Contributions to the theory of optimal stopping

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Contributions to the theory of optimal stopping

Sören Christensen

Abstract

This thesis deals with the explicit solution of optimal stopping problems with infinite time horizon.

To solve Markovian problems in continuous time we introduce an approach that gives rise to explicit results in various situations. The main idea is to characterize the optimal stopping set as the union of the maximum points of explicitly given functions involving the harmonic functions for the underlying stochastic process. This provides elementary solutions for a variety of optimal stopping problems and answers questions concerning the geometric shape of the optimal stopping set. The approach is shown to work well for one- and multidimensional diffusion processes, spectrally negative Lévy processes and problems containing the running maxima process.
Furthermore we introduce a new class of problems, which we call problems with guarantee. For continuous one-dimensional driving processes and certain Lévy processes we prove that the optimal strategies are of two-sided type and establish first-order ODEs that characterize the solution.

In the second part we consider optimal stopping problems for autoregressive processes in discrete time. This class of processes is intensively studied in statistics and other fields of applied probability. We establish elementary conditions to ensure that the optimal stopping time is of threshold type and find the joint distribution of the threshold-time and the overshoot for a wide class of innovations. Using the principle of continuous fit this leads to explicit solutions.

Zusammenfassung

Gegenstand dieser Arbeit ist die explizite Lösung von Problemen des optimalen Stoppens mit unendlichem Zeithorizont.

Im ersten Teil führen wir zur Lösung Markovscher Probleme in stetiger Zeit einen Ansatz ein, der in einer Vielzahl von Situationen zu expliziten Ergebnissen führt. Unter Benutzung der harmonischen Funktionen des zugrunde liegenden Prozesses charakterisieren wir dazu zunächst das Stoppgebiet als Menge von Maximalstellen konkret gegebener Funktionen. Dies führt in vielen Fällen zu elementaren Lösungen und ermöglicht Aussagen zur geometrischen Form des Stoppgebiets. Der Ansatz ist anwendbar auf ein- und mehrdimensionale Diffusionen, spektral-negative Lévyprozesse und Probleme, die den Supremumsprozess enthalten.
Des Weiteren führen wir eine neue Klasse von Problemen ein, die wir Stoppprobleme mit Garantien nennen. Für stetige eindimensionale Prozesse zeigen wir mithilfe des obigen Ansatzes, dass die optimalen Strategien zweiseitig sind und charakterisieren die optimalen Grenzen mittels gewöhnlicher Differentialgleichungen erster Ordnung. Diese Ergebnisse übertragen wir anschließend auf Lévyprozesse.

Im zweiten Teil beschäftigen wir uns mit Problemen des optimalen Stoppens autoregressiver Folgen, welche zur Beantwortung von Fragen in der Statistik und in anderen Feldern der angewandten Mathematik untersucht werden. Wir geben elementare Bedingungen an, die sicherstellen, dass die optimalen Stoppzeiten Erstübertrittszeiten sind und bestimmen die gemeinsame Verteilung von Erstübertrittszeit und Overshoot für eine große Klasse von Innovationen. Mithilfe des Prinzips des stetigen Übergangs erhält man explizite Lösungen.

Contents

Chapter 1

Introduction

In this thesis we focus on the solution of optimal stopping problems. As a motivation of these we consider the following four examples:

1. We drive a car along a street with parking spaces on the way to our destination. What is the best time to take a free parking space if we want to minimize the walking distance without turning around?
2. We sequentially observe a random process that depends on an unknown parameter and want to test a hypothesis about this parameter. What is the best time to decide as soon and as accurate as possible whether the hypothesis is true or not?
3. We observe the data of a certain early warning system. What is the "right" time to send out an alarm?
4. We own an American option at a financial market, i.e. we have the opportunity to exercise an option at any time up to a maturity. What exercise time should we use to maximize our expected payoff?

Abstracting from these examples our problem can be formulated as follows:

What time should we choose for a particular action to maximize the expected payoff (resp. minimize the expected costs)?

As a restriction for a solution we assume that we are no clairvoyants, i.e. we can only use stopping strategies that are based on information available so far. Furthermore we assume the observed process to have a random structure. Such kind of problems are called optimal stopping problems.
They appear, for example, in mathematical statistics, derivative pricing and portfolio

optimization, stochastic control theory and in the general theory of probability and its connection to problems in analysis. Hence solutions to these problems are important both for theory and applications. Early textbook treatments are [CRS71] for the discrete case and [Shi78] for Markovian problems; a recent monograph is [PS06].

In the following section we give a formal definition of the problem and a short overview on the development of the theory including some basic facts. In Section 1.2 we describe the outline of this thesis and point out the main contributions.

1.1 A brief overview of optimal stopping theory

Optimal stopping problems can be seen in the light of classical calculus of variations: Following this line, problems of optimal control in stochastics were investigated in the 1940s and 50s by Bellman's work in dynamic programming (see [Bel57]).
Problem 2 above can be seen as another early motivation for studying problems of optimal stopping: Wald and Wolfowitz transformed the problem of finding Bayesian procedures about the distribution of an observed sequence of i.i.d. random variables to an optimal stopping problem and described solutions for these problems ([WW48], [WW50]). This work was extended in many directions and the research is still going on. We refer to [Lai01] for a detailed overview. Motivated by this development Snell formulated the general problem in [Sne51]. We state it as follows:

Given a time parameter set $\mathfrak{T} \subseteq [0,\infty)$, and an $(\mathcal{F}_t)_{t\in\mathfrak{T}}$-adapted process $(Y_t)_{t\in\mathfrak{T}}$ (the payoff process) find an $(\mathcal{F}_t)_{t\in\mathfrak{T}}$-stopping time τ^* such that

$$E(Y_{\tau^*}) = \sup_{\tau} E(Y_\tau)$$

and give an explicit expression for the optimal expected payoff. Here the supremum is taken over all $(\mathcal{F}_t)_{t\in\mathfrak{T}}$-stopping times τ (the admissible strategies) such that the expectation exists.

Snell proved that in discrete time – i.e. $\mathfrak{T} = \mathbb{N}_0$ – under natural conditions the stopping time

$$\tau^* = \inf\{n \in \mathbb{N}_0 : Y_n = Z_n\}$$

is optimal, where $(Z_n)_{n\in\mathbb{N}_0}$ is the minimal regular supermartingale dominating $(Y_n)_{n\in\mathbb{N}_0}$. Furthermore this "Snell envelope" $(Z_n)_{n\in\mathbb{N}_0}$ is given by

$$Z_n = \operatorname*{ess\,sup}_{\tau\geq n} E(Y_\tau|\mathcal{F}_n).$$

The same results hold for continuous time parameter sets under minimal conditions. For more details including proofs we refer to [PS06, Chapter I].

For a general process $(Y_t)_{t\in\mathfrak{T}}$ the explicit determination of optimal stopping times via the Snell envelope is successful only for few examples. For a finite time parameter set $\mathfrak{T}$ one can use the method of backward induction as described in [CRS71, Chapter 3 and 4] but in this thesis we concentrate on infinite – and especially on unbounded – time parameter sets.
In this case a Markovian structure simplifies the problem. Dynkin initialized this development in [Dyn63]. The payoff process is assumed to have the form $Y_t = g(X_t)$, where $(X_t)_{t\in\mathfrak{T}}$ is a Markov process with state space E and $g : E \to \mathbb{R}$ is a measurable function. The solution to the optimal stopping problem depends on the starting point of the Markov process and we consider the value function v defined by

$$v(x) = \sup_{\tau} E_x(g(X_\tau)), \qquad x \in E.$$

This function v can be characterized via superharmonic functions: Under minimal conditions v is the smallest superharmonic function dominating g. Furthermore the stopping time

$$\tau^* = \inf\{t \in \mathfrak{T} : X_t \in S\}$$

is optimal for all starting points whenever an optimal stopping time exists; here

$$S = \{x \in E : g(x) = v(x)\}.$$

Therefore the set S is called the optimal stopping set. Hence if the smallest superharmonic majorant of g is known the problem is completely solved. But the concrete determination is not an easy task in general. As before we refer to [PS06, Chapter I].

For the solution of optimal stopping problems in the Markovian setting and continuous time the connection to free boundary problems was discovered by different authors in the 1950s and 1960s, see [PS06, Chapter IV]. We also refer to Subsection 2.4.1. If a solution to the associated free boundary problem is found, verification theorems are used to guarantee that this candidate is indeed a solution to the optimal stopping problem. This approach can be applied to a wide variety of problems. Explicit solutions can be expected for one-dimensional diffusion processes as well as for some jump processes, maximum processes and multidimensional diffusion processes.
Another more direct approach introduced by Beibel and Lerche gives rise to explicit solutions in various situations, which we will discuss in Section 2.4.

1.2 Thesis outline

The contribution of this thesis to the theory of optimal stopping can be described as follows:
We consider Markovian problems with unbounded time parameter sets $\mathfrak{T} = [0, \infty)$, resp. $\mathfrak{T} = \mathbb{N}_0$. For the continuous case $\mathfrak{T} = [0, \infty)$ we introduce an approach that gives rise to explicit solutions in different situations. The main idea is to characterize the optimal stopping set as a union of the maximum points of explicitly given functions. This leads to elementary solutions of optimal stopping problems in different situations and provides a simple way to answer conjectures on the optimal stopping set. Furthermore we introduce a new class of problems, called problems with guarantee, and solve them for different types of underlying processes.
Furthermore in the case $T = \mathbb{N}_0$ we develop a method for the solution of optimal stopping problems driven by autoregressive processes.

The detailed structure of the thesis is as follows:
Chapter 2 provides some – mostly well-known – results that will be used in the following. First we introduce two classes of Markov processes for the purposes of this thesis: One-dimensional diffusion processes and Lévy processes. In Subsection 2.3 we shortly describe the idea of the Choquet representation as a tool for the following chapters. We end the chapter with an overview on martingale techniques for optimal stopping that were introduced by Beibel and Lerche ([BL97]) and modified by Irle and Paulsen ([IP04]), and we describe the connection to the free boundary approach illustrated by an example.

In Chapter 3 – 5 we study Markovian problems in continuous time, i.e. $\mathfrak{T} = [0, \infty)$.
In Section 3.1 we motivate the further development. In the remainder of Chapter 3 we apply this idea to optimal stopping problems driven by a one-dimensional Markov process. First we study optimal stopping problems for diffusion processes with discounting and show that our idea works well under minimal conditions. We illustrate its applicability by different examples and obtain useful corollaries.
Afterwards we study problems with linear cost structure instead of discounting and prove that our approach works well in this situation too. As a generalization of the previous results we give some remarks concerning a random cost structure.
To end this chapter we turn our attention to jump processes. We consider spectrally negative Lévy processes and describe situations where threshold-times are optimal and show how the optimal threshold can be determined.

In Chapter 4 we introduce a new class of optimal stopping problems in which the gain

explicitly depends on the starting point. From a financial point of view this structure can be seen as a guarantee for the holder of an option. It turns out that the optimal strategies are of two-sided type.
If the driving process is a diffusion we will use the theory developed in the Chapter 3 to obtain general results. For an explicit solution we derive two differential equations that characterize the optimal strategies. Furthermore we study this type of stopping problems for Lévy processes. An explicit solution is obtained for spectrally negative processes.

Chapter 5 deals with optimal stopping problems where the driving process is multidimensional. We again use our approach to obtain explicit solutions. After some remarks on Martin boundary theory for this setting we consider optimal stopping problems when the driving process is a two-dimensional geometric Brownian motion and the gain function is homogeneous.
Then we turn our attention to the problem of optimal investment that was studied from different points of view. We disprove a conjecture of Hu and Øksensdal ([HØ98]) on the form of the optimal stopping set using our method.
In the last section we illustrate that our approach works for optimal stopping problems driven by running maxima processes as well.

In the final Chapter 6 we change our focus and consider problems in discrete time, i.e. $\mathfrak{T} = \mathbb{N}_0$. In this case overshoot plays a fundamental role. We assume the driving process to be of autoregressive type. This class of processes is intensively studied in statistics and other fields of applied stochastics, but only few results are known for optimal stopping problems.
We determine the overshoot and threshold-time distribution when the upward innovations are in the class of phasetype distributions which provides a dense class widely used in queueing theory. This result is valuable on its own. Using elementary arguments and the principle of continuous fit it leads to explicit solutions.

Chapter 2

Preliminaries

2.1 One-dimensional diffusions

One-dimensional diffusions form a class of stochastic processes that is known to be sufficiently wide both for theory and applications. These processes are closely connected to stochastic differential equations and this point of view gives rise to a heuristic interpretation that is often used for modeling problems in a wide range of applications such as mathematical finance, mathematical biology, stochastic control and economics.
There are various approaches in literature to the definition of a diffusion process. We follow the approach given in [RY99, Chapter VII, §3] that is based on the work of Feller and Itô and McKean (cf. [IM74]):

Definition 2.1. *Let $I \subseteq \mathbb{R}$ be an interval. A (time-homogeneous) strong Markov process $(X_t)_{t\geq 0}$ with state space I is called a one-dimensional diffusion process on I, if $(X_t)_{t\geq 0}$ has continuous sample path and can be killed only at the boundary points of I that do not belong to I.*

To prevent that the interval I can be decomposed into disjoint subintervals from which $(X_t)_{t\geq 0}$ cannot exit, we always assume that all diffusions are regular, that is

$$P_x(X_t = y \text{ for some } t \geq 0) > 0 \qquad \text{for all } x \in \text{int}(I),\ y \in I.$$

Here and in the following we do not mention the underlying filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, P)$ and the ingredients for a Markov process – i.e. the shift operator θ and the transition function – explicitly.
Now we state some well-known results that will be frequently used in this thesis, especially in Chapter 3. All proofs can be found in [RY99, Chapter VII, §3].

Proposition 2.1. *Let $J \subseteq \mathrm{int}(I)$ be a compact subinterval and $\tau \leq \inf\{t \geq 0 : X_t \notin J\}$ be a stopping time. Then*

$$E_x(\tau) < \infty \quad \textit{for all } x \in I.$$

Remark 2.2. *Stopping times τ as described in the proposition play an important role in Section 3.5 and are called regular .*

Here and in the following we often write $\tau_x = \inf\{t \geq 0 : X_t = x\}$ for a fixed $x \in I$.

Proposition and Definition 2.3. *(Scale function)*

(i) *There exists a continuous, strictly increasing function $s : I \to \mathbb{R}$ such that*

$$P_x(\tau_b < \tau_a) = \frac{s(x) - s(a)}{s(b) - s(a)} \qquad \textit{for all } a, b, x \in I \textit{ with } a < x < b.$$

(ii) *s is unique in the sense that if $\tilde{s}$ is another function with the same properties, then there exist $\alpha > 0, \beta \in \mathbb{R}$ such that $\tilde{s} = \alpha s + \beta$.*
s is called the scale function of $(X_t)_{t \geq 0}$.

(iii) *A locally bounded, strictly increasing Borel function f is a scale function if and only if $(f(X_{t \wedge \tau_{b_l} \wedge \tau_{b_r} \wedge \zeta}))_{t \geq 0}$ is a local martingale, where b_l, b_r are the boundary points of I and ζ is the life time.*

Remark 2.4. *Statement (iii) is a corrected version of Proposition (3.5) in [RY99, Chapter VII]. The proof given there is correct.*

A diffusion is characterized by the scale function and a Radon-measure on $\mathrm{int}(I)$, the so-called speed measure: See [RY99, p. 304].

In other references, diffusion processes are characterized via the form of their generator: It is assumed the the generator is an elliptic second-order differential operator of the form

$$A = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}$$

for some continuous functions $\sigma > 0$, μ. This representation can be used for a heuristic interpretation. Surprisingly a remarkable theorem going back to Dynkin states that this assumption on the form of the generator is not very restrictive, see [BS02, II.9]: If the scale function is continuously differentiable and the speed measure is absolutely continuous with respect to the Lebesgue measure, then the generator is an elliptic second-order differential operator as above. In this case the scale function is given by

$$s(x) = c\int_a^x \exp\left(-\int_a^z \frac{2\mu(y)}{\sigma^2(y)}dy\right)dz \quad \text{for some } a \in \mathrm{int}(I),\ c > 0$$

and is the (up to affine transformation) unique solution to the differential equation

$$Af = 0.$$

For the remainder of this section we fix $r > 0$ and define the "minimal r-harmonic functions" ψ_+ and ψ_- by

$$\psi_+(x) = \begin{cases} E_x(e^{-r\tau_a}\mathbb{1}_{\{\tau_a<\infty\}}), & x \le a \\ [E_a(e^{-r\tau_x}\mathbb{1}_{\{\tau_x<\infty\}})]^{-1}, & x > a \end{cases}$$

and

$$\psi_-(x) = \begin{cases} [E_a(e^{-r\tau_x}\mathbb{1}_{\{\tau_x<\infty\}})]^{-1}, & x \le a \\ E_x(e^{-r\tau_a}\mathbb{1}_{\{\tau_a<\infty\}}), & x > a \end{cases}$$

for a fixed point $a \in \operatorname{int}(I)$. Obviously ψ_+ is increasing and ψ_- is decreasing. Furthermore they are positive, continuous and can be used to characterize the boundary behavior of $(X_t)_{t\ge0}$. For results in this direction we refer to [IM74, Section 4.6]. The functions are especially important for us, because they are r-harmonic, i.e. for all $a < b \in \operatorname{int}(I)$, $x \in I$ and $\tau = \inf\{t \ge 0 : X_t \notin (a,b)\}$ it holds that

$$\psi_+(x) = E_x(e^{-r\tau}\psi_+(X_\tau)) \text{ and } \psi_-(x) = E_x(e^{-r\tau}\psi_-(X_\tau))$$

and all other positive r-harmonic functions are linear combinations of ψ_+ and ψ_-. For the moment write

$$h_a(x) = \frac{\psi_+(x)\psi_-(b) - \psi_+(b)\psi_-(x)}{\psi_+(a)\psi_-(b) - \psi_+(b)\psi_-(a)}, \quad h_b(x) = \frac{\psi_+(a)\psi_-(x) - \psi_+(x)\psi_-(a)}{\psi_+(a)\psi_-(b) - \psi_+(b)\psi_-(a)}.$$

It holds that $h_a(a) = h_b(b) = 1$ and $h_a(b) = h_b(a) = 0$. Since $X_\tau \in \{a,b\}$ under P_x for $x \in [a,b]$ we obtain

Proposition 2.5. *For all $a \le x \le b$ with $a,b \in \operatorname{int}(I)$ it holds that*

$$E_x(e^{-r\tau_a}\mathbb{1}_{\{\tau_a<\tau_b\}}) = E_x(e^{-r\tau}h_a(X_\tau)) = h_a(x)$$

and

$$E_x(e^{-r\tau_b}\mathbb{1}_{\{\tau_b<\tau_a\}}) = E_x(e^{-r\tau}h_b(X_\tau)) = h_b(x).$$

Furthermore

$$E_x(e^{-r\tau_b}\psi_+(X_{\tau_b})\mathbb{1}_{\{\tau_b<\infty\}}) = \psi_+(x)$$

and

$$E_x(e^{-r\tau_a}\psi_-(X_{\tau_a})\mathbb{1}_{\{\tau_a<\infty\}}) = \psi_-(x)$$

If A denotes the generator as above, ψ_+ resp. ψ_- are the (up to a constant factor) unique increasing resp. decreasing solutions of

$$Af - rf = 0.$$

2.2 Lévy processes

Lévy processes may be seen as a stochastic analogon to linear functions in the deterministic world:
Linear functions $X : \mathbb{R}_{\geq 0} \to \mathbb{R}$ can be characterized by the conditions $X_0 = 0$ and X growths steadily, i.e.

$$X_t - X_s \text{ depends only on } t - s.$$

The stochastic translation of this condition to a stochastic process $(X_t)_{t\geq 0}$ is that the distribution of $X_t - X_s$ only depends on $t - s$, and – to avoid feedback between successive parts – one assumes the increments to be independent:

Definition 2.2. *(Lévy process)*
An $(\mathcal{F}_t)_{t\geq 0}$-adapted real-valued process $(X_t)_{t\geq 0}$ is called Lévy process if

(i) $X_0 = 0$ *P-a.s.*

(ii) $(X_t)_{t\geq 0}$ *has independent increments, i.e.* $X_t - X_s$ *is independent of* $\mathcal{F}_s$ *for all* $0 \leq s \leq t$.

(iii) $(X_t)_{t\geq 0}$ *has stationary increments, i.e.* $P^{X_t - X_s} = P^{X_{t-s}}$ *for all* $0 \leq s \leq t$.

(iv) $(X_t)_{t\geq 0}$ *has cádlág sample paths.*

One immediately sees that Lévy processes are the continuous time analogon to random walks in discrete time. A Lévy process is both a semimartingale and a Markov process and can be seen as a standard example for both this classes in continuous time. The class of Lévy processes contains compound Poisson processes and the Brownian motion, but the class is much richer and very flexible for modeling. More details and the following standard facts can be found in the monographs [Ber96], [Kyp06], [Sat99] and [App04].
For a Lévy process $(X_t)_{t\geq 0}$ the function

$$\Psi_X : \mathbb{R} \to \mathbb{C}, u \mapsto -\log E(e^{iuX_1})$$

is called the characteristic exponent of $(X_t)_{t\geq 0}$ and this function characterizes its distribution by the Lévy-Kintchin formula (see [Sat99, Chapter 1, Theorem 8.1]):

Proposition 2.6. *(i) If $(X_t)_{t\geq 0}$ is a Lévy process, then there exist unique $a \in \mathbb{R}$, $\sigma \geq 0$ and a Lévy measure π such that*

$$\Psi_X(u) = iau + \frac{\sigma^2}{2}u^2 + \int (1 - e^{iux} + iux\mathbb{1}_{\{|x|<1\}})\pi(dx) \quad \textit{for all } u \in \mathbb{R}.$$

(ii) *On the other hand if $a \in \mathbb{R}$, $\sigma \geq 0$ and π is a Lévy measure, then there exists a Lévy process $(X_t)_{t\geq 0}$ with characteristic exponent as given above.*

Recall that a Lévy measure π is a measure on $\mathbb{R}$ concentrated on $\mathbb{R} \setminus \{0\}$ such that $\int (1 \wedge x^2)\pi(dx) < \infty$. The triple (a, σ, π) is called the characteristic triple of $(X_t)_{t\geq 0}$.

In many situations further assumptions on the jump structure of a Lévy process offer a significant advantage. Here it is often convenient to assume that the process has only one-sided jumps:
If $\pi(0, \infty) = 0$ and $(X_t)_{t\geq 0}$ has non-monotone paths, then $(X_t)_{t\geq 0}$ is called spectrally negative. Areas of application for this processes range from the theory of dams, insurance risk and branching processes to finance. All the following results can be found in [Kyp06, Chapter 8] and the references therein.
If $(X_t)_{t\geq 0}$ is spectrally negative then the Laplace exponent

$$\psi : [0, \infty) \to \mathbb{R}, \lambda \mapsto \log E(e^{\lambda X_1})$$

is well-defined and is often used instead of the characteristic exponent. ψ is infinitely differentiable on $(0, \infty)$, is strictly convex and

$$\psi(0) = 0, \quad \lim_{x\to\infty} \psi(x) = \infty, \quad \psi'(0) \in [-\infty, \infty),$$

A great advantage for spectrally negative Lévy processes is that there exists a function that can be seen as an analogon to the scale function for diffusion processes defined in the previous section:
For fixed $r \geq 0$ we define the scale function $W = W^{(r)} : \mathbb{R} \to [0, \infty)$ by $W(x) = 0$ for $x < 0$ and $W|_{[0,\infty)}$ to be the unique continuous increasing function on $[0, \infty)$ whose Laplace transform satisfies

$$\int_0^\infty e^{-\lambda t} W(x)dx = \frac{1}{\psi(\lambda) - r} \quad \text{for all } \lambda > \Phi(r),$$

where Φ denotes the right inverse of ψ. Furthermore we define the function $Z = Z^{(r)} : \mathbb{R} \to \mathbb{R}$ by

$$Z(x) = 1 + r \int_0^x W(y)dy.$$

A great advantage in the use of spectrally negative Lévy processes is that semi-explicit fluctuation identities containing the function W and Z can be proved (see [AKP04, Proposition 1]):

Proposition 2.7. *Let $(X_t)_{t\geq 0}$ be a spectrally negative Lévy processes, $r \geq 0$, $a, b \in \mathbb{R}$ and $x \in (a, b)$. Write*

$$\tau_a^- = \inf\{t > 0 : X_t \leq a\} \quad \textit{and} \quad \tau_b^+ = \inf\{t > 0 : X_t \geq b\}.$$

Then

$$E_x(e^{-r\tau_a^-}\mathbb{1}_{\{\tau_a^-<\infty\}}) = Z(x-a) - \frac{r}{\Phi(r)}W(x-a),$$
$$E_x(e^{-r\tau_b^+}\mathbb{1}_{\{\tau_b^+<\tau_a^-\}}) = \frac{W(x-a)}{W(b-a)}$$

and

$$E_x(e^{-r\tau_a^-}\mathbb{1}_{\{\tau_a^-<\tau_b^+\}}) = Z(x-a) - W(x-a)\frac{Z(b-a)}{W(b-a)}.$$

Note that the second equation for $r = 0$ may be seen as an analogon to the result for diffusions in Proposition 2.3, (i). This justifies the name scale function for W.

In Section 4.4.2 it will be helpful to know under what kind of conditions W and Z are C^2-functions on $(0,\infty)$. This problem was studied quite recently in [CKS10]. We just want to mention that this is the case whenever $(X_t)_{t\geq 0}$ has a Gaussian part, i.e. $\sigma \neq 0$ in the Lévy triple.

2.3 Results of Choquet-type and minimal functions

As a motivation for the next results recall the classical theorem, that goes back to Hermann Minkowski:

If K is a compact convex subset of a finite-dimensional vector space then each $x \in K$ can be represented as a convex combination of extreme points of K.

Gustave Choquet generalized this result from a functional analytic point of view in the following way:

Theorem 2.8. *(Choquet)*
Let K be a metrizable compact convex subset of a locally convex space. Then for each $x \in K$ there exists a probability measure μ on K that is supported by the extreme points of K such that

$$x = \int_K y\mu(dy).$$

This result can be generalized in many directions. A good overview is given in the monograph [Phe01]. We apply the idea given in the Choquet theorem in different situations throughout this thesis. To demonstrate it with a well-known example we restate the representation theorem for r-superharmonic functions given in [Sal85]:
Here K denotes the set of all positive r-superharmonic functions w.r.t. a one-dimensional regular diffusion process $(X_t)_{t\geq 0}$ on an interval I with boundary points b_l and b_r. I.e. K

consists of all lower semicontinuous functions $f : \mathbb{R} \to [0, \infty)$ such that

$$E_x(e^{-r\tau} f(X_\tau)\mathbb{1}_{\{\tau<\infty\}}) \leq f(x) \quad \text{for all } x \in I \text{ and stopping times } \tau.$$

K is obviously a convex cone. Theorem 2.7 in [Sal85] states that all extreme points are given by the functions

$$k_z(\cdot) = \min\left\{\frac{\psi_+(\cdot)}{\psi_+(z)}, \frac{\psi_-(\cdot)}{\psi_-(z)}\right\} \qquad \text{for all } z \in (b_l, b_r)$$

and $k_{b_r} = \psi_+$, $k_{b_l} = \psi_-$. Variants of the Choquet theorem for this situation yield that all r-superharmonic functions can be written in the form

$$\int_{[b_l,b_r]} k_a(\cdot)\nu(da)$$

for an appropriate measure ν (cf. [Sal85, (3.1)]). This can be used to obtain structural results about all r-superharmonic functions. In this context the extreme points are called “minimal functions” and the analogous results are referred to as “Martin-boundary theory” (see [Pin95, Chapter 7] for an introduction).

Another example for the Choquet situation was given in Section 2.1. Recall that ψ_+ and ψ_- are the “minimal” r-harmonic functions. This can be justified from the point of view of Choquet's theorem:
Consider the set K of all positive r-harmonic functions. K is obviously convex. The result for r-superharmonic functions given above imply that ψ_+ and ψ_- are extremal points of K; now it is not surprising that all other r-harmonic functions are linear combinations of ψ_+ and ψ_-.

2.4 Martingale techniques for optimal stopping

In this section we present two related approaches to the solution of optimal stopping problems. Both are based on martingale techniques; the first one was initially used in the article [Ler86] concerning the repeated significance test and was applied to other problems in sequential statistics. Later the approach was extended to other classes of problems related to mathematical finance by Beibel and Lerche in [BL97] and [BL00]; see [LU07] for an overview, further examples and references. In the following this idea will be referred to as the Beibel-Lerche approach (BL approach). The second approach can be found in [IP04] and transfers the idea of the BL approach to an additive setting. We call this idea the Beibel-Lerche-Irle-Paulsen approach (BLIP approach). Now we give the basic ideas in a general setting:

Let $(Z_t)_{t\geq 0}$ be a stochastic process. The aim is to find a stopping time τ^* such that

$$E(Z_{\tau^*}) = \sup_{\tau\in\mathcal{T}_E} E(Z_\tau),$$

where $\mathcal{T}_E$ denotes the set of all stopping time such that the expectation exists.

1. The BL approach:
   Find a positive local martingale $(M_t)_{t\geq 0}$ such that there exists $B \in [0,\infty)$ with

   - $B \geq \sup_{t\geq 0} \frac{Z_t}{M_t}$ a.s. and
   - the stopping time $\tau^* := \inf\{t \geq 0 : \frac{Z_t}{M_t} = B\}$ fulfills
   $$E(M_{\tau^*}\mathbb{1}_{\{\tau^*<\infty\}}) = E(M_0).$$

   Then for all $\tau \in \mathcal{T}_E$ we obtain

   $$\begin{aligned} E(Z_\tau \mathbb{1}_{\{\tau<\infty\}}) &= E\left(\frac{Z_\tau}{M_\tau} M_\tau \mathbb{1}_{\{\tau<\infty\}}\right) \leq BE(M_\tau \mathbb{1}_{\{\tau<\infty\}}) \\ &\leq BE(M_0), \end{aligned}$$

   where we used the optional sampling theorem for positive supermartingales in the last step (cf. [KS91, Theorem 3.22]). Furthermore we have equality for $\tau = \tau^*$, i.e. τ^* is optimal and

   $$E(Z_{\tau^*}\mathbb{1}_{\{\tau<\infty\}}) = BE(M_0).$$

   Since there is a one-to-one correspondence between positive martingales (with $M_0 = 1$) and changes of measure the approach can be seen as a change-of-measure-technique (see [LU07]).

2. The BLIP approach:
   Find a local martingale $(M_t)_{t\geq 0}$ such that there exists $B \in \mathbb{R}$ with

   - $B = \sup_{t\geq 0}(Z_t - M_t)$ a.s. and
   - the stopping time $\tau^* := \inf\{t \geq 0 : Z_t - M_t = B\}$ fulfills
   $$E(M_{\tau^*}) = E(M_0).$$

   Then for all finite stopping times $\tau \in \mathcal{T}_E$ with $E(M_\tau) \leq E(M_0)$ it holds that

   $$\begin{aligned} E(Z_\tau) &= E(Z_\tau - M_\tau) + E(M_\tau) \leq B + E(M_\tau) \\ &\leq B + E(M_0) \end{aligned}$$

   with equality for τ^*. If $\{\tau \in \mathcal{T}_E : E(M_\tau) \leq E(M_0)\}$ is rich enough (in an appropriate sense), then τ^* is optimal. Moreover

   $$\sup_{\tau\in\mathcal{T}_E} E(Z_\tau) = B + E(M_0).$$

We would like to emphasize that both approaches do not use a Markovian structure of the process.

2.4.1 Connection to the free-boundary approach

As described in the introduction most solvable optimal stopping problems in continuous time are Markovian problems with one-dimensional regular diffusions as driving processes. Here the BL approach works fine for problems with discounting, i.e. stopping problems of the form

$$v(x) = \sup_{\tau \in \mathcal{T}} E_x(e^{-r\tau} g(X_\tau) \mathbb{1}_{\{\tau < \infty\}}),$$

see [BL00]. In [IP04] problems with constant costs of observation were considered via the BLIP approach, i.e. problems of the form

$$v(x) = \sup_{\tau \in \mathcal{T}} E_x(g(X_\tau) - c\tau), \quad c > 0.$$

Another common method for solving such problems explicitly is the free-boundary approach, see the discussion in [PS06, VI.8]. The connection between free-boundary problems and the BL approach is discussed in the forthcoming article [GL09]. In the following we give a short discussion for the case of constant costs of observation, i.e. we consider the situation that X is a regular diffusion process on an open interval I with generator

$$A = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}.$$

Furthermore we assume $g : I \to \mathbb{R}$ to be continuously differentiable, $c > 0$, $x \in I$ and we would like to find an optimal stopping time τ^* for

$$v(x) = \sup_{\tau \in \mathcal{T}} E_x(g(X_\tau) - c\tau).$$

Because of the cost-structure the optimal stopping time is expected to be of two-sided type, i.e. of the form

$$\tau^* = \inf\{t \geq 0 : X_t \in \{a, b\}\} \;\text{ for some } a, b \in I,\; a < x < b.$$

The free-boundary approach suggests to find the optimal thresholds a and b together with the value function v as a solution to the following free-boundary problem:

$$Av(y) = c \quad \text{for all } y \in (a, b) \tag{2.1}$$

$$v(y) > g(y) \quad \text{for all } y \in (a, b) \tag{2.2}$$

$$v(a+) = g(a) \quad \text{and} \quad v(b-) = g(b). \tag{2.3}$$

To guarantee uniqueness of the solution further conditions are needed. To this end it is often appropriate to use the smooth fit condition, i.e.

$$v'(a+) = g'(a) \text{ and } v'(b-) = g'(b), \tag{2.4}$$

see [PS06, p. 149 ff.] for a discussion of the condition. The smooth fit condition does not seem to be very natural at the first view. Kolmogorov's argument for it to hold was that "diffusions do not like angels".

To solve (2.1)-(2.4) using the general theory of second order linear ODEs, it is natural to fix an arbitrary solution u to $Au = c$ and make the ansatz

$$v(x) = u(x) + \gamma s(x) + d \text{ for some } \gamma, d \in \mathbb{R},$$

where s denotes the scale function. This reduces the free-boundary problem to the following:
Find constants γ, d and $a < x < b$ such that

$$\begin{aligned} (u + \gamma s + d)(a) = g(a) &\text{ and } (u + \gamma s + d)(b) = g(b) \\ (u + \gamma s + d)'(a) = g'(a) &\text{ and } (u + \gamma s + d)'(b) = g'(b) \\ (u + \gamma s + d)(y) > g(y) &\quad \text{for all } y \in (a, b) \end{aligned}$$

i.e.

$$(g - u - \gamma s)(a) = d \text{ and } (g - u - \gamma s)(b) = d \tag{2.5}$$

$$(g - u - \gamma s)'(a) = 0 \text{ and } (g - u - \gamma s)'(b) = 0 \tag{2.6}$$

$$(g - u - \gamma s)(y) < d \quad \text{for all } y \in (a, b) \tag{2.7}$$

In [IP04] the BLIP approach was used with the local martingale

$$M_t^\lambda = u(X_t) + \lambda s(X_t) - ct, \quad t \geq 0$$

for some $\lambda \in \mathbb{R}$ to be determined. The BLIP approach suggests to find $\lambda \in \mathbb{R}$ and $a < x < b$ such that

$$a, b \in \arg\max(g - u - \lambda s). \tag{2.8}$$

a and b does not have to be unique. We choose a, b with minimal distance to x. Then a necessary condition for (2.8) to hold is that

$$(g - u - \lambda s)(a) = (g - u - \lambda s)(b) \tag{2.9}$$

$$(g - u - \lambda s)'(a) = 0 \text{ and } (g - u - \lambda s)'(b) = 0 \tag{2.10}$$

$$(g - u - \lambda s)(y) < (g - u - \lambda s)(a) \quad \text{for all } y \in (a, b) \tag{2.11}$$

Now note that (2.5)-(2.7) and (2.9)-(2.11) are equivalent.
Furthermore the smooth-fit condition (2.6) turns out to be a natural necessary condition for a maximum point in (2.10). But note that this is no sufficient condition. This gives rise to examples where a solution for the free-boundary problem is no solution of the optimal stopping problem:

2.4.2 Example

To see an explicit example let X be a standard Brownian motion starting in 0 and $c = 1$. The scale function of X is given by $s(x) = x$ and u given by $u(x) = x^2$ is a solution to $Au = 1$. Let $p : \mathbb{R} \to \mathbb{R}$ be any C^2-function with $p(x) = p(-x)$ for all $x \in \mathbb{R}$ such that $p|_{[0,1]}$ is strictly increasing from 0 to 1, $p|_{[1,2]}$ is decreasing, $p|_{[2,3]}$ increasing to 2 and $p|_{[3,\infty)}$ decreases with $\lim_{x\to\infty} \frac{p(x)}{x^2} = -1$. Furthermore write $g(x) = p(x) + x^2$ for all $x \in \mathbb{R}$ and consider the optimal stopping problem

$$v(x) = \sup_{\tau \in \mathcal{T}_E} E_x(g(X_\tau) - \tau) \qquad \text{for } x = 0.$$

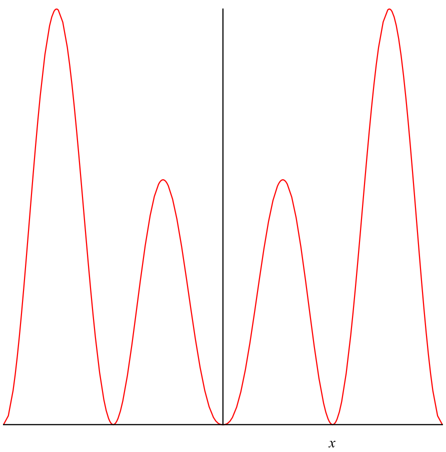

Figure 2.1: Graph of the function p

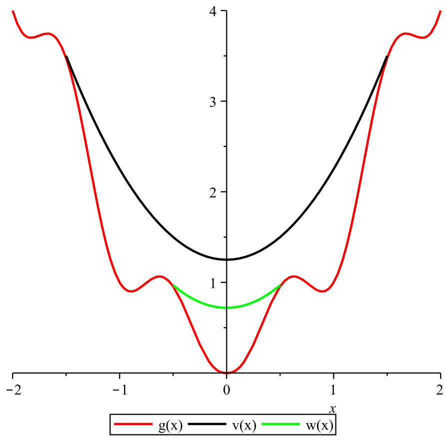

Figure 2.2: The gain function g, the value function v and the candidate w

Note that

$$g - u - 0 \cdot s = p$$

has maximum points -3 and 3 and the BLIP approach suggests to consider the stopping time

$$\tau^* = \inf\{t \geq 0 : |X_t| = 3\}$$

that gives the expected reward

$$E(g(X_{\tau^*}) - \tau^*) = E(p(X_{\tau^*})) + E(X_{\tau^*}^2 - \tau^*) = E(p(X_{\tau^*})) = p(3) = 2.$$

Let us mention that all conditions of [IP04, Theorem 3.4] are fulfilled and this stopping time is indeed optimal, i.e. $v(0) = 2$.

On the other hand write

$$-\tilde{a} := \tilde{b} := 1 \text{ and } w : (-1, 1) \to \mathbb{R}, x \mapsto x^2 + 1.$$

Then one immediately checks that $(\tilde{a}, \tilde{b}, w)$ solves the free-boundary problem (2.1) – (2.4), but

$$E(g(X_{\tilde{\tau}}) - \tilde{\tau}) = w(0) = 1 < 2 = v(0),$$

where $\tilde{\tau} = \inf\{t \geq 0 : |X_t| = 1\}$, so that it is no solution to the optimal stopping problem.

Chapter 3

An approach for optimal stopping of one-dimensional Markov processes

3.1 Motivation

We consider the problem of finding a stopping time τ^* that maximizes

$$v(x) = \sup_{\tau \in \mathcal{T}} E_x(e^{-r\tau} g(X_\tau) \mathbb{1}_{\{\tau<\infty\}}), \quad x \in E$$

where X is a strong Markov process in continuous time with state-space E. Furthermore let $\mathcal{T}$ denote the set of all stopping times, $g : E \to \mathbb{R}$ is measurable and $r \geq 0$ a constant. If $g \leq 0$, then $\tau \equiv \infty$ is optimal and the solution is trivial. Hence we assume that $\sup(g) > 0$ and that the probability of reaching a point y with $g(y) > 0$ is positive for each starting point. Then v is strictly positive.

As a motivation for our approach we summarize the BL approach – as described Section 2.4 – for this Markovian setting with discounting. Fix $x \in E$. The technique is based on finding a positive function h such that $M = (e^{-rt} h(X_t))_{t \geq 0}$ is a martingale and the function $\frac{g}{h}$ attains its maximum. After normalizing if necessary, M defines a change of measure via $M_t = \frac{dQ_x|_{\mathcal{F}_t}}{dP_x|_{\mathcal{F}_t}}$. For each stopping time τ it holds that

$$\begin{aligned} E_x(e^{-r\tau} g(X_\tau) \mathbb{1}_{\{\tau<\infty\}}) &= E_{Q_x}(\frac{g}{h}(X_\tau) \mathbb{1}_{\{\tau<\infty\}}) \\ &\leq \max_{y \in E} \frac{g}{h}(y). \end{aligned}$$

The BL approach works fine if $\tau^* = \inf\{t \geq 0 : X_t \in \arg\max(\frac{g}{h})\}$ is a.s finite under Q_x, since then we have equality and τ^* is optimal. But for many functions h one cannot expect this property to hold. Nonetheless the Beibel-Lerche approach immediately shows that, if the starting point x is a maximum point of $\frac{g}{h}$, then it is in the optimal stopping set, even if M is a supermartingale:

Lemma 3.1. *Let $x \in E$ and h a positive function such that $M = (e^{-rt}h(X_t))_{t\geq 0}$ is a supermartingale under P_x. Assume that x is a maximum point of $\frac{g}{h}$.*
Then x is in the optimal stopping set, i.e. $g(x) = v(x)$.

Proof. For all stopping times τ using the optional sampling theorem for positive supermartingales we obtain

$$\begin{aligned} E_x(e^{-r\tau}g(X_\tau)\mathbb{1}_{\{\tau<\infty\}}) &= E_x(M_\tau \frac{g}{h}(X_\tau)\mathbb{1}_{\{\tau<\infty\}}) \\ &\leq \max_{y\in E} \frac{g}{h}(y) E_x(M_\tau \mathbb{1}_{\{\tau<\infty\}}) \leq \frac{g}{h}(x)h(x) = g(x). \end{aligned}$$

□

The question arises whether each point in the optimal stopping set is a maximum point of $\frac{g}{h}$ for an appropriate function $h \in H$ in an easy to handle class H of r-superharmonic functions.
We will show that a positive answer to this question provides a very efficient way of finding optimal stopping times. In this case, the problem in fact reduces to finding maximum points of explicitly given functions. A benefit compared to the general approach of characterizing the optimal stopping set via the smallest r-superharmonic majorant is that the functions in H often have an explicit expression (e.g. a Martin boundary representation) as we will see later. Furthermore one does not have to guess the structure of the stopping set as needed for the applicability of the free-boundary approach. Compared to the martingale technique described above, a benefit is that the characterization of the optimal stopping set is global in nature and one does not have to treat each starting point x separately. Furthermore our approach makes use of the Markovian structure of the problem. All these properties will be illustrated by examples.

In this chapter we concentrate on the case that $(X_t)_{t\geq 0}$ is a one-dimensional Markov process. The structure is as follows:
In Section 3.2 we show that in the case of one-dimensional diffusion processes with discounting the idea described above is applicable. This yields an easy way to solve problems of optimal stopping explicitly under weak assumptions on the process and the gain function. This result is illustrated by some examples in Section 3.3. These first sections are extended versions of the results in [CI10].
In Section 3.4 we discuss some implications of our point of view, e.g. we give easy to handle conditions that ensure the optimal stopping set to be an interval.

Section 3.5 deals with the problem of optimal stopping with constant costs. The idea described above also works in this situation and leads to generalizations of the results in [IP04]. Examples are given in Section 3.6. We conclude the discussion of diffusion processes in Section 3.7 by describing generalizations of the previous results to random cost structures.
In the last section we illustrate that our approach works well for jump processes too. To be more precise we consider spectrally negative Lévy processes and describe situations where threshold-times are optimal and characterize the optimal threshold.

3.2 Optimal stopping with discounting

The problem of optimal stopping for one-dimensional diffusion processes with discounting was considered from different points of view in a variety of articles. Let us just mention [Muc79], [Sal85], [BL00] and [DK03]. In the recent paper [HS10] the authors solved the problem by embedding it into a linear programming problem over a space of measures.
To make things work without technical difficulties we assume g to be lower semi-continuous and $r > 0$. Let X be a regular one-dimensional diffusion process on an interval I as state space with boundary points $-\infty \leq b_l < b_r \leq \infty$ that is not killed in the interior as described in Section 2.1.
Most of the references mentioned above make use of the two minimal r-harmonic functions ψ_+ and ψ_- introduced in Section 2.1. It seems natural to apply our idea to the set H of all positive r-harmonic function, i.e. to the positive linear combinations of ψ_+ and ψ_-. This set has a simple structure, but is rich enough to characterize the optimal stopping set in the following sense:

Theorem 3.2. *A point $x \in I$ is in the optimal stopping set if and only if there exists a positive r-harmonic function h such that $x \in \arg\max \frac{g}{h}$.*

Proof. The first implication is immediate from Lemma 3.1 since $(e^{-rt}h(X_t))_{t\geq 0}$ is a positive local martingale and hence a supermartingale.
So let x be in the stopping set, i.e. $g(x) = v(x)$. Under the stated conditions it is well-known that v is an r-superharmonic function, see [Dyn63]. As discussed in Section 2.3 there exists a measure σ^v on $[b_l, b_r]$ such that

$$v(y) = \int k_z(y)\sigma^v(dz) \text{ for all } y \in I,$$

where

$$k_z(\cdot) = \min\left\{\frac{\psi_+(\cdot)}{\psi_+(z)}, \frac{\psi_-(\cdot)}{\psi_-(z)}\right\} \text{ for all } z \in (b_l, b_r)$$

and $k_{b_r} = \psi_+$, $k_{b_l} = \psi_-$.
Define

$$c = \sigma^v(\{b_r\}) + \int_{(x,b_r)} \frac{1}{\psi_+(z)} \sigma^v(dz),$$
$$d = \sigma^v(\{b_l\}) + \int_{(b_l,x]} \frac{1}{\psi_-(z)} \sigma^v(dz)$$

and $h = c\psi_+ + d\psi_-$. Then we have for all $y \in I$

$$\int_{(b_l,b_r)} \min\left\{\frac{\psi_+(y)}{\psi_+(z)}, \frac{\psi_-(y)}{\psi_-(z)}\right\}\sigma^v(dz) \leq \int_{(b_l,x]} \frac{\psi_-(y)}{\psi_-(z)}\sigma^v(dz) + \int_{(x,b_r)} \frac{\psi_+(y)}{\psi_+(z)}\sigma^v(dz).$$

We obtain

$$\begin{aligned} v(y) &= \sigma^v(\{b_r\})\psi_+(y) + \sigma^v(\{b_l\})\psi_-(y) + \int_{(b_l,b_r)} \min\left\{\frac{\psi_+(y)}{\psi_+(z)}, \frac{\psi_-(y)}{\psi_-(z)}\right\}\sigma^v(dz) \\ &\leq \sigma^v(\{b_r\})\psi_+(y) + \sigma^v(\{b_l\})\psi_-(y) + \int_{(b_l,x]} \frac{\psi_-(y)}{\psi_-(z)}\sigma^v(dz) + \int_{(x,b_r)} \frac{\psi_+(y)}{\psi_+(z)}\sigma^v(dz) \\ &= c\psi_+(y) + d\psi_-(y) = h(y), \end{aligned}$$

with equality for $y = x$. Summarizing we obtain $\frac{g}{h}(x) = \frac{v}{h}(x) = 1$ and $\frac{g}{h}(y) \leq \frac{v}{h}(y) \leq 1$ for all $y \in I$. $\square$

Remark 3.3. *If g and ψ_+, ψ_- are continuously differentiable and $x \in \mathrm{int}(I)$ we can determine the parameters c and d in the calculation explicitly. They are uniquely given by $g(x) = h(x)$ and $g'(x) = h'(x)$, i.e.*

$$c = \frac{-g(x)\psi_-'(x) + g'(x)\psi_-(x)}{w(x)} \text{ and } d = \frac{g(x)\psi_+'(x) - g'(x)\psi_+(x)}{w(x)},$$

where $w(x) = \psi_+'(x)\psi_-(x) - \psi_+(x)\psi_-'(x) \geq 0$ denotes the Wronskian. Let us emphasize that we do not obtain these conditions using the smooth fit principle but simply as necessary conditions for a maximum.

Remark 3.4. *The case $r = 0$ can be treated in an analogous way with some restrictions on the boundary behavior of X, cf. [Sal85, Theorem 2.10] for the tools to modify the proof.*

Up to now we just characterized the optimal stopping set and have no explicit form of the value function. But if we have the stopping set then one can obtain v as the solution to a usual boundary problem where it is well-known how to find solutions. In many situations it may be useful not to construct the value function and the stopping set simultaneously but to treat the problems separately.
A useful tool for the explicit determination of v is the following

Corollary 3.5. *The value function v is the (pointwise) infimum of all g-majorizing r-harmonic functions and if $v(x) < \infty$, then $v(x)$ is attained.*

Proof. Since v is the smallest r-superharmonic majorant of g it holds that $v \leq h$ for each r-harmonic h. On the other hand for each $x \in I$ the proof of Theorem 3.2 provides an r-harmonic function $h \geq v$ with $h(x) = v(x)$. □

3.3 Examples

3.3.1 One-sided boundaries – Taylor

To illustrate the applicability of the theorem we consider a classical example that goes back to Taylor (cf. [Tay68]) for $\alpha = 1$. Let $(X_t)_{t\geq 0}$ be a Brownian motion with drift μ; we treat the problem

$$\sup_{\tau \in \mathcal{T}} E_x(e^{-r\tau}((X_\tau)^+)^\alpha 1_{\{\tau<\infty\}}) \quad (\alpha \geq 1).$$

The minimal r-harmonic functions are given by $\psi_+(x) = e^{\beta_1 x}$ and $\psi_-(x) = e^{\beta_2 x}$, where $\beta_1 > 0 > \beta_2$ are the solutions to the quadratic equation $\frac{1}{2}\beta^2 + \mu\beta - r = 0$. Write $x^* = \frac{\alpha}{\beta_1}$, then $x^* \in \arg\max_{y\in\mathbb{R}} \frac{(y^+)^\alpha}{e^{\beta_1 y}}$. Hence x^* is in the optimal stopping set.

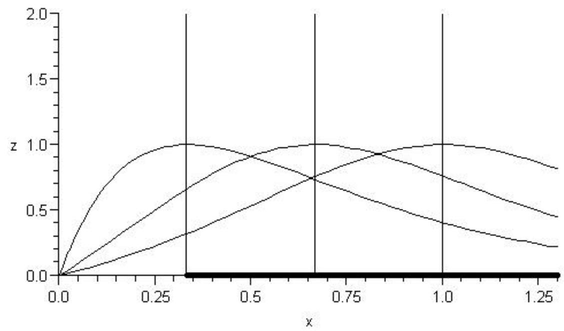

Figure 3.1: Taylor-example: Some graphs of $\frac{g}{h}$ for different h

Let $x < x^*$. Assume that x is in the stopping set, hence $x > 0$. Then there would exist $c, d \geq 0$ such that for $h = c\psi_+ + d\psi_-$ the function f given by $f(y) = \frac{(y^+)^\alpha}{h(y)}$ attains its maximum in x. As ψ_- is decreasing we necessarily have $c > 0$, and w.l.o.g. $c = 1$. Hence we would have $f'(x) = 0$, i.e. $\alpha(\psi_+(x) + d\psi_-(x)) - x(\psi_+'(x) + d\psi_-'(x)) = 0$. Therefore

$$\alpha e^{\beta_1 x} - x\beta_1 e^{\beta_1 x} = d(\beta_2 e^{\beta_2 x} - \alpha e^{\beta_2 x}).$$

Because $x < x^*$ the left hand side is positive, but the right hand side is ≤ 0; but this is a a contradiction, i.e. x is in the continuation set.

For $x > x^*$ the formulas of Remark 3.3 provide parameters $c, d > 0$ for which the r-harmonic function $h := c\psi_+ + d\psi_-$ has a critical point in x. One immediately checks that this is a maximum. Hence x is in the optimal stopping set which thus is the interval $[x^*, \infty)$. Therefore we obtain $v(x) = x^\alpha$ for $x \geq x^*$. Write $h^*(x) = qe^{\beta_1 x}$, where $q = \left(\frac{\alpha}{\beta_1 e}\right)^\alpha$ so that we get $h^*(x^*) = g(x^*)$. We claim that $v = h^*$ on $(-\infty, x^*)$. Therefore we have to show that h^* is the smallest r-harmonic majorant of g on this set.
So let $h = c\psi_+ + d\psi_-$ be any other r-harmonic function that dominates g. Because h^* also dominates g we can assume that $c \leq q$ because else we could scale h down and it still dominates g. We know that $h(x^*) \geq h^*(x^*)$ and we have that $h' - h^{*\prime} = (c-q)\psi_+' + d\psi_-' \leq 0$. We obtain $h \geq h^*$ on $(-\infty, x^*)$.
Summarizing the results gives

$$v(x) = \begin{cases} x^\alpha & \text{if } x \geq x^* = \frac{\alpha}{\beta_1} \\ qe^{\beta_1 x} & \text{else .} \end{cases}$$

Note that in this example the first entrance time into the stopping set τ_S is the optimal stopping time although $P_x(\tau_S < \infty) < 1$ for $x < x^*$ and $\mu < 0$.

We would like to remark that in this example each function $\frac{g}{h}$ has at most one critical point and this point is a maximum point. Furthermore the optimal stopping set is an interval. This is a general phenomenon – see Proposition 3.7 in the next section.
Because the dependence on x is additive for Brownian motions the interval structure of the optimal stopping set is furthermore implied by the elementary arguments in Section 6.2.1 for $\alpha \in \mathbb{N}$ (cf. Remark 6.3).

3.3.2 Two-sided boundaries – Stock with guarantee 1

Let X be a geometric Brownian motion with drift μ and volatility σ, i.e. the dynamic of X is given by

$$dX_t = X_t(\mu dt + \sigma dW_t), \quad X_0 = x > 0.$$

We consider the problem of optimal stopping of X when it is guaranteed to get at least the initial value X_0, i.e. the problem

$$\sup_\tau E_x(e^{-r\tau} \max\{X_\tau, X_0\} \mathbb{1}_{\{\tau<\infty\}}).$$

At first glance the problem does not fit into the context of this chapter because the gain functions depends on x. But this problem can easily be transformed via

$$E_x(e^{-r\tau} \max\{X_\tau, x\} \mathbb{1}_{\{\tau<\infty\}}) = xE_x\left(e^{-r\tau} \max\left\{\frac{X_\tau}{x}, 1\right\} \mathbb{1}_{\{\tau<\infty\}}\right).$$

Hence it is enough to solve the problem $\sup_\tau E_z(e^{-r\tau}\max\{X_\tau,1\}1\!\!1_{\{\tau<\infty\}})$; evaluation at $z=1$ yields the solution for the original problem.
We assume $r>\mu$; the other parameters are treated in the following subsection. The minimal r-harmonic functions are given by $\psi_+(z)=z^a$ and $\psi_-(z)=z^b$, where $a>1>0>b$ are the solutions to

$$w^2+\left(\frac{2\mu}{\sigma^2}-1\right)w-\frac{2r}{\sigma^2}=0.$$

By Theorem 3.2 the optimal stopping set is the set of maximum point of $z\mapsto\frac{\max\{z,1\}}{cz^a+dz^b}$, $c,d\geq 0$. Since multiplying by a constant does not change the maximum points it is enough to consider the maximum points of the functions f_λ defined via

$$f_\lambda(z)=f_{\lambda,1}(z)=\frac{1}{\lambda z^a+(1-\lambda)z^b}\quad\text{for } z\leq 1,$$
$$f_\lambda(z)=f_{\lambda,2}(z)=\frac{z}{\lambda z^a+(1-\lambda)z^b}\quad\text{for } z>1$$

where $0\leq\lambda\leq 1$; to this end note that the enumerator $\max\{z,1\}$ is $=1$ if $z\leq 1$ and $=z$ else. In the following we consider $f_{\lambda,1}$ and $f_{\lambda,2}$ as functions on $(0,\infty)$.
Direct computation immediately yields that the functions $f_{\lambda,1}$ resp. $f_{\lambda,2}$ have unique maximum points given by

$$z_{\lambda,1}=\left(-\frac{b(1-\lambda)}{a\lambda}\right)^{1/(a-b)}\quad\text{resp.}\quad z_{\lambda,2}=\left(\frac{(1-b)(1-\lambda)}{(a-1)\lambda}\right)^{1/(a-b)}.$$

It is obvious that $f_{\lambda,1}(z_{\lambda,1})\geq f_{\lambda,2}(z_{\lambda,2})$ implies $z_{\lambda,1}\in(0,1]$ and $z_{\lambda,1}$ is a maximum point of f_λ. On the other hand $f_{\lambda,1}(z_{\lambda,1})\leq f_{\lambda,2}(z_{\lambda,2})$ implies $z_{\lambda,2}\in[1,\infty)$ and $z_{\lambda,2}$ is a maximum point of f_λ. Note that

$$f_{\lambda,1}(z_{\lambda,1})=\lambda^{-1}\left(\frac{1-\lambda}{\lambda}\right)^{-a/(a-b)}\frac{1}{(-\frac{b}{a})^{a/(a-b)}+(-\frac{b}{a})^{b/(a-b)}}$$

and

$$f_{\lambda,2}(z_{\lambda,2})=\lambda^{-1}\left(\frac{1-\lambda}{\lambda}\right)^{-a/(a-b)}\frac{(\frac{1-b}{a-1}\frac{1-\lambda}{\lambda})^{1/(a-b)}}{(\frac{1-b}{a-1})^{a/(a-b)}+(\frac{1-b}{a-1})^{b/(a-b)}}.$$

After some algebra we obtain

$$f_{\lambda,1}(z_{\lambda,1})\geq f_{\lambda,2}(z_{\lambda,2})\ \text{ if and only if }\ \frac{1-\lambda}{\lambda}\leq\frac{a}{b}\left(\frac{a-1}{a}\right)^{1-a}\left(\frac{b}{b-1}\right)^{1-b}=:\frac{1-\lambda^*}{\lambda^*}$$

and

$$z_{\lambda^*,1}=\left(\frac{a-1}{a}\right)^{\frac{1-a}{a-b}}\left(\frac{b}{b-1}\right)^{\frac{1-b}{a-b}}\in(0,1],$$
$$z_{\lambda^*,2}=\left(\frac{a-1}{a}\right)^{-\frac{a}{a-b}}\left(\frac{b}{b-1}\right)^{-\frac{b}{a-b}}\in[1,\infty).$$

Hence the optimal stopping set is

$$S = (0, z_{\lambda^*,1}] \cup [z_{\lambda^*,1}, \infty).$$

We obtain that for each $x > 0$ the optimal stopping time for the problem of stopping with guarantee is given by

$$\tau_x = \inf\{t \geq 0 : X_t \leq xz_1 \text{ or } X_t \geq xz_2\}.$$

3.3.3 No optimal stopping time – Stock with guarantee 2

Even if optimal stopping times do not exist, Corollary 3.5 might lead to a simple solution in this case as will be illustrated in this subsection.
Again we consider the problem described in the previous subsection that was reduced to the problem

$$\sup_\tau E_z(e^{-r\tau} \max\{X_\tau, 1\} 1\!\!1_{\{\tau<\infty\}}).$$

Proposition 3.6 in the next section implies that $v \equiv \infty$ for $r < \mu$ and no optimal stopping time exists; the case $r > \mu$ was solved in the previous subsection. Hence we assume $r = \mu$, i.e. $\psi_+(z) = z$ and $\psi_-(z) = z^b$, where $b = -\frac{2r}{\sigma^2}$.
Collecting the maximum points of $\frac{g}{h}$ as in the previous subsection we find that the points $(0, z_1]$ are the optimal stopping set, where $z_1 = \frac{b}{b-1} \in (0, 1)$. Note that the optimal stopping set is one-sided and the general theory does not guarantee the first hitting time of the optimal stopping set to be an optimal stopping time. Indeed for all $z > 1$ it holds that

$$E_z(e^{-r\tau_{z_1}} g(X_{\tau_{z_1}}) 1\!\!1_{\{\tau_{z_1}<\infty\}}) \leq g(z_1) = 1 < g(z)$$

where $\tau_{z_1} = \inf\{t \geq 0 : X_t \in (0, z_1]\}$, so the first hitting time of the optimal stopping set is not optimal.
Corollary 3.5 states that v is the minimum taken over all positive r-harmonic functions $h(z) = cz + dz^b$ majorizing g. Note that an r-harmonic function majorizes $g(z) = \max\{z, 1\}$ in $[1, \infty)$ iff $c \geq 1$, i.e. v is the minimum taken over all r-harmonic functions h with $c \geq 1$ majorizing $z \mapsto 1$ in $(0, 1]$. The function h^* given by $h^*(z) = z + \frac{b}{1-b}\left(\frac{b}{b-1}\right)^b z^b$ is the only r-harmonic function majorizing g, touching g in z_1 and is asymptotic to g for $z \to \infty$. Now let h be any other r-harmonic function majorizing g. Then there exists $z_2 > 1$ such that $h(z) \geq h^*(z)$ for all $z \geq z_2$; furthermore $h(z_1) \geq g(z_1) = h^*(z_1)$. This implies

$$\begin{aligned} h(z) &= h(z_1)P_z(\tau_{z_1} < \tau_{z_2}) + h(z_2)P_z(\tau_{z_2} < \tau_{z_1}) \\ &\geq h^*(z_1)P_z(\tau_{z_1} < \tau_{z_2}) + h^*(z_2)P_z(\tau_{z_2} < \tau_{z_1}) \\ &= h^*(z) \end{aligned}$$

for all $z \in (z_1, z_2)$. Hence h^* is the minimum taken over r-harmonic function majorizing g on $[z_1, \infty)$. Since $1 > z_1$ we obtain

$$v(x) = x \sup_\tau E_1(e^{-r\tau} \max\{X_\tau, 1\} \mathbb{1}_{\{\tau<\infty\}}) = h^*(1)x,$$

where $h^*(1) = 1 + \frac{b}{1-b} \left(\frac{b}{b-1}\right)^b$.

General solutions for problems of optimal stopping with guarantee and their application to finance are investigated in Chapter 4.

3.4 Consequences of the main theorem

In this section we prove structural results that are helpful for treating several problems. Because this case is the most interesting and boundary problems are avoided we assume that $I = (b_l, b_r)$ and that both boundary points are natural. In this case

$$\lim_{x \to b_l} \psi_+(x) = \lim_{x \to b_r} \psi_-(x) = 0 \quad \text{and} \quad \lim_{x \to b_r} \psi_+(x) = \lim_{x \to b_l} \psi_-(x) = \infty,$$

cf. [BS02, p.19]. Nonetheless we would like to mention that different boundary behaviors may be handled in a similar way.

First we clarify under what kind of conditions the value function is finite (cf. [BL00, Section 3]). Corollary 3.5 provides a short proof.

Proposition 3.6. *The following conditions are equivalent:*

(i) $v \equiv \infty$.

(ii) There exists $x \in I$ *such that* $v(x) = \infty$.

(iii) $\limsup_{y \to b_l} \frac{g^+(y)}{\psi_-(y)} = \infty$ *or* $\limsup_{y \to b_r} \frac{g^+(y)}{\psi_+(y)} = \infty$.

Proof. Obviously (i) implies (ii).

Now let $x \in I$ such that $v(x) = \infty$. Corollary 3.5 states that v is the infimum taken over all positive r-harmonic functions h majorizing g, so that this set must be empty since $h(x) < \infty$ for all h. This implies that $\sup \frac{g^+}{h} = \infty$ where $h = \psi_+ + \psi_-$. Since $\frac{g^+}{h}$ is bounded on compact sets we obtain

$$\limsup_{y \to b_l} \frac{g^+}{\psi_+ + \psi_-}(y) = \infty \quad \text{or} \quad \limsup_{y \to b_r} \frac{g^+}{\psi_+ + \psi_-}(y) = \infty,$$

i.e. (with $\frac{c}{0} := \infty$ for $c > 0$)

$$\liminf_{y \to b_l} \left(\frac{\psi_+}{g^+} + \frac{\psi_-}{g^+} \right)(y) = 0 \quad \text{or} \quad \liminf_{y \to b_r} \left(\frac{\psi_+}{g^+} + \frac{\psi_-}{g^+} \right)(y) = 0,$$

hence

$$\liminf_{y\to b_l}\frac{\psi_-}{g^+}(y)=0 \quad\text{or}\quad \liminf_{y\to b_r}\frac{\psi_+}{g^+}(y)=0$$

and this implies (iii).
Now let (iii) hold. W.l.o.g we can assume $\limsup_{y\to b_l}\frac{g^+(y)}{\psi_-(y)}=\infty$. Let $h=c\psi_++d\psi_-$ be any positive r-harmonic function. Note that $\psi_+(y)\le\psi_-(y)$ for y near b_l. We obtain

$$\limsup_{y\to b_l}\frac{g^+}{c\psi_++d\psi_-}(y)\ge\limsup_{y\to b_l}\frac{g^+}{(c+d)\psi_-}(y)=\infty,$$

i.e. no positive r-harmonic function majorizes g and Corollary 3.5 implies (i). $\square$

In many applications it is helpful to have elementary conditions that guarantee an easy geometric shape of the stopping set. For example it is good to have conditions at hand to check that the optimal stopping set is an interval. This kind of conditions will be discussed in Section 6.2 for autoregressive processes. The approach of this chapter gives rise to results in this direction for diffusion processes.

Proposition 3.7. *Let g,ψ_+,ψ_- be continuously differentiable such that for each $\lambda\in(0,1)$ the function $\frac{g}{\lambda\psi_++(1-\lambda)\psi_-}$ has a unique critical point x_λ and this is a maximum point. Then the optimal stopping S set is an interval.*

Proof. Write $f_\lambda=\frac{g}{\lambda\psi_++(1-\lambda)\psi_-}$ for all $\lambda\in[0,1]$. The implicit function theorem yields that the function $(0,1)\to I,\lambda\mapsto x_\lambda$ is continuous, hence the set $\tilde S:=\{x_\lambda:\lambda\in(0,1)\}$ is connected, i.e. it is an interval. We have the pointwise convergence

$$f_\lambda\to f_1 \text{ for } \lambda\to1 \quad\text{and}\quad f_\lambda\to f_0 \text{ for } \lambda\to0.$$

First we note that the set of maximum points of $\frac{g}{\psi_+}$ resp. $\frac{g}{\psi_-}$ is connected:
Let $x<z<y$. Since for each $\lambda\in(0,1)$ f_λ has a unique critical point x_λ and this is a maximum point it holds that $f_\lambda(z)\ge f_\lambda(x)\wedge f_\lambda(y)$. If $x<y$ are maximum points of $\frac{g}{\psi_+}$, then we have $f_1(z)\ge f_1(x)\wedge f_1(y)=\max f_1$ and the same argument holds for f_0.
Furthermore if the set $\arg\max\frac{g}{\psi_+}$ resp. $\arg\max\frac{g}{\psi_-}$ is not empty, then one boundary point coincides with a boundary point of $\tilde S$: If x is a maximum points of $\frac{g}{\psi_+}$, $x\notin\tilde S$ and z is on the connecting line of z and $\tilde S$ it holds that $f_\lambda(x)\le f_\lambda(z)$ for all $\lambda\in(0,1)$ and pointwise convergence yields $f_1(x)\le f_1(z)$. The same argument holds for f_0.
We can conclude that the set $\tilde S\cup\arg\max\frac{g}{\psi_+}\cup\arg\max\frac{g}{\psi_-}$ is connected and this set is the optimal stopping set by Theorem 3.2. $\square$

To end this section we give an easy condition for the optimality of possibly two-sided stopping times. This condition will be needed in Chapter 4.

Proposition 3.8. *Let $x \in I$ and assume there exist $y_1 \le x \le y_2$ and $\lambda_1, \lambda_2 \in [0,1]$ such that*

$$y_i = \arg\max \frac{g}{\lambda_i \psi_+ + (1-\lambda_i)\psi_-} \qquad \text{for } i = 1,2.$$

Then there exist $x_1 \le x \le x_2$ such that $v(x) = E_x(e^{-r\tau} g(X_\tau))$, where

$$\tau = \inf\{t \ge 0 : X_t \le x_1 \text{ or } X_t \ge x_2\}.$$

Proof. By Theorem 3.2 y_1 and y_2 are in the stopping set S. Hence

$$x_1 := \sup\{y \in S : y \le x\} \text{ and } x_2 := \inf\{y \in S : y \ge x\}$$

are in S too, i.e. $\tau_S = \tau$ under P_x and since the interval $[x_1, x_2]$ is compact the assertion holds. □

3.5 Optimal stopping with constant costs of observation

Problems of optimal stopping with discounting – as discussed in the previous sections – often arise in finance. In sequential analysis another cost structure is natural, namely linear costs of observation. I.e. problems of the form

$$v(x) = \sup_{\tau \in \mathcal{T}_E} E_x(g(X_\tau) - c\tau)$$

are of interest, starting with the work of Wald and Wolfowitz [WW50]. Furthermore this kind of problem arises in portfolio optimization, see [MP95].
Here $g : I \to \mathbb{R}$ is continuous and bounded from below, $c > 0$ is constant, $\mathcal{T}_E$ denotes the set of all finite stopping times such that the expectation exists and $(X_t)_{t\ge0}$ is a regular diffusion process on an open interval I with generator

$$A = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}$$

for continuous $\sigma > 0$ and μ. We fix a point $a \in I$ and take the scale function s with normalization $s(a) = 0$, $s'(a) = 1$. Furthermore we write

$$u(x) = c\int_a^x s'(y)\int_a^y \frac{2}{s'(z)\sigma^2(z)}dzdy \quad \text{for all } x \in E,$$

so that u is the unique solution to

$$Af = c, \quad f(a) = f'(a) = 0.$$

As described in Section 2.4 the BLIP approach suggests to fix a point x and find a function h such that

$$E_x(h(X_\tau) - c\tau) = h(x) \quad \text{for a wide class of stopping times } \tau$$

and

$$g - h \text{ has two maximum points } y_l < x < y_r.$$

On the other hand if x is a maximum point of $g - h$, then it holds that

$$\begin{aligned} E_x(g(X_\tau) - c\tau) &= E_x(g(X_\tau) - h(X_\tau)) + E_x(h(X_\tau) - c\tau) \\ &= E_x(g(X_\tau) - h(X_\tau)) + h(x) \\ &\leq \sup(g - h) + h(x) \\ &= g(x) \end{aligned}$$

and if the class of stopping times is rich enough, then x is in the optimal stopping set. Following the analogous idea to that described in Section 3.1 we can hope that each point in the optimal stopping set arises as a maximum point of $g - h$ for some appropriate h. This leads to an elegant solution under minimal assumptions:
As discussed in [IP04] the following assumption is natural to guarantee finiteness of the solution:

$$\text{For each } x \in I \text{ there exists } \epsilon > 0 \text{ such that } E_x(\sup_{t\geq 0}(g(X_t) - (c - \epsilon)t)) < \infty. \tag{3.1}$$

The following Proposition is taken from [IP04] (cf. 2.1 there). In the following we call a stopping time τ regular if there exists a compact interval J such that $\tau \leq \tau_J$, where $\tau_J = \inf\{t \geq 0 : X_t \notin J\}$.

Proposition 3.9. *Assume (3.1).*
Then $v(x) < \infty$ for all $x \in I$ and can be obtained by maximizing over all regular stopping times.
Furthermore the first entrance time τ_S into the optimal stopping set is optimal.

We need the following

Lemma 3.10. *Assume (3.1).*
Let $c < x < d \in I$ and $\sigma = \inf\{t \geq 0 : X_t \in \{c, d\}\}$. Then

$$v(x) \geq E_x(v(X_\sigma) - c\sigma).$$

Proof. This is a standard fact for the value function from the general theory, but one has to be a bit careful since the function $k(x,t) = g(x) - ct$ is not bounded below. But nonetheless using the strong Markov property we obtain

$$\begin{aligned} E_x(v(X_\sigma) - c\sigma) &= E_x(E_{X_\sigma}(g(X_{\tau_S}) - c\tau_S) - c\sigma) = E_x(E_x((g(X_{\tau_S}) - c\tau_S) \circ \theta_\sigma | \mathcal{F}_\sigma) - c\sigma) \\ &= E_x(g(X_{\sigma + \tau_S \circ \theta_\sigma}) - c(\sigma + \tau_S \circ \theta_\sigma)) \leq \sup_{\tau \in \mathcal{T}_E} E_x(g(X_\tau) - c\tau) \\ &= v(x), \end{aligned}$$

where θ denotes the shift operator. □

Proposition 3.9 shows that determining the optimal stopping set solves the problem. To this end we define the analogon to r-harmonic functions in the situation of constant costs:

$$H := \{h_\lambda : \lambda \in \mathbb{R}\}, \quad \text{where} \quad h_\lambda = u + \lambda s.$$

In this situation the idea of our approach also works:

Theorem 3.11. *Assume (3.1).*
A point $x \in I$ is in the optimal stopping set if and only if there exists $h \in H$ such that $x \in \arg\max(g - h)$.

Proof. First let x be a maximum point of $g - h$ for some $h \in H$. For all regular stopping times τ it holds that

$$\begin{aligned} E_x(g(X_\tau) - c\tau) &= E_x(g(X_\tau) - h(X_\tau)) + E_x(h(X_\tau) - c\tau) \\ &= E_x(g(X_\tau) - h(X_\tau)) + h(x) \leq \sup(g - h) + h(x) = g(x) \end{aligned}$$

using Dynkin's identity and optional sampling. Now Proposition 3.9 leads to $v(x) \leq g(x)$. For the other implication let $x \in S$, i.e. $g(x) = v(x)$. Write

$$\sigma_{c,d} = \tau_c \wedge \tau_d, \text{ where } \tau_c = \inf\{t \geq 0 : X_t = c\}, \quad \tau_d = \inf\{t \geq 0 : X_t = d\}.$$

Using Lemma 3.10 for all $c < y < d \in I$ it holds that

$$\begin{aligned} v(y) \geq E_y(v(X_{\sigma_{c,d}}) - c\sigma_{c,d}) &= E_y(v(X_{\sigma_{c,d}}) - u(X_{\sigma_{c,d}})) + E_y(u(X_{\sigma_{c,d}}) - c\sigma_{c,d}) \\ &= (v(c) - u(c))P_y(\tau_c < \tau_d) + (v(d) - u(d))P_y(\tau_d < \tau_c) + u(y). \end{aligned}$$

Using the definition of the scale function

$$v(y) - u(y) \geq (v(c) - u(c))\frac{s(d) - s(y)}{s(d) - s(c)} + (v(d) - u(d))\frac{s(y) - s(c)}{s(d) - s(c)},$$

i.e. – noting that s is strictly increasing – $(v-u)\circ s^{-1}$ is concave. Now take any affine function $y \mapsto \lambda y + d$ that is tangent to $(v-u)\circ s^{-1}$ in $s(x)$. This implies

$$v - u \leq \lambda s + b \quad \text{and} \quad (v-u)(x) = \lambda s(x) + b.$$

Since $g(x) = v(x)$ and $g \leq v$ this leads to

$$g - h \leq b \quad \text{and} \quad (g-h)(x) = b,$$

where $h := u + \lambda s$, i.e. x is a maximum point of $g - h$.

□

In [IP04] assumptions on the boundary behavior of $\frac{s(x)}{u(x)}$ and $\frac{g(x)}{u(x)}$ were necessary. Furthermore g was assumed to be twice continuously differentiable (except perhaps in one special point). As seen above these assumptions can be relaxed naturally by considering the problem from our point of view.
Furthermore we can again expect global solutions, i.e. we do not need to consider each starting point separately.

3.6 Examples

3.6.1 Wald's type optimal stopping for a Wiener process

Let $(X_t)_{t\geq 0}$ be a standard Brownian motion and let the gain function be given by $g(x) = |x|^p$ for some $p \in (0,2)$. This problem was studied in [GP97]. In this case the scale function s and the function u (w.r.t $a = 0$) fulfill

$$s(x) = x, \quad u(x) = cx^2.$$

Now write

$$f_\lambda(x) = |x|^p - cx^2 - \lambda x \qquad \text{for } \lambda, x \in \mathbb{R}.$$

One immediately sees that f_0 has maximum points $-x_{0,1} = x_{0,2} = \left(\frac{p}{2c}\right)^{\frac{1}{2-p}}$ and all other function f_λ have unique maximum points x_λ with $x_\lambda < x_{0,1}$ for $\lambda > 0$ and $x_\lambda > x_{0,2}$ for $\lambda < 0$. Furthermore $x_\lambda \to -\infty$ for $\lambda \to \infty$ and $x_\lambda \to \infty$ for $\lambda \to -\infty$. By the intermediate value theorem the optimal stopping set is given

$$S = \left(-\infty, -\left(\frac{p}{2c}\right)^{\frac{1}{2-p}}\right] \cup \left[\left(\frac{p}{2c}\right)^{\frac{1}{2-p}}, \infty\right).$$

3.6.2 Portfolio optimization model by Morton and Pliska

In [MP95] a portfolio optimization problem under constant costs of observation was considered. This led to problems of optimal stopping for a diffusion on $(0, 1)$ with generator given by

$$A = \frac{1}{2}x^2(1-x)^2\frac{d^2}{dx^2} + x(1-x)(\frac{1}{2} - x)\frac{d}{dx}.$$

We consider the gain function $g(x) = |\log(\frac{x}{1-x})|$.
The scale function and the function u are

$$s(x) = \log\left(\frac{x}{1-x}\right), \qquad u(x) = \left(\log\frac{x}{1-x}\right)^2$$

and we again define the family of functions

$$f_\lambda = g - u - \lambda s, \qquad \lambda \in \mathbb{R}.$$

By standard calculus we obtain that the maximum points of f_λ are

$$x_\lambda = \frac{\exp(\frac{1-\lambda}{2c})}{1+\exp(\frac{1-\lambda}{2c})} \quad \text{for } \lambda < 0 \quad \text{and} \quad x_\lambda = \frac{\exp(\frac{-1-\lambda}{2c})}{1+\exp(\frac{-1-\lambda}{2c})} \quad \text{for } \lambda > 0,$$

as well as the two points

$$x_{0,1} = \frac{\exp(\frac{1}{2c})}{1+\exp(\frac{1}{2c})} \quad \text{and} \quad x_{0,2} = \frac{\exp(\frac{-1}{2c})}{1+\exp(\frac{-1}{2c})} \quad \text{for } \lambda = 0,$$

so that the optimal stopping set is

$$S = \left(0, \frac{\exp(\frac{-1}{2c})}{1+\exp(\frac{-1}{2c})}\right] \cup \left[\frac{\exp(\frac{1}{2c})}{1+\exp(\frac{1}{2c})}, 1\right).$$

3.7 Some remarks on random cost structure

In [BL00] and [Day08] a more general optimal stopping problem is considered, namely

$$v(x) = \sup_{\tau\in\mathcal{T}} E_x(e^{-A_\tau}g(X_\tau)\mathbb{1}_{\{\tau<\infty\}}), \quad x \in I,$$

where $(A_t)_{t\geq 0}$ is a non-negative continuous additive functional of the regular one-dimensional diffusion $(X_t)_{t\geq 0}$, i.e. $(A_t)_{t\geq 0}$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted, continuous, non decreasing, $A_0 = 0$ and $A_{s+t} = A_s + A_t \circ \theta_s$ for all $t, s \geq 0$. Here θ denotes the shift-operator.
Let $\sigma.$ denote the generalized inverse of A, i.e.

$$\sigma_t := \inf\{s \geq 0 : A_s > t\} \quad \text{for all } t \geq 0.$$

The process given by

$$\tilde{X}_t = X_{\sigma_t}, \quad \tilde{\mathcal{F}}_t = \mathcal{F}_{\sigma_t} \quad \text{for all } t \geq 0.$$

induces a strong Markov process as explained in [RW94, III.21]. For stochastic optimization the additive functional given by $A_t = \int_0^t f(X_s)ds$ is of special interest, where $f : I \to (0, \infty)$ is assumed to be continuous and bounded away from 0. We assume this structure in the following. If $(X_t)_{t\geq 0}$ is generated by a second order differential operator, then so is the time-changed process $(\tilde{X}_t)_{t\geq 0}$, see again [RW94, III.21.4].

In the following Proposition the problem with random discounting is reduced to the problem we considered before.

Proposition 3.12. *For all $x \in I$ and $g : I \to \mathbb{R}$ continuous it holds that*

$$\sup_{\tau\in\mathcal{T}} E_x(e^{-A_\tau} g(X_\tau)\mathbb{1}_{\{\tau<\infty\}}) = \sup_{\tau\in\tilde{\mathcal{T}}} E_x(e^{-\tau} g(\tilde{X}_\tau)\mathbb{1}_{\{\tau<\infty\}}),$$

and

$$\sup_{\tau\in\mathcal{T}_E} E_x(g(X_\tau) - A_\tau) = \sup_{\tau\in\tilde{\mathcal{T}}_E} E_x(g(\tilde{X}_\tau) - \tau),$$

where $\tilde{\mathcal{T}}$ denotes the stopping times with respect to $(\tilde{\mathcal{F}}_t)_{t\geq 0}$.

Proof. Since both arguments are the same, we only prove the second statement.
For each $(\tilde{\mathcal{F}}_t)_{t\geq 0}$-stopping time $\tilde{\tau}$ the random variable $\sigma_{\tilde{\tau}}$ is an $(\mathcal{F}_t)_{t\geq 0}$-stopping time and on the other hand for each $(\mathcal{F}_t)_{t\geq 0}$-stopping time τ the random variable A_τ is an $(\tilde{\mathcal{F}}_t)_{t\geq 0}$-stopping time, cf. [RW94, III.21].
Let $\tau_0 \in \mathcal{T}_E$, then $\tilde{\tau}_0 := A_{\tau_0} \in \tilde{\mathcal{T}}$ and we obtain

$$E_x(g(X_{\tau_0}) - A_{\tau_0}) = E_x(g(X_{\sigma_{\tilde{\tau}_0}}) - \tilde{\tau}_0) = E_x(g(\tilde{X}_{\tilde{\tau}_0}) - \tilde{\tau}_0),$$

i.e. the left hand side of the assertion is smaller than or equal the right hand side. Now let $\tilde{\tau}_0 \in \tilde{\mathcal{T}}_E$, then $\tau_0 := \sigma_{\tau_0} \in \mathcal{T}$ and we obtain

$$E_x(g(\tilde{X}_{\tilde{\tau}_0}) - \tilde{\tau}_0) = E_x(g(X_{\tau_0}) - A_{\tau_0})$$

and we obtain the converse inequality. $\square$

To illustrate the time-change method let us consider the Wald problem with state dependent costs of the form

$$v(x) = \sup_{\tau\in\mathcal{T}_E} E_x(|X_\tau|^p - \int_0^\tau (X_s^2 + 1)ds), \quad x \in \mathbb{R},$$

where X denotes a standard Brownian motion. For simplicity we consider $p = 3$. Note that this parameter leads to a trivial solution in the problem with constant costs. The time changed process $\tilde{X}$ has a generator $\tilde{A}$ given by

$$\tilde{A}w(x) = \frac{1}{2(x^2+1)} w''(x),$$

see [RW94, III.21.4, Formula (21.6)(i)]. We are faced with the problem

$$v(x) = \sup_{\tau \in \tilde{\mathcal{T}}_E} E_x(|\tilde{X}_\tau|^p - \tau), \quad x \in \mathbb{R},$$

that can be solved using Theorem 3.11. One immediately checks that the functions h_λ are given by

$$h_\lambda(x) = -\frac{1}{6}x^4 + x^2 + \lambda x, \quad \lambda \in \mathbb{R},$$

and the optimal stopping set is the union of the maximum points of $x \mapsto |x|^3 - h_\lambda(x)$. For $\lambda = 0$ these are given by

$$x_1^* = -\frac{9+\sqrt{33}}{4} \quad \text{and} \quad x_2^* = \frac{9+\sqrt{33}}{4}.$$

All other functions have unique maximum points x_λ with $x_\lambda < x_1^*$ for $\lambda > 0$ and $x_\lambda > x_2^*$ for $\lambda < 0$. Furthermore $x_\lambda \to -\infty$ for $\lambda \to \infty$ and $x_\lambda \to \infty$ for $\lambda \to -\infty$. By the intermediate value theorem the optimal stopping set is

$$S = (-\infty, x_1^*] \;\cup\; [x_2^*, \infty)\,.$$

3.8 Optimal stopping of spectrally negative Lévy processes

In the previous sections we considered one-dimensional Markov processes with continuous paths. Now we change our focus to jump processes. As described in Section 2.2 an important subclass of Lévy processes is the class of spectrally negative processes and we study the problem

$$v(x) = \sup_{\tau \in \mathcal{T}} E_x(e^{-r\tau} g(X_\tau) \mathbb{1}_{\{\tau<\infty\}}), \quad x \in \mathbb{R},\ r > 0$$

for these processes using our approach for a wide set of gain functions that includes the Novikov-Shiryaev problem – $g(x) = (x^+)^\alpha$ – for example.

To introduce an easy to handle set of functions we use the Laplace exponent ψ as introduced in Section 2.2. Because of convexity there exists a unique $\theta > 0$ such that $\psi(\theta) = r$. For each $\lambda \in \mathbb{R}$ we consider

$$h_\lambda : \mathbb{R} \to \mathbb{R}, x \mapsto e^{\lambda\theta x}$$

and write

$$H := \{h_\lambda : \lambda \in [0,1]\}.$$

For our considerations we need the following elementary

Lemma 3.13. *For all $\lambda \in [0,1]$ the process $M^\lambda = (e^{-rt}h_\lambda(X_t))_{t\geq 0}$ is a positive supermartingale under each measure P_x. Furthermore M^1 is a martingale.*

Proof. Since $(X_t)_{t\geq 0}$ has no upward jumps it has arbitrary positive exponential moments, so that M_t^λ is integrable for each $t \geq 0$ and we obtain by independence of the increments

$$\begin{aligned} E_x(M_t|\mathcal{F}_s) &= e^{\lambda\theta X_s - rs}E_x(e^{\lambda\theta(X_t-X_s)}|\mathcal{F}_s)e^{-r(t-s)} \\ &= M_s e^{(\psi(\lambda\theta)-\psi(\theta))(t-s)} \leq M_s \end{aligned}$$

for each $x \in \mathbb{R}$, $0 \leq s \leq t$. For the last inequality we used that $\psi(\lambda\theta) \leq \psi(\theta)$ by convexity of ψ. □

By Lemma 3.1 we obtain that the set

$$\tilde{S} = \left\{x \in \mathbb{R} : \text{ There exists } h \in H \text{ such that } x \in \arg\max \frac{g}{h}\right\}$$

is a subset of the optimal stopping set S. To obtain a condition for equality we use an argument inspired by the Beibel-Lerche approach:

Theorem 3.14. *Assume $\frac{g}{h_1}$ has a maximum point x_0 and $\tilde{S}$ is an interval of the form $\tilde{S} = [y, \infty)$.*
Then the stopping time

$$\tau^* = \inf\{t \geq 0 : X_t \geq x_0\}$$

is optimal and

$$v(x) = \begin{cases} g(x) & \text{if } x \geq x_0, \\ \frac{g(x_0)}{h_1(x_0)}h_1(x) & \text{otherwise }. \end{cases}$$

Proof. As $x_0 \in \tilde{S}$ it holds that $x \in \tilde{S}$ for all $x \geq x_0$ by assumption on $\tilde{S}$. This proves the claim for all $x \geq x_0$. Now let $x < x_0$. We define the change of measure by

$$\frac{dQ_x|_{\mathcal{F}_t}}{dP_x|_{\mathcal{F}_t}} = \frac{1}{h_1(x)}e^{-rt}h_1(X_t)$$

and remark that $(X_t)_{t\geq 0}$ is a spectrally negative Lévy process that drifts to ∞ under Q_x (see [Kyp06, p. 213–214]). For each stopping time τ we obtain

$$\begin{aligned} E_x(e^{-r\tau}g(X_\tau)\mathbb{1}_{\{\tau<\infty\}}) &= h_1(x)E_{Q_x}(\frac{g}{h_1}(X_\tau)\mathbb{1}_{\{\tau<\infty\}}) \\ &\leq h_1(x)\frac{g}{h_1}(x_0)Q_x(\tau < \infty) \leq \frac{g}{h_1}(x_0)h_1(x). \end{aligned}$$

Since $Q_x(\tau^* < \infty) = 1$ and $\frac{g}{h_1}(X_{\tau^*}) = \frac{g}{h_1}(x_0)$ under Q_x, the stopping time τ^* is optimal and the value function is given as above. $\square$

To illustrate the previous result we generalize the results of Subsection 3.3 to spectrally negative Lévy processes, i.e. we consider $g(x) = (x^+)^\alpha$ for some $\alpha > 0$. Then the function $\frac{g}{h_\lambda}(x) = \frac{x^\alpha}{e^{\lambda\theta x}}$, $x \geq 0$ has a unique maximum point at $\frac{\alpha}{\lambda\theta}$ so that

$$\tilde{S} = [\frac{\alpha}{\theta}, \infty)$$

and $\tau^* = \inf\{t \geq 0 : X_t \geq \frac{\alpha}{\theta}\}$ is optimal. This is a special case of the general solution established in [NS07], but our result does not rely on the special structure of g.

Observation 3.15. *Note that in this example the smooth fit condition* $v'(x_0) = g'(x_0)$ *holds. This is not surprising since for all spectrally negative Lévy processes* 0 *is regular for* $(0, \infty)$ *(see [Kyp06, Theorem 6.5]). For a detailed discussion of this phenomenon we refer to [PS06, IV, 9.1 and 9.2] and to [CI09] for the special case of Lévy processes.*

Chapter 4

American options with guarantee

4.1 Introduction

Options with guarantee provide a safety belt against a substantial loss of fortune. Particularly in highly volatile markets the usage of these options can become reasonable for risk averse investors, since they guarantee a payoff that is a fraction of the starting price. I.e. we consider American options with payoff $g(X_\tau) \vee k(X_0)$; here $(X_t)_{t\geq 0}$ is the stock price process, $k \leq g$ are increasing functions, τ is the (random) time to exercise the option and $\vee$ denotes the maximum. To our knowledge such American options have not been treated in the literature. This problem is connected to the solution to the optimal stopping problem

$$v(x) = \sup_{\tau \in \mathcal{T}} E_x(e^{-r\tau}(g(X_\tau) \vee k(x))1\!\!1_{\{\tau<\infty\}}),$$

where r is the discounting factor. This motivates to consider such problems for a one-dimensional Markov process $(X_t)_{t\geq 0}$. But the optimal stopping theory for one-dimensional processes – as described for diffusions in the previous section – does not apply immediately since the payoff depends on the starting point. One possibility is to embed the problem into the two-dimensional problem

$$v(x,y) = \sup_{\tau \in \mathcal{T}} E_{(x,y)}(e^{-r\tau}(g(X_\tau) \vee k(Y_\tau))1\!\!1_{\{\tau<\infty\}}),$$

where $Y_t \equiv y$; from this point of view, structural results can be obtained but explicit calculations seem to be hard to handle.

Before starting let us establish a heuristic on the structure of the optimal stopping rule: In many situations optimal stopping rules for problems with discounting and increasing gain function have the form $\tau^* = \inf\{t \geq 0 : X_t \geq a\}$ for some a. Such kind of strategies does not seem to be appropriate in our setting: Assume that you want to stop at a reasonable high level a and you are far below it, then you have the option to stop and accept the guarantee. Therefore it seems to be better to consider stopping rules that are two-sided.

In this chapter we prove that this intuition is correct in many situations and give formulas for an explicit computation of these optimal values. In Section 4.2 this is established for the case that $(X_t)_{t\geq 0}$ is a one-dimensional diffusion process by using the theory developed in the previous chapter. The explicit computation of these optimal values and the value function v for all starting points x at once leads to a system of two coupled first-order ODEs. We derive this equation and illustrate all results by an example in Section 4.3. Afterwards we assume $(X_t)_{t\geq 0}$ to be a Lévy process in Section 4.4. In this case overshoot plays a fundamental role. For the special case $g(x) = (x-K)^+$ we demonstrate how results about the shape of the optimal stopping set can be established for general Lévy processes. Here explicit results can not be expected. Therefore we restrict our interest to spectrally negative Lévy processes and prove that the optimal strategies are also two-sided for arbitrary gain functions g. We again establish two first-order ODEs for the solution.

4.2 Diffusion processes

Let $(X_t)_{t\geq 0}$ be a regular diffusion on an open interval $I = (b_l, b_r)$ as introduced in Section 2.1 and $r > 0$. For convenience we assume that the boundary points are natural, so that

$$\lim_{x\to b_l} \psi_+(x) = \lim_{x\to b_r} \psi_-(x) = 0 \quad \text{and} \quad \lim_{x\to b_r} \psi_+(x) = \lim_{x\to b_l} \psi_-(x) = \infty.$$

Let g, k be as in the introduction, i.e. g, k increasing, $g \geq k$ and we assume $g : I \to \mathbb{R}_{\geq 0}$ to be continuous and to avoid trivialities $\sup k > 0$. We consider the optimization problem

$$v(x) = \sup_{\tau\in\mathcal{T}} E_x(e^{-r\tau}(g(X_\tau) \vee k(x))\mathbb{1}_{\{\tau<\infty\}}), \quad x \in I,$$

i.e. for each $x \in I$ we try to find a stopping time τ_x^* such that

$$v(x) = E_x(e^{-r\tau_x^*}(g(X_{\tau_x^*}) \vee k(x))\mathbb{1}_{\{\tau_x^*<\infty\}}).$$

In this setting we see that the optimal strategies are indeed of two-sided type:

Theorem 4.1. *The following two assertions hold true:*

(i) $v \equiv \infty$ *if and only if* $\limsup_{y\to b_r} \frac{g(y)}{\psi_+(y)} = \infty$.

(ii) Assume $\lim_{y\to b_r} \frac{g(y)}{\psi_+(y)} = 0$. *Then for each* $x \in I$ *with* $k(x) > 0$ *there exist two constants* $a_x, b_x \in I$ *with* $a_x \leq x \leq b_x$ *such that*

$$v(x) = E_x(e^{-r\tau_x^*}(g(X_{\tau_x^*}) \vee k(x))\mathbb{1}_{\{\tau_x^*<\infty\}}),$$

where $\tau_x^* = \inf\{t \geq 0 : X_t = a_x \text{ or } X_t = b_x\}$.

Proof. For fixed $x \in I$ we consider the problem

$$v_x(y) = \sup_{\tau \in \mathcal{T}} E_y(e^{-r\tau}(g(X_\tau) \vee k(x))\mathbb{1}_{\{\tau<\infty\}}), \qquad y \in I, \tag{OSP_x}$$

so that $v(x) = v_x(x)$ and use the theory developed in Chapter 3.

(i) Since

$$\limsup_{y \to b_l} \frac{g(y) \vee k(x)}{\psi_-(y)} \leq g(x) \limsup_{y \to b_l} \frac{1}{\psi_-(y)} = 0 < \infty$$

the assertion holds by Proposition 3.6.

(ii) Using the assumption on the boundary behavior we observe that

$$\sup_{y \leq x} \frac{g \vee k(x)}{\psi_+}(y) \geq k(x) \sup_{y \leq x} \frac{1}{\psi_+}(y) = \infty > \sup_{y \geq x} \frac{g(y)}{\psi_+(y)} = \sup_{y \geq x} \frac{g \vee k(x)}{\psi_+}(y) \tag{4.1}$$

and

$$\sup_{y \geq x} \frac{g \vee k(x)}{\psi_-}(y) = \sup_{y \geq x} \frac{g(y)}{\psi_-(y)} > \frac{g(x)}{\psi_-(x)} = \sup_{y \leq x} \frac{g(y)}{\psi_-(y)}. \tag{4.2}$$

The functions

$$\lambda \mapsto \inf_{y \leq x} \frac{\lambda\psi_+(y) + (1-\lambda)\psi_-(y)}{g(y) \vee k(x)} \quad \text{and} \quad \lambda \mapsto \inf_{y \geq x} \frac{\lambda\psi_+(y) + (1-\lambda)\psi_-(y)}{g(y) \vee k(x)}$$

are continuous as infima taken over linear functions. Using the equations (4.1) and (4.2) we obtain that for λ near 1

$$\sup_{y \leq x} \frac{g(y) \vee k(x)}{\lambda\psi_+(y) + (1-\lambda)\psi_-(y)} > \sup_{y \geq x} \frac{g(y) \vee k(x)}{\lambda\psi_+(y) + (1-\lambda)\psi_-(y)}$$

and for λ near 0

$$\sup_{y \leq x} \frac{g(y) \vee k(x)}{\lambda\psi_+(y) + (1-\lambda)\psi_-(y)} < \sup_{y \geq x} \frac{g(y) \vee k(x)}{\lambda\psi_+(y) + (1-\lambda)\psi_-(y)}.$$

Therefore by assumption on the boundary behavior there exist $\tilde{a}_x \leq x \leq \tilde{b}_x$ and λ_1, λ_2 such that

$$\tilde{a}_x = \arg\max_{y \in I} \frac{g(y) \vee k(x)}{\lambda_1\psi_+(y) + (1-\lambda_1)\psi_-(y)} \text{ and } \tilde{b}_x = \arg\max_{y \in I} \frac{g(y) \vee k(x)}{\lambda_2\psi_+(y) + (1-\lambda_2)\psi_-(y)}.$$

The assertions follows from Proposition 3.8. □

Remark 4.2. *(i) The condition* $\lim_{y \to b_r} \frac{g(y)}{\psi_+(y)} = 0$ *in (ii) may be relaxed by a slight refinement of the proof. But example 3.3.3 shows that it cannot be omitted at all.*

(ii) If $k(x) \leq 0$, then by monotonicity $k(y) \leq 0$ for all $y \leq x$. For all these values the optimal stopping problem reduces to an ordinary optimal stopping problem since no guarantee occurs.

Theorem 4.1 states that for the question of optimal stopping with guarantee it is optimal to choose boundaries depending on the starting point, and to stop at reaching these boundaries. This provides a way to solve the optimization problem (OSP_x) for each fixed x. In ordinary problems of optimal stopping for diffusions (without an explicit starting point dependence for the gain) the Markov property gives rise to an explicit solution for all starting points in a connected component by the knowledge of an optimal stopping time for one fixed x via the optimal stopping set.

Because of starting point dependence this is not so easy in our problem since the optimal thresholds obviously depend on the starting point. But in Example 3.3.1 – i.e. $g(x) = x$ and geometric Brownian motion as the driving process – we reduced the problem to a standard optimal stopping problem by homogeneity, so that it was enough to solve the problem for the starting point $x = 1$.

The purpose of the following is to generalize this: We want to reduce the general problem such that it is enough to solve the problem for one (or two) starting points. Then the other boundary points can be computed as solutions of a certain system of two first order ODEs.

To this end we assume that g is twice continuously differentiable and $g = k$ in the following. For fixed $x \in I$ we again consider the problem

$$v_x(y) = \sup_{\tau \in \mathcal{T}} E_y(e^{-r\tau}(g(X_\tau) \vee g(x))\mathbb{1}_{\{\tau<\infty\}}) \quad \text{for all } y \in I.$$

Writing $\tau_a = \inf\{t \geq 0 : X_t = a\}$, $\tau_b = \inf\{t \geq 0 : X_t = b\}$ and $\tau_{a,b} = \tau_a \wedge \tau_b$ for all a, b Proposition 2.5 yields for $a \leq x \leq b$

$$\begin{aligned} &E_x(e^{-r\tau_{a,b}}(g(X_{\tau_{a,b}}) \vee g(x))\mathbb{1}_{\{\tau_{a,b}<\infty\}}) \\ &= g(x)E_x(e^{-r\tau_a}\mathbb{1}_{\{\tau_a<\tau_b\}}) + g(b)E_x(e^{-r\tau_b}\mathbb{1}_{\{\tau_a<\tau_b\}}) \\ &= g(x)\frac{\psi_+(x)\psi_-(b) - \psi_+(b)\psi_-(x)}{\psi_+(a)\psi_-(b) - \psi_+(b)\psi_-(a)} + g(b)\frac{\psi_+(a)\psi_-(x) - \psi_+(x)\psi_-(a)}{\psi_+(a)\psi_-(b) - \psi_+(b)\psi_-(a)} \\ &=: F(x, a, b). \end{aligned}$$

For fixed x Proposition 4.1 provides the existence of (a_x, b_x) that is a maximum point of $(a, b) \mapsto F(x, a, b)$ and $v(x) = F(x, a_x, b_x)$. In the following we assume that ψ_+ and ψ_- are C^2. As discussed in Section 2.1 this is no hard assumption since it holds for all diffusion processes that are generated by a second order differential operator. In this case the function F is C^2 too and it holds that

$$D_{2,3}F(x, a_x, b_x) = 0,$$

where $D_{2,3}$ is the total differential for F as a function of the second and third argument. Write $G = D_{2,3}F : \{(x, a, b) \in I^3 : a \le x \le b\} \to \mathbb{R}^2$ and assume that (a_x, b_x) is a unique critical point of F, i.e. it is a unique zero of G for each $x \in I$. Hence $x \mapsto (a_x, b_x)$ is implicitly defined by

$$G(x, a, b) = 0.$$

The elementary implicit differential formula yields that $x \mapsto (a_x, b_x)$ solves the following system of first-order ODEs

$$\frac{d}{dx}(a_x, b_x)^T = -(D_{2,3}G(x, a_x, b_x))^{-1} D_1 G(x, a_x, b_x), \tag{4.3}$$

where ^{T} means transposition.

4.3 Stock with guarantee revisited

In this section we treat the example studied in Subsection 3.3.2 again, i.e. we consider a geometric Brownian motion as a driving process and $g(x) = x$.

Although the reduction is so easy in this case it is instructional to study it via differential equations to see how the approach works without technical difficulties. Recall that the function ψ_+ and ψ_- are power functions in this case. Using this fact one immediately checks that the function

$$F(x, a, b) = x\frac{\psi_+(x)\psi_-(b) - \psi_+(b)\psi_-(x)}{\psi_+(a)\psi_-(b) - \psi_+(b)\psi_-(a)} + b\frac{\psi_+(a)\psi_-(x) - \psi_+(x)\psi_-(a)}{\psi_+(a)\psi_-(b) - \psi_+(b)\psi_-(a)}$$

fulfills

$$F(x, a, b) = xF(1, \frac{a}{x}, \frac{b}{x}) \quad \text{for all } a \le x \le b.$$

Using this identity the chain rule yields

$$\begin{aligned} D_1F(x, a, b) &= \frac{\partial}{\partial x}(xF(1, \frac{a}{x}, \frac{b}{x})) \\ &= F(1, \frac{a}{x}, \frac{b}{x}) + xD_{2,3}F(1, \frac{a}{x}, \frac{b}{x}) \cdot (-\frac{a}{x^2}, -\frac{b}{x^2})^T \\ &= \frac{1}{x}F(x, a, b) - \frac{1}{x}D_{2,3}F(x, a, b) \cdot (a, b)^T, \end{aligned}$$

hence

$$\begin{aligned} &-[{D_{2,3}}^2F(x, a, b)]^{-1} \cdot D_1D_{2,3}F(x, a, b) \\ &\quad = -[D_{2,3}^2F(x, a, b)]^{-1}\frac{1}{x}D_{2,3}F(x, a, b) \\ &\quad\quad + \frac{1}{x}[D_{2,3}^2F(x, a, b)]^{-1}(\frac{\partial}{\partial a}, \frac{\partial}{\partial b})(D_{2,3}F(x, a, b) \cdot (a, b)^T) \\ &\quad = \frac{1}{x}(a, b)^T \end{aligned}$$

and the differential equation (4.3) reads as

$$\frac{d}{dx}(a_x, b_x) = \frac{1}{x}(a_x, b_x),$$

which proves that a_x and b_x are affine. Furthermore one immediately sees that $a_x, b_x \to 0$ for $x \to 0$, so that the problem is reduced to an ordinary optimal stopping problem for an arbitrary starting point x.

4.4 Lévy-driven market

In the last years processes with jumps have become more and more popular as models for financial markets. One standard model consists of two assets:
One deterministic bond $(B_0 e^{rt})_{t\geq 0}$, $B_0, r > 0$ and the risky asset given by

$$S_t = S_0 e^{X_t}, \qquad S_0 > 0,\ t \geq 0,$$

where $(X_t)_{t\geq 0}$ is a Lévy process. If $(X_t)_{t\geq 0}$ is a Brownian motion we get the standard Black-Scholes model that is extensively studied, but this model fails to satisfy the stylized facts for financial data, such as skewness, asymmetry and heavy tails. To deal with this problem $(X_t)_{t\geq 0}$ is assumed to be a general Lévy process. Unfortunately this class is too wide to provide explicit results. For more details we refer to [Sch03].
In the first subsection we investigate results for optimal stopping with guarantee in a general Lévy market.
In the second subsection we assume the Lévy process to be spectrally negative. This leads to tractable formulas using the scale function (cf. Section 2.2) and is flexible enough for fitting financial data (cf. [CW02]). We show that the scale function leads to similar results as investigated for diffusion processes.

4.4.1 General Lévy processes

We assume that an optimal stopping time τ_x^* exists. Then it is given as the first entrance time into the optimal stopping set S_x for all starting points x in the problem

$$v_x(y) = \sup_{\tau\in\mathcal{T}} E_y(e^{-r\tau}(g(X_\tau) \vee g(x))\mathbb{1}_{\{\tau<\infty\}}), \quad y \in \mathbb{R}.$$

As described in the introduction this is true under natural conditions. Hence it is enough to consider stopping times of the form

$$\tau_S = \inf\{t \geq 0 : X_t \in S\} \quad \text{for } S \subseteq \mathbb{R}.$$

A priori the optimal stopping set may have a very complex form and – since Lévy processes do not have continuous paths – τ_x^* cannot be expected to be easily manageable. Nonetheless the guarantee provides a simple form of $S_x \cap (-\infty, x]$:

Proposition 4.3. *(i) For all $x \in \mathbb{R}$ there exists $a_x \in [-\infty, x]$ such that*

$$S_x \cap (-\infty, x] = (-\infty, a_x].$$

(ii) For all $x \in \mathbb{R}$ it holds that

$$S_x \cap [x, \infty) \neq \emptyset.$$

Proof. (i) Write $a_x = \sup\{a \in S_x : a \leq x\}$. The case $a_x = -\infty$ is trivial, therefore we assume $a_x > -\infty$. Since S_x is closed $a_x \in S_x$. For all $y \leq a_x$ and all stopping times τ it holds that

$$\begin{aligned} E(e^{-r\tau}(g(X_\tau + y) \vee g(x))\mathbb{1}_{\{\tau<\infty\}}) &\leq E(e^{-r\tau}(g(X_\tau + a_x) \vee g(x))\mathbb{1}_{\{\tau<\infty\}}) \\ &\leq g(a_x) \vee g(x) = g(x). \end{aligned}$$

Therefore

$$v_x(y) = \sup_{\tau \in \mathcal{T}} E(e^{-r\tau}(g(X_\tau + y) \vee g(x))\mathbb{1}_{\{\tau<\infty\}}) \leq g(a_x) \vee g(x),$$

i.e. $y \in S_x$. Here we used the fact that the starting point dependence of Lévy processes is explicitly given by adding the starting point to the process started in 0.

(ii) Write $b_x = \inf\{b \in S_x : b \geq x\}$ and assume that $b_x = \infty$. Since $x \notin S_x$ and S_x is closed we obtain $a_x < x$ and since $g(X_{\tau_x^*}) \vee g(x) = g(x)$ under P_x we would have

$$E_x(e^{-r\tau_x^*}(g(X_{\tau_x^*}) \vee g(x))\mathbb{1}_{\{\tau_x^*<\infty\}}) = g(x)E_x(e^{-r\tau_x^*}\mathbb{1}_{\{\tau_x^*<\infty\}}) < g(x),$$

a contradiction to the optimality of τ_x^*. □

The structure of $S_x \cap [x, \infty)$ is much harder to handle and one can not expect it to be an interval in general. But a stopping set of the form $S_x = (-\infty, a_x] \cup [b_x, \infty)$ is necessary for semi-explicit results. In some examples elementary arguments provide this structure. To demonstrate this we consider the gain function $g(x) = (x - K)^+$ for some $K \in \mathbb{R}$ and fix some $x \geq K$. We write $b_x = \inf\{b \in S_x : b \geq x\}$. Then for all $y = b_x + h \geq b_x$ and all $\tau \in \mathcal{T}$ it holds that

$$\begin{aligned} E(e^{-r\tau}[g(X_\tau + y) \vee g(x)]\mathbb{1}_{\{\tau<\infty\}}) &\leq E(e^{-r\tau}[((X_\tau + b_x - K)^+ + h) \vee g(x)]\mathbb{1}_{\{\tau<\infty\}}) \\ &\leq E(e^{-r\tau}((X_\tau + b_x - K)^+ \vee g(x) + h)\mathbb{1}_{\{\tau<\infty\}}) \\ &= E(e^{-r\tau}(g(X_\tau + b_x) \vee g(x))\mathbb{1}_{\{\tau<\infty\}}) + hE(e^{-r\tau}\mathbb{1}_{\{\tau<\infty\}}) \\ &\leq g(b_x) + h = y - K. \end{aligned}$$

For the second inequality we used the elementary fact $(a+h) \vee b \leq a \vee b + h$ for all $h \geq 0, a, b$. This implies that S_x has the form $S_x = (-\infty, a_x] \cup [b_x, \infty)$. In this case results for the general representation of r-superharmonic functions provide a representation of the optimal value function in terms of the running infimum and supremum as described in [MS07].

But this result does not give rise to an explicit determination of the boundaries and the value function. Even for the problem without guarantee no nontrivial, explicit results for the two-sided case are known to our knowledge. But with further assumptions on the jump-structure explicit calculations are possible. This is worked out in the next subsection.

4.4.2 Spectrally negative Lévy processes

Now we assume that the Lévy process is spectrally negative, see Section 2.2. The following results show that the guarantee structure leads to optimal stopping times of the same simple form as in the diffusion-case.

Theorem 4.4. *For each $x \in \mathbb{R}$ with $g(x) > 0$ there exist $-\infty < a_x \leq x \leq b_x < \infty$ such that*

$$\tau_x^* = \inf\{t \geq 0 : X_t \leq a_x \textit{ or } X_t = b_x\}.$$

Proof. Write $a_x = \sup\{a \in S_x : a \leq x\}$ and $b_x = \inf\{b \in S_x : b \geq x\}$. By Theorem 4.3 the optimal stopping set has the form $S_x = (-\infty, a_x] \cup S_x^*$ for some $S_x^* \subseteq [x, \infty)$ and furthermore $b_x < \infty$. Since $(X_t)_{t \geq 0}$ has no positive jumps τ_x^* is given by

$$\tau_x^* = \inf\{t \geq 0 : X_t \leq a_x \text{ or } X_t = b_x\}.$$

It remains to prove that $a_x > -\infty$. So assume $a_x = -\infty$, i.e. $S_x \cap (-\infty, x] = \emptyset$. For all $y \leq x$ it holds that

$$g(x) = g(x) \vee g(y) < E_y(e^{-r\tau_x^*} g(X_{\tau_x^*}) \mathbb{1}_{\{\tau_x^* < \infty\}}) = g(b_x) E_y(e^{-r\tau_x^*} \mathbb{1}_{\{\tau_x^* < \infty\}})$$

and on the other hand

$$\begin{aligned} E_y(e^{-r\tau_x^*} \mathbb{1}_{\{\tau_x^* < \infty\}}) &= E_{y-b_x}(e^{-r\tau_0} \mathbb{1}_{\{\tau_0 < \infty\}}) \\ &= P_{y-b_x}(e_r > \tau_0) = P_0(\underline{X}_{e_r} + y - b_x > 0), \end{aligned}$$

where e_r is an Exp(r)-distributed random variable independent of everything else and $\underline{X}$ denotes the running minimum process. We obtain

$$E_y(e^{-r\tau_x^*} \mathbb{1}_{\{\tau_x^* < \infty\}}) \to 0 \quad \text{as } y \to -\infty.$$

This is a contradiction to $g(x) > 0$. Therefore $S_x \cap (-\infty, x] \neq \emptyset$. $\square$

For fixed x the problem is reduced to the maximization of

$$v_{a,b}(x) := E_x(e^{-r\tau_{a,b}}(g(X_{\tau_{a,b}}) \vee g(x))\mathbb{1}_{\{\tau_{a,b}<\infty\}})$$

in $a \leq x \leq b$, where $\tau_{a,b} = \inf\{t \geq 0 : X_t \leq a \text{ or } X_t = b\}$. To this end we need an explicit expression for $v_{a,b}(x)$, that is given in the following

Lemma 4.5. *It holds that*

$$v_{a,b}(x) = g(x)Z(x-a) + d_{a,b}W(x-a) \quad \textit{for all } a \leq x \leq b,$$

where $d_{a,b} = \frac{g(b)-Z(b-a)}{W(b-a)}$ *and* Z, W *are the functions defined in Section 2.2.*

Proof. Write $\tau_a = \inf\{t \geq 0 : X_t \leq a\}$ and $\tau_b = \inf\{t \geq 0 : X_t = b\}$. Using Proposition 2.7 we obtain that

$$\begin{aligned}
&E_x(e^{-r\tau_{a,b}}(g(X_{\tau_{a,b}}) \vee g(x))\mathbb{1}_{\{\tau_{a,b}<\infty\}}) \\
=&E_x(e^{-r\tau_a}g(x)\mathbb{1}_{\{\tau_a<\tau_b\}}) + E_x(e^{-r\tau_b}g(b)\mathbb{1}_{\{\tau_b<\tau_a\}}) \\
=&g(x)E_x(e^{-r\tau_a}\mathbb{1}_{\{\tau_a<\tau_b\}}) + g(b)E_x(e^{-r\tau_b}\mathbb{1}_{\{\tau_b<\tau_a\}}) \\
=&g(x)\left(Z(x-a) - W(x-a)\frac{Z(b-a)}{W(b-a)}\right) + g(b)\frac{W(x-a)}{W(b-a)} \\
=&g(x)Z(x-a) + d_{a,b}W(x-a).
\end{aligned}$$

□

Putting pieces together we obtain

Theorem 4.6. *For each $x \in \mathbb{R}$ let (a_x, b_x) be a maximum point of the function given by*

$$(a,b) \mapsto g(x)Z(x-a) + d_{a,b}W(x-a).$$

Then

$$\tau_{a_x,b_x} = \inf\{t \geq 0 : X_t \leq a_x \textit{ or } X_t = b_x\}$$

is an optimal stopping time and the value function is given by

$$v(x) = g(x)Z(x-a_x) + d_{a_x,b_x}W(x-a_x).$$

As for diffusion processes we would like to find a differential equation that provides the opportunity to find all optimal thresholds at once. Again we may use the implicit functions theorem. Write $F(x,a,b) = g(x)Z(x-a) + d_{a,b}W(x-a)$ for all $x, a, b \in \mathbb{R}$ with $a \leq x \leq b$. We again assume that g is a C^2-function. For F to be C^2 too we assume that Z is C^2 on $(0,\infty)$. This assumption was discussed recently in [CKS10]. We just mention that this is the case under quite general conditions, e.g. whenever $(X_t)_{t\geq 0}$ has a Gaussian part, i.e.

$\sigma \neq 0$ in the Lévy triple. Note that if Z is C^2, then W is C^3.
We assume that (a_x, b_x) is the unique critical point of $(a,b) \mapsto F(x,a,b)$, i.e. the solution to $G(x,a,b) := D_{2,3}F(x,a,b) = 0 \quad (a < x < b)$. Implicit differentiation again yields the following two coupled first-order ODEs

$$\frac{d(a_x, b_x)}{dx} = -\left(D^2_{2,3}F(x,a,b)\right)^{-1} D_1 D_{2,3} F(x, a_x, b_x).$$

Chapter 5

Multidimensional optimal stopping problems

In this chapter we apply the idea described in Section 3.1 to optimal stopping problems involving multidimensional processes. Before starting let us emphasize that our aim is to find explicit solutions. This seems to be hopeless for general multidimensional processes; even if the driving process is $(t, X_t)_{t\geq 0}$, where $(X_t)_{t\geq 0}$ is an "easy" process (such as a one-dimensional Brownian motion) no explicit results can be expected in general. For example problems with gain function given by $g(t, x) = x^+ - c(t)$ for a deterministic function c were studied in detail in [IKP01], but one cannot expect more than asymptotic results. Therefore we consider different special settings and apply our method to find explicit results.

The structure of this chapter is as follows: In Section 5.1 we recall some facts of Martin boundary theory and identify the minimal r-harmonic functions for a multidimensional geometric Brownian motion that will be used in the next sections. The case of two-dimensional geometric Brownian motion is treated in Section 5.2, where we concentrate on the interesting case of homogeneous gain functions of arbitrary degree. In this setting our approach works as shown in Theorem 5.5. We illustrate this general result by examples and show that all parameters can be treated simultaneously, even if the geometric structure of the stopping set changes. In Section 5.3 we discuss a generalization to gain functions of other type.
In Section 5.4 we discuss the problem of optimal investment for $d \geq 3$ underlying components. Using our method we disprove a conjecture given in [OS92] and [HØ98] about the shape of the optimal stopping region. These first sections are extended versions of the results in [CI10].
Section 5.5 shows that our approach is applicable for two-dimensional optimal stopping

problems involving the maximum process and leads to easy solutions.

5.1 Martin boundary theory

In Chapter 3 the minimal r-harmonic functions ψ_+ and ψ_- play a major role. Therefore it seems natural to determine this functions in more general settings. If X is a multidimensional diffusion process the space of r-harmonic functions cannot expected to be finite-dimensional. But Martin boundary theory gives a possibility to represent r-harmonic functions as integrals taken over the family of all minimal r-harmonic functions. See [Pin95, p. 285 ff.] for an overview of this theory.

In the case that X is a (multidimensional) Brownian motion with drift or a geometric Brownian motion the minimal r-harmonic functions – and hence all r-harmonic functions – can be represented explicitly as stated in the following result.

Proposition 5.1. *Let X be a d-dimensional Brownian motion on $\mathbb{R}^d$ with covariance matrix (σ_{ij}) and drift $\mu = (\mu_1, .., \mu_d)$, i.e. the generator of X is given by*

$$L = \frac{1}{2}\sum_{i,j}\sigma_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i \mu_i \frac{\partial}{\partial x_i}.$$

Write

$$A = \{a \in \mathbb{R}^d : \frac{1}{2}\sum_{i,j=1}^{d}\sigma_{ij}a_i a_j + \sum_{i=1}^{d}\mu_i a_i - r = 0\}.$$

Then

$$\{x \mapsto \exp(a \bullet x) : a \in A\}$$

is the set of all minimal positive r-harmonic functions, where $\bullet$ denotes the usual scalar product.

Proof. As the generator of X is given by

$$L = \frac{1}{2}\sum_{i,j}\sigma_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i \mu_i \frac{\partial}{\partial x_i},$$

one immediately sees that the functions are r-harmonic. The minimality is proved in [Pin95, p.348]. There it is also proved that these are indeed all minimal positive r-harmonic functions. The main tool for the proof is Harnack's inequality. □

Since Brownian motion and geometric Brownian motion are closely related we get the analogous result for geometric Brownian motion.

Corollary 5.2. *Let X be a d-dimensional geometric Brownian motion on $(0,\infty)^d$ with generator*

$$L = \frac{1}{2}\sum_{i,j}\sigma_{ij}x_ix_j\frac{\partial^2}{\partial x_i\partial x_j} + \sum_i \mu_i x_i \frac{\partial}{\partial x_i},$$

and let

$$A = \{a \in \mathbb{R}^d : \frac{1}{2}\sum_{i,j}^{d}\sigma_{ij}a_ia_j + \sum_{i=1}^{d}(\mu_i - \frac{\sigma_i^2}{2})a_i - r = 0\},$$

where $\sigma_i^2 = \sigma_{ii}$. Then

(i) $\{x \mapsto x^a : a \in A\}$ is the set of all minimal r-harmonic functions, where $x^a := x_1^{a_1} \cdot ... \cdot x_d^{a_d}$.

(ii) For each positive r-harmonic function h there exists a finite measure μ on A such that

$$h(x) = \int_A x^a \mu(da) \qquad \textit{for all } x \in (0,\infty)^d.$$

Proof. (i) For each function $f : (0,\infty)^d \to \mathbb{R}$ define the function $\tilde{f} : \mathbb{R}^d \to \mathbb{R}$ by $\tilde{f}(y) = f(\exp(y))$, where the exponential function is applied componentwise and define the operator $\tilde{L}$ by

$$\tilde{L} = \frac{1}{2}\sum_{i,j}\sigma_{ij}\frac{\partial^2}{\partial y_i\partial y_j} + \sum_i(\mu_i - \frac{\sigma_i^2}{2})\frac{\partial}{\partial y_i},$$

so that $\tilde{L}$ is the generator of a Brownian motion on $\mathbb{R}^d$. We have

$$\tilde{L}(\tilde{f})(y) = L(f)(e^y) \text{ for all } y \in \mathbb{R}^d \text{ and } f \in C^2((0,\infty)^d).$$

This immediately implies that all functions $x \mapsto x^a$, $a \in A$, are r-harmonic. To prove the minimality let $a \in A$ and $f : (0,\infty)^d \to \mathbb{R}, x \mapsto x^a$. If $f = cf_1 + df_2$ for some $c, d \geq 0$ and f_1, f_2 r-harmonic with respect to L, then $\tilde{f} = c\tilde{f}_1 + d\tilde{f}_2$ and $\tilde{f}_i$ is r-harmonic with respect to $\tilde{L}$. Because $\tilde{f}$ is minimal r-harmonic w.r.t. $\tilde{L}$ we have $c = 0$ or $d = 0$, hence f is minimal.

It remains to prove that each r-harmonic function is a power function for an exponent in A. Let f be r-harmonic w.r.t. L. $\tilde{f}$ is r-harmonic w.r.t. $\tilde{L}$. Proposition 5.1 yields that there exists $a \in A$ such that $\tilde{f}(x) = e^{a \bullet x}$ for all $x \in \mathbb{R}^d$. Consequently it holds that $f(y) = \tilde{f}(\log(y_1), ..., \log(y_d)) = y^a$ for all $y \in (0,\infty)^d$.

(ii) This result is not surprising keeping Choquet's theorem in mind. Indeed it immediately follows from (i) by the Martin representation theorem (see [Pin95, 7.1, Theorem 1.2]).

□

The case of one-dimensional diffusions suggests to consider the set H of all positive r-harmonic functions and the candidate set

$$\tilde{S} = \{x \in E : \exists h \in H \text{ such that } x \in \arg\max(\frac{g}{h})\}.$$

This set is a subset of the optimal stopping set by Lemma 3.1 and equal to the stopping set in the one-dimensional case. So the main question is: What can be said about equality in multidimensional problems?

5.2 Homogeneous gain functions

In this section we consider homogeneous gain functions of arbitrary degree for two-dimensional geometric Brownian motions and show that our approach is applicable in this situation (i.e. $\tilde{S}$ is the stopping set). As examples we consider an extended version of the the classical Margrabe option problem and a two-sided problem. Let X, Y be geometric Brownian motions with dynamics

$$dX_t = \mu_1 X_t dt + \sigma_1 X_t dW_t^{(1)}, dY_t = \mu_2 Y_t dt + \sigma_2 Y_t dW_t^{(2)},$$

where $W^{(1)}$ and $W^{(2)}$ are standard Brownian motions with covariation $[W_t^{(1)}, W_t^{(2)}] = \rho t$. Furthermore let $g : (0, \infty)^2 \to \mathbb{R}$ be homogeneous of degree $\kappa > 0$, i.e. we assume for this section that

$$g(\lambda x, \lambda y) = \lambda^\kappa g(x, y) \quad \text{for all } (x, y) \in (0, \infty)^2 \text{ and all } \lambda > 0.$$

To avoid trivialities we assume that $\sup(g) > 0$. Obviously $v > 0$.

In the working paper [AV05] the authors considered the optimal stopping problem for homogeneous gain functions that were assumed to be twice continuously differentiable. Under further assumptions on the monotonicity of g they solved the problem using excessive functions. Their method works for the case that the continuation set and the optimal stopping set are both connected. Our approach is not restricted to this case as demonstrated by the examples in the subsections at the end of this section.

Lemma 5.3. *Let h be a positive r-harmonic function such that $\frac{g}{h}$ has a maximal point. Then there exist $a, b \in \mathbb{R}$ and $c, d \geq 0$ such that $h(x, y) = cx^a y^{\kappa-a} + dx^b y^{\kappa-b}$ for all $(x, y) \in (0, \infty)^2$ and a, b are solutions of the quadratic equation*

$$\frac{\sigma^2}{2} w^2 + (\mu - \frac{\sigma^2}{2}) w + \gamma = 0, \tag{5.1}$$

where $\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho$, $\mu = (\kappa-1)(\sigma_1\sigma_2\rho - \sigma_2^2) + \mu_1 - \mu_2$ and $\gamma = \kappa(\kappa-1)\frac{\sigma_2^2}{2} + \kappa\mu_2 - r$. In particular h is homogeneous of degree $\kappa > 0$.

Proof. By the Martin representation theorem and Corollary 5.2 there exists a finite measure μ on

$$A = \{a \in \mathbb{R}^2 : \frac{1}{2}(\sum_{i=1}^{2} \sigma_i a_i^2 + 2\sigma_1\sigma_2\rho a_1 a_2) + \sum_{i=1}^{2}(\mu_i - \frac{\sigma_i^2}{2})a_i - r = 0\}$$

such that $h(x,y) = \int_A x^{a_1} y^{a_2} \mu(da)$. Hence

$$\frac{g}{h}(x,y) = \frac{g(\frac{x}{y},1)}{\int_A (\frac{x}{y})^{a_1} y^{a_1+a_2-\kappa} \mu(da)}.$$

If there exists $(a_1,a_2) \in supp(\mu)$ such that $a_1 + a_2 - \kappa \neq 0$ then

$$\lim \sup_{\|(x,y)\|\to 0} \frac{g}{h}(x,y) = \infty \text{ or } \lim \sup_{\|(x,y)\|\to\infty} \frac{g}{h}(x,y) = \infty$$

and $\frac{g}{h}$ has no maximum point in $(0,\infty)^2$.
The only elements of the form $(a_1, \kappa - a_1) \in A$ are $(a, \kappa - a)$ and $(b, \kappa - b)$ as described above, hence $h(x,y) = cx^a y^{\kappa-a} + dx^b y^{\kappa-b}$ for some $c, d \geq 0$. $\square$

We need the following well known result concerning the Laplace transform of first exit times of (geometric) Brownian motions (cf. Section 2.1 and (for this special result) for example [Won08, Lemma 1]).

Lemma 5.4. *Let Z be a geometric Brownian motion with drift μ and volatility σ. Let $\gamma \in \mathbb{R}$, $0 < l < q$, $\tau_l = \inf\{t \geq 0 : Z_t = l\}$, $\tau_q = \inf\{t \geq 0 : Z_t = q\}$ and $\tau = \tau_l \wedge \tau_q$.*

(i) If the quadratic equation (5.1) has real solutions a and b, then

$$E_z(e^{\gamma\tau}\mathbb{1}_{\{\tau_l<\tau_q\}}) = \frac{z^a q^b - z^b q^a}{l^a q^b - l^b q^a}$$

for all $z \in (l,q)$.

(ii) If the quadratic equation (5.1) has no real solutions, then

$$E_z(e^{\gamma\tau}\mathbb{1}_{\{\tau_l<\tau_q\}}) = \infty$$

for all $z \in (l,q)$.

Now we come to the main result of this section:

Theorem 5.5. *The pair (x,y) is in the stopping set if and only if there exists an r-harmonic function h such that $(x,y) \in \arg\max\left(\frac{g}{h}\right)$.*

Proof. Let $(x, y) \in (0, \infty)^2$. For $0 < l < \frac{x}{y} < q$ set

$$\tau_l = \inf\left\{t \geq 0 : \frac{X_t}{Y_t} = l\right\}, \quad \tau_q = \inf\left\{t \geq 0 : \frac{X_t}{Y_t} = q\right\}$$

and $\tau = \tau_l \wedge \tau_q$. Because g is homogeneous of degree κ, so is v. Write $v(x, y) = y^\kappa \tilde{v}(\frac{x}{y})$, where $\tilde{v}(\cdot) = v(\cdot, 1)$. It holds that

$$\begin{aligned}
v(x, y) &\geq E_{(x,y)}(e^{-r\tau} v(X_\tau, Y_\tau) \mathbb{1}_{\{\tau < \infty\}}) \\
&= \tilde{v}(l) E_{(x,y)}(e^{-r\tau} Y_\tau^\kappa \mathbb{1}_{\{\tau_l < \tau_q\}}) + \tilde{v}(q) E_{(x,y)}(e^{-r\tau} Y_\tau^\kappa \mathbb{1}_{\{\tau_q < \tau_l\}}) \\
&= \tilde{v}(l) y^\kappa E^Q_{(x,y)}(e^{\gamma\tau} \mathbb{1}_{\{\tau_l < \tau_q\}}) + \tilde{v}(q) y^\kappa E^Q_{(x,y)}(e^{\gamma\tau} \mathbb{1}_{\{\tau_q < \tau_l\}}),
\end{aligned}$$

where Q is defined by

$$\frac{dQ|_{\mathcal{F}_t}}{dP|_{\mathcal{F}_t}} = \frac{1}{y^\kappa} e^{\gamma t} e^{\kappa\sigma_2 W_t^{(2)} - (1/2)\kappa^2\sigma_2^2 t}, \qquad \gamma = \kappa(\kappa - 1)\frac{\sigma_2^2}{2} + \kappa\mu_2 - r.$$

Girsanov's theorem yields that $(W^{(1)}, W^{(2)})$ is a Brownian motion with drift $(\kappa\sigma_2\rho, \kappa\sigma_2)$ under Q. Hence Itô's lemma shows that the stochastic process $(\frac{X}{Y})$ is a geometric Brownian motion with drift $\mu = (\kappa - 1)(\sigma_1\sigma_2\rho - \sigma_2^2) + \mu_1 - \mu_2$ and volatility σ under Q, where $\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho$.
Using Lemma 5.4 we obtain

$$E^Q_{(x,y)}(e^{\gamma\tau} \mathbb{1}_{\{\tau_l < \tau_q\}}) = \begin{cases} \frac{(\frac{x}{y})^a q^b - (\frac{x}{y})^b q^a}{l^a q^b - l^b q^a} & , \text{ equation (5.1) has real solutions } a \text{ and } b \\ \infty & , \text{ otherwise} \end{cases}.$$

Case 1: Equation (5.1) has no real solutions.

Then $v(x, y) = \infty$, i.e. the stopping set is empty. On the other hand, Lemma 5.3 yields that under this conditions there exists no r-harmonic function h such that $\frac{g}{h}$ has a maximum point, i.e. $\tilde{S} = \emptyset$. That proves the assertion in this case.

Case 2: Equation (5.1) has real solutions a and b.

Then the equations above yield that

$$v(x, y) \geq \tilde{v}(l) y^\kappa \frac{(\frac{x}{y})^a q^b - (\frac{x}{y})^b q^a}{l^a q^b - l^b q^a} + \tilde{v}(q) y^\kappa \frac{(\frac{x}{y})^b l^a - (\frac{x}{y})^a q^b}{l^a q^b - l^b q^a},$$

i.e.

$$\tilde{v}(\frac{x}{y}) \geq \tilde{v}(l) \frac{(\frac{x}{y})^a q^b - (\frac{x}{y})^b q^a}{l^a q^b - l^b q^a} + \tilde{v}(q) \frac{(\frac{x}{y})^b l^a - (\frac{x}{y})^a l^b}{l^a q^b - l^b q^a}$$

for all $l < \frac{x}{y} < q$. Hence

$$\frac{\tilde{v}(z)}{z^a} \geq \frac{\tilde{v}(l)}{l^a} \frac{q^{b-a} - z^{b-a}}{q^{b-a} - l^{b-a}} + \frac{\tilde{v}(q)}{q^a} \frac{z^{b-a} - l^{b-a}}{q^{b-a} - l^{b-a}}$$

for all z, l, q with $0 < l < z < q$. By changing the variables this shows that the function $z \mapsto \frac{\tilde{v}(z^{1/(b-a)})}{z^{a/(b-a)}}$ is concave. This implies that for $z_0 := \frac{x}{y}$ there exist $c, d \geq 0$ such that $\frac{\tilde{v}(z_0^{1/(b-a)})}{z_0^{a/(b-a)}} = dz_0 + c$ and $\frac{\tilde{v}(z^{1/(b-a)})}{z^{a/(b-a)}} \leq dz + c$ for all $z \in (0, \infty)$. By a further change of variables we obtain

$$\tilde{v}(z_0) = cz_0^a + dz_0^b \text{ and } \tilde{v}(z) \leq cz^a + dz^b \text{ for all } z \in (0, \infty).$$

Define the function h by $h(x, y) = cx^a y^{\kappa - a} + dx^b y^{\kappa - b}$. Corollary 5.2 shows that h is r-harmonic, $h \geq v$ and $h(x, y) = v(x, y)$. If (x, y) is in the stopping set, this implies $h \geq g$ and $h(x, y) = g(x, y)$ thus $(x, y) \in \arg\max \frac{g}{h}$. $\square$

5.2.1 Power exchange options

In this section we study the special functions $g : (0, \infty)^2 \to \mathbb{R}, (x, y) \mapsto \max(x - y, 0)^\alpha$, i.e. we study the problem

$$\sup_{\tau \in \mathcal{T}} E_{(x,y)}(e^{-r\tau}((X_\tau - Y_\tau)^+)^\alpha \mathbb{1}_{\{\tau < \infty\}}), \quad x, y > 0. \tag{Exch}$$

For $\alpha = 1$ this problem is connected with pricing an exchange option in a Black-Scholes market and with the timing of an investment (cf. [MS86]). The problem was studied from different points of view and was solved for the whole range of parameters quite recently (cf. [Won08] and the references therein). Our methods provide an immediate solution.
Use γ, μ, σ^2 as in the previous section. Let a, b be the (possibly complex) solutions to (5.1) (Lemma 5.3), assuming $a \geq b$ in the case that a, b are real.

Proposition 5.6. *(Solution to (Exch))*
In the problem (Exch) the stopping set S is given by

(i) $S = \{(x, y) \in (0, \infty) : \frac{a}{a-\alpha} \leq \frac{x}{y} \leq \frac{b}{b-\alpha}\}$, *if a, b are real and $a, b > \alpha$.*

(ii) $S = \{(x, y) \in (0, \infty) : \frac{a}{a-\alpha} \leq \frac{x}{y}\}$, *if a, b are real and $a > \alpha \geq b$.*

(iii) $S = \emptyset$ *otherwise.*

Proof. The results in the previous section yield that S consists of the maximum points of the mappings

$$(x, y) \mapsto \frac{((x - y)^+)^\kappa}{cx^a y^{\kappa - a} + dx^b y^{\kappa - b}} = \frac{((\frac{x}{y} - 1)^+)^\kappa}{c(\frac{x}{y})^a + d(\frac{x}{y})^b} \quad c, d \geq 0.$$

Because the maximum points are scale-invariant it is enough to find the maximum points of the functions

$$f_\lambda : (0,\infty) \to \mathbb{R}, z \mapsto \frac{((z-1)^+)^\kappa}{\lambda z^a + (1-\lambda)z^b}, \quad \lambda \in [0,1].$$

In the following arguments we use the explicit formula

$$f'(z) = \frac{\kappa z^{\kappa-1}(\lambda z^a + (1-\lambda)z^b) - z^\kappa(\lambda a z^{a-1} + (1-\lambda)bz^{b-1})}{(\lambda z^a + (1-\lambda)z^b)^2} \quad \text{for } z > 1.$$

First let $a, b > \kappa$. Then f_1 resp. f_0 has a maximum point at $\frac{a}{a-\kappa}$ resp. $\frac{b}{b-\kappa}$. By the implicit function theorem the maximum points of f_λ are continuous in λ and the intermediate value theorem yields that

$$\{(x,y) \in (0,\infty) : \frac{a}{a-\kappa} \leq \frac{x}{y} \leq \frac{b}{b-\kappa}\} \subseteq S.$$

Now let $z \notin [\frac{a}{a-\kappa}, \frac{b}{b-\kappa}]$. If $1 < z < \frac{a}{a-\kappa}$ resp. $z > \frac{b}{b-\kappa}$, then $f_0'(z), f_1'(z) > 0$ resp. $f_0'(z), f_1'(z) < 0$. If we would have $f_\lambda'(z) = 0$, then $(1-\lambda)f_0'(z)z^{2b} = -\lambda f_1'(z)z^{2a}$, a contradiction. This proves (i).

Now let $a > \kappa, b \leq \kappa$. It holds that $f_0'(z) > 0$ for $z > 1$. Fix $z \geq \frac{a}{a-\kappa}$ and define $\lambda = \frac{f_1'(z)}{f_1'(z)-f_0'(z)}$. Because $f_1'(z) \leq 0$ we have $\lambda \in [0,1]$. Computing f_λ' shows that $f_\lambda'(z) = 0$ and f_λ has a maximum point in z. This shows that the right hand side in (ii) is a subset of the optimal stopping set. On the other hand for each $z > 1$ we have $f_0'(z) > 0$, furthermore $f_1'(z) > 0$ for $z \in (1, \frac{a}{a-\kappa})$. Therefore the same argument as in case (i) yields that there cannot be more maximum points - thus (ii).

Finally let $a, b \leq \kappa$. Then f_λ has no maximum points and therefore $S = \emptyset$. If a, b are not real, then there exists no positive r-harmonic function of degree κ, therefore $S = \emptyset$, and (iii) is proved. □

Remark 5.7. *The conditions on the solutions a and b of the quadratic equation can be translated into conditions on the parameter of the process. For example, the conditions for (ii) in the case $\alpha = 1$ are given by*

$$r > \mu_1 \text{ or } (r = \mu_1 \text{ and } r < \mu_2 - \frac{1}{2}(\sigma_1^2 - \sigma_2^2 - 2\sigma_1\sigma_2\rho)).$$

This special case will play an important role in the next section.

5.2.2 Straddle exchange option

As a second example we consider a case where the continuation set is connected but the stopping set is not, i.e. we consider a two-sided situation. Let the gain function be given

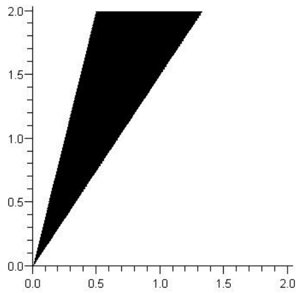

Figure 5.1: Exchange-Option in case (i)

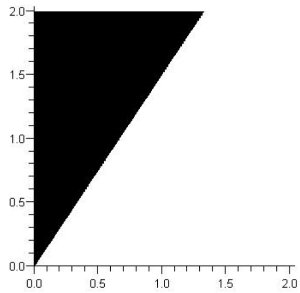

Figure 5.2: Exchange-Option in case (ii)

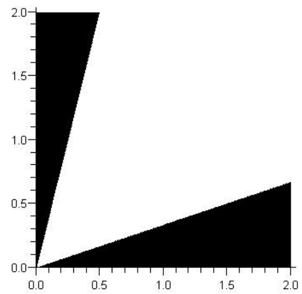

Figure 5.3: Straddle exchange option

by $g(x, y) = (x - y) \vee (y - x)$, where $\vee$ denotes the maximum.

As in the previous example we have to find the maximum points of the functions f_λ given by $f_\lambda(z) = f_{\lambda,1}(z) = \frac{1-z}{\lambda z^a + (1-\lambda)z^b}$ for $z \in (0, 1]$ and $f_\lambda(z) = f_{\lambda,2}(z) = \frac{z-1}{\lambda z^a + (1-\lambda)z^b}$ for $z \in (1, \infty)$. For $a < 1$ or $b > 0$ it is easily seen that the problem becomes trivial. The case $a = 1$ or $b = 0$ is interesting but the rather lengthy arguments are omitted and we assume that $a > 1$, $b < 0$.

We have to consider $\lambda \in (0, 1)$. By direct computation one immediately sees that $f_{\lambda,1}$ and $f_{\lambda,2}$ have unique maximum points $z_{\lambda,1} \in (0, 1)$ and $z_{\lambda,2} \in (1, \infty)$; furthermore $\sup(f_{\lambda,1})$ is continuously increasing in λ, while $\sup(f_{\lambda,2})$ is decreasing and $\sup(f_{1,1}) = \infty = \sup(f_{0,2})$. Therefore there exists a unique point λ^* such that $f_{\lambda^*,1}(z_{\lambda^*,1}) = f_{\lambda^*,2}(z_{\lambda^*,2})$.

Furthermore $z_{\lambda,1}$ decreases to 0 as $\lambda \to 1$ and $z_{\lambda,2}$ increases to ∞ as $\lambda \to 0$. By the implicit functions theorem we get that the set of maximum points of the functions f_λ is $(0, z_{1,\lambda^*}] \cup [z_{\lambda^*,2}, \infty)$. By Theorem 5.5 the stopping set is given by

$$S = \{(x, y) : \frac{x}{y} \le z_{\lambda^*,1} \text{ or } \frac{x}{y} \ge z_{\lambda^*,2}\}.$$

Furthermore the values $z_{\lambda^*,1}, z_{\lambda^*,2}$ and λ^* are uniquely determined by the following system of equations

$$\begin{aligned} f'_{\lambda^*,1}(z_{\lambda^*,1}) &= 0 \\ f'_{\lambda^*,2}(z_{\lambda^*,2}) &= 0 \\ f_{\lambda^*,1}(z_{\lambda^*,1}) &= f_{\lambda^*,2}(z_{\lambda^*,2}). \end{aligned}$$

5.3 Extensions to non-homogeneous gain functions

Although many interesting cases are covered by homogeneous gain functions, other functions are of interest too. In this section we describe how our method can be extended to other situations too. In [Mak08] the gain functions g given by $g(x, y) = e^x y$ is studied under constant costs of observation and a solution is given under certain conditions. In generalization of this example we consider gain functions of the form $g(x, y) = f(x)y^\alpha$ for an upper semicontinuous function f and a fixed $\alpha > 0$ in our setting. From a financial point of view an interesting example is given by the case that the option on Y can be exercised only if X fulfills a certain condition, that is $f(x) = \mathbb{1}_{\{x\in A\}}$ for some closed set A. As before we have

Theorem 5.8. *The pair (x, y) is in the stopping set if and only if there exists an r-harmonic function h such that $(x, y) \in \arg\max\left(\frac{g}{h}\right)$.*

Proof. Because g is homogeneous of degree α as a function in y, so is v. Write $v(x, y) = \tilde{v}(x)y^\alpha$, where $\tilde{v}(\cdot) = v(\cdot, 1)$. Using the stopping times $\tau_l = \inf\{t \geq 0 : X_t = l\}$, $\tau_q = \inf\{t \geq 0 : X_t = q\}$ and $\tau = \tau_l \wedge \tau_q$ the proof works exactly as the proof of Theorem 5.5. □

5.4 The problem of optimal investment

In [OS92] and [HØ98] the situation of subsection 5.2.1 was extended to a multidimensional problem $(Invest)$ in the form

$$\sup_{\tau\in\mathcal{T}} E_x(e^{-r\tau}(X_\tau^{(1)} - X_\tau^{(2)} - ... - X_\tau^{(d)})\mathbb{1}_{\{\tau<\infty\}}), \quad x \in (0, \infty)^d.$$

Here $X^{(1)}, X^{(2)}, ..., X^{(d)}$ are geometric Brownian motions with dynamic

$$dX_t^{(i)} = X_t^{(i)}(\mu_i dt + dW_t^{(i)}),$$

where $W^{(1)}, ..., W^{(d)}$ are Brownian motions with covariances $[W^{(i)}, W^{(j)}]_t = \sigma_{ij}t$ for all $i, j = 1, .., d$ and we write $\sigma_i^2 = \sigma_{ii}$ for short. We assume that the associated covariance matrix is not singular.

One can interpret this problem as the question about the best time to invest when one has to pay $X_t^{(2)} + ... + X_t^{(d)}$ to get $X_t^{(1)}$ (cf. [HØ98]). The difficulty in solving this problem stems from the fact that the structure is not multiplicative but additive and the sum of geometric Brownian motions is not easy to handle.

Remark 5.9. *For $d = 2$ the difference between $(Exch)$ and $(Invest)$ is that in $(Exch)$ the positive part of the difference is considered while in $(Invest)$ the difference itself is treated. Because the stopping times τ may be infinite the values and optimal stopping times are the same.*

We now consider the set

$$\begin{aligned} A =& \{a \in \mathbb{R}^d : x \mapsto x^{-a} \ \text{ is } r\text{-harmonic}\} \\ =& \{a \in \mathbb{R}^d : p(a) = 0\} \end{aligned}$$

using the notations

$$p(a) = \frac{1}{2}\sum_{i,j}\sigma_{ij}a_ia_j - \sum_i(\mu_i - \frac{\sigma_i^2}{2})a_i - r$$

and $z^w := \prod_{i=1}^d z_i^{w_i}$ and $z, w \in (0,\infty)^d$. We switch from a, Corollary 5.2, to $-a$ in the definition of A to facilitate some later computations. Note that the matrix associated with the quadratic form p is positive definite as a non singular correlation matrix. Hence the level sets of p are ellipsoids.

In the previous section a special role was played by the functions $(x, y) \mapsto x^{-a_1}y^{-a_2}$ where $-a_2 = \kappa + a_1$, see Lemma 5.3. This motivates to consider parameters $a \in A$ where just the first and one more component i is not 0 and it holds that $a_1 + a_i = -1$ (note that $\kappa = 1$ in our situation). For any i, the existence of such parameters is ensured by Remark 5.7 if and only if

$$r - \mu_1 > 0 \text{ or } (r - \mu_1 = 0 \text{ and } r < \mu_i - \frac{1}{2}(\sigma_1^2 + \sigma_i^2 - \sigma_{1i}) \text{ for all } i > 1)$$

and then there are exactly two of them as solutions to a quadratic equation. We assume this condition in the following. Then there exist unique $\lambda_i > 1$ such that $u^i := (-\lambda_i, 0, .., 0, \lambda_i - 1, 0, ..., 0) \in A$, where λ_i is at position i for $i = 2, ..., d$. In [OS92] and [HØ98] the following conjecture was stated and was supported by numerical calculations:

Conjecture. *The halfspace*

$$S^H := \{x \in (0,\infty)^d : x_1 \geq \sum_{i=2}^d \frac{\lambda_i}{\lambda_i - 1}x_i\}$$

is the stopping set S for a wide range of the parameter-space.

The following theorem proves that this conjecture is not true in the above situation.

Theorem 5.10. *If $d > 2$, then S^H is a strict subset of the optimal stopping set S.*

Proof. (i) A short calculation yields that for each $a \in \mathbb{R}^d$ the function

$$g_a : (0,\infty)^d \to \mathbb{R}, x \mapsto x^a(x_1 - \sum_{j=2}^d x_j)$$

has a maximum point if and only if

$$\sum_{i=1}^{d} a_i = -1, \quad a_1 < 0, \quad \text{and } a_j > 0 \text{ for } j > 1;$$

in this case it holds that $\arg\max(g_a) = \{\lambda(-a_1, a_2, a_3, ..., a_d) : \lambda > 0\}$.

(ii) If $p(a) \leq 0$, then applying the generator

$$A = \frac{1}{2}\sum_{ij} \sigma_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i (\mu_i - \frac{\sigma_i^2}{2}) \frac{\partial}{\partial x_i}$$

shows that $x \mapsto x^{-a}$ is r-superharmonic. Thus Lemma 3.1 can be applied and yields that $\mathbb{R}_{>0} \cdot (-a_1, a_2, a_3, ..., a_d)$ is in the optimal stopping set S if condition (i) holds, i.e.

$$S' := \{\lambda \cdot x : x \in (0,\infty)^d, p(\Phi(x)) \leq 0, x_1 - \sum_{j=1}^{d} x_j = 1, \lambda > 0\} \subseteq S,$$

where the function $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ is given by $\Phi(x) = (-x_1, x_2, x_3, ..., x_d)$. Note that Φ is self-inverse.

(iii) Let U be the hyperplane $U := \{x \in (0,\infty)^d : x_1 - \sum_{j=2}^{d} x_j = 1\}$ and define $S_U = S \cap U$, $S_U^H = S^H \cap U$ and $S'_U = S' \cap U$. Thus S'_U is the intersection of an ellipsoid in U with a quadrant hence it is strictly convex. Furthermore $p(u^i) = 0$ holds for all $i = 2, ..., d$ by definition and hence

$$\{x \in U : x_1 = \sum_{j=2}^{d} \frac{\lambda_j}{\lambda_j - 1} x_j\} = conv(\Phi(u^2), ..., \Phi(u^d)) \subseteq S',$$

where $conv(\cdot)$ denotes the convex hull. If $y \in S_U^H$ then $y_1 - \sum_{j=2}^{d} y_j = 1$ and $y_1 - \sum_{j=2}^{d} \frac{\lambda_j}{\lambda_j - 1} y_j \geq 0$ and equivalently $y_1 - \sum_{j=2}^{d} y_j = 1$ and $1 \geq \sum_{j=2}^{d} \frac{1}{\lambda_j - 1} y_j$. Thus there exists $\lambda \in (0, 1]$ such that $1 = \frac{1}{\lambda} \sum_{j=2}^{d} \frac{1}{\lambda_j - 1} y_j$.
Define $x = (x_1, ..., x_d)$ by $x_j = \frac{1}{\lambda} y_j$ for $j \neq 1$ and $x_1 = 1 + \sum_{j=2}^{d} x_j$. Then

$$1 = \sum_{j=2}^{d} \frac{1}{\lambda_j - 1} x_j \text{ and } x_1 - \sum_{j=2}^{d} x_j = 1$$

and equivalently

$$x_1 = \sum_{j=2}^{d} \frac{\lambda_j}{\lambda_j - 1} x_j \text{ and } x_1 - \sum_{j=2}^{d} x_j = 1$$

i.e. $x \in conv(\Phi(u^2), ..., \Phi(u^d)) \cap U \subseteq S'_U$. Because $p(\Phi(1, 0, ..., 0)) = \mu_1 - r \leq 0$ it also holds that $(1, 0, ..., 0) \in S'_U$. We have that $y = \lambda x + (1 - \lambda)(1, 0, ..., 0) \in S'_U$ as a convex combination of elements in S'_U. This yields $S_U^H \subseteq S'_U \subseteq S_U$. Because S'_U is strictly convex, but S_U^H is not, the sets cannot be equal.

□

Remark 5.11. *After presenting and submitting our result K. Nishide and L.C.G. Rogers found an independent proof for the case $d = 3$, see [NR10]. Their approach is based on using the elementary inequality*

$$X_t^{(2)} + X_t^{(3)} \geq (X_t^{(2)}/p)^p (X_t^{(3)}/q)^q \quad \text{for all } p \in [0,1],\ q = 1-p.$$

Noting that the right hand side defines a one-dimensional geometric Brownian motion, this leads to a lower bound for the value function.

5.5 On optimal stopping of the maximum process

Another two-dimensional process where explicit results can be expected is the process $(M_t, X_t)_{t\geq 0}$, where $(X_t)_{t\geq 0}$ is a diffusion process and $M_t = \sup_{s\leq t} X_s \vee M_0$ denotes its running maximum. Optimal stopping problems involving the running maxima were introduced in finance by Shepp and Shiryaev in [SS93] and this so called "Russian options" were discussed in different settings. Most authors use a free-boundary approach and verification theorems for a solution. Using this way an interesting class of examples in the Black-Scholes setting was recently treated in [GZ10]. Using the martingale approach the Russian-option problem in the Black-Scholes market was studied in [LU07, Section 4]. An overview and further references are given in [PS06, Section 26].

In this section we demonstrate that our idea for a solution is applicable for these kind of problems and gives rise to easy solutions. We want to illustrate it by the following example:

Let $(X_t)_{t\geq 0}$ be a one-dimensional Brownian motion with drift μ and volatility $\sigma > 0$ and $(M_t)_{t\geq 0}$ its running maximum. The process $(M_t, X_t)_{t\geq 0}$ is a Markov process with state space $\{(m, x) : x \leq m\}$. We consider the problem

$$v(m, x) = \sup_{\tau \in \mathcal{T}} E_{(m,x)}(e^{-r\tau}(M_\tau - X_\tau)^\alpha \mathbb{1}_{\{\tau<\infty\}}), \qquad m \geq x,\ \alpha > 0.$$

For $\alpha = 1$ this problem is connected to pricing a lookback option in the Bachelier model. To apply our idea it is necessary to have a simple class H of suitable functions h such that the sets $\arg\max_{(m,x)} \frac{g(m,x)}{h(m,x)}$ are subsets of the optimal stopping set, where $g(m, x) = m - x$. To this end note that M_t is constant as long as X_t does not reach its maximum. Therefore it seems natural to use the r-harmonic functions for $(X_t)_{t\geq 0}$, i.e.

$$\psi_-(x) = e^{\beta_1 x}, \quad \psi_+(x) = e^{\beta_2 x}.$$

Here $\beta_1 < 0 < \beta_2$ are the solutions to the quadratic equation

$$\frac{1}{2}\beta^2 + \mu\beta - r = 0;$$

for simplicity we assume w.l.o.g. that $\sigma = 1$. Furthermore we want to obtain a "large" set of maximum points for each function $h \in H$, so that we consider functions that are constant in $m - x$. This leads to the following

Lemma 5.12. *For all $\lambda \geq 1$ write*

$$h_\lambda : \{(m,x) : m \geq x\} \to \mathbb{R}, (m,x) \mapsto \lambda c e^{-\beta_1 m} e^{\beta_1 x} + e^{-\beta_2 m} e^{\beta_2 x},$$

where $c = -\frac{\beta_2}{\beta_1}$. Then

(i) The process $(e^{-rt} h_\lambda(M_t, X_t))_{t\geq 0}$ is a positive supermartingale for each $\lambda \geq 1$.

(ii) The process $(e^{-rt} h_1(M_t, X_t))_{t\geq 0}$ is a positive local martingale.

(iii) The union of all maximum points of $\frac{g}{h_\lambda}$, $\lambda \geq 1$, is given by

$$\tilde{S} = \{(m,x) : m \geq x + z^*\},$$

where $z^* > 0$ is the unique solution to

$$1 = \frac{\frac{\alpha}{\beta_2} + z}{\frac{\alpha}{\beta_1} + z} e^{(\beta_1 - \beta_2) z}.$$

Proof.
(i) and (ii): We apply Itô's lemma to the process $(M_t, X_t)_{t\geq 0}$. To this end note that $(M_t)_{t\geq 0}$ is non-decreasing and hence of locally bounded variation. Denote the generator of $(X_t)_{t\geq 0}$ by A. We obtain

$$\begin{aligned} e^{-rt} h_\lambda(M_t, X_t) &= h_\lambda(M_0, X_0) + \int_{[0,t]} e^{-rs} (A - r) h_\lambda(M_s - X_s) ds \\ &\quad + \int_{[0,t]} e^{-rs} \frac{\partial}{\partial m} h_\lambda(M_s, X_s) dM_s + \text{"local martingale"} \\ &= h_\lambda(M_0, X_0) + \int_{[0,t]} e^{-rs} \frac{\partial}{\partial m} h_\lambda(M_s, X_s) dM_s + \text{"local martingale"} \end{aligned}$$

since $(A - r) h_\lambda = 0$. Furthermore

$$\frac{\partial}{\partial m} h_\lambda(m,x) = -\lambda c \beta_1 e^{-\beta_1 m} e^{\beta_1 x} - \beta_2 e^{-\beta_2 m} e^{\beta_2 x},$$

hence

$$\frac{\partial}{\partial m} h_\lambda(m,m) = -\lambda c \beta_1 - \beta_2 = (\lambda - 1)\beta_2 \geq 0.$$

Since M_t increases only if $X_t = M_t$ the process $(e^{-rt} h_\lambda(M_t, X_t))_{t\geq 0}$ is a positive local supermartingale, i.e. a positive supermartingale. For $\lambda = 1$ the second summand vanishes

too and $(e^{-rt}h_\lambda(M_t, X_t))_{t\geq 0}$ is a positive local martingale.
(iii): We define

$$f_\lambda : [0,\infty) \to \mathbb{R}, z \mapsto \frac{z^\alpha}{\lambda c e^{-\beta_1 z} + e^{-\beta_2 z}} \quad \text{for all } \lambda \geq 1$$

so that

$$\frac{g}{h_\lambda}(m,x) = f_\lambda(m-x), \qquad m \geq x.$$

One immediately checks that f_λ has a unique maximum point z_λ that is the unique solution to

$$\lambda = \frac{\frac{\alpha}{\beta_2} + z}{\frac{\alpha}{\beta_1} + z} e^{(\beta_1 - \beta_2)z}.$$

Since the right hand side is increasing to ∞ in z we obtain that z_λ is increasing in λ from z^* to ∞, i.e.

$$\{z_\lambda : \lambda \geq 1\} = [z^*, \infty).$$

This proves the assertion. □

Write

$$H = \{h_\lambda : \lambda \geq 1\},$$

where h_λ is defined in the previous lemma. Now we can solve the optimal stopping problem:

Proposition 5.13. *The point (m,x) is in the optimal stopping set if and only if there exists $h \in H$ such that $(m,x) \in \arg\max \frac{g}{h}$.*
Furthermore $\tau^* = \inf\{t \geq 0 : M_t - X_t \geq z^*\}$ is an optimal stopping time, where z^* is given in the previous lemma and

$$v(m,x) = h_1(m,x) \qquad \textit{for all } (m,x) \textit{ with } m - x < z^*.$$

Proof. By Lemma 5.12 and Lemma 3.1 we know that $\tilde{S}$ is a subset of the optimal stopping set S. Now we prove that τ^* is optimal when the process is started in $(m,x) \notin \tilde{S}$:
Since in this case $(M_{\tau^*}, X_{\tau^*}) \in \arg\max \frac{g}{h_1}$ on $\{\tau^* < \infty\}$ it seems natural to use the idea of the Beibel-Lerche approach similarly to the arguments in [LU07, Section 4]. We obtain

$$\begin{aligned} E_{(m,x)}(e^{-r\tau}(M_\tau - X_\tau)^\alpha \mathbb{1}_{\{\tau<\infty\}}) &= E_{(m,x)}(e^{-r\tau} h_1(M_\tau, X_\tau) \frac{g(M_\tau, X_\tau)}{h_1(M_\tau, X_\tau)} \mathbb{1}_{\{\tau<\infty\}}) \\ &\leq \max \frac{g}{h_1} E_{(m,x)}(e^{-r\tau} h_1(M_\tau, X_\tau) \mathbb{1}_{\{\tau<\infty\}}) \\ &\leq \max \frac{g}{h_1} h_1(m,x) \end{aligned}$$

for all stopping times τ, so that it is enough to prove

$$E_{(m,x)}(e^{-r\tau^*} h_1(M_{\tau^*}, X_{\tau^*}) \mathbb{1}_{\{\tau^*<\infty\}}) = h_1(m,x).$$

To this end note that by Lemma 5.12 *(ii)* the process $(e^{-rt}h_1(M_t, X_t))_{t\geq 0}$ is indeed a local martingale. Now take a localizing sequence $(\sigma_n)_{n\in\mathbb{N}}$ of bounded stopping times for $(e^{-rt}h_1(M_t, X_t))_{t\geq 0}$. Note that

$$\begin{aligned} h_1(M_{\tau^*\wedge\sigma_n}, X_{\tau^*\wedge\sigma_n}) &= c\exp(-\beta_1(M_{\tau^*\wedge\sigma_n} - X_{\tau^*\wedge\sigma_n})) + \exp(-\beta_2(M_{\tau^*\wedge\sigma_n} - X_{\tau^*\wedge\sigma_n})) \\ &\leq ce^{-\beta_1 z^*} + 1 \end{aligned}$$

for all $n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} e^{-r(\tau^*\wedge\sigma_n)} h_1(M_{\tau^*\wedge\sigma_n}, X_{\tau^*\wedge\sigma_n}) = e^{-r\tau^*} h_1(M_{\tau^*}, X_{\tau^*}) \mathbb{1}_{\{\tau^*<\infty\}}$$

dominated convergence and optional stopping yields

$$\begin{aligned} E_{(m,x)}(e^{-r\tau^*} h_1(M_{\tau^*}, X_{\tau^*}) \mathbb{1}_{\{\tau^*<\infty\}}) &= \lim_{n\to\infty} E_{(m,x)}(e^{-r(\tau^*\wedge\sigma_n)} h_1(M_{\tau^*\wedge\sigma_n}, X_{\tau^*\wedge\sigma_n})) \\ &= h_1(m, x). \end{aligned}$$

This yields $v(m, x) = h_1(m, x)$ if $m - x < z^*$ and $v(m, x) = m - x$ if $m - x \geq z^*$, i.e. $\tilde{S}$ is the optimal stopping set and τ^* is optimal. □

Remark 5.14. *Note that the special form of g was used for convenience only to determine the maximum points in Lemma 5.12 (iii). The same arguments can be carried over to other classes of functions $g(m, x) = \tilde{g}(m-x)$ too, if one assumes that the set of maximum points has the same form as in Lemma 5.12, (iii).*

Chapter 6

Explicit results for optimal stopping of autoregressive processes

6.1 Introduction

Let $0 < \lambda \leq 1$, $(Z_n)_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables on a probability space $(\Omega, \mathcal{A}, P)$ and let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be the filtration generated by $(Z_n)_{n\in\mathbb{N}}$. Define the autoregressive process of order 1 (AR(1)-process) $(X_n)_{n\in\mathbb{N}_0}$ by

$$X_n = \lambda X_{n-1} + Z_n \quad \text{for all } n \in \mathbb{N}$$

i.e.

$$X_n = \lambda^n X_0 + \sum_{k=0}^{n-1} \lambda^k Z_{n-k}.$$

The random variables $(Z_n)_{n\in\mathbb{N}}$ are called the innovations of $(X_n)_{n\in\mathbb{N}_0}$. Using the difference notation the identity $X_n = \lambda X_{n-1} + Z_n$ can be written as

$$\Delta X_n = -(1-\lambda)X_{n-1}\Delta n + \Delta L_n,$$

where $\Delta X_n = X_n - X_{n-1}$, $\Delta n = n - (n-1) = 1$ and $\Delta L_n = \sum_{k=1}^{n} Z_k - \sum_{k=1}^{n-1} Z_k = Z_n$. This shows that AR(1)-processes are the discrete-time analogon to (Lévy-driven) Ornstein-Uhlenbeck processes. We just want to mention that many arguments in the following can be carried over to Ornstein-Uhlenbeck processes as well.
Autoregressive processes were studied in detail in the last years. The joint distribution of the threshold-time

$$\tau_b = \inf\{n \in \mathbb{N}_0 : X_n \geq b\}$$

and the overshoot $X_{\tau_b} - b$ over a fixed level b was of special interest. This first passage problem was considered in different applications, such as signal detection and surveillance analysis, cf. [FS06]. If $\lambda = 1$ the process $(X_n)_{n\in\mathbb{N}_0}$ is a random walk and many results about this distributions are well known. Most of them are based on techniques using the Wiener Hopf-factorization – see [Fel66, Chapter VII] for an overview. Unfortunately no analogon to the Wiener-Hopf factorization is known for AR(1)-processes, so that other ideas are necessary. To get rid of well-studied cases we assume that $\lambda < 1$ in the following.

Most known results about the distribution of τ_b are based on martingales defined by using integrals of the form

$$\int_0^\infty e^{uy-\phi(u)}u^{v-1}du \tag{6.1}$$

where ϕ is the logarithm of the Laplace transform of the stationary distribution discussed in Section 6.4. For the integral to be well defined it is necessary that $E(e^{uZ_1}) < \infty$ for all $u \in [0,\infty)$ – cf. [NK08] and the references therein.

On the other hand if one wants to obtain explicit results about the joint distribution of τ_b and the overshoot it is useful to assume Z_1 to be exponentially distributed. In this case explicit results are given in [CIN10, Section 3] by setting up and solving differential equations. Unfortunately in this case not all exponential moments of Z_1 exist and the integral described above cannot be used.

The contribution of this chapter is twofold:

1. We find the joint distribution of τ_b and the overshoot for a wide class of innovations: We assume that $Z_1 = S_1 - T_1$, where S_1 has a phasetype distribution and $T_1 \geq 0$ is arbitrary. This generalizes the assumption of exponentially distributed innovations to a much wider class. In Section 6.3 we establish that for this class τ_b and the overshoot are – conditioned on certain events – independent and we find the distribution of the overshoot. In Section 6.4 we use a series inspired by the integral (6.1) to construct martingales with the objective of finding the distribution of τ_b.

2. As an application we consider the (Markovian) problem of optimal stopping for $(X_n)_{n\in\mathbb{N}_0}$ with discounted non-negative continuous gain function g, i.e. we study the optimization problem

$$v(x) = \sup_{\tau\in\mathcal{T}} E_x(\rho^\tau g(X_\tau)) = \sup_{\tau\in\mathcal{T}} E(\rho^\tau g(\lambda^\tau x + X_\tau)), \quad x \in \mathbb{R}, \quad 0 < \rho < 1,$$

where $\mathcal{T}$ denotes the set of stopping times with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$; to simplify notation here and in the following we set the payoff equal to 0 on $\{\tau = \infty\}$. Just very few results are known for this problem. In [Nov09] and [CIN10] the innovations are assumed to be exponentially distributed and in [Fin82] asymptotic results were

given for $g(x) = x$. Our way to a solution is the following:
First we find easy conditions to ensure that the optimal stopping time is a threshold-time; this is done in Section 6.2. In a second step we use the joint distribution of τ_b and the overshoot to find the optimal threshold. One way is described in Section 6.5. Another way is the principle of continuous fit, that is established in Section 6.6. An example is given in Section 6.7.

6.2 Simple conditions for the optimality of threshold-times

To tackle the optimal stopping problem described above it is useful to reduce the (infinite dimensional) set of stopping times to a finite dimensional subclass. In Section 3.4 we gave easy conditions for the optimal stopping time to be of threshold type in the case of diffusion processes. But we proved them after characterizing the optimal stopping time in a convenient way. Because we do not have such a characterization on hand for AR(1)-processes we have two possibilities to proceed in a different way:

(a) We use elementary arguments to reduce the set of potential optimal stopping times to the subclass of threshold-times, i.e. to stopping times of the form

$$\tau_b = \inf\{t \geq 0 : X_t \geq b\}$$

for some $b \in \mathbb{R}$. Then we find the optimal threshold. An example for this type of argument is given in Subsection 6.2.1. Cf. [CIN10, Section 2] for more results in this direction.

(b) We make the ansatz that the optimal stopping time is of threshold type, identify the optimal threshold and use a verification theorem to prove that this stopping time is indeed optimal. Such a verification theorem is given in Subsection 6.2.2.

We want to remark that under very general conditions it holds that $\tau_b < \infty$ P_x-a.s. for all $x, b \in \mathbb{R}$ – cf. Theorem 2 in [NK08] and the discussion on the existence of a stationary distribution in Section 6.4. These conditions are fulfilled for all cases of interest in the following.

6.2.1 The Novikov-Shiryaev-problem for AR(1)-processes

In the random walk case, the problem of optimal stopping for

$$g(x) = (x^+)^\alpha,\ \alpha > 0$$

has been of particular interest and was completely solved in [NS04]. The solution is based on the use of Appell polynomials and on the Wiener-Hopf factorization. After establishing the solution it comes out that the optimal stopping time is of threshold type. In this subsection we go the inverse way by giving elementary arguments to determine the form of the optimal stopping time. The results of this subsection are included in the article [CIN10].

If we consider AR(1)-sequences with nonnegative innovations we take $[0,\infty)$ as the state space of $(X_n)_{n\in\mathbb{N}_0}$ and consider

$$g(x) = x^\alpha,\ \alpha > 0.$$

Then for any $x > 0,\ 0 < \delta \leq 1,\ z \geq 0$

$$\frac{g(\delta x + z)}{g(x)} = \left(\delta + \frac{z}{x}\right)^\alpha \text{ is decreasing in } x$$

hence so is

$$\frac{v(x)}{g(x)} = \sup_{\tau\in\mathcal{T}} E\left(\rho^\tau\left(\lambda^\tau + \frac{X_\tau}{x}\right)^\alpha\right).$$

Thus if x is in the optimal stopping set, then y is in the optimal stopping set too for all $y \geq x$, and the optimal stopping problem

$$\sup_{\tau\in\mathcal{T}} E_x(\rho^\tau X_\tau^\alpha)$$

is solved by a threshold-time for any $\alpha > 0$. Unfortunately this easy argument does not work for general innovations so that another approach is necessary:

Without further assumptions on the innovations we consider

$$g(x) = (x^+)^\alpha,\ \alpha > 0$$

and denote the associated optimal stopping set by S_α. Under minimal assumptions it holds that $0 < v(x) < \infty$ for all x and we have $S_\alpha \subset (0,\infty)$. This yields

$$x \in S_\alpha \quad \Leftrightarrow \quad E(\rho^\tau(\frac{X_\tau}{x} + \lambda^\tau)^\alpha) \leq 1 \text{ for all } \tau \in \mathcal{T}_x,$$

where $\mathcal{T}_x$ is the set of all stopping times with $X_\tau + \lambda^\tau x \geqslant 0$ P-a.s.

The following observation about the dependence of the stopping sets on α is useful to establish the threshold structure of the stopping set:

Lemma 6.1. *For all $0 < \beta \leq \alpha$ it holds that $S_\alpha \subseteq S_\beta$.*

Proof. Let $x \in S_\alpha$ and $\tau \in \mathcal{T}_x$. Jensen's inequality yields

$$\begin{aligned}1 &\geq E(\rho^\tau(\frac{X_\tau}{x} + \lambda^\tau)^\alpha) = E((\rho^{\frac{\beta}{\alpha}\tau}(\frac{X_\tau}{x} + \lambda^\tau)^\beta)^{\alpha/\beta}) \\ &\geq (E(\rho^{\frac{\beta}{\alpha}\tau}(\frac{X_\tau}{x} + \lambda^\tau)^\beta))^{\alpha/\beta} \geq (E(\rho^\tau(\frac{X_\tau}{x} + \lambda^\tau)^\beta))^{\alpha/\beta},\end{aligned}$$

i.e. $E(\rho^\tau(\frac{X_\tau}{x} + \lambda^\tau)^\beta) \leq 1$. □

Now we are prepared to show that the optimal stopping time is of threshold type:

Proposition 6.2. *Fix $n \in \mathbb{N}$ and assume the first hitting time τ of S_n to be optimal. Then τ is of threshold type, i.e. there exists $x_n \in (0, \infty)$ such that $S_n = [x_n, \infty)$.*

Proof. Write $x := x_n := \inf(S_n)$. Since S_n is closed $x \in S_n$.
Let $m \in \mathbb{N}$ such that $mx \in S_n$. We first prove that $y = (m + c)x \in S_n$ for all $c \in [0, 1]$. Let τ be the optimal stopping time for the process started in y under P, i.e.

$$\tau = \inf\{m \in \mathbb{N}_0 : X_m + \lambda^m y \in S_n\}.$$

We have $X_\tau + \lambda^\tau y \geq x$ P-a.s. and we obtain

$$X_\tau + \lambda^\tau mx \geq x - \lambda^\tau cx \geq 0,$$

i.e. $\tau \in \mathcal{T}_{mx}$. Furthermore it holds that

$$\begin{aligned}v(y) &= E(\rho^\tau(X_\tau + \lambda^\tau y)^n) = E(\rho^\tau((X_\tau + \lambda^\tau mx) + \lambda^\tau cx)^n) \\ &= \sum_{k=0}^{n} \binom{n}{k} E(\rho^\tau(\lambda^\tau cx)^{n-k}(X_\tau + \lambda^\tau mx)^k)\end{aligned}$$

Since mx is in the stopping set we obtain

$$\begin{aligned}v(y) &= \sum_{k=0}^{n} \binom{n}{k} E(\rho^\tau(\lambda^\tau cx)^{n-k}(X_\tau + \lambda^\tau mx)^k) \\ &\leq \sum_{k=0}^{n} \binom{n}{k} (cx)^{n-k} E(\rho^\tau(X_\tau + \lambda^\tau mx)^k) \\ &\leq \sum_{k=0}^{n} \binom{n}{k} (cx)^{n-k}(mx)^k = (cx + mx)^n = y^n.\end{aligned}$$

By applying this to $c = 1$ we get by induction that mx is in the stopping set for all positive integers m. Thus we obtain the statement for all y by applying the calculation to general c. We remark that the induction is necessary to guarantee that $X_\tau + \lambda^\tau mx$ is always non-negative. □

Remark 6.3. *Note that all arguments work for random walks and Lévy processes as well. This can be used for another approach to the well-studied optimal stopping problem for these processes.*

6.2.2 Verification theorem

To use the second approach described in the introduction of this section the following easy verification theorem is useful.

Lemma 6.4. *Let $b^* \in \mathbb{R}$, write $v^*(x) = E_x(\rho^{\tau_{b^*}} g(X_{\tau_{b^*}}))$ and assume that*

(a) $v^*(x) \geq g(x)$ for all $x < b^*$.

(b) $E(\rho v^*(\lambda x + Z_1)) \leq v^*(x)$ for all $x \in \mathbb{R}$.

Then $v = v^*$ and τ_{b^*} is optimal.

Proof. By the independence of $(Z_n)_{n\in\mathbb{N}}$ property (b) implies that $(\rho^n v^*(X_n))_{n\in\mathbb{N}}$ is a supermartingale under each measure P_x. Since it is positive the optional sampling theorem leads to

$$v^*(x) \geq \sup_{\tau\in\mathcal{T}} E_x(\rho^\tau v^*(X_\tau)) \geq \sup_{\tau\in\mathcal{T}} E_x(\rho^\tau g(X_\tau)) \quad \text{for all } x \in \mathbb{R},$$

where the second inequality holds by (a) since $v^*(x) = g(x)$ for all $x \geq b$. On the other hand $v^*(x) \leq v(x)$, i.e. $v^*(x) = v(x)$ and τ_{b^*} is optimal. □

6.3 Innovations of phasetype

In this section we recall some basic properties of phasetype distributions and identify the connection to AR(1)-processes.
In the first subsection we establish the terminology and state some well-known results, that are of interest for our purpose. All results can be found in [Asm03] discussed from the perspective of queueing theory.
In the second subsection we concentrate on the threshold-time distribution for autoregressive processes when the positive part of the innovations is of phasetype. The key result for the next sections is that – conditioned to certain events – the threshold-time is independent of the overshoot and the overshoot is phasetype distributed as well.

6.3.1 Definition and some properties

Let $m \in \mathbb{N}$, $E = \{1, ..., m\}$, $\Delta = m + 1$ and $E_\Delta = E \cup \{\Delta\}$.
In this subsection we consider a Markov chain $(J_t)_{t\geq 0}$ in continuous time with state space

E_Δ. The states $1, ..., m$ are assumed to be transient and Δ is absorbing. Denote the generator of $(J_t)_{t\geq 0}$ by $\hat{Q} = (q_{ij})_{i,j\in E_\Delta}$, i.e.

$$\hat{q}_{ij}(h) := P(J_{t+h} = j | J_t = i) = q_{ij}h + o(h) \text{ for all } i \neq j \in E_\Delta \text{ and}$$
$$\hat{q}_{ii}(h) := P(J_{t+h} = i | J_t = i) = 1 + q_{ii}h + o(h) \text{ for all } i \in E_\Delta \text{ and } h \to 0, t \geq 0.$$

If we write $\hat{Q}(h) = (\hat{q}_{ij}(h))_{i,j\in E_\Delta}$ for all $h \geq 0$, then $(\hat{Q}(h))_{h\geq 0}$ is a semigroup and the general theory yields that

$$\hat{Q}(h) = e^{\hat{Q}h} \text{ for all } h \geq 0.$$

Since Δ is assumed to be absorbing $\hat{Q}$ has the form

$$\hat{Q} = \begin{pmatrix} Q & -Q\mathbf{1} \\ 0...0 & 0 \end{pmatrix}$$

for an $m \times m$-matrix Q, where $\mathbf{1}$ denotes the column-vector with entries 1.
We consider the survival time of $(J_t)_{t\geq 0}$, i.e. the random variable

$$\eta = \inf\{t \geq 0 : J_t = \Delta\}.$$

Let $\hat{\alpha} = (\alpha, 0)$ be an initial distribution of $(J_t)_{t\geq 0}$. Here and in the following $\alpha = (\alpha_1, ..., \alpha_m)$ is assumed to be a row-vector.

Definition 6.1. *$P^\eta_{\hat{\alpha}}$ is called a distribution of phasetype with parameters (Q, α) and we write $P^\eta_{\hat{\alpha}} = PH(Q, \alpha)$ for short.*

Let $m = 1$ and $Q = (-\beta)$ for a parameter $\beta > 0$. In this case it is well-known that η is exponentially distributed with parameter β. This special case will be the key example we often think of. Furthermore let us mention that the class of phasetype distributions is stable under convolutions and mixtures. This shows that the important classes of Erlang- and hyperexponential distributions are of phasetype.
Exponential distributions have a very special structure, but phasetype distributions are flexible:

Proposition 6.5. *The distributions of phasetype are dense in the space of all probability measures on $(0, \infty)$ with respect to convergence in distribution.*

Proof. See [Asm03, III, Theorem 4.2]. □

The definition of phasetype distributions does not give rise to an obvious calculus with these distributions, but the theory of semigroups leads to simple formulas for the density and the Laplace-transform as the next lemma shows. All the formulas contain matrix exponentials. The explicit calculation of such exponentials can be complex in higher dimensions, but many algorithms are available for a numerical approximation.

Proposition 6.6. *(a) The eigenvalues of Q have negative real part.*

(b) The distribution function of $PH(Q, \alpha)$ is given by

$$H_\alpha(s) := P_\alpha(\eta \leq s) = 1 - \alpha e^{Qs}\mathbf{1}, \quad s \geq 0.$$

(c) The density is given by

$$h_\alpha(s) = \alpha e^{Qs} q, \qquad s \geq 0$$

where $q = -Q\mathbf{1}$.

(d) For all $s \in \mathbb{C}$ with $E_{\hat{\alpha}}(e^{\Re(s)\eta}) < \infty$ it holds that

$$\hat{H}_\alpha(s) := E_{\hat{\alpha}}(e^{s\eta}) = \alpha(-sI - Q)^{-1}q,$$

where I is the $m \times m$-identity matrix.
In particular $\hat{H}_\alpha$ is a rational function.

Proof. See [Asm03, II, Corollary 4.9 and III, Theorem 4.1]. □

An essential property for the applicability of the exponential distribution in modeling and examples is the memoryless property, which even characterizes the exponential distribution. The next lemma can be seen as a generalization of this property to distributions of phasetype.

Lemma 6.7. *Let $t \geq 0$ and write*

$$H^t_\alpha(s) = P_{\hat{\alpha}}(\eta \leq s + t | \eta \geq t) \quad \textit{for all } s \geq 0.$$

Then H^t_α is a distribution function of a phasetype distribution with parameters (Q, π^t), where $\pi^t_i = P_{\hat{\alpha}}(J_t = i | \eta \geq t)$ for all $i = 1, ..., m$.

Proof. By Proposition 6.6 the random variable η has a continuous distribution. Therefore using the Markov-property of $(J_t)_{t \geq 0}$ we obtain

$$\begin{aligned}
P_{\hat{\alpha}}(\eta \leq t + s | \eta \geq t) &= P_{\hat{\alpha}}(\eta \leq t + s | \eta > t) \\
&= \sum_{i \in E} P_{\hat{\alpha}}(\eta \leq t + s, J_t = i) \frac{1}{P_{\hat{\alpha}}(\eta > t)} \\
&= \sum_{i \in E} P_{\hat{\alpha}}(\eta \leq t + s | J_t = i) \frac{P_{\hat{\alpha}}(J_t = i)}{P_{\hat{\alpha}}(\eta > t)} \\
&= \sum_{i \in E} P_{e_i}(\eta \leq s) \pi^t_i = \sum_{i \in E} \pi^t_i H_{e_i}(s),
\end{aligned}$$

where e_i is the i-th unit vector. □

For the application to autoregressive processes we need the generalization of the previous lemma to the case that the random variable is not necessarily positive.

Lemma 6.8. *Let $S, T \geq 0$ be stochastically independent random variables, where S is $PH(Q, \alpha)$-distributed. Furthermore let $r \geq 0$ and $Z = S - T$. Then*

$$P_{\hat{\alpha}}(r \leq Z \leq r+s) = \sum_{i \in E} \lambda_i(r) H_{e_i}(s), \quad s \geq 0,$$

where $\lambda_i(r) = \int P_{\hat{\alpha}}(J_{r+t} = i) P_{\hat{\alpha}}^T(dt)$.

Proof. Using Lemma 6.7 it holds that

$$\begin{aligned} P_{\hat{\alpha}}(r \leq Z \leq r+s) &= \int P_{\hat{\alpha}}(r+t \leq S \leq r+s+t) P_{\hat{\alpha}}^T(dt) \\ &= \sum_{i \in E} \int H_{e_i}(s) P_{\hat{\alpha}}(J_{t+r} = i) P_{\hat{\alpha}}^T(dt). \end{aligned}$$

□

6.3.2 Phasetype distributions and overshoot of AR(1)-processes

We again consider the situation of Section 6.1. In addition we assume the innovations to have the following structure:

$$Z_n = S_n - T_n \qquad \text{for all } n \in \mathbb{N},$$

where S_n and T_n are non-negative and independent and S_n is $PH(Q, \alpha)$-distributed. In this context we remark that each probability measure Q on $\mathbb{R}$ with $Q(\{0\}) = 0$ can be written as $Q = Q_+ * Q_-$ where Q_+ and Q_- are probability measures with $Q_+((-\infty, 0)) = Q_-((0, \infty)) = 0$ and $*$ denotes convolution (cf. [Fel66, p.383]).

As a motivation we consider the case of exponentially distributed innovations. If Z_n is exponentially distributed then it holds that for all $\rho \in (0, 1]$ and measurable $g : \mathbb{R} \to [0, \infty)$

$$E_x(\rho^\tau g(X_\tau)) = E_x(\rho^\tau) E(g(R+b)), \tag{6.2}$$

where $\tau = \tau_b$ is a threshold-time, $x < b$ and R is exponentially distributed with the same parameter as the innovations (cf. [CIN10, Theorem 3.1]). This fact is well known for random walks, cf. [Fel66, Chapter XII.]. The representation of the joint distribution of overshoot and τ reduces to finding a explicit expression of the Laplace-transform of τ. In this subsection we prove that a generalization of this phenomenon holds in our more general situation.

To this end we use an embedding of $(X_n)_{n\in\mathbb{N}_0}$ into a stochastic process in continuous time as follows:
For all $n \in \mathbb{N}$ denote the Markov chain which generates the phasetype-distribution of S_n by $(J_t^{(n)})_{t\geq 0}$ and write

$$J_t = J^{(n_t+1)}_{t-\sum_{k=1}^{n_t} S_k}, \text{ where } n_t = \max\{n \in \mathbb{N}_0 : \sum_{k=1}^{n} S_k \leq t\} \text{ for all } t \geq 0 .$$

Hence the process $(J_t)_{t\geq 0}$ is constructed by compounding the processes $J^{(n)}$ restricted to their lifetime. Obviously $(J_t)_{t\geq 0}$ is a continuous time Markov chain with state space E, as one immediately checks (cf. [Asm03, III, Proposition 5.1]). Furthermore we define a process $(Y_t)_{t\geq 0}$ by

$$Y_t = \lambda X_{n_t} - T_{n_t+1} + t - \sum_{k=1}^{n_t} S_k.$$

See Figure 6.1 for an illustration. It holds that

$$X_n = Y_{(S_1+\ldots+S_n)-} \quad \text{for all } n \in \mathbb{N},$$

so that we can find $(X_n)_{n\in\mathbb{N}_0}$ in $(Y_t)_{t\geq 0}$. Now let $\hat{\tau}$ be the threshold-time of the process $(Y_t)_{t\geq 0}$ over the threshold b, i.e.

$$\hat{\tau} = \inf\{t \geq 0 : Y_t \geq b\}.$$

By definition of $(Y_t)_{t\geq 0}$ it holds that

$$Y_t = b \quad \Leftrightarrow \quad t = -\lambda X_{n_t} + T_{n_t+1} + \sum_{k=1}^{n_t} S_k + b \text{ for all } t \geq 0. \tag{6.3}$$

For the following result we need the event that the associated Markov chain is in state i when $(Y_t)_{t\geq 0}$ crosses $b \geq 0$, i.e. the event

$$G_i = \{J_{\hat{\tau}} = i\} \quad \text{for } i \in E.$$

For the following considerations we fix the threshold $b \geq 0$.
In generalization of the result for exponential distributed innovations the following theorem states that – conditioned on G_i – the threshold-time and the overshoot are independent and the overshoot is phasetype distributed as well.

Theorem 6.9. *Let $x < b$, $n \in \mathbb{N}$, $y \geq 0$ and write*

$$\tau = \tau_b = \inf\{n \in \mathbb{N}_0 : X_n \geq b\}.$$

Then

$$P_x(X_\tau - b \leq y, \tau = n) = \sum_{i\in E} H_{e_i}(y) P_x(\tau = n, G_i).$$

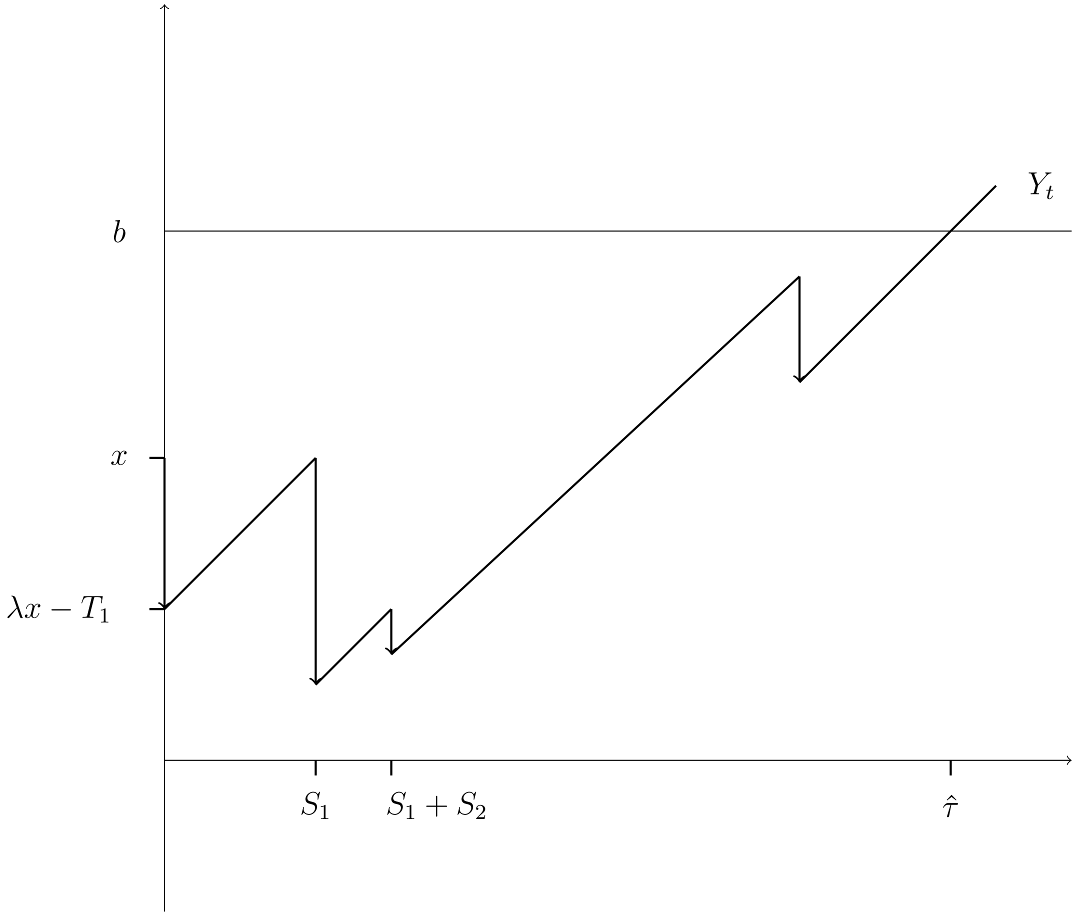

Figure 6.1: A path of $(Y_t)_{t\geq 0}$

Proof. Using Lemma 6.8 and the identity (6.3) we obtain

$$\begin{aligned} P_x(X_\tau - b \leq y, \tau = n) &= E_x(\mathbb{1}_{\{\tau\geq n\}} P_x(X_n \geq b, X_n - b \leq y | \mathcal{F}_{n-1})) \\ &= E_x(\mathbb{1}_{\{\tau\geq n\}} P_x(b \leq X_n \leq b + y | \mathcal{F}_{n-1})) \\ &= E_x(\mathbb{1}_{\{\tau\geq n\}} P_x(b - \lambda X_{n-1} \leq Z_n \leq b + y - \lambda X_{n-1} | \mathcal{F}_{n-1})) \\ &= E_x(\mathbb{1}_{\{\tau\geq n\}} \sum_{i\in E} H_{e_i}(y) P_x(J^{(n)}_{b-\lambda X_{n-1}+T_n} = i | \mathcal{F}_{n-1})) \\ &= \sum_{i\in E} H_{e_i}(y) P_x(\tau = n, J_{\hat{\tau}} = i). \end{aligned}$$

□

This immediately implies a generalization of (6.2) to the case of general phasetype distributions:

Corollary 6.10. *It holds that*

$$E_x(\rho^\tau g(X_\tau)) = \sum_{i\in E} E_x(\rho^\tau \mathbb{1}_{G_i}) E(g(b + R^i)),$$

where R^i is a $Ph(Q, e_i)$-distributed random variable (under P).

Hence we reduced the problem of finding $E_x(\rho^\tau g(X_\tau))$ to the problem of finding $E_x(\rho^\tau \mathbb{1}_{G_i})$ for all $i \in E$. This problem is treated in the following section:

6.4 Construction of appropriate martingales

The aim of this section is to construct martingales of the form $(\rho^{n\wedge\tau} h(X_{n\wedge\tau}))_{n\in\mathbb{N}}$ as a tool for the explicit representation of $\Phi_i^b(x) = \Phi_i(x) = E_x(\rho^\tau \mathbb{1}_{G_i})$ for $\tau = \tau_b$ and $b > x$. To this end some definitions are necessary.
We assume the setting of the previous section, i.e. we assume that the innovations can be written in the form

$$Z_n = S_n - T_n \quad \text{for all } n \in \mathbb{N},$$

where S_n and T_n are non-negative and independent and S_n is $PH(Q, \alpha)$-distributed.

Let $\exp(\psi)$ be the Laplace-transform of Z_1, i.e. $\psi(u) = \log E(e^{uZ_1})$ for all $u \in \mathbb{C}_+ := \{z \in \mathbb{C} : \Re(z) \geq 0\}$ with real part $\Re(u)$ so small that the expectation exists. Since $E(e^{uZ_1}) = E(e^{uS_1})E(e^{-uT_1})$ and $T_1 \geq 0$ Proposition 6.6 yields the existence of $\psi(u)$ for all u with $\Re(u)$ smaller then the smallest eigenvalue of $-Q$. ψ is analytic on this stripe and – because of independence – it holds that

$$\psi(u) = \psi_1(u) + \psi_2(u),$$

where $\exp(\psi_1)$ denotes the Laplace-transform of S_1 and $\exp(\psi_2)$ is the Laplace-transform of $-T_1$. ψ_2 is analytic on $\mathbb{C}_+$ and ψ_1 can be analytically extended to $\mathbb{C}_+ \setminus Sp(-Q)$ by Proposition 6.6. Here $Sp(\cdot)$ denotes the spectrum, i.e. the set of all eigenvalues. Hence ψ can be extended to $\mathbb{C}_+ \setminus Sp(-Q)$ as well and this extension is again denoted by ψ. Note that this extension can not be interpreted from a probabilistic point of view because $E(e^{uZ_1})$ does not exist for $u \in \mathbb{C}_+$ with too large real part.
To guarantee the convergence of $(X_n)_{n\in\mathbb{N}_0}$ we assume a weak integrability condition – the well known Vervaat condition

$$E(\log(1 + |Z_1|)) < \infty, \tag{6.4}$$

see [GM00, Theorem 2.1] for a characterization of such conditions in the theory of perpetuities. We do not go into details here, but just want to use the fact that $(X_n)_{n\in\mathbb{N}_0}$ converges to a (finite) random variable θ in distribution, that fulfills the stochastic fixed point equation

$$P^\theta = P^{\lambda\theta} * P^{Z_1},$$

where $*$ denotes convolution. Since the AR(1)-process has the representation

$$X_n = \lambda^n X_0 + \sum_{k=0}^{n-1} \lambda^k Z_{n-k}$$

and convergence in distribution is equivalent to the pointwise convergence of the Laplace-transforms the Laplace-transform $\exp(\phi)$ of θ fulfills

$$\phi(u) = \sum_{k=0}^{\infty} \psi(\lambda^k u) \tag{6.5}$$

for all $u \in \mathbb{C}_+$ such that the Laplace-transform of S_1 exists.
The right hand side defines a holomorphic function on $\mathbb{C}_+ \setminus \hat{P}$ that is also denoted by ϕ, where we write $\hat{P} = \bigcup_{n\in\mathbb{N}_0} Sp(-\lambda^{-n}Q)$. For the convergence of the series note that – as described above – it converges for all $u \in \mathbb{C}_+$ such that $E(e^{uZ_1}) < \infty$. For all other $u \in \mathbb{C}_+$ the series also converges since there exists k_0 such that $E(e^{\lambda^k u Z_1}) < \infty$ for all $k \geq k_0$.
Furthermore the identity

$$\phi(u) = \phi(\lambda u) + \psi(u) \tag{6.6}$$

holds, whenever $u, \lambda u$ are in the domain of ϕ. To avoid problems concerning the applicability of (6.6) we assume that

$$Sp(\lambda^n Q) \cap Sp(Q) = \emptyset \quad \text{for all } n \in \mathbb{N}. \tag{6.7}$$

We would like to mention, that the function ϕ was used and studied in [NK08] as well.

The next two lemmas are helpful in the construction of the martingales.

Lemma 6.11. *Let $\delta \in \mathbb{C}_+$ such that $E(e^{\delta S_1})$ exists. Then for all $x < b$ it holds that*

$$\rho E(e^{\delta(\lambda x + Z_1)} \mathbb{1}_{\{\lambda x + Z_1 \geq b\}}) = \alpha_\delta e^{-\lambda x Q} q,$$

where $\alpha_\delta = \rho\alpha(-\delta I - Q)^{-1} e^{(\delta I + Q)b + \psi_2(-Q)}$.

Proof. In the following calculation we use the fact that all matrices commutate and that all eigenvalues of $\delta I + Q$ have negative real part. It holds

$$\begin{aligned}
E(e^{\delta(\lambda x + Z_1)} \mathbb{1}_{\{\lambda x + S_1 \geq b\}}) &= e^{\delta\lambda x} \int_0^\infty E(e^{\delta(S-t)} \mathbb{1}_{\{\lambda x + S_1 - t \geq b\}}) P^T(dt) \\
&= e^{\delta\lambda x} \int_0^\infty \int_{b+t-\lambda x}^\infty e^{\delta(s-t)} \alpha e^{Qs} q ds P^T(dt) \\
&= e^{\delta\lambda x} \int_0^\infty e^{-\delta t} \alpha \int_{b+t-\lambda x}^\infty e^{(\delta I + Q)s} ds q P^T(dt) \\
&= -e^{\delta\lambda x} \int_0^\infty e^{-\delta t} \alpha (\delta I + Q)^{-1} e^{(\delta I + Q)(b+t-\lambda x)} q P^T(dt).
\end{aligned}$$

We obtain

$$\begin{aligned}\rho E(e^{\delta(\lambda x+Z_1)}\mathbb{1}_{\{\lambda x+Z_1\geq b\}}) &= -\rho e^{\delta\lambda x}\alpha\int_0^\infty e^{Qt}P^T(dt)(\delta I+Q)^{-1}e^{(\delta I+Q)(b-\lambda x)}q\\ &= -\rho\alpha\int_0^\infty e^{-Qs}P^{-T}(ds)(\delta I+Q)^{-1}e^{-Q\lambda x}e^{(\delta I+Q)b}q\\ &= \alpha_\delta e^{-Q\lambda x}q.\end{aligned}$$

□

We write $Q_\gamma = -\gamma Q$ for short. For all $\gamma \in \mathbb{C}_+$ fulfilling

$$Sp(\lambda^n Q_\gamma)\cap\hat{P}=\emptyset \quad \text{for all } n\in\mathbb{N} \tag{6.8}$$

we define the function

$$f_\gamma:\mathbb{R}\to\mathbb{C}^{m\times m}, x\mapsto\sum_{n\in\mathbb{N}}e^{x\lambda^n Q_\gamma-\phi(\lambda^n Q_\gamma)}\rho^{n-1}.$$

This series converges because $e^{x\lambda^n Q_\gamma-\phi(\lambda^n Q_\gamma)}$ is bounded in n. Note that the summand of this series is similar to the integrand in (6.1).

Lemma 6.12. *There exists $\delta>0$ such that for all $x\in\mathbb{R}$ and $\gamma\in\mathbb{C}_+$ with $|\gamma|<\delta$ it holds that*

$$\rho E(f_\gamma(\lambda x+Z_1)) = f_\gamma(x) - e^{\lambda x Q_\gamma-\phi(\lambda Q_\gamma)}.$$

Proof. For all $\gamma\in\mathbb{C}_+$ with $|\gamma|$ sufficiently small the expected value $E(e^{Q_\gamma\lambda^n Z_1})$ exists for all $n\in\mathbb{N}$ since Q has (finitely many) negative eigenvalues. This leads to

$$\begin{aligned}E(f_\gamma(\lambda x+Z_1)) &= E(\sum_{n\in\mathbb{N}}e^{(\lambda x+Z_1)\lambda^n Q_\gamma-\phi(\lambda^n Q_\gamma)}\rho^{n-1})\\ &= \sum_{n\in\mathbb{N}}e^{(\lambda x)\lambda^n Q_\gamma-\phi(\lambda^n Q_\gamma)}E(e^{\lambda^n Q_\gamma Z_1})\rho^{n-1}\\ &= \sum_{n\in\mathbb{N}}e^{x\lambda^{n+1}Q_\gamma-\phi(\lambda^n Q_\gamma)+\psi(\lambda^n Q_\gamma)}\rho^{n-1}\\ &\overset{(6.6)}{=} \frac{1}{\rho}\sum_{n\in\mathbb{N}}e^{x\lambda^{n+1}Q_\gamma-\phi(\lambda^{n+1}Q_\gamma)}\rho^n\\ &= \frac{1}{\rho}(f_\gamma(x)-e^{\lambda Q_\gamma x-\phi(\lambda Q_\gamma)}).\end{aligned}$$

□

The next step is to find a family of equations characterizing

$$\Phi(x) = (\Phi_1(x),...,\Phi_m(x)) = (E_x(\rho^\tau\mathbb{1}_{G_1}),...,E_x(\rho^\tau\mathbb{1}_{G_m}))$$

using martingale techniques where $\tau = \tau_b$, $x < b$. To this end we consider

$$h_{\gamma,\delta} : \mathbb{R} \to \mathbb{C}, x \mapsto e^{\delta x} \mathbb{1}_{\{x \geq b\}} + \beta_{\gamma,\delta} f_\gamma(x) q,$$

for all $\delta \in \mathbb{C}_+ \setminus SP(-Q)$ and γ fulfilling (6.8) where $\beta_{\gamma,\delta} = \alpha_\delta e^{\phi(\lambda Q_\gamma)}$. For the special value $\gamma = 1$ we write $h_\delta = h_{\gamma,\delta}$ and this function is well-defined by (6.7).

Putting together the results of Lemma 6.11 and Lemma 6.12 we obtain the equation

$$\rho E_x(h_{\gamma,\delta}(X_1)) - h_{\gamma,\delta}(x) = \alpha_\delta e^{-\lambda x Q} q - \alpha_\delta e^{-\lambda \gamma x Q} q \quad \text{for all } x < b \tag{6.9}$$

for all $\gamma, \delta \in \mathbb{C}_+$ with sufficiently small modulus.

Before stating the equations we need one more technical result.

Lemma 6.13. *Let R^i be a $PH(Q, e_i)$-distributed random variable and denote by $\psi^i(\cdot) = \log(e_i(-\cdot I - Q)^{-1} q)$ the holomorphic extension of the logarithmized Laplace-transform of R_i. Here e_i denotes the i-th unit vector. Let $|\gamma|, |\delta|$ be so small that $E(h_{\gamma,\delta}(b + R^i))$ exists. Then it holds that*

$$E(h_{\gamma,\delta}(b + R^i)) = e^{\delta b} \alpha_{\gamma,i} (-\delta I - Q)^{-1} q =: \eta_{\gamma,\delta,i},$$

where

$$\alpha_{\gamma,i} = e_i + \alpha e^{Qb + \psi_2(-Q) + \phi(\lambda Q_\gamma)} \sum_{n \in \mathbb{N}} e^{b \lambda^n Q_\gamma - \phi(\lambda^n Q_\gamma) + \psi^i(\lambda^n Q_\gamma)} \rho^n.$$

Proof. Simple calculus yields the result:

$$\begin{aligned}
E(h_{\gamma,\delta}(b + R^i)) &= E(e^{\delta(b+R^i)} + \beta_{\gamma,\delta} \sum_{n \in \mathbb{N}} e^{(b+R^i)\lambda^n Q_\gamma - \phi(\lambda^n Q_\gamma)} \rho^{n-1} q) \\
&= e^{\delta b} e_i (-\delta I - Q)^{-1} q + \beta_{\gamma,\delta} \sum_{n \in \mathbb{N}} e^{b \lambda^n Q_\gamma - \phi(\lambda^n Q_\gamma)} E(e^{\lambda^n Q_\gamma R^i}) \rho^{n-1} q \\
&= e^{\delta b} e_i (-\delta I - Q)^{-1} q \\
&\quad + \rho \alpha (-\delta I - Q)^{-1} e^{(\delta I + Q) b + \psi_2(-Q) + \phi(\lambda Q_\gamma)} (\sum_{n \in \mathbb{N}} e^{b \lambda^n Q_\gamma - \phi(\lambda^n Q_\gamma) + \psi^i(\lambda^n Q_\gamma)} \rho^{n-1}) q \\
&= e^{\delta b} (e_i + \alpha e^{Qb + \psi_2(-Q) + \phi(\lambda Q_\gamma)} \sum_{n \in \mathbb{N}} e^{b \lambda^n Q_\gamma - \phi(\lambda^n Q_\gamma) + \psi^i(\lambda^n Q_\gamma)} \rho^n) (-\delta I - Q)^{-1} q.
\end{aligned}$$

□

Theorem 6.14. *For all $x < b$ and $\delta \in \mathbb{C}_+ \setminus Sp(-Q)$ it holds that*

$$\sum_{i=1}^m \eta_{\delta,i} \Phi_i(x) = h_\delta(x),$$

where $\eta_{\delta,i} = \eta_{1,\delta,i}$ is given in the previous Lemma.

Proof. Write $h := h_{\gamma,\delta}$ for $\delta, \gamma \in \mathbb{C}_+$ with $|\delta|, |\gamma|$ so small that $E(h(Z_1)) < \infty$. The discrete version of Itô's formula yields

$$\begin{aligned}&\rho^n h(X_n) - \sum_{i=0}^{n-1} \rho^i(\rho E_x(h(X_{i+1})|X_i) - h(X_i))\\ =&h(X_0) + \sum_{i=0}^{n-1} \rho^{i+1}(h(X_{i+1}) - E_x(h(X_{i+1})|X_i)) =: M_n\end{aligned}$$

and $(M_n)_{n\in\mathbb{N}}$ is a martingale. The optional sampling theorem applied to $\tau = \tau_b$ yields

$$\begin{aligned}h(x) = E_x(M_{\tau\wedge n}) &= E_x(\rho^{n\wedge\tau} h(X_{n\wedge\tau})) - E_x\left(\sum_{i=0}^{n\wedge\tau-1} \rho^i(\rho E_x(h(X_{i+1})|X_i) - h(X_i))\right)\\ &= E_x(\rho^{n\wedge\tau} h(X_{n\wedge\tau})) - E_x\left(\sum_{i=0}^{n\wedge\tau-1} \rho^i(\alpha_\delta e^{-\lambda X_i Q} q - \alpha_\delta e^{-\lambda\gamma X_i Q} q)\right)\end{aligned}$$

using equality (6.9). The dominated convergence theorem shows that

$$h_{\delta,\gamma}(x) = E_x(\rho^\tau h_{\delta,\gamma}(X_\tau)) - E_x\left(\sum_{i=0}^{\tau-1} \rho^i(\alpha_\delta e^{-\lambda X_i Q} q - \alpha_\delta e^{-\lambda\gamma X_i Q} q)\right);$$

note that the dominated convergence theorem is applicable to both summands since Q has negative eigenvalues and so e^{-sQ} is bounded in s for s with $\Re(s)$ being bounded above. Corollary 6.10 leads to

$$h_{\gamma,\delta}(x) = \sum_{i=1}^{m} E_x(\rho^\tau \mathbb{1}_{G_i}) E(h_{\gamma,\delta}(R_i + b)) - E_x\left(\sum_{i=0}^{\tau-1} \rho^i(\alpha_\delta e^{-\lambda X_i Q} q - \alpha_\delta e^{-\lambda\gamma X_i Q} q)\right),$$

where R_i is $PH(Q, e_i)-$distributed and the previous Lemma implies

$$E(h_{\gamma,\delta}(R_i + b)) = e^{\delta b} \alpha_{\gamma,i}(-\delta I - Q)^{-1} q.$$

Since $\sum_{i=0}^{\tau-1} \rho^i(\alpha_\delta e^{-\lambda X_i Q} q - \alpha_\delta e^{-\lambda\gamma X_i Q} q)$ is bounded both sides of the equation

$$h_{\delta,\gamma}(x) = \sum_{i=1}^{m} \eta_{\gamma,\delta,i}\Phi_i(x) - E_x\left(\sum_{i=0}^{\tau-1} \rho^i(\alpha_\delta e^{-\lambda X_i Q} q - \alpha_\delta e^{-\lambda\gamma X_i Q} q)\right)$$

are holomorphic in $\{\gamma \in \mathbb{C}_+ : \hat{P} \cap Sp(-\gamma\lambda^n Q) = \emptyset \text{ for all } n \in \mathbb{N}\}$ and the identity theorem for holomorphic functions yields that these extensions agree on their domains. Keeping (6.7) in mind we especially obtain for $\gamma = 1$

$$h_\delta(x) = \sum_{i=1}^{m} \eta_{\delta,i}\Phi_i(x).$$

Furthermore both sides of the equations are again holomorphic functions in δ on $\mathbb{C}_+ \setminus Sp(-Q)$. Another application of the identity theorem proves the assertion. □

The equation in the theorem above appears useful and flexible enough for the explicit solution as shown in the next subsections.

6.4.1 The case of exponential positive innovations

As described above the case of Exp(μ)-distributed positive innovations is of special interest. In this case we obtain the solution directly from the results above. Hence let $m = 1$, $\alpha = 1$, $Q = -\mu$ and $q = \mu$. It is not relevant which δ we take; because the expressions simplify a bit we choose $\delta = 0$. Then we obtain

$$\begin{aligned}\eta_{0,1} &= 1 + e^{\psi_2(\mu)-\mu b+\phi(\lambda\mu)} \sum_{n\in\mathbb{N}} e^{\lambda^n\mu b-\phi(\lambda^n\mu)+\psi_1(\lambda^n\mu)}\rho^n \\ &\overset{(6.6)}{=} 1 + e^{\psi_2(\mu)-\mu b+\phi(\lambda\mu)} \sum_{n\in\mathbb{N}} e^{\lambda^n\mu b-\phi(\lambda^{n+1}\mu)-\psi_2(\lambda^n\mu)}\rho^n \\ &= e^{\psi_2(\mu)-\mu b+\phi(\lambda\mu)} \sum_{n\in\mathbb{N}_0} e^{\lambda^n\mu b-\phi(\lambda^{n+1}\mu)-\psi_2(\lambda^n\mu)}\rho^n\end{aligned}$$

and

$$h_0(x) = e^{\psi_2(\mu)-\mu b+\phi(\lambda\mu)} \sum_{n\in\mathbb{N}} e^{\lambda^n\mu x-\phi(\lambda^n\mu)}\rho^n$$

for $x < b$. Theorem 6.14 yields

Theorem 6.15.

$$E_x(\rho^\tau) = \frac{h_0(x)}{\eta_{0,1}} = \frac{\sum_{n\in\mathbb{N}} e^{\lambda^n\mu x-\phi(\lambda^n\mu)}\rho^n}{\sum_{n\in\mathbb{N}_0} e^{\lambda^n\mu b-\phi(\lambda^{n+1}\mu)-\psi_2(\lambda^n\mu)}\rho^n} \quad \textit{for all } x < b.$$

In [CIN10] the special case of positive exponential distributed innovations was treated by finding and solving ordinary differential equations for $E_x(\rho^\tau)$. For this case – i.e. $T_1 = 0$ – we obtain

$$E_x(\rho^\tau) = \frac{\sum_{n\in\mathbb{N}} e^{\lambda^n\mu x-\phi(\lambda^n\mu)}\rho^n}{\sum_{n\in\mathbb{N}_0} e^{\lambda^n\mu b-\phi(\lambda^{n+1}\mu)}\rho^n}.$$

To get more explicit results we need a simple expression for $\phi(\lambda^n\mu)$. Using identity (6.5) we find such an expression as

$$\begin{aligned}e^{\phi(\lambda^n\mu)} &= \prod_{k=0}^{\infty} e^{\psi_1(\lambda^{n+k}\mu)} = \prod_{k=0}^{\infty} \frac{\mu}{\mu-\lambda^{n+k}\mu} = \prod_{k=0}^{\infty} \frac{1}{1-\lambda^{n+k}} \\ &= \frac{\prod_{k=1}^{n-1}(1-\lambda^k)}{\prod_{k=1}^{\infty}(1-\lambda^k)} = \frac{(\lambda,\lambda)_{n-1}}{\phi_e(\lambda)},\end{aligned}$$

where $(a,q)_n = \prod_{k=1}^{n-1}(1-aq^{k-1})$ denotes the q-Pochhammer-symbol and $\phi_e(q) = (q,q)_\infty$ denotes the Euler function. This leads to

$$E_x(\rho^\tau) = \frac{\sum_{n\in\mathbb{N}} \frac{\rho^n}{(\lambda,\lambda)_{n-1}} e^{\lambda^n\mu x}}{\sum_{n\in\mathbb{N}_0} \frac{\rho^n}{(\lambda,\lambda)_n} e^{\lambda^n\mu b}}$$

and the numerator is given by

$$\begin{aligned}
\sum_{n\in\mathbb{N}} \frac{\rho^n}{(\lambda,\lambda)_{n-1}} e^{\lambda^n \mu x} &= \sum_{k\in\mathbb{N}_0} \frac{(\mu x)^k}{k!} \sum_{n\in\mathbb{N}} \frac{(\rho\lambda^k)^n}{(\lambda,\lambda)_{n-1}} \\
&= \rho \sum_{k\in\mathbb{N}_0} \frac{(\mu x\lambda)^k}{k!} \sum_{n\in\mathbb{N}} \frac{(\rho\lambda^k)^{n-1}}{(\lambda,\lambda)_{n-1}} \\
&= \rho \sum_{k\in\mathbb{N}_0} \frac{(\mu x\lambda)^k}{k!} \frac{1}{(\rho\lambda^k,\lambda)_\infty} \\
&= \frac{\rho}{(\rho,\lambda)_\infty} \sum_{k\in\mathbb{N}_0} \frac{(\rho,\lambda)_k(\mu\lambda x)^k}{k!}.
\end{aligned}$$

Note that we used the q-binomial-theorem in the third step (see [GR04, (1.3.15)] for a proof). An analogous calculation for the denominator yields

$$\sum_{n\in\mathbb{N}_0} \frac{\rho^n}{(\lambda,\lambda)_n} e^{\lambda^n \mu b} = \frac{1}{(\rho,\lambda)_\infty} \sum_{k\in\mathbb{N}_0} \frac{(\rho,\lambda)_k(\mu b)^k}{k!}$$

and we obtain

Theorem 6.16. *If S_1 is $\mathrm{Exp}(\mu)$-distributed and $T_1 = 0$ it holds that*

$$E_x(\rho^{\tau_b}) = \rho \cdot \frac{\sum_{k\in\mathbb{N}_0}(\rho,\lambda)_k \frac{(\mu x\lambda)^k}{k!}}{\sum_{k\in\mathbb{N}_0}(\rho,\lambda)_k \frac{(\mu b)^k}{k!}} \qquad \text{for all } x < b.$$

For the special case $\mu = 1$ this formula was obtained by an approach via differential equations based on the generator in [Nov09, Theorem 3].
Noting that

$$E_b(\rho^{\tau_{b+}}) = \rho \cdot \frac{\sum_{k\in\mathbb{N}_0}(\rho,\lambda)_k \frac{(\mu b\lambda)^k}{k!}}{\sum_{k\in\mathbb{N}_0}(\rho,\lambda)_k \frac{(\mu b)^k}{k!}}$$

by direct calculation we find

$$\frac{d}{db} E_x(\rho^{\tau_b}) = E_x(\rho^{\tau_b})\mu(E_b(\rho^{\tau_{b+}}) - 1).$$

This reproduces Theorem 3.3 in [CIN10]. Note that in that article the stopping time $\tilde{\tau}_b = \inf\{n \in \mathbb{N}_0 : X_t > b\}$ was considered. But this leads to analogous results since

$$\tau_{b+} = \tilde{\tau}_b \quad \text{under } P_b$$

and

$$\tau_b = \tilde{\tau}_b \quad P_x \text{ a.s. for all } x \neq b.$$

6.4.2 The general case

Theorem 6.14 gives a powerful tool for the explicit calculation of Φ in many cases of interest as follows:
By Lemma 6.13 we see that $\eta_{\delta,i}$ is a rational function of δ with poles in $Sp(-Q)$ for all $i = 1, .., m$. We assume for simplicity that all eigenvalues are pairwise different (for the general case see the remark at the end of the section). Then partial fraction decomposition yields the representation

$$\eta_{\delta,i} = \sum_{j=1}^{m} \frac{a_{i,j}}{\mu_j - \delta} \quad \text{for some } a_{i,1}, ..., a_{i,m}$$

and since $h_\delta(x)$ is rational in δ with the same poles we may write

$$h_\delta(x) = \sum_{j=1}^{m} \frac{c_j(x)}{\mu_j - \delta} \text{ for some } c_1(x), ..., c_m(x).$$

Theorem 6.14 reads

$$\sum_{j=1}^{m} \frac{\sum_{i=1}^{m} a_{ij}\Phi_i(x)}{\mu_j - \delta} = \sum_{j=1}^{m} \frac{c_j(x)}{\mu_j - \delta}$$

and the uniqueness of the partial fraction decomposition yields

$$\sum_{i=1}^{m} a_{ij}\Phi_i(x) = c_j(x),$$

i.e.

$$A\Phi(x) = c(x),$$

where $A = (a_{ij})_{i,j=1}^m$, $c(x) = (c_j(x))_{j=1}^m$. This leads to

Theorem 6.17. *If A is invertible, then $\Phi(x)$ is given by*

$$\Phi(x) = A^{-1}c(x) \quad \text{for all } x < b.$$

Remark 6.18. *Note that the assumption of distinct eigenvalues was made for simplicity only. When it is not fulfilled we can use the general partial fraction decomposition formula and obtain the analogous result.*

6.5 On the explicit solution of the problem

Now we are prepared to solve the optimal optimal stopping problem

$$v(x) = \sup_\tau E_x(\rho^\tau g(X_\tau)), \quad x \in \mathbb{R}.$$

Section 6.2 gives conditions for the optimality of threshold-times. In this cases we can simplify the problem to

$$v(x) = \sup_b E_x(\rho^{\tau_b} g(X_{\tau_b})), \quad x \in \mathbb{R}.$$

Now take an arbitrary starting point x. Then we have to maximize the real function

$$\Psi_x : (x, \infty) \to \mathbb{R}, b \mapsto E_x(\rho^{\tau_b} g(X_{\tau_b})) = \sum_{i=1}^{m} \Phi_i^b(x) E(g(b + R^i)),$$

where R^i is $Ph(e_i, Q)$-distributed, $b \geq 0$. The results of the previous section give rise to an explicit calculation of $\Phi_i^b(x)$ and of Ψ_x.
Hence we are faced with the well-studied maximization problem for real functions, that can – e.g. – be solved using the standard tools from differential calculus.
If we have found a maximum point b^* of Ψ_x and $\Psi_x(b^*) > g(x)$, then

$$\tau^* = \inf\{n \in \mathbb{N}_0 : X_n \geq b^*\}$$

is an optimal stopping time when $(X_n)_{n \in \mathbb{N}_0}$ is started in x.
A more elegant approach for finding the optimal threshold b^* is the principle of continuous fit:

6.6 The principle of continuous fit

The principles of smooth and continuous fit play an important role in the study of many optimal stopping problems. The principle of smooth fit was already introduced in [Mik56] and has been applied in a variety of problems, ranging from sequential analysis to mathematical finance – cf. Subsection 2.4.1 for the case of diffusions. The principle of continuous pasting is more recent and was introduced in [PS00] as a variational principle to solve sequential testing and disorder problems for the Poisson process. For a discussion in the case of Lévy processes and further references we refer to [CI09]. Another overview is given in [PS06, Chapter IV.9] and one may summarize, see the above reference, p. 49:

"If X enters the interior of the stopping region S immediately after starting on ∂S, then the optimal stopping point x^* is selected so that the value function v is smooth in x^*. If X does not enter the interior of the stopping region immediately, then x^* is selected so that v is continuous in x^*."

Most applications of this principle involve processes in continuous time. In discrete time an immediate entrance is of course not possible, so that one can not expect the smooth-fit principle to hold. In this section we prove that the continuous-fit principle holds in

our setting and illustrate how it can be used for an easy determination of the optimal threshold.
We keep the notations and assumptions of the previous sections and – as before – we assume that the optimal stopping set is an interval of the form $[b^*, \infty)$ and consider the optimal stopping time $\tau_{b^*} = \tau = \inf\{n \in \mathbb{N}_0 : X_n \geq b^*\}$.
Furthermore we assume that

$$\lim_{\epsilon \searrow 0} \Phi_i^{b^*}(b^* - \epsilon) = \lim_{\epsilon \searrow 0} \Phi_i^{b^*+\epsilon}(b^*) \quad \text{for all } i = 1, ..., m. \tag{6.10}$$

Note that this condition is obviously fulfilled in the cases discussed above. If g is continuous under an appropriate integrability condition it furthermore holds that

$$E(g(R_i + \epsilon + b^*)) \to E(g(R_i + b^*)) \quad \text{as } \epsilon \to 0, \quad i = 1, ..., m. \tag{6.11}$$

Proposition 6.19. *Assume (6.10) and (6.11). Then it holds that*

$$\lim_{b \nearrow b^*} v(b) = g(b^*).$$

Proof. Let $\epsilon > 0$. First note that $v(b) > g(b)$ for all $b < b^*$ so that

$$\liminf_{b \nearrow b^*} v(b) \geq \liminf_{b \nearrow b^*} g(b) = g(b^*).$$

Furthermore using Corollary 6.10

$$\begin{aligned} v(b^* - \epsilon) - g(b^*) &= E_{b^*-\epsilon}(\rho^\tau g(X_\tau)) - v(b^*) \\ &\leq E_{b^*-\epsilon}(\rho^\tau g(X_\tau)) - E_{b^*}(\rho^{\tau_{b^*+\epsilon}} g(X_{\tau_{b^*+\epsilon}})) \\ &= \sum_{i=1}^m \left(\Phi_i^{b^*}(b^* - \epsilon) E(g(R_i + b^*)) - \Phi_i^{b^*+\epsilon}(b^*) E(g(R_i + \epsilon + b^*)) \right) \to 0 \end{aligned}$$

as $\epsilon \searrow 0$. This proves $\limsup_{b \nearrow b^*} v(b) \geq g(b^*)$. □

Figure 6.2 illustrates how the continuous fit principle can be used: We consider the candidate solutions

$$v(b, x) = \begin{cases} \Psi_x(b) & , \ x < b \\ g(x) & , \ x \geq b \end{cases}$$

and solve the equation $\Psi_{b-}(b) = g(b)$, where $\Psi_\cdot$ is defined as in the previous section. If the equation has a unique solution we can conclude that this solution must be the optimal threshold as illustrated in the following Section.

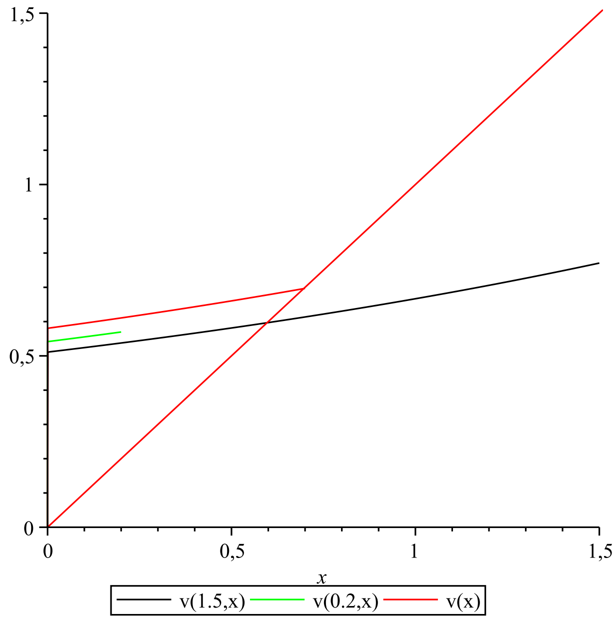

Figure 6.2: Some candidate solutions for different thresholds in the case $g(x) = x$

6.7 Example

We consider the gain function $g(x) = x$ and Exp(μ)-distributed innovations; in this setting we always assume $(X_n)_{n\in\mathbb{N}_0}$ to have values in $[0,\infty)$. Subsection 6.2.1 guarantees that the optimal stopping time is of threshold-type. The optimal threshold can be found by the continuous fit principle described in the previous section:
The problem is solved if we find a unique $b^* \in [0,\infty)$ that solves the equation

$$b = \Psi_{b-}(b) = \Phi_{b-}(b)(b + \frac{1}{\mu}) = \frac{\rho \sum_{k\in\mathbb{N}_0} (\rho,\lambda)_k \frac{(\mu b \lambda)^k}{k!}}{\sum_{k\in\mathbb{N}_0} (\rho,\lambda)_k \frac{(\mu b)^k}{k!}}(b + \frac{1}{\mu}),$$

where we used Theorem 6.16 in the last step. This equation is equivalent to

$$\sum_{k=0}^{\infty} (\rho,\lambda)_k \frac{\mu^k}{k!} b^{k+1} = \sum_{k=0}^{\infty} \rho(\rho,\lambda)_k \frac{\mu^k \lambda^k}{k!} b^{k+1} + \sum_{k=0}^{\infty} \frac{\rho}{\mu}(\rho,\lambda)_k \frac{\mu^k \lambda^k}{k!} b^k$$

$$\text{i.e.} \quad \frac{\rho}{\mu} - \sum_{k=0}^{\infty} (\rho,\lambda)_k \frac{\mu^k}{k!}(1 - \rho\lambda^k - \frac{\rho}{\mu}(1-\rho\lambda^k)\frac{\mu\lambda^{k+1}}{k+1}) b^{k+1} = 0$$

$$\text{i.e.} \quad f(b) = 0,$$

where

$$f(b) = \frac{\rho}{\mu} - \sum_{k=0}^{\infty} (\rho,\lambda)_{k+1} \frac{\mu^k}{k!}(1 - \frac{\rho\lambda^{k+1}}{k+1}) b^{k+1}.$$

Note that $f(0) = \frac{\rho}{\mu} > 0$ and

$$f'(b) = -\sum_{k=0}^{\infty} (\rho,\lambda)_{k+1} \frac{\mu^k}{k!}(1 - \frac{\rho\lambda^{k+1}}{k+1})(k+1)b^k < 0 \text{ for all } b \in [0,\infty).$$

Since $f(b) \leq \frac{\rho}{\mu} - (1-\rho)(1-\rho\lambda)b$ we furthermore obtain $f(b) \to -\infty$ for $b \to \infty$. Hence there exists a unique solution b^* of the transcendental equation $f(b) = 0$.
The optimal stopping time is

$$\tau^* = \inf\{n \in \mathbb{N} : X_n \geq b^*\}$$

and the value function is given by

$$v(x) = \begin{cases} (b + \frac{1}{\mu})\rho \cdot \frac{\sum_{k\in\mathbb{N}_0} (\rho,\lambda)_k \frac{(\mu x\lambda)^k}{k!}}{\sum_{k\in\mathbb{N}_0} (\rho,\lambda)_k \frac{(\mu b^*)^k}{k!}} & , \ x < b^* \\ x & , \ x \geq b^*. \end{cases}$$

In Figure 6.2 v is plotted for the parameters $\mu = 1$, $\rho = \lambda = 1/2$.

Notations

$\mathbb{N}$	$\{1, 2, 3, ...\}$
$\mathbb{N}_0$	$\{0, 1, 2, 3, ...\}$
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{C}_+$	set of complex numbers with nonnegative real part
$\Re(z)$, $\Im(z)$	real, imaginary part of a complex number z
$x \wedge y$	minimum of x and y
$x \vee y$	maximum of x and y
x^+	maximum of x and 0
log	main branch of complex logarithm
$f\|_G$	restriction of a function f to G
$f(t-)$, $f(t+)$	left-hand, right-hand limit of a real function f in t
$\mathbb{1}_A$	indicator function of a set A
$(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, P)$	the underlying filtered probability space
P^X	distribution of a random variable X
P_x	$P(\cdot \| X_0 = x)$
ψ_+, ψ_-	the minimal r-harmonic functions
$\sigma.$	time shift of a Markov process
$\mathrm{Exp}(\lambda)$	exponential distribution with mean $\frac{1}{\lambda}$
$PH(Q, \alpha)$	phasetype distribution with parameters Q, α
$\operatorname{ess\,sup}_{i\in I} X_i$	essential supremum of $(X_i)_{i\in I}$
$\mathrm{int}(A)$	the interior of a set A

List of Figures

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Index

Erklärung

Ich habe die vorliegende Arbeit abgesehen von der Beratung durch den Betreuer meiner Promotion unter Einhaltung guter wissenschaftlicher Praxis selbstständig angefertigt und keine anderen als die angegebenen Hilfsmittel verwendet; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Diese Arbeit hat weder ganz noch zum Teil an anderer Stelle im Rahmen eines Prüfungsverfahrens vorgelegen. Des Weiteren habe ich noch keinen Promotionsversuch unternommen.
Teile dieser Arbeit wurden in folgenden Artikeln vorab zur Veröffentlichung eingereicht:

[CI10] Sören Christensen and Albrecht Irle. A harmonic-function technique for the optimal stopping of diffusions. Submitted to Stochastics, 2010.

[CIN10] Sören Christensen, Albrecht Irle, and Alexander A. Novikov. An Elementary Approach to Optimal Stopping Problems for AR(1) Sequences. Submitted to Sequential analysis, 2010.

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