Sections of Simplices and Cylinders

Volume Formulas and Estimates

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Abstract

We investigate sections of simplices and generalized cylinders. We are interested in the volume of sections of these bodies with affine subspaces and give explicit formulas and estimates for these volumes.

For the regular n-simplex we state a general formula to compute the volume of the intersection with some k-dimensional subspace. A formula for central hyperplane sections was given by S. Webb. He also showed that the hyperplane through the centroid containing $n-1$ vertices gives the maximal volume. We generalize the formula to arbitrary dimensional sections that do not necessarily have to contain the centroid. And we show that, for a prescribed small distance of a hyperplane to the centroid, still the hyperplane containing $n-1$ vertices is volume maximizing. The proof also yields a new and short argument for Webb's result. The minimal hyperplane section is conjectured to be the one parallel to a face. We show that this hyperplane section is indeed minimal for dimensions $n = 2, 3, 4$ and that it is a local minimum in general dimension. Using results by Brehm e.a. we compute the average hyperplane section volume. For k-dimensional sections we give an upper bound. Finally we modify our volume formula to compute the section volume of irregular simplices. As an application we show that in odd dimensions larger than 4 there exist irregular simplices whose maximal section is not a face.

A generalized cylinder is the Cartesian product of a n-dimensional cube and a m-dimensional ball of radius r. This body has not been considered in the literature so far. We study the behavior of the hyperplane section volume depending on the radius of the cylinder. First we show for the three-dimensional cylinder that always a truncated ellipse gives the maximal volume. This is done by elementary geometric considerations and calculus. For the generalized cylinder we use the Fourier transform to derive an explicit formula. Then we estimate this by Hölder's inequality. Finally, it remains to prove an integral inequality that is similar to the inequality of K. Ball for the cube.

Zusammenfassung

In dieser Arbeit untersuchen wir Schnitte von Simplexen und verallgemeinerten Zylindern. Wir sind am Volumen der Schnitte dieser Körper mit affinen Teilräumen interessiert und geben explizite Volumenformeln und -abschätzungen an.

Für den regulären n-Simplex beweisen wir eine allgemeine Formel, um das Volumen der Schnitte mit k-dimensionalen Teilräumen zu berechnen. Für zentrale Hyperebenenschnitte stammt eine Formel von S. Webb. Er zeigt auch, dass der maximale Schnitt durch den Schwerpunkt des Simplex durch eine Ebene, die $n-1$ Eckpunkte enthält, gegeben ist. Wir verallgemeinern diese Formel auf beliebig dimensionale Schnitte, die den Schwerpunkt nicht mehr notwendigerweise enthalten. Anschließend zeigen wir, dass der maximale Schnitt immer noch $n-1$ Eckpunkte enthält, wenn wir den Abstand der Schnittebene zum Schwerpunkt fixieren und dieser klein ist. Der Beweis dieser Aussage ergibt auch ein neues und kurzes Argument für das Resultat von Webb. Es besteht die Vermutung, dass das minimale Schnittvolumen durch eine Ebene parallel zu einer Facette des Simplex angenommen wird. Wir zeigen, dass dieser Schnitt tatsächlich lokal ein Minimum liefert und dass für Dimensionen $n = 2, 3, 4$ dieser Schnitt auch global minimal ist. Außerdem berechnen wir unter Benutzung von Resultaten von Brehm u.a. das mittlere Schnittvolumen. Für k-dimensionale Schnitte geben wir eine obere Abschätzung an. Abschließend modifizieren wir unsere Volumenformel, sodass auch Hyperebenenschnitte irregulärer Simplexe berechnet werden können. Als Anwendung zeigen wir, dass es in ungeraden Dimensionen größer als vier Simplexe gibt, sodass alle Facetten kleineres Volumen haben als einer der zentralen Schnitte.

Ein verallgemeinerter Zylinder ist das kartesische Produkt aus einem n-dimensionalen Würfel und einer m-dimensionalen Kugel mit Radius r. Schnitte dieses Körpers wurden bisher in der Literatur nicht betrachtet. Wir studieren das Verhalten des Volumens von Hyperebenenschnitten für variablen Radius des Zylinders. Wir beginnen mit dem gewöhnlichen dreidimensionalen Zylinder, für den das maximale Schnittvolumen stets durch eine angeschnittene Ellipse angenommen wird. Dies wird mit elementargeometrischen Betrachtungen und Differentialrechnung in einer Veränderlichen bewiesen. Für den verallgemeinerten Zylinder leiten wir eine Formel unter Verwendung der Fouriertransformation her. Diese wird dann mit der Hölderungleichung abgeschätzt. Schließlich ist eine Integralungleichung vergleichbar zu K. Balls Integralungleichung zu beweisen.

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Contents

1. Introduction

Given a convex body $K \subset \mathbb{R}^n$ and some subspace H, how to compute the volume of the intersection $H \cap K$? How to find the maximal or minimal sections? These questions have been considered for various convex bodies.

A first example is the unit cube intersected with central hyperplanes. The first explicit formulas were already found by Laplace. Around 1812 he computed the volume of the section orthogonal to the main diagonal. Later Pólya gave a general formula for central hyperplane sections [Pol13]. The first investigations on bounds for central sections are found in [Hen79]. D. Hensley showed that the volume is between 1 and 5, where 1 is optimal but 5 is not. The optimal upper bound for sections was proved by K. Ball to be $\sqrt{2}$ in 1986 [Bal86]. This was also Hensley's conjecture. Thereafter K. Ball continued his investigations by considering k-dimensional sections of the cube [Bal89].

Since then many other bodies and modified questions have been considered. For example, ℓ_p-balls in [MP88] and [Kol05], complex cubes in [OP00]; also non-central sections [MSZZ13] as well as taking other than Lebesgue measures [KK13] have been investigated.

Of course the question of how to compute volumes of sections of bodies and estimating these volumes is a question of its own interest. But there are links to other questions. The investigation of volumes of sections originated in the search for counterexamples for conjectures like the Busemann-Petty problem. There is also a link to probability theory, since the involved functions define probability densities.

In the literature there are basically two different approaches to prove inequalities for section volumes. One is analytical, the second has a probabilistic flavour. For the analytic approach investigate, for a given convex body $K \subset \mathbb{R}^n$ with its centroid in the origin, the function

$$A(a,t) := \mathrm{vol}_{n-1}(H_a^t \cap K),$$

where $H_a^t \cap K$ is a hyperplane section orthogonal to a at distance t from the origin. The first aim is to find an analytic volume formula. Such a formula can be derived by applying the Fourier transform and the inversion formula in t to the function $A(a,t) = \int_{H_a^t} \chi_K(x)\mathrm{d}x$. For example, the formula for the cube is

$$A(a,0) = \int_{\mathbb{R}} \prod_{j=1}^{n} \frac{\sin(a_j t)}{a_j t} \mathrm{d}t,$$

up to constants. This formula is bounded by Hölder's inequality in the first step. Then a real integral inequality has to be estimated. Examples for this approach can be found in [Bal86], for the upper bound for the cube, or in [Web96], for the upper bound for the simplex.

The idea of the second approach is the following: For a fixed direction a, we define a function

$$f_a \colon \mathbb{R} \to \mathbb{R}_{\geq 0},\ t \mapsto \mathrm{vol}_{n-1}(H_a^t \cap K),$$

where $K \subset \mathbb{R}^n$ has volume 1 and its centroid in the origin. The function f_a describes the volume of the section with the hyperplane that is shifted through the body. It defines a probability density function. By Brunn-Minkowski's inequality the function is log-concave. For all a the functions f_a have several invariants: $\int_{\mathbb{R}} f_a(t)\mathrm{d}t = \mathrm{vol}_n(K)$, $\int_{\mathbb{R}} f_a(t)t\mathrm{d}t = 0$ and, for certain bodies, $\int_{\mathbb{R}} f_a(t)t^2\mathrm{d}t =: L_K^2$, called the isotropic constant. For central sections we are interested in $f_a(0)$. This allows to establish inequalities involving the quantities listed above. Examples for this approach can be found in [Hen79] for the lower bound for sections of the cube, in [Web96] for the upper bound for sections of the simplex and in [Brz13] for a non-optimal lower bound for sections of the simplex.

Here we are interested in the regular simplex and in generalized cylinders. S. Webb [Web96] gave a formula for central hyperplane sections of the simplex. He also proved that the maximal central section is the one containing $n-1$ vertices and the centroid. The question of the minimal central hyperplane section is not completely solved yet. P. Filliman stated that his methods can be used to prove that the section parallel to a face is minimal [Fil92]. But he gave no precise arguments. P. Brzezinski proved a lower bound which differs from the conjectured minimal volume by a factor of approximately 1.27 [Brz13]. Also A. Barvinok considered sections of the regular simplex [Bar09], but his results differ by at least an order of dimension from our results.

For cylinders there are no results that are known to us. But it turns out that Brzezinski's results resp. his tools for generalized cubes are helpful in this case.

Results for the simplex

We briefly describe our main results. We start by giving a general formula for the volume of sections of simplices (Theorem 3.1). While Webb's formula was restricted to hyperplane sections through the centroid, we give formulas for k-dimensional and not necessarily central sections in Chapter 3. For hyperplanes we show that, for fixed small distances from the centroid to the hyperplane, the one containing $n-1$ vertices still gives maximal volume (Theorem 4.3). We prove the local minimality of the conjectured minimal section and prove its global minimality for dimension 2, 3 and 4 (Theorem 4.4 and Theorem 4.7). In Chapter 5 we compute the average hyperplane section volume (Theorem 5.1). For k-dimensional sections we give an upper bound in Chapter 6 (Theorem 6.1). Additionally, we give a formula for hyperplane sections of non-regular simplices (Theorem 7.1). As an application, we show that there is a simplex whose central section is larger than all its facets, if the dimension of the simplex is even and larger than 5 (Theorem 7.2). This result is originally by [Phi72]. This is remarkable since the statement is not true for dimensions up to 5.

Results for cylinders

In the second part of this thesis we deal with cylinders. We start with the usual three-dimensional cylinder with varying radius in Chapter 8. We prove a formula by geometric considerations and find the maximal volume by calculus (Theorem 8.3). The result is that always some truncated ellipse, if r is large enough, or the rectangular section, if r is small, is maximal. In the final Chapter 9 we define a generalized cylinder and prove a volume formula (Theorem 9.1). Again we investigate the behavior for varying radius. We find an upper bound that is sharp for large r (Theorem 9.6). In contrast to the three-dimensional setting the maximum is attained by a cylindrical section. The main difficulty is the proof of a certain integral inequality (Theorem 9.3). Similar inequalities also appear in the investigation of sections of cubes [Bal86] and generalized cubes [Brz11].

2. Preliminaries

We introduce **some notations**. Let $A, B \subset \mathbb{R}^n$ and $x, y \in \mathbb{R}^n$.
The Euclidean norm is denoted by $\|x\|$, the standard scalar product by $\langle x, y\rangle$.
The distance of two sets is given by $\operatorname{dist}(A, B) := \inf\{\|a - b\| \mid a \in A, b \in B\}$, especially $\operatorname{dist}(x, A) := \operatorname{dist}(\{x\}, A)$.
For $a \in \mathbb{R}^n$ with $\|a\| = 1$ and $t \in \mathbb{R}$, let $H_a^t := \{x \in \mathbb{R}^n \mid \langle a, x\rangle = t\} = H_a + t \cdot a$ be a translated hyperplane, especially $H_a := H_a^0$. For $\|a\| \neq 1$ the hyperplane H_a is still well defined.
If H is a k-dimensional (affine) subspace and $A \subset H$, the k-volume of A is the standard induced Lebesgue volume of the subspace, denoted by $\operatorname{vol}_k(A)$.
The Euclidean ball is denoted by $B_2^n := \{x \in \mathbb{R}^n \mid \|x\|_2 \leq 1\}$, with $\operatorname{vol}_n(B_2^n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$.
The characteristic function of the set A is denoted by χ_A.
The transpose of a matrix $T \in \mathbb{R}^{n\times n}$ is denoted by T^*.

The volume formula for cylinders contains the gamma function and Bessel functions. We give some basic facts and references. The proof of our theorem on large cylinder sections uses some technical estimates on these functions that we also state in this chapter.

2.1. The Gamma function

For $x > 0$ define

$$\Gamma(x) := \int_0^\infty t^{x-1} \exp(-t) \mathrm{d}t.$$

The gamma function satisfies the functional equation $\Gamma(x + 1) = x\Gamma(x)$.

Lemma 2.1 (Legendre's duplication formula)**.** *For $x > 0$ we have*

$$\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right) = \frac{\sqrt{\pi}}{2^{x-1}}\Gamma\left(x\right).$$

The definition and a proof of this lemma is found in [Kol05, Section 2.4]. Using the next lemma, one can approximate the gamma function. We refer to [Kön90, Section 18.3].

Lemma 2.2 (Stirling's formula)**.** *Let* $x > 0$*. Then there exists some* $\mu(x) \in \left(0, \frac{1}{12x}\right)$ *such that*

$$\Gamma(x) = \sqrt{2\pi} x^{x-\frac{1}{2}} \exp(-x) \exp(\mu(x)).$$

We prove two further technical lemmas to estimate quotients of gamma functions.

Lemma 2.3. *For* $x \geq 2$ *we have*

$$\frac{\Gamma(x)}{\Gamma\left(x - \frac{1}{2}\right)} > \frac{\sqrt{x}}{2}.$$

Proof. We estimate the gamma functions by Stirling's formula:

$$\begin{aligned}
\frac{\Gamma(x)}{\Gamma\left(x - \frac{1}{2}\right)} &\geq \left(\frac{x}{x - \frac{1}{2}}\right)^x \frac{x - \frac{1}{2}}{\sqrt{x}} \frac{1}{\exp(\frac{1}{2})} \frac{1}{\exp(\frac{1}{24})} \\
&\geq \frac{x - \frac{1}{2}}{\sqrt{x}} \frac{1}{\exp(\frac{1}{24})} \\
&= \left(\sqrt{x} - \frac{1}{2\sqrt{x}}\right) \exp\left(-\frac{1}{24}\right)
\end{aligned}$$

Note that $\left(\frac{x}{x-\frac{1}{2}}\right)^x$ strictly decreases to $\exp(\frac{1}{2})$. Additionally $\left(\sqrt{x} - \frac{1}{2\sqrt{x}}\right) \exp(-\frac{1}{24})$ increases faster than $\frac{\sqrt{x}}{2}$ and the inequality holds for $x = 2$. This proves the Lemma. □

Lemma 2.4. *Let* $m \geq 5$*. Then we have*

$$\frac{\Gamma\left(\frac{m}{2} + 1\right)^2 \Gamma(m)}{\Gamma\left(\frac{m}{2} + \frac{1}{2}\right)^2 \Gamma\left(m + \frac{1}{2}\right)} \leq \frac{m+2}{m+1} \frac{\sqrt{m}}{2}.$$

Proof. Using Legendre's duplication formula, we find

$$\begin{aligned}
\frac{\Gamma\left(\frac{m}{2} + 1\right)^2 \Gamma(m)}{\Gamma\left(\frac{m}{2} + \frac{1}{2}\right)^2 \Gamma\left(m + \frac{1}{2}\right)} &= \frac{\left(\frac{m}{2}\right)^2 \Gamma\left(\frac{m}{2}\right)^2 \Gamma\left(\frac{m}{2}\right)^2 \quad \Gamma(m)\Gamma(m)}{\left[\Gamma\left(\frac{m}{2} + \frac{1}{2}\right)^2 \Gamma\left(\frac{m}{2}\right)^2\right] \left[\Gamma\left(m + \frac{1}{2}\right) \Gamma(m)\right]} \\
&= \frac{m^2}{4} \frac{\Gamma\left(\frac{m}{2}\right)^4 \Gamma(m)^2}{\left[\frac{\sqrt{\pi}}{2^{m-1}} \Gamma(m)\right]^2 \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma(2m)} \\
&= 2^{4m-5} \frac{m^2}{\pi^{\frac{3}{2}}} \frac{\Gamma\left(\frac{m}{2}\right)^4}{\Gamma(2m)}.
\end{aligned}$$

So we need to show

$$q(m) := \frac{2^{4m-4}}{\pi^{\frac{3}{2}}} \frac{m^{\frac{3}{2}}(m+1)}{m+2} \frac{\Gamma\left(\frac{m}{2}\right)^4}{\Gamma(2m)} \leq 1.$$

By application of Stirling's formula we find

$$\begin{aligned} q(m) &\leq \frac{2^{4m-4}}{\pi^{\frac{3}{2}}} \frac{m^{\frac{3}{2}}(m+1)}{m+2} \frac{\left(\sqrt{2\pi}\left(\frac{m}{2}\right)^{\frac{m}{2}-\frac{1}{2}} \exp\left(-\frac{m}{2}\right) \exp\left(\frac{1}{6m}\right)\right)^4}{\sqrt{2\pi}(2m)^{2m-\frac{1}{2}} \exp(-2m)} \\ &= \frac{m+1}{m+2} \exp\left(\frac{2}{3m}\right) =: \tilde{q}(m). \end{aligned}$$

As a function on $\mathbb{R}_{\geq 0}$, the derivative of $\tilde{q}(m)$ only has a zero in $m = 3 + \sqrt{13} > 6$. Note that $\tilde{q}(5) = \frac{6}{7} \exp\left(\frac{2}{15}\right) < 1$ and $\tilde{q}(6) = \frac{7}{8} \exp\left(\frac{2}{18}\right) < \tilde{q}(5)$. Obviously $\tilde{q}(m) \to 1$ for $m \to \infty$. Therefore $\tilde{q}(m)$ is increasing for $m \geq 7$. This proves $\tilde{q}(m) < 1$ for all $m \geq 5$. □

2.2. Bessel functions

A classical introduction to Bessel functions is [Wat66]. Let $s \geq 0$, $\nu \in \mathbb{C}$ with the real part of ν larger then $\frac{1}{2}$. Define

$$J_\nu(s) := \frac{\left(\frac{1}{2}s\right)^\nu}{\Gamma\left(\nu + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-1}^{1} \left(1 - t^2\right)^{\nu - \frac{1}{2}} \cos(st) \mathrm{d}t.$$

J_ν is called the Bessel function of order ν. We defined them by the so-called *Poisson representation formula*. The *normalized* Bessel function of order ν is given by

$$j_\nu(s) := 2^\nu \Gamma(\nu + 1) \frac{J_\nu(s)}{s^\nu} \quad \text{for } s > 0 \text{ and } j_\nu(0) := 1.$$

The normalized Bessel function j_ν is continuous in 0.

Using Bessel functions, we can compute the Fourier transform of the indicator function of the unit ball.

Lemma 2.5 (Fourier transform of $\chi_{B_2^n}$)**.** *Let $s > 0$, $a \in \mathbb{R}^n$. Then*

$$\int_{B_2^n} \exp\left(-\mathrm{i}s \langle x, a \rangle\right) \mathrm{d}x = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)} j_{\frac{n}{2}}(s \|a\|).$$

Proof. First observe that the integral is invariant under rotations of a. So we may assume

$\langle x, a\rangle = x_1 \|a\|$. Then

$$\begin{aligned}
\int_{B_2^n} \exp\left(-\mathrm{i}s\langle x, a\rangle\right) \mathrm{d}x &= \int_{B_2^n} \exp\left(-\mathrm{i}sx_1\|a\|\right) \mathrm{d}x \\
&= \int_{-1}^{1} \int_{\{\sum_{j=2}^n x_j^2 \le 1 - x_1^2\}} \exp\left(-\mathrm{i}sx_1\|a\|\right) \mathrm{d}(x_2, \ldots, x_n) \mathrm{d}x_1 \\
&= \int_{-1}^{1} \exp\left(-\mathrm{i}sx_1\|a\|\right) \mathrm{vol}_{n-1}\left(\left\{\sum_{j=2}^n x_j^2 \le 1 - x_1^2\right\}\right) \mathrm{d}x_1 \\
&= \int_{-1}^{1} \exp\left(-\mathrm{i}sx_1\|a\|\right) \left(1 - x_1^2\right)^{\frac{n-1}{2}} \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2} + 1\right)} \mathrm{d}x_1 \\
&= \frac{(2\pi)^{\frac{n}{2}}}{s\|a\|^{\frac{n}{2}}} J_{\frac{n}{2}}(s\|a\|) \\
&= \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)} j_{\frac{n}{2}}(s\|a\|).
\end{aligned}$$

□

The computation of the Fourier transform of the indicator function of the unit cube is a consequence of the previous lemma.

Lemma 2.6 (Fourier transform of $\chi_{\frac{1}{2}B_\infty^n}$)**.** *Let $s \in \mathbb{R}$, $a \in \mathbb{R}^n$. Then we have*

$$\int_{[-\frac{1}{2}, \frac{1}{2}]^n} \exp\left(-\mathrm{i}s\langle x, a\rangle\right) \mathrm{d}x = \prod_{j=1}^n \frac{\sin\left(\frac{a_j s}{2}\right)}{\frac{a_j s}{2}}.$$

Proof. For $n = 1$ the previous lemma yields $\int_{[-1,1]} \exp\left(-\mathrm{i}sx\|a\|\right) \mathrm{d}x = 2j_{\frac{1}{2}}(s\|a\|)$. Using Fubini's theorem and the substitution $x = \frac{1}{2}y$ we get the result. □

A bound for the absolute value of Bessel functions follows from [GR07, (8.479)]:

Lemma 2.7 (Bounds for Bessel functions)**.** *Let $\nu \ge \frac{1}{2}$ and $x > \nu$. Then*

$$|J_\nu(x)| \le \sqrt{\frac{2}{\pi}} \frac{1}{(x^2 - \nu^2)^{\frac{1}{4}}}.$$

For the normalized Bessel functions this reads as:

Lemma 2.8. *Let $m \in \mathbb{N}$ and $s > \frac{m}{2}$. Then*

$$|j_{\frac{m}{2}}(s)| \leq \frac{2^{\frac{m+1}{2}}\Gamma(\frac{m}{2}+1)}{\sqrt{\pi}} \frac{1}{\left(s^2 - \frac{m^2}{4}\right)^{\frac{1}{4}}} \frac{1}{s^{\frac{m}{2}}}.$$

Proof. This is a direct consequence of Lemma 2.7 and the definition of the normalized Bessel functions. □

More elaborated estimates were used in several contexts. We collect a few results that we need later.

Lemma 2.9. *Let $m \geq 2$ and $s \in [0, \frac{m}{2}+3]$ resp. let $m = 1$ and $s \in [0, 3.38]$. Then*

$$|j_{\frac{m}{2}}(s)| \leq \exp\left(-\frac{s^2}{2m+4} - \frac{s^4}{4(m^2+2m+4)(m+4)}\right)$$

Proof. This is found in [KK01, p. 19]. □

Lemma 2.10. *Let $m \geq 5$ and $s \in [0, m]$. Then we have*

$$|j_{\frac{m}{2}}(s)| \leq \exp\left(-\frac{s^2}{2m+4}\right).$$

Proof. Let $m = 5$ or $m = 6$. Then the inequality follows directly from Lemma 2.9, since $\frac{m}{2} + 3 \geq m$. The same lemma shows the inequality for $m \geq 7$ and $s \in [0, \frac{m}{2}+3]$. In [Brz11, Lemma 3.17] it is proved that for all $m \geq 7$ and $s \in [\frac{m}{2}+3, m]$ the claimed inequality also holds. Brzezinski's proof uses the estimate from Lemma 2.8. □

Lemma 2.11. *Let $m \in \mathbb{N}$ and $s \geq \frac{m}{2}+3$. Then*

$$|j_{\frac{m}{2}}(s)| \leq 2^{\frac{m+1}{2}} \frac{\Gamma(\frac{m}{2}+1)}{\sqrt{\pi}} \frac{\sqrt{m+6}}{\sqrt[4]{12m+36}} \frac{1}{s^{\frac{m+1}{2}}}.$$

Proof. For $s \geq \frac{m}{2}+3$:

$$\begin{aligned}\left(s^2 - \frac{m^2}{4}\right)^{-\frac{1}{4}} s^{-\frac{m}{2}} &= \left(1 - \frac{m^2}{4s^2}\right)^{-\frac{1}{4}} s^{-\frac{m+1}{2}} \\ &\leq \frac{\sqrt{m+6}}{\sqrt[4]{12m+36}}\end{aligned}$$

The estimate follows together with Lemma 2.8. □

We also need a lower bound on $\left|j_{\frac{m}{2}}(\cdot)\right|$.

Lemma 2.12. *For all $m \in \mathbb{N}$ and $s \in [0,1]$ we have*

$$\left| j_{\frac{m}{2}}(s) \right| \geq \exp\left(-\frac{s^2}{2m+4} - s^4 \right).$$

Proof. This is found in [Brz11, Lemma 3.5, part 2]. The estimate there is even stronger. □

Part I.

Sections of Simplices

3. Volume Formulas

First we derive formulas to compute the volume of the intersection of a regular simplex with some k-dimensional subspace. In the case of hyperplane sections and for slabs we give further formulas.

We use the representation of the n-dimensional simplex with $n+1$ vertices in $\mathbb{R}^{n+1}$, also called the embedded n-simplex. Let

$$S := \left\{ x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{j=1}^{n+1} x_j = 1,\ x_j \geq 0 \right\}.$$

S is the convex hull of the canonical unit vector basis $e_1, \ldots, e_{n+1}$. In this representation S has side length $\sqrt{2}$ and n-volume $\frac{\sqrt{n+1}}{n!}$. The centroid c of the simplex is at $\left(\frac{1}{n+1}, \ldots, \frac{1}{n+1}\right)$.

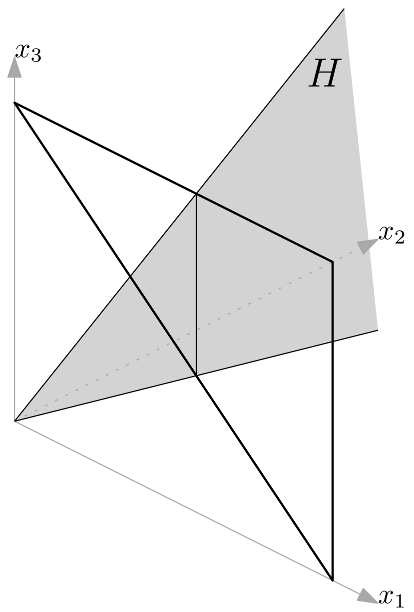

Figure 3.1.: The embedded 2-simplex and a subspace H

3.1. General formula

We prove the following general volume formula.

Theorem 3.1. *Let H be a k-dimensional subspace of $\mathbb{R}^{n+1}$ and a^l, $l = 1, \dots, n+1-k$ some orthonormal basis of $H^\perp$. Then*

$$\begin{aligned}&\operatorname{vol}_{k-1}(H \cap S)\\ &= \frac{\sqrt{n+1-\sum_{l=1}^{n+1-k}\left(\sum_{j=1}^{n+1} a_j^l\right)^2}}{(k-1)!} \frac{1}{(2\pi)^{n+1-k}} \int_{\mathbb{R}^{n+1-k}} \prod_{j=1}^{n+1} \frac{1}{1+\mathrm{i}\left(\sum_{l=1}^{n+1-k} a_j^l s^l\right)} \mathrm{d}s.\end{aligned} \tag{3.1}$$

Note that the formula is indeed independent of the choice of the orthonormal basis of $H^\perp$. The left-hand side of (3.1) only depends on H, but not on the choice of the orthonormal basis of $H^\perp$. We claim that also $\sum_{l=1}^{n+1-k}\left(\sum_{j=1}^{n+1} a_j^l\right)^2$ and the integral on the right-hand side do not depend on the basis. Let b^l, $l = 1, \dots, n+1-k$ be another orthonormal basis of $H^\perp$. Then there exists an orthogonal matrix

$$U = \left(u_l^m\right)_{m,l=1}^{n+1-k}$$

such that $b^l = \sum_{m=1}^{n+1-k} u_l^m a^m$ for every $l = 1, \dots, n+1-k$. Using $\sum_{l=1}^{n+1-k} u_l^p u_l^q = \delta_{pq}$, we compute

$$\begin{aligned}\sum_{l=1}^{n+1-k}\left(\sum_{j=1}^{n+1} b_j^l\right)^2 &= \sum_{i,j=1}^{n+1}\sum_{l=1}^{n+1-k} b_i^l b_j^l\\ &= \sum_{i,j=1}^{n+1}\sum_{p,q=1}^{n+1-k}\left(\sum_{l=1}^{n+1-k} u_l^p u_l^q\right) a_i^p a_j^q\\ &= \sum_{l=1}^{n+1-k}\left(\sum_{j=1}^{n+1} a_j^l\right)^2.\end{aligned}$$

Since $|\det U| = 1$, by the transformation theorem also the integrals in the formula, $\int_{\mathbb{R}^{n+1-k}} \prod_{j=1}^{n+1} \frac{1}{1+\mathrm{i}\left(\sum_{l=1}^{n+1-k} a_j^l s^l\right)} \mathrm{d}s$ and $\int_{\mathbb{R}^{n+1-k}} \prod_{j=1}^{n+1} \frac{1}{1+\mathrm{i}\left(\sum_{l=1}^{n+1-k} b_j^l s^l\right)} \mathrm{d}s$, are equal.

If H is a k-dimensional subspace of $\mathbb{R}^{n+1}$, the intersection $H \cap S$ is $(k-1)$-dimensional, if it is not degenerated. For the computations it is convenient to work with

$$\bar{S} := \left\{ x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \;\Big|\; \sum_{j=1}^{n+1} x_j \leq 1,\ x_j \geq 0 \right\}$$

which is an irregular $(n+1)$-simplex with an orthogonal corner at the origin, i.e. all edges are pairwise orthogonal. It can also be written as $\bar{S} = \operatorname{conv}(0, S)$. $\bar{S}$ is a proper $(n+1)$-dimensional body in $\mathbb{R}^{n+1}$. The intersection $H \cap \bar{S}$ is a pyramid with base $H \cap S$ and apex 0. Its height is given by $\operatorname{dist}(0, H \cap \tilde{S})$, where $\tilde{S} := \{x \in \mathbb{R}^{n+1} \mid \sum_{j=1}^{n+1} x_j = 1\}$. The set $H \cap \bar{S}$ is k-dimensional.

We start with the computation of the height of the pyramid $H \cap \bar{S}$.

Lemma 3.2. *Let $a^l, l = 1, \ldots, n+1-k$ be an orthonormal basis of $H^\perp$. Then*

$$\operatorname{dist}(H \cap \tilde{S}, 0) = \frac{1}{\sqrt{n+1-\sum_{l=1}^{n+1-k}\left(\sum_{j=1}^{n+1} a_j^l\right)^2}}.$$

Proof. We minimize $\|x\|$ under the constraints $\langle x, a^l \rangle = 0$ for all $l = 1, \ldots, n+1-k$, and $\sum_{j=1}^{n+1} x_j = 1$. Define the Lagrange function

$$\Lambda(x, \lambda, \mu) := \sum_{j=1}^{n+1} x_j^2 + \lambda \left(\sum_{j=1}^{n+1} x_j - 1 \right) + \sum_{l=1}^{n+1-k} \mu^l \left(\sum_{j=1}^{n+1} a_j^l x_j \right).$$

For the derivative with respect to x_J we find $\frac{\partial \Lambda}{\partial x_J} = 2x_J + \lambda + \sum_{l=1}^{n+1-k} \mu^l a_J^l$, so for a critical vector x:

$$0 = 2x_J + \lambda + \sum_{l=1}^{n+1-k} \mu^l a_J^l. \tag{3.2}$$

Summing (3.2) over J and using $\sum_{j=1}^{n+1} x_j = 1$ leads to

$$0 = 2 + (n+1)\lambda + \sum_{l=1}^{n+1-k} \mu^l \sum_{J=1}^{n+1} a_J^l. \tag{3.3}$$

Multiplying (3.2) with x_J and then summing over J gives

$$0 = 2\sum_{J=1}^{n+1} x_J^2 + \lambda. \tag{3.4}$$

Multiplying (3.2) with a_J^L for a fixed L and then summing over J we find

$$0 = \lambda \sum_{J=1}^{n+1} a_J^L + \sum_{l=1}^{n+1-k} \mu^l \sum_{J=1}^{n+1} a_J^l a_J^L = \lambda \sum_{J=1}^{n+1} a_J^L + \mu^L. \tag{3.5}$$

This implies that

$$\sum_{J=1}^{n+1} x_J^2 = \frac{1}{n+1-\sum_{L=1}^{n+1-k}\left(\sum_{J=1}^{n+1} a_J^L\right)^2}$$

is as a necessary condition for an extremum. Obviously this critical point is a minimum. Therefore

$$\operatorname{dist}(H \cap \tilde{S}) = \frac{1}{\sqrt{n+1-\sum_{l=1}^{n+1-k}\left(\sum_{j=1}^{n+1} a_j^l\right)^2}}. \tag{3.6}$$

□

To compute the k-volume of $H \cap \bar{S}$ we use the following lemma which is a variant of a lemma of M. Meyer and A. Pajor from [MP88].

Lemma 3.3. *Let H be a k-dimensional subspace of $\mathbb{R}^{n+1}$. Then*

$$\operatorname{vol}_k(H \cap \bar{S}) = \frac{1}{k!} \int_{H \cap \mathbb{R}_{\geq 0}^{n+1}} \exp\left(-\sum_{j=1}^{n+1} x_j\right) \mathrm{d}x, \tag{3.7}$$

where we integrate with respect to the induced Lebesgue measure of H.

Proof. Let H be defined by some orthonormal basis of $H^\perp$, say $a^1, \dots, a^{n+1-k}$. Let $K \subset H$. For $\epsilon \geq 0$ we define

$$K(\epsilon) := \left\{ x + \sum_{l=1}^{n+1-k} t^l a^l \;\middle|\; x \in K,\ t^l \in [-\epsilon, \epsilon] \right\}.$$

The set $K(\epsilon)$ and therefore the computations below depend on the choice of the orthonormal basis. But finally we consider limits for $\epsilon \to 0$. They are independent of the choice of $a^1, \dots, a^{n+1-k}$.

The k-volume of K is given by $\lim_{\epsilon \to 0} \frac{1}{(2\epsilon)^{n+1-k}} \operatorname{vol}_{n+1}(K(\epsilon))$. For $c > 0$ we have

$$\operatorname{vol}_{n+1}(K(c\epsilon)) = c^{n+1-k} \operatorname{vol}_{n+1}(K(\epsilon)). \tag{3.8}$$

We consider the following integral with respect to the Lebesgue measure on $\mathbb{R}^{n+1}$:

$$g(\epsilon) := \frac{1}{(2\epsilon)^{n+1-k}} \int_{H(\epsilon)\cap\mathbb{R}^{n+1}_{\geq 0}} \exp\left(-\sum_{j=1}^{n+1} x_j\right) \mathrm{d}x.$$

Then

$$g(\epsilon) = \frac{1}{(2\epsilon)^{n+1-k}} \int_{H(\epsilon)\cap\mathbb{R}^{n+1}_{\geq 0}} \int_{\sum_{j=1}^{n+1} x_j}^{\infty} \exp(-t)\,\mathrm{d}t\ \mathrm{d}x.$$

Integrating on the level sets of $\sum_{j=1}^{n+1} x_j$ we find

$$g(\epsilon) = \frac{1}{(2\epsilon)^{n+1-k}} \int_0^\infty \mathrm{vol}_n \left(\left\{ x \in \mathbb{R}^{n+1}_{\geq 0} \;\middle|\; x \in H(\epsilon), \sum_{j=1}^{n+1} x_j = s \right\}\right) \int_s^\infty \exp(-t)\,\mathrm{d}t\ \mathrm{d}s.$$

A special case of Fubini's theorem for an integrable function f states $\int_a^b \int_x^b f(x,y)\mathrm{d}y\ \mathrm{d}x = \int_a^b \int_a^y f(x,y)\mathrm{d}x\ \mathrm{d}y$. So we get

$$\begin{aligned} g(\epsilon) &= \frac{1}{(2\epsilon)^{n+1-k}} \int_0^\infty \int_0^t \mathrm{vol}_n \left(\left\{ x \in \mathbb{R}^{n+1}_{\geq 0} \;\middle|\; x \in H(\epsilon), \sum_{j=1}^{n+1} x_j = s \right\}\right) \exp(-t)\,\mathrm{d}s\ \mathrm{d}t \\ &= \frac{1}{(2\epsilon)^{n+1-k}} \int_0^\infty \mathrm{vol}_{n+1} \left(\left\{ x \in \mathbb{R}^{n+1}_{\geq 0} \;\middle|\; x \in H(\epsilon), \sum_{j=1}^{n+1} x_j \leq t \right\}\right) \exp(-t)\,\mathrm{d}t. \end{aligned}$$

For the set $\left\{ x \in \mathbb{R}^{n+1}_{\geq 0} \;\middle|\; x \in H(\epsilon), \sum_{j=1}^{n+1} x_j \leq t \right\}$ we have

$$\begin{aligned} &\left\{ x \in \mathbb{R}^{n+1}_{\geq 0} \;\middle|\; x \in H(\epsilon), \sum_{j=1}^{n+1} x_j \leq t \right\} \\ &= \left\{ x \in \mathbb{R}^{n+1}_{\geq 0} \;\middle|\; \left|\left\langle a^l, x \right\rangle\right| \leq \epsilon, l = 1, \ldots, n+1-k,\ \sum_{j=1}^{n+1} x_j \leq t \right\} \\ &= \left\{ x \in \mathbb{R}^{n+1}_{\geq 0} \;\middle|\; \left|\left\langle a^l, \frac{x}{t} \right\rangle\right| \leq \frac{\epsilon}{t}, l = 1, \ldots, n+1-k,\ \sum_{j=1}^{n+1} \frac{x_j}{t} \leq 1 \right\} \\ &= \left\{ t \cdot x \in \mathbb{R}^{n+1}_{\geq 0} \;\middle|\; \left|\left\langle a^l, x \right\rangle\right| \leq \frac{\epsilon}{t}, l = 1, \ldots, n+1-k,\ \sum_{j=1}^{n+1} x_j \leq 1 \right\} \\ &= \left\{ t \cdot x \in \mathbb{R}^{n+1}_{\geq 0} \;\middle|\; x \in H\left(\frac{\epsilon}{t}\right), \sum_{j=1}^{n+1} x_j \leq 1 \right\}. \end{aligned}$$

By homogeneity of vol_{n+1} we get

$$g(\epsilon) = \frac{1}{(2\epsilon)^{n+1-k}} \int_0^\infty t^{n+1} \mathrm{vol}_{n+1} \left(\left\{ x \in \mathbb{R}^{n+1}_{\geq 0} \;\middle|\; x \in H\left(\frac{\epsilon}{t}\right), \sum_{j=1}^{n+1} x_j \leq 1 \right\} \right) \exp(-t)\,\mathrm{d}t.$$

Since $\left\{ x \in \mathbb{R}^{n+1}_{\geq 0} \;\middle|\; x \in H\left(\frac{\epsilon}{t}\right), \sum_{j=1}^{n+1} x_j \leq 1 \right\} \subset \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; x \in (H \cap \bar{S})\left(\frac{\epsilon}{t}\right) \right\}$ and by (3.8) we have

$$\begin{aligned} g(\epsilon) &\leq \frac{1}{(2\epsilon)^{n+1-k}} \int_0^\infty t^k \mathrm{vol}_{n+1} \left(\left\{ x \in \mathbb{R}^{n+1} \;\middle|\; x \in (H \cap \bar{S})(\epsilon) \right\} \right) \exp(-t)\,\mathrm{d}t \\ &= \frac{1}{(2\epsilon)^{n+1-k}} \mathrm{vol}_{n+1} \left(\left\{ x \in \mathbb{R}^{n+1}_{\geq 0} \;\middle|\; x \in (H \cap \bar{S})(\epsilon) \right\} \right) \Gamma(1+k) \end{aligned}$$

For $\epsilon \to 0$ this tends to $\Gamma(1+k)\,\mathrm{vol}_k(H \cap \bar{S})$.

For $\delta \geq 0$ let M_δ be the subset of $H \cap \bar{S}$ such that $M_\delta(\delta) \subset \bar{S}$. Since the following inclusion holds

$$M_{\frac{\epsilon}{t}}\left(\frac{\epsilon}{t}\right) \subset \left\{ x \in \mathbb{R}^{n+1}_{\geq 0} \;\middle|\; x \in H\left(\frac{\epsilon}{t}\right), \sum_{j=1}^{n+1} x_j \leq 1 \right\},$$

we get by (3.8):

$$g(\epsilon) \geq \frac{1}{(2\epsilon)^{n+1-k}} \int_0^\infty t^k \mathrm{vol}_{n+1} \left(M_{\frac{\epsilon}{t}}(\epsilon) \right) \exp(-t)\,\mathrm{d}t.$$

Note that $\lim_{\epsilon \to 0} \frac{1}{(2\epsilon)^{n+1-k}} \mathrm{vol}_{n+1} \left(M_{\frac{\epsilon}{t}}(\epsilon) \right) = \mathrm{vol}_k(M_0) = \mathrm{vol}_k(H \cap \bar{S})$. So for $\epsilon \to 0$ we have

$$\frac{1}{(2\epsilon)^{n+1-k}} \int_0^\infty t^k \mathrm{vol}_{n+1} \left(M_{\frac{\epsilon}{t}}(\epsilon) \right) \exp(-t)\,\mathrm{d}t \quad \longrightarrow \quad \Gamma(1+k)\,\mathrm{vol}_k(H \cap \bar{S}),$$

since we may interchange the limit and the integral due to dominated convergence.

Therefore

$$\lim_{\epsilon \to 0} g(\epsilon) = \Gamma(1+k)\,\mathrm{vol}_k(H \cap \bar{S}).$$

□

To get an explicit formula it is classical to use the Fourier transformation and the Fourier inversion theorem, e.g. [Bal86], [OP00].

Lemma 3.4. *Let H be a k-dimensional subspace of $\mathbb{R}^{n+1}$ and a^l, $l = 1, \dots, n+1-k$ some orthonormal basis of $H^\perp$. Then*

$$\int\limits_{H\cap\mathbb{R}^{n+1}_{\geq 0}} \exp\left(-\sum_{j=1}^{n+1} x_j\right) \mathrm{d}x = \frac{1}{(2\pi)^{n+1-k}} \int\limits_{\mathbb{R}^{n+1-k}} \prod_{j=1}^{n+1} \frac{1}{1+\mathrm{i}\left(\sum_{l=1}^{n+1-k} a_j^l s^l\right)} \mathrm{d}s. \tag{3.9}$$

Proof. First rewrite the integral

$$\int\limits_{H\cap\mathbb{R}^{n+1}_{\geq 0}} \exp\left(-\sum_{j=1}^{n+1} x_j\right) \mathrm{d}x = \int_{\forall l,j:\ \left\langle x,a^l\right\rangle=0,\ x_j\geq 0} \exp\left(-\sum_{j=1}^{n+1} x_j\right) \mathrm{d}x$$

and define a map $F\colon \mathbb{R}^{n+1-k} \to \mathbb{R}$ by

$$t = (t^1, \dots, t^{n+1-k}) \mapsto \int_{\forall l,j:\ \left\langle x,a^l\right\rangle=t^l,\ x_j\geq 0} \exp\left(-\sum_{j=1}^{n+1} x_j\right) \mathrm{d}x.$$

This function is integrable, since $\int_{\mathbb{R}^{n+1-k}} F(t)\mathrm{d}t = \int_{\mathbb{R}^{n+1}_{\geq 0}} \exp\left(-\sum_{j=1}^{n+1} x_j\right) \mathrm{d}x = 1$. We apply the Fourier transform:

$$\begin{aligned}
&(2\pi)^{\frac{n+1-k}{2}} \hat{F}(\tau) \\
&= \int_{\mathbb{R}^{n+1-k}} F(t) \exp\left(-\mathrm{i}\left\langle t,\tau\right\rangle\right) \mathrm{d}t \\
&= \int_{\mathbb{R}^{n+1-k}} \left(\int_{\forall l,j:\ \left\langle x,a^l\right\rangle=t^l,\ x_j\geq 0} \exp\left(-\sum_{j=1}^{n+1} x_j\right) \ \mathrm{d}x\right) \exp\left(-\mathrm{i}\sum_{l=1}^{n+1-k} \left\langle x,a^l\right\rangle \tau^l\right) \mathrm{d}t \\
&= \int_{\mathbb{R}^{n+1}_{\geq 0}} \exp\left(-\sum_{j=1}^{n+1} x_j\right) \exp\left(-\mathrm{i}\sum_{j=1}^{n+1}\left(\sum_{l=1}^{n+1-k} a_j^l \tau^l\right) x_j\right) \mathrm{d}x \\
&= \int_{\mathbb{R}^{n+1}_{\geq 0}} \exp\left(-\sum_{j=1}^{n+1}\left(1+\mathrm{i}\sum_{l=1}^{n+1-k} a_j^l \tau^l\right) x_j\right) \mathrm{d}x \\
&= \prod_{j=1}^{n+1} \frac{1}{1+\mathrm{i}\left(\sum_{l=1}^{n+1-k} a_j^l \tau^l\right)}.
\end{aligned}$$

This function is continuous and integrable. By the Fourier inversion formula we get

$$\begin{aligned} F(0) &= \frac{1}{(2\pi)^{(n+1-k)/2}} \int_{\mathbb{R}^{n+1-k}} \hat{F}(\tau)\mathrm{d}\tau \\ &= \frac{1}{(2\pi)^{n+1-k}} \int_{\mathbb{R}^{n+1-k}} \prod_{j=1}^{n+1} \frac{1}{1+\mathrm{i}\left(\sum_{l=1}^{n+1-k} a_j^l s^l\right)} \mathrm{d}s. \end{aligned}$$

□

Now we put together Lemmas 3.2, 3.3 and 3.4 and use the formula for the k-volume of a pyramid

$$\mathrm{vol}_k\left(H \cap \bar{S}\right) = \frac{1}{k}\,\mathrm{dist}(H \cap \tilde{S}, 0)\,\mathrm{vol}_{k-1}(H \cap S).$$

This proves Theorem 3.1.

Remark: At first sight the appearance of the imaginary factor i in a formula for a volume looks strange. But the integral is indeed real. One can expand the product and observe that summands having an odd power of i also have an odd power of s. So by integration the complex part of the integral vanishes. A different argument for the integral being real can be taken from [Bra78, p. 14]: The Fourier transform of a real and asymmetric function is complex but hermitian, i.e. the real part is even and the imaginary part is odd.

3.2. Formulas for hyperplane sections

In this subsection H is a subspace of dimension n, so it is a hyperplane. For a vector $a \in \mathbb{R}^{n+1}$ with $\|a\| = 1$ and $t \geq 0$ let $H_a^t := \{x \in \mathbb{R}^{n+1} \mid \langle x, a\rangle = t\}$ and $H_a := H_a^0$. With this notation Theorem 3.1 states

Corollary 3.5. *Let $a \in \mathbb{R}^{n+1}$ with $\|a\| = 1$. Then*

$$\mathrm{vol}_{n-1}(H_a \cap S) = \frac{\sqrt{n+1-(\sum_{j=1}^{n+1} a_j)^2}}{(n-1)!} \frac{1}{2\pi} \int_{\mathbb{R}} \prod_{j=1}^{n+1} \frac{1}{1+\mathrm{i}a_j s}\mathrm{d}s. \tag{3.10}$$

If we require the normal vector a to satisfy

$$\sum_{j=1}^{n+1} a_j = 0 \tag{3.11}$$

the section $H_a \cap S$ contains the centroid of the simplex $c = \left(\frac{1}{n+1}, \ldots, \frac{1}{n+1}\right)$. In this case formula (3.10) is the same as in Webb's paper, cf. [Web96].

Arbitrary sections of the simplex can either be written as $H_a^t \cap S$ with a satisfying (3.11) and $\|a\| = 1$ and t giving the distance of H_a to the centroid, more precisely

$$\operatorname{dist}(c, H_a^t \cap S) = |t|.$$

Or we take a suitable $b \in \mathbb{R}^{n+1}$, not necessarily with $\sum_{j=1}^{n+1} b_j \neq 0$, such that $H_a^t \cap S = H_b \cap S$, see Figure 3.2 resp. 3.3.

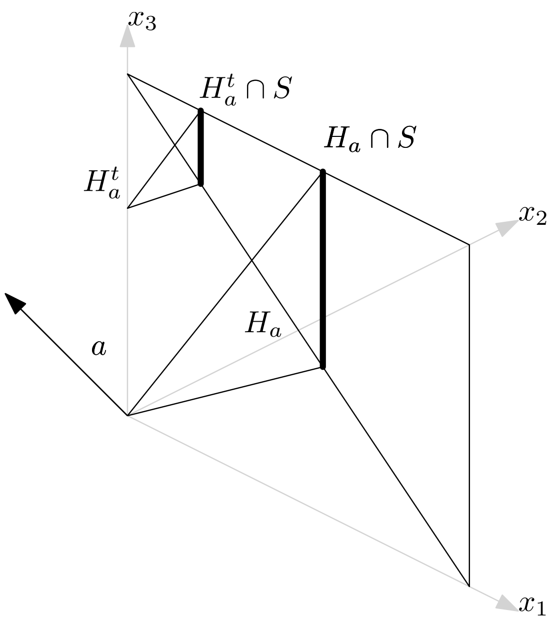

Figure 3.2.: Section $H_a^t \cap S$

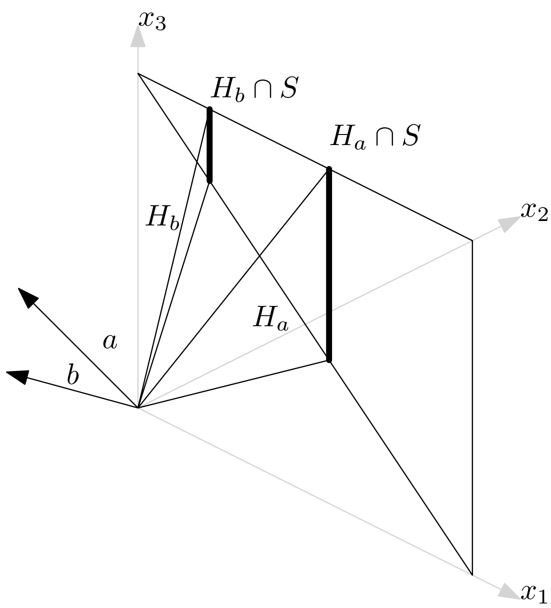

Figure 3.3.: Section $H_b \cap S$

One can convert the representations into each other in the following way:

From Figure 3.2 to 3.3: Given $a = (a_1, \dots, a_{n+1})$ with $\|a\| = 1$, $\sum_{j=1}^{n+1} a_j = 0$ and $t \in \mathbb{R}$ set

$$b_j := \frac{a_j - t}{\sqrt{1 + (n+1)t^2}}$$

for $j = 1, \dots, n+1$. Then b has norm 1. Let $x \in S$. Then $\langle x, a \rangle = t$ is equivalent to $\sqrt{1+(n+1)t^2}\, \langle x, b \rangle = \sum_{j=1}^{n+1} (a_j x_j - t x_j) = \langle a, x \rangle - t \sum_{j=1}^{n+1} x_j = 0$. So

$$H_a^t \cap S = H_b \cap S.$$

From Figure 3.3 to 3.2: On the other hand, given $b = (b_1, \dots, b_{n+1})$ with $\|b\| = 1$ set

$$a_j = \sqrt{\frac{n+1}{n+1-(\sum_{j=1}^{n+1} b_j)^2}} \cdot b_j - \frac{\sum_{j=1}^{n+1} b_j}{\sqrt{(n+1)(n+1-\left(\sum_{j=1}^{n+1} b_j)^2\right)}} \quad \text{and}$$

$$t = -\frac{\sum_{j=1}^{n+1} b_j}{\sqrt{(n+1)\left(n+1-(\sum_{j=1}^{n+1} b_j)^2\right)}} \tag{3.12}$$

for $j = 1, \dots, n+1$. Then a has norm 1 and fulfills (3.11). For $x \in S$ we have $\langle x, a\rangle - t = \sum_{j=1}^{n+1} x_j(a_j - t) = \sqrt{\frac{n+1}{n+1-(\sum_{j=1}^{n+1} b_j)^2}} \sum_{j=1}^{n+1} x_j b_j$. Therefore the condition $\langle x, b\rangle = 0$ is equivalent to $\langle x, a\rangle = t$. So we have

$$H_b \cap S = H_a^t \cap S.$$

Due to (3.12) we also know

$$\operatorname{dist}(c, H_b \cap S) = \frac{\left|\sum_{j=1}^{n+1} b_j\right|}{\sqrt{(n+1)\left(n+1-(\sum_{j=1}^{n+1} b_j)^2\right)}}. \tag{3.13}$$

Substituting the corresponding a, t and b one gets from (3.10) the formula

Corollary 3.6. *Let $a \in \mathbb{R}^{n+1}$ with $\|a\| = 1$, $\sum_{j=1}^{n+1} a_j = 0$ and $t \in \mathbb{R}$. Then*

$$\operatorname{vol}_{n-1}(H_a^t \cap S) = \frac{\sqrt{n+1}}{(n-1)!} \frac{1}{2\pi} \int_{\mathbb{R}} \prod_{j=1}^{n+1} \frac{1}{1+\mathrm{i}(a_j - t)s} \mathrm{d}s. \tag{3.14}$$

In the case of hyperplane sections the formulas resp. the integrals can be evaluated by the residue theorem. For the special case of central sections this was suggested by Webb [Web96] and done by Brzezinski [Brz11]. One gets an explicit formula without integrals. This still works for non-central sections.

The proof of the following lemma can be found in [Brz11, Satz 6.5].

Lemma 3.7. *For $x_1, \dots, x_{n+1} \in \mathbb{R}$, such that all $x_j > 0$ are pairwise distinct and there is at least one $x_j > 0$ and one $x_j < 0$ we have*

$$\frac{1}{2\pi} \int_{\mathbb{R}} \prod_{j=1}^{n+1} \frac{1}{1+\mathrm{i}x_j s} \mathrm{d}s = \sum_{x_j > 0} \frac{1}{x_j} \prod_{k \neq j} \frac{x_j}{x_j - x_k}.$$

The integral is also equal to $\sum_{x_j<0} \frac{1}{x_j} \prod_{k \neq j} \frac{x_j}{x_j - x_k}$ *and equal to* $\frac{1}{2} \sum_{j=1}^{n+1} \frac{1}{|x_j|} \prod_{k \neq j} \frac{x_j}{x_j - x_k}$.

We use this to give explicit formulas for both representations (3.10) and (3.14).

Corollary 3.8. *Given* $a \in \mathbb{R}^{n+1}$ *with* $\|a\| = 1$, $\sum_{j=1}^{n+1} a_j = 0$ *and* $t \in \mathbb{R}$ *or* $b \in \mathbb{R}^{n+1}$ *with* $\|b\| = 1$ *and* $H_a^t \cap S = H_b \cap S$ *we have*

$$\mathrm{vol}_{n-1}(H_b \cap S) = \frac{\sqrt{n+1-(\sum_{j=1}^{n+1} b_j)^2}}{(n-1)!} \sum_{b_j>0} \frac{1}{b_j} \prod_{k \neq j} \frac{b_j}{b_j - b_k}, \tag{3.15}$$

$$\mathrm{vol}_{n-1}(H_a^t \cap S) = \frac{\sqrt{n+1}}{(n-1)!} \sum_{a_j - t>0} \frac{1}{a_j - t} \prod_{k \neq j} \frac{a_j - t}{a_j - a_k}, \tag{3.16}$$

as long as a *and* b *satisfy that all positive coordinates are pairwise distinct and at least one coordinate is positive and at least one coordinate is negative.*

The formulas are also true if we write $\sum_{b_j<0}$ *or* $\frac{1}{2}\sum_{j=1}^{n+1}$ *as in Lemma 3.7.*

The map $a \mapsto \mathrm{vol}_{n-1}(H_a \cap S)$ is continuous. So the restriction on the coordinates of the normal vector is not strict, in the sense that one can always extend the formulas continuously.

Remark on an alternative approach: There is a different way to derive volume formula (3.16) using the theory of splines. We just indicate the idea. Already in the classical paper [CS66] on spline functions one finds the idea. Roughly speaking, a spline is a piecewise polynomial function. The pieces are fitted together at “knots”. It is proved in [CS66] that

$$\mathrm{vol}_{n-1}(H_a^t \cap S) = \sqrt{n+1}\ \operatorname{Spline}(t \mid a_{n+1}, \ldots, a_1).$$

These spline functions can be evaluated using the finite difference representation. This also leads to formula (3.16). A different description is given in [Mic95].

3.3. Formula for slabs

A symmetric slab of width t is given by

$$\mathrm{slab}(a,t) := \{x \in S \mid |\langle x, a\rangle| \leq t\}.$$

We get a formula by integrating (3.16).

Corollary 3.9. *Let $a \in \mathbb{R}^{n+1}$ with $\|a\| = 1$ and $\sum_{j=1}^{n+1} a_j = 0$, such that all positive a_j are pairwise distinct and $t \in [0, \max_j |a_j|)$. Then*

$$\operatorname{vol}_n(\operatorname{slab}(a,t)) = \frac{\sqrt{n+1}}{2n!} \sum_{j=1}^{n+1} \left(\prod_{k \neq j} \frac{1}{a_j - a_k} \right) \left((a_j + t)^n + (a_j - t)^n \right).$$

Proof. We use formula (3.16):

$$\begin{aligned}
\operatorname{vol}_n(\operatorname{slab}(a,t)) &= \int_{-t}^{t} \operatorname{vol}_{n-1}(H_a^\tau \cap S) \mathrm{d}\tau \\
&= \frac{\sqrt{n+1}}{(n-1)!} \frac{1}{2} \sum_{j=1}^{n+1} \left(\prod_{k \neq j} \frac{1}{a_j - a_k} \right) \int_{-t}^{t} \frac{(a_j - \tau)^n}{|a_j - \tau|} \mathrm{d}\tau \\
&= \frac{\sqrt{n+1}}{(n-1)!} \frac{1}{2} \sum_{j=1}^{n+1} \left(\prod_{k \neq j} \frac{1}{a_j - a_k} \right) \frac{1}{n} \left((a_j + t)^n + (a_j - t)^n \right).
\end{aligned}$$

□

4. Bounds for Hyperplane Sections

In this section we give bounds for the volume of hyperplane sections. S. Webb found the maximal *central* section [Web96]. We give the maximal section also for hyperplanes of fixed small distance to the centroid. On the way we provide an alternative proof for the maximal central section.

Filiman [Fil92] claimed that the minimal central section is given by the one parallel to a face, but he did not give precise arguments. Brzezinski [Brz11] proved this up to a constant. We prove the conjecture for 2-, 3- and 4-simplices. Furthermore we show that for all dimensions n the hyperplane that would give the conjectured minimum is at least some local minimum.

We denote two special directions:

$$a_{\min} := \left(\sqrt{\frac{n}{n+1}}, -\frac{1}{\sqrt{n(n+1)}}, \ldots, -\frac{1}{\sqrt{n(n+1)}}\right) \in \mathbb{R}^{n+1}$$
$$a_{\max} := \left(\frac{1}{\sqrt{2}}, 0, \ldots, 0, -\frac{1}{\sqrt{2}}\right) \in \mathbb{R}^{n+1}$$

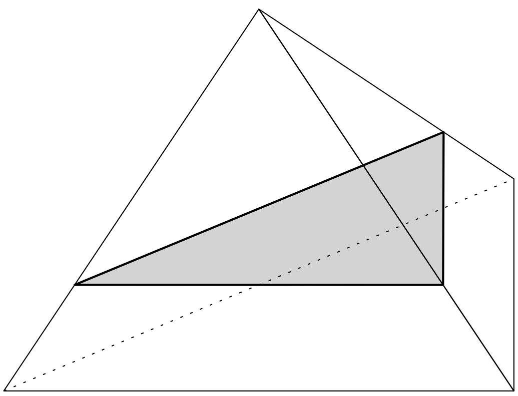

Figure 4.1.: Section orthogonal to $a_{\min}$

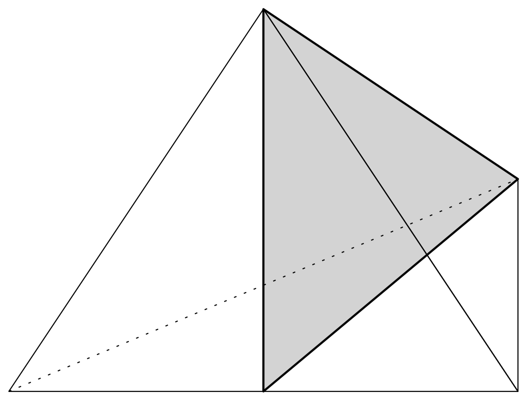

Figure 4.2.: Section orthogonal to $a_{\max}$

Note that, for the moment, $a_{\min}$ and $a_{\max}$ are just names for these vectors. The volumes for these directions are computed with the formulas from Section 3.2. One finds

$$\operatorname{vol}_{n-1}\left(H_{a_{\min}} \cap S\right) = \frac{\sqrt{n+1}}{(n-1)!}\left(\frac{n}{n+1}\right)^{n-\frac{1}{2}},$$
$$\operatorname{vol}_{n-1}\left(H_{a_{\max}} \cap S\right) = \frac{\sqrt{n+1}}{(n-1)!}\frac{1}{\sqrt{2}}.$$

For $n \to \infty$ the term $\left(\frac{n}{n+1}\right)^{n-\frac{1}{2}}$ decreases to $\frac{1}{e}$. The hyperplane $H_{a_{\min}}$ is parallel to one of the faces of the simplex. The hyperplane $H_{a_{\max}}$ contains $n-1$ vertices and the midpoint of the remaining two vertices, see Figures 4.1 and 4.2.

4.1. First estimates

We investigate the volume of a hyperplane section with fixed distance to the centroid. For a normal vector a set $K := \sum_{j=1}^{n+1} a_j$. Due to (3.13), fixed distance means K is fixed. We may assume $K \geq 0$, since a and $-a$ determine the same hyperplane. By continuity of $a \mapsto \operatorname{vol}_{n-1}(H_a \cap S)$ we may assume that $a_j \neq 0$. Furthermore we assume that $a = (a_1, \ldots, a_P, a_{P+1}, \ldots, a_{n+1})$ with $a_j > 0$ for $j = 1, \ldots, P$ and $a_j < 0$ for $j = P+1, \ldots, n+1$. This does not change the volume, since permutations of the coordinates correspond to the symmetries of the simplex. We work with formula (3.15). So we estimate

$$F(a) := \sum_{j=1}^{P} \frac{1}{a_j} \prod_{k=1, k\neq j}^{n+1} \frac{1}{1-\frac{a_k}{a_j}}$$

as long as all positive a_j are pairwise distinct. The estimates rely on the following inequalities, valid for $N \in \mathbb{N}$ and $x_1, \ldots, x_N > 0$. These inequalities are a version of Bernoulli's inequality resp. of the Arithmetic-Geometric-Mean inequality:

$$1 + \sum_{j=1}^{N} x_j \leq \prod_{j=1}^{N}(1+x_j) \leq \left(1 + \frac{\sum_{j=1}^{N} x_j}{N}\right)^N. \tag{4.1}$$

The idea for our estimates is the following: We modify a given vector a such that the sum of the coordinates, their square sum and their signs do not change. Preserving the sign geometrically means the polytopal structure of the section is preserved (see Subsection 4.3 below). Preserving the sum of the coordinates means preserving the distance of the hyperplane to the centroid.

Lemma 4.1. *Let* $0 \leq K \leq 1$, $a \in \mathbb{R}^{n+1}$ *with* $\|a\| = 1$ *and* $\sum_{j=1}^{n+1} a_j = K$. *Then we have*

$$F(a) \leq \frac{1}{\sqrt{2 - K^2}}$$

with equality for $a = \left(\frac{K}{2} + \sqrt{\frac{1}{2} - \frac{K^2}{4}}, 0, \dots, 0, \frac{K}{2} - \sqrt{\frac{1}{2} - \frac{K^2}{4}}, 0, \dots, 0\right)$.

Proof. For a given a we define

$$\tilde{a} := \left(\gamma a_1, \dots, \gamma a_P, \beta \sum_{j>p} a_j, 0, \dots\right),$$

with $\gamma \geq 0, \beta \geq 0$ such that $\|\tilde{a}\| = 1$ and $\sum_{j=1}^{n+1} \tilde{a}_j = \sum_{j=1}^{n+1} a_j$. Then we have

$$\gamma \sum_{j \leq P} a_j + \beta \sum_{j>P} a_j = \sum_{j=1}^{n+1} a_j, \tag{4.2}$$

$$\gamma^2 \sum_{j \leq P} a_j^2 + \beta^2 \left(\sum_{j>P} a_j\right)^2 = 1. \tag{4.3}$$

We show that β and γ in $[0, 1]$ with these properties exist:
Equation (4.2) describes a line $\beta_g(\gamma)$ in the (γ, β)-plane with positive slope, and values $\beta_g(0) = \frac{\sum_{j=1}^{n+1} a_j}{\sum_{j>P} a_j} \leq 0$ and $\beta_g(1) = 1$, since $\sum_{j=1}^{n+1} a_j \geq 0$. Equation (4.3) describes an ellipse. Solving this for the positive part of the ellipse, we get $\beta_e(0) = \sqrt{\frac{1}{(\sum_{j>P} a_j)^2}} > 0$ and $\beta_e(1) = \sqrt{\frac{1 - \sum_{j \leq p} a_j^2}{(\sum_{j>p} a_j)^2}} = \sqrt{\frac{\sum_{j>P} a_j^2}{(\sum_{j>p} a_j)^2}} \leq 1$. Therefore $\beta_g(0) \leq 0 \leq \beta_e(0)$ and $\beta_g(1) = 1 \geq \beta_e(1)$. The two functions $\beta_g(\cdot)$ and $\beta_e(\cdot)$ are continuous on $\mathbb{R}_{\geq 0}$. Due to the intermediate value theorem they intersect, so there are $\beta, \gamma \in [0, 1]$ with the desired properties.

We show that $F(\tilde{a}) \geq F(a)$:

$$\begin{aligned}
F(\tilde{a}) &= \sum_{j=1}^{P} \frac{1}{\gamma a_j} \left(\prod_{k=1, k \neq j}^{P} \frac{1}{1 - \frac{\gamma a_k}{\gamma a_j}}\right) \frac{1}{1 - \frac{\beta \sum_{l>P} a_l}{\gamma a_j}} \\
&= \sum_{j=1}^{P} \frac{1}{a_j} \left(\prod_{k=1, k \neq j}^{P} \frac{1}{1 - \frac{a_k}{a_j}}\right) \frac{1}{\gamma - \frac{\beta \sum_{l>P} a_l}{a_j}} \\
&\geq \sum_{j=1}^{P} \frac{1}{a_j} \left(\prod_{k=1, k \neq j}^{P} \frac{1}{1 - \frac{a_k}{a_j}}\right) \prod_{l>P} \frac{1}{1 - \frac{a_l}{a_j}} \\
&= F(a)
\end{aligned}$$

The last inequality is due to the left-hand side of (4.1), i.e. Bernoulli's inequality, and the fact that $\beta, \gamma \in [0,1]$. More precisely, for all $j = 1 \ldots P$, we have

$$\gamma - \frac{\beta \sum_{l>P} a_l}{a_j} \leq 1 + \sum_{l>P} \frac{|a_l|}{a_j} \leq \prod_{l>P} \left(1 + \frac{|a_l|}{a_j}\right) = \prod_{l>P} \left(1 - \frac{a_l}{a_j}\right).$$

We do the same trick for $j = P+1, \ldots, n+1$. For a vector $a = (a_1, \ldots, a_P, 0, \ldots, 0, a_{n+1})$, with $a_j > 0$ for $j = 1, \ldots, P$ and $a_{n+1} < 0$ we define

$$\tilde{a} := \left(\gamma \sum_{j=1}^{P} a_j, 0, \ldots, 0, \beta a_{n+1}\right)$$

such that $\|\tilde{a}\| = 1$ and $\sum_{j=1}^{n+1} \tilde{a}_j = \sum_{j=1}^{n+1} a_j$. So we have the following conditions

$$\gamma \sum_{j\leq P} a_j + \beta a_{n+1} = \sum_{j=1}^{n+1} a_j, \tag{4.4}$$

$$\gamma^2 \left(\sum_{j\leq P} a_j\right)^2 + \beta^2 a_{n+1}^2 = 1. \tag{4.5}$$

Again we consider the solutions in the (γ, β)-plane. Equation (4.4) defines a line with $\beta_g(0) < 0$ and $\beta_g(1) = 1$. Note that for $\gamma_g := \frac{\sum_{j=1}^{n+1} a_j}{\sum_{j\leq P} a_j}$ we have $\beta_g(\gamma_g) = 0$.

The second condition (4.5) defines an ellipse with $\beta_e(0) > 0$. For $\gamma_e := \frac{1}{\sum_{j=1}^{P} a_j}$ we have $\beta_e(\gamma_e) = 0$. Since we assumed $K \leq 1$, we know that $\gamma_e = \frac{1}{\sum_{j\leq P} a_j} \geq \frac{\sum_{j=1}^{n+1} a_j}{\sum_{j\leq P} a_j} = \gamma_g$. Therefore there is an intersection of the two curves in the first quadrant, i.e. $\beta, \gamma \geq 0$.

If $\sum_{j\leq P} a_j \leq 1$ we compute $\beta_e(1) = \sqrt{\frac{1-(\sum_{j\leq P} a_j)^2}{a_{n+1}}}$. Then we have $\beta_e(1) \leq 1$, since

$$1 = \sum_{j\leq P} a_j^2 + a_{n+1}^2 \leq \left(\sum_{j=1}^{P} a_j\right)^2 + a_{n+1}^2$$

and therefore $\sqrt{\frac{1-(\sum_{j\leq P} a_j)^2}{a_{n+1}}} \leq 1$. If $\sum_{j\leq P} a_j > 1$ we have $\gamma_e \leq 1$ and $\beta_e(\gamma_e) = 0$. In both cases the intersection of the two curves satisfies $\beta, \gamma \leq 1$.

All together we found $\beta, \gamma \in [0,1]$ with (4.4) and (4.5). Now we compare $F(-a)$ and $F(-\tilde{a})$. Note that the function F now only has one summand. The estimates are the same as in the first step.

So F attains its maximum at a vector of the form $a = (a_1, 0, \ldots, 0, a_{P+1}, 0, \ldots, 0)$.

Using $\|a\| = 1$ and $\sum_{j=1}^{n+1} a_j = K$, we conclude that

$$a_1 = \frac{K}{2} + \sqrt{\frac{1}{2} - \frac{K^2}{4}}, \qquad a_{P+1} = \frac{K}{2} - \sqrt{\frac{1}{2} - \frac{K^2}{4}}$$

and $F(a) = \frac{1}{\sqrt{2-K^2}}$. □

A similar lemma can be obtained for an estimate in the converse direction. Instead of *concentrating* the coordinates we *balance* them. This estimate is weaker, since we can only balance the negative half of the coordinates of a, whereas we could concentrate the positive and the negative coordinates in the above lemma. Note that we do not need the restriction $K \leq 1$ in this lemma.

One might conjecture that the minimum of $F(a)$ without changing the signs of a_j is attained in a vector of the form $\tilde{a} = (\xi, \dots, \xi, \eta, \dots, \eta)$, for some $\xi \geq 0, \eta < 0$. But for a vector of this form the formula from Corollary 3.8 and the estimates from the previous lemma do not work anymore. The example in Lemma 4.6 shows that this conjecture is even false for higher dimensions.

Lemma 4.2. *Let* $0 \leq K$, $a \in \mathbb{R}^{n+1}$ *with* $\|a\| = 1$ *and* $\sum_{j=1}^{n+1} a_j = K$. *Then we have*

$$F(a) \geq F(\tilde{a})$$

where $\tilde{a} := \left(\gamma a_1, \dots, \gamma a_P, \beta \frac{\sum_{j=P+1}^{n+1} a_j}{N}, \dots, \beta \frac{\sum_{j=P+1}^{n+1} a_j}{N}\right)$ *with* $N := n+1-P$ *and* $\gamma, \beta \geq 0$ *such that* $\|\tilde{a}\| = 1$ *and* $\sum_{j=1}^{n+1} \tilde{a}_j = K$.

Proof. As in the last lemma we modify the vector a to obtain an estimate for F. This time we choose $\beta, \gamma \geq 1$ such that for

$$\tilde{a} = \left(\gamma a_1, \dots, \gamma a_P, \beta \frac{\sum_{j=P+1}^{n+1} a_j}{N}, \dots, \beta \frac{\sum_{j=P+1}^{N} a_j}{N}\right)$$

the equations

$$\gamma \sum_{j \leq P} a_j + \beta \sum_{j > P} a_j = \sum_{j=1}^{n+1} a_j, \tag{4.6}$$

$$\gamma^2 \sum_{j \leq P} a_j^2 + \beta^2 \frac{\left(\sum_{j>P} a_j\right)^2}{N} = 1 \tag{4.7}$$

hold. Equation (4.6) is the same as (4.2) from the previous proof with the same implications $\beta_g(0) \leq 0$ and $\beta_g(1) = 1$. Note that the slope of $\beta_g(\cdot)$ is larger than 1.

Equation (4.7) again defines an ellipse. With $\beta_e(\gamma) = \sqrt{N}\sqrt{\frac{1-\gamma^2\sum_{j\leq P} a_j^2}{\left(\sum_{j>P} a_j\right)^2}}$ we find that $\beta_e(1) = \sqrt{N}\sqrt{\frac{\sum_{j>p} a_j^2}{\left(\sum_{j>P} a_j\right)^2}} \geq 1$. Furthermore $\beta_e(\gamma) = 0$ for $\gamma = \frac{1}{\sqrt{\sum_{j\leq P} a_j^2}} \geq 1$.

By the intermediate value theorem, there are $\beta \geq 1$ and $\gamma \geq 1$ with (4.6) and (4.7). Since the slope of the line defined by (4.6) is greater than 1, also $\beta \geq \gamma$.

We compare $F(a)$ and $F(\tilde{a})$ and obtain

$$F(\tilde{a}) = \sum_{j=1}^{P} \frac{1}{\gamma a_j} \prod_{k=1,k\neq j}^{P} \frac{1}{1-\frac{a_k}{a_j}} \prod_{k=P+1}^{n+1} \frac{1}{1-\frac{\beta\sum_{l>P} a_l}{\gamma N a_j}}$$
$$\leq \sum_{j=1}^{P} \frac{1}{a_j} \prod_{k=1,k\neq j}^{P} \frac{1}{1-\frac{a_k}{a_j}} \left(\frac{1}{1-\frac{\beta\sum_{l>P} a_l}{\gamma N a_j}}\right)^N.$$

Now we use $\frac{\beta}{\gamma} \geq 1$ and the right inequality of (4.1), which is the AGM inequality:

$$\left(1-\frac{\beta\sum_{l>P} a_l}{\gamma N a_j}\right)^N = \left(1+\frac{\beta}{\gamma}\frac{\sum_{l>P} |a_l|}{N a_j}\right)^N \geq \prod_{l>P}\left(1+\frac{|a_l|}{a_j}\right) = \prod_{l>P}\left(1-\frac{a_l}{a_j}\right)$$

Finally, we find that

$$F(\tilde{a}) \leq \sum_{j=1}^{P} \frac{1}{a_j} \prod_{k=1,k\neq j}^{P} \frac{1}{1-\frac{a_k}{a_j}} \prod_{l=P+1}^{n+1} \frac{1}{1-\frac{a_l}{a_j}} = F(a).$$

□

4.2. Maximal sections close to the centroid

Here we consider hyperplane sections close to the centroid. Close means precisely that the distance between the hyperplane H_a and the centroid c is at most the distance from a face to the centroid. The distance of a face to the centroid is attained by a point of the form $\tilde{c} := \left(\frac{1}{n}, \ldots, \frac{1}{n}, 0\right)$. So the distance of a face to the centroid is equal to $\|c - \tilde{c}\| = \frac{1}{\sqrt{(n+1)n}}$. By (3.13) the condition $\operatorname{dist}(H_a \cap S, c) \leq \frac{1}{\sqrt{(n+1)n}}$ is equivalent to

$$\frac{\left|\sum_{j=1}^{n+1} a_j\right|}{\sqrt{(n+1)\left(n+1-(\sum_{j=1}^{n+1} a_j)^2\right)}} \leq \frac{1}{\sqrt{(n+1)n}},$$

and this is equivalent to $|\sum_{j=1}^{n+1} a_j| \leq 1$.

By Lemma 4.1 we immediately get the maximal section for a fixed distance close to the centroid. The maximal section always contains $n-1$ of the vertices of the simplex.

Theorem 4.3. *Let $0 \leq K \leq 1$. For all $a \in \mathbb{R}^{n+1}$ with $\|a\| = 1$ and $\sum_{j=1}^{n+1} a_j = K$ we have*

$$\mathrm{vol}_{n-1}(H_a \cap S) \leq \frac{\sqrt{n+1-K^2}}{(n-1)!} \frac{1}{\sqrt{2-K^2}}$$

with equality for $a = \left(\frac{K}{2} + \sqrt{\frac{1}{2} - \frac{K^2}{4}}, \frac{K}{2} - \sqrt{\frac{1}{2} - \frac{K^2}{4}}, 0, \ldots, 0\right)$.

Remarks: (i) For $K = 0$ we recover the result by S. Webb for the maximal central section. Lemma 4.1 is even simpler to prove if one assumes $K = 0$. In that case one does not have to introduce β and γ and the proof is basically just applying Bernoulli's inequality.

(ii) The bound in Theorem 4.3 is increasing in K. For $K = 1$ the maximal section is one of the faces and the corresponding normal vector a is equal to $(1, 0, \ldots, 0)$.

4.3. Locally small sections

From Lemma 4.2 we get the absolute minimum for $n = 2$. For general dimensions Lemma 4.2 only implies that $a_{\min}$ is a *local* minimum of the function $a \mapsto \mathrm{vol}_{n-1}(H_a \cap S)$. We have

Theorem 4.4. *The volume of the section $H_{a_{min}}$ is locally minimal, more precisely for all $a \in \mathbb{R}^{n+1}$ with $\|a\| = 1$, $\sum_{j=1}^{n+1} a_j = 0$ and $a_1 \geq 0 \geq a_2, \ldots, a_{n+1}$ we have*

$$\mathrm{vol}_{n-1}(H_a \cap S) \geq \mathrm{vol}_{n-1}(H_{a_{min}} \cap S).$$

4.4. Polytopal structure of $H \cap S$

We take a closer look to the geometric structure of $H_a \cap S$. For general a we cannot say too much about $H_a \cap S$. The section is some polytope without obvious regularity. But if we assume a to have the form $a = (a_1, \ldots, a_P, \beta, \ldots, \beta)$ for some $\beta < 0$, as in Lemma 4.2, then we have some regularity and we get an additional volume formula. At least for small dimensions this leads to a solution of the minimal section problem.

Let $a = (a_1, \ldots, a_P, a_{P+1}, \ldots, a_{n+1})$ with $a_1, \ldots, a_P > 0$ and $a_{P+1}, \ldots, a_{n+1} < 0$. P is the number of positive coordinates, $N := n+1-P$ is the number of negative coordinates. The points v_{ij} with $i \in \{1, \ldots, P\}$ and $j \in \{P+1, \ldots, n+1\}$ are the intersection points of $H_a \cap S$ with the edges $[e_i, e_j]$ of the simplex S. They are of the form

$$v_{ij} = \Big(0, \dots, 0, \underbrace{\frac{-a_j}{a_i - a_j}}_{i\text{-th coordinate}}, 0, \dots, 0, \underbrace{\frac{a_i}{a_i - a_j}}_{j\text{-th coordinate}}, 0, \dots, 0\Big).$$

The section polytope $H_a \cap S$ is the convex hull of these $P \cdot N$ points v_{ij}. Let us further decompose the section polytope $H_a \cap S$. It is the convex hull of polytopes of the form

$$K_i := \operatorname{conv}\{v_{ij}, j = P+1, \dots, n+1\}$$

for $i = 1, \dots, P$.

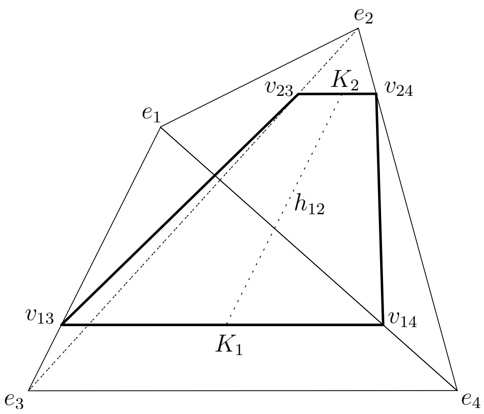

Figure 4.3.: The case $n = 3$, $P = 2$

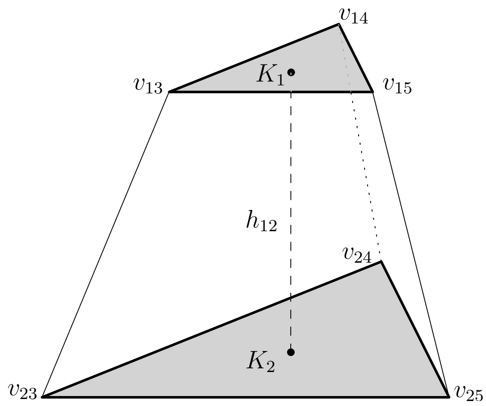

Figure 4.4.: $H_a \cap S$ for the case $n = 4$, $P = 2$

Let $a_{P+1} = \cdots = a_{n+1} = \beta$ and $\beta < 0$, see Figures 4.3 and 4.4. In this case the K_i are regular simplices with N vertices and side length $l_i := \sqrt{2}\frac{a_i}{a_i - \beta}$, since for fixed i and for all $j, \tilde{j} \in \{P+1, \dots, n+1\}$:

$$\left\| v_{ij} - v_{i\tilde{j}} \right\| = l_i.$$

The $(N-1)$-volume of these simplices is

$$\operatorname{vol}_{N-1}(K_i) = \frac{\sqrt{N}}{(N-1)!} \left(\frac{a_i}{a_i - \beta} \right)^{N-1}.$$

Two simplices $K_i, K_{\tilde{i}}$ lie in parallel subspaces, since their edges $[v_{ij}, v_{i\hat{j}}]$ and $[v_{\tilde{i}j}, v_{\tilde{i}\hat{j}}]$ are parallel. The distance of K_i and $K_{\tilde{i}}$ is given by the distance of their centroids:

$$\begin{aligned} h_{i,\tilde{i}} &= \left\| \frac{1}{N} \sum_{j=P+1}^{n+1} v_{ij} - \frac{1}{N} \sum_{j=P+1}^{n+1} v_{\tilde{i}j} \right\| = \left\| \frac{1}{N} \sum_{j=P+1}^{n+1} v_{ij} - v_{\tilde{i}j} \right\| \\ &= \left\| \left(0, \frac{-\beta}{a_i - \beta}, 0, \frac{\beta}{a_{\tilde{i}} - \beta}, 0, \ldots, 0, \frac{1}{N} \left(\frac{a_i}{a_i - \beta} - \frac{a_{\tilde{i}}}{a_{\tilde{i}} - \beta} \right), \ldots, \frac{1}{N} \left(\frac{a_i}{a_i - \beta} - \frac{a_{\tilde{i}}}{a_{\tilde{i}} - \beta} \right) \right) \right\| \\ &= \sqrt{\frac{\beta^2}{(a_i - \beta)^2} + \frac{\beta^2}{(a_{\tilde{i}} - \beta)^2} + \frac{1}{N} \left(\frac{a_i}{a_i - \beta} - \frac{a_{\tilde{i}}}{a_{\tilde{i}} - \beta} \right)^2}. \end{aligned}$$

Note that l_i and $h_{i\tilde{i}}$ are independent from the scaling of a. The value β is determined by the constraints for the normal vector, up to scaling. We have $\beta = -\frac{1}{N} \sum_{j=1}^{P} a_j$. Two simplices K_i and $K_{\tilde{i}}$ constitute a truncated pyramid, if $n = 4$, resp. a frustum with a regular $(N-1)$-simplex as its base and another regular $(N-1)$-simplex as its top.

Let $P = 2$. The section $H_a \cap S$ is the convex hull of two parallel $(N-1)$-simplices, i.e. a frustum. Let V_1, V_2 be the $(N-1)$-volume of the top resp. bottom and h_1, h_2 the relative heights. By homogeneity we know $\frac{h_1}{h_2} = \frac{V_1^{\frac{1}{N-1}}}{V_2^{\frac{1}{N-1}}}$. Set $\lambda := \frac{h_1}{V_1^{\frac{1}{N-1}}}$ (then $\lambda = \frac{h_2}{V_2^{\frac{1}{N-1}}}$). The volume V of $H_a \cap S$ is computed as

$$\begin{aligned} V &= \frac{1}{N} \left(h_1 V_1 - h_2 V_2 \right) \\ &= \frac{\lambda}{N} \left(V_1^{1+\frac{1}{N-1}} - V_2^{1+\frac{1}{N-1}} \right) \\ &= \frac{\lambda}{N} \left(V_1^{\frac{N}{N-1}} - V_2^{\frac{N}{N-1}} \right). \end{aligned}$$

Note that for $x, y > 0$ and $N \in \mathbb{N}$: $x^N - y^N = (x - y) \left(\sum_{m=0}^{N-1} x^{N-1-m} y^m \right)$. Therefore

$$\begin{aligned} V &= \frac{\lambda \left(V_1^{\frac{1}{N-1}} - V_2^{\frac{1}{N-1}} \right)}{N} \sum_{m=0}^{N-1} V_1^{\frac{N-1-m}{N-1}} V_2^{\frac{m}{N-1}} \\ &= \frac{h_1 - h_2}{N} \sum_{m=0}^{N-1} V_1^{\frac{N-1-m}{N-1}} V_2^{\frac{m}{N-1}}. \end{aligned}$$

The volume of the section can be expressed by

$$\mathrm{vol}_{n-1}(H_a \cap S) = \frac{1}{N}\; h \sum_{m=0}^{n-2} \mathrm{vol}_{n-2}(K_1)^{\frac{n-2-m}{n-2}}\; \mathrm{vol}_{n-2}(K_2)^{\frac{m}{n-2}},$$

with $h := h_{1,2}$.

Under the assumption $P = 2$, the normal vector a only depends on one variable. Since the formulas are independent of scaling the vector a, it is sufficient to consider a of the form

$$a = a(x) := \left(x, 1-x, -\frac{1}{N}, \dots, -\frac{1}{N}\right)$$

with $x \in (0,1)$.

So the volume is given by

$$\begin{aligned} V(x) &:= \mathrm{vol}_{n-1}(H_{a(x)} \cap S) \\ &= \frac{\sqrt{N}}{N!} \sqrt{\frac{1}{(Nx+1)^2} + \frac{1}{(N(1-x)+1)^2} + N\left(\frac{x}{Nx+1} - \frac{1-x}{N(1-x)+1}\right)^2} \\ &\quad \cdot \sum_{m=0}^{N-1} \left(\frac{Nx}{Nx+1}\right)^{N-1-m} \left(\frac{N(1-x)}{N(1-x)+1}\right)^{m}. \end{aligned} \tag{4.8}$$

For $x = 0$ and $x = 1$ the geometric arguments do not work, since the involved simplices become degenerated. However, this function is still well defined. Note that for $x = 0$ and $x = 1$ the vector $a(x)$ corresponds to the vector $(a_{\min}, 0) \in \mathbb{R}^{n+1}$ with $a_{\min} \in \mathbb{R}^n$. The values of $V(0)$ and $V(1)$ equal the volume of the section for such $a(x)$, computed at the beginning of Chapter 4. Therefore the formula may be extended to $[0,1]$.

Summarizing we proved

Lemma 4.5. *Let $a \in \mathbb{R}^{n+1}$, $\|a\| = 1$ and $\sum_{j=1}^{n+1} a_j = 0$. Additionally $a = (a_1, a_2, \beta, \dots, \beta)$ for some $\beta < 0$ and $a_1, a_2 \geq 0$. Then with the definition (4.8):*

$$\mathrm{vol}_{n-1}\left(H_a \cap S\right) = V\left(\frac{a_1}{a_1 + a_2}\right).$$

For normal vectors of this special form we now determine the minimal section volume.

Lemma 4.6. *Let V be as defined above. Then for $N = 2, 3, 4$ and $x \in [0,1]$*

$$V(x) \geq V\left(\frac{1}{2}\right).$$

But for $N = 5$ we have $V(\frac{1}{2}) > V(0)$.

Proof. The function V is differentiable. So one finds the extrema by finding the zeros of its derivative. The calculations are elementary and we just give the results.

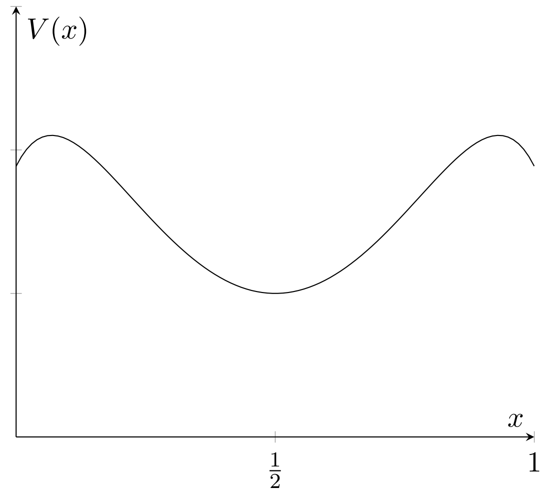

Figure 4.5.: $V(x)$ for $N = 2$

Note that it is suffient to consider $x \in \left[0, \frac{1}{2}\right]$ due to the symmetry of the function. For $N = 2, 3, 4$ the derivative of V has a zero in $\frac{1}{2}$ and one more zero in $\left(0, \frac{1}{2}\right)$. The other zeros are complex or not in $\left(0, \frac{1}{2}\right)$. We list them with their function value together with the value of V in 0 in Table 4.6.

$N = 2$		$N = 3$		$N = 4$	
x	$V(x)$	x	$V(x)$	x	$V(x)$
0	$\frac{2\sqrt{6}}{9}$	0	$\frac{3}{64}\sqrt{15}$	0	$\frac{16}{1875}\sqrt{30}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{9\sqrt{6}}{125}$	$\frac{1}{2}$	$\frac{8}{243}\sqrt{2}$
$\frac{1}{2}(1 - \sqrt{\sqrt{33} - 5})$	~ 0.555	$\frac{1}{2} - \frac{1}{6}\sqrt{5}$	~ 0.1886	~ 0.16177	~ 0.04899

Figure 4.6.: Extremal values of V

For $N = 5$ we obtain a different behavior. V still has a minimum in $\frac{1}{2}$, but this is not the global minimum anymore:

$$V(0) = \frac{125}{186624}\sqrt{5}\sqrt{42} < \frac{625}{201684}\sqrt{10} = V\left(\frac{1}{2}\right)$$

□

4.5. Minimal sections for small dimension ($n \leq 4$)

Now we prove that $a_{\min}$ indeed is the global minimum for dimensions $n = 2, 3, 4$.

Theorem 4.7. *Let $n \in \{2,3,4\}$ and $a \in \mathbb{R}^{n+1}$ with $\|a\| = 1$ and $\sum_{j=1}^{n+1} a_j = 0$. Then*

$$\mathrm{vol}_{n-1}\left(H_a \cap S\right) \geq \mathrm{vol}_{n-1}\left(H_{a_{min}} \cap S\right) = \frac{\sqrt{n+1}}{(n-1)!}\left(\frac{n}{n+1}\right)^{n-\frac{1}{2}}.$$

Proof. First recall that multiplying the normal vector by -1 or permuting the coordinates does not change the volume of the section. If $a = (a_1, \ldots, a_{n+1})$ with $a_1 > 0$ and $a_2, \ldots, a_{n+1} < 0$, by Theorem 4.3 we have $\mathrm{vol}_{n-1}(H_a \cap S) \geq \mathrm{vol}_{n-1}(H_{a_{\min}} \cap S)$.

For $n = 2$ this is already sufficient.

Let $n = 3$ and $a_1, a_2 > 0$ and $a_3, a_4 < 0$. Then by Lemma 4.6 with $N = 2$ we know

$$\mathrm{vol}_{n-1}\left(H_a \cap S\right) \geq \mathrm{vol}_{n-1}\left(H_{\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)} \cap S\right) = \frac{1}{2} > \mathrm{vol}_{n-1}\left(H_{a_{\min}} \cap S\right).$$

This solves the case $n = 3$.

Finally let $n = 4$. For $a_1, a_2 > 0$ and $a_3, a_4, a_5 < 0$ Lemma 4.6 with $N = 3$ yields

$$\mathrm{vol}_{n-1}\left(H_a \cap S\right) \geq \mathrm{vol}_{n-1}\left(H_{a'} \cap S\right) = \frac{9\sqrt{6}}{125} > \mathrm{vol}_{n-1}\left(H_{a_{\min}} \cap S\right)$$

with $a' = \sqrt{\frac{6}{5}}\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$ □

Remark: For ℓ_p-balls with $0 < p \leq 2$ the hyperplane section volume is given by

$$\mathrm{vol}_{n-1}\left(B_p^n \cap H_a\right) = C_p \int_0^\infty \prod_{j=1}^n \gamma_p(ta_j)\mathrm{d}t,$$

where C_p is some constant depending on p and n and γ_p is the Fourier transform of the function $\exp(-|\cdot|^p)$ on $\mathbb{R}$, see [Kol05, Thm. 7.7]. Koldobsky used the fact that

$\ln(\gamma_p(\sqrt{\cdot}))$ is convex on $\mathbb{R}_{\geq 0}$ to find the minimum and the maximum. The maximum is attained if one coordinate is 1 and the rest is zero. The minimum is attained if all coordinates are equal.

This technique cannot be used for the simplex. The analogue to $t \mapsto \gamma_p(t)$ is the function $t \mapsto \frac{1}{1+\mathrm{i}t}$. This function is complex-valued. Furthermore the example in Lemma 4.6 shows that for the simplex the *balanced* vector cannot be minimal in general.

5. Average Volume of Hyperplane Sections

In this chapter we ask for the average volume of central hyperplane sections. Using results from U. Brehm, P. Hinow, H. Vogt and J. Voigt we compute this average volume, cf. [Voi00], [BV00], [BHVV02].

The probabilistic setting is the following: Consider a to be a randomly choosen vector from $\{a \in \mathbb{R}^{n+1} \mid \|a\| = 1,\ \sum_{j=1}^{n+1} a_j = 0\}$ with uniform probability and compute the expected value $\mathbb{E}\left[\mathrm{vol}_{n-1}(H_a \cap S)\right]$. We find the following result.

Theorem 5.1. *For all $n \geq 3$ we have*

$$\mathbb{E}\left[\mathrm{vol}_{n-1}(H_a \cap S)\right] \geq \frac{\sqrt{n+1}}{(n-1)!}\frac{1}{\sqrt{2\pi}}. \tag{5.1}$$

Furthermore for $n \to \infty$

$$\frac{\mathbb{E}\left[\mathrm{vol}_{n-1}(H_a \cap S)\right]}{\frac{\sqrt{n+1}}{(n-1)!}\frac{1}{\sqrt{2\pi}}} \to 1. \tag{5.2}$$

The main Lemma 5.2 requires that the simplex is isotropic. A convex body $K \subset \mathbb{R}^n$ is called *isotropic* if it has volume 1, its centroid in the origin and there exists a constant $L_K > 0$ such that

$$\int_K \langle x, \theta \rangle^2 \,\mathrm{d}x = L_K^2$$

for every $\theta \in \mathbb{R}^n$ with $\|\theta\| = 1$. L_K is called the isotropic constant of K. For every convex body $K \subset \mathbb{R}^n$ there exists a linear transformation T such that $T(K)$ is isotropic. Furthermore the isotropic constant can also be computed by

$$\frac{1}{n}\int_K \|x\|^2 \,\mathrm{d}x = L_K^2. \tag{5.3}$$

These results and a detailed description is found in [Gia03, Chapter 1].

Since we apply results from the papers mentioned above, it is convenient to work with a different representation of the n-simplex. So let $S^n \subset \mathbb{R}^n$ be the regular n-simplex with n-dimensional volume 1 and with its centroid in the origin. If we scale the embedded n-simplex S by a factor $\left(\frac{n!}{\sqrt{n+1}}\right)^{\frac{1}{n}}$ we get a regular simplex with volume 1. The normalized simplex is known to be isotropic. The isotropic constant is given by (see [Voi00, p. 240]):

$$L_{S^n} = \frac{(n!)^{\frac{1}{n}}}{\sqrt{(n+1)(n+2)}(n+1)^{\frac{1}{2n}}} < \frac{1}{e}.$$

In the same paper we find that for all simplices S^n the inequality

$$1 \leq \int_{S^n} \|x\|^2 \,\mathrm{d}x \left(\int_{S^n} \frac{1}{\|x\|} \mathrm{d}x \right)^2 \tag{5.4}$$

holds true and for $n \to \infty$

$$\int_{S^n} \|x\|^2 \,\mathrm{d}x \left(\int_{S^n} \frac{1}{\|x\|} \mathrm{d}x \right)^2 \to 1. \tag{5.5}$$

More precisely this is implied by [Voi00, Prop. 1.1] and [Voi00, Section 2].

The following lemma on average section volume of any convex body K in isotropic position holds, as a special case of [BHVV02, Lemma 1.2].

Lemma 5.2. *For a convex body $K \subset \mathbb{R}^n$ in isotropic position we have*

$$\mathbb{E}\left[\mathrm{vol}_{n-1}(K \cap H)\right] = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \int_K \frac{1}{\|x\|} \mathrm{d}x.$$

Using this lemma and (5.3) and (5.4) we obtain

$$\begin{aligned} \mathbb{E}\left[\mathrm{vol}_{n-1}(H_a \cap S^n)\right] &= \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(\frac{n-1}{2})} \int_{S^n} \frac{1}{\|x\|} \mathrm{d}x \\ &\geq \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(\frac{n-1}{2})} \frac{1}{\left(\int_{S^n} \|x\|^2 \mathrm{d}x\right)^{\frac{1}{2}}} \\ &= \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(\frac{n-1}{2})} \frac{1}{\sqrt{n}} \frac{\sqrt{(n+1)(n+2)}(n+1)^{\frac{1}{2n}}}{(n!)^{\frac{1}{n}}}. \end{aligned}$$

The last term tends to $\frac{e}{\sqrt{2\pi}}$ for $n \to \infty$. We scale the normalized simplex S^n by a factor $\left(\frac{\sqrt{n+1}}{n!}\right)^{\frac{1}{n}}$ to obtain the results for the simplex S with side length $\sqrt{2}$. So

$$\mathbb{E}\left[\mathrm{vol}_{n-1}(H_a \cap S)\right] = \left(\frac{\sqrt{n+1}}{n!}\right)^{\frac{n-1}{n}} \mathbb{E}\left[\mathrm{vol}_{n-1}(H_a \cap S^n)\right]$$

$$
\begin{aligned}
&\geq \left(\frac{\sqrt{n+1}}{n!}\right)^{\frac{n-1}{n}} \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(\frac{n-1}{2})} \frac{1}{\sqrt{n}} \frac{\sqrt{(n+1)(n+2)}(n+1)^{\frac{1}{2n}}}{(n!)^{\frac{1}{n}}} \\
&= \frac{\sqrt{n+1}}{(n-1)!} \frac{\sqrt{(n+1)(n+2)}}{n\sqrt{n}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(\frac{n-1}{2})} \\
&\geq \frac{\sqrt{n+1}}{(n-1)!} \frac{1}{\sqrt{2\pi}}.
\end{aligned}
$$

This proves the first part of Theorem 5.1.

The application of (5.5) shows the second part of Theorem 5.1. □

6. Bounds for k-dimensional Sections

The maximal central hyperplane section of the simplex is the one that contains $n-1$ vertices and the midpoint of the opposite edge. The natural generalization of this section to lower dimensional sections is

$$H \cap S = \operatorname{conv}\left\{ e_1, \ldots, e_{k-1}, \frac{1}{n+2-k} \sum_{j=k}^{n+1} e_j \right\},$$

where $H \subset \mathbb{R}^{n+1}$ is a suitable k-dimensional subspace. This polytope contains $k-1$ vertices and the centroid of the opposite face. It has $(k-1)$-volume $\frac{\sqrt{n+1}}{(k-1)!} \frac{1}{\sqrt{n+2-k}}$, which is computed by the elementary volume formula for cones.

With an extra condition we prove that this is indeed maximal. Without additional conditions we prove a bound that differs from this conjectured maximum by some factor depending on the dimension resp. codimension of the intersecting subspace. The main tool for the proof is the Brascamp-Lieb inequality.

Our result is

Theorem 6.1. *Let H be a k-dimensional subspace of $\mathbb{R}^{n+1}$ that contains the centroid of S. Then we have*

$$\operatorname{vol}_{k-1}(H \cap S) \leq \frac{\sqrt{n+1}}{(k-1)!} \frac{\left(\sqrt{k}\right)^{\frac{k}{n+1}}}{\sqrt{n+1}}. \tag{6.1}$$

If additionally $\operatorname{dist}(H, e_j)^2 \leq \frac{n+1-k}{n+2-k}$ *for all $j = 1, \ldots, n+1$, then*

$$\operatorname{vol}_{k-1}(H \cap S) \leq \frac{\sqrt{n+1}}{(k-1)!} \frac{1}{\sqrt{n+2-k}} \tag{6.2}$$

holds. In this case the estimate is sharp.

Remark on the accuracy of the bounds:
Fix $k \in \mathbb{N}$. Then the quotient of the proven bound (6.1) and the conjectured optimal bound (6.2) tends to 1 for $n \to \infty$:

$$\frac{\frac{\sqrt{n+1}}{(k-1)!} \frac{(\sqrt{k})^{\frac{k}{n+1}}}{\sqrt{n+1}}}{\frac{\sqrt{n+1}}{(k-1)!} \frac{1}{\sqrt{n+2-k}}} = \sqrt{\frac{n+2-k}{n+1}} \sqrt{k}^{\frac{k}{n+1}} \longrightarrow 1 \tag{6.3}$$

So this is asymptotically optimal.

Now we fix the codimension of the section, i.e. fix $d \in \mathbb{N}$ and let $k = n - d$. Then for $n \to \infty$

$$\frac{\frac{\sqrt{n+1}}{(k-1)!} \frac{(\sqrt{n-d})^{\frac{n-d}{n+1}}}{\sqrt{n+1}}}{\frac{\sqrt{n+1}}{(k-1)!} \frac{1}{\sqrt{2+d}}} = \sqrt{2+d} \frac{\sqrt{n-d}^{\frac{n-d}{n+1}}}{\sqrt{n+1}} \longrightarrow \sqrt{2+d}. \tag{6.4}$$

We state the Brascamp-Lieb inequality in a normalized form. This is found in [Bal89].

Lemma 6.2 (Brascamp-Lieb). *Let $u_1, \ldots, u_{n+1}$ be unit vectors in $H \subset \mathbb{R}^{n+1}$, H a k-dimensional subspace and $d_1, \ldots, d_{n+1} > 0$ satisfying $\sum_{j=1}^{n+1} d_j \langle \cdot , u_j \rangle u_j = \mathrm{Id}_H$. Then for integrable functions $f_1, \ldots, f_{n+1} : \mathbb{R} \to [0, \infty)$ we have*

$$\int_H \prod_{j=1}^{n+1} f_j \left(\langle u_j, x \rangle\right)^{d_j} \mathrm{d}x \le \prod_{j=1}^{n+1} \left(\int_{\mathbb{R}} f_j(t) \mathrm{d}t \right)^{d_j}.$$

Proof of Theorem 6.1. Let P be the orthogonal projection onto H, $d_j := \|Pe_j\|$ and $u_j := \frac{Pe_j}{d_j}$. Then $\sum_{j=1}^{n+1} d_j^2 \langle \cdot , u_j \rangle u_j = Id_H$. Furthermore $\sum_{j=1}^{n+1} d_j^2$ is just the trace resp. the rank of the projection P, so $\sum_{j=1}^{n+1} d_j^2 = k$. With the operator norm of the projection we observe

$$\|Pe_j\| \le \|P\|_{\mathrm{op}} \|e_j\| \le 1.$$

And if we require H to contain the centroid c we have by Pythagoras

$$\|Pe_j\|^2 = \|c + P(e_j - c)\|^2 = \|c\|^2 + \|P(e_j - c)\|^2 \ge \frac{1}{n+1}.$$

Note that $\langle c, P(e_j - c) \rangle = \langle c, e_j - c \rangle = \langle c, e_j \rangle - \langle c, c \rangle = \frac{1}{n+1} - \frac{1}{n+1} = 0$.

So we get the conditions

$$\sum_{j=1}^{n+1} d_j^2 = k, \quad d_j \le 1, \quad d_j \ge \frac{1}{\sqrt{n+1}} \text{ for } j = 1, \ldots, n+1. \tag{6.5}$$

Now we consider the additional assumption of our theorem. Let $\mathrm{dist}(H, e_j)^2 \le \frac{n+1-k}{n+2-k}$ for all $j = 1, \ldots, n+1$. For each $x \in \mathbb{R}^{n+1}$ we have

$$\mathrm{dist}(H, x) = \|Px - x\| = \left\|AA^t x\right\|,$$

where A is the matrix whose columns are the vectors $a^1, \ldots, a^{n+1-k}$.

If $\left\|AA^t e_j\right\| \leq \sqrt{\frac{n+1-k}{n+2-k}}$ then $\|Pe_j\| = \left\|1 - AA^t e_j\right\| \geq \sqrt{\frac{1}{n+2-k}}$. Therefore

$$d_j \geq \frac{1}{\sqrt{n+2-k}} \quad \text{for } j = 1, \ldots, n+1. \tag{6.6}$$

Now we apply Brascamp-Lieb. For $x \in H$ we have

$$x_j = \langle x, e_j \rangle = \langle Px, e_j \rangle = \langle x, Pe_j \rangle = \langle x, d_j u_j \rangle .$$

Therefore

$$\begin{aligned}
\int_{H, x_j \geq 0} \prod_{j=1}^{n+1} \exp\left(-x_j\right) \mathrm{d}x &= \int_{H, x_j \geq 0} \prod_{j=1}^{n+1} \exp\left(- \langle x, d_j u_j \rangle\right) \mathrm{d}x \\
&= \int_{H, x_j \geq 0} \prod_{j=1}^{n+1} \left(\exp\left(-\frac{1}{d_j} \langle x, u_j \rangle\right)\right)^{d_j^2} \mathrm{d}x \\
&\leq \prod_{j=1}^{n+1} \left(\int_0^\infty \exp\left(-\frac{1}{d_j} s\right) \mathrm{d}s\right)^{d_j^2} \\
&= \prod_{j=1}^{n+1} d_j^{d_j^2}.
\end{aligned}$$

We maximize $\prod_{j=1}^{n+1} d_j^{d_j^2}$ under the constraints given in (6.5). We consider the equivalent problem to maximize $F(x) := \sum_{j=1}^{n+1} x_j \ln x_j$ under the constraints

$$\sum_{j=1}^{n+1} x_j = k, \;\; \frac{1}{n+1} \leq x_j \leq 1.$$

The function $x \mapsto x \ln x$ is convex on $(0,1)$. Therefore also $\sum_{j=1}^{n+1} x_j \ln x_j$ is convex on $(0,1)^{n+1}$. The set

$$\left\{ x \in [0,1]^{n+1} \;\middle|\; \sum_{j=1}^{n+1} x_j = k, \;\; \frac{1}{n+1} \leq x_j \leq 1 \right\}$$

is also convex, so the maximum is attained in some extremal point of the set. The extremal points are permutations of the point

$$x = \Big(\underbrace{1, \quad \ldots \quad, 1}_{k-1}, \underbrace{\frac{1}{n+1}, \ldots, \frac{1}{n+1}}_{n+1-k}, \frac{k}{n+1}\Big).$$

For d_j this means

$$(d_1, \ldots, d_{n+1}) = \Big(\underbrace{1, \quad \ldots \quad, 1}_{k-1}, \underbrace{\frac{1}{\sqrt{n+1}}, \ldots, \frac{1}{\sqrt{n+1}}}_{n+1-k}, \sqrt{\frac{k}{n+1}}\Big).$$

Therefore the maximum is $\prod_{j=1}^{n+1} d_j^{d_j^2} = \frac{\sqrt{k}^{\frac{k}{n+1}}}{\sqrt{n+1}}$.

With the additional condition (6.6) we find that

$$(d_1, \ldots, d_{n+1}) = \Big(\underbrace{1, \quad \ldots \quad, 1}_{k-1}, \underbrace{\frac{1}{\sqrt{n+2-k}}, \ldots, \frac{1}{\sqrt{n+2-k}}}_{n+2-k}\Big)$$

is maximal and $\prod_{j=1}^{n+1} d_j^{d_j^2} = \frac{1}{\sqrt{n+2-k}}$. □

Remark on the optimality: Brascamp-Lieb is a sharp inequality and its application gives sharp estimates. For example in the consideration of sections of ℓ_p-balls one gets sharp results [Bar01], [Bal89].

The reason why we do not get a sharp bound here is the additional restriction $c \in H$. If we do not assume the subspace H to contain the centroid or even fulfill some extra condition, there is no lower bound on d_j as in (6.5) or (6.6). Then the integral is simply bounded by 1. This bound is attained by $H = \operatorname{span}(e_1, \ldots, e_k)$. The corresponding section of S is a $(k-1)$-face which has distance $\frac{1}{\sqrt{k}}$ from the origin.

7. Irregular Simplices

Here we give a formula for sections of arbitrary simplices. As an application of the formula we construct a simplex such that all its faces are smaller than some central section. This is only possible in dimensions larger than 5.

7.1. Volume formula

Every simplex is an affine image of the regular simplex. The idea for the formula is to use the transformation theorem. One has to pay attention to use the transformation theorem in the appropriate subspace, since $H \cap S$ is $(n-1)$-dimensional. Here we denote the regular simplex by S_{reg} and a general simplex by S.

Theorem 7.1. *Let $S = \text{conv}\{v^l \mid l = 1, \dots, n+1\}$ be an arbitrary simplex in $\mathbb{R}^{n+1}$. Without loss of generality we may assume that $\sum_{j=1}^{n+1} v_j^l = 1$ for all $l = 1, \dots, n+1$, i.e. all vertices lie in the affine hyperplane defined by the regular simplex. Let T be the linear transformation that maps S to the regular simplex S_{reg}. Let $a \in \mathbb{R}^{n+1}$ with $\|a\| = 1$. Then we have*

$$\text{vol}_{n-1}(H_a \cap S) = \frac{\det\left(v^1 \dots v^{n+1}\right)}{\|T^{-1*}a\|} \frac{\sqrt{n+1-\left(\sum_{j=1}^{n+1} a_j\right)^2}}{\sqrt{n+1-\left(\sum_{j=1}^{n+1} \tilde{a}_j\right)^2}} \text{vol}_{n-1}(H_{\tilde{a}} \cap S_{reg}) \quad (7.1)$$

where $\tilde{a} := \frac{T^{-1*}a}{\|T^{-1*}a\|}$.

Proof. The idea is the following. For $(n+1)$-dimensional subsets we have the transformation theorem in $\mathbb{R}^{n+1}$. The set $H \cap S$ is $(n-1)$-dimensional. We enlarge this set in two directions. We thicken $H \cap S$ in direction a and we take the convex hull of this enlarged set and $\{0\}$. Then we consider the image of this $(n+1)$-dimensional set and use the transformation theorem.

The transformation T maps S to the regular simplex bijectively. Therefore T^{-1} is given by

$$T^{-1} = (v^1 \dots v^{n+1}),$$

where the v^j are column vectors. We analyze how the $(n-1)$-volume behaves under T.

(i) The hyperplane H_a is mapped to the hyperplane $H_{\frac{\tilde{a}}{\|\tilde{a}\|}}$, where $\tilde{a} = \frac{T^{-1*}a}{\|T^{-1*}a\|}$:

$$\begin{aligned} T(H_a) &= \left\{ y \mid \left\langle T^{-1}y, a \right\rangle = 0 \right\} \\ &= \left\{ y \mid \left\langle y, T^{-1*}a \right\rangle = 0 \right\} \\ &= \left\{ y \;\middle|\; \left\langle y, \frac{T^{-1*}a}{\|T^{-1*}a\|} \right\rangle = 0 \right\} = H_{\tilde{a}} \end{aligned}$$

(ii) We enlarge $H_a \cap S$ orthogonally in the hyperplane defined by S. For $\epsilon > 0$ define

$$(H_a \cap S)(\epsilon) = H_a \cap S + \left\{ \tau \cdot a \;\middle|\; |\tau| \leq \frac{1}{2}\epsilon \right\}.$$

The n-volume of $(H_a \cap S)(\epsilon)$ is given by $\epsilon \|a\| \operatorname{vol}_{n-1}(H_a \cap S)$. For the image we get

$$T((H_a \cap S)(\epsilon)) = T(H_a \cap S) + \left\{ \tau \cdot Ta \;\middle|\; |\tau| \leq \frac{1}{2}\epsilon \right\},$$

which is a shear of the set $T(H_a \cap S) + \{\tau \cdot \tilde{a} \mid |\tau| \leq \delta\}$ for a suitable δ. We compute δ as the *height* of $T(H_a \cap S) + \{\tau \cdot T(a) \mid |\tau| \leq \epsilon\}$:

$$\delta = \langle \tilde{a}, \epsilon Ta \rangle = \frac{\epsilon}{\|T^{-1*}a\|} \left\langle T^{-1*}a, Ta \right\rangle = \frac{\epsilon}{\|\tilde{a}\|}.$$

So we have $\operatorname{vol}_n \left(T((H_a \cap S)(\epsilon)) \right) = \operatorname{vol}_{n-1}(H_{\tilde{a}} \cap S_{\text{reg}}) \frac{\epsilon}{\|T^{-1*}a\|}$.

(iii) Now consider the cone $P(\epsilon) := \operatorname{conv}(0, (H_a \cap S)(\epsilon))$ and its image $T(P(\epsilon))$. Then

$$\begin{aligned} \operatorname{vol}_{n+1}(P(\epsilon)) &= \frac{1}{n+1} \operatorname{vol}_n((H_a \cap S)(\epsilon)) \operatorname{dist}(0, ((H_a \cap S)(\epsilon))), \\ \operatorname{vol}_{n+1}(T(P(\epsilon))) &= \frac{1}{n+1} \operatorname{vol}_n(T((H_a \cap S)(\epsilon))) \operatorname{dist}(0, T((H_a) \cap S(\epsilon))). \end{aligned}$$

The set $P(\epsilon)$ is a $(n+1)$-dimensional set, so we may also use the transformation theorem to get $\operatorname{vol}_{n+1}(T(P(\epsilon))) = \det(T) \operatorname{vol}_{n+1}(P(\epsilon))$. So we find

$$\operatorname{vol}_n((H_a \cap S)(\epsilon)) = \frac{1}{\det T} \frac{\operatorname{dist}(0, T((H_a \cap S)(\epsilon))}{\operatorname{dist}(0, H_a \cap S(\epsilon))} \operatorname{vol}_{n-1}(H_{\tilde{a}} \cap S_{\text{reg}}) \cdot \frac{\epsilon}{\|T^{-1*}a\|}.$$

(iv) If we consider the limit for $\epsilon \to 0$, then $\operatorname{dist}(0, H_a \cap S(\epsilon)) \to \operatorname{dist}(0, H_a \cap S)$, and similarly $\operatorname{dist}(0, T((H_a \cap S)(\epsilon)) \to \operatorname{dist}(0, H_{\tilde{a}} \cap S_{\text{reg}})$. Now we use the formula from Lemma 3.2. It is still valid for irregular simplices S, since the proof only used that $x \in S$ implies $\sum_{j=1}^{n+1} x_j = 1$.

Finally for $\epsilon \to 0$
$$\frac{\mathrm{vol}_n((H_a \cap S)(\epsilon))}{\epsilon} \to \mathrm{vol}_{n-1}(H_a \cap S).$$
□

7.2. A simplex with a large cross section

For the regular simplex the maximal hyperplane section is a face, if the distance to the centroid is not prescribed. This can be proven by a theorem of Fradelizi about isotropic convex bodies. He proves that the maximal section of a cone in isotropic position is its base [Fra99, Corollay 3 (2.), p. 169]. This is still true for deformed and therefore non-isotropic simplices in dimensions $2, 3$ and 4. The case $n = 2$ is obvious. The case $n = 3$ was considered by [EEE63]. The first proof that this phenomenon does not generalize to all dimensions appears in [Wal68]. The full solution, i.e. $n = 4$ affirmative and $n \geq 5$ negative, was given by [Phi72]. He uses Walkup's construction for $n = 5$ and concluded by induction.

With our formula from Theorem 7.1 we give a direct proof that for odd dimensions not some of the faces is the maximal section. The constructed example uses Walkup's idea.

Theorem 7.2. *For odd dimension $n \geq 5$ there is a simplex whose largest hyperplane section is not one of its faces.*

Proof. We describe the idea. We consider the regular simplex with $n+1$ vertices, where $n+1$ is even. Then we take a hyperplane such that on both sides of the hyperplane the vertices build a $\frac{n+1}{2}$-simplex parallel to the hyperplane. Now we compress the simplex along the normal vector of this hyperplane, see Figure 7.1. The intersection with the simplex remains the same but the faces of the compressed simplex become smaller. With our formula (7.1) we compute the volumes and show that for $n \geq 5$ the compressed simplex has the desired property for a certain degree of compression.

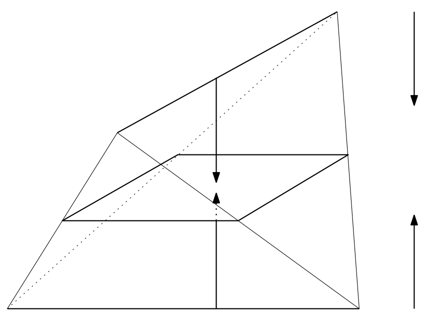

Figure 7.1.: Idea of the construction

We make this construction explicit. For $-\frac{1}{n+1} < \delta \le 0$ we give the matrix T^{-1} whose columns define the vertices of a deformed simplex $S(\delta)$:

$$T^{-1} = \left(\begin{array}{cccc|cccc} 1+\delta & \delta & \dots & \delta & -\delta & \dots & \dots & -\delta \\ \delta & 1+\delta & & \vdots & \vdots & & & \vdots \\ \vdots & & \ddots & \delta & \vdots & & & \vdots \\ \delta & \dots & \delta & 1+\delta & -\delta & \dots & \dots & -\delta \\ \hline -\delta & \dots & \dots & -\delta & 1+\delta & \delta & \dots & \delta \\ \vdots & & & \vdots & \delta & \ddots & & \vdots \\ \vdots & & & \vdots & \vdots & & \ddots & \delta \\ -\delta & \dots & \dots & -\delta & \delta & \dots & \delta & 1+\delta \end{array}\right)$$

Note that $S(0) = S_{\text{reg}}$ and $S\left(-\frac{1}{n+1}\right)$ is a degenerate simplex which is a $(n-1)$-dimensional set with n-dimensional volume equal to 0.

With $v^* = \sqrt{\delta}\,(1,\dots,1,-1,\dots,-1)$, we may also write $T^{-1} = (\mathrm{Id} + vv^*)$. Using basic matrix theory we know that $T = \mathrm{Id} - \frac{1}{1+v^*v} vv^*$ and $\det(T^{-1}) = \det \mathrm{Id}(1+v^*v)$. So

$$\det(T^{-1}) = 1 + (n+1)\delta.$$

Let

$$\begin{aligned} a &:= \frac{1}{\sqrt{n+1}}(1,\dots,1,-1,\dots,-1), \\ b &:= \frac{1}{\sqrt{(1+n\delta)^2 + n\delta^2}}(1+n\delta, -\delta,\dots,-\delta,\delta,\dots,\delta). \end{aligned}$$

We compute the vectors $\tilde{a}$ and $\tilde{b}$ according to Theorem 7.1. We find $\tilde{a} = a$ and $\tilde{b} = (1,0,\dots,0)$. The intersections $H_{\tilde{a}} \cap S_{\text{reg}}$ resp. $H_a \cap S(\delta)$ are the central sections described at the beginning of the proof. The intersection $H_{\tilde{b}} \cap S_{\text{reg}}$ is a face of the regular simplex. Therefore $H_b \cap S(\delta)$ is a face of the simplex $S(\delta)$, since T maps faces to faces. All faces of the simplex are of this form and therefore have the same volume.

It remains to compute the section volumes using formula (3.10) and analyze the behavior for $\delta \to -\frac{1}{n+1}$. Note that $\left\|T^{-1*}a\right\| = 1 + (n+1)\delta$ and $\left\|T^{-1*}b\right\| = \frac{1+(n+1)\delta}{\sqrt{(1+n\delta)^2+n\delta^2}}$. Using [GR07, (3.249)] we get

$$\begin{aligned} \mathrm{vol}_{n-1}(H_{\tilde{a}} \cap S_{\text{reg}}) &= \frac{\sqrt{n+1}}{(n-1)}\frac{1}{2\pi}\int_{\mathbb{R}} \left(1 + \left(\frac{s}{\sqrt{n+1}}\right)^2\right)^{-\frac{n+1}{2}} \mathrm{d}s \\ &= \frac{n+1}{(n-1)!}\frac{(n-2)!!}{2(n-1)!!} \\ &= \frac{n+1}{2((n-1)!!)^2} \end{aligned}$$

and

$$\mathrm{vol}_{n-1}(H_{\tilde{b}} \cap S_{\mathrm{reg}}) = \frac{\sqrt{n}}{(n-1)!}.$$

Therefore

$$\mathrm{vol}_{n-1}(H_a \cap S(\delta)) = \frac{n+1}{2((n-1)!!)^2},$$

$$\mathrm{vol}_{n-1}(H_b \cap S(\delta)) = \sqrt{(1+n\delta)^2 + n\delta^2}\,\frac{\sqrt{n+1-\left(\frac{1+(n+1)\delta}{\sqrt{(1+n\delta)^2+n\delta^2}}\right)^2}}{\sqrt{n}}\,\frac{\sqrt{n}}{(n-1)!}.$$

If $\delta \to -\frac{1}{n+1}$, then $\mathrm{vol}_{n-1}(H_b \cap S(\delta)) \to \frac{1}{(n-1)!}$. Finally we compute the ratio of the volumes:

$$\frac{\mathrm{vol}_{n-1}(H_a \cap S)}{\mathrm{vol}_{n-1}(H_b \cap S)} = \frac{(n+1)(n-1)!}{2((n-1)!!)^2} > 1.$$

For odd $n \geq 5$ this quotient is larger than 1. So for some $\delta \in \left(-\frac{1}{n+1}, 0\right)$ the simplex $S(\delta)$ has the desired property. □

Part II.

Sections of Cylinders

8. Three-dimensional Case

As a second class of bodies we consider cylinders. First we investigate the usual three-dimensional case. In Chapter 9 we define a generalized cylinder. This is a Cartesian product of a n-dimensional cube and a m-dimensional ball with radius r. We are interested in the $(n+m-1)$-volume of hyperplane sections of the cylinder. We vary the radius r and analyze the behavior of the maximal section.

We start with the three-dimensional case. First we derive a volume formula by geometric considerations. For fixed r this formula only depends on one variable and the maximum can be found by calculus.

For $r > 0$ let

$$Z := \left[-\frac{1}{2}, \frac{1}{2}\right] \times r \cdot B_2^2,$$

where $B_2^2 = \{x \in \mathbb{R}^2 \mid \|x\| = 1\}$. Z is an ordinary cylinder with height 1 and radius r. Since Z is rotational symmetric to the x_1-axis and symmetric to the origin, it suffices to consider sections orthogonal to $a = \left(\sqrt{1-\alpha^2}, \alpha, 0\right)$ for $\alpha \in [0,1]$. It is well known that plane intersections of a cylinder are either an ellipse or a truncated ellipse, including a circle and a rectangle as the extremal cases, see Figure 8.1. By intuition, if r is small enough the maximum should be attained by a rectangular section. We determine this critical r. And we show that for large r always some truncated ellipse is maximal. This result is formulated in Theorem 8.3.

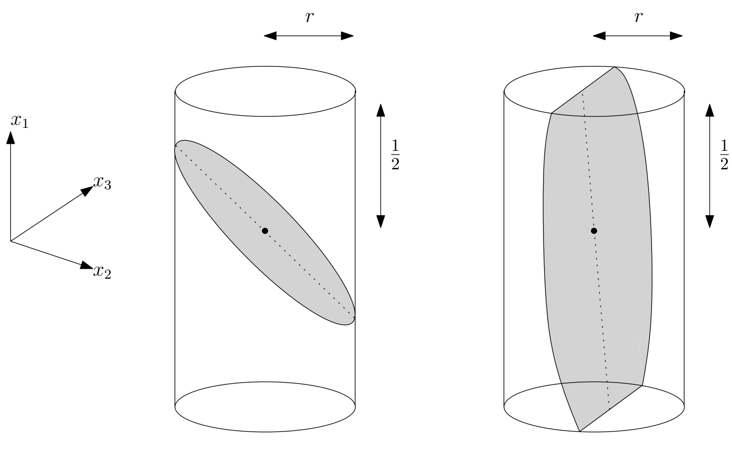

Figure 8.1.: Sections of the cylinder: “ellipse” and “truncated ellipse”

We start with the volume formula.

Lemma 8.1. *Let Z be the three-dimensional cylinder with radius $r > 0$. For $\alpha \in [0,1]$ let $a = \left(\sqrt{1-\alpha^2}, \alpha, 0\right)$. Then the 2-volume (area) of the section $H_a \cap Z$ is given by the function $A\colon [0,1] \to \mathbb{R}$,*

$$A(\alpha) = \begin{cases} \pi r \frac{r}{\sqrt{1-\alpha^2}} & \text{for } 0 \le \alpha \le \frac{1}{\sqrt{1+4r^2}} \\ \frac{r}{\alpha}\sqrt{1 - \frac{1-\alpha^2}{4\alpha^2 r^2}} + \frac{2r^2}{\sqrt{1-\alpha^2}} \arcsin\left(\frac{\sqrt{1-\alpha^2}}{2\alpha r}\right) & \text{for } \frac{1}{\sqrt{1+4r^2}} < \alpha < 1 \\ 2r & \text{for } \alpha = 1. \end{cases}$$

Proof. In general, the area $\mathcal{A}$ of an ellipse, defined by the lengths of its semi-axes e and f, is given by the formula $\mathcal{A} = \pi e f$. For a truncated ellipse, i.e. a set in $\mathbb{R}^2$ defined by $\left\{(x,y) \in \mathbb{R}^2 \;\middle|\; \frac{x^2}{e^2} + \frac{y^2}{f^2} = 1, x \in [-\hat{e}, \hat{e}]\right\}$ with $\hat{e} < e$, the area is given by

$$\begin{aligned} \mathcal{A} = 4\int_0^{\hat{e}} f\sqrt{1-\frac{x^2}{e^2}}\mathrm{d}x &= 2f\left[x\sqrt{1-\frac{x^2}{e^2}} + e\arcsin\left(\frac{x}{e}\right)\right]_0^{\hat{e}} \\ &= 2f\left(\hat{e}\sqrt{1-\frac{\hat{e}^2}{e^2}} + e\arcsin\left(\frac{\hat{e}}{e}\right)\right). \end{aligned} \tag{8.1}$$

We apply this to our setting. In the (x_1, x_2)-plane we get one of the two pictures in Figure 8.2, which corresponds to the picture in Figure 8.1. We compute the quantities

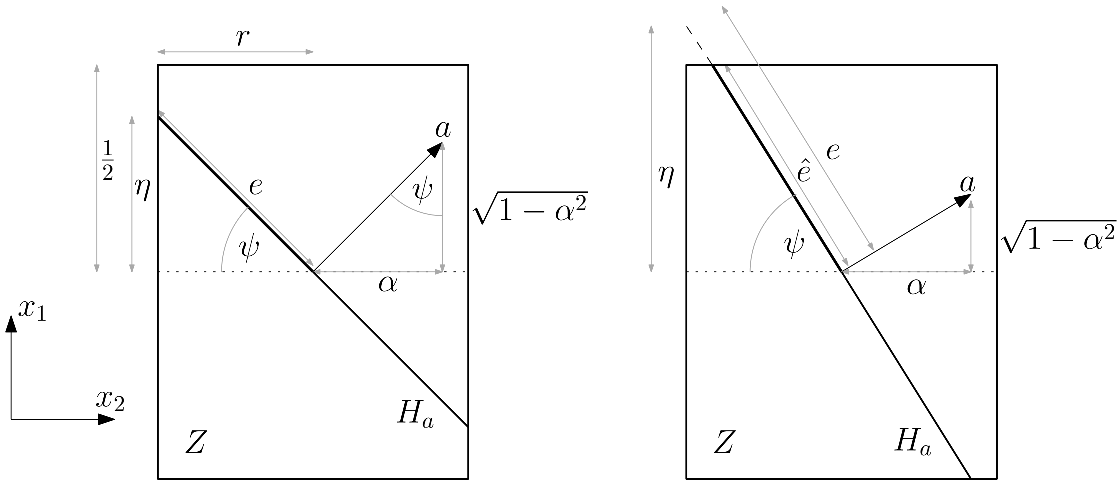

Figure 8.2.: Sectional image of cylinder, $x_3 = 0$

shown there for $\alpha \in (0,1)$. In both cases we find $\cos\psi = \sqrt{1-\alpha^2}$ and on the other hand $\cos\psi = \frac{r}{e}$, so $e = \frac{r}{\sqrt{1-\alpha^2}}$. Similarly $\sin\psi = \frac{\eta}{e}$ and $\sin\psi = \alpha$. So we find $\eta = \frac{\alpha r}{\sqrt{1-\alpha^2}}$. Additionally in the right picture $\sin\psi = \frac{1}{2\hat{e}}$, therefore $\hat{e} = \frac{1}{2\alpha}$.

If $\alpha \le \sqrt{\frac{1}{4r^2+1}}$, then $\eta \le \frac{1}{2}$. So $H_a \cap Z$ is an ellipse with semi-axes of length e and r, where r is independent of α. So the area of the ellipse is given by

$$\pi r \frac{r}{\sqrt{1-\alpha^2}}.$$

If $\alpha > \sqrt{\frac{1}{4r^2+1}}$, then $H_a \cap Z$ forms a truncated ellipse. With formula (8.1) we find

$$A(\alpha) = 2r\left(\frac{1}{2\alpha}\sqrt{1-\frac{1-\alpha^2}{4\alpha^2 r^2}} + \frac{r}{\sqrt{1-\alpha^2}}\arcsin\left(\frac{\sqrt{1-\alpha^2}}{2\alpha r}\right)\right).$$

If $\alpha = 0$ the section is a disk with area πr^2. If $\alpha = 1$ the section is a rectangle with side lengths 1 and $2r$. □

Lemma 8.2. *The function A defined in Lemma 8.1 is differentiable in $(0,1)$, particularly for $\alpha^* = \frac{1}{\sqrt{1+4r^2}}$, and $A'(\alpha^*) > 0$ for all $r > 0$.*

Proof. For $0 < \alpha < \alpha^*$ we have $A'(\alpha) = \frac{\pi r^2 \alpha}{(1-\alpha^2)^{\frac{3}{2}}}$. This is larger than 0 for all $r > 0$. For the left derivative in α^* we get $A'_-(\alpha^*) = \frac{\pi(1+4r^2)}{8r}$.

For $\alpha^* < \alpha < 1$ we find

$$\begin{aligned} A'(\alpha) = {} & \frac{1}{4r\alpha^4}\frac{1}{\sqrt{1+\frac{1}{4r^2}-\frac{1}{4r^2\alpha^2}}} - \frac{r}{\alpha^2}\sqrt{1+\frac{1}{4r^2}-\frac{1}{4r^2\alpha^2}} \\ & + \frac{2\alpha r^2}{(1-\alpha^2)^{\frac{3}{2}}}\arcsin\left(\frac{\sqrt{1-\alpha^2}}{2\alpha r}\right) - \frac{r}{\alpha^2(1-\alpha^2)}\frac{1}{\sqrt{1+\frac{1}{4r^2}-\frac{1}{4r^2\alpha^2}}}. \end{aligned} \tag{8.2}$$

Compute the limit of (8.2) for $\alpha \to \alpha^*$, $\alpha > \alpha^*$. Note that for $\alpha = \alpha^*$ we have $\sqrt{1+\frac{1}{4r^2}-\frac{1}{4r^2\alpha^2}} = 0$. The sum of the first and the last summand of (8.2) tends to 0 by L'Hôpital's rule. The second summand tends to 0 as well. The third summand tends to $\frac{\pi(1+4r^2)}{8r}$, which coincides with the left derivative in α^*. So A is differentiable in $(0,1)$ with $A'(\alpha^*) = \frac{\pi(1+4r^2)}{8r} > 0$ for all $r > 0$. □

Theorem 8.3. *Let $Z := [-\frac{1}{2}, \frac{1}{2}] \times r \cdot B_2^2$. Depending on r we have:*

(i) For $r > \frac{1}{2\sqrt{3}}$ a section orthogonal to $(\sqrt{1-\alpha^2}, \alpha, 0)$ for some $\alpha \in \left(\sqrt{\frac{1}{4r^2+1}}, 1\right)$ is maximal. So the maximal section is a truncated ellipse.

(ii) For $r \leq \frac{1}{2\sqrt{3}}$ *the section orthogonal to* $(0,1,0)$ *is maximal. So the maximal section is a rectangle.*

Proof. The function A is defined on the closed interval $[0,1]$, it is continuous and so it attains its maximum. From Lemma 8.2 we know in particular that A' is positive on $(0,\alpha^*]$. So A is maximal for some $\alpha \in (\alpha^*, 1]$. The maximum is attained for some $\alpha < 1$ if and only if $A'(\alpha)$ has a zero in $(\alpha^*, 1)$. Otherwise the function A is monotonously increasing from 0 to 1 and attains the maximum for $\alpha = 1$.

For $\alpha \in (\alpha^*, 1)$ the equation $A'(\alpha) = 0$ is equivalent to the following equation, using (8.2):

$$\begin{aligned}\frac{2\alpha r^2}{(1-\alpha^2)^{\frac{3}{2}}} \arcsin\left(\frac{\sqrt{1-\alpha^2}}{2\alpha r}\right) = {} & \frac{r}{\alpha^2(1-\alpha^2)} \frac{1}{\sqrt{1+\frac{1}{4r^2}-\frac{1}{4r^2\alpha^2}}} \\ & + \frac{r}{\alpha^2}\sqrt{1+\frac{1}{4r^2}-\frac{1}{4r^2\alpha^2}} \\ & - \frac{1}{4r\alpha^4}\frac{1}{\sqrt{1+\frac{1}{4r^2}-\frac{1}{4r^2\alpha^2}}}.\end{aligned}$$

Multiplying this by $\frac{1-\alpha^2}{r}$ and adding the first and the third summand on the right-hand side this simplifies to

$$\frac{\arcsin\left(\frac{\sqrt{1-\alpha^2}}{2\alpha r}\right)}{\frac{\sqrt{1-\alpha^2}}{2\alpha r}} = \frac{2-\alpha^2}{\alpha^3}\sqrt{\alpha^2+\frac{\alpha^2}{4r^2}-\frac{1}{4r^2}}. \tag{8.3}$$

Set $x := \frac{\sqrt{1-\alpha^2}}{2\alpha r}$, then $\alpha = \frac{1}{\sqrt{1+4r^2x^2}}$. Equation (8.3) reads as

$$\frac{\arcsin(x)}{x} = (1+8r^2x^2)\sqrt{1-x^2}. \tag{8.4}$$

So $A'(\alpha) = 0$ for some $\alpha \in (\alpha^*, 1)$ is equivalent to (8.4) for some $x \in (0,1)$.

Let $x \in (0,1)$. We estimate both sides of equation (8.4) by Taylor's theorem. There exists some $\xi \in (0,x)$ such that

$$\frac{\arcsin(x)}{x} = 1 + \frac{1}{6}x^2 + \frac{3}{40}\xi^4,$$

and therefore $\frac{\arcsin(x)}{x} > 1 + \frac{1}{6}x^2$. On the other hand there is some $\zeta \in (0,x)$ such that

$$(1+8r^2x^2)\sqrt{1-x^2} = 1 + \left(8r^2 - \frac{1}{2}\right)x^2 - \left(2r^2+\frac{1}{8}\right)\zeta^4.$$

Let $r \leq \frac{1}{2\sqrt{3}}$. Then $8r^2 - \frac{1}{2} < \frac{1}{6}$, and therefore $(1 + 8r^2x^2)\sqrt{1-x^2} < 1 + \frac{1}{6}x^2$. This implies

$$\frac{\arcsin(x)}{x} > \left(1 + 8r^2x^2\right)\sqrt{1-x^2},$$

for all $x \in (0,1)$, so (8.4) cannot have a root in $(0,1)$.

Now let $r > \frac{1}{2\sqrt{3}}$. First note that for $x = 1$ we have

$$\frac{\arcsin(x)}{x} = \frac{\pi}{2} > 0 = (1 + 8r^2x^2)\sqrt{1-x^2}.$$

On the other hand, we compute the first two derivatives in zero of the two functions:

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}\frac{\arcsin(x)}{x}\right|_{x=0} = 0,$$
$$\left.\frac{\mathrm{d}}{\mathrm{d}x}(1 + 8r^2x^2)\sqrt{1-x^2}\right|_{x=0} = 0,$$

and

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\arcsin(x)}{x}\right|_{x=0} = \frac{1}{3},$$
$$\left.\frac{\mathrm{d}^2}{\mathrm{d}x^2}(1 + 8r^2x^2)\sqrt{1-x^2}\right|_{x=0} = 16r^2 - 1 > \frac{1}{3},$$

where we used $r > \frac{1}{2\sqrt{3}}$. So there exists $x > 0$ such that

$$\frac{\arcsin(x)}{x} < (1 + 8r^2x^2)\sqrt{1-x^2}.$$

By the intermediate value theorem there is a root of equation (8.4) within the interval $(0,1)$.

This proves that A' has a zero if and only if $r > \frac{1}{2\sqrt{3}}$. □

9. Generalized Cylinder

We consider a generalization of the three-dimensional cylinder. Let

$$Z := \frac{1}{2}B_\infty^n \times rB_2^m \subset \mathbb{R}^{n+m}$$

for $r > 0$, $n, m \in \mathbb{N}$, $B_\infty^n := [-1,1]^n$ and $B_2^m := \{x \in \mathbb{R}^m \mid \|x\|_2 \leq 1\}$.

We are interested in the volume of central sections, i.e. in the quantity

$$\mathrm{vol}_{n+m-1}(H_a \cap Z)$$

for $a \in \mathbb{R}^{n+m}, \|a\| = 1$. We may always assume $a = (a_1, \ldots, a_n, a_{n+1}, 0, \ldots, 0)$, with $a_1, \ldots, a_{n+1} \geq 0$, since Z is rotationally symmetric with respect to the coordinates $n+1, \ldots, n+m$ and symmetric with respect to the origin.

We are interested in upper bounds on the volume, depending on the radius. In Section 9.1, we derive a volume formula. We apply Hölder's inequality to this formula, so the problem reduces to prove a real integral inequality that involves Bessel functions. This is done in Section 9.2, with three slightly different approaches depending on the dimension m. Finally, we provide the estimates on the volume of hyperplane sections in Section 9.3.

9.1. Volume formula

Theorem 9.1. *For the cylinder $Z \subset \mathbb{R}^{n+m}$, with $m, n \in \mathbb{N}$, $r > 0$, and a normal vector $a \in \mathbb{R}^{n+m}$ the volume of the hyperplane section $H_a \cap Z$ is given by*

$$\mathrm{vol}_{n+m-1}(H_a \cap Z) = r^m \frac{\pi^{\frac{m}{2}-1}}{\Gamma\left(\frac{m}{2}+1\right)} \int_0^\infty \prod_{j=1}^n \frac{\sin(\frac{a_j s}{2})}{\frac{a_j s}{2}} \cdot j_{\frac{m}{2}}(a_{n+1} r s)\mathrm{d}s. \tag{9.1}$$

Proof. Define

$$A(a,t) := \mathrm{vol}_{n+m-1}(H_a^t \cap Z) \quad \text{for } a \in \mathbb{R}^{n+m}, \|a\| = 1, \text{ and } t \geq 0,$$

in particular $A(a) := A(a,0)$. We apply the Fourier transformation and the inversion formula to the function $t \mapsto A(a,t)$. With Fubini's theorem and the integrals computed in Lemmas 2.5 and 2.6 we have

$$\begin{aligned}
&(2\pi)^{\frac{1}{2}}\,\hat{A}(a,s)\\
&=\int_{\mathbb{R}} A(a,t)\exp\left(-\mathrm{i}st\right)\mathrm{d}t\\
&=\int_{\mathbb{R}}\int_{\langle x,a\rangle=t}\chi_{[-\frac{1}{2},\frac{1}{2}]^n}((x_1,\dots,x_n))\ \chi_{rB_2^m}((x_{n+1},\dots,x_{n+m}))\ \exp\left(-\mathrm{i}st\right)\mathrm{d}x\mathrm{d}t\\
&=\int_{\mathbb{R}^{n+m}}\chi_{[-\frac{1}{2},\frac{1}{2}]^n}((x_1,\dots,x_n))\ \chi_{rB_2^m}((x_{n+1},\dots,x_{n+m}))\exp\left(-\mathrm{i}s\left\langle x,a\right\rangle\right)\mathrm{d}x\\
&=\int\limits_{[-\frac{1}{2},\frac{1}{2}]^n}\exp\left(-\mathrm{i}s\sum_{j=1}^{n}a_jx_j\right)\mathrm{d}(x_1,\dots,x_n)\int\limits_{rB_2^m}\exp\left(-\mathrm{i}s\cdot\sum_{j=n+1}^{n+m}a_jx_j\right)\mathrm{d}(x_{n+1},\dots,x_{n+m})\\
&=\prod_{j=1}^{n}\frac{\sin(\frac{a_js}{2})}{\frac{a_js}{2}}\cdot r^m\frac{\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}+1\right)}j_{\frac{m}{2}}(rsa_{n+1}).
\end{aligned}$$

Finally, by the Fourier inversion formula we get the formula stated in the theorem. □

The formula for the three-dimensional cylinder from Lemma 8.1 can also be obtained as a special case from this general formula for $n=1$ and $m=2$. By [GR07, 6.693 (4), p. 720], for $Re(\nu)>0$, $\nu\neq 1$ and $a,b>0$ we have

$$\int_0^\infty J_\nu(ax)\sin(bx)\frac{\mathrm{d}x}{x^2}=\begin{cases}\frac{\sqrt{a^2-b^2}\sin\left(\nu\arcsin\left(\frac{b}{a}\right)\right)}{\nu^2-1}-\frac{b\cos\left(\nu\arcsin\left(\frac{b}{a}\right)\right)}{\nu(\nu^2-1)}, & 0<b<a\\ \frac{-a^\nu\cos(\nu\pi/2)(b+\nu\sqrt{b^2+a^2})}{\nu(\nu^2+1)(b+\sqrt{b^2+a^2})^\nu}, & 0<a<b.\end{cases}$$

By L'Hôpital's rule one can compute the limit of the right-hand side for $\nu\to 1$. Then we find the same explicit formula as in Lemma 8.1.

As a special case, we get the volume formula for the cube if $m=1$ and $r=\frac{1}{2}$. Since $j_{\frac{1}{2}}(s)=\frac{\sin s}{s}$ the formula coincides with the volume formula of K. Ball [Bal86].

Lemma 9.2. *Let $a\in\mathbb{R}^{n+m}$ be a normal vector. Then*

$$\mathrm{vol}_{n+m-1}(H_a\cap Z)\le r^m\frac{\pi^{\frac{m}{2}-1}}{\Gamma\left(\frac{m}{2}+1\right)}\prod_{j=1}^{n}\left(2\mathcal{J}_1\left(\frac{1}{a_j^2}\right)\right)^{a_j^2}\ \left(\frac{1}{r}\mathcal{J}_m\left(\frac{1}{a_{n+1}^2}\right)\right)^{a_{n+1}^2},$$

where

$$\mathcal{J}_m(p):=\sqrt{p}\left(\int_0^\infty\left|j_{\frac{m}{2}}(u)\right|^p du\right).$$

Proof. We apply Hölder's inequality to formula (9.1) and then substitute $u=\frac{a_js}{2}$ resp.

$u = a_{n+1}rs$:

$$\begin{aligned}
&\mathrm{vol}_{n+m-1}(H_a \cap Z) \\
&\leq r^m \frac{\pi^{\frac{m}{2}-1}}{\Gamma(\frac{m}{2}+1)} \prod_{j=1}^{n} \left(\frac{2}{a_j} \int_0^\infty \left| \frac{\sin u}{u} \right|^{\frac{1}{a_j^2}} du \right)^{a_j^2} \left(\frac{1}{r a_{n+1}} \int_0^\infty \left| j_{\frac{m}{2}}(u) \right|^{\frac{1}{a_{n+1}^2}} du \right)^{a_{n+1}^2} \\
&= r^m \frac{\pi^{\frac{m}{2}-1}}{\Gamma(\frac{m}{2}+1)} \prod_{j=1}^{n} \left(2\mathcal{J}_1 \left(\frac{1}{a_j^2} \right) \right)^{a_j^2} \left(\frac{\mathcal{J}_m \left(\frac{1}{a_{n+1}^2} \right)}{r} \right)^{a_{n+1}^2}. \qquad (9.2)
\end{aligned}$$

□

9.2. Integral inequality

We state the classical inequality due to K. Ball [Bal86]. The "cube-part" of (9.2) is estimated by the following inequality. For $p \geq 2$,

$$\mathcal{J}_1(p) = \sqrt{p} \int_0^\infty \left| \frac{\sin(u)}{u} \right|^p \mathrm{d}u \leq \frac{\pi}{\sqrt{2}}, \qquad (9.3)$$

and $\lim_{p\to\infty} \mathcal{J}_1(p) = \sqrt{\frac{3}{2}\pi}$.

For the "B_2^m-ball part" of (9.2), we show a generalization of Ball's inequality.

Theorem 9.3. *For all $m \in \mathbb{N}, m \geq 2$, and $p \in \mathbb{R}, p \geq 2$, we have*

$$\mathcal{J}_m(p) = \sqrt{p} \int_0^\infty \left| j_{\frac{m}{2}}(s) \right|^p \mathrm{d}s \leq \sqrt{\pi} \sqrt{\frac{m}{2}+1}$$

and $\lim_{p\to\infty} \mathcal{J}_m(p) = \sqrt{\pi}\sqrt{\frac{m}{2}+1}$.

Remark: Note that $\mathcal{J}_1(2) > \lim_{p\to\infty} \mathcal{J}_1(p)$ in contrast to $\mathcal{J}_m(2) \leq \lim_{p\to\infty} \mathcal{J}_m(p)$ for $m \geq 2$. So for $m = 1$, equality holds for $p = 2$ in contrast to $m \geq 2$, where equality holds for $p = \infty$.

Similar integral inequalities were established for complex cubes and for generalized cubes, see [OP00] and [Brz11]. We briefly describe the setting for the complex cubes. We identify $\mathbb{C}^n$ and $\mathbb{R}^{2n}$. Then the hyperplane sections of the complex cube have real dimension $2n - 2$. The integral inequality needed for this case is

$$\sqrt{p} \int_0^\infty |j_1(s)|^p \, s \, \mathrm{d}s \leq \frac{4}{p} \qquad (9.4)$$

for $p \geq 2$. Note that compared to (9.3) there is an additional factor s in front of ds. For generalized cubes one has to consider a similar integral with some higher power of s in front of ds.

We prove Theorem 9.3 by applying the following lemma due to Nazarov and Podkorytov. They used this lemma to simplify the proof of K. Ball for inequality (9.3). The oscillating behavior of the function $\sin(s)/s$ is a main difficulty. By the Nazarov-Podkorytov lemma one avoids the oscillations by considering the distribution functions. These functions are decreasing. [NP00]

For a function $f : X \to \mathbb{R}_{\geq 0}$ on a measure space (X, μ), define the cumulative distribution function $F : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ by

$$F(y) := \mu(\{x \in X \mid f(x) > y\}).$$

Lemma 9.4 (Nazarov-Podkorytov)**.** *Let h, g be non-negative measurable functions on a measure space (X, μ). Let H, G be their distribution functions. Assume that $H(y)$, $G(y)$ are finite for all $y > 0$. Also assume that*

(N1) there is some $y_0 > 0$ such that $G(y) \leq H(y)$ for all $y < y_0$ and $G(y) \geq H(y)$ for all $y > y_0$, i.e. the difference $G - H$ changes its sign exactly once from $-$ to $+$;

(N2) for some $p_0 > 0$: $\int_X h^{p_0} d\mu = \int_X g^{p_0} d\mu$.

Then

$$\int_X h^p d\mu \leq \int_X g^p d\mu$$

for all $p > p_0$ as long as the integrals exist.

9.2.1. The limit of the integral

We prove the asymptotic result of the integral inequality from Theorem 9.3. Using Lemmas 2.9 and 2.11, we estimate

$$\begin{aligned}
&\sqrt{p} \int_0^\infty \left| j_{\frac{m}{2}}(s) \right|^p \mathrm{d}s \\
\leq &\sqrt{p} \int_0^{\frac{m}{2}+3} \exp\left(-\frac{s^2}{2m+4}\right)^p \mathrm{d}s \\
&+ \sqrt{p} \, \left(2^{\frac{m+1}{2}} \frac{\Gamma(\frac{m}{2}+1)}{\sqrt{\pi}}\right)^p \left(\frac{\sqrt{m+6}}{\sqrt[4]{12m+36}}\right)^p \int_{\frac{m}{2}+3}^\infty s^{-p\frac{m+1}{2}} \mathrm{d}s \\
= &\sqrt{p} \int_0^\infty \exp\left(-\frac{ps^2}{2m+4}\right) \mathrm{d}s
\end{aligned}$$

$$+ \sqrt{p} \left(2^{\frac{m+1}{2}} \frac{\Gamma(\frac{m}{2}+1)}{\sqrt{\pi}}\right)^p \left(\frac{\sqrt{m+6}}{\sqrt[4]{12m+36}}\right)^p \frac{1}{p^{\frac{m+1}{2}}-1} \left(\frac{m}{2}+3\right)^{-p\frac{m+1}{2}+1}.$$

For $p \to \infty$, the first summand tends to $\sqrt{\pi}\sqrt{\frac{m}{2}+1}$ since $\int_0^\infty \exp(-x^2/K)\mathrm{d}x = \sqrt{K\pi}/2$ for $K > 0$. Comparing the exponents, the second summand tends to 0 for $p \to \infty$.

On the other hand, using Lemma 2.12, by the substitution $u = \sqrt{p}s$ and by the series expansion of the exponential function we have

$$\begin{aligned}
\sqrt{p}\int_0^\infty \left|j_{\frac{m}{2}}(s)\right|^p \mathrm{d}s &\geq \sqrt{p}\int_0^1 \exp\left(-\frac{ps^2}{2m+4} - ps^4\right)\mathrm{d}s \\
&= \int_0^{\sqrt{p}} \exp\left(-\frac{u^2}{2m+4} - \frac{u^4}{p}\right)\mathrm{d}u \\
&\geq \int_0^{\sqrt{p}} \exp\left(-\frac{u^2}{2m+4}\right)\left(1-\frac{u^4}{p}\right)\mathrm{d}u \\
&\geq \int_0^{\sqrt{p}} \exp\left(-\frac{u^2}{2m+4}\right)\mathrm{d}u - \frac{1}{p}\int_0^{\sqrt{p}} u^4 \exp\left(-\frac{u^2}{2m+4}\right)\mathrm{d}u \\
&\geq \int_0^{\sqrt{p}} \exp\left(-\frac{u^2}{2m+4}\right)\mathrm{d}u - \frac{1}{p}\int_0^{p} u \exp\left(-\frac{u}{2m+4}\right)\mathrm{d}u.
\end{aligned}$$

For $p \to \infty$, we observe that the first summand again tends to $\sqrt{\pi}\sqrt{\frac{m}{2}+1}$, and the second summand vanishes since $\int_0^\infty x\exp(-x)\mathrm{d}x = 1$. By the sandwich lemma we have found the limit as claimed in Theorem 9.3.

9.2.2. The case $m = 2$

For $m = 2$ the integral inequality from Theorem 9.3 is similar to Oleskiewicz's and Pełczyński's inequality to estimate the section volume of complex cubes, see (9.4). They used a different technique than we do. We use the Nazarov-Podkorytov lemma. This proof is a modification of an unpublished proof of Oleskiewicz's and Pełczyński's inequality by H. König [Kön14b].

We apply the Nazarov-Podkorytov lemma 9.4 to the functions

$$h(s) := |j_1(s)| = \left|\frac{2J_1(s)}{s}\right| \quad \text{and} \quad g(s) := \exp\left(-\frac{s^2}{8}\right),$$

see Figure 9.1.

By H resp. G we denote the distribution functions with respect to the Lebesgue measure λ on $\mathbb{R}_{\geq 0}$. We check the two conditions of Lemma 9.4.

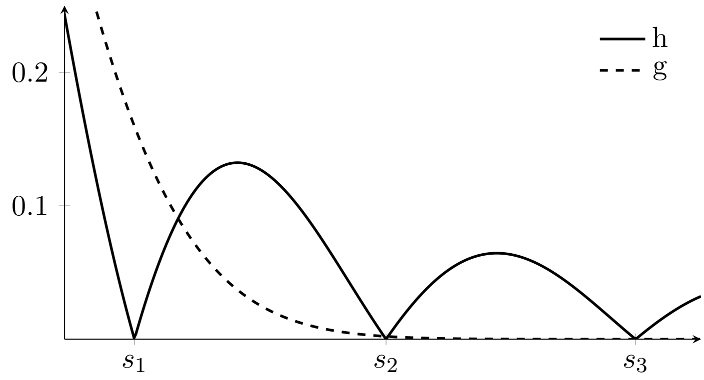

Figure 9.1.: The functions h and g

Condition (N2): Independently of p we have

$$\sqrt{p}\int_0^\infty g(s)^p \mathrm{d}s = \sqrt{2\pi}.$$

For $p = 2$, we evaluate the other integral explicitly, using [Wat66, p. 403]:

$$\sqrt{2}\int_0^\infty h(s)^2 \mathrm{d}s = \frac{\sqrt{2}\cdot 3}{16\pi} < \sqrt{2\pi}.$$

By [AS84, (9.2.1)] we know the asymptotic behavior of Bessel functions:

$$J_\nu(s) = \sqrt{\frac{2}{\pi s}}\cos\left(s - \left(\frac{1}{2}\nu - \frac{1}{4}\right)\pi\right) + O\left(s^{-\frac{3}{2}}\right). \tag{9.5}$$

So $\sqrt{p}\int_0^\infty h(s)^p \mathrm{d}s$ diverges for $p \to \frac{2}{3}$. By the intermediate value theorem, there is $p_0 \in \left(\frac{2}{3}, 2\right)$ such that

$$\sqrt{p_0}\int_0^\infty h(s)^{p_0} \mathrm{d}s = \sqrt{p_0}\int_0^\infty g(s)^{p_0} \mathrm{d}s. \tag{9.6}$$

Condition (N1): We investigate the two distribution functions H and G.

The distribution function G is given by the inverse of g, since g is a decreasing and bijective function $\mathbb{R}_{\geq 0} \to (0, 1]$. So for $y \geq 1$, $G(y) = 0$ and for $s \in (0, 1)$ we write explicitly

$$G(y) = \lambda\left(\left\{s \;\middle|\; y < \exp\left(-\frac{s^2}{8}\right)\right\}\right) = \lambda\left(\left\{s \;\middle|\; s < \sqrt{8\ln\left(\frac{1}{y}\right)}\right\}\right) = \sqrt{8\ln\left(\frac{1}{y}\right)}.$$

Its derivative is

$$G'(y) = -\frac{\sqrt{2}}{y\ln\left(\frac{1}{y}\right)}. \tag{9.7}$$

Later, we need that $\frac{1}{|G'(y)|}$ is decreasing for $0 \le y \le \frac{1}{\sqrt{e}}$.

Now we investigate H. The function h is oscillating. Denote the k-th local maximum of h by $y_k := \max\{h(s) \mid s \in (s_k, s_{k+1})\}$, with s_k the k-th zero of the Bessel function J_1 and $s_0 = 0$. The approximation of the first zeros is taken from [Wat66, p. 748: Table VII]; $s_1 = 3.832, s_2 = 7.016, s_3 = 10.173$.

Step (i): There is at least one intersection of G and H.
From Lemma 2.9 we know $h(s) = |j_1(s)| \le \exp\left(-\frac{s^2}{8}\right) = g(s)$ for $s \in [0, 4]$. So for $y \ge y_1$:

$$\begin{aligned} H(y) &= \lambda\left(\{x \in [0,\infty) \mid h(x) > y\}\right) \\ &= \lambda\left(\{x \in [0,s_1] \mid h(x) > y\}\right) \\ &\le \lambda\left(\{x \in [0,s_1] \mid g(x) > y\}\right) \\ &= G(y). \end{aligned}$$

So $G - H \ge 0$ for $y \in (y_1, \infty)$. Consider (9.6) and observe that by Fubini and substitution

$$\begin{aligned} 0 &= \int_0^\infty (g(s)^{p_0} - h(s)^{p_0})\mathrm{d}s \\ &= \int_0^\infty \left(G(y^{\frac{1}{p_0}}) - H(y^{\frac{1}{p_0}})\right)\mathrm{d}y \\ &= p_0 \int_0^\infty y^{p_0-1}\left(G(y) - H(y)\right)\mathrm{d}y. \end{aligned}$$

So $G - H$ has to change its sign at least once.

Step (ii): There is at most one intersection of G and H.
If we prove that $G - H$ is increasing on $(0, y_1)$, this implies $G - H$ changes its sign only once. We show this by proving that for each interval (y_{k+1}, y_k), the quotient $\frac{|H'|}{|G'|}$ is strictly larger than 1. The distribution functions are decreasing, so their derivatives are negative (or 0). So $\frac{|H'|}{|G'|} > 1$ implies $H' < G'$ and therefore $G - H$ is increasing.

Step (iii): Estimate the local maxima of H.
From [Sch08, p. 116] we know the approximate position of the zeros of the Bessel

functions:

$$s_k \in (k\pi, (k+1/4)\pi). \tag{9.8}$$

In [Kön14a, p. 32] it is noted that the successive maxima of $\left|\sqrt{\frac{2}{\pi}}\sqrt{s}J_1(s)\right|$ are decreasing to 1. This implies

$$2\sqrt{\frac{2}{\pi}}\frac{1}{(s_{k+1})^{\frac{3}{2}}} \leq y_k \leq 2\sqrt{\frac{2}{\pi}}\frac{1}{(s_k)^{\frac{3}{2}}}.$$

In particular, together with (9.8), we get

$$\frac{2\sqrt{2}}{\pi^2}\frac{1}{(k+\frac{5}{4})^{\frac{3}{2}}} \leq y_k \leq \frac{2\sqrt{2}}{\pi^2}\frac{1}{k^{\frac{3}{2}}}. \tag{9.9}$$

Step (iv): Compute H.
For $y \neq y_k$ we claim that

$$|H'(y)| = \sum_{s>0, h(s)=y} \frac{1}{|h'(s)|}.$$

To see this, note that for a bijective function f, the distribution function F is given by $F = f^{-1}$ and $F' = \frac{1}{f'}$. Now H can be decomposed into the sum of the bijective parts of h, where $H(y)$ is the length of the dashed intervals, see Figure 9.2. The figure is taken from [NP00, p. 6]. The equation $h(s) = y$ has one root in $(0, s_1)$ and two roots in each interval (s_k, s_{k+1}) for $1 \leq k \leq K$, with some $K \in \mathbb{N}$ depending on y.

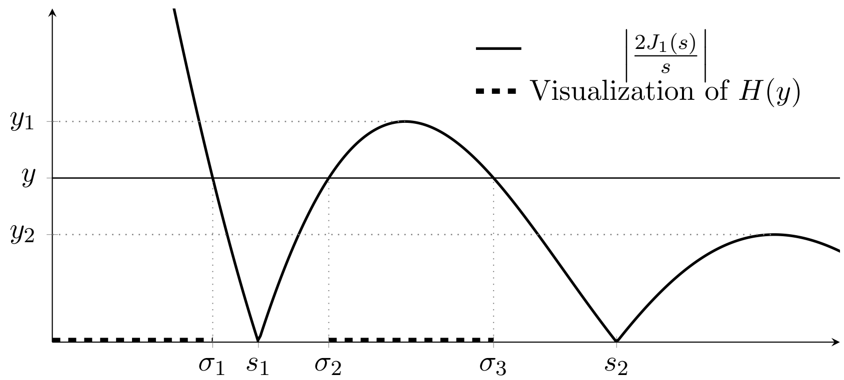

Figure 9.2.: Visualization of $H(y)$

Step (v): Estimate h'.
We estimate $h'(s)$ at these roots. By the recurrence relation for Bessel functions, we

have $|h'(s)| = \left(\frac{|2J_1(s)|}{s}\right)' = 2\frac{|J_2(s)|}{s}$. We approximate J_2 with [GR07, (8.479)] and find $\left|\frac{2J_2(s)}{s}\right| \leq 2\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt[4]{s^2-4}}\frac{1}{s}$ for $s \geq 2$. Additionally for $s \geq 3$, $\frac{1}{\sqrt[4]{s^2-4}}\frac{1}{s} \leq \frac{1}{s^{\frac{3}{2}}}\sqrt{\frac{\pi}{2}}$. So for $s \geq 3$ we estimate

$$|h'(s)| \leq 2\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt[4]{s^2-4}}\frac{1}{s} \leq 2\frac{1}{s^{\frac{3}{2}}}. \tag{9.10}$$

This holds in particular for $s \in (s_k, s_{k+1})$, $k \geq 1$, since $s_1 \geq 3$. Therefore

$$|h'(s)| \leq 2\frac{1}{s_k^{\frac{3}{2}}} \leq 2\frac{1}{(\pi k)^{\frac{3}{2}}}.$$

For $s \in [0, s_1)$, a rough estimate is sufficient:

$$|h'(s)| \leq 0.4.$$

Step (vi): Estimate H'/G'.
Fix k and let $y \in (y_{k+1}, y_k)$. Then

$$|H'(y)| \geq \left(2.5 + 2\cdot\frac{1}{2}\pi^{3/2}\sum_{l=1}^{k} l^{3/2}\right).$$

Since $y_k \leq y_1 < \frac{1}{\sqrt{e}}$, we may use (9.7) and (9.9) to estimate

$$\frac{1}{|G'(y)|} \geq \frac{1}{|G'(y_{k+1})|} \geq \frac{1}{\left|G'\left(\frac{\pi^2}{2\sqrt{2}}\left(k+\frac{9}{4}\right)^{3/2}\right)\right|}.$$

For the quotient we get

$$\frac{|H'(y)|}{|G'(y)|} \geq \left(2.5 + \pi^{3/2}\sum_{l=1}^{k} l^{3/2}\right)\frac{2}{\pi^2\,(k+9/4)^{3/2}}\sqrt{\ln\left(\frac{\pi^2}{2\sqrt{2}}\left(k+\frac{9}{4}\right)^{3/2}\right)} =: Q(k).$$

We estimate $\sum_{l=1}^{k} l^{\frac{3}{2}} \geq \int_0^k l^{\frac{3}{2}}\,\mathrm{d}l = \frac{2}{5}k^{\frac{5}{2}}$. Using this estimate, note that $Q(k)$ is increasing in k. By evaluation, $Q(2) > 1$, so $Q(k) > 1$ for all $k \geq 2$.

Since $Q(1) < 1$, we need to be more precise for $k = 1$. Let $y \in (y_2, y_1)$. The equation $h(s) = y$ has three solutions, see Figure 9.2. Denote them by $\sigma_1, \sigma_2, \sigma_3$ in ascending order. We estimate these roots numerically, using the roots of $h(s) = y_2$. Then we use Lemma 2.7 to estimate $|h'|$. We find $\sigma_1 \in (3.3050, s_1)$, so $\frac{1}{|h'(\sigma_1)|} \geq \frac{1}{0.298}$. And $\sigma_2 \in (4.1896, s_2)$, so $\frac{1}{|h'(\sigma_2)|} \geq \frac{1}{0.199}$, as well as $\sigma_3 \in (4.1896, s_2)$, so $\frac{1}{|h'(\sigma_3)|} > \frac{1}{0.199}$. The corresponding estimate for G' is $\frac{1}{|G'(y)|} \geq \frac{1}{|G'(y_2)|} \geq 0.077$. Therefore we get for all

$y \in (y_2, y_1)$:

$$\frac{|H'(y)|}{|G'(y)|} > 1.$$

Thus we have shown that $\frac{|H'(y)|}{|G'(y)|} > 1$ for all $y \in (0, y_1)$.

This finishes the proof of condition (N1) and therefore the proof of Theorem 9.3 for $m = 2$. □

9.2.3. The case $m \geq 5$

The previous proof relied on the approximate knowledge of the zeros of the Bessel function. Here we use a different approach. The idea is due to [Brz11]. The aim is to simplify $j_{\frac{m}{2}}$ and get rid of the oscillating behavior. Due to the rougher estimates this only works for $m \geq 5$. We define

$$\tilde{j}_{\frac{m}{2}}(s) := \begin{cases} \left| j_{\frac{m}{2}}(s) \right|, & s \in [0, m) \\ 2^{\frac{m+1}{2}} \frac{\Gamma(\frac{m}{2}+1)}{\sqrt{\pi}} \left(s^2 - \frac{m^2}{4} \right)^{-\frac{1}{4}} s^{-\frac{m}{2}}, & s \in [m, \infty), \end{cases} \tag{9.11}$$

see Figure 9.3.

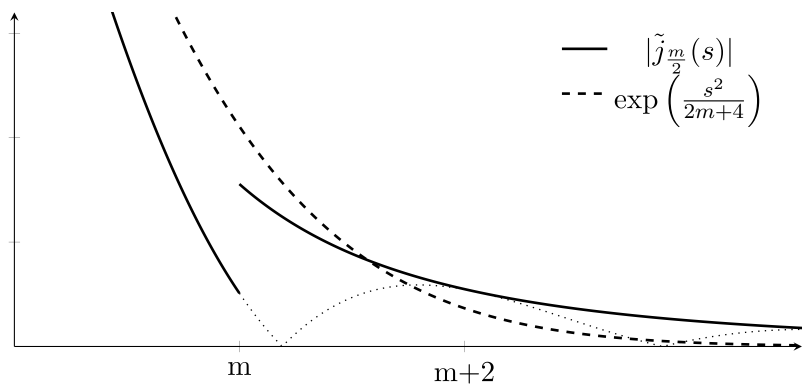

Figure 9.3.: The functions $\tilde{j}_{\frac{m}{2}}(s)$ and $\exp\left(\frac{s^2}{2m+4}\right)$

For this simplification, by Lemma 2.8 it is true that for all $s \geq 0$

$$j_{\frac{m}{2}}(s) \leq \tilde{j}_{\frac{m}{2}}(s). \tag{9.12}$$

So it is sufficient to prove the inequality for this simplification of $j_{\frac{m}{2}}$. We apply the Nazarov-Podkorytov lemma 9.4.

Check condition (N1): We compare $\tilde{j}_{\frac{m}{2}}(s)$ and $g(s) := \exp\left(-\frac{s^2}{2m+4}\right)$. We claim

$$\tilde{j}_{\frac{m}{2}}(s) < g(s), \quad s \in [0, m], \tag{9.13}$$
$$\tilde{j}_{\frac{m}{2}}(s) > g(s), \quad s \in (m+2, \infty), \tag{9.14}$$
$$\tilde{j}_{\frac{m}{2}}(s) = g(s), \quad \text{for exactly one } s \in (m, m+2). \tag{9.15}$$

Inequality (9.13) corresponds to Lemma 2.10. Inequality (9.14) is [Brz11, Lemma 3.19]; note that the lemma there is also true for $m = 5$ by exactly the same argument. Property (9.15) is from [Brz11, Lemma 3.18]; this does not include $m = 5$ and $m = 6$, but one can easily check the statement by hand with analogous arguments.

Since g and $\tilde{j}_{\frac{m}{2}}$ are bounded by 1, for $y \geq 1$ we have $G(y) = 0 = \tilde{J}_{\frac{m}{2}}(y)$, where $\tilde{J}_{\frac{m}{2}}$ is the distribution function of $\tilde{j}_{\frac{m}{2}}$. The functions g and $\tilde{j}_{\frac{m}{2}}$ intersect exactly once, so the difference of the cumulative distribution functions changes its sign exactly once as well. This shows (N1).

For condition (N2) we show:

$$\text{for } p \to \frac{2}{m+1}, \quad \sqrt{p}\int_0^\infty \tilde{j}_{\frac{m}{2}}(s)^p \mathrm{d}s \longrightarrow \infty, \tag{9.16}$$
$$\sqrt{2}\int_0^\infty \tilde{j}_{\frac{m}{2}}(s)^2 \mathrm{d}s < \sqrt{\pi}\sqrt{\frac{m}{2}+1}, \tag{9.17}$$
$$\exists p_0 \in \left(\frac{2}{m+1}, 2\right] : \sqrt{p_0}\int_0^\infty \tilde{j}_{\frac{m}{2}}(s)^{p_0} \mathrm{d}s = \sqrt{p_0}\int_0^\infty g(s)^{p_0} \mathrm{d}s = \sqrt{\pi}\sqrt{\frac{m}{2}+1}. \tag{9.18}$$

For large s the function $\tilde{j}_{\frac{m}{2}}$ is asymptotically $\frac{2^{\frac{m+1}{2}}}{\sqrt{\pi}}\Gamma\left(\frac{m}{2}+1\right)s^{-\frac{m+1}{2}}$. Therefore $\tilde{j}_{\frac{m}{2}}(\cdot)^p$ is integrable for $p > \frac{2}{m+1}$, and $\int_0^\infty \tilde{j}_{\frac{m}{2}}(s)^p \mathrm{d}s$ diverges for $p \to \frac{2}{m+1}$; this is (9.16).

For inequality (9.17) evaluate the integral. We have

$$\begin{aligned}
&\sqrt{2}\int_0^\infty \tilde{j}_{\frac{m}{2}}(s)^2 \mathrm{d}s \\
=&\sqrt{2}\int_0^m \left|j_{\frac{m}{2}}(s)\right|^2 \mathrm{d}s + \sqrt{2}\int_m^\infty \left(2^{\frac{m+1}{2}}\frac{\Gamma(\frac{m}{2}+1)}{\sqrt{\pi}}\left(s^2 - \frac{m^2}{4}\right)^{-\frac{1}{4}} s^{-\frac{m}{2}}\right)^2 \mathrm{d}s \\
\leq&\sqrt{2}\int_0^\infty \left|j_{\frac{m}{2}}(s)\right|^2 \mathrm{d}s + \sqrt{2}\int_m^\infty \left(2^{\frac{m+1}{2}}\frac{\Gamma(\frac{m}{2}+1)}{\sqrt{\pi}}\left(s^2 - \frac{m^2}{4}\right)^{-\frac{1}{4}} s^{-\frac{m}{2}}\right)^2 \mathrm{d}s \\
=&\sqrt{2}\int_0^\infty 2^m\Gamma\left(\frac{m}{2}+1\right)^2 \frac{J_{\frac{m}{2}}(s)^2}{s^m}\mathrm{d}s + 2^{m+\frac{3}{2}}\frac{\Gamma\left(\frac{m}{2}+1\right)^2}{\pi}\int_m^\infty \left(s^2 - \frac{m^2}{4}\right)^{-\frac{1}{2}} s^{-m}\mathrm{d}s.
\end{aligned}$$

The first integral is evaluated by [GR07, 6.575 (2)] and then estimated by Lemma 2.4:

$$\begin{aligned}\sqrt{2}\int_0^\infty 2^m\Gamma\left(\frac{m}{2}+1\right)^2\frac{J_{\frac{m}{2}}(s)^2}{s^m}\mathrm{d}s &= \sqrt{2}\sqrt{\pi}\frac{\Gamma\left(\frac{m}{2}+1\right)^2\Gamma(m)}{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(\frac{m}{2}+\frac{1}{2}\right)^2}\\ &\leq \sqrt{\pi}\frac{m+2}{m+1}\frac{\sqrt{m}}{\sqrt{2}}.\end{aligned}$$

For the second summand, estimate the integrand by $\left(s^2-\frac{m^2}{4}\right)^{-\frac{1}{2}}s^{-m}\leq\sqrt{\frac{4}{3}}s^{-m-1}$, which is true for $s\geq m$. Then use again Stirling's formula for $\Gamma\left(\frac{m}{2}+1\right)^2$ and get

$$\begin{aligned}2^{m+\frac{3}{2}}\frac{\Gamma\left(\frac{m}{2}+1\right)^2}{\pi}\int_m^\infty\left(s^2-\frac{m^2}{4}\right)^{-\frac{1}{2}}s^{-m}\mathrm{d}s &\leq 2^{m+\frac{3}{2}}\frac{\Gamma\left(\frac{m}{2}+1\right)^2}{\pi}\sqrt{\frac{4}{3}}\int_m^\infty s^{-m-1}\mathrm{d}s\\ &= 2^{m+\frac{3}{2}}\sqrt{\frac{4}{3}}\frac{\Gamma\left(\frac{m}{2}+1\right)^2}{\pi}m^{-(m+1)}\\ &\leq \frac{\sqrt{2}^5}{\sqrt{3}}\exp\left(\frac{1}{3m}\right)\exp(-m).\end{aligned}$$

It remains to show

$$\sqrt{\pi}\frac{m+2}{m+1}\frac{\sqrt{m}}{\sqrt{2}}+\frac{\sqrt{2}^5}{\sqrt{3}}\exp\left(\frac{1}{3m}\right)\exp(-m)\leq\sqrt{\pi}\sqrt{\frac{m}{2}+1}. \tag{9.19}$$

This follows if we prove the stronger inequality

$$3.65\exp(-m)\leq\sqrt{\pi}\left(\sqrt{\frac{m}{2}+1}-\frac{m+2}{m+1}\frac{\sqrt{m}}{2}\right). \tag{9.20}$$

Note that $\exp(m)\left(\sqrt{m+2}-\frac{m+2}{m+1}\sqrt{m}\right)\geq\exp(m)\frac{1}{3}m^{-\frac{3}{2}}$, and $\exp(m)\frac{1}{3}m^{-\frac{3}{2}}$ is increasing in m. For $m=5$, inequality (9.20) is true, and so it is true for all $m\geq 5$. This proves (9.17).

Now (9.18) follows by the intermediate value theorem.

Thus we proved (N1) and (N2), so the Nazarov-Podkorytov lemma gives the desired result. □

9.2.4. The case $m\in\{3,4\}$

The estimates made above by the simplification of $j_{\frac{m}{2}}$ in (9.11) are too rough for $m<5$. So we need a different approach here that involves numerical estimates. Therefore one has to treat the cases $m\in\{3,4\}$ separately. The idea is basically given in [Brz11], and it is a generalization of [OP00]. This approach also works for $m = 2$.

We first prove the following integral inequality, which is not only true for $m \in \{3, 4\}$.

Lemma 9.5. *For $p > 0$ and $m \in \mathbb{N}$ we have*

$$\int_0^\infty e^{-\frac{ps^2}{2m+4} - \frac{ps^4}{4(m+2)^2(m+4)}} \mathrm{d}s \leq \frac{1}{\sqrt{p}} \sqrt{\frac{m}{2} + 1} \sqrt{\pi} \left(1 - \frac{3}{4} \frac{1}{p(m+4)} + \frac{105}{16} \frac{1}{2p^2(m+4)^2}\right).$$

Proof. Substitute $u := \frac{ps^2}{2m+4}$. Then

$$\int_0^\infty e^{-\frac{ps^2}{2m+4} - \frac{ps^4}{4(m+2)^2(m+4)}} \mathrm{d}s = \frac{1}{2} \sqrt{\frac{2m+4}{p}} \int_0^\infty e^{-u} e^{-\frac{u^2}{p(m+4)}} u^{-\frac{1}{2}} \mathrm{d}u.$$

We estimate the exponential function by its series expansion

$$\exp\left(-\frac{u^2}{p(m+4)}\right) \leq 1 - \frac{u^2}{p(m+4)} + \frac{u^4}{2p^2(m+4)^2}.$$

We split the integral and compute

$$\begin{aligned}
&\int_0^\infty e^{-\frac{ps^2}{2m+4} - \frac{ps^4}{4(m+2)^2(m+4)}} \mathrm{d}s \\
&\leq \frac{1}{2} \sqrt{\frac{2m+4}{p}} \left(\int_0^\infty e^{-u} u^{-\frac{1}{2}} \mathrm{d}u - \frac{1}{p(m+4)} \int_0^\infty e^{-u} u^{\frac{3}{2}} \mathrm{d}u + \frac{1}{2p^2(m+4)^2} \int_0^\infty e^{-u} u^{\frac{7}{2}} \mathrm{d}u\right) \\
&= \frac{1}{2} \sqrt{\frac{2m+4}{p}} \left(\Gamma\left(\frac{1}{2}\right) - \frac{1}{p(m+4)} \Gamma\left(\frac{5}{2}\right) + \frac{1}{2p^2(m+4)^2} \Gamma\left(\frac{9}{2}\right)\right) \\
&= \frac{1}{2} \sqrt{\frac{2m+4}{p}} \sqrt{\pi} \left(1 - \frac{3}{4} \frac{1}{p(m+4)} + \frac{105}{16} \frac{1}{2p^2(m+4)^2}\right) \\
&= \frac{1}{\sqrt{p}} \sqrt{\frac{m}{2} + 1} \sqrt{\pi} \left(1 - \frac{3}{4} \frac{1}{p(m+4)} + \frac{105}{16} \frac{1}{2p^2(m+4)^2}\right).
\end{aligned}$$

□

With the previous lemma, we prove the original integral inequality from Theorem 9.3 for $m \in \{3, 4\}$. Split the integral into two parts and estimate them separately:

$$\int_0^\infty \left|j_{\frac{m}{2}}(s)\right|^p \mathrm{d}s = \int_0^{\frac{m}{2}+3} \left|j_{\frac{m}{2}}(s)\right|^p \mathrm{d}s + \int_{\frac{m}{2}+3}^\infty \left|j_{\frac{m}{2}}(s)\right|^p \mathrm{d}s$$

For the first integral, we use the pointwise estimate Lemma 2.9 and then Lemma 9.5.

$$\int_0^{\frac{m}{2}+3} \left|j_{\frac{m}{2}}(s)\right|^p \mathrm{d}s \leq \frac{\sqrt{\pi}}{\sqrt{p}}\sqrt{\frac{m}{2}+1}\left(1-\frac{3}{4}\frac{1}{p(m+4)}+\frac{105}{16}\frac{1}{2p^2(m+4)^2}\right) \tag{9.21}$$

For the second integral, we estimate the integrand pointwise by Lemma 2.11. This gives

$$\begin{aligned}\int_{\frac{m}{2}+3}^{\infty} \left|j_{\frac{m}{2}}(s)\right|^p \mathrm{d}s &\leq \left(2^{\frac{m+1}{2}}\frac{\Gamma(\frac{m}{2}+1)}{\sqrt{\pi}}\frac{\sqrt{m+6}}{\sqrt[4]{12m+36}}\right)^p \int_{\frac{m}{2}+3}^{\infty} s^{-\frac{m+1}{2}p}\mathrm{d}s \\ &= \left(2^{\frac{m+1}{2}}\frac{\Gamma(\frac{m}{2}+1)}{\sqrt{\pi}}\frac{\sqrt{m+6}}{\sqrt[4]{12m+36}}\right)^p \frac{2}{(m+1)p-2}\left(\frac{m}{2}+3\right)^{1-\frac{m+1}{2}p}.\end{aligned} \tag{9.22}$$

With the estimates (9.21) and (9.22) of the two parts of the integral, it remains to prove the following inequality for $p \geq 2$ and $m \in \{3, 4\}$

$$\begin{aligned}&\frac{\sqrt{\pi}}{\sqrt{p}}\sqrt{\frac{m}{2}+1}\left(1-\frac{3}{4}\frac{1}{p(m+4)}+\frac{105}{16}\frac{1}{2p^2(m+4)^2}\right) \\ &+\left(2^{\frac{m+1}{2}}\frac{\Gamma(\frac{m}{2}+1)}{\sqrt{\pi}}\frac{\sqrt{m+6}}{\sqrt[4]{12m+36}}\right)^p \frac{2}{(m+1)p-2}\left(\frac{m}{2}+3\right)^{1-\frac{m+1}{2}p} \\ &\leq \frac{\sqrt{\pi}}{\sqrt{p}}\sqrt{\frac{m}{2}+1}.\end{aligned}$$

Subtract $\frac{\sqrt{\pi}}{\sqrt{p}}\sqrt{\frac{m}{2}+1}$ from both sides. For $m = 3$ this reads as

$$\frac{\sqrt{\frac{5}{2}\pi}}{\sqrt{p}}\left(\frac{15}{224p^2}-\frac{3}{28p}\right)+\left(\frac{9}{\sqrt[4]{2}\sqrt{6}}\right)^p\frac{2}{4p-2}\left(\frac{9}{2}\right)^{1-2p} \leq 0.$$

Multiplying by $p^{\frac{5}{2}}(4p-2)$ and simplifying, this reduces to show

$$-p^2\frac{3}{7}\sqrt{\frac{2}{5}\pi}+p\frac{27}{56}\sqrt{\frac{2}{5}\pi}-\frac{30}{224}\sqrt{\frac{2}{5}\pi}+\left(\frac{4}{9\sqrt[4]{2}\sqrt{6}}\right)^p 9p^{\frac{5}{2}} \leq 0. \tag{9.23}$$

The last summand of the left-hand side of (9.23) is decreasing in p for $p \geq 2$ and its value for $p = 2$ is less than $\frac{32}{27}$. So we estimate the left-hand side of (9.23) by a quadratic function and get

$$\begin{aligned}
&- p^2\frac{3}{7}\sqrt{\frac{2}{5}\pi} + p\frac{27}{56}\sqrt{\frac{2}{5}\pi} - \frac{30}{224}\sqrt{\frac{2}{5}\pi} + \left(\frac{4}{9\sqrt[4]{2}\sqrt{6}}\right)^p 9p^{\frac{5}{2}} \\
&\le -p^2\frac{3}{7}\sqrt{\frac{2}{5}\pi} + p\frac{27}{56}\sqrt{\frac{2}{5}\pi} - \frac{30}{224}\sqrt{\frac{2}{5}\pi} + \frac{32}{27}.
\end{aligned}$$

This function has its maximum in $p = \frac{9}{16}$, so it is decreasing for $p \geq 2$. For $p = 2$ the value is $-\frac{99}{224}\sqrt{10}\sqrt{\pi} + 32/27 < 0$. This proves the inequality.

This argument works analogously for $m = 4$. □

9.3. Bounds for sections

Finally, we give the main theorem on sections of cylinders.

Theorem 9.6. *Let $n > 1, m > 1$ and $r > 0$. Then for all $a \in \mathbb{R}^{n+m}$ with $\|a\| = 1$,*

$$\mathrm{vol}_{n+m-1}(H_a \cap Z) \leq \begin{cases} r^m \frac{\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}+1\right)} \cdot \sqrt{2}, & r \geq \frac{\Gamma\left(\frac{m}{2}+1\right)}{\Gamma\left(\frac{m}{2}+\frac{1}{2}\right)} \frac{1}{\sqrt{\pi}} \\ r^{m-1} \frac{\pi^{\frac{m-1}{2}}}{\Gamma\left(\frac{m-1}{2}+1\right)} \cdot \sqrt{2}, & r < \frac{\Gamma\left(\frac{m}{2}+1\right)}{\Gamma\left(\frac{m}{2}+\frac{1}{2}\right)} \frac{1}{\sqrt{\pi}}. \end{cases} \tag{9.24}$$

For $r \geq \frac{\Gamma\left(\frac{m}{2}+1\right)}{\Gamma\left(\frac{m}{2}+\frac{1}{2}\right)} \frac{1}{\sqrt{\pi}}$, the bound is attained for $a = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \dots, 0\right)$.

Remarks: (i) We did not touch the question if the distinction of the cases in (9.24) is sharp. In Theorem 8.3 the distinction of the cases is sharp. In this theorem, for $n = 1$ and $m = 2$ the critical radius would be equal to $\frac{\Gamma(2)}{\Gamma(3/2)} \frac{1}{\sqrt{\pi}} \frac{2}{\pi}$, which is much larger than the critical radius $\frac{1}{2\sqrt{3}}$ from Theorem 8.3.

(ii) For the three-dimensional cylinder we found that a *truncated* ellipse gives maximal volume for large r. For the generalized cylinder there is a different behavior. The volume-maximal section of the cylinder is the Cartesian product of the maximal section of the cube and a ball of dimension m. For example, for a 4-dimensional cylinder, i.e. $n = 2 = m$, for large r the maximal section is a three-dimensional cylinder of height $\sqrt{2}$ and radius r.

(iii) We conjecture that, if r is sufficiently small, the section orthogonal to $a = (0, \dots, 0, 1, 0, \dots, 0)$ is maximal, where the $(n+1)$-th coordinate of a is 1. The volume of this section is equal to

$$\mathrm{vol}_n\left(\frac{1}{2}B_\infty^n\right) \mathrm{vol}_{m-1}\left(rB_2^{m-1}\right) = r^{m-1}\frac{\pi^{\frac{m-1}{2}}}{\Gamma\left(\frac{m-1}{2}+1\right)}.$$

Comparing this to the bound from (9.24), there is an error of $\sqrt{2}$.
Numerical experiments suggest that for medium sized r, some non-standard direction is

maximal.

(iv) As Theorem 8.3 shows, there is a critical value of the radius that originates in the geometry of the cylinder. For generalized cylinders an additional distinction comes from the method, and this does not give the geometric distinction as in Theorem 8.3.

Proof of Theorem 9.6. We have to distinguish between normal vectors without a dominating coordinate (case 1) and with a dominating coordinate (cases 2 and 3). The integral inequality (9.3) and those from Theorem 9.3 may only be applied in case 1. For cases 2 and 3, we use a different estimate that is also used in Ball's proof for example [Bal86].

Case 1: Let $|a_j| \leq \frac{1}{\sqrt{2}}$ for all $j = 1, \ldots, n+1$, so there is no dominating coordinate. We apply the integral inequalities (9.3) and those from Theorem 9.3 to Lemma 9.2. For the third inequality, note that $\frac{1}{r}\sqrt{\pi}\sqrt{\frac{m}{2}+1} < \sqrt{2}\pi$ if and only if $r > \frac{\sqrt{\frac{m}{2}+1}}{\sqrt{2}\pi}$, so

$$\begin{aligned}
\mathrm{vol}_{n+m-1}(H_a \cap Z) &\leq r^m \frac{\pi^{\frac{m}{2}-1}}{\Gamma(\frac{m}{2}+1)} \prod_{j=1}^{n} \left(\sqrt{2}\pi\right)^{a_j^2} \cdot \left(\frac{1}{r}\mathcal{J}_m\left(\frac{1}{a_{n+1}^2}\right)\right)^{a_{n+1}^2} \\
&\leq \frac{\pi^{\frac{m}{2}-1}}{\Gamma(\frac{m}{2}+1)} \left(\sqrt{2}\pi\right)^{\sum_{j=1}^{n} a_j^2} \left(\frac{1}{r}\sqrt{\pi}\sqrt{\frac{m}{2}+1}\right)^{a_{n+1}^2} \\
&\leq \begin{cases} r^m \frac{\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2}+1)}\sqrt{2}, & r > \frac{\sqrt{\frac{m}{2}+1}}{\sqrt{2}\pi} \\ r^{m-1} \frac{\pi^{\frac{m-1}{2}}}{\Gamma(\frac{m}{2}+1)}\sqrt{\frac{m}{2}+1}, & r \leq \frac{\sqrt{\frac{m}{2}+1}}{\sqrt{2}\pi}. \end{cases}
\end{aligned}$$

Case 2: Let $|a_j| > \frac{1}{\sqrt{2}}$ for some $j = 1, \ldots, n$. Let P be the projection onto the hyperplane $\{x_j = 0\}$. Since $P(H \cap Z) \subset P(Z)$, we have $\mathrm{vol}(P(H \cap Z)) \leq \mathrm{vol}(P(Z))$. The projected cylinder $P(Z)$ is isomorphic to $\frac{1}{2}B_\infty^{n-1} \times rB_2^m$, so the volume can be computed elementary. Furthermore,

$$\mathrm{vol}_{n+m-1}(H_a \cap Z) = \frac{1}{|a_j|}\,\mathrm{vol}_{n+m-1}(P(H_a \cap Z)).$$

Therefore

$$\begin{aligned}
\mathrm{vol}_{n+m-1}(H_a \cap Z) &< \sqrt{2}\,\mathrm{vol}_{n+m-1}(P(Z)) \\
&= \sqrt{2} r^m \frac{\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}+1\right)}.
\end{aligned}$$

Case 3: Let $|a_j| > \frac{1}{\sqrt{2}}$ for $j = n+1$. We consider the projection onto $\{x_{n+1} = 0\}$. Now $P(Z)$ is isomorphic to $\frac{1}{2}B_\infty^n \times B_2^{m-1}$. By the same argument as in case 2,

$$\mathrm{vol}_{n+m-1}(H_a \cap Z) < \sqrt{2} r^{m-1} \frac{\pi^{\frac{m-1}{2}}}{\Gamma(\frac{m-1}{2}+1)}. \tag{9.25}$$

We summarize the estimates. Note that by Lemma 2.3 for $m \geq 2$:

$$\frac{\Gamma\left(\frac{m}{2}+1\right)}{\Gamma\left(\frac{m}{2}+\frac{1}{2}\right)} \frac{1}{\sqrt{\pi}} > \frac{\sqrt{\frac{m}{2}+1}}{\sqrt{2\pi}}. \tag{9.26}$$

Let $r \geq \frac{\Gamma\left(\frac{m}{2}+1\right)}{\Gamma\left(\frac{m}{2}+\frac{1}{2}\right)} \frac{1}{\sqrt{\pi}}$. Due to (9.26), also $r > \frac{\sqrt{\frac{m}{2}+1}}{\sqrt{2\pi}}$. So in all three cases, we have $\mathrm{vol}_{n+m-1}(H_a \cap Z) \leq r^m \frac{\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}+1\right)} \sqrt{2}$. This bound is attained for the normal vector $a = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \ldots, 0\right)$.

If $r < \frac{\Gamma\left(\frac{m}{2}+1\right)}{\Gamma\left(\frac{m}{2}+\frac{1}{2}\right)} \frac{1}{\sqrt{\pi}}$, then the bound from case 3 is the largest.

□

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Erklärung

Hiermit erkläre ich, dass diese Abhandlung - abgesehen von der Beratung durch meinen Betreuer Herrn Prof. König - nach Inhalt und Form meine eigene Arbeit ist und dass diese Abhandlung unter Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft entstanden ist.
Diese Arbeit hat weder ganz noch zum Teil an anderer Stelle im Rahmen eines Prüfungsverfahrens vorgelegen. Teile der Kapitel 3 und 4 dieser Arbeit wurden bereits auf *www.arxiv.org*[1] veröffentlicht. Davon abgesehen wurde diese Arbeit nicht veröffentlicht oder zur Veröffentlichung eingereicht.

Kiel, den 15.12.2015

Hauke Dirksen

[1]Dirksen, Hauke: Sections of the regular Simplex - Volume formulas and estimates, arXiv preprint, arXiv:1509.06408 (Version 1 am 21.09.2015)