

Before proving the theorem, we state some consequences. In particular, (3.3) holds never for $c = 4$. For $c = 3$, it holds exactly if d is odd.

Corollary 3.4. *(a) Let $d \geq 3$ be an odd number. Then* $\operatorname{disc}(\Delta^d \mathcal{H}, 3) \leq \operatorname{disc}(\mathcal{H}, 3)$ *holds for any hypergraph $\mathcal{H}$.*

(b) Let $d \geq 2$ be an even number and $c = 3l$, $l \in \mathbb{N}$. There exists a hypergraph $\mathcal{H}$ with arbitrary large discrepancy that fulfills $\operatorname{disc}(\Delta^d \mathcal{H}, c) \geq \frac{1}{6} \operatorname{disc}(\mathcal{H}, c)^2$.

Proof. Obviously $3 |\, k!$ for all $k \geq 3$. Since $S(d,2) = 2^{d-1} - 1$, we have $3 |\, S(d,2)$ if and only if d is odd. Indeed, $2^{3-1} - 1 = 3$, $2^{4-1} - 1 = 7$ and if $d = k + 2$, then $2^{d-1} - 1 = 4(2^{k-1} - 1) + 3$, hence $3 |\, (2^{d-1} - 1)$ if and only if $3 |\, (2^{k-1} - 1)$. Hence Theorem 3.3 proves both claims. ☐

Corollary 3.5. *Let $l \in \mathbb{N}$ and $c = 4l$. For all $d \geq 2$ there exists a hypergraph $\mathcal{H}$ with arbitrary large discrepancy such that* $\operatorname{disc}(\Delta^d \mathcal{H}, c) \geq \frac{1}{6} \operatorname{disc}(\mathcal{H}, c)^2$.

Proof. As $S(d,2) = 2^{d-1} - 1$ is an odd number, we have $4 \not|\, 2!\, S(d,2)$. Applying Theorem 3.3 concludes the proof. ☐

Corollary 3.6. *Let $c \geq 3$ be an odd number and $d \geq 2$. We have*

$$\operatorname{disc}(\Delta^d \mathcal{H}, c) \leq \operatorname{disc}(\mathcal{H}, c) \quad \text{for all hypergraphs } \mathcal{H} \tag{3.5}$$

if and only if we have

$$\operatorname{disc}(\Delta^d \mathcal{H}, 2c) \leq \operatorname{disc}(\mathcal{H}, 2c) \quad \text{for all hypergraphs } \mathcal{H}\,. \tag{3.6}$$

Proof. According to Theorem 3.3, (3.5) is equivalent to the statement that $c |\, k!\, S(d,k)$ for all $k \in \{2, \ldots, d\}$. But, since $2 |\, k!$ for all $k \geq 2$ and c is odd, this is equivalent to $2c |\, k!\, S(d,k)$ for all $k \in \{2, \ldots, d\}$, which is equivalent to (3.6). ☐

Our proof of Theorem 3.3 uses the following lemma.

Lemma 3.7. *Let $c, d \in \mathbb{N}$. For all $m \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ having the following property: For each c–coloring $\chi : [n]^d \to [c]$ we find a subset $T \subseteq [n]$ with $|T| = m$ such that for all $l \in [d]$ each l–dimensional simplex in T^d is monochromatic with respect to χ.*

Proof of Lemma 3.7. The proof is based on an argument from Ramsey theory. First we verify the statement of Lemma 3.7 for a fixed simplex. Then, by induction over the number of all simplices, we prove the complete assertion of Lemma 3.7.

Claim: For all $m \in \mathbb{N}$, all $l \in [d]$, all $\sigma \in S_l$, and all $J \in P_l(d)$, there is an $n \in \mathbb{N}$ such that for all $N \subseteq \mathbb{N}$ with $|N| = n$ and each c–coloring $\chi : N^d \to [c]$ there is a subset $T \subseteq N$ with $|T| = m$ and $S_J^\sigma(T)$ is monochromatic with respect to χ.

Proof of the claim: By Ramsey's theorem (see, e.g. [GMRT04], Section 1.2), for every $l \in [d]$ there exists an n such that for each c–coloring $\psi : \binom{[n]}{l} \to [c]$ there is a subset T of $[n]$ with $|T| = m$ and $\binom{T}{l}$ is monochromatic with respect to ψ. Let $N \subseteq \mathbb{N}$ with $|N| = n$. We can assume $N = [n]$ by renaming the elements of N and preserving their order. Let $\chi : [n]^d \to [c]$ be an arbitrary c–coloring. We define $\chi_{l,\sigma,J} : \binom{[n]}{l} \to [c]$ by $\chi_{l,\sigma,J}(\{x_1, \ldots, x_l\}_<) = \chi(\sum_{i=1}^{l} x_{\sigma(i)} f_i)$, where the $f_i = f_i(J)$ are the vectors corresponding to the partition J introduced in Definition 3.1. By the Ramsey theory argument there is a $T \subseteq N$ with $|T| = m$ and $\chi_{l,\sigma,J}$ is constant on $\binom{T}{l}$. Hence, $S_J^\sigma(T)$ is monochromatic with respect to χ. This proves the claim.

Now we derive Lemma 3.7 from the claim. Each simplex is uniquely determined by a pair

$$(\sigma, J) \in \bigcup_{l=1}^{d} (S_l \times P_l(d)) .$$

Let $(\sigma_i, J_i)_{i \in [s]}$ be an enumeration of all these pairs. Put $n_0 := m$. We proceed by induction. Let $i \in [s]$ be such that n_{i-1} is already defined and has the property that for any $N \subseteq \mathbb{N}$, $|N| = n_{i-1}$ and any coloring $\chi : N^d \to [c]$ there is a $T \subseteq N$, $|T| = m$ such that for all $j \in [i-1]$, $S_{J_j}^{\sigma_j}(T)$ is monochromatic. Using the claim, we choose n_i large enough such that for each $N \subseteq \mathbb{N}$ with $|N| = n_i$ and for each c–coloring $\varphi : N^d \to [c]$ there exists a subset T of N with $|T| = n_{i-1}$ and $S_{J_i}^{\sigma_i}(T)$ is monochromatic with respect to φ. Note that there is a $T' \subseteq T$, $|T| = m$ such that $S_{J_j}^{\sigma_j}(T')$ is monochromatic for all $j \in [i]$. Choosing $n := n_s$ proves the lemma. ☐

Related to Lemma 3.7 is a result of Gravier, Maffray, Renault and Trotignon [GMRT04]. They have shown that for any $m \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that any collection of n different sets contains an induced subsystem on m points such that one of the following holds: (a) each vertex forms a singleton, (b) for each vertex there is a set containing all m

points except this one, or (c) by sufficiently ordering the points $p_1, \ldots, p_m$ we have that all sets $\{p_1, \ldots, p_\ell\}, \ell \in [m]$, are contained in the system.[1]

In our language, this means that any $0, 1$ matrix having n distinct rows contains a $m \times m$ submatrix that can be transformed through row and column permutations into a matrix that is (a) a diagonal matrix, (b) the inverse of a diagonal matrix, or (c) a triangular matrix.

Hence this result is very close to the assertion of Lemma 3.7 for dimension $d = 2$ and $c = 2$ colors. It is stronger in the sense that not only monochromatic simplices are guaranteed, but also a restriction to 3 of the 8 possible color combinations for the 3 simplices is given. Of course, this stems from the facts that (a) column and row permutations are allowed, (b) not a submatrix with index set T^2 is provided but only one of type $S \times T$, and (c) the assumption of having different sets ensures sufficiently many entries in both colors.

Proof of Theorem 3.3. Let us first consider the case that $c \mid k!\, S(d, k)$ for all $k \in \{2, \ldots, d\}$. Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph and let $\psi : V \to [c]$ such that $\operatorname{disc}(\mathcal{H}, \psi) = \operatorname{disc}(\mathcal{H}, c)$. For $X \subseteq V$, put $D(X) = \{(x, \ldots, x) \mid x \in X\}$. We define the following c–coloring $\chi : V^d \to [c]$. For $(v, \ldots, v) \in D(V)$, set $\chi(v, \ldots, v) = \psi(v)$. For the remaining vertices, let χ be such that all simplices are monochromatic, and for each k there are exactly $\frac{1}{c} k! S(d, k)$ monochromatic k–dimensional simplices in each color.

Let $E \in \mathcal{E}$ and put $R(E) := E^d \setminus D(E)$. For any $k \in \{2, \ldots, d\}$ and any two k–dimensional simplices S, S' we have $|S \cap R(E)| = |S' \cap R(E)|$. Therefore, our choice of χ implies $|\chi^{-1}(i) \cap R(E)| = \frac{1}{c}|R(E)|$ for all $i \in [c]$. Hence

$$\begin{aligned}
&\max_{i \in [c]} \left| |\chi^{-1}(i) \cap E^d| - \frac{|E^d|}{c} \right| \\
&= \max_{i \in [c]} \left| |\chi^{-1}(i) \cap R(E)| - \frac{|R(E)|}{c} + |\chi^{-1}(i) \cap D(E)| - \frac{|D(E)|}{c} \right| \\
&= \max_{i \in [c]} \left| |\chi^{-1}(i) \cap D(E)| - \frac{|D(E)|}{c} \right| = \max_{i \in [c]} \left| |\psi^{-1}(i) \cap E| - \frac{|E|}{c} \right|.
\end{aligned}$$

This calculation establishes $\operatorname{disc}(\Delta^d \mathcal{H}, c) \leq \operatorname{disc}(\mathcal{H}, c)$.

Let us now consider the case that $c \nmid k!\, S(d, k)$ for some $k \in \{2, \ldots, d\}$. Let m be large enough to fulfill

$$\frac{1}{2}\binom{m}{\kappa} - \sum_{l=0}^{\kappa-1} l!\, S(d, l)\binom{m}{l} \geq \frac{1}{3\, k!} m^k$$

for all $\kappa \in \{k, \ldots, d\}$. (This can obviously be done, since the left hand side of the last inequality is of the form $m^\kappa / 2\kappa! + O(m^{\kappa-1})$ for $m \to \infty$.) Using Lemma 3.7, we choose

[1] To be precise, the authors also have the empty set contained in cases (a) and (c) and the whole set in case (b). It is obvious that by altering m by one, one can transform one result into the other.

$n \in \mathbb{N}$ such that for any c–coloring $\chi : [n]^d \to [c]$ there is an m–point set $T \subseteq [n]$ with all simplices in T^d being monochromatic with respect to χ.

We show that $\mathcal{K} = \left([n], \binom{[n]}{m}\right)$ satisfies our claim. Let χ be any c–coloring of $\mathcal{K}$, choose T as in Lemma 3.7. Let $\kappa \in \{k, \ldots, d\}$ be such that for each $l \in \{\kappa+1, \ldots, d\}$ there is the same number of l–dimensional simplices in T in each color but not so for the κ–dimensional simplices. With

$$\mathcal{S} := \bigcup_{l=\kappa}^{d} \bigcup_{J \in P_l(d)} \bigcup_{\sigma \in S_l} S_J^\sigma(T)$$

we obtain

$$\begin{aligned}
&\operatorname{disc}(\Delta^d \mathcal{K}, \chi) \\
&\geq \max_{i \in [c]} \left| |\chi^{-1}(i) \cap T^d| - \frac{|T^d|}{c} \right| \\
&\geq \max_{i \in [c]} \left\{ \left| |\chi^{-1}(i) \cap \mathcal{S}| - \frac{|\mathcal{S}|}{c} \right| - \left| |\chi^{-1}(i) \cap (T^d \setminus \mathcal{S})| - \frac{|T^d \setminus \mathcal{S}|}{c} \right| \right\} \\
&\geq \max_{i \in [c]} \left| \sum_{J \in P_\kappa(d), \sigma \in S_\kappa} |\chi^{-1}(i) \cap S_J^\sigma(T)| - \frac{\kappa!\, S(d,\kappa)}{c} \binom{m}{\kappa} \right| \\
&\quad - \frac{c-1}{c} \left(m^d - \sum_{l=\kappa}^{d} l!\, S(d,l) \binom{m}{l} \right) \\
&\geq \frac{1}{2} \binom{m}{\kappa} - \sum_{l=0}^{\kappa-1} l!\, S(d,l) \binom{m}{l} \geq \frac{1}{3\, k!} m^k .
\end{aligned}$$

This establishes $\operatorname{disc}(\Delta^d \mathcal{K}, c) \geq \frac{1}{3\,k!} m^k$. Note that our choice of n implies $\operatorname{disc}(\mathcal{K}, c) = \left(1 - \frac{1}{c}\right) m$. □

3.3 Further Upper Bounds

Besides the first part of Theorem 3.3, there are more ways to obtain upper bounds.

Theorem 3.8. *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. Let p be a prime number, $q \in \mathbb{N}$ and $c = p^q$. Furthermore, let $d \geq c$ and $s = d - (p-1)p^{q-1}$. Then* $\operatorname{disc}(\Delta^d \mathcal{H}, c) \leq \operatorname{disc}(\Delta^s \mathcal{H}, c)$.

Corollary 3.9. *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph.*

(a) If c is a prime number, $q \in \mathbb{N}$ and $d = c^q$, then $\operatorname{disc}(\Delta^d \mathcal{H}, c) \leq \operatorname{disc}(\mathcal{H}, c)$.

(b) For arbitrary $d \in \mathbb{N}$ there holds $\operatorname{disc}(\Delta^d \mathcal{H}, 2) \leq \operatorname{disc}(\mathcal{H}, 2)$.

Statement (a) of the corollary follows from the identity $c^q = 1 + (c-1)\sum_{j=0}^{q-1} c^j$ and the (repeated) use of Theorem 3.8. Conclusion (b) follows also from Theorem 3.8. Note that Theorem 3.3 implies that in both parts of Corollary 3.9 we have $c \,|\, k!\, S(d,k)$ for all $k \in \{2,\ldots,d\}$. Hence Corollary 3.9 could also have been proven by analyzing the Stirling numbers.

Proof of Theorem 3.8. Let $|V| = n$. Then, without loss of generality, we can assume that $V = [n]$. Let us define the shift operator $S : [n]^d \to [n]^d$ by

$$S(x_1,\ldots,x_c,x_{c+1},\ldots,x_d) = (x_2,\ldots,x_c,x_1,x_{c+1},\ldots,x_d)\,.$$

It induces an equivalence relation $\sim$ on $[n]^d$ by $x \sim y$ if and only if there exists a $k \in [c]$ with $S^k x = y$. Now let $x \in [n]^d$ and denote its equivalence class by $\langle x\rangle$. Put $k = |\langle x\rangle|$. Obviously k is the minimal integer in $[c]$ with $S^k x = x$. A standard argument from elementary group theory ("group acting on a set") shows that $k \,|\, c$. Thus either $k = c$ or $S^{p^{q-1}} x = x$. Define $D = \{y \in [n]^d \,|\, |\langle y\rangle| < c\}$. Then

$$\psi : D \to [n]^s\,,\; y \mapsto (y_1,\ldots,y_{p^{q-1}},y_{c+1},\ldots,y_d)$$

is a bijection. For a given c–coloring χ of $[n]^s$, we define a c–coloring $\tilde\chi$ of $[n]^d$ in the following way: We choose a system of representatives R for $\sim$. If $x \in R$ with $|\langle x\rangle| = c$, we put $\tilde\chi(S^i x) = i$ for all $i \in [c]$. If $|\langle x\rangle| < c$, then $\tilde\chi(y) = (\chi \circ \psi)(y)$ for all $y \in \langle x\rangle$.

Let $E \in \mathcal{E}$. Notice that $x \in E^d$ implies $\langle x\rangle \subseteq E^d$, and $x \in D$ implies $\langle x\rangle \subseteq D$. Furthermore, the restriction of ψ to $E^d \cap D$ is a bijection onto E^s. Thus

$$\begin{aligned}\max_{i\in[c]}\left||\tilde\chi^{-1}(i)\cap E^d| - \frac{|E^d|}{c}\right| &\le \max_{i\in[c]}\left||\tilde\chi^{-1}(i)\cap (E^d\cap D)| - \frac{|E^d\cap D|}{c}\right| \\ &\quad + \max_{i\in[c]}\left||\tilde\chi^{-1}(i)\cap (E^d\setminus D)| - \frac{|E^d\setminus D|}{c}\right| \\ &\le \max_{i\in[c]}\left||\chi^{-1}(i)\cap E^s| - \frac{|E^s|}{c}\right| + 0\,.\end{aligned}$$

Hence $\mathrm{disc}(\Delta^d\mathcal{H},c) \le \mathrm{disc}(\Delta^s\mathcal{H},c)$. □

The following is an extension of the first statement of Theorem 3.3.

Theorem 3.10. *Let c, $d \in \mathbb{N}$, and let $d' \in \{2.\ldots,d\}$. If $c \,|\, k!\, S(d',k)$ for all $k \in \{2,\ldots,d'\}$, then*

$$\mathrm{disc}(\Delta^d\mathcal{H},c) \le \mathrm{disc}(\Delta^{d-d'+1}\mathcal{H},c) \tag{3.7}$$

holds for every hypergraph $\mathcal{H}$.

Proof of Theorem 3.10. Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. W.l.o.g., let $V = [n]$. Let $\chi : [n]^{d-d'+1} \to [c]$ be an arbitrary c–coloring. We define a c–coloring $\widetilde{\chi} : [n]^d \to [c]$. Let $z \in [n]^d$, $x = (z_1, \ldots, z_{d'})$, and $y = (z_{d'+1}, \ldots, z_d)$. If $z_1 = \ldots = z_{d'} =: \zeta$, put $\widetilde{\chi}(z) = \chi(\zeta, z_{d'+1}, \ldots, z_d)$. Otherwise we find $k \in \{2, \ldots, d'\}$, $J \in P_k(d')$ and $\sigma \in S_k$ with $x \in S_J^\sigma([n])$. Since $c \,|\, k!\, S(d', k)$, we can color the set $D := \{(z_{\tau(1)}, \ldots, z_{\tau(d')}, y) \,|\, \tau \in S_{d'}\}$ of cardinality $k!\, S(d', k)$ evenly by our coloring $\widetilde{\chi} : [n]^d \to [c]$. A similar calculation as the one at the end of the proof of Theorem 3.8 establishes $\operatorname{disc}(\Delta^d \mathcal{H}, \widetilde{\chi}) \leq \operatorname{disc}(\Delta^{d-d'+1} \mathcal{H}, \chi)$. □

Remark 3.11. *The condition in Theorem 3.10 is only sufficient but not necessary for the validity of (3.7), as the following example shows:*

Let $c = 4$, $d \geq c$ *and* $d' = 3$. *According to Theorem 3.8, we get for each hypergraph* $\mathcal{H}$ *that* $\operatorname{disc}(\Delta^d \mathcal{H}, c) \leq \operatorname{disc}(\Delta^{d-2} \mathcal{H}, c) = \operatorname{disc}(\Delta^{d-d'+1} \mathcal{H}, c)$. *But we have* $2!\, S(d', 2) = 6 = 3!\, S(d', 3)$ *and* $4 \nmid 6$.

This example shows also that the methods used in the proofs of Theorem 3.8 and Theorem 3.10 are different.

Chapter 4

Discrepancy and Declustering

The declustering problem is to allocate given data on parallel working storage devices in such a manner that typical requests find their data evenly distributed on the devices. Using deep results from discrepancy theory, we improve previous work of several authors concerning rectangular queries to higher-dimensional data. We give a declustering scheme with an additive error of $O_d(\log^{d-1} M)$ independent of the data size, where d is the dimension, M the number of storage devices and $d-1$ does not exceed the smallest prime power in the canonical decomposition of M into prime powers. In particular, our schemes work for arbitrary M in two and three dimensions, and arbitrary M that is a power of two and at least $d-1$. For a lower bound, we show that a recent proof of a $\Omega_d(\log^{\frac{d-1}{2}} M)$ bound contains an error. We close the gap in the proof and thus establish the bound. Parts of the results of this chapter can be found in [DHW05, DHW04a, DHW04b].

4.1 Introduction

The last decade saw dramatic improvements in computer processing speeds and storage capacities. Nowadays, the bottleneck in data-intensive applications is the time needed to retrieve typically large amounts of data from external storage devices. One idea to overcome this obstacle is to distribute the data on disks of multi-disk systems so that it can be retrieved in parallel. Hopefully, this declustering reduces the retrieval time by a factor equal to the number of disks. The data allocation is determined by so-called declustering schemes. The schemes should allocate the data in such a manner that typical requests find their data evenly distributed on the disks.

We consider the problem of declustering uniform multi-dimensional data that is arranged in a multi-dimensional grid. There are many data-intensive applications that deal with this kind of data, e.g. multi-dimensional databases as remote-sensing databases [CMA+97].

A range query Q requests the data blocks that are associated with a hyper-rectangular subspace of the grid. Since we will not deal with syntactic issues of queries, we may identify a query with the set of requested block. In consequence, $|Q|$ denotes the number of requested blocks.

The response time of a query Q is (proportional to) the maximum number of blocks of Q that are assigned to the same disk (hence we assume identical disks). In an ideal declustering scheme for a system with M disks, this would be $|Q|/M$ for all queries Q. As we will see, this aim cannot be achieved. The quality of a declustering scheme is measured by the worst case (over all queries Q) additive deviation of the response time from the ideal value $|Q|/M$.

Declustering is an intensively studied problem and a number of schemes [CBS03, PAGAA98, AP00, DS82, FB93] have been developed in the last twenty years. It was an important turning point when discrepancy theory was connected to declustering.

Before the use of discrepancy theory, no provable performance bounds were known for arbitrary dimension d. Such bounds existed only for a few declustering schemes in two dimensions: The known results for these schemes considered only special cases, e. g., for the scheme proposed in [CBS03] a proof for the average performance is given if the number M of disks is a Fibonacci number, and for the construction of the scheme in [AP00] M has to be a power of 2.

A breakthrough was marked by noting that the declustering problem is a discrepancy problem. Sinha, Bhatia and Chen [SBC03] and Anstee, Demetrovics, Katona and Sali [ADKS00] developed declustering schemes for all M for two dimensional problems and proved their asymptotically optimal behavior. The schemes of Sinha et al. [SBC03] are based on two dimensional low discrepancy point sets. They also give generalizations to arbitrary dimension d, but without bounds on the error. Both papers show a lower bound of $\Omega(\log M)$ for the additive error of any declustering scheme in dimension two. The result of Anstee et al. [ADKS00] applies to Latin square type colorings only, but their proof can easily be extended to the general case as well. Sinha et al. [SBC03] claim a bound of $\Omega_d(\log^{\frac{d-1}{2}} M)$ for arbitrary dimension d, but their proof contains an error for $d \geq 3$ (cf. Section 4.3).

The first non-trivial upper bounds for declustering schemes in arbitrary dimension were proposed by Chen and Cheng [CC02], who present two schemes for the d–dimensional declustering problem. The first one has an additive error of $O_d(\log^{d-1} M)$, but works only if $M = p^k$ for some $k \in \mathbb{N}$ and p a prime such that $d \leq p$. The second works for arbitrary M, but the error increases with the size of the data. (Note that all other bounds stated in this chapter are independent of the data size.)

Our Results: We work both on upper and lower bounds. For the upper bound, we present an improved scheme that yields an additive error of $O_d(\log^{d-1} M)$ for all values of M (independent of the data size) and all d such that $d \leq q_1 + 1$, where q_1 is the smallest factor in the canonical decomposition of M into prime powers. Thus, in particular, our schemes work for M being a power of two such that $M \geq d-1$ and for all M in dimension 2 and 3, which is very useful from the viewpoint of application. We also show that the Latin hypercube construction used by Chen and Cheng [CC02] and in our work is much better than proven there. Where they show that the final scheme has an error of at most 2^d times the one of the Latin hypercube coloring, we show that both errors are the same.

For the lower bound, we present the first correct proof of the $\Omega_d(\log^{\frac{d-1}{2}} M)$ bound.

4.2 Discrepancy Theory

In this section, we sketch the connection between the declustering problem and discrepancy theory.

4.2.1 Combinatorial Discrepancy

Recall that the declustering problem is to assign data blocks from a multi-dimensional grid to M storage devices (disks) in a balanced manner. The aim is that range queries use all storage devices in a similar amount. More precisely, our grid is $V = [n_1] \times \cdots \times [n_d]$ for some positive integers $n_1, \ldots, n_d$.[1] A query Q requests the data assigned to a *rectangle* (or *box*) $[x_1..y_1] \times \cdots \times [x_d..y_d]$ for some integers $1 \leq x_i \leq y_i \leq n_i$. We identify a query with the set of blocks it requests, i.e., $Q = [x_1..y_1] \times \cdots \times [x_d..y_d]$.

We assume that the time to process a query is proportional to the maximum number of requested data blocks that are stored on a single device. We represent the assignment of data blocks to devices through a mapping $\chi : V \to [M]$. The processing time of the query Q then is $\max_{i \in [M]} |\chi^{-1}(i) \cap Q|$. Clearly, no declustering scheme can do better than $|Q|/M$. Hence a natural performance measure is the additive deviation from this lower bound. We are interested in the worst-case behavior. Thus we are looking for declustering schemes such that $\max_Q \max_{i \in [M]} |\chi^{-1}(i) \cap Q|$ is small.

This makes the problem a combinatorial discrepancy problem in M colors. Denote by $\mathcal{E}$ the set of all rectangles in V. Then $\mathcal{H} = (V, \mathcal{E})$ is a hypergraph. Summarizing the discussion above, we have

Theorem 4.1. *The additive error of an optimal declustering scheme for range queries is* $\operatorname{disc}^+(\mathcal{H}, M)$.

Since a central result of this chapter are discrepancy bounds independent of the size of the grid, we usually work with the hypergraph $\mathcal{H}_N^d = ([N]^d, \mathcal{E}_N^d)$, $\mathcal{E}_N^d = \{\prod_{i=1}^d [x_i..y_i] \,|\, 1 \leq x_i \leq$

[1] We use the notations $[n] := \{1, 2, \ldots, n\}$ and $[n..m] := \{n, \ldots, m\}$ for $n, m \in \mathbb{N}$, $n \leq m$.

$y_i \leq N\}$ for some sufficiently large integer N. Furthermore, we regard only the case that $M \geq 3$. For $M = 2$, a checkerboard coloring yields a declustering scheme with an additive error of $1/2$. We prove the following result.

Theorem 4.2. *Let $M \geq 3$ and $d \geq 2$ be integers and q_1 the smallest prime power in the canonical factorization of M into prime powers. Then*

(i) $\operatorname{disc}^+(\mathcal{H}_N^d, M) = O_d(\log^{d-1} M)$ *for $d \leq q_1 + 1$, independently of $N \in \mathbb{N}$,*

(ii) $\operatorname{disc}^+(\mathcal{H}_N^d, M) = \Omega_d(\log^{\frac{d-1}{2}} M)$ *for $N \geq M$,*

(iii) $\operatorname{disc}^+(\mathcal{H}_N^d, M) = \Theta(\log M)$ *for $d = 2$.*

4.2.2 Geometric Discrepancy

As mentioned before, the use of geometric discrepancies in [SBC03, ADKS00] in the analysis of declustering problems was a major breakthrough in this area. We refer to the recent book of Matoušek [Mat99] for both a great introduction and a thorough treatment of geometric discrepancies.

4.3 The Lower Bound

To prove our lower bounds, we use classical lower bounds for geometric discrepancies. Roth's [Rot54] famous lower bound for the L_2 discrepancy of the axis parallel boxes immediately implies the following.

Theorem 4.3 (Roth's lower bound)**.** *Let $d \geq 2$. There exists a constant $k > 0$ (depending on d) such that for any n–point set $\mathcal{P}$ in the unit cube $[0,1)^d$, there is an axis-parallel box R in $[0,1)^d$ with*

$$D(\mathcal{P}, R) \geq k \log^{\frac{d-1}{2}} n.$$

It was Schmidt [Sch72] who came up with the sharp lower bound in two dimensions.

Theorem 4.4 (Schmidt's lower bound)**.** *There is a constant $k > 0$ such that for any n–point set $\mathcal{P}$ in the unit square $[0,1)^2$, there is an axis-parallel rectangle R in $[0,1)^2$ with*

$$D(\mathcal{P}, R) \geq k \log n.$$

The general idea in the proofs of the lower bound for declustering schemes in Sinha et al. [SBC03] and Anstee et al. [ADKS00] (for $d = 2$ only) is the following.

Any M–coloring of $[M]^d$ produces a monochromatic set of M^{d-1} vertices. By scaling, this yields an M^{d-1}–point set $\mathcal{P}$ in $[0,1)^d$. The lower bounds above give a box R with polylogarithmic discrepancy. Round R to a box R_r with corners in $\{0, \frac{1}{M}, \cdots, \frac{M-1}{M}, 1\}^d$ in such a way that $R \cap \mathcal{P} = R_r \cap \mathcal{P}$. Then R and R_r have similar volume and hence similar discrepancy. Rescaling R_r yields a hyperedge $\hat{R}$ with combinatorial discrepancy equal to the geometric one of R_r.

The small, but crucial mistake in the proof of Sinha et al. [SBC03] is in the transfer from the geometric discrepancy setting back to the combinatorial one. Unlike in dimension $d = 2$, rounding R to R_r does not yield a constant change in the discrepancy in higher dimensions. The volume difference $|\operatorname{vol}(R) - \operatorname{vol}(R_r)|$ is still $O_d(\frac{1}{M})$. However, since the number of points is M^{d-1}, the discrepancy may change by something of order $\Theta_d(M^{d-2})$. This is way too large for $d > 2$.

For this reason, a straight generalization of the proof of Anstee et al. [ADKS00] of the lower bound in two dimensions (as attempted in [SBC03]) is not possible. We solve this problem by finding a *small* box R having large discrepancy. This keeps the change of volume due to rounding small enough so that the discrepancies of R and R_r differ by at most a constant.

Proof of Theorem 4.2 (ii). We show the claim for $N = M$, which clearly implies the result for arbitrary $N \geq M$. Let $\chi : [M]^d \to [M]$ be an M–coloring of $\mathcal{H}_M^d$. Choose an $s \in [M^{-\frac{d-2}{d-1}}, 2M^{-\frac{d-2}{d-1}}) \cap [0,1]$ such that s is a multiple of $\frac{1}{M}$. Such an s exists since $M^{-\frac{d-2}{d-1}} > \frac{1}{M}$. Without loss of generality, we may assume that $|\chi^{-1}(1) \cap [1..sM]^d| \geq s^d M^{d-1}$. By Theorem 4.3 for every n–point set $\mathcal{P}$ in the unit cube $[0,1)^d$, there is an axis-parallel box R in $[0,1)^d$ with $D(\mathcal{P}, R) \geq k \log^{\frac{d-1}{2}} n$.

If $|\chi^{-1}(1) \cap [1..sM]^d| \geq s^d M^{d-1} + \frac{k}{2} \left(\frac{1}{d-1}\right)^{\frac{d-1}{2}} \log^{\frac{d-1}{2}} M$, we clearly have $\operatorname{disc}(\mathcal{H}_M^d, \chi) \geq \big| |\chi^{-1}(1) \cap [1..sM]^d| - s^d M^{d-1} \big| \geq \frac{k}{2} \left(\frac{1}{d-1}\right)^{\frac{d-1}{2}} \log^{\frac{d-1}{2}} M$. Therefore, we may assume

$$|\chi^{-1}(1) \cap [1..sM]^d| < s^d M^{d-1} + \frac{k}{2} \left(\tfrac{1}{d-1}\right)^{\frac{d-1}{2}} \log^{\frac{d-1}{2}} M.$$

For every $z = (z_1, z_2, \ldots, z_d) \in \chi^{-1}(1) \cap [1..sM]^d$ we define $x_z := \left(\frac{2z_1-1}{2M}, \frac{2z_2-1}{2M}, \ldots, \frac{2z_d-1}{2M}\right)$. Let $\mathcal{P} := \{x_z \mid z \in \chi^{-1}(1) \cap [1..sM]^d\}$ and $n := |\mathcal{P}|$. Then $\widetilde{\mathcal{P}} := \frac{1}{s}(\mathcal{P})$ is an n–point set in the unit cube $[0,1)^d$. Estimating the cardinality of $\widetilde{\mathcal{P}}$, we get $n \geq s^d M^{d-1} \geq M^{\frac{1}{d-1}}$. By Theorem 4.3, there exists a box $\widetilde{R}$ in $(0,1)^d$ with

$$\left| |\widetilde{R} \cap \widetilde{\mathcal{P}}| - n \operatorname{vol}(\widetilde{R}) \right| \geq k \log^{\frac{d-1}{2}} n \geq k \left(\frac{1}{d-1}\right)^{\frac{d-1}{2}} \log^{\frac{d-1}{2}} M. \tag{4.1}$$

Let $R := s\widetilde{R} = \prod_{i=1}^d [x_i, y_i)$ be the corresponding box in $[0,s)^d$. Now we construct a box R_r by rounding the x_i and y_i to the nearest multiple of $\frac{1}{M}$. In case of ties, we round down. This ensures $\mathcal{P} \cap R_r = \mathcal{P} \cap R$.

Since we have chosen a small box R, this rounding changes the volume not too much. Using $s \leq 2M^{-\frac{d-2}{d-1}}$, we get

$$|\operatorname{vol}(R) - \operatorname{vol}(R_r)| \leq 2d\tfrac{1}{2M}s^{d-1} < d2^{d-1}M^{-(d-1)}.$$

The combinatorial counterpart of R_r is the box

$$\hat{R} := \left\{x \in [M]^d \,\middle|\, \left(\tfrac{2x_1-1}{2M}, \ldots, \tfrac{2x_d-1}{2M}\right) \in R_r \right\}.$$

One easily checks that $M^d \operatorname{vol}(R_r) = |\hat{R}|$. By construction and (4.1),

$$\begin{aligned}
\operatorname{disc}(\mathcal{H}_M^d, \chi) &\geq \left| |\chi^{-1}(1) \cap \hat{R}| - \tfrac{1}{M}|\hat{R}| \right| \\
&= \left| |\mathcal{P} \cap R| - M^{d-1} \operatorname{vol}(R_r) \right| \\
&= \big| |\widetilde{\mathcal{P}} \cap \widetilde{R}| - n \operatorname{vol}(\widetilde{R}) + \left(n - s^d M^{d-1}\right) \operatorname{vol}(\widetilde{R}) \\
&\quad + M^{d-1} \left(\operatorname{vol}(R) - \operatorname{vol}(R_r)\right) \big| \\
&\geq \tfrac{k}{2} \left(\frac{1}{d-1}\right)^{\frac{d-1}{2}} \log^{\frac{d-1}{2}} M - O(1) = \Omega_d\left(\log^{\frac{d-1}{2}} M\right).
\end{aligned}$$

Thus, $\operatorname{disc}(\mathcal{H}_M^d, M) = \Omega_d(\log^{\frac{d-1}{2}} M)$. It remains to show that this bound also holds for the positive discrepancy. To this end, let us assume that the discrepancy of the box $\hat{R}$ in color 1 is caused by a lack of vertices in color 1. Since $|\chi^{-1}(1)| \geq M^{d-1}$, the complement of $\hat{R}$ in $[M]^d$ has at least the same discrepancy as $\hat{R}$, but caused by an excess of vertices in color 1. Though this complement is not a box, it is the union of at most $2d$ boxes. Therefore, one of these boxes has a positive discrepancy that is at least $\frac{1}{2d}$ times the discrepancy of $\hat{R}$ in color 1. □

This last argument increases the implicit constant of the lower bound by a factor of $\frac{3^d}{2d}$ compared to the approach of Sinha et al. [SBC03].

We briefly show how to use the above to prove the $\Omega(\log M)$ bound for dimension $d = 2$. For this bound, two not completely satisfying bounds exist. Anstee et al. [ADKS00] only treated Latin square type colorings of $[M]^2$ and posed it an open problem to extend their result to arbitrary colorings. The proof in [SBC03] does not have this restriction, but is not very precise, which in particular helped to hide the error for $d > 2$.

As a simple and clean proof we therefore propose the following: Use the same reasoning as in the case of arbitrary dimension $d \geq 2$, but apply Schmidt's lower bound instead of Roth's. The parameter s can be chosen as 1. In dimension $d = 2$ we do not need small boxes, because the roundoff error has an effect on the discrepancy which is of order $O(1)$.

4.4 The Upper Bound

In this section, we present a declustering scheme showing our upper bound. As in previous work, we use low discrepancy point sets to construct the declustering scheme. In the following we use the notation of Niederreiter [Nie87]. For an integer $b \geq 2$, an *elementary interval* in base b is an interval of the form $E = \prod_{i=1}^{d} \left[a_i b^{-d_i}, (a_i+1) b^{-d_i}\right)$, with integers $d_i \geq 0$ and $0 \leq a_i < b^{d_i}$ for $1 \leq i \leq d$. For integers t, m such that $0 \leq t \leq m$, a (t,m,d)*–net* in base b is a point set of b^m points in $[0,1)^d$ such that all elementary intervals with volume b^{t-m} contain exactly b^t points.

Note that any elementary interval with volume b^{t-m} has discrepancy zero in a (t,m,d)–net. Since any subset of an elementary interval of volume b^{t-m} has discrepancy at most b^t and any box can be packed with elementary intervals in a way that the uncovered part can be covered by $O_d(\log^{d-1} b^m)$ elementary intervals of volume b^{t-m}, the following is immediate:

Theorem 4.5. *A (t,m,d)–net $\mathcal{P}_{net}$ in base b with $n = b^m$ points has discrepancy*

$$D(\mathcal{P}_{net}, \mathcal{R}_d) = O_d(\log^{d-1} n).$$

The central argument in our proof of the upper bound is the following result of Niederreiter [Nie87] on the existence of $(0,m,d)$–nets. From the view-point of application it is important that his proof is constructive.

Theorem 4.6. *Let $b \geq 2$ be an arbitrary base and $b = q_1 q_2 \ldots q_u$ be the canonical factorization of b into prime powers such that $q_1 < \cdots < q_u$. Then for any $m \geq 0$ and $d \leq q_1 + 1$ there exists a $(0,m,d)$–net in base b.*

We use $(0,m,d)$–nets to construct an M–coloring of $\mathcal{H}_M^d$ in Lemma 4.7. For the definition of these colorings, we need the following special elements of $\mathcal{E}_M^d$: A set $\prod_{j=1}^{d} I_j \in \mathcal{E}_M^d$ is called a *row* of $[M]^d$ if there is an $i \in [d]$ with $I_i = [1..M]$ and $|I_j| = 1$ for all $j \neq i$. In Lemma 4.8 we use the M–coloring of $\mathcal{H}_M^d$ to construct an M–coloring of $\mathcal{H}_N^d$ with same discrepancy.

Lemma 4.7. *Let $\mathcal{P}_{net}$ be a $(0, d-1, d)$–net in base M in $[0,1)^d$. Then there is an M–coloring χ_M of $\mathcal{H}_M^d = ([M]^d, \mathcal{E}_M^d)$ such that all rows of $[M]^d$ contain every color exactly once*[2] *and*

$$\operatorname{disc}(\mathcal{H}_M^d, \chi_M) \leq D(\mathcal{P}_{net}, \mathcal{R}_d).$$

Proof. The net $\mathcal{P}_{net}$ consists of M^{d-1} points and all elementary intervals with volume M^{-d+1} contain exactly one point. In particular, all subsets $\prod_{j=1}^{d} I_j$ of $[0,1)^d$ such that

[2]Some authors call this a permutation scheme for $[M]^d$

there is an $i \in [d]$ with $I_i = [0,1)$ and for all $j \neq i$ there exist $a_j \in [0..M-1]$ with $I_j = [\frac{a_j}{M}, \frac{a_j+1}{M})$, contain exactly one point.

We construct a coloring χ_M of $\mathcal{H}_M^d = ([M]^d, \mathcal{E}_M^d)$ corresponding to the set $\mathcal{P}_{net}$. Let $\hat{\mathcal{P}} := \left\{ x \in [M]^d \,\middle|\, \mathcal{P}_{net} \cap \prod_{i=1}^d [\frac{x_i-1}{M}, \frac{x_i}{M}) \neq \emptyset \right\}$. Then each row of $[M]^d$ contains exactly one point of $\hat{\mathcal{P}}$. We define the coloring $\chi_M : [M]^d \to [M]$ by $\chi_M(y, x_2, \dots, x_d) = i$ for all $x = (x_1, x_2, \dots, x_d) \in \hat{\mathcal{P}}$, $i, y \in [M]$ such that $y \equiv x_1 + (i-1) \mod M$. Hence $\hat{\mathcal{P}}$ receives color 1, color class 2 is obtained from shifting $\hat{\mathcal{P}}$ along the first coordinate and so on. This defines an M–coloring χ_M of $\mathcal{H}_M^d = ([M]^d, \mathcal{E}_M^d)$ such that each row of $\mathcal{H}_M^d$ contains every color exactly once.

For this coloring it is sufficient to calculate $\max_{\hat{R} \in \mathcal{E}_M^d} \left| |\chi_M^{-1}(1) \cap \hat{R}| - \frac{1}{M}|\hat{R}| \right|$, because for each color $i \in [M]$ and each box $\hat{R} \in \mathcal{E}_M^d$ we get the same discrepancy for the box $\hat{R}'$, which is a copy of $\hat{R}$ shifted along the first dimension by $i-1$ and wrapped around perhaps, with respect to the color 1. If $\hat{R}'$ is wrapped around, it is the union of two boxes. Since whole rows have discrepancy zero, the discrepancy of those boxes is the same as the discrepancy of the the box between them, and we have

$$\operatorname{disc}(\mathcal{H}_M^d, \chi_M) = \max_{\hat{R} \in \mathcal{E}_M^d} \left| |\hat{\mathcal{P}} \cap \hat{R}| - \frac{1}{M}|\hat{R}| \right|.$$

Let $\hat{R} = \prod_{i=1}^d [x_i..y_i]$ an arbitrary hyperedge of $\mathcal{H}_M^d$. The associated box in $[0,1)^d$ is $R = \prod_{i=1}^d \left[\frac{x_i-1}{M}, \frac{y_i}{M}\right)$. Then $|\hat{\mathcal{P}} \cap \hat{R}| = |\mathcal{P}_{net} \cap R|$ and $|\hat{R}| = M^d \operatorname{vol}(R)$. Thus the combinatorial discrepancy of $\hat{R}$ equals the geometric one of R. We have

$$\left| |\chi_M^{-1}(1) \cap \hat{R}| - \frac{1}{M}|\hat{R}| \right| = \left| |\mathcal{P}_{net} \cap R| - M^{d-1} \operatorname{vol}(R) \right| \leq D(\mathcal{P}_{net}, \mathcal{R}_d).$$

Hence we get $\operatorname{disc}(\mathcal{H}_M^d, \chi_M) \leq D(\mathcal{P}_{net}, \mathcal{R}_d)$. □

Lemma 4.8. *Let χ_M be an M–coloring of $\mathcal{H}_M^d$ such that all rows of $[M]^d$ contain every color exactly once and χ a coloring of $\mathcal{H}_N^d$ defined by $\chi(x_1, \dots, x_d) = \chi_M(y_1, \dots, y_d)$ with $x_i \equiv y_i \mod M$ for $i \in [d]$, $x_i \in [N]$, $y_i \in [M]$. Then*

$$\operatorname{disc}(\mathcal{H}_N^d, \chi) = \operatorname{disc}(\mathcal{H}_M^d, \chi_M).$$

Proof. Let $\hat{R} = \prod_{i=1}^d [x_i..y_i]$ be an arbitrary hyperedge of $\mathcal{H}_N^d$. For all $i \in [d]$ there exist unique $\widetilde{x}_i, \widetilde{y}_i \in [M]$ with $x_i \equiv \widetilde{x}_i \mod M$ respectively $y_i \equiv \widetilde{y}_i \mod M$. Set $\bar{x}_i := \min\{\widetilde{x}_i, \widetilde{y}_i\}$ and $\bar{y}_i := \max\{\widetilde{x}_i, \widetilde{y}_i\}$ for all $i \in [d]$. We have $\operatorname{disc}(\hat{R}, \chi) = \operatorname{disc}([\bar{x}_1..\bar{y}_1] \times [x_2..y_2] \times \ldots \times [x_d..y_d], \chi)$, since whole rows have discrepancy zero. Applying this successively in every coordinate we get

$$\operatorname{disc}(\hat{R}, \chi) = \operatorname{disc}(\prod_{i=1}^d [\bar{x}_i..\bar{y}_i], \chi) = \operatorname{disc}(\prod_{i=1}^d [\bar{x}_i..\bar{y}_i], \chi_M).$$

□

Lemma 4.8 is a remarkable improvement of Theorem 4.2 in [CC02], where $\operatorname{disc}(\mathcal{H}_N^d, \chi) \leq 2^d \operatorname{disc}(\mathcal{H}_M^d, \chi_M)$ is shown. Note that this reduces the implicit constant in the upper bound by factor of 2^d.

It remains to show that the upper bound in Theorem 4.2 follows from Lemma 4.7 and Lemma 4.8.

Theorem 4.2(i). Let $M \geq 3$ and $d \geq 2$ be positive integers and $d \leq q_1 + 1$, where q_1 is the smallest prime power in the canonical factorization of M into prime powers. Theorem 4.6 provides a $(0, d-1, d)$–net $\mathcal{P}_{net}$ in base M in $[0,1)^d$. Using Lemma 4.7 , we get an M–coloring χ_M of $\mathcal{H}_M^d$ such that all rows contain each color exactly once and $\operatorname{disc}(\mathcal{H}_M^d, \chi_M) \leq D(\mathcal{P}_{net}, \mathcal{R}_d)$. With Lemma 4.8 and Theorem 4.5, we have $\operatorname{disc}(\mathcal{H}_N^d, M) \leq D(\mathcal{P}_{net}, \mathcal{R}_d) = O_d(\log^{d-1} M)$. □

4.5 Alternative Approach for the Lower Bound

In this section we give an alternative proof of the lower bound in Theorem 4.2. We use Theorem 4.9 which is a stronger version of a geometrical discrepancy theorem that can be found in [BC87]. Following the notation introduced in Beck and Chen [BC87], the cube $[-s, s]^d$ has side s, we show

Theorem 4.9. *For any n–point set $\mathcal{P}$ in the unit cube $[0,1)^d$, there is an axis-parallel cube Q with side at most $n^{-\frac{(2d-3)d}{(d-1)^2(2d+1)}}$ fully contained in $[0,1)^d$ with*

$$D(\mathcal{P}, Q) = \Omega(\log^{\frac{d-1}{2}} n).$$

We first deduce Theorem 4.2 (ii) from Theorem 4.9.

Theorem 4.2 (ii). We show the claim for $N = M$, which clearly implies the result for arbitrary $N \geq M$. Let $\chi : [M]^d \to [M]$ be a M–coloring of $\mathcal{H}_M^d$. Without loss of generality we may assume $|\chi^{-1}(1)| \geq M^{d-1}$. In the case $|\chi^{-1}(1)| \geq M^{d-1} + \frac{k}{2} \log^{\frac{d-1}{2}} M$, where k is the constant implicitly given in Theorem 4.9, we have $\operatorname{disc}(\mathcal{H}_M^d, \chi) \geq \big||\chi^{-1}(1)| - M^{d-1}\big| \geq \frac{k}{2} \log^{\frac{d-1}{2}} M$. Therefore, we may assume $|\chi^{-1}(1)| < M^{d-1} + \frac{k}{2} \log^{\frac{d-1}{2}} M$. For every vertex $z = (z_1, z_2, \ldots, z_d) \in \chi^{-1}(1)$ we define $x_z := \big(\frac{2z_1-1}{2M}, \frac{2z_2-1}{2M}, \ldots, \frac{2z_d-1}{2M}\big)$. Let $\mathcal{P} := \{x_z \mid z \in \chi^{-1}(1)\}$ and $n := |\mathcal{P}|$. By Theorem 4.9, there is a cube $Q = \prod_{i=1}^d [x_i, x_i + 2s)$ such that the side s is at most $n^{-\frac{(2d-3)d}{(d-1)^2(2d+1)}}$ and

$$D(\mathcal{P}, Q) = \big||\mathcal{P} \cap Q| - n \operatorname{vol}(Q)\big| \geq k \log^{\frac{d-1}{2}} M.$$

Now we construct a box B by rounding the x_i and $x_i + 2s$ to the nearest multiple of $\frac{1}{M}$. We ensure $\mathcal{P} \cap B = \mathcal{P} \cap Q$ by rounding up $x_i + 2s$ if $x_i + 2s = \frac{h}{2M}$ and rounding x_i down if $x_i = \frac{h}{2M}$ for an odd h.

Since we have chosen a relatively small cube Q, our rounding changes the volume not to much. Using $n \geq M^{d-1}$, we get

$$|\operatorname{vol}(Q) - \operatorname{vol}(B)| \leq 2d\tfrac{1}{2M}(\tfrac{1}{M} + 2s)^{d-1} < d3^{d-1}M^{-(d-1)}.$$

The combinatorial counterpart of B is the box

$$\hat{B} := \left\{x \in [M]^d \,\middle|\, \left(\tfrac{2x_1-1}{2M}, \ldots, \tfrac{2x_d-1}{2M}\right) \in B\right\}.$$

One can easily check that $M^d \operatorname{vol}(B) = |\hat{B}|$. By construction,

$$\begin{aligned}
\operatorname{disc}(\mathcal{H}_M^d, \chi) &\geq \left| |\chi^{-1}(1) \cap \hat{B}| - \tfrac{1}{M}|\hat{B}| \right| \\
&= \left| |\mathcal{P} \cap Q| - M^{d-1} \operatorname{vol}(B) \right| \\
&= \big| (|\mathcal{P} \cap Q| - n \operatorname{vol}(Q)) + \big(n \operatorname{vol}(Q) - M^{d-1} \operatorname{vol}(Q)\big) \\
&\quad + M^{d-1} (\operatorname{vol}(Q) - \operatorname{vol}(B)) \big| \\
&\geq \tfrac{k}{2} \log^{\frac{d-1}{2}} M - O(1) = \Omega\left(\log^{\frac{d-1}{2}} M\right).
\end{aligned}$$

Thus, $\operatorname{disc}(\mathcal{H}_M^d, M) = \Omega(\log^{\frac{d-1}{2}} M)$. It remains to show that this bound also holds for the positive discrepancy. To this end, let us assume that the discrepancy of the box $\hat{B}$ in color 1 is caused by a lack of vertices in color 1. Since $|\chi^{-1}(1)| \geq M^{d-1}$, the complement of $\hat{B}$ in $[M]^d$ has at least the same discrepancy as $\hat{B}$, but caused by an excess of vertices in color 1.

Though this complement is not a box, it is the union of at most $2d$ boxes. Therefore, one of these boxes has a positive discrepancy that is at least $\frac{1}{2d}$ times the discrepancy of $\hat{B}$ in color 1. □ □

This last argument increases the implicit constant of the lower bound by a factor of $\frac{3^d}{2d}$ compared to the approach of Sinha et al. [SBC03].

For the proof of Theorem 4.9, we need some notions from Fourier analysis. For the set $\mathcal{P} := \{p_1, p_2, \ldots, p_n\} \subseteq \mathbb{R}^d$ we define $\nu := \sum_{i=1}^n \delta_{p_i} - n\mu$, where δ_{p_i} denotes the Dirac measure concentrated on p_i and μ is the d–dimensional Lebesgue measure on $[0,1]^d$ with $\mu([0,1]^d) = 1$. For any $\lambda \in (0,1]$ and $g \in L^2(\mathbb{R}^d)$ write $g_\lambda(x) := g(\lambda^{-1}x)$ for all $x \in \mathbb{R}^d$. Put $F_g := g * \nu$. Then we have

$$F_g(x) = \int_{\mathbb{R}^d} g(x-y)d\nu(y) = \sum_{i=1}^n g(x-p_i) - n \int_{\mathbb{R}^d} g(x-y)d\mu(y).$$

Let $\mathbb{1}_r$ be the characteristic function of the cube $[-r,r]^d$. Then $|F_{\mathbb{1}_r}(x)|$ is the discrepancy of $Q_r(x) := \big(x + [-r,r]^d\big) \cap [0,1]^d$ with respect to the set $\mathcal{P}$:

$$|F_{\mathbb{1}_r}(x)| = \big||\mathcal{P} \cap Q_r(x)| - n\,\mathrm{vol}(Q_r(x))\big| = \mathrm{disc}(\mathcal{P}, Q_r(x)).$$

Let $\Delta_1(g) := \int\limits_{\mathbb{R}^d} |F_g(x)|^2 dx$ and $\Delta_2(g) := \int\limits_0^1 \int\limits_{\mathbb{R}^d} |F_{g_\lambda}(x)|^2 dx d\lambda$. Using Parseval's theorem for Fourier transforms and the fact that $\widehat{f * g} = \widehat{f}\widehat{g}$, we can show $\Delta_1(g) = \int\limits_{\mathbb{R}^d} |\hat{g}(t)|^2 |\hat{\nu}(t)|^2 dt$ and $\Delta_2(g) = \int\limits_{\mathbb{R}^d} \left(\int\limits_0^1 |\hat{g}_\lambda(t)|^2 d\lambda\right) |\hat{\nu}(t)|^2 dt$. Here $\hat{f}$ denotes the Fourier transform

$$\hat{f} : \mathbb{R}^d \to \mathbb{C}, t \mapsto \hat{f}(t) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int\limits_{\mathbb{R}^d} f(x) e^{-ix\cdot t} dx$$

of $f : \mathbb{R}^d \to \mathbb{C}$. Let $m := n^{\frac{(2d-3)d}{(d-1)^2(2d+1)}}$. Note that $m > 1$. For the proof of Theorem 4.9 we need the following main lemma, which determines an average discrepancy for all cubes of side at most $\frac{1}{m}$ that intersect the unit cube $[0,1]^d$.

Lemma 4.10. *We have* $\Delta_2(\mathbb{1}_{\frac{1}{m}}) = \Omega(\log^{d-1} n)$.

Let us first derive Theorem 4.9 from Lemma 4.10.

Theorem 4.9. We distinguish two cases. Either there exists some $r \in [0, \frac{1}{m}]$ and $x_0 \in \mathbb{R}^d$ with $|F_{\mathbb{1}_r}(x_0)| > 2n(\frac{2}{m})^d$ or there does not. In the former case, the cube Q_0 with center x_0 and side r has discrepancy at least $2n(\frac{2}{m})^d$, as we have mentioned above. This cube may cross the border of $[0,1]^d$, but we can find a cube Q with side $\frac{1}{m}$ and $Q_0 \cap [0,1)^d \subseteq Q$ fully contained in $[0,1)^d$. With $n\,\mathrm{vol}(Q_0) = n(2r)^d \leq n(\frac{2}{m})^d$, we see that the discrepancy of Q_0 must be caused by the excess of points in Q_0. Therefore we have

$$D(\mathcal{P}, Q) \geq |\mathcal{P} \cap Q| - n\,\mathrm{vol}(Q) \geq n(\tfrac{2}{m})^d = 2^d n^{\frac{1}{(d-1)^2(2d+1)}} = \Omega(\log^{\frac{d-1}{2}} n).$$

Let us assume the latter case. Lemma 4.10 gives us a lower bound for the average square discrepancy of all cubes of side at most $\frac{1}{m}$. Since the contribution of cubes intersecting the border of $[0,1]^d$ to this average square discrepancy is

$$O\left(\tfrac{1}{m}\left(n(\tfrac{1}{m})^d\right)^2\right) = O\left(n^{-\frac{d-2}{(d-1)^2}}\right) = O(1),$$

there is a cube Q with side at most $\frac{1}{m}$ and discrepancy $\Omega(\log^{\frac{d-1}{2}} n)$ fully contained in $[0,1]^d$. □ □

It remains to prove Lemma 4.10. We set for all $l = (l_1, l_2, \ldots, l_d) \in \mathbb{Z}^d$

$$h_l(x) := \prod_{i=1}^{d} \exp(-\tfrac{1}{2} l_i^2 x_i^2).$$

By the fact that $\hat{f}(t) = a^{-1} \exp(-\frac{t^2}{2a^2})$ for $f(x) = \exp(-\frac{1}{2}a^2 x^2)$, the Fourier transform of h_l is $\hat{h}_l(t) = \prod_{i=1}^{d} \frac{1}{l_i} \exp\left(-\frac{t_i^2}{2l_i^2}\right)$. Now let L be the integer power of 2 satisfying $4(2\pi)^{\frac{d}{2}} n \le L < 8(2\pi)^{\frac{d}{2}} n$ and

$$\mathbb{Z}^d(L,m) := \left\{ l \in \mathbb{Z}^d \mid l_i = 2^{s_i} \ge m, s_i \in \mathbb{Z}, \prod_{i=1}^{d} l_i = L \right\}.$$

The following three lemmas yield the Lemma 4.10.

Lemma 4.11. *$|\mathbb{Z}^d(L,m)| > \Omega(\log^{d-1} n)$.*

Proof. Set $L' := \log_2 L$ and $m' := \lceil \log_2 m \rceil$. Then $|\mathbb{Z}^d(L,m)|$ is the number of integral lattice points $(s_1, s_2, \ldots, s_d)$ with $\sum_{i=1}^{d} s_i = L'$ and $s_i \ge m'$ for all $1 \le i \le d$. Hence

$$|\mathbb{Z}^d(L,m)| = \binom{L' - (m'-1)d - 1}{d-1} \ge \frac{(L' - m'd + 1)^{d-1}}{(d-1)!}.$$

With $L' \ge \log_2 \left(4(2\pi)^{\frac{d}{2}} n\right) > \log_2 n + d + 1$ and $m' < \frac{(2d-3)d \log_2 n}{(d-1)^2(2d+1)} + 1$ we get

$$|\mathbb{Z}^d(L,m)| = \Omega(\log^{d-1} n).$$

□ □

The following two lemmas are taken from Beck and Chen [BC87]:

Lemma 4.12 ([BC87], Lemma 6.3)**.** *$\Delta_2(\mathbb{1}_{\frac{1}{m}}) = \Omega(\sum_{l \in \mathbb{Z}^d(L,m)} \Delta_1(h_l))$.*

Lemma 4.13 ([BC87], Lemma 6.4)**.** *For every $l \in \mathbb{Z}^d(L,m)$ we have*

$$\Delta_1(h_l) = \Omega(1).$$

Now Lemma 4.10 is a direct consequence of Lemma 4.11, 4.12 and 4.13. We get

$$\Delta_2(\mathbb{1}_{\frac{1}{m}}) = \Omega\left(\sum_{l \in \mathbb{Z}^d(L,m)} \Delta_1(h_l) \right) = \sum_{l \in \mathbb{Z}^d(L,m)} \Omega(1) = \Omega(\log^{d-1} n).$$

It remains to prove the lower bound of Theorem 4.2 (iii). Anstee et al. [ADKS00] only treated Latin square type colorings of $[M]^2$. However, the proof is easily extended through the triangle inequality argument used in the proof of Theorem 4.2 (ii).

4.6 Conclusion

We gave lower and upper bounds for the declustering problem. This chapter contains the first complete proof of the lower bound $\Omega_d(\log^{\frac{d-1}{2}} M)$ for arbitrary values of M and d.

We propose a declustering scheme that has an additive error of $O_d(\log^{d-1} M)$ with the sole condition that $d \leq q_1+1$, where q_1 is the smallest prime power in the canonical factorization of M into prime powers. This improves the former best declustering schemes of Chen and Cheng [CC02], where either bounds depend on the data size N^d or $M = p^t$ and $p \geq d$ was required for a prime p and $t \in \mathbb{N}$. Furthermore, Lemma 4.8 improves the analysis of Chen and Cheng [CC02] of the discrepancy of Latin square colorings by a factor of 2^{-d}.

The natural problem arising from this work is to close the gap between the lower and upper bound. However, this is probably a very hard one. The reason is that the corresponding problem of geometric discrepancies of boxes is extremely difficult. Closing the gap between the $\Omega_d(\log^{\frac{d-1}{2}} n)$ lower and the $O_d(\log^{d-1} n)$ upper bound for $D(n, \mathcal{R}_d)$ was baptized 'the great open problem' already in Beck and Chen [BC87]. Since then no further progress has been made for the general problem. Note that in the proof of a slight improvement due to Baker [Bak99] recently a serious error was found, so that the result was withdrawn by the author [talk of József Beck, Oberwolfach Seminar on Discrepancy Theory and Applications, March 2004].

Chapter 5

Positive Discrepancy of Linear Hyperplanes in Finite Vector Spaces

In this chapter we investigate the c–color discrepancy, $c \geq 2$ and the positive c–color discrepancy of linear hyperplanes in the finite vector space $\mathbb{F}_q^r$. We denote by $\mathcal{H}$ the hypergraph with vertex set $\mathbb{F}_q^r$ and all linear hyperplanes (i.e., subspaces of codimension 1) in $\mathbb{F}_q^r$ as hyperedges. We show that the discrepancy of $\mathcal{H}$ is $\Theta_q\left(\sqrt{nz(1-z)}\right)$, where $n = |\mathbb{F}_q^r|$ and $z = \frac{(q-1) \mod c}{c}$. With Fourier analysis on $\mathbb{F}_q^r$ we further prove that the positive discrepancy of $\mathcal{H}$ is bounded from below by $\Omega_q\left(\sqrt{\frac{nz(1-z)}{c}}\right)$, which differs from the upper bound by a factor of $\sqrt{c}$. For large c, i.e., $c \geq qn^{1/3}$ we close the gap proving a $\Theta_q\left(\sqrt{\frac{n}{c}}\right)$ behavior of the positive discrepancy. All together this exhibits a new example for a hypergraph with (almost) tight discrepancy bounds. The results of this chapter can be found in [HSS05].

5.1 Introduction

Let $\mathbb{F}_q$ be the field of q elements, where $q = p^k$ is a prime power and $V = \mathbb{F}_q^r$ is the r–dimensional vector space over $\mathbb{F}_q$. Let $\mathcal{E}$ be the set of linear hyperplanes of V. This means that $\mathcal{E}$ is the set of all subspaces of codimension 1. For a set $S \subseteq \mathbb{F}_q^r$ we define $S^\sharp := S \setminus \{0\}$. With $n := |V| = q^r$, $\mathcal{H} = (V, \mathcal{E})$ is an $\frac{n}{q}$–uniform hypergraph with n vertices and $|\mathcal{E}| = \frac{n-1}{q-1}$ hyperedges.

The Results. Let $z = \frac{(q-1) \ mod \ c}{c}$. For $z = 0$, i.e., $c|(q-1)$, the c–color discrepancy and positive discrepancy both are exactly $\frac{c-1}{c}$. For $z \neq 0$ we prove the upper bound for both, the c–color discrepancy and the positive c–color discrepancy:

$$\operatorname{disc}^+(\mathcal{H}, c) \leq \operatorname{disc}(\mathcal{H}, c) \leq \alpha\sqrt{nz(1-z)} \tag{5.1}$$

for a constant $\alpha > 0$. Our lower bound for the c–color discrepancy is

$$\frac{1}{q}\sqrt{nz(1-z)} - 1 \leq \operatorname{disc}(\mathcal{H}, c). \tag{5.2}$$

For $r \geq 6$ the lower bound for the positive c–color discrepancy is

$$\frac{\sqrt{z(1-z)}}{4q(q-1)}\sqrt{\frac{n}{c}} - 1 \leq \operatorname{disc}^+(\mathcal{H}, c). \tag{5.3}$$

For constant q, by (5.1) and (5.2) the c–color discrepancy is $\Theta(\sqrt{nz(1-z)})$ while the lower bound for the positive c–color discrepancy in (5.3) differs from the upper bound $O(\sqrt{nz(1-z)})$ in (5.1) by a factor of $\sqrt{\frac{1}{c}}$. We can close this gap in a special situation. In fact, for a large number of colors c, i.e., $c \geq qn^{1/3}$ and if $r \geq 4$ we get the lower bound

$$\frac{1}{22\sqrt{q}}\sqrt{\frac{n}{c}} - 1 \leq \operatorname{disc}^+(\mathcal{H}, c). \tag{5.4}$$

Thus, in the case $c \geq qn^{1/3}$ the positive c–color discrepancy is exactly of the order $\Theta_q(\sqrt{\frac{n}{c}})$.

All together we have a new hypergraph with tight discrepancy behaviour.

Methods. The methods in proving the bounds are of combinatorial as well as of analytic type. The bounds for the discrepancy function are obtaint with standard methods (eigenvalue calculation, bounded Vapnik-Červonenkis dimension).

The hard work is to prove the lower bound for the positive discrepancy function. There we invoke Fourier analysis on the additive group $\mathbb{F}_q^r$ and the pigeon hole principle at many times. First of all, we observe that the positive discrepancy $\operatorname{disc}^+(\mathcal{H}, \chi, c)$ for a c–coloring χ of V can be written as

$$\operatorname{disc}^+(\mathcal{H}, \chi, c) = \max_{i \in [c]} \max_{E \in \mathcal{E}} \left(\frac{\hat{A}_i(E^\perp)}{q} + \frac{|A_i|}{q} - \frac{n}{qc} \right), \tag{5.5}$$

where $\hat{A}_i$ is the Fourier transform of the indicator function $\mathbb{1}_{A_i}$ in $\mathbb{F}_q^r$. The exact definition is given in subsection 5.3.1. We prove that for any set $A \subseteq \mathbb{F}_q^r$ there exists a $E \in \mathcal{E}$ with $\hat{A}(E^\perp) \geq -1$, so $\hat{A}(.)$ the Fourier Transform cannot be too small. Now a certain tradeoff between the size of A_i and $\hat{A}_i(E^\perp)$ in (5.5) can be shown.

5.2 Discrepancy of $\mathcal{H}$

In this section we give both, a lower and an upper bound for the c–color discrepancy of the hypergraph $\mathcal{H}$. We can assume that the number of colors c does not exceed the number of vertices n. Otherwise the discrepancy is at most 1. Therefore the factor $\left(\frac{c}{n}\right)^{\frac{1}{2(r-1)}}$ is a decreasing function in the upper bound. First we consider the case $c|(q-1)$.

Proposition 5.1. *If* $c|(q-1)$ *we have*

$$\mathrm{disc}^+(\mathcal{H},c) = \mathrm{disc}(\mathcal{H},c) = \frac{c-1}{c}.$$

Proof. Let $\mathcal{E}_1$ denote the set of all one-dimensional subspaces of V. It holds

$$V = \{0\} \cup \bigcup_{W\in\mathcal{E}_1} (W \setminus \{0\}) \tag{5.6}$$

and this union is disjoint. For every $W \in \mathcal{E}_1$ the set $W \setminus \{0\}$ contains $q-1$ elements. Using $c|(q-1)$ we can color $W \setminus \{0\}$ in an exactly balanced way. We just color $\frac{q-1}{c}$ vertices of $W \setminus \{0\}$ in each color. Doing this for every $W \in \mathcal{E}_1$ and coloring the origin with the color 1 we get a c–coloring $\chi : V \to [c]$. For every linear hyperplane $E \in \mathcal{E}$ we denote by $\mathcal{E}_1(E)$ the set of all one-dimensional subspaces of E. Then (5.6) holds as well for E. For every $E \in \mathcal{E}$ and every $i \in [c]$ we have

$$\begin{aligned}\left||A_i \cap E| - \tfrac{1}{c}|E|\right| &= \left||A_i \cap \{0\}| - \tfrac{1}{c}|\{0\}| + \sum_{W\in\mathcal{E}_1} \left(|A_i \cap (W\setminus\{0\})| - \tfrac{1}{c}|(W\setminus\{0\})|\right)\right| \\ &= \left||A_i \cap \{0\}| - \tfrac{1}{c}\right| \\ &= \begin{cases} \frac{c-1}{c} & : \ i = 1, \\ \frac{1}{c} & : \ \textit{otherwise}. \end{cases}\end{aligned}$$

Thus we have

$$\mathrm{disc}^+(\mathcal{H},c) \le \mathrm{disc}(\mathcal{H},c) \le \frac{c-1}{c}.$$

Let $\chi : V \to [c]$ be a c–coloring and $E \in \mathcal{E}$ a hyperedge. Then there exists an $i \in [c]$ with $|A_i \cap E| \ge \left\lceil \frac{q^{r-1}}{c} \right\rceil = \frac{q^{r-1}-1}{c} + 1$. This yields

$$\mathrm{disc}(\mathcal{H},c,\chi) \ge \mathrm{disc}^+(\mathcal{H},c,\chi) \ge |E \cap A_i| - \tfrac{1}{c}|E| \ge \frac{q^{r-1}-1}{c} + 1 - \frac{q^{r-1}}{c} = \frac{c-1}{c}.$$

□

In the rest of the chapter we assume that $c \nmid (q-1)$.

Theorem 5.2. *Let* $z := \frac{(q-1) \mod c}{c}$. *There exists a constant* $\alpha > 0$ *such that*

$$\frac{1}{q}\sqrt{nz(1-z)} - 1 \le \mathrm{disc}(\mathcal{H},c) \le \alpha\sqrt{nz(1-z)}.$$

5.2.1 The Lower Bound

Calculating a lower bound for the discrepancy of the hypergraph $\mathcal{H}$ becomes much easier, if we delete the origin from the vector space. Let us define $V' := V \setminus \{0\}$, $\mathcal{E}' := \{E \cap V' \mid E \in \mathcal{E}\}$, and $\mathcal{H}' := (V', \mathcal{E}')$. It is obvious that $\operatorname{disc}(\mathcal{H}, c) \geq \operatorname{disc}(\mathcal{H}', c) - 1$, thus we can focus on $\mathcal{H}'$. The hypergraph $\mathcal{H}'$ has an interesting property. For two arbitrary elements $x, y \in V'$ let us denote by $d(x, y)$ the number of hyperedges $E \in \mathcal{E}'$ with $x, y \in E$ and call it the pair-degree of x and y. One can check that

$$d(x,y) := \begin{cases} \frac{q^{r-1}-1}{q-1} & , \text{ if } x \text{ and } y \text{ are linearly dependent,} \\ \frac{q^{r-2}-1}{q-1} & , \text{ otherwise.} \end{cases}$$

Let M be the incidence matrix of $\mathcal{H}'$: the rows of M correspond to the hyperedges of $\mathcal{H}'$, the columns to the vertices of $\mathcal{H}'$ and the component $M_{i,j}$ of M is 1 if $v_j \in E_i$ and is 0 otherwise. W.l.o.g. we can assume that the columns of M are arranged according to the one-dimensional subspaces of V. Let us denote by $\left(M^T M\right)_{x,y}$ the component of the matrix $M^T M$ in the row that corresponds to the vertex $x \in V'$ and the column that corresponds to the vertex $y \in V'$. Then we have $\left(M^T M\right)_{x,y} = d(x, y)$, and the matrix $M^T M$ is of the following form: on the diagonal there are $\frac{n-1}{q-1}$ copies of the $(q-1) \times (q-1)$–matrix with all components equal to $\frac{q^{r-1}-1}{q-1}$, and all other components of M are equal to $\frac{q^{r-2}-1}{q-1}$.

In this situation we can use the following lemma in which we extend Theorem 2.8 and Corollary 2.9 from [BS95] to block diagonal matrices and the c–color discrepancy. For simplicity we focus on block diagonal matrices consisting of blocks of same size and identical elements.

Lemma 5.3. *Let $\mathcal{H} = (V, \mathcal{E})$ be a finite hypergraph, set $m := |\mathcal{E}|$ and let M be the incidence matrix of $\mathcal{H}$. If for some block diagonal matrix D consisting of k copies of an $l \times l$–matrix X with all entries equal to $x > 0$, the matrix $M^T M - D$ is positive semidefinite, then with $z := \frac{l \mod c}{c}$*

$$\operatorname{disc}(\mathcal{H}, c) \geq \left(\frac{kxz(1-z)}{m}\right)^{\frac{1}{2}}.$$

Proof. We follow the approach to c-color discrepancies in [DS03]. The color $i \in [c]$ is described by a vector $m^{(i)} \in \mathbb{R}^c$ with

$$m_j^{(i)} := \begin{cases} \frac{c-1}{c} & : \quad i = j, \\ -\frac{1}{c} & : \quad \textit{otherwise.} \end{cases}$$

We define $M_c := \{m^{(i)} \mid i \in [c]\}$ and fix an optimal c–coloring $\chi : V \to M_c$ with respect to the c–color discrepancy. Furthermore, for all $i \in [l]$ we denote the coloring of the set V_i of vertices corresponding to the i–th block of D by $\chi_i : V_i \to M_c$. Then

$$\begin{aligned}
\operatorname{disc}(\mathcal{H}, c) &= \|(M \otimes I_c)\chi\|_\infty \\
&\geq \frac{1}{\sqrt{cm}}\|(M \otimes I_c)\chi\|_2 \\
&= \left(\frac{1}{cm}\chi^T(M \otimes I_c)^T(M \otimes I_c)\chi\right)^{\frac{1}{2}} \\
&= \left(\frac{1}{cm}\chi^T(M^TM \otimes I_c)\chi\right)^{\frac{1}{2}} \\
&= \left(\frac{1}{cm}\chi^T[(M^TM - D) \otimes I_c + D \otimes I_c]\chi\right)^{\frac{1}{2}} \\
&\geq \left(\frac{1}{cm}\chi^T(D \otimes I_c)\chi\right)^{\frac{1}{2}} \\
&= \left(\frac{1}{cm}\sum_{i \in [k]} \left(\chi_i^T(X \otimes I_c)\chi_i\right)\right)^{\frac{1}{2}} \\
&\geq \left(\frac{kx}{cm}\left(cz(1-z)^2 + c(1-z)z^2\right)\right)^{\frac{1}{2}} \\
&= \left(\frac{kxz(1-z)}{m}\right)^{\frac{1}{2}}.
\end{aligned}$$

□

Now we can apply Lemma 5.3. Here the block diagonal matrix D consists of $\frac{n-1}{q-1}$ block matrices on the diagonal. All of them are $(q-1) \times (q-1)$–matrices with all components equal to $\frac{q^{r-1}-1}{q-1} - \frac{q^{r-2}-1}{q-1} = q^{r-2}$. Hence the lower bound in Theorem 5.2 is established. Note that the term -1 is caused by the origin.

5.2.2 The Upper Bound

For the proof of the upper bound in Theorem 5.2 we invoke the VC-dimension of the hypergraph $\mathcal{H}$.

Definition 5.4. Let $\mathcal{H} = (V, \mathcal{E})$ be a finite hypergraph and set $n := |V|$. The VC-dimension of $\mathcal{H}$ is defined by

$$\dim(\mathcal{H}) := \max\{m \in [n] \mid \exists A \subseteq V : |A| = m,\ \mathcal{E} \cap A = 2^A\},$$

where $\mathcal{E} \cap A := \{E \cap A \mid E \in \mathcal{E}\}$.

Lemma 5.5. $\dim(\mathcal{H}) = r - 1$.

Proof. First of all each $A \subseteq V$ with the property $\mathcal{E} \cap A = 2^A$ is linearly independent. Furthermore a set A with $\mathcal{E} \cap A = 2^A$ and maximal cardinality cannot contain a complete basis, because a complete basis is not contained in any linear hyperplane. Thus the VC-dimension of $\mathcal{H}$ is at most $r - 1$.
On the other hand we show that every subset containing up to $r - 1$ linearly independent elements of V is shattered by $\mathcal{H}$. Let $\{v_1, v_2, \ldots, v_r\}$ be a basis of V, $A := \{v_1, v_2, \ldots, v_{r-1}\}$, $B \subseteq A$ and $k := |B|$. W.l.o.g. we can assume $B = \{v_1, v_2, \ldots, v_k\}$. We define a linear hyperplane E of V with $E \cap A = B$ by

$$E := \langle v_1, v_2, \ldots, v_k, v_{k+1} - v_r, v_{k+2} - v_r, \ldots, v_{r-1} - v_r \rangle.$$

One can easily check that the $r - 1$ vectors generating E are linearly independent. Thus $E \in \mathcal{E}$. Therefore it remains to show $v_{k+1}, \ldots, v_{r-1} \notin E$. Let $j \in \{k + 1, k + 2, \ldots, r - 1\}$. Assume that $v_j \in E$. Then there are $\lambda_i \in \mathbb{F}_q$, $i \in [r - 1]$ with

$$v_j = \sum_{i=1}^{k} \lambda_i v_i + \sum_{i=k+1}^{r-1} \lambda_i (v_i - v_r),$$

therefore

$$0 = \sum_{\substack{i=1 \\ i \neq j}}^{r-1} \lambda_i v_i + (\lambda_j - 1) v_j - \left(\sum_{i=k+1}^{r-1} \lambda_i \right) v_r.$$

With the linear independency of $v_1 v_2, \ldots, v_r$ we get $\lambda_j = 1$, $\lambda_i = 0$ for all $i \in [r - 1] \setminus \{j\}$ and

$$\sum_{i=k+1}^{r-1} \lambda_i = 0 \neq 1$$

which is a contradiction. Hence $E \cap A = B$ and we are done. □

After determing the VC-dimension we are now able to give an upper bound for the c–color discrepancy of $\mathcal{H}$. Recall that $z = \frac{(q-1) \mod c}{c}$.

Lemma 5.6.

$$\operatorname{disc}(\mathcal{H}, c) = O\left(\left(\frac{nz}{q} \right)^{\frac{1}{2} - \frac{1}{2(r-1)}} \right).$$

Proof. In the case $z = 0$ there is nothing left to prove. Thus we can assume $z \neq 0$, i.e., $(q - 1) \mod c \neq 0$. Let W be an arbitrary one-dimensional subspace of V. It is obvious that for every $E \in \mathcal{E}$ either $W \subseteq E$ or $W \cap E = \{0\}$. Thus for any two non-trivial $x, y \in W$ there exists no $E \in \mathcal{E}$ with $x \in E$ and $y \notin E$. Therefore we can give colors to all but $(q-1) \mod c$ non-trivial elements of W in such a way that every color is useded in the same amount. We do the same for every one-dimensional subspace of V. One can check that the sub-hypergraph induced by the precolored vertices has discrepancy 0. Let $\mathcal{H}' = (V', \mathcal{E}')$ be the sub-hypergraph induced by the other $n' := \frac{n-1}{q-1}x + 1$ vertices of V. Because every one-dimensional subspace of V contains at least one non-trivial element that is not precolored it is easy to see that $\mathcal{H}'$ has VC-dimension $dim(\mathcal{H}') = dim(\mathcal{H}) = r-1$. With Lemma 5.9 from [Mat99] we get

$$\pi_{\mathcal{E}'}(k) = O\left(k^{r-1}\right),$$

for all $k \in [n']$, where $\pi_{\mathcal{E}'}$ is the primal shatter function of the set system $\mathcal{E}'$ defined in the beginning of chapter 5 of [Mat99]. Now applying Theorem 3.19 from [DS03], which is a c–color extension of Theorem 5.3 from [Mat99], we have

$$\operatorname{disc}(\mathcal{H}, c) = O\left(\left(\frac{n'}{c}\right)^{\frac{1}{2}-\frac{1}{2(r-1)}}\right) = O\left(\left(\frac{nz}{q}\right)^{\frac{1}{2}-\frac{1}{2(r-1)}}\right).$$

□

Proof of the upper bound in Theorem 5.2. We apply Lemma 5.6. Thus, we have to prove $\left(\frac{nz}{q}\right)^{\frac{1}{2}-\frac{1}{2(r-1)}} \leq \sqrt{nz(1-z)}$. First we show $\frac{1}{q} \leq (1-z)$. If $c < q$ then $(1-z) = \frac{c-x}{c} \geq \frac{1}{c} > \frac{1}{q}$ is obvious. Thus we can assume $c \geq q$. We have $x = (q-1) \mod c = q-1$, hence $x = q-1 \leq \frac{q-1}{q}c$. Therefore

$$(1-z) = \frac{c-x}{c} \geq \frac{1}{q}.$$

Now the upper bound in Theorem 5.2 follows from

$$\begin{aligned} \left(\frac{nz}{q}\right)^{\frac{1}{2}-\frac{1}{2(r-1)}} &\leq (nz(1-z))^{\frac{1}{2}-\frac{1}{2(r-1)}} \\ &\leq \sqrt{nz(1-z)}. \end{aligned}$$

□

5.3 Positive Discrepancy of $\mathcal{H}$

Obviously the upper bound in Theorem 5.2 for the c–color discrepancy is also an upper bound for the positive c–color discrepancy $\operatorname{disc}^+(\mathcal{H}, c)$. But the lower bound needs to be investigated separately as we have mentioned at the end of section 5.1. The main theorem is:

Theorem 5.7. *Let* $z := \frac{(q-1) \bmod c}{c}$, $z \neq 0$. *If* $r \geq r_0(q) = \begin{cases} 6 & : \quad q = 2, \\ 5 & : \quad q = 3, 4, 5, \\ 4 & : \quad \textit{otherwise}, \end{cases}$

Then we have $\mathrm{disc}^+(\mathcal{H}, c) \geq \frac{\sqrt{z(1-z)}}{4q(q-1)}\sqrt{\frac{n}{c}} - 1$.

For the proof of Theorem 5.7 we need some lemmas for the Fourier transform on $\mathbb{F}_q^r$. For basics for the Fourier transform in finite groups we follow [LN94].

5.3.1 The Fourier Transform and Facts about $\mathbb{F}_q^r$

Let $f : V \to \mathbb{C}$ be a function. To define the Fourier Transform of f, we need the absolute trace function

$$\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p} : \mathbb{F}_q \to \mathbb{F}_p, \alpha \mapsto \alpha + \alpha^p + \alpha^{p^2} + \ldots + \alpha^{p^{k-1}}.$$

If it is clear which p and q are considered, we just write $\mathrm{Tr}(.)$. Theorem 2.23 (iii) in [LN94] states that the function $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$ is linear and onto. Now the Fourier Transform $\hat{f}$ is defined by

$$\hat{f} : V \to \mathbb{C}, z \mapsto \sum_{x \in V} f(x) e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle x, z\rangle)},$$

where $\langle x, z\rangle = x_1 z_1 + x_2 z_2 + \ldots + x_r z_r$ is the common inner product in $\mathbb{F}_q^r$. Furthermore we set for all $W \leq V$:

$$\hat{f}(W) := \sum_{z \in W^\sharp} \hat{f}(z),$$

where $W^\sharp = W \setminus \{0\}$. For convenience let $\hat{A} := \hat{\mathbb{1}}_A$ be the Fourier transform of the indicator function $\mathbb{1}_A$ $(A \subseteq V)$. For all $E \in \mathcal{E}$ there is a unique subspace $E^\perp$ of V of dimension one which is orthogonal to E. Lemma 5.10 together with the following definition shows the benefit of the Fourier transform for the positive discrepancy.

Definition 5.8. For every $A \subseteq V$ and every hyperedge $E \in \mathcal{E}$ we define

$$d^+(A, E) \quad := \quad |A \cap E| - \frac{|A|}{q}.$$

We need the following orthogonality relation.

Lemma 5.9. *For each* $z \in V^\sharp$ *and* $a \in V$ *we have*

$$\frac{1}{q}\sum_{x \in \langle z\rangle} e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle a, x\rangle)} = \begin{cases} 0 & : \quad \langle a, z\rangle \neq 0, \\ 1 & : \quad \langle a, z\rangle = 0. \end{cases}$$

Proof. If $\langle a, z\rangle = 0$, then $\langle a, x\rangle = 0$ for every $x \in \langle z\rangle$, hence

$$\begin{aligned}
\frac{1}{q}\sum_{x\in\langle z\rangle} e^{\frac{2\pi i}{p}\operatorname{Tr}(\langle a,x\rangle)} &= \frac{1}{q}\sum_{x\in\langle z\rangle} e^{\frac{2\pi i}{p}\operatorname{Tr}(0)} \\
&= \frac{1}{q}\sum_{x\in\langle z\rangle} 1 \\
&= \frac{1}{q}|\langle z\rangle| = 1.
\end{aligned}$$

Now we assume $\langle a, z\rangle \neq 0$. Then $\langle a, x\rangle$ passes through the whole field $\mathbb{F}_q$, if x passes through the whole subspace $\langle z\rangle$. Because the trace function Tr a onto non-trivial maps, there exists some $y \in \langle z\rangle$ with $e^{\frac{2\pi i}{p}\operatorname{Tr}(\langle a,y\rangle)} \neq 1$. Since the function $f : \langle z\rangle \to \langle z\rangle, x \mapsto x + y$ is bijective

$$\begin{aligned}
e^{\frac{2\pi i}{p}\operatorname{Tr}(\langle a,y\rangle)}\left(\frac{1}{q}\sum_{x\in\langle z\rangle} e^{\frac{2\pi i}{p}\operatorname{Tr}(\langle a,x\rangle)}\right) &= \frac{1}{q}\sum_{x\in\langle z\rangle} e^{\frac{2\pi i}{p}\operatorname{Tr}(\langle a,x+y\rangle)} \\
&= \frac{1}{q}\sum_{x\in\langle z\rangle} e^{\frac{2\pi i}{p}\operatorname{Tr}(\langle a,x\rangle)}.
\end{aligned}$$

Thus with $e^{\frac{2\pi i}{p}\operatorname{Tr}(\langle a,y\rangle)} \neq 1$ we get $\frac{1}{q}\sum_{x\in\langle z\rangle} e^{\frac{2\pi i}{p}\operatorname{Tr}(\langle a,x\rangle)} = 0$. □

With Lemma 5.9 we establish the first link between positive discrepancy and Fourier analysis.

Lemma 5.10. *For every subset $A \subseteq V$ and every hyperedge $E \in \mathcal{E}$*

$$d^{+}(A, E) = \frac{1}{q}\hat{A}(E^{\perp}).$$

Proof. Let $z \in E^{\perp} \setminus \{0\}$. Using Lemma 5.9 we have

$$\begin{aligned}
|A \cap E| &= \sum_{a \in A} \delta_{\langle a,z\rangle,0} \\
&= \sum_{a \in A} \frac{1}{q} \sum_{x \in \langle z\rangle} e^{\frac{2\pi i}{p}\operatorname{Tr}(\langle a,x\rangle)} \\
&= \frac{1}{q} \sum_{x \in E^{\perp}} \sum_{a \in A} e^{\frac{2\pi i}{p}\operatorname{Tr}(\langle a,x\rangle)} \\
&= \frac{1}{q} \sum_{x \in E^{\perp}} \hat{A}(x) \\
&= \frac{1}{q}\hat{A}(E^{\perp}) + \frac{1}{q}\hat{A}(0) \\
&= \frac{1}{q}\hat{A}(E^{\perp}) + \frac{1}{q}|A|.
\end{aligned}$$

Thus

$$d^{+}(A,E) = |A \cap E| - \frac{1}{q}|A| = \frac{1}{q}\hat{A}(E^{\perp}).$$

□

Note that Lemma 5.10 immediately implies the statement 5.5 in the introduction. We fix two parameters. Let $\alpha := |\mathcal{E}|$ and $\beta := |\{E \in \mathcal{E} \mid v \in E\}|$ for a $v \in V^{\sharp} = V \setminus \{0\}$.

Lemma 5.11. *We have*

(i) $\alpha = \frac{q^r-1}{q-1}$,

(ii) $\beta = \frac{q^{r-1}-1}{q-1}$.

Proof. (i) Due to orthogonality, there is a one-to-one correspondence between the subspaces of dimension one and codimension one in V. So the number of linear hyperplanes in V is the number of basis of a one-dimensional subspace of V divided by the number of basis of a fixed one-dimensional subspace of V. Thus we have $\alpha = \frac{q^r-1}{q-1}$.

(ii) Double-counting gives

$$\begin{aligned}
\beta(q^r - 1) + \alpha &= \left(\sum_{v \in V^{\sharp}} |\{E \in \mathcal{E} \mid v \in E\}|\right) + |\{E \in \mathcal{E} \mid 0 \in E\}| \\
&= \sum_{v \in V} |\{E \in \mathcal{E} \mid v \in E\}| \\
&= \sum_{E \in \mathcal{E}} |E| = \alpha q^{r-1}.
\end{aligned}$$

Thus we have $\beta = \alpha \frac{q^{r-1}-1}{q^r-1} = \frac{q^{r-1}-1}{q-1}$.

□

The three statements in the next lemma will be useful for our calculations. For convenience we define $M_2 := \left\{(a_1, a_2, k_1, k_2) \in A \times A \times \mathbb{F}_q^\sharp \times \mathbb{F}_q^\sharp \mid k_1 a_1 + k_2 a_2 = 0\right\}$ and $M_3 := \left\{(a_1, a_2, a_3, k_1, k_2, k_3) \in A^3 \times \left(\mathbb{F}_q^\sharp\right)^3 \mid k_1 a_1 + k_2 a_2 + k_3 a_3 = 0\right\}$. Recall that $\mathcal{E}_1$ denotes the set of all one-dimensional subspaces of V.

Lemma 5.12. *Let $A \subseteq V$. It holds*

(i) $\sum\limits_{E \in \mathcal{E}} \hat{A}(E^\perp) = n\, \mathbb{1}_A(0) - |A|$.

(ii) $\sum\limits_{E \in \mathcal{E}} \hat{A}(E^\perp)^2 = \frac{n}{q-1} |M_2| - (q-1)|A|^2$.

(iii) $\sum\limits_{E \in \mathcal{E}} \hat{A}(E^\perp)^3 = \frac{n}{q-1} |M_3| - (q-1)^2|A|^3$.

Proof. (i) Every vector space is the disjoint union of all its one-dimensional subspaces without the origin and the origin itself. Thus we have

$$\begin{aligned}
\sum_{E \in \mathcal{E}} \hat{A}(E^\perp) &= \sum_{x \in V} \hat{A}(x) \\
&= \sum_{x \in V^\perp} \sum_{a \in A} e^{\frac{2\pi i}{q} \operatorname{Tr}(\langle a, x \rangle)} \\
&= \sum_{a \in A} \sum_{x \in V} e^{\frac{2\pi i}{q} \operatorname{Tr}(\langle a, x \rangle)} - |A| \\
&= n\, \mathbb{1}_A(0) - |A|
\end{aligned}$$

(ii) For every $E \in \mathcal{E}$ we fix a non-trivial element $v_{E^\perp} \in E^\perp$ of the one-dimensional subspace $E^\perp$. Then

$$\begin{aligned}
\sum_{E\in\mathcal{E}} \hat{A}(E^\perp)^2 &= \sum_{E\in\mathcal{E}} \left(\sum_{a\in A} \sum_{x\in E^\perp\setminus\{0\}} e^{\frac{2\pi i}{p}\operatorname{Tr}(\langle a,x\rangle)} \right)^2 \\
&= \sum_{E\in\mathcal{E}} \left(\sum_{a_1\in A} \sum_{x_1\in E^\perp\setminus\{0\}} e^{\frac{2\pi i}{p}\operatorname{Tr}(\langle a_1,x_1\rangle)} \right) \left(\sum_{a_2\in A} \sum_{x_2\in E^\perp\setminus\{0\}} e^{\frac{2\pi i}{p}\operatorname{Tr}(\langle a_2,x_2\rangle)} \right) \\
&= \sum_{E\in\mathcal{E}} \sum_{a_1,a_2\in A} \sum_{x_1,x_2\in E^\perp\setminus\{0\}} e^{\frac{2\pi i}{p}\operatorname{Tr}(\langle a_1,x_1\rangle+\langle a_2,x_2\rangle)} \\
&= \sum_{E\in\mathcal{E}} \sum_{a_1,a_2\in A} \sum_{k_1,k_2\in\mathbb{F}_q^\sharp} e^{\frac{2\pi i}{p}\operatorname{Tr}(\langle a_1,k_1v_{E^\perp}\rangle+\langle a_2,v_{E^\perp}\rangle)} \\
&= \sum_{E\in\mathcal{E}} \sum_{a_1,a_2\in A} \sum_{k_1,k_2\in\mathbb{F}_q^\sharp} e^{\frac{2\pi i}{p}\operatorname{Tr}(\langle k_1a_1+k_2a_2,v_{E^\perp}\rangle)} \\
&= \frac{1}{q-1} \sum_{E\in\mathcal{E}} \sum_{v\in(E^\perp)^\sharp} \sum_{a_1,a_2\in A} \sum_{k_1,k_2\in\mathbb{F}_q^\sharp} e^{\frac{2\pi i}{p}\operatorname{Tr}(\langle k_1a_1+k_2a_2,v\rangle)} \\
&= \frac{1}{q-1} \sum_{v\in V^\sharp} \sum_{a_1,a_2\in A} \sum_{k_1,k_2\in\mathbb{F}_q^\sharp} e^{\frac{2\pi i}{p}\operatorname{Tr}(\langle k_1a_1+k_2a_2,v\rangle)} \\
&= \frac{1}{q-1} \sum_{v\in V} \sum_{a_1,a_2\in A} \sum_{k_1,k_2\in\mathbb{F}_q^\sharp} e^{\frac{2\pi i}{p}\operatorname{Tr}(\langle k_1a_1+k_2a_2,v\rangle)} - (q-1)|A|^2 \\
&= \frac{n}{q-1}\,|M_2| - (q-1)|A|^2
\end{aligned}$$

(iii) For every one-dimensional subspace $W \in \mathcal{E}_1$ we fix an element $x_W \in W^\sharp$. Then

$$\begin{aligned}
\sum_{E\in\mathcal{E}} \hat{A}(E^\perp)^3 &= \sum_{E\in\mathcal{E}} \left(\sum_{x\in E^\perp\setminus\{0\}} \sum_{a\in A} e^{\frac{2\pi i}{p}\operatorname{Tr}(\langle a,x\rangle)} \right)^3 \\
&= \sum_{W\in\mathcal{E}_1} \sum_{a_1,a_2,a_3\in A} \sum_{x_1,x_2,x_3\in W^\sharp} e^{\frac{2\pi i}{p}\operatorname{Tr}(\langle a_1,x_1\rangle+\langle a_2,x_2\rangle+\langle a_3,x_3\rangle)} \\
&= \sum_{W\in\mathcal{E}_1} \sum_{a_1,a_2,a_3\in A} \sum_{k_1,k_2,k_3\in\mathbb{F}_q^\sharp} e^{\frac{2\pi i}{p}\operatorname{Tr}(\langle k_1a_1+k_2a_2+k_3a_3,x_W\rangle)} \\
&= \frac{1}{q-1} \sum_{v\in V^\sharp} \sum_{a_1,a_2,a_3\in A} \sum_{k_1,k_2,k_3\in\mathbb{F}_q^\sharp} e^{\frac{2\pi i}{p}\operatorname{Tr}(\langle k_1a_1+k_2a_2+k_3a_3,v\rangle)}
\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{E\in\mathcal{E}} \hat{A}(E^{\perp})^3 &= \frac{1}{q-1}\sum_{v\in V^\sharp}\sum_{a_1,a_2,a_3\in A}\sum_{k_1,k_2,k_3\in\mathbb{F}_q^\sharp} e^{\frac{2\pi i}{p}\operatorname{Tr}(\langle k_1a_1+k_2a_2+k_3a_3,v\rangle)} \\ &= \frac{1}{q-1}\sum_{v\in V}\sum_{a_1,a_2,a_3\in A}\sum_{k_1,k_2,k_3\in\mathbb{F}_q^\sharp} e^{\frac{2\pi i}{p}\operatorname{Tr}(\langle k_1a_1+k_2a_2+k_3a_3,v\rangle)} - (q-1)^2|A|^3 \\ &= \frac{n}{q-1}\,|M_3| - (q-1)^2|A|^3\end{aligned}$$

□

Calculating $|M_2|$ gives a Parseval type equation.

Corollary 5.13. *For $A \subseteq V$*

$$\sum_{E\in\mathcal{E}} |\hat{A}(E^{\perp})|^2 = n(q-1)\mathbb{1}_A(0) - (q-1)|A|^2 + n\sum_{W\in\mathcal{E}_1}|A\cap W^\sharp|^2$$

and in particular

$$\sum_{E\in\mathcal{E}} |\hat{A}(E^{\perp})|^2 \geq |A|(n-(q-1)|A|).$$

We proceed to the discussion of the positive discrepancy and show first lower bounds.

Lemma 5.14. *Let $A \subseteq V$. Then there exists an $E \in \mathcal{E}$ with*

$$d^+(A,E) \geq -1.$$

Proof. Using Lemma 5.10 we have

$$\begin{aligned}
\sum_{E\in\mathcal{E}} d^+(A,E) &= \frac{1}{q}\sum_{E\in\mathcal{E}} \hat{A}(E^\perp) \\
&= \frac{1}{q}\sum_{E\in\mathcal{E}}\sum_{z\in(E^\perp)^\sharp}\sum_{a\in A} e^{\frac{2\pi i}{p}\operatorname{Tr}(\langle a,z\rangle)} \\
&= \frac{1}{q}\sum_{a\in A}\sum_{z\in V^\sharp}\sum_{\substack{E\in\mathcal{E}\\ z\in E^\perp}} e^{\frac{2\pi i}{p}\operatorname{Tr}(\langle a,z\rangle)} \\
&= \frac{1}{q}\left(\sum_{a\in A^\sharp}\sum_{z\in V^\sharp} e^{\frac{2\pi i}{p}\operatorname{Tr}(\langle a,z\rangle)} + \mathcal{X}_A(0)\sum_{z\in V^\sharp} e^{\frac{2\pi i}{p}\operatorname{Tr}(\langle 0,z\rangle)}\right) \\
&= \frac{1}{q}\left(\sum_{a\in A^\sharp}(-1) + \mathcal{X}_A(0)(n-1)\right) \\
&= \frac{1}{q}\left(\mathcal{X}_A(0)(n-1) - |A^\sharp|\right) \\
&= \frac{1}{q}\left(\mathcal{X}_A(0)(n) - |A|\right).
\end{aligned}$$

Using $|\mathcal{E}| = \frac{n-1}{q-1}$ there is an $E\in\mathcal{E}$ with

$$d^+(A,E) \;\geq\; \frac{q-1}{n-1}\frac{1}{q}\left(\mathcal{X}_A(0)n - |A|\right).$$

For $0\in A$ we get $d^+(A,E)\geq 0$. Thus we can assume $0\notin A$. It holds

$$\begin{aligned}
d^+(A,E) &\geq -\frac{q-1}{n-1}\frac{|A|}{q} \\
&\geq -\frac{q-1}{n-1}\frac{n-1}{q} \\
&= -\frac{q-1}{q}.
\end{aligned}$$

□

Our strategy is the following: first we find a large color-class A_i. For this color-class the term $\frac{|A_i|}{q} - \frac{|E|}{c}$ is large. In the second step Lemma 5.14 gives us a hyperedge $E\in\mathcal{E}$ with $\frac{\hat{A}_i(E^\perp)}{q} \geq -1$. The next lemma will be used to find a large color-class A_i.

Lemma 5.15. *Let $k\in\mathbb{R}^+$ with $\frac{1}{q^{r-1}-q^{\frac{r}{2}}-1} \leq k \leq \frac{1}{3q}$ and $z = \frac{(q-1) \mod c}{c}$. Either there exists a color $i\in[c]$ and a hyperedge $E\in\mathcal{E}$ with*

$$|A_i\cap E| - \frac{|E|}{c} > k\frac{\sqrt{n}}{c}$$

or we have

$$\sum_{i\in[c]}\sum_{E\in\mathcal{E}}|\hat{A}_i(E^\perp)|^2 \geq \frac{n(n-1)c}{q-1}\left(z(1-z)-\tfrac{k}{c^2}(q-1)^2\right).$$

Proof. Let $A \subseteq V$. Then

$$\sum_{E\in\mathcal{E}}|A\cap E| = \sum_{a\in A}\sum_{\substack{E\in\mathcal{E}\\ a\in E}} 1 \geq \sum_{a\in A}\beta = |A|\beta.$$

Hence for every $A\subseteq V$ there exists an $E_0\in\mathcal{E}$ with

$$|A\cap E_0| \geq |A|\frac{\beta}{\alpha} = |A|\frac{q^{r-1}-1}{q^r-1}. \tag{5.7}$$

Recall that $\mathcal{E}_1$ is the set of all one-dimensional subspaces of V. Using Corollary 5.13 we get

$$\sum_{i\in[c]}\sum_{E\in\mathcal{E}}|\hat{A}_i(E^\perp)|^2 = n(q-1) - \sum_{i\in[c]}(q-1)|A_i|^2 + n\sum_{i\in[c]}\sum_{W\in\mathcal{E}_1}|A\cap W^\sharp|^2. \tag{5.8}$$

We distinguish the following two cases.

Case 1: There exists an $i\in[c]$ with $|A_i| > \frac{q^r-1}{c(q^{r-1}-1)}\left(kq^{\frac{r}{2}}+q^{r-1}\right)$

Then (5.7) yields the existence of an $E\in\mathcal{E}$ with

$$|A_i\cap E| - \frac{|E|}{c} \geq |A_i|\frac{q^{r-1}-1}{q^r-1} - \frac{|E|}{c} \geq k\frac{\sqrt{n}}{c}.$$

Case 2: For all $i\in[c]$ we have $|A_i| \leq \frac{q^r-1}{c(q^{r-1}-1)}\left(kq^{\frac{r}{2}}+q^{r-1}\right)$

Put $x := (q-1) \mod c$ and $y := \lfloor \frac{q-1}{c} \rfloor = \frac{(q-1)-x}{c}$ and recall $n = q^r$. (5.8) yields

$$\begin{aligned}
\sum_{i\in[c]} \sum_{E\in\mathcal{E}} |\hat{A}_i(E^\perp)|^2 &= n(q-1) - (q-1)\sum_{i\in[c]} |A_i|^2 \\
&\quad + n \sum_{i\in[c]} \sum_{W\in\mathcal{E}_1} |A_i \cap W^\sharp|^2 \\
&\geq n(q-1) - (q-1)\frac{q^r-1}{c(q^{r-1}-1)}\left(kq^{\frac{r}{2}} + q^{r-1}\right)\sum_{i\in[c]} |A_i| \\
&\quad + n \sum_{W\in\mathcal{E}_1} \left((c-x)y^2 + x(y+1)^2\right) \\
&= n(q-1) - (q-1)n^2 \frac{q^r-1}{c(q^{r-1}-1)}\left(\frac{k}{q^{\frac{r}{2}}} + \frac{1}{q}\right) \\
&\quad + n\frac{q^r-1}{q-1}\left(cy^2 + 2xy + x\right) \\
&\geq -n(n-1)(q-1)\frac{n-1}{c(q^{r-1}-1)}\left(\frac{k}{q^{\frac{r}{2}}} + \frac{1}{q}\right) \\
&\quad + n\frac{n-1}{q-1}\left(\frac{((q-1)-x)^2}{c} + \frac{2x((q-1)-x)}{c} + x\right) \\
&= n(n-1)\left(\frac{(q-1)^2 - x^2 + cx}{c(q-1)} - \frac{(q-1)(q^r-1)}{c(q^{r-1}-1)}\left(\frac{k}{q^{\frac{r}{2}}} + \frac{1}{q}\right)\right).
\end{aligned}$$

Now $q^{r-1} - q^{\frac{r}{2}} - 1 \geq \frac{1}{k}$ yields $-kq^{\frac{3}{2}r-1} + kq^r + kq^{\frac{r}{2}} + q^{\frac{r}{2}} \leq 0$. Using this inequality we get

$$\begin{aligned}
(q^r-1)(k + q^{\frac{r}{2}-1}) - (1+k)(q^{r-1}-1)q^{\frac{r}{2}} &= kq^r - k - q^{\frac{r}{2}-1} - kq^{\frac{3}{2}r-1} - q^{\frac{r}{2}} - kq^{\frac{r}{2}} \\
&\leq -kq^{\frac{3}{2}r-1} + kq^r + kq^{\frac{r}{2}} + q^{\frac{r}{2}} \leq 0,
\end{aligned}$$

and hence

$$\frac{q^r-1}{q^{r-1}-1}\left(\frac{k}{q^{\frac{r}{2}}} + \frac{1}{q}\right) \leq 1 + k. \tag{5.9}$$

Thus, we have

$$\begin{aligned}
\sum_{i\in[c]} \sum_{E\in\mathcal{E}} |\hat{A}_i(E^\perp)|^2 &\geq n(n-1)\left(\frac{cx + (q-1)^2 - x^2}{c(q-1)} - \frac{(q-1)(q^r-1)}{c(q^{r-1}-1)}\left(\frac{k}{q^{\frac{r}{2}}} + \frac{1}{q}\right)\right) \\
&\overset{(5.9)}{\geq} n(n-1)\left(\frac{cx + (q-1)^2 - x^2}{c(q-1)} - \frac{(q-1)}{c}(1+k)\right) \\
&= n(n-1)\frac{cx - k(q-1)^2 - x^2}{c(q-1)} \\
&= \frac{n(n-1)c}{q-1}\left(z(1-z) - \tfrac{k}{c^2}(q-1^2)\right).
\end{aligned}$$

□

5.3.2 Proof of the Main Theorem

For the proof of Theorem 5.7 we need some notations. Put for all $E \in \mathcal{E}$

$$\begin{aligned} I^+(E) &:= \{i \in [c] \mid \hat{A}(E^\perp) \geq 0\}, \\ I^-(E) &:= [c] \setminus I^+(E), \\ M &:= \max_{i\in[c]} \max_{E\in\mathcal{E}} |\hat{A}_i(E^\perp)|. \end{aligned}$$

It is straightforward to see that there is an $E \in \mathcal{E}$ with

$$\sum_{i\in I^+(E)} \hat{A}_i(E^\perp) = -\sum_{i\in I^-(E)} \hat{A}_i(E^\perp) \geq M. \tag{5.10}$$

Proof of Theorem 5.7. Let $r \geq r_0(q)$ and $\chi : V \to [c]$ be a c–coloring of $\mathcal{H}$. Set $k := \frac{\sqrt{cz(1-z)}}{4q(q-1)}$. We have to show

$$\mathrm{disc}^+(\mathcal{H}, c) \geq k\frac{\sqrt{n}}{c}.$$

For convenience we fix another constant $\xi := \sqrt{\frac{nc}{8q}\left(z(1-z) - \frac{k}{c^2}(q-1)^2\right)}$. Straightforward calculation shows that the radian is positive. We want to show

$$\frac{\xi}{c} \geq k\frac{\sqrt{n}}{c}, \tag{5.11}$$

which we will use later on. Both numbers are positive, hence it suffices to prove $\frac{\xi^2}{n} - k^2 \geq 0$. We have

$$\begin{aligned} \frac{\xi^2}{n} - k^2 &= \frac{c}{8q}\left(z(1-z) - \frac{\sqrt{cz(1-z)}(q-1)}{4qc^2}\right) - \frac{cz(1-z)}{16q^2(q-1)^2} \\ &= \frac{c}{8q}\left[z(1-z) - \frac{\sqrt{cz(1-z)}(q-1)}{4qc^2} - \frac{z(1-z)}{2q(q-1)^2}\right] \\ &\geq \frac{c}{8q}\left[z(1-z) - \frac{z(1-z)}{4} - \frac{z(1-z)}{4}\right] \\ &= \frac{cz(1-z)}{16q} \geq 0. \end{aligned}$$

We make the following nested case distinctions.

Case 1: $\sum_{i\in I^+(E)}\left(|A_i \cap E| - \frac{|E|}{c}\right) \geq \xi$ for an $E \in \mathcal{E}$.

In this case we have

$$c\ \mathrm{disc}^+(\mathcal{H}, c, \chi) \geq \sum_{i\in I^+(E)}\left(|A_i \cap E| - \frac{|E|}{c}\right) \geq \xi,$$

hence

$$\mathrm{disc}^+(\mathcal{H}, c, \chi) \geq \frac{\xi}{c} \overset{(5.11)}{\geq} k\frac{\sqrt{n}}{c}. \tag{5.12}$$

Case 2: $\sum_{i\in I^+(E)} \left(|A_i \cap E| - \frac{|E|}{c}\right) < \xi$ for all $E \in \mathcal{E}$.

It follows

$$\sum_{i\in I^-(E)} \left(|A_i \cap E| - \frac{|E|}{c}\right) = -\sum_{i\in I^+(E)} \left(|A_i \cap E| - \frac{|E|}{c}\right) > -\xi \tag{5.13}$$

for each $E \in \mathcal{E}$.

Case 2.1: $M \geq 2q\xi$.

By (5.5) and (5.10) we have for an appropriate $E \in \mathcal{E}$:

$$\begin{aligned}
\frac{1}{q}\sum_{i\in I^-(E)} |A_i| - |I^-(E)|\frac{|E|}{c} &= \sum_{i\in I^-(E)} \left(|A_i \cap E| - \frac{|E|}{c}\right) - \frac{1}{q}\sum_{i\in I^-(E)} \hat{A}_i(E^\perp) \\
&\overset{(5.13)}{>} -\xi + \frac{1}{q}M \\
&\geq -\xi + \frac{2q\xi}{q} \\
&= \xi.
\end{aligned}$$

Hence there exists an $i_0 \in I^-(X)$ with

$$\frac{1}{q}|A_{i_0}| - \frac{|E|}{c} \geq \frac{\xi}{c} \overset{(5.11)}{\geq} k\frac{\sqrt{n}}{c}.$$

Lemma 5.14 ensures the existence of an $E_0 \in \mathcal{E}$ with

$$\frac{1}{q}\hat{A}_{i_0}(E_0^\perp) \geq -1,$$

thus

$$\begin{aligned}
\mathrm{disc}^+(\mathcal{H}, c, \chi) \geq |A_{i_0} \cap E_0| - \frac{|E_0|}{c} &= \frac{1}{q}\hat{A}_{i_0}(E_0^\perp) + \frac{1}{q}|A_{i_0}| - \frac{|E_0|}{c} \\
&\geq k\frac{\sqrt{n}}{c} - 1. \qquad (5.14)
\end{aligned}$$

Case 2.2: $M < 2q\xi$.

To use Lemma 5.15 we have to verify $\frac{1}{q^{r-1}-q^{\frac{r}{2}}-1} \leq k \leq \frac{1}{3q}$. It holds $cz \leq q-1$ and $1 - z \leq 1$. Therefore we get

$$k = \frac{\sqrt{cz(1-z)}}{4q(q-1)} \leq \frac{\sqrt{q-1}}{4q(q-1)} < \frac{1}{3q}.$$

Using $z \neq 0$ we get a lower bound for k:

$$k \geq \frac{\sqrt{\frac{c-1}{c}}}{4q(q-1)} \geq \frac{1}{4\sqrt{2}q(q-1)} > \frac{1}{6q(q-1)}.$$

The only thing that is left to prove is $\frac{1}{6q(q-1)} \geq \frac{1}{q^{r-1}-q^{\frac{r}{2}}-1}$. But it is easy to check that $r \geq r_0(q)$ assures this inequality. Thus we can apply Lemma 5.15. Hence either there exists a color $i \in [c]$ and an $E \in \mathcal{E}$ such that

$$|A_i \cap E| - \frac{|E|}{c} > k\frac{\sqrt{n}}{c} \tag{5.15}$$

or we have

$$\sum_{i\in[c]} \sum_{E\in\mathcal{E}} |\hat{A}_i(E^\perp)|^2 \geq \frac{n(n-1)c}{q-1} \left(z(1-z) - \frac{k}{c^2}(q-1)^2 \right). \tag{5.16}$$

In the first case we get a lower bound for the positive discrepancy as desired. Therefore we can assume that (2) holds. Then there exists an $E \in \mathcal{E}$ with

$$\sum_{i\in[c]} |\hat{A}_i(E^\perp)|^2 \geq nc \left(z(1-z) - \frac{k}{c^2}(q-1)^2 \right).$$

Hence we get

$$\begin{aligned} 2M \left(\sum_{i\in I^-(E)} |\hat{A}_i(E^\perp)| \right) &= M \left(\sum_{i\in I^-(E)} |\hat{A}_i(E^\perp)| + \sum_{i\in I^+(E)} |\hat{A}_i(E^\perp)| \right) \\ &= M \sum_{i\in[c]} |\hat{A}_i(E^\perp)| \\ &\geq \sum_{i\in[c]} |\hat{A}_i(E^\perp)|^2 \\ &\geq nc \left(z(1-z) - \frac{k}{c^2}(q-1)^2 \right). \end{aligned}$$

Thus, we get

$$\begin{aligned} \sum_{i\in I^-(E)} |\hat{A}_i(E^\perp)| &\geq \frac{nc}{2M} \left(z(1-z) - \frac{k}{c^2}(q-1)^2 \right) \\ &> \frac{nc}{4q\xi} \left(z(1-z) - \frac{k}{c^2}(q-1)^2 \right) = 2\xi \end{aligned}$$

and (5.5) yields

$$\begin{aligned} \frac{1}{q}\sum_{i\in I^-(E)} |A_i| - |I^-(E)|\frac{|E|}{c} &= \sum_{i\in I^-(E)} \left(|A_i \cap E| - \frac{|E|}{c}\right) - \frac{1}{q}\sum_{i\in I^-(E)} \hat{A}_i(E^\perp) \\ &\overset{(5.13)}{>} -\xi + 2\xi = \xi. \end{aligned}$$

Hence there exists an $i_0 \in I^-(X)$ such that

$$\frac{1}{q}|A_{i_0}| - \frac{|E|}{c} > \frac{\xi}{c} \overset{(5.11)}{\geq} k\frac{\sqrt{n}}{c}.$$

Lemma 5.14 provides the existence of an $E_0 \in \mathcal{E}$ with

$$\frac{1}{q}\hat{A}_{i_0}(E_0^\perp) \geq -1,$$

and we have

$$\begin{aligned} \mathrm{disc}^+(\mathcal{H}, c, \chi) &\geq |A_{i_0} \cap E_0| - \frac{|E_0|}{c} \\ &= \frac{1}{q}\hat{A}_{i_0}(E_0^\perp) + \frac{1}{q}|A_{i_0}| - \frac{|E|}{c} \\ &\geq k\frac{\sqrt{n}}{c} - 1. \end{aligned} \tag{5.17}$$

We have shown

$$\mathrm{disc}^+(\mathcal{H}, c, \chi) \geq k\frac{\sqrt{n}}{c} - 1 = \frac{\sqrt{z(1-z)}}{4q(q-1)} - 1.$$

□

5.4 Positive Discrepancy for a Large Number of Colors

One factor in the lower bound of the positive discrepancy is the term $\sqrt{z(1-z)}$ with $z = \frac{(q-1) \mod c}{c}$. Therefore we get no lower bound if $c|(q-1)$ as we have already mentioned. Moreover, if $(q-1) \mod c$ is either small (almost zero) or large (almost c), the lower bound is of order $\Omega(\frac{\sqrt{n}}{c})$. This is a factor of $\sqrt{c}$ smaller than the upper bound $O(\sqrt{\frac{n}{c}})$. This problem arises for instance if the number of colors c is greater or equal than q. In this situation we have $\sqrt{z(1-z)} \leq \sqrt{\frac{q-1}{c}}$ and the lower bound in Theorem 5.7 becomes less or equal than

$$\frac{1}{4q\sqrt{q-1}}\frac{\sqrt{n}}{c} - 1.$$

Thus it is only of order $\Omega(\frac{\sqrt{n}}{c})$. But if the number of colors is large enough, precisely if $c \geq qn^{\frac{1}{3}}$, we can close this gap.

Theorem 5.16. *Let $c \geq qn^{1/3}$ and $r \geq 4$. Then*

$$\mathrm{disc}^+(\mathcal{H}, c) \geq \frac{1}{22\sqrt{q}}\sqrt{\frac{n}{c}} - 1.$$

In particular

$$\mathrm{disc}^+(\mathcal{H}, c) = \Theta(\sqrt{\tfrac{n}{c}}).$$

The key for the proof of Theorem 5.16 is the following lemma.

Lemma 5.17. *Let $A \subseteq V$ with $|A| \leq \frac{1}{2}q^{r-1}$ and $0 \in A$. There exists an $E \in \mathcal{E}$ with*

$$d^+(A, E) \geq \min\left\{\frac{1}{16(q-1)^2}\frac{n}{|A|}, \frac{1}{3\sqrt{q}}\sqrt{|A|}\right\}.$$

Proof. Corollary 5.13 yields

$$\sum_{E\in\mathcal{E}} |\hat{A}(E^\perp)|^2 \geq |A|(n-(q-1)|A|) \geq \tfrac{1}{2}n|A|.$$

Let us denote by $\mathcal{E}^+$ the set of all $E \in \mathcal{E}$ with $\hat{A}(E^\perp) \geq 0$ and by $\mathcal{E}^-$ the set of all $E \in \mathcal{E}$ with $\hat{A}(E^\perp) < 0$. Furthermore define $M := \max_{E\in\mathcal{E}} \hat{A}(E^\perp)$. Recall that we have defined $\alpha = |\mathcal{E}| = \frac{n-1}{q-1} \leq 2q^{r-1}$. Lemma 5.12 (i) gives

$$\begin{aligned}
\sum_{E\in\mathcal{E}^-} \left|\hat{A}(E^\perp)\right| &= -\sum_{E\in\mathcal{E}} \hat{A}(E^\perp) + \sum_{E\in\mathcal{E}^+} \hat{A}(E^\perp) \\
&= |A| - n\chi_A(0) + \sum_{E\in\mathcal{E}^+} \hat{A}(E^\perp) \\
&\leq \alpha M.
\end{aligned}$$

Using Lemma 5.12 (iii) in the same way, we get

$$\sum_{E\in\mathcal{E}^-} \left|\hat{A}(E^\perp)\right|^3 = \alpha M^3 + (q-1)^2|A|^2$$

Thus, the Cauchy-Schwarz inequality gives

$$\begin{aligned}
\tfrac{1}{2}n|A| &\leq \sum_{E\in\mathcal{E}^+} |\hat{A}(E^\perp)|^2 + \sum_{E\in\mathcal{E}^-} |\hat{A}(E^\perp)|^2 \\
&\leq \alpha M^2 + \sum_{E\in\mathcal{E}^-} |\hat{A}(E^\perp)|^{\frac{1}{2}} |\hat{A}(E^\perp)|^{\frac{3}{2}} \\
&\leq \alpha M^2 + \left(\sum_{E\in\mathcal{E}^-} |\hat{A}(E^\perp)|\right)^{\frac{1}{2}} \left(\sum_{E\in\mathcal{E}^-} |\hat{A}(E^\perp)|^3\right)^{\frac{1}{2}} \\
&\leq \alpha M^2 + (\alpha M)^{\frac{1}{2}} \left((q-1)^2|A|^3 + \alpha M^3\right)^{\frac{1}{2}}.
\end{aligned}$$

Case 1: $(q-1)^2|A|^3 < \alpha M^3$

Here we get

$$\begin{aligned}
\frac{1}{2}n|A| &\leq \alpha M^2 + (\alpha M)^{\frac{1}{2}}(2\alpha M^3)^{\frac{1}{2}} \\
&\leq (1+\sqrt{2})\alpha M^2,
\end{aligned}$$

and thus $M \geq \sqrt{\frac{q|A|}{10}}$.

Case 2: $(q-1)^2|A|^3 \geq \alpha M^3$

We distinguish two sub-cases. We first assume that $\alpha M^2 \geq (\alpha M)^{\frac{1}{2}} \left(2(q-1)^2|A|^3\right)^{\frac{1}{2}}$. Then we have

$$M \geq \sqrt{\frac{n|A|}{4\alpha}} \geq \sqrt{\frac{q|A|}{8}}.$$

The other case is $\alpha M^2 < (\alpha M)^{\frac{1}{2}} \left(2(q-1)^2|A|^3\right)^{\frac{1}{2}}$. Here we have

$$\frac{1}{2}n|A| \leq \left(\alpha M 2(q-1)^2|A|^3\right)^{\frac{1}{2}},$$

and altogether

$$M \geq \frac{q^{2r}}{|A|8\alpha(q-1)^2} \geq \frac{q}{16(q-1)^2}\frac{n}{|A|}.$$

□

Now we are able to prove Theorem 5.16.

Proof of Theorem 5.16. Let $\chi : V \to [c]$ be a c–coloring of $\mathcal{H}$. Let $A_i := \chi^{-1}(i)$ for all $i \in [c]$. There exists at least one color-class A_i with $|A_i| \geq \frac{n}{c}$. In the case that there is no color-class A_i with $\frac{n}{c} \leq |A_i \cup \{0\}| \leq \frac{n}{c} + \frac{1}{3}\sqrt{\frac{n}{c}}$ there must be a color-class A_{i_0} with $|A_{i_0} \cup \{0\}| > \frac{n}{c} + \frac{1}{3}\sqrt{\frac{n}{c}}$. Using Lemma 5.14 we get

$$\text{disc}^+(\mathcal{H}, c, \chi) \geq \max_{E \in \mathcal{E}} \left(d^+\left(A_{i_0} \cup \{0\}, E\right) + \frac{|A_{i_0} \cup \{0\}|}{q} - \frac{|E|}{c} \right) - 1 \geq \frac{1}{3q}\sqrt{\frac{n}{c}} - 1.$$

Therefore we can assume the existence of a color-class A_{i_0} with $\frac{n}{c} \leq |A_{i_0} \cup \{0\}| \leq \frac{n}{c} + \frac{1}{3}\sqrt{\frac{n}{c}}$. To apply Lemma 5.17 we have to assure $|A_{i_0} \cup \{0\}| < \frac{1}{2}q^{r-1}$. We get

$$|A_{i_0} \cup \{0\}| \leq \frac{n}{c} + \frac{1}{3}\sqrt{\frac{n}{c}} \leq q^{r-\frac{4}{3}-1} + \frac{1}{3}q^{\frac{r}{2}-\frac{2}{3}-\frac{1}{2}} \leq q^{r-1}\left(2^{-\frac{4}{3}} + \frac{1}{3}2^{-\frac{13}{6}}\right) < \frac{1}{2}q^{r-1}.$$

Hence there exists an $E \in \mathcal{E}$ with

$$d^+(A_{i_0} \cup \{0\}, E) \geq \min\left\{ \frac{1}{16(q-1)^2} \frac{n}{|A_{i_0} \cup \{0\}|}, \frac{1}{3\sqrt{q}}\sqrt{|A_{i_0} \cup \{0\}|} \right\} \geq \frac{1}{22\sqrt{q}}\sqrt{\frac{n}{c}}.$$

Thus, we have

$$\text{disc}^+(\mathcal{H}, c, \chi)) \geq d^+(A_{i_0} \cup \{0\}, E) + \frac{|A_{i_0}|}{q} - \frac{|E|}{c} - 1 \geq \frac{1}{22\sqrt{q}}\sqrt{\frac{n}{c}} - 1.$$

□

5.5 Conclusion

We have shown tight bounds for the c-color discrepancy of the hypergraph of linear hyperplanes in $\mathbb{F}_q^r$. For the positive c-color discrepancy of the same hypergraph we have given a lower and an upper bound which differ by a factor a $\sqrt{c}$. We have closed this gap in the case that the number of colors c is large ($c \geq qn^{\frac{1}{3}}$). The challenging algorithmic problem still is to construct an $O(\sqrt{n})$–discrepancy coloring for the linear hyperplanes in $\mathbb{F}_q^r$. Furthermore, a more detailed study of positive discrepancy is desirable.

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