Indecomposable Modules and AR-Components of Domestic Finite Group Schemes

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Abstract

In representation theory one studies modules to get an insight of the linear structures in a given algebraic object. Thanks to the theorem of Krull-Remak-Schmidt, any finite-dimensional module over a finite-dimensional algebra can be decomposed in a unique way into indecomposable modules. In this way, one reduces this problem to the study of indecomposable modules. In this work we are interested in representation theory of the group algebra of a finite group scheme. Examples of these algebras are given by group algebras of ordinary groups or by universal enveloping algebras of Lie algebras. Our main interest lies in the finite group schemes of domestic representation type. By definition, in each dimension all but finitely many indecomposable modules of their group algebras are parametrized by a bounded number of parameters. One of the main results in this work provides a full classification of the indecomposable modules for a certain subclass of the domestic finite group schemes.

Based on this classification we will make some observations regarding the Auslander-Reiten quiver and geometric invariants which lead us to more general results. With the Auslander-Reiten quiver of an algebra one can describe its indecomposable modules and their irreducible morphisms. The vertices of this quiver are the isomorphism classes of indecomposable modules and the arrows correspond to irreducible morphisms between these modules. The shape of these quivers is well understood for the algebras we are investigating in this work. We will give a concrete description of the Euclidean components with respect to the McKay quiver of a certain binary polyhedral group scheme. The McKay quiver of these group schemes consists of their simple modules and the arrows are determined by tensor products with a given module.

An important fact for this work is that the module category of a group scheme is closed under taking tensor products. Besides the classification and the McKay-quivers we also use them for obtaining geometric invariants of a group scheme and its modules. Friedlander and Suslin proved that the even cohomology ring of a finite group scheme is a finitely generated commutative algebra. The variety defined by this algebra is the cohomological support variety of the group scheme. These varieties contain many interesting information about the representation theory of the group schemes they are assigned to. In this work we will study the ramification index of a morphism between two support varieties. As we will see, this number has a connection to the ranks of the tubes in the Auslander-Reiten quivers.

Zusammenfassung

In der Darstellungstheorie studiert man Moduln, um einen Einblick in die linearen Strukturen eines gegebenen algebraischen Objektes zu erhalten. Laut dem Satz von Krull-Remak-Schmidt kann über einer endlich-dimensionalen Algebra jeder endlich-dimensionale Modul in eindeutiger Weise in unzerlegbare Moduln zerlegt werden. Auf diese Weise reduziert man dieses Problem auf das Studium unzerlegbarer Moduln. In dieser Arbeit interessieren wir uns für die Darstellungstheorie der Gruppenalgebra eines endlichen Gruppenschemas. Beispiele für diese Algebren sind Gruppenalgebren von gewöhnlichen Gruppen sowie die universelle einhüllende Algebra einer Lie-Algebra. Unser Hauptinteresse liegt in den endlichen Gruppenschemata von domestischem Darstellungstyp. Per Definition sind in jeder Dimension alle bis auf endlich viele unzerlegbare Moduln dieser Gruppenalgebren durch eine beschränkte Anzahl von Parametern parametrisiert. Eines der Hauptergebnisse dieser Arbeit liefert eine vollständige Klassifikation der unzerlegbaren Moduln für eine gewisse Unterklasse von domestischen endlichen Gruppenschemata.

Basierend auf dieser Klassifikation machen wir einige Beobachtungen bezüglich des Auslander-Reiten-Köchers und geometrischen Invarianten, welche uns auch zu allgemeineren Ergebnissen führen. Mit dem Auslander-Reiten-Köcher einer Algebra kann man die unzerlegbaren Moduln sowie ihre irreduziblen Morphismen beschreiben. Die Punkte dieses Köchers sind die Isomorphieklassen von unzerlegbaren Moduln und seine Pfeile entsprechen den irreduziblen Abbildungen zwischen diesen Moduln. Für die Algebren, die wir in dieser Arbeit untersuchen, ist die Form dieses Köchers allgemein bekannt. Wir werden eine konkrete Beschreibung der Euklidischen Komponenten in Bezug auf den McKay-Köcher eines gewissen binären Polyeder-Gruppenschemas geben. Der McKay Köcher dieses Gruppenschemas besteht aus seinen einfachen Moduln und die Pfeile sind durch das Tensorprodukt mit einem bestimmten Modul festgelegt.

Eine wichtige Tatsache in dieser Arbeit ist, dass die Modulkategorie eines Gruppenschemas abgeschlossen unter der Bildung von Tensorprodukten ist. Neben der Klassifikation und den McKay-Köchern benutzen wir sie um geometrische Invarianten eines Gruppenschemas und seiner Moduln zu erhalten. Friedlander und Suslin haben gezeigt, dass der gerade Kohomologiering eines endlichen Gruppenschemas eine endlich erzeugte kommutative Algebra ist. Die durch diese Algebra definierte Varietät ist die kohomologische Trägervarietät des Gruppenschemas. Diese Varietäten enthalten viele interessante Informationen über die Darstellungstheorie der Gruppenschemata, welchen sie zugeordnet sind. In dieser Arbeit studieren wir den Verzweigungsindex eines Morphismus zwischen zwei Trägervarietäten. Wie wir sehen werden, hat diese Zahl eine Verbindung zu den Rängen der Röhren in den Auslander-Reiten-Köchern.

Contents

1. Preliminaries

1.1. Introduction

In representation theory we have a trichotomy of representation types. Over an algebraically closed field any algebra A has either finite, tame or wild representation type. We say that A has finite representation type if A possesses only finitely many isomorphism classes of indecomposable modules. An algebra A has tame representation type if it is not of finite representation type and if in each dimension almost all isomorphism classes of indecomposable modules occur in only finitely many one-parameter families.
The representation type of group algebras of finite groups was determined in [4]. Let k be a field of characteristic $p > 0$ and G be a p-group. Then the group algebra kG has tame representation type if and only if $p = 2$ and G is a dihedral, semidihedral or generalized quaternion group. The classification of modules for a tame algebra can be a hard endeavour. For example up to now there is no such classification for the quaternion group Q_8.
Another class of examples are tame hereditary algebras ([8]). These algebras have the additional property, that the number of one-parameter families is uniformly bounded. In general an algebra with this property is called of domestic representation type. The only p-group with domestic group algebra is the Klein four group. Its representation theory is clearly related to that of the 2-Kronecker quiver.
In the setting of group algebras for finite group schemes there occur more domestic group algebras. We call a finite group scheme $\mathcal{G}$ domestic if its group algebra $k\mathcal{G} := (k[\mathcal{G}])^*$ is domestic.
In [17] Farnsteiner classified the domestic finite group schemes over a field of characteristic $p > 2$. Let $\mathcal{G}$ be a domestic finite group scheme, then the principal block of $k\mathcal{G}$ is Morita-equivalent to the trivial extension of a radical square zero tame hereditary algebra. Moreover, the principal blocks of these group schemes are isomorphic to the principal blocks of certain domestic finite group schemes, the so-called amalgamated polyhedral group schemes.
The goal of this work is the classification of the indecomposable modules of the amalgamated polyhedral group schemes. A foundation for this is Premets work ([41]) on the representation theory of the restricted Lie algebra $\mathfrak{sl}(2)$. Farnsteiner started in [15] to extend these results to the infinitesimal case, the group schemes $SL(2)_1T_r$ for $r \geq 1$. These results will be summarized in section 4.1. The missing part was a realization of the periodic $SL(2)_1T_r$-modules. This gap will be closed in section 4.3:

Let $\mathcal{G}$ be a finite group scheme and $\mathcal{N}$ a normal subgroup scheme of $\mathcal{G}$ such that $\mathcal{G}/\mathcal{N}$ is infinitesimal. For an $\mathcal{N}$-module Z Voigt [51] introduced a filtration

$$Z = N_0 \subseteq N_1 \subseteq N_2 \subseteq \ldots \subseteq N_{n-1} \subseteq k\mathcal{G} \otimes_{k\mathcal{N}} Z$$

by $\mathcal{N}$-modules and used it to give a generalized version of Clifford theory in form of a splitting criterion of the short exact sequences

$$0 \to N_{l-1} \longrightarrow N_l \longrightarrow N_l/N_{l-1} \to 0.$$

We develop in section 4.2 a criterion which ensures that none of these sequences split. For certain quasi-simple modules Z this result implies that all constituents of the filtration belong to the same AR-component. In section 4.3 we show when these assumptions are true for modules over $SL(2)_1$. Therefore we obtain new realizations of those modules and are able to show when they have an $SL(2)_1T_r$-module structure. These are exactly those modules which were missing in [15].
We then turn to a different topic in section 5.1. If $\mathcal{G}$ is a finite group scheme then $G := \mathcal{G}(k)$ acts on the projectivized rank variety $\mathbb{P}(V_{\mathfrak{g}})$ where $\mathfrak{g}$ denotes the restricted Lie algebra of $\mathcal{G}$. This action gives nice properties for the stabilizers of periodic modules. If the action of G on $\mathbb{P}(V_{\mathfrak{g}})$ is faithful and the variety $\mathbb{P}(V_{\mathfrak{g}})$ is smooth and irreducible, then the stabilizer G_M of any periodic module M is contained in $GL_r(k)$, where r is the dimension of $\mathbb{P}(V_{\mathfrak{g}})$. Especially, if the connected component $\mathcal{G}^0$ of $\mathcal{G}$ is tame, we obtain that the stabilizer G_M is cyclic.
The goal of section 5.2 is to prove a generalized Clifford theory decomposition result of induced modules for certain group schemes. For this we need a normal subgroup scheme $\mathcal{N}$ of a finite group scheme $\mathcal{G}$ which is contained in $\mathcal{G}^0$ such that $\mathcal{G}^0/\mathcal{N}$ is multiplicative. The indecomposable $\mathcal{N}$-modules in consideration need to be restrictions of $\mathcal{G}$-modules and under the assumption of an additional stability criterion the decomposition of the induced module corresponds to the decomposition of $k(\mathcal{G}/\mathcal{N})$ into projective indecomposable modules.
In section 2.4 we pick up results of [32] about the application of Clifford theory over strongly group graded algebras to Auslander-Reiten quivers. We analyse the effects of the restriction functor between components of the occurring Auslander-Reiten quivers for cyclic groups. Especially, if the components are tubes, we can give a relation between their ranks.
Section 5.3 combines the results of prior sections to describe the structure of the amalgamated polyhedral group schemes. Now we are able to give a complete classification of the indecomposable modules for these group schemes in chapter 8.

Thanks to this classification, we obtain many new examples of modules for finite group schemes. Consequently we can use them to test conjectures or to search for new general results of their representation theory.
In this work we will use these results to get a better understanding of the Auslander-Reiten components of a domestic finite group scheme. As noted above, any such group scheme can be associated to an amalgamated polyhedral group scheme and the non-simple blocks of an amalgamated polyhedral group scheme are Morita-equivalent to a radical square zero tame hereditary algebra. In this way the components of the Auslander-Reiten quiver of these group schemes are classified abstractly. Our goal is to describe these components in a direct way by using tensor products, McKay-quivers and ramification indices of certain morphisms.

In chapter 7 we will describe the Euclidean components. For this purpose, we show in chapter 6 how to extend certain almost split sequences over a normal subgroup scheme $\mathcal{N} \subseteq \mathcal{G}$ to almost split sequences over $\mathcal{G}$ if the group scheme $\mathcal{G}/\mathcal{N}$ is linearly reductive.

Moreover, for a simple $\mathcal{G}/\mathcal{N}$-module S we will show that the tensor functor $-\otimes_k S$ sends these extended almost split sequences to almost split sequences. In chapter 7 we will use these results to show that for any amalgamated polyhedral group scheme $\mathcal{G}$ there is a finite linearly reductive subgroup scheme $\tilde{\mathcal{G}} \subseteq SL(2)$ such that the Euclidean components of $\Gamma_s(\mathcal{G})$ can be explicitly described by the McKay-quiver $\Upsilon_{L(1)}(\tilde{\mathcal{G}})$.
Of great importance for the proofs of these results is the fact that the category of $\mathcal{G}$-modules is closed under taking tensor products over the field k. This comes into play in the definition of the McKay quiver, for the construction of new almost split sequences and in the description of the Euclidean components. Thanks to this property, we are also able to introduce geometric invariants for the representation theory of $\mathcal{G}$. If $\mathcal{G}$ is any finite group scheme, one can endow the even cohomology ring $H^\bullet(\mathcal{G},k)$ with the structure of a commutative graded k-algebra. Thanks to the Friedlander-Suslin-Theorem ([22]), this algebra is finitely-generated. Therefore, the maximal ideal spectrum $\mathcal{V}_\mathcal{G}$ of $H^\bullet(\mathcal{G},k)$ is an affine variety. As $H^\bullet(\mathcal{G},k)$ is graded, we can also consider its projectivized variety $\mathbb{P}(\mathcal{V}_\mathcal{G})$.
Now let us again assume that $\mathcal{N} \subseteq \mathcal{G}$ is a normal subgroup scheme such that $\mathcal{G}/\mathcal{N}$ is linearly reductive. Then the ramification indices of the restriction morphism $\mathbb{P}(\mathcal{V}_\mathcal{N}) \to \mathbb{P}(\mathcal{V}_\mathcal{G})$ will give upper bounds for the ranks of the corresponding tubes in the Auslander-Reiten quiver. Here a tube $\mathbb{Z}/(r)[A_\infty]$ of rank r can be regarded as a quiver which is arranged on an infinite tube with circumference r. Moreover, if $\mathcal{G}$ is an amalgamated polyhedral group scheme and $\mathcal{N} = \mathcal{G}_1$ is its first Frobenius kernel, the ranks are equal to the corresponding ramification indices. Hence we will prove the following:

Theorem. *Let $\mathcal{G}$ be an amalgamated polyhedral group scheme and Θ a component of the stable Auslander-Reiten quiver $\Gamma_s(\mathcal{G})$. Then the following hold:*

(i) If Θ is Euclidean, then there is a component Q of the separated quiver $\Upsilon_{L(1)}(\tilde{\mathcal{G}})_s$ and a concrete isomorphism $\Theta \cong \mathbb{Z}[Q]$.

(ii) Let Θ be a tube and e_Θ the ramification index of the restriction morphism $\mathbb{P}(\mathcal{V}_{\mathcal{G}_1}) \to \mathbb{P}(\mathcal{V}_\mathcal{G})$ at the corresponding point x_Θ. Then $\Theta \cong \mathbb{Z}/(e_\Theta)[A_\infty]$.

There seems to be a connection to a result of Crawley-Boevey ([6]), which states that a finite-dimensional tame algebra has only finitely many non-homogeneous tubes. On the other hand, the restriction morphism $\mathbb{P}(\mathcal{V}_\mathcal{N}) \to \mathbb{P}(\mathcal{V}_\mathcal{G})$ is finite and has constant ramification on an open dense subset of $\mathbb{P}(\mathcal{V}_\mathcal{N})$. Hence, there are only finitely many exceptional ramification points. In our situation, all but finitely many points will be unramified and a tube can only be non-homogeneous, if it belongs to the image of a ramification point.

1.2. Notation and Prerequisites

If not otherwise mentioned, k will always denote an algebraically closed field of characteristic $p > 2$ and all modules and algebras occurring in this work are supposed to be finite-dimensional over k.

In the following sections we will give a short introduction to some concepts and results that are used in this work. Introductions to representation theory can be found in [1], [2] and [3]. Throughout this work we will use tools from homological algebra and category theory. For these topics we refer the reader to [45], [53] and [31]. Moreover, some knowledge in algebraic geometry is helpful. Thorough introductions to this topic may be found in [23], [7] and [25].

1.3. Group graded algebras

In this section we will give a short overview of the theory of group graded algebras. For more details, we refer the reader to [26]. In the following, k will always denote an arbitrary field.

Definition 1.3.1. Let G be a group and A be a k-algebra which admits a decomposition $A = \bigoplus_{g \in G} A_g$ as k-vector spaces. Then A is called G-graded if for all $g, h \in G$ we have $A_g A_h \subseteq A_{gh}$. If we always have equality, the algebra A is called strongly G-graded.

Remark 1.3.2. Let G be a group and A be a G-graded k-algebra.

1. Let $H \subseteq G$ be a subgroup. Then the subalgebra $A_H := \bigoplus_{g \in H} A_g$ is H-graded. If A is strongly G-graded, then A_H is strongly H-graded.

2. Let $N \subseteq G$ be a normal subgroup of G. Then A can be regarded as a G/N-graded algebra via $A_{gN} := \bigoplus_{x \in gN} A_x$ for all $g \in G$. If A is strongly G-graded, then it is also strongly G/N-graded.

3. For all $g \in G$, the space A_g is an (A_1, A_1)-bimodule.

Example 1.3.3. If G acts on a k-algebra A by algebra automorphisms, we let $A * G$ be a free A-module with basis G and multiplication

$$(rg)(sh) = rg(s)gh \quad \text{for all } r, s \in A \text{ and } g, h \in G.$$

This algebra is a strongly G-graded algebra and called the skew group algebra of G over A. If the operation of G is trivial we get the group algebra AG of G over A.

Definition 1.3.4. Let G be a group, A be a G-graded k-algebra and $H \leq U \leq G$ be subgroups. Then

$$\mathrm{ind}_H^U := \mathrm{ind}_{A_H}^{A_U} : \mathrm{mod}\, A_H \to A_U, M \mapsto A_U \otimes_{A_H} M$$

is the induction functor and

$$\mathrm{res}_H^U := \mathrm{res}_{A_H}^{A_U} : \mathrm{mod}\, A_U \to A_H, M \mapsto M|_{A_H}$$

is the restriction functor. For $H = \{1\}$ we will write ind_1^U and res_1^U.

Definition 1.3.5. Let G be a group, A be a G-graded algebra and M be an A_1-module. For $g \in G$ we denote by M^g the A_1-module $A_g \otimes_{A_1} M$.
The subgroup $G_M := \{g \in G \mid M^g \cong M\}$ is called the stabilizer of M. If $G = G_M$ we say that M is G-invariant.

Remark 1.3.6. If $A * G$ is a skew group algebra and M an A-module, then M^g can be identified as a k-space with M and A_1-action twisted by g^{-1}, i.e.

$$a.m := g^{-1}(a)m \quad \text{for all } a \in A_1 \text{ and } m \in M.$$

To conclude this section, we will give some results concerning this topic which will come up later in this work.

Lemma 1.3.7 ([38, Corollary 2.10])**.** *Let G be a finite group and A be a strongly G-graded k-algebra. Then A is self-injective if and only if A_1 is self-injective.*

Theorem 1.3.8 ([26, 4.5.2])**.** *Let G be a finite group and A be a finite-dimensional strongly G-graded k-algebra. Let M be an A_1-module and $\operatorname{ind}_1^{G_M} M = \bigoplus_{i=1}^n M_i$ be a decomposition into indecomposable A_{G_M}-modules. Then $\operatorname{ind}_1^G M = \bigoplus_{i=1}^n \operatorname{ind}_{G_M}^G M_i$ is a decomposition into indecomposable A-modules. Moreover, $\operatorname{ind}_{G_M}^G M_i \cong \operatorname{ind}_{G_M}^G M_j$ if and only if $M_i \cong M_j$.*

Corollary 1.3.9. *Let G be a finite group, $H \subseteq G$ a subgroup and A be a finite-dimensional strongly G-graded k-algebra. Let N be an A_1-module with $G_N \subseteq H$ and M be an indecomposable direct summand of $\operatorname{ind}_1^G N$. Then there is an indecomposable direct summand V of $\operatorname{res}_H^G M$ such that $\operatorname{ind}_H^G V \cong M$.*

Proof. Let $\operatorname{ind}_1^{G_N} N = \bigoplus_{i=1}^n U_i$ be a decomposition into indecomposable A_{G_N}-modules. By 1.3.8 this yields a decomposition $\operatorname{ind}_1^G N = \bigoplus_{i=1}^n \operatorname{ind}_{G_N}^G U_i$ into indecomposable A-modules and a decomposition $\operatorname{ind}_1^H N = \bigoplus_{i=1}^n \operatorname{ind}_{G_N}^H U_i$ into indecomposable A_H-modules. Assume $M = \operatorname{ind}_{G_N}^G U_i$. Then $V := \operatorname{ind}_{G_N}^H U_i$ is an indecomposable direct summand of $\operatorname{res}_H^G M$ with $\operatorname{ind}_H^G V = M$. □

Proposition 1.3.10 ([26, 4.5.15, 4.5.17])**.** *Let k be an algebraically closed field of characteristic p, G be a finite cyclic group of order n such that $p \nmid n$, A be a strongly G-graded k-algebra and M be a finite-dimensional indecomposable G-invariant A_1-module. Then $\operatorname{ind}_1^G M$ has a decomposition $\bigoplus_{i=1}^n N_i$ into indecomposable A-modules such that $\operatorname{res}_1^G N_i = M$ for all $i \in \{1, \dots, n\}$.*

1.4. Hopf algebras and Hopf-Galois extensions

We start this section by giving a short introduction to the theory of Hopf algebras. After that, we will introduce Hopf-Galois extensions. These extensions are a generalization of strongly group graded algebras. We will also include an overview of some properties of these extensions, which we will use later in this work. For more details we refer the reader to [33] and [47].

Definition 1.4.1. Let k be a field. A tuple $(H, m, u, \Delta, \varepsilon, \eta)$ is called Hopf algebra if the following holds:

1. The tuple (H, m, u) is a k-algebra with multiplication $m : H \otimes_k H \to H$ and unit $u : k \to H$.
2. The tuple (H, Δ, ε) is a k-coalgebra, i.e. $\Delta : H \to H \otimes_k H$ and $\varepsilon : H \to k$ are k-linear maps such that
   a) $(\mathrm{id}_H \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}_H) \circ \Delta$ and
   b) $(\mathrm{id}_H \otimes \varepsilon) \circ \Delta = \mathrm{id}_H = (\varepsilon \otimes \mathrm{id}_H) \circ \Delta$.

   The map Δ is called comultiplication and the map ε is called counit.
3. The maps Δ and ε are k-algebra homomorphisms (Or equivalently, the maps m and u are k-coalgebra homomorphisms).
4. The map $\eta : H \to H$ is k-linear such that

$$u \circ \varepsilon(h) = \sum_{(h)} h_{(1)} \eta(h_{(2)}) = \sum_{(h)} \eta(h_{(1)}) h_{(2)} \quad \text{for all } h \in H.$$

   The map η is called the antipode of H.

Remark 1.4.2. In the last property we used the Sweedler notation. For each $h \in H$ we write $\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}$.

Definition 1.4.3. Let H be a Hopf algebra and $\tau : H \otimes_k H \to H \otimes_k H$ with $\tau(a \otimes b) = b \otimes a$ for $a, b \in H$. Then H is called cocommutative if $\tau \circ \Delta = \Delta$.

Definition 1.4.4. Let H be a Hopf algebra. A subalgebra $K \subseteq H$ of the k-algebra H is called Hopf subalgebra of H if:

1. $\Delta(K) \subseteq K \otimes_k K$.
2. $\eta(K) \subseteq K$.

Definition 1.4.5. Let H be a Hopf algebra. An ideal $I \subseteq H$ of the k-algebra H is called Hopf ideal of H if:

1. $\Delta(I) \subseteq I \otimes_k H + H \otimes_k I$.
2. $\varepsilon(I) = 0$.
3. $\eta(I) \subseteq I$.

Remark 1.4.6.

1. The ideal $H^\dagger := \ker \varepsilon$ is a Hopf ideal of H. It is called the augmentation ideal of H.
2. If I is a Hopf ideal of H, then H/I is a Hopf algebra.

Definition 1.4.7. Let H be a Hopf algebra. The map $\mathrm{Ad}_l : H \to \mathrm{End}_k(H)$ with $\mathrm{Ad}_l(h)(x) = \sum_{(h)} h_{(1)} x \eta(h_{(2)})$ for $h, x \in H$ is called the left adjoint representation of H. Dually, the map $\mathrm{Ad}_r : H \to \mathrm{End}_k(H)$ with $\mathrm{Ad}_r(h)(x) = \sum_{(h)} \eta(h_{(1)}) x h_{(2)}$ for $h, x \in H$ is called the right adjoint representation of H.
A Hopf subalgebra $K \subseteq H$ is called normal, if it is invariant under both adjoint representations.

Remark 1.4.8. If K is a normal Hopf subalgebra of a Hopf algebra H, then $HK^\dagger$ is a Hopf ideal of H.

Definition 1.4.9. Let H be a Hopf algebra. A k-vector space M is called an H-comodule, if there is a k-linear map $\rho_M : M \to M \otimes_k H$ such that

1. $(\mathrm{id}_M \otimes \Delta) \circ \rho_M = (\rho_M \otimes \mathrm{id}_H) \circ \rho_M$, and
2. $(\mathrm{id}_M \otimes \varepsilon) \circ \rho_M = \mathrm{id}_M \otimes 1$.

If M is an H-comodule, then the subspace

$$M^{\mathrm{co}H} := \{m \in M \mid \rho_M(m) = m \otimes 1\}$$

is called the space of H-coinvariants in M.
If M is an H-module, then the subspace

$$M^H := \{m \in M \mid h.m = \varepsilon(h)m \text{ for all } h \in H\}$$

is called the space of H-invariants in M.

Definition 1.4.10. Let H be a Hopf algebra over k and A be a k-algebra.

1. The algebra A is called an H-comodule algebra if it is an H-comodule such that the comodule map $\rho_A : A \to A \otimes_k H$ is an algebra homomorphism. Denote by $B := A^{\mathrm{co}H}$ the coinvariants of H. Then $A : B$ is called an H-extension.
2. An H-extension $A : B$ is called H-Galois if the map $\beta : A \otimes_B A \to A \otimes_k H$ with $\beta(a \otimes b) = a\rho_A(b)$ is bijective.
3. The algebra A is called an H-module algebra if
   a) A is an H-module,
   b) $h.(ab) = \sum_{(h)} (h_{(1)}.a)(h_{(2)}.b)$ for all $h \in H$ and $a, b \in A$, and
   c) $h.1 = \varepsilon(h)1$ for all $h \in H$.
4. Let A be an H-module algebra. Then the smash product $A\#H$ is the algebra with underlying space $A \otimes_k H$ and multiplication

   $$(a\#h)(b\#k) = \sum_{(h)} a(h_{(1)}b)\#h_{(2)}k$$

   for all $a, b \in A$ and $h, k \in H$.

Remark 1.4.11. Let $L : k$ be a field extension, $G \subseteq \mathrm{Aut}_k(L)$ a finite subgroup and $K = L^G$ the subfield of G-invariants. If $H = k[G] = (kG)^*$, then L is a H-comodule algebra. One can show (c.f. [34, 2.3]) that $L : K$ is a Galois extension in the classical sense if and only if $L : K$ is an H-Galois extension.

Example 1.4.12.
1. The smash product algebra $A\#H$ gives rise to an H-Galois extension $A\#H : A$.
2. If $H = kG$ is the group algebra of a group, then the smash product $A\#H$ is isomorphic to the skew group algebra $A * G$.
3. Let H be a Hopf algebra with normal Hopf subalgebra $K \subseteq H$. Set $\bar{H} := H/(HK^\dagger)$. Then $H : K$ is an $\bar{H}$-Galois extension.

Let $A : B$ be an H-Galois extension and M be an A-module. Then $\mathrm{End}_B(M)$ is an H-module algebra via

$$(h.f)(m) = \sum_{i=1}^{n} a_i f(b_i m)$$

for $h \in H$, $f \in \mathrm{End}_B(M)$, $m \in M$ and $\sum_{i=1}^n a_i \otimes b_i = \beta^{-1}(1 \otimes \eta(h)) \in A \otimes_B A$.
The following result is an analogue of [50, 2.3] for left modules which itself is a generalization of [26, 4.5.4] from the group graded case:

Lemma 1.4.13. *Let H be a finite-dimensional Hopf-algebra, $A : B$ be a H-Galois extension of k-algebras and M be a B-module. Then* $\mathrm{End}_A(A \otimes_B M) \cong \mathrm{End}_B(M)\#H$.

Definition 1.4.14. Let H be a Hopf algebra over k and A be an H-module algebra. We define the following two radicals of A:

1. $\mathrm{Rad}^H(A) := \{x \in \mathrm{Rad}(A) \mid hx \in \mathrm{Rad}(A) \text{ for all } h \in H\}$ and
2. $\mathrm{Rad}_H(A) := \mathrm{Rad}(A\#H) \cap A$.

Proposition 1.4.15 ([55, 3.2, 3.3], [5, 4.3])**.** *Let H be a Hopf algebra over k and A be an H-module algebra. Then the following statements hold:*

1. $\mathrm{Rad}_H(A) \subseteq \mathrm{Rad}^H(A)$, *with equality if H is finite-dimensional,*
2. $\mathrm{Rad}_H(A)\#H \subseteq \mathrm{Rad}(A\#H)$, *and*
3. *if $H = kG$ is the group algebra of a finite group G, then* $\mathrm{Rad}(A)\#kG \subseteq \mathrm{Rad}(A\#kG)$.

Definition 1.4.16. We say that an extension $A : B$ of k-algebras is separable if the multiplication $A \otimes_B A \to A$ is a split surjective homomorphism of (A, A)-bimodules.

Remark 1.4.17. Let $A : B$ be a separable extension of k-algebras. Using basic properties of separable extensions ([39, 10.8]), we obtain for every indecomposable A-module M an indecomposable direct summand N of the B-module $\mathrm{res}^A_B M$ such that M is a direct summand of $\mathrm{ind}^A_B N$.

Proposition 1.4.18 ([9, 3.15])**.** *Let H be a finite-dimensional semisimple Hopf algebra over k and $A : B$ be an H-Galois extension. Then $A : B$ is separable.*

Definition 1.4.19. We say that an extension $A : B$ of k-algebras is a free Frobenius extension of first kind, if

(a) A is a finitely generated free B-module, and

(b) there is an (A, B)-bimodule isomorphism $A \to \mathrm{Hom}_B(A, B)$.

If $A : B$ is an extension of rings, the functor

$$\mathrm{coind}_B^A : \mathrm{mod}\, B \to \mathrm{mod}\, A,\ M \mapsto \mathrm{Hom}_B(A, M)$$

is called coinduction functor.

Theorem 1.4.20 ([36, 2.1])**.** *Let $A : B$ be a free Frobenius extension of first kind. Then the induction and coinduction functors are equivalent.*

Theorem 1.4.21 ([30, 1.7(5)])**.** *Let H be a Hopf algebra over k and $A : B$ be an H-Galois extension. Then $A : B$ is a free Frobenius extension of first kind.*

1.5. Finite group schemes

The goal of this section is to give a short introduction to affine group schemes. We will mainly concentrate on finite group schemes. For more details and more general results, we refer the reader to [25], [7] and [16].
For two commutative k-algebras R and S we denote by $\mathrm{Alg}_k(R, S)$ the set of k-algebra homomorphisms from R to S. Denote by M_k the category of commutative k-algebras, by Sets the category of sets and by Grp the category of groups.

Definition 1.5.1.

1. A functor $F : M_k \to \mathrm{Sets}$ is called representable if there is a commutative k-algebra A and a natural equivalence $F \simeq \mathrm{Alg}_k(A, -)$.

2. A representable functor $\mathcal{G} : M_k \to \mathrm{Grp}$ is called an (affine) group scheme. By Yonedas Lemma, the commutative k-algebra A with $\mathcal{G} \simeq \mathrm{Alg}_k(A, -)$ is uniquely determined, up to isomorphism. This algebra is called the coordinate ring of $\mathcal{G}$ and will be denoted by $k[\mathcal{G}]$.

3. Let $\mathcal{G}$ be a group scheme. A subfunctor $\mathcal{H} \subseteq \mathcal{G}$ is called subgroup scheme if there is an Hopf ideal $I \subseteq H$ such that

$$\mathcal{H}(R) = \{g \in \mathcal{G}(R) \mid g(I) = (0)\}$$

for every commutative k-algebra R. A subgroup scheme $\mathcal{N} \subseteq \mathcal{G}$ is called normal, if $\mathcal{N}(R)$ is a normal subgroup of $\mathcal{G}(R)$ for every commutative k-algebra R.

4. Let $k : k'$ be an extension of fields and $\mathcal{F} : M_k \to M_{k'}$ be the forgetful functor. If $\mathcal{G}$ is a group scheme over k', then $\mathcal{G}_k := \mathcal{G} \circ \mathcal{F}$ is the base change to k. A group scheme $\mathcal{H}$ over k is said to be defined over k', if there is a group scheme $\mathcal{G}$ over k' with $\mathcal{H} \cong \mathcal{G}_k$.

5. A group scheme $\mathcal{G}$ is called finite, if $k[\mathcal{G}]$ is finite-dimensional.

6. A group scheme $\mathcal{G}$ is called reduced, if $k[\mathcal{G}]$ is reduced.

Remark 1.5.2. If $\mathcal{G}$ is a group scheme, its coordinate ring $k[\mathcal{G}]$ is a commutative Hopf algebra. In particular, if $\mathcal{G}$ is finite, the k-linear dual $k[\mathcal{G}]^*$ is a cocommutative Hopf algebra.

Definition 1.5.3. Let $\mathcal{G}$ be a finite group scheme.

1. The algebra $k\mathcal{G} := k[\mathcal{G}]^*$ is called the group algebra of $\mathcal{G}$.

2. We call $|\mathcal{G}| := \dim_k k\mathcal{G}$ the order of $\mathcal{G}$.

3. If $k[\mathcal{G}]$ is local, we call $\mathcal{G}$ infinitesimal.

4. If $k\mathcal{G}$ is semisimple, we call $\mathcal{G}$ linearly reductive.

5. The group $X(\mathcal{G})$ of k-algebra homomorphisms $k\mathcal{G} \to k$ with multiplication given by the convolution product
$$(f * g)(h) = \sum_{(h)} f(h_{(1)})g(h_{(2)}) \qquad \forall f, g \in X(\mathcal{G}), h \in k\mathcal{G}$$
is called the character group of $\mathcal{G}$.

6. The group scheme $\mathcal{G}$ is called diagonalizable if the coordinate ring $k[\mathcal{G}]$ is isomorphic to the group algebra $kX(\mathcal{G})$. Over an algebraically closed field these group schemes are also called multiplicative.

Example 1.5.4. For $r \in \mathbb{N}$ let $\mu_{(r)}$ be the group scheme given by
$$\mu_{(r)}(R) = \{x \in R \mid x^r = 1\}$$
for every commutative k-algebra R. Then $k[\mu_{(r)}] \cong k[T]/(T^r - 1)$ and $k\mu_{(r)} \cong k^r$ as k-algebras. Hence $\mu_{(r)}$ is a linearly reductive finite group scheme. Now assume that k is a field of characteristic $p > 0$. If $p \nmid r$, then $\mu_{(r)}$ is reduced and if $r = p^n$, then $\mu_{(r)}$ is infinitesimal.

Definition 1.5.5. Let $\mathcal{G}$ be a group scheme and M be a k-vector space. Consider the functor $M_a : M_k \to \text{Sets},\ R \mapsto M \otimes_k R$. Then M is called a $\mathcal{G}$-module if there is a natural transformation $\rho : \mathcal{G} \times M_a \to M_a$ such that $\rho_R : \mathcal{G}(R) \times M_a(R) \to M_a(R)$ is an R-linear group action of $\mathcal{G}(R)$ on $M \otimes_k R$ for any commutative k-algebra R.
Let M be a $\mathcal{G}$-module. We denote by $M^{\mathcal{G}}$ the subspace of $\mathcal{G}$-invariants of M given by
$$M_a^{\mathcal{G}}(R) = \{m \in M_a(R) \mid g.m = m \text{ for all } g \in \mathcal{G}(R)\}$$
for every commutative k-algebra R.

Remark 1.5.6 ([25, I.5.5(6)]). Let $\mathcal{G}$ be a finite group scheme and $\mathcal{N} \subseteq \mathcal{G}$ be a normal subgroup scheme. Then the quotient $\mathcal{G}/\mathcal{N}$ is given by the coordinate ring $k[\mathcal{G}/\mathcal{N}] := k[\mathcal{G}]^{\mathcal{N}}$.

Proposition 1.5.7 ([16, I.5.2]). *Let $\mathcal{G}$ be a finite group scheme and $\mathcal{N} \subseteq \mathcal{G}$ be a normal subgroup scheme. Then $k\mathcal{G}k\mathcal{N}^\dagger$ is a Hopf ideal of $k\mathcal{G}$ such that $k(\mathcal{G}/\mathcal{N}) \cong k\mathcal{G}/(k\mathcal{G}k\mathcal{N}^\dagger)$.*

Remark 1.5.8. Let $\mathcal{G}$ be a finite group scheme and ε the counit of the group algebra $k\mathcal{G}$. As an algebra, $k\mathcal{G}$ has a block decomposition

$$k\mathcal{G} = \mathcal{B}_0 \oplus \ldots \oplus \mathcal{B}_n,$$

where we assume that $\mathcal{B}_0$ is the block belonging to the trivial module k defined by ε. This block is called the principal block of $k\mathcal{G}$ and will be denoted by $\mathcal{B}_0(\mathcal{G})$.

Proposition 1.5.9 ([13, 1.1]). *Let $\mathcal{G}$ be a finite group scheme and $\mathcal{G}_{lr}$ be the largest linearly reductive normal subgroup scheme of $\mathcal{G}$. Then the canonical projection $k\mathcal{G} \to k(\mathcal{G}/\mathcal{G}_{lr})$ induces an isomorphism $\mathcal{B}_0(\mathcal{G}) \cong \mathcal{B}_0(\mathcal{G}/\mathcal{G}_{lr})$.*

Remark 1.5.10. Any finite group scheme decomposes into a semi-direct product $\mathcal{G}^0 \rtimes \mathcal{G}_{red}$ with an infinitesimal normal subgroup scheme $\mathcal{G}^0$ and a reduced group scheme $\mathcal{G}_{red}$. The group algebra $k\mathcal{G}$ is isomorphic to the skew group algebra $(k\mathcal{G}^0) * G$ where $G = \mathcal{G}(k)$. The subgroup scheme $\mathcal{G}^0$ is called the connected component of $\mathcal{G}$. Its coordinate ring $k[\mathcal{G}^0]$ is the principal block of the Hopf algebra $k[\mathcal{G}]$.

As in the case of group graded algebras one can define the stabilizer for a module of a normal subgroup scheme of a finite group scheme. The following construction – which will be used in some situations in this work – shows how these notions are connected in certain cases:

Lemma 1.5.11. *Let $\mathcal{G}$ be a finite group scheme, $\mathcal{N} \subseteq \mathcal{G}$ be a normal subgroup scheme with $\mathcal{G}^0 \subset \mathcal{N}$ and set $G := (\mathcal{G}/\mathcal{N})(k)$. Then $k\mathcal{G}$ has the structure of a G-graded k-algebra with $(k\mathcal{G})_1 = k\mathcal{N}$ and if M is an $\mathcal{N}$-module there is a unique subgroup scheme $\mathcal{G}_M$ of $\mathcal{G}$ with $k\mathcal{G}_M = (k\mathcal{G})_{G_M}$.*

Proof. As above, the group algebra $k\mathcal{G}$ is isomorphic to the skew group algebra $k\mathcal{G}^0 * \mathcal{G}(k)$. As $\mathcal{G}^0 \subseteq \mathcal{N}$, we therefore obtain that $k\mathcal{G}$ has the structure of a G-graded k-algebra with $(k\mathcal{G})_1 = k\mathcal{N}$. Let M be an $\mathcal{N}$-module. Then the Hopf-subalgebra $(k\mathcal{G})_{G_M}$ of $k\mathcal{G}$ determines a unique subgroup scheme $\mathcal{G}_M$ of $\mathcal{G}$ with $k\mathcal{G}_M = (k\mathcal{G})_{G_M}$. □

Theorem 1.5.12 ([15, 2.1.2]). *Let $\mathcal{G}$ be an infinitesimal group scheme and $\mathcal{N} \subseteq \mathcal{G}$ be a normal subgroup scheme of $\mathcal{G}$. Then the restriction functor*

$$\mathrm{res}^{\mathcal{G}}_{\mathcal{N}} : \mathrm{mod}\,\mathcal{G} \to \mathrm{mod}\,\mathcal{N},\ M \mapsto M|_{\mathcal{N}}$$

sends indecomposable modules to indecomposable modules.

Definition 1.5.13. Let $\mathcal{G}$ be a group scheme over a field k of characteristic $p > 0$. For $r \geq 0$ we denote by $k[\mathcal{G}]^{(r)}$ the k-algebra with same underlying space but with scalar multiplication given by

$$\alpha.f := \alpha^{p^{-r}} f \qquad \text{for all } \alpha \in k,\ f \in k[\mathcal{G}].$$

We denote by $\mathcal{G}^{(r)}$ the group scheme with coordinate ring $k[\mathcal{G}^{(r)}] = k[\mathcal{G}]^{(r)}$.
The k-algebra homomorphism $k[\mathcal{G}]^{(r)} \to k[\mathcal{G}],\ f \mapsto f^{p^r}$ induces a morphism $F_r : \mathcal{G} \to \mathcal{G}^{(r)}$ of group schemes. This morphism is called the r-th Frobenius morphism of $\mathcal{G}$. The group scheme $\mathcal{G}_r := \ker F_r$ is called the r-th Frobenius kernel of $\mathcal{G}$.

Remark 1.5.14 ([25, I.9])**.**

1. The group scheme $\mathcal{G}_r$ is infinitesimal.

2. If $\mathcal{G}$ is defined over $\mathbb{F}_p$, then there is an isomorphism $\mathcal{G} \cong \mathcal{G}^{(r)}$ of group schemes.

Proposition 1.5.15 ([16, I.3.5])**.** *Let $\mathcal{G}$ be a finite group scheme. Then $\mathcal{G}$ is infinitesimal if and only if there is an $r \geq 0$ with $\mathcal{G} = \mathcal{G}_r$.*

Definition 1.5.16. Let $\mathcal{G}$ be an infinitesimal group scheme. The number

$$\operatorname{ht}(\mathcal{G}) := \min\{r \in \mathbb{N}_0 \mid \mathcal{G} = \mathcal{G}_r\}$$

is called the height of $\mathcal{G}$.

Definition 1.5.17. Let k be a field of characteristic $p > 0$, $\mathcal{G}$ be a group scheme which is defined over $\mathbb{F}_p$ and M be a $\mathcal{G}$-module. Denote by $\rho : \mathcal{G} \to \mathrm{GL}(M)$ the corresponding representation. As $\mathcal{G}$ is defined over $\mathbb{F}_p$, we can regard the r-th Frobenius morphism as a morphism $F_r : \mathcal{G} \to \mathcal{G}$. We denote by $M^{[r]}$ the $\mathcal{G}$-module corresponding to the representation $\rho \circ F_r$. The module $M^{[r]}$ is called the r-th Frobenius twist of M.

Remark 1.5.18. Let $\mathcal{G}$ be a finite group scheme over a field of characteristic $p > 0$ and Δ be the comultiplication of $k\mathcal{G}$. Then the p-restricted Lie algebra

$$\operatorname{Lie}(\mathcal{G}) := \{x \in k\mathcal{G} \mid \Delta(x) = x \otimes 1 + 1 \otimes x\}$$

is called the Lie algebra of $\mathcal{G}$.

1.6. Support and rank varieties

Support varieties are very helpful geometric invariants, which enable us to use geometric methods in the study of finite group schemes and their representation theory. As we will only give a short overview to this topic, we refer the reader to [16],[20],[21] and [18] for further details.
Let k be an algebraically closed field of characteristic $p > 0$ and $(\mathfrak{g}, [p])$ be a restricted Lie algebra. We denote by $V_{\mathfrak{g}} = \{x \in \mathfrak{g} \mid x^{[p]} = 0\}$ the nullcone of $\mathfrak{g}$. For any $x \in V_{\mathfrak{g}}$ the

algebra $U_0(kx)$ is a subalgebra of $U_0(\mathfrak{g})$. For any $U_0(\mathfrak{g})$-module M we define its rank variety by

$$V_{\mathfrak{g}}(M) := \{x \in V_{\mathfrak{g}} \mid M|_{U_0(kx)} \text{ is not projective}\} \cup \{0\}.$$

The dimension of this rank variety is equal to the complexity $\mathrm{cx}_{\mathfrak{g}}(M)$ of the module M, i.e. the polynomial rate of growth of the dimensions of a minimal projective resolution of M (c.f. [19]).

Example 1.6.1. For $\mathfrak{g} = \mathfrak{sl}(2)$ the nullcone is given by

$$V_{\mathfrak{sl}(2)} = \{\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \mid a^2 + bc = 0\}.$$

Let $\mathcal{G}$ be a finite group scheme and M be a $\mathcal{G}$-module. We denote by

$$H^n(\mathcal{G}, M) := \mathrm{Ext}^n_{\mathcal{G}}(k, M)$$

the n-th cohomology of $\mathcal{G}$ with coefficients in M. We define

$$H^\bullet(\mathcal{G}, k) := \begin{cases} \bigoplus_{n \geq 0} H^{2n}(\mathcal{G}, k) & \text{if } p > 2 \\ \bigoplus_{n \geq 0} H^{n}(\mathcal{G}, k) & \text{if } p = 2. \end{cases}$$

Then the Yoneda product endows $H^\bullet(\mathcal{G}, k)$ with the structure of a commutative, graded k-algebra and $\mathrm{Ext}^*_{\mathcal{G}}(M, M)$ with the structure of an $H^\bullet(\mathcal{G}, k)$-module.

Theorem 1.6.2 (Friedlander-Suslin, [22])**.** *Let $\mathcal{G}$ be a finite group scheme and M be a finite-dimensional $\mathcal{G}$-module. Then*

1. *$H^\bullet(\mathcal{G}, k)$ is a finitely generated k-algebra.*
2. *$\mathrm{Ext}^*_{\mathcal{G}}(M, M)$ is a finitely generated $H^\bullet(\mathcal{G}, k)$-module.*

Definition 1.6.3. Let $\mathcal{G}$ be a finite group scheme and M be a finite-dimensional $\mathcal{G}$-module. Then the spectrum $\mathcal{V}_{\mathcal{G}} = \mathrm{Maxspec}\, H^\bullet(\mathcal{G}, k)$ of maximal ideals of $H^\bullet(\mathcal{G}, k)$ is called the cohomological support variety of $\mathcal{G}$. The projectivization of the cohomological support variety will be denoted by $\mathbb{P}(\mathcal{V}_{\mathcal{G}})$.
There is a natural homomorphism $\Phi_M : H^\bullet(\mathcal{G}, k) \to \mathrm{Ext}^*_{\mathcal{G}}(M, M)$ of graded k-algebras. The cohomological support variety of the module M is then defined as the subvariety $\mathcal{V}_{\mathcal{G}}(M) = \mathrm{Maxspec}(H^\bullet(\mathcal{G}, k)/\ker \Phi_M)$ of $\mathcal{V}_{\mathcal{G}}$.

For a subgroup scheme $\mathcal{H}$ of $\mathcal{G}$ let $\iota_{*,\mathcal{H}} : \mathbb{P}(\mathcal{V}_{\mathcal{H}}) \to \mathbb{P}(\mathcal{V}_{\mathcal{G}})$ be the morphism which is induced by the canonical inclusion $\iota : k\mathcal{H} \to k\mathcal{G}$.

Theorem 1.6.4 ([20, 5.6],[18, 3.3])**.** *Let $\mathcal{G}$ be a finite group scheme and M be a $\mathcal{G}$-module. Then the following holds:*

1. *If $\mathcal{G}$ is infinitesimal of height 1 and $\mathfrak{g} = \mathrm{Lie}(\mathcal{G})$, then $\mathcal{V}_{\mathcal{G}}$ and $V_{\mathfrak{g}}$ are homeomorphic.*
2. *Let $\mathcal{H} \subseteq \mathcal{G}$ be a subgroup scheme. Then $\iota^{-1}_{*,\mathcal{H}}(\mathcal{V}_{\mathcal{G}}(M)) = \mathcal{V}_{\mathcal{H}}(\mathrm{res}^{\mathcal{G}}_{\mathcal{H}} M)$.*
3. *If M is indecomposable, then $\mathbb{P}(\mathcal{V}_{\mathcal{G}}(M))$ is connected.*
4. *$\dim \mathcal{V}_{\mathcal{G}}(M) = \mathrm{cx}_{\mathcal{G}}(M)$.*

2. Auslander-Reiten theory

In Auslander-Reiten theory one studies the representations of an algebra with the help of so-called almost split sequences. These sequences give rise to a very powerful combinatorial invariant of the representation theory of an algebra, the so-called Auslander-Reiten quiver. This quiver describes almost all indecomposable modules and their irreducible morphisms. In the following sections we will introduce almost split sequences and the stable Auslander-Reiten quiver of a self-injective algebra. In the end we will give an alternate introduction via a functorial approach. For further details we refer to [2] and [1].

2.1. Almost split sequences

In this section k is an arbitrary field and all modules and algebras are supposed to be finite-dimensional over k.

Definition 2.1.1. Let A be a finite-dimensional k-algebra and let M, N and E be finite-dimensional A-modules.

1. An A-module homomorphism $\varphi : M \to N$ is called irreducible if
    a) φ is neither a split monomorphism nor a split epimorphism and
    b) if $\varphi = \varphi_1 \circ \varphi_2$ then either φ_1 is a split epimorphism or φ_2 is a split monomorphism.
2. A short exact sequence
$$0 \to N \xrightarrow{\varphi} E \xrightarrow{\psi} M \to 0$$
of A-modules is called almost split, if φ and ψ are both irreducible.

Theorem 2.1.2 ([2, V.1.15])**.** *Let A be a finite-dimensional k-algebra and M be a non-projective indecomposable A-module. Then there exists an almost split sequence*

$$0 \to N \xrightarrow{\varphi} E \xrightarrow{\psi} M \to 0$$

which is unique up to equivalence of short exact sequences.

Remark 2.1.3. The module N is uniquely determined up to isomorphism. In the following we will denote it by $\tau_A(M)$ and τ_A is called the Auslander-Reiten translation of A. If A is a symmetric k-algebra, then $\tau_A = \Omega_A^2$, where Ω_A denotes the Heller shift of $\operatorname{mod} A$ (c.f. [3, 4.12.8]).

Proposition 2.1.4 ([2, V.2.2])**.** *Let A be a finite-dimensional k-algebra, M be a non-projective indecomposable A-module and $\mathcal{E} : 0 \to \tau_A(M) \longrightarrow E \xrightarrow{\varphi} M \to 0$ a non-split short exact sequence. Then $\mathcal{E}$ is almost split if and only if each non-isomorphism $\psi : M \to M$ factors through φ.*

For future reference we record the following consequence of the previous proposition:

Lemma 2.1.5 ([2, V.2.4])**.** *Let A be a finite-dimensional k-algebra and M be a non-projective indecomposable A-module such that $\mathrm{End}_A(M) \cong k$. Then every short exact sequence $0 \to \tau_A(M) \longrightarrow E \longrightarrow M \to 0$ is either split or almost split.*

Definition 2.1.6. Let A be a finite-dimensional k-algebra. For indecomposable A-modules M and N we define the radical of $\mathrm{Hom}_A(M, N)$ as

$$\mathrm{Rad}_A(M, N) := \{\varphi \in \mathrm{Hom}_A(M, N) \mid \varphi \text{ is not an isomorphism}\}.$$

Moreover, we define the k-vector spaces

$$\mathrm{Rad}_A^2(X, M) := \{\alpha \mid \exists Z \in \mathrm{mod}\, A, \varphi \in \mathrm{Rad}_A(X, Z), \psi \in \mathrm{Rad}_A(Z, M) : \alpha = \psi \circ \varphi\}$$

and $\mathrm{Irr}_A(X, M) := \mathrm{Rad}_A(X, M) / \mathrm{Rad}_A^2(X, M)$.

Proposition 2.1.7. *Let A be a finite-dimensional k-algebra, M and N be indecomposable A-modules and $\varphi : M \to N$ be A-linear. Then φ is an irreducible morphism if and only if $\varphi \in \mathrm{Rad}_A(M, N) \setminus \mathrm{Rad}_A^2(M, N)$.*

2.2. Auslander-Reiten quiver

Definition 2.2.1. Let A be a finite-dimensional self-injective k-algebra. The stable Auslander-Reiten quiver $\Gamma_s(A)$ is the stable translation quiver given by the following data:

- The vertices are the isomorphism classes of non-projective indecomposable finite-dimensional A-modules.
- The arrows between two classes $[M]$ and $[N]$ are in bijective correspondence to a k-basis of $\mathrm{Irr}_A(M, N)$.
- The translation is the Auslander-Reiten translation τ_A of A.

Definition 2.2.2. Let Q be a quiver. We denote by $\mathbb{Z}[Q]$ the translation quiver with underlying set $\mathbb{Z} \times Q$, arrows $(n, x) \to (n, y)$ and $(n+1, y) \to (n, x)$ for any arrow $x \to y$ in Q and translation $\tau : \mathbb{Z}[Q] \to \mathbb{Z}[Q]$ given by $\tau(n, x) = (n+1, x)$.

Theorem 2.2.3 (Struktursatz of Riedtmann [43])**.** *Let $\Theta \subseteq \Gamma_s(A)$ be a connected component. Then there is an isomorphism of stable translation quivers $\Theta \cong \mathbb{Z}[T_\Theta]/\Pi$, where T_Θ denotes a directed tree and Π is an admissible subgroup of $\mathrm{Aut}(\mathbb{Z}[T_\Theta])$.*

Remark 2.2.4. The underlying undirected tree $\overline{T}_\Theta$ is called the tree class of Θ. If Θ has tree class A_∞, then there is for each vertex M only one sectional path to the end of the component ([2, VII.2]). The length of this path is called the quasi-length $\mathrm{ql}(M)$ of M. The modules of quasi-length 1 are also called quasi-simple. Components of the form $\mathbb{Z}[A_\infty]/(\tau^n)$, $n \geq 1$, are called tubes of rank n. These components contain for each $l \geq 1$ exactly n modules of quasi-length l. Tubes of rank 1 are also called homogeneous tubes and all other tubes are called exceptional tubes.

Remark 2.2.5. Let Θ be a homogeneous tube in the stable Auslander-Reiten quiver $\Gamma_s(A)$ and denote by V_l the module in Θ of quasi-length l. If V_1 is a brick (i.e. $\mathrm{End}_A(V_1) \cong k$), then basic properties of almost split sequences ([2, V.1]) imply that $\dim_k \mathrm{Hom}_A(V_i, V_j) = \min\{i, j\}$.

If $\mathcal{G}$ is a finite group scheme, we denote the Auslander-Reiten quiver $\Gamma_s(k\mathcal{G})$ also by $\Gamma_s(\mathcal{G})$.

Proposition 2.2.6 ([14, 3.1])**.** *Let $\mathcal{G}$ be a finite group scheme and Θ be a connected component of $\Gamma_s(\mathcal{G})$. If A and B are $\mathcal{G}$-modules which belong to Θ, then $\mathcal{V}_{\mathcal{G}}(A) = \mathcal{V}_{\mathcal{G}}(B)$.*

Thanks to 2.2.6 we can define $\mathcal{V}_{\mathcal{G}}(\Theta) := \mathcal{V}_{\mathcal{G}}(A)$ for some $\mathcal{G}$-module A belonging to Θ.

Proposition 2.2.7 ([14, 3.3(3)])**.** *Let $\mathcal{G}$ be a finite group scheme and Θ be a connected component of $\Gamma_s(\mathcal{G})$. Then $|\mathbb{P}(\mathcal{V}_{\mathcal{G}}(\Theta))| = 1$ if and only if $\bar{T}_\Theta$ is a finite Dynkin diagram or if Θ is a tube.*

Lemma 2.2.8. *Let $\mathcal{G}$ be an infinitesimal group scheme of height 1, M a $\mathcal{G}$-module which belongs to a homogeneous tube and $\mathcal{E} : 0 \to M \longrightarrow E \longrightarrow M \to 0$ the almost split sequence ending in M. Then E possesses no non-zero projective direct summand.*

Proof. Let B be the block of M and assume that E has a non-zero projective indecomposable direct summand P. By [1, IV.3.11] the sequence $\mathcal{E}$ is equivalent to $0 \to \mathrm{Rad}(P) \longrightarrow \mathrm{Rad}(P)/\mathrm{Soc}(P) \oplus P \longrightarrow P/\mathrm{Soc}(P) \to 0$. We obtain an isomorphism $M \cong P/\mathrm{Soc}(P)$ and therefore $\mathrm{cx}_B(P/\mathrm{Soc}(P)) = \mathrm{cx}_B(M) = 1$. Let $(P_i)_{i\geq 0}$ be a projective resolution of $\mathrm{Soc}(P)$ and set $Q_i := P_{i+1}, Q_0 = P$. Then $(Q_i)_{i\geq 0}$ is a projective resolution of $P/\mathrm{Soc}(P)$. Therefore the simple module $\mathrm{Soc}(P)$ has complexity 1. Now [12, 3.2(2)] yields that B is a Nakayama algebra and therefore representation finite. Hence M belongs to a finite component, a contradiction. □

2.3. Functorial approach

In this section we will give a short overview to the functorial approach of almost split sequences. For further details we refer the reader to [1, IV.6].
Let A be a finite-dimensional k-algebra. Denote by $\mathcal{F}un^{\mathrm{op}}\, A$ and $\mathcal{F}un\, A$ the categories of contravariant and covariant k-linear functors from $\mathrm{mod}\, A$ to $\mathrm{mod}\, k$. A functor F in $\mathcal{F}un^{\mathrm{op}}\, A$ is finitely generated if the functor F is isomorphic to a quotient of $\mathrm{Hom}_A(-, M)$ for some $M \in \mathrm{mod}\, A$. A functor F in $\mathcal{F}un^{\mathrm{op}}\, A$ is finitely presented if there is an exact sequence

$$\mathrm{Hom}_A(-, M) \to \mathrm{Hom}_A(-, N) \to F \to 0$$

of functors in $\mathcal{F}un^{\mathrm{op}}\, A$ for some $M, N \in \mathrm{mod}\, A$. The full subcategory of $\mathcal{F}un^{\mathrm{op}}\, A$ consisting of the finitely presented functors will be denoted by $\mathrm{mmod}\, A$. Up to isomorphism the finitely generated projective functors in $\mathcal{F}un^{\mathrm{op}}\, A$ are exactly the functors of the form $\mathrm{Hom}_A(-, M)$. Such a functor is indecomposable if and only if the A-module M is indecomposable.

The functor $\operatorname{Rad}_A(-, M)$ is a subfunctor of $\operatorname{Hom}_A(-, M)$ and we define the functor $S^M := \operatorname{Hom}_A(-, M)/\operatorname{Rad}_A(-, M)$. Up to isomorphism the simple functors in $\mathcal{Fun}^{\mathrm{op}} A$ are exactly the functors of the form S^M with an indecomposable A-module M. The projective cover of S^M is $\operatorname{Hom}_A(-, M)$. Let N be an indecomposable A-module. An A-module homomorphism $g : M \to N$ is (minimal) right almost split if and only if the induced sequence

$$\operatorname{Hom}_A(-, M) \to \operatorname{Hom}_A(-, N) \to S^N \to 0$$

of functors in $\mathcal{Fun}^{\mathrm{op}} A$ is a (minimal) projective presentation of S^N.
A functor $F : \operatorname{mod} A \to \operatorname{mod} B$ induces a functor $F : \operatorname{mmod} A \to \operatorname{mmod} B$ via $F(\operatorname{Hom}_A(-, M)) = \operatorname{Hom}_B(-, F(M))$. There are dual notions and results for left almost split morphisms and functors in $\mathcal{Fun}\, A$.

Remark 2.3.1 ([2, V.1])**.** A short exact sequence

$$0 \to N \xrightarrow{\varphi} E \xrightarrow{\psi} M \to 0$$

is almost split if and only if φ is left almost split and ψ is right almost split.

2.4. Auslander-Reiten quiver of group graded algebras

The Auslander-Reiten quiver of a group algebra of a finite group has been studied by Kawata in [27] and [28]. These results where generalized in [32] to the context of strongly group graded algebras. In this section we will present some of these results and we investigate how the restriction functor behaves for algebras which are graded by a cyclic group.

Definition 2.4.1. A morphism $\sigma : (\Gamma, \tau_\Gamma) \to (\Lambda, \tau_\Lambda)$ of stable translation quivers is a morphism of quivers which commutes with the translation $\sigma \circ \tau_\Gamma = \tau_\Lambda \circ \sigma$. For a stable translation quiver (Γ, τ_Γ) we will denote by $\operatorname{Aut}(\Gamma) = \operatorname{Aut}(\Gamma, \tau_\Gamma)$ its automorphism group.

Remark 2.4.2. If $\Gamma = \mathbb{Z}[A_\infty]/(\tau^n)$ is an exceptional tube of rank n, then the group $\operatorname{Aut}(\Gamma) = \langle \tau_\Gamma \rangle$ has order n.

Let G be a group and A be a finite-dimensional strongly G-graded k-algebra such that A_1 is self-injective. (Thanks to 1.3.7 this implies that A is self-injective, too.) The group G acts on the module category $\operatorname{mod} A_1$ via equivalences of categories

$$\operatorname{mod} A_1 \to \operatorname{mod} A_1,\ M \mapsto M^g \quad \text{for } g \in G.$$

Since these equivalences commute with the Auslander-Reiten translation of $\Gamma_s(A_1)$, each $g \in G$ induces an automorphism t_g of the quiver $\Gamma_s(A_1)$. As t_g permutes the components of $\Gamma_s(A_1)$, we can conclude that G acts on the set of components of $\Gamma_s(A_1)$. For a component Θ we write $\Theta^g = t_g(\Theta)$ and let $G_\Theta = \{g \in G \mid \Theta^g = \Theta\}$ be the stabilizer of Θ. If $g \in G$ and Θ is a component, then we have $\Theta = \Theta^g$ or $\Theta \cap \Theta^g = \emptyset$. Hence, if M is an A_1-module which belongs to the component Θ, this implies $G_M \subseteq G_\Theta$.

Lemma 2.4.3. *Let Θ be a component of $\Gamma_s(A_1)$ with finite automorphism group $\operatorname{Aut}(\Theta)$ such that $|G_\Theta|$ and $|\operatorname{Aut}(\Theta)|$ are relatively prime. Let M be an A_1-module which belongs to Θ. Then $G_M = G_\Theta$.*

Proof. The action of G_Θ on Θ induces a homomorphism $\psi : G_\Theta \to \operatorname{Aut}(\Theta)$ of groups. The kernel of this homomorphism is given by $\ker \psi = \bigcap_{N \in \Theta} G_N$. Since $|G_\Theta|$ and $|\operatorname{Aut}(\Theta)|$ have no common divisor, the homomorphism ψ is trivial. Hence $G_\Theta = \bigcap_{N\in\Theta} G_N$ and thus $G_M = G_\Theta$. $\square$

Now let N be an indecomposable non-projective A_1-module and Ξ the corresponding component in $\Gamma_s(A_1)$. Assume there is an indecomposable non-projective direct summand M of $\operatorname{ind}_1^G N$ and let Θ be the corresponding component in $\Gamma_s(A)$. Since G_N is contained in G_Ξ, 1.3.9 provides an indecomposable direct summand U of $\operatorname{res}^G_{G_\Xi} M$ such that $\operatorname{ind}^G_{G_\Xi} U = M$. Denote by Ψ the component of U in $\Gamma_s(A_{G_\Xi})$.

Lemma 2.4.4 ([32, 4.5.8])**.** *Let W be an indecomposable A-module which belongs to Θ. Then every indecomposable direct summand of $\operatorname{res}_1^G W$ belongs to $\bigcup_{g\in G} \Xi^g$.*

Theorem 2.4.5 ([32, 4.5.10])**.**

1. *Let V be an A_{G_Ξ}-module which belongs to Ψ. Then $\operatorname{ind}^G_{G_\Xi} V$ is indecomposable.*
2. *The functor $\operatorname{ind}^G_{G_\Xi} : \operatorname{mod} A_{G_\Xi} \to \operatorname{mod} A$ induces an isomorphism of stable translation quivers $\operatorname{ind}^G_{G_\Xi} : \Psi \to \Theta$.*

For the proof of the following result we will need the following:

Theorem 2.4.6 ([48, Theorem 6])**.** *Let k be a field, G be a finite group such that $|G|$ is invertible in k and A be a strongly G-graded finite-dimensional k-algebra. Then the induction functor $\operatorname{ind}_1^G : A_1 \to A$ (or the restriction functor $\operatorname{res}_1^G : A \to A_1$) sends almost split sequences over A_1 (or over A, respectively) to direct sums of almost split sequences over A (or over A_1, respectively).*

Proposition 2.4.7. *Let k be an algebraically closed field. Suppose that all A_1-modules which belong to Ξ are G_Ξ-stable and that G_Ξ is a cyclic group such that $\operatorname{char} k \nmid |G_\Xi|$. Then the following hold:*

(a) $\operatorname{res}_1^{G_\Xi} : \Psi \to \Xi$, $[X] \mapsto [\operatorname{res}_1^{G_\Xi} X]$ is a morphism of stable translation quivers,

(b) for all $[Y] \in \Xi$ we have $|(\operatorname{res}_1^{G_\Xi})^{-1}([Y])| \leq |G_\Xi|$,

(c) if Ξ and Ψ have tree class A_∞, then $\operatorname{res}_1^{G_\Xi} : \Psi \to \Xi$ preserves the quasi-length, and

(d) if Ξ and Ψ are tubes of finite rank n and m, then $m \leq |G_\Xi| n$.

Proof. We first show that under our assumptions the restriction of every A_{G_Ξ}-module in Ψ is an indecomposable A_1-module which belongs to Ξ. Let V be an indecomposable A_{G_Ξ}-module in Ψ and let $\operatorname{res}_1^{G_\Xi} V = \bigoplus_{i=1}^n U_i$ be its decomposition into indecomposable A_1-modules. Applying 2.4.4 to the G_Ξ-graded algebra A_{G_Ξ} yields that all these modules belong to $\bigcup_{g \in G_\Xi} \Xi^g = \Xi$ and therefore are G_Ξ-stable. Let $r = |G_\Xi|$. Since G_Ξ is cyclic, $\operatorname{char} k \nmid |G_\Xi|$ and k is algebraically closed, we get due to 1.3.10 a decomposition $\operatorname{ind}_1^{G_\Xi} U_i = \bigoplus_{j=1}^r W_{i,j}$ into indecomposable A_{G_Ξ}-modules of dimension $\dim_k W_{i,j} = \dim_k U_i$ for all $i \in \{1, \ldots, n\}$. In particular, the restriction $\operatorname{res}_1^{G_\Xi} W_{i,j}$ is isomorphic to U_i. As the ring extension $A_{G_\Xi} : A_1$ is separable, the module V is a direct summand of $\operatorname{ind}_1^{G_\Xi} \operatorname{res}_1^{G_\Xi} V = \bigoplus_{i=1}^n \bigoplus_{j=1}^r W_{i,j}$ and therefore isomorphic to one of the $W_{i,j}$. In particular, the module $\operatorname{res}_1^{G_\Xi} V \cong \operatorname{res}_1^{G_\Xi} W_{i,j} \cong U_i$ is indecomposable.

(a) Let $X \to Y$ be an arrow in Ψ. Then there is an almost split sequence of A_{G_Ξ}-modules

$$\mathcal{E} :\ 0 \to \tau_\Psi(Y) \longrightarrow E \longrightarrow Y \to 0$$

such that X is a direct summand of E and the indecomposable A_1-modules $\operatorname{res}_1^{G_\Xi} X$ and $\operatorname{res}_1^{G_\Xi} Y$ belong to Ξ. By 2.4.6, the sequence $\operatorname{res}_1^{G_\Xi} \mathcal{E}$ is a direct sum of almost split sequences. Since $\operatorname{res}_1^{G_\Xi} Y$ and $\operatorname{res}_1^{G_\Xi} \tau_\Psi(Y)$ are indecomposable, the sequence $\operatorname{res}_1^{G_\Xi} \mathcal{E}$ is almost split. In particular, $\operatorname{res}_1^{G_\Xi} \tau_\Psi(Y) \cong \tau_\Xi(\operatorname{res}_1^{G_\Xi} Y)$. Moreover, this gives us an arrow $\operatorname{res}_1^{G_\Xi} X \to \operatorname{res}_1^{G_\Xi} Y$. Therefore, $\operatorname{res}_1^{G_\Xi} : \Psi \to \Xi$, $[X] \mapsto [\operatorname{res}_1^{G_\Xi} X]$ is a morphism of stable translation quivers.

(b) Let $[Y] \in \Xi$ and $[X] \in \Psi$ with $\operatorname{res}_1^{G_\Xi}([X]) = [Y]$. As before, we have a decomposition $\operatorname{ind}_1^{G_\Xi} Y = \bigoplus_{i=1}^r Y_i$ into indecomposable A_{G_Ξ}-modules and X is a direct summand of $\operatorname{ind}_1^{G_\Xi} \operatorname{res}_1^{G_\Xi} X = \operatorname{ind}_1^{\mathcal{G}_\Xi} Y = \bigoplus_{i=1}^r Y_i$. Therefore the number of preimages of $[Y]$ is bounded by r.

(c) Let $[M] \in \Psi$. If $[N] \in \Psi$ is a successor of $[M]$ in Ψ then $\operatorname{res}_1^{G_\Xi}[N]$ is a successor of $\operatorname{res}_1^{G_\Xi}[M]$ in Ξ. Hence, we only need to show that $\operatorname{res}_1^{G_\Xi} : \Psi \to \Xi$ sends quasi-simple modules to quasi-simple modules. Let Y be a quasi-simple module in Ψ and let

$$0 \to \tau_\Psi(Y) \longrightarrow E \longrightarrow Y \to 0$$

be the almost split sequence ending in Y. As shown in (a), the sequence $\operatorname{res}_1^{G_\Xi} \mathcal{E}$ is almost split. As Y is quasi-simple, the module E is the direct sum $X \oplus P$ of an indecomposable module X and a projective module P. Since X belongs to Ψ, the module $\operatorname{res}_1^{G_\Xi} X$ is indecomposable, so that $\operatorname{res}_1^{G_\Xi} E = \operatorname{res}_1^{G_\Xi} X \oplus \operatorname{res}_1^{G_\Xi} P$ is the direct sum of an indecomposable and a projective module. Hence $\operatorname{res}_1^{\mathcal{G}_\Xi} Y$ is quasi-simple.

(d) Let $Y_1, \ldots, Y_n$ be the quasi-simple modules in Ξ. As $\operatorname{res}_1^{\mathcal{G}_\Xi}$ preserves the quasi-length, every module belonging to $(\operatorname{res}_1^{\mathcal{G}_\Xi})^{-1}([Y_i])$ is quasi-simple. Applying (b) yields that Ψ has, up to isomorphism, at most rn quasi-simple modules.

$\square$

3. Domestic Finite Group Schemes

In representation theory we have a trichotomy of representation types for finite dimensional algebras. Any such algebra is either of finite, tame or wild representation type. The class of algebras having tame representation type consists of those algebras, which have up to isomorphism infinitely many indecomposable modules such that in each dimension all but finitely many indecomposable modules are parametrized by a finite number of parameters. The algebras of domestic representation type are those with a common bound of this number for all dimensions.
The finite group schemes of domestic representation type were described in [13] and [17]. Any such group scheme can be associated to an amalgamated polyhedral group scheme. The goal of this chapter is to introduce the amalgamated polyhedral group schemes and to explain how they relate to the domestic finite group schemes.

3.1. Representation Type

Definition 3.1.1. Let A be a finite dimensional k-algebra.

1. The algebra A is referred to be of finite representation if it admits only finitely many isomorphism classes of indecomposable A-modules. Otherwise it is referred to be of infinite representation type.

2. The algebra A is of tame representation type if it is of infinite representation type and if for any $d \in \mathbb{N}$ there are $(A, k[T])$-bimodules $M_1, \dots, M_{n(d)}$ which are free $k[T]$-modules of rank d such that all but finitely many indecomposable A-modules are isomorphic to $M_i \otimes_{k[T]} k[T]/(T - \lambda)$ for some $1 \leq i \leq n(d)$ and $\lambda \in k$. For $d \in \mathbb{N}$ denote by $\mu_A(d)$ the smallest possible choice for the number $n(d)$.

3. The algebra A is of domestic representation type if it is of tame representation type and if there is $m \in \mathbb{N}$ such that $\mu_A(d) \leq m$ for all $d \in \mathbb{N}$.

4. If $\mathcal{G}$ is a finite group scheme we say that $\mathcal{G}$ is domestic (or tame), if the algebra $k\mathcal{G}$ is of domestic (or tame) representation type.

5. The algebra A is of wild representation type if there is a $(A, k\langle X, Y\rangle)$-bimodule M which is a finitely generated free right $k\langle X, Y\rangle$-module, such that the functor $M \otimes_{k\langle X,Y\rangle} - : \operatorname{mod} k\langle X, Y\rangle \to \operatorname{mod} A$ preserves indecomposables and reflects isomorphism classes.

As already mentioned, we have the following trichotomy for finite dimensional algebras:

Theorem 3.1.2 (Drozd [10])**.** *Let A be a finite dimensional algebra over an algebraically closed field k. Then exactly one of the following cases occurs:*

1. *A is of finite representation type.*

2. *A is of tame representation type.*

3. *A is of wild representation type.*

In the case of group algebras of finite groups we have a complete classification of the groups having a certain representation type.

Theorem 3.1.3 ([3, 4.4.4])**.** *Let G be a finite group and k be an infinite field of characteristic p. Then the following holds:*

1. *The group algebra kG has finite representation type if and only if the p-Sylow subgroups of G are cyclic.*

2. *The group algebra kG has domestic representation type if and only if $p = 2$ and the 2-Sylow subgroups of G are isomorphic to the Klein four group.*

3. *The group algebra kG has tame representation type if and only if $p = 2$ and the 2-Sylow subgroups of G are isomorphic to a dihedral, semidihedral or generalized quaternion group.*

4. *In all other cases the group algebra kG is of wild representation type.*

For later use we mention at this the point the following results concerning the representation type of group graded algebras and group schemes.

Lemma 3.1.4 ([17, 4.1.3])**.** *Let k be a field of characteristic p and G be a finite group with $p \nmid |G|$. Let A be a G-module algebra. Then A has domestic representation type if and only if $A * G$ has domestic representation type.*

Proposition 3.1.5 ([13, 6.2.1])**.** *Let k be an algebraically closed field of characteristic $p \geq 3$ and $\mathcal{G}$ be a finite group scheme with tame principal block $\mathcal{B}_0(\mathcal{G})$. Then $p \nmid |\mathcal{G}(k)|$.*

3.2. Amalgamated polyhedral group schemes

Let k be an algebraically closed field of characteristic $p > 2$. In this section we will give first examples of domestic finite group schemes, the so-called amalgamated polyhedral group schemes. Every such group scheme is of the following form:
Let $\mathcal{Z}$ be the center of the group scheme $SL(2)$ and $\tilde{\mathcal{G}}$ be a binary polyhedral subgroup scheme of $SL(2)$. Then the group scheme $SL(2)_1\tilde{\mathcal{G}}/\mathcal{Z}$ is an amalgamated polyhedral group scheme. If $\tilde{\mathcal{G}}$ is reduced, we say that $SL(2)_1\tilde{\mathcal{G}}/\mathcal{Z}$ is an amalgamated reduced-polyhedral group scheme. Analogously, if $\tilde{\mathcal{G}}$ is not reduced, we say that $SL(2)_1\tilde{\mathcal{G}}/\mathcal{Z}$ is an amalgamated non-reduced-polyhedral group scheme. The binary polyhedral group schemes were classified in [13, Section 3] and are given as follows:

For $m \in \mathbb{N}$ consider the subgroup scheme $T_{(m)} \subseteq SL(2)$ given by

$$T_{(m)}(R) := \{\left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix}\right) \mid x \in \mu_{(m)}(R)\},$$

for any commutative k-algebra R. Then $T_{(2m)}$ is a binary cyclic group scheme.
Let $\mathcal{H}_4$ be the reduced subgroup scheme of $N_{SL(2)}(T)$ with $\mathcal{H}_4(k) = \langle w_0 \rangle$. Then there is

$h_4 \in GL(2)(k)$ such that $\mathcal{H}_4 = h_4 T_{(4)} h_4^{-1}$. For $m \geq 2$ the group scheme $N_{(m)} := T_{(m)}\mathcal{H}_4$ is a binary dihedral group scheme.
For $m \geq 1$ with $(p,m) = 1$ let $\zeta_m \in k$ be a m-th primitive root of unity. We define the following elements of $SL(2)(k)$:

$$x(\zeta_{2m}) := \begin{pmatrix} \zeta_{2m} & 0 \\ 0 & \zeta_{2m}^{-1} \end{pmatrix}, \quad y(\zeta_4) := \frac{1}{\zeta_4 - 1}\begin{pmatrix} 1 & 1 \\ \zeta_4 & -\zeta_4 \end{pmatrix}, \quad y(\zeta_5) := \frac{1}{\zeta_5^2 - \zeta_5^3}\begin{pmatrix} \zeta_5+\zeta_5^4 & 1 \\ 1 & -(\zeta_5+\zeta_5^4) \end{pmatrix}.$$

By [13, 3.2] there are unique reduced subgroup schemes $\hat{T}$, $\hat{O}$ and $\hat{I}$ of $SL(2)$ such that

$$\hat{T}(k) = \langle w_0, x(\zeta_4), y(\zeta_4)\rangle, \quad \hat{O}(k) = \langle w_0, x(\zeta_8), y(\zeta_4)\rangle$$

for $p \neq 2,3$ and

$$\hat{I}(k) = \langle w_0, x(\zeta_5), y(\zeta_5)\rangle$$

for $p \neq 2,3,5$. The group schemes $\hat{T}, \hat{O}$ and $\hat{I}$ are called binary tetrahedral group scheme, binary octahedral group scheme and binary icosahedral group scheme, respectively.

Definition 3.2.1. The following are the amalgamated polyhedral group schemes:

- For $m \in \mathbb{N}$ the group schemes $P\mathcal{SC}_{(m)} := SL(2)_1 T_{(2m)}/\mathcal{Z}$ are the amalgamated cyclic group schemes.
- For $m \geq 2$ the group schemes $P\mathcal{SQ}_{(m)} := SL(2)_1 N_{(2m)}/\mathcal{Z}$ are the amalgamated dihedral group schemes.
- $P\mathcal{S}\hat{T} := SL(2)_1\hat{T}/\mathcal{Z}$ is the amalgamated tetrahedral group scheme.
- $P\mathcal{S}\hat{O} := SL(2)_1\hat{O}/\mathcal{Z}$ is the amalgamated octahedral group scheme.
- $P\mathcal{S}\hat{I} := SL(2)_1\hat{I}/\mathcal{Z}$ is the amalgamated icosahedral group scheme.

3.3. The McKay quiver of a binary polyhedral group scheme

The binary polyhedral group schemes can be classified with the help of their McKay quivers in the following way:
Let $\mathcal{H}$ be a finite linearly reductive group scheme, $S_1, \ldots, S_n$ a complete set of pairwise non-isomorphic simple $\mathcal{H}$-modules and L be an $\mathcal{H}$-module. For each $1 \leq j \leq n$ there are $a_{ij} \geq 0$ such that

$$L \otimes_k S_j \cong \bigoplus_{i=1}^{n} a_{ij} S_i.$$

The McKay quiver $\Upsilon_L(\mathcal{H})$ of $\mathcal{H}$ relative to L is the quiver with underlying set $\{S_1, \ldots, S_n\}$ and a_{ij} arrows from S_i to S_j.

Proposition 3.3.1 ([13, Section 3])**.** *Let $\mathcal{H}$ be a finite linearly reductive group scheme. Then the following holds:*

1. *If L is a faithful $\mathcal{H}$-module, then $\Upsilon_L(\mathcal{H})$ is connected.*

2. *If L is two-dimensional and self-dual, then the matrix (a_{ij}) is symmetric.*

Set $A := (a_{ij})$ and assume that L is two-dimensional and self-dual. Then $C := 2I_n - A$ is a generalized Cartan matrix. In this situation the valued graph $\bar{\Upsilon}_L(\mathcal{H})$ associated to C is called the McKay graph of $\mathcal{H}$ relative to L.
The next theorem characterizes the finite linearly reductive subgroup schemes of $SL(2)$ with respect to their McKay graph. The diagrams occurring in the table can be found in the appendix.

Theorem 3.3.2 ([13, 3.3])**.** *Let k be an algebraically closed field of characteristic $p > 2$ and $\mathcal{H}$ be a finite linearly reductive subgroup scheme of $SL(2)$. Denote by L the two-dimensional standard module of $\mathcal{H}$. Then there is $g \in SL(2)(k)$ such that $g\mathcal{H}g^{-1}$ and its McKay graph $\bar{\Upsilon}_L(\mathcal{H})$ belong to the following list:*

$g\mathcal{H}g^{-1}$	$\bar{\Upsilon}_L(\mathcal{H})$
e_k	$\tilde{L}_0$
$T_{(np^r)}$	$\tilde{A}_{np^r-1}$
$N_{(np^r)}$	$\tilde{D}_{np^r+2}$
$\hat{T}$	$\tilde{E}_6$
$\hat{O}$	$\tilde{E}_7$
$\hat{I}$	$\tilde{E}_8$

where $(n,p) = 1$, $n + r \neq 1$ and r is the height of $\mathcal{G}^0$.

3.4. Classification of domestic finite group schemes

For a finite group scheme $\mathcal{G}$ denote by $\mathcal{G}_{lr}$ the largest linearly reductive normal subgroup scheme of $\mathcal{G}$. The domestic finite group schemes are well understood in the following way:

Theorem 3.4.1 ([17, 4.3.2])**.** *Let $\mathcal{G}$ be a finite group scheme over an algebraically closed field of characteristic $p > 2$. The following statements are equivalent:*

1. *$\mathcal{G}$ is domestic.*
2. *The principal block $\mathcal{B}_0(\mathcal{G})$ of $k\mathcal{G}$ is of domestic representation type.*
3. *The principal block $\mathcal{B}_0(\mathcal{G})$ is Morita-equivalent to the trivial extension of a radical square zero tame hereditary algebra.*
4. *The group scheme $\mathcal{G}/\mathcal{G}_{lr}$ is isomorphic to an amalgamated polyhedral group scheme.*

Proposition 3.4.2 ([13, 7.4.1])**.** *Let $\mathcal{G}$ be a finite group scheme with tame principal block $\mathcal{B}_0(\mathcal{G})$ over an algebraically closed field of characteristic $p > 2$. Then $k\mathcal{G}$ is symmetric.*

Remark 3.4.3. 1. Let $\mathcal{G}$ be a finite group scheme over an algebraically closed field of characteristic $p > 2$ with tame principal block $\mathcal{B}_0(\mathcal{G})$. If $\mathcal{G}_{lr}$ is trivial, then all non-simple blocks of $k\mathcal{G}$ are Morita-equivalent to the principal block $\mathcal{B}_0(\mathcal{G})$. (see [13, 7.3.2])

2. The Auslander-Reiten theory of trivial extension of radical square zero tame hereditary algebra is well understood (c.f. [24, V.3.2]). Let Q be a Euclidean diagram, A be the trivial extension of a radical square zero tame hereditary algebra of type Q and $(n_1, \ldots, n_l)$ the tubular type of Q (c.f. [44, 3.6(5)]). Then the Auslander-Reiten quiver of A has two Euclidean components $\mathbb{Z}[Q]$, for each $i \in \{1, \ldots, l\}$ two exceptional tubes of rank n_i and infinitely many homogeneous tubes.

4. Modules for the infinitesimal amalgamated cyclic group schemes

Each infinitesimal amalgamated cyclic group scheme is isomorphic to one of the group schemes $SL(2)_1T_r$ for some $r \geq 1$. In the case $r = 1$, Premet gave a complete characterization of its indecomposable modules. His work was extended in [15] to the case $r > 1$. The classification given there lacks a concrete realization of the modules belonging to the homogeneous tubes of its Auslander-Reiten quiver. The goal of this section is to complete the classification by developing a new method to realize the missing modules. We start by giving an overview of the results from [41] and [15]. After that we will use a filtration of induced modules, which was introduced by Voigt ([51]), to describe the modules in the homogeneous tubes.

4.1. The modules and Auslander-Reiten quiver of $SL(2)_1T_r$

Let k be an algebraically closed field of characteristic $p > 2$. The group algebra $kSL(2)_1$ is isomorphic to the restricted universal enveloping algebra $\mathrm{U}_0(\mathfrak{sl}(2))$ of the restricted Lie algebra $\mathfrak{sl}(2)$. There are one-to-one correspondences between the representations of $SL(2)_1$, $\mathrm{U}_0(\mathfrak{sl}(2))$ and the restricted representations of $\mathfrak{sl}(2)$. The indecomposable representations of the restricted Lie algebra $\mathfrak{sl}(2)$ were classified by Premet in [41]. In [15, 4.1] Farnsteiner incorporated these results into the Auslander-Reiten theory of this algebra. Let $T \subseteq SL(2)$ be the standard torus of diagonal matrices. Following [15, 4.1], we will give here an overview of the representation theory of the group schemes $SL(2)_1T_r$, which is based on Premet's work.
Let $\{e, f, h\}$ denote the standard basis of $\mathfrak{sl}(2)$. For $d \in \mathbb{N}_0$ we consider the $(d+1)$-dimensional Weyl module $V(d)$ of highest weight d. These are rational $SL(2)$-modules which are obtained by twisting the 2-dimensional standard module with the Cartan involution $(x \mapsto -x^{tr})$ and taking its d-th symmetric power. Each of these modules $V(d)$ possesses a k-basis $v_0, \dots, v_d$ such that

$$e.v_i = (i+1)v_{i+1}, \quad f.v_i = (d-i+1)v_{i-1}, \quad h.v_i = (2i-d)v_i.$$

For $d \leq p-1$ we obtain in this way exactly the simple $\mathrm{U}_0(\mathfrak{sl}(2))$-modules.
For $s \in \mathbb{N}$, $a \in \{0, \dots, p-2\}$, and $d = sp + a$ Premet introduced the sp-dimensional maximal $\mathrm{U}_0(\mathfrak{sl}(2))$-submodule $W(d)$ of $V(d)$ generated by $v_{a+1}, \dots, v_d$. These modules are stable under the action of the standard Borel subgroup $B \subseteq SL(2)$ of upper triangular matrices.
The group $SL(2,k)$ operates on $\mathrm{U}_0(\mathfrak{sl}(2))$ via the adjoint representation and for each element $g \in SL(2,k)$ the space $g.W(d)$ is a $\mathrm{U}_0(\mathfrak{sl}(2))$-module which is isomorphic to $W(d)^g$, the space $W(d)$ with action twisted by g^{-1}. For each $g \in SL(2,k)$ the rank variety of $g.W(d)$ can be computed as $V_{\mathfrak{sl}(2)}(g.W(d)) = k(geg^{-1})$.

Let $\mathfrak{b}$ be the Borel subalgebra of $\mathfrak{sl}(2)$ which is generated by h and e. For each $i \in \{0, \dots, p-1\}$ let k_i be the one-dimensional $U_0(\mathfrak{b})$-module with $h.1 = i$ and $e.1 = 0$. Then the induced $\mathrm{U}_0(\mathfrak{sl}(2))$-module $Z(i) := \mathrm{U}_0(\mathfrak{sl}(2)) \otimes_{U_0(\mathfrak{b})} k_i$ is called a baby Verma module of highest weight i.

Lemma 4.1.1 ([15, 4.1.2])**.** *Let $s \in \mathbb{N}$, $a \in \{0, \dots, p-2\}, d = sp + a$ and $g \in SL(2, k)$. Then the AR-component $\Theta \subseteq \Gamma_s(\mathfrak{sl}(2))$ containing $g.W(d)$ is a homogeneous tube with quasi-simple module $Z(a)^g$. Moreover, we have $\mathrm{ql}(g.W(d)) = s$.*

The Auslander-Reiten quiver of each non-simple block of $kSL(2)_1$ consists of two components of type $\mathbb{Z}[\tilde{A}_{1,1}]$ and infinitely many homogeneous tubes $\mathbb{Z}[A_\infty]/(\tau)$. Thanks to [15, 4.1], each of the $p-1$ Euclidean components $\Theta(i)$ contains exactly one simple $SL(2)_1$-module $L(i)$ with $0 \leq i \leq p-2$. This component is then given by

$$\Theta(i) = \{\Omega^{2n}(L(i)), \Omega^{2n+1}(L(p-2-i)) \mid n \in \mathbb{Z}\}$$

with almost split sequences

$$0 \to \Omega^{2n+2}(L(i)) \longrightarrow \Omega^{2n+1}(L(p-2-i)) \oplus \Omega^{2n+1}(L(p-2-i)) \longrightarrow \Omega^{2n}(L(i)) \to 0.$$

The Auslander-Reiten quiver of each block of $kSL(2)_1T_r$ consists of two components of type $\mathbb{Z}[\tilde{A}_{p^{r-1},p^{r-1}}]$, four exceptional tubes $\mathbb{Z}[A_\infty]/(\tau^{p^{r-1}})$ and infinitely many homogeneous tubes $\mathbb{Z}[A_\infty]/(\tau)$.
Denote by $w_0 := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ the standard generator of the Weyl group of $SL(2)$. The spaces $W(sp+a)$ and $w_0.W(sp+a)$ are stable under the action of $SL(2)_1T_r$ and therefore already $SL(2)_1T_r$-modules. These modules (and certain twists of them) belong to the exceptional tubes. There was no realization given in [15] of the modules belonging to homogeneous tubes. But the following was shown:

Lemma 4.1.2 ([15, 4.3], [15, 4.2.3])**.** *For each $l \in \mathbb{N}$, $g \in SL(2) \setminus (B \cup w_0B)$ and $i \in \{0, \dots, p-2\}$ there is, up to isomorphism, a unique indecomposable $SL(2)_1T_r$-module $X(i,g,l)$ with $\mathrm{res}_{SL(2)_1} X(i,g,l) \cong g.W(lp^r + i)$. Moreover, we have an isomorphism $X(i,g,1) \cong kSL(2)_1T_r \otimes_{kSL(2)_1} Z(i)^g$ of $SL(2)_1T_r$-modules.*

We will see in section 4.3 how to realize these modules. The $SL(2)_1T_r$-modules are then classified in the following way:

Theorem 4.1.3 ([15, 4.3.1])**.** *Let $C \subseteq SL(2,k)$ be a set of representatives of $SL(2,k)/B$ such that $\{1, w_0\} \subseteq C$ and M be a non-projective indecomposable $SL(2)_1T_r$-module. Then M is isomorphic to a module of the following list of pairwise non-isomorphic $SL(2)_1T_r$-modules:*

- *$V(d) \otimes_k k_\lambda, V(d)^* \otimes_k k_\lambda$, $V(i) \otimes_k k_\lambda$ for $d \geq p$, $\lambda \in X(\mu_{(p^{r-1})})$, $d \not\equiv -1 \pmod p$ and $0 \leq i \leq p-1$. (Modules belonging to Euclidean components)*
- *$w_0^j.W(d) \otimes_k k_\lambda$ for $j \in \{0,1\}$, $d = sp + a$ with $a \in \{0, \dots, p-2\}$, $s \in \mathbb{N}$ and $\lambda \in X(\mu_{(p^{r-1})})$. (Modules belonging to exceptional tubes)*
- *$X(i,g,l)$ for $g \in C \setminus \{1, w_0\}$ and $d = sp + a$ with $l \in \mathbb{N}$ and $i \in \{0, \dots, p-2\}$. (Modules belonging to homogeneous tubes)*

4.2. Filtrations of induced modules

Let k be a field of characteristic $p > 0$, $\mathcal{G}$ a finite group scheme and $\mathcal{N} \subseteq \mathcal{G}$ a normal subgroup scheme such that $\mathcal{G}/\mathcal{N}$ is infinitesimal. Let J be the kernel of the canonical projection $k[\mathcal{G}] \to k[\mathcal{N}]$. The algebra $k[\mathcal{G}]^{\mathcal{N}} \cong k[\mathcal{G}/\mathcal{N}]$ is local and consequently the ideal $I := k[\mathcal{G}]^{\mathcal{N}} \cap J$ is nilpotent. Moreover, I is the augmentation ideal of $k[\mathcal{G}]^{\mathcal{N}}$ and therefore $J = Ik[\mathcal{G}]$ by [52, 2.1]. Hence J is also nilpotent. Thus setting

$$H_l := (J^{l+1})^{\perp} = \{v \in k\mathcal{G} \mid v(J^{l+1}) = (0)\}$$

gives us an ascending filtration of $k\mathcal{G}$ consisting of $(k\mathcal{N}, k\mathcal{N})$-bimodules

$$(0) = H_{-1} \subseteq k\mathcal{N} = H_0 \subseteq H_1 \subseteq \ldots \subseteq H_n = k\mathcal{G}.$$

Now let Z be an $\mathcal{N}$-module. Due to [51, 9.5], the canonical maps $\iota_l : H_l \otimes_{k\mathcal{N}} Z \to k\mathcal{G} \otimes_{k\mathcal{N}} Z$ are injective. Set $N_l := \operatorname{im} \iota_l$. In [51, 9] Voigt introduced the following ascending filtration by $\mathcal{N}$-modules of the $\mathcal{G}$-module $N_n = k\mathcal{G} \otimes_{k\mathcal{N}} Z$:

$$(0) = N_{-1} \subseteq Z \cong N_0 \subseteq N_1 \subseteq \ldots \subseteq N_n.$$

The algebra $k\mathcal{G}$ becomes a $(k[\mathcal{G}]^{\mathcal{N}}, k\mathcal{N})$-bimodule via $(x.h)(y) = h(yx)$ and $h \bullet h' = hh'$ for all $h \in k\mathcal{G}, h' \in k\mathcal{N}, x \in k[\mathcal{G}]^{\mathcal{N}}$ and $y \in k[\mathcal{G}]$. Hence the induced module $k\mathcal{G} \otimes_{k\mathcal{N}} Z$ has also a $k[\mathcal{G}]^{\mathcal{N}}$-module structure. Voigt has given an alternative description of the modules occurring in the above filtration:

Proposition 4.2.1 ([51, 9.6])**.** *In the above situation we get the following equality:*

$$N_l = \{n \in N_n \mid \forall f \in I^{l+1} : f.n = 0\}.$$

Moreover, the $\mathcal{N}$-module N_l/N_{l-1} is isomorphic to a direct sum of $\dim_k H_l/H_{l-1}$ *copies of Z.*

Let $f_1, \ldots, f_{q_l}$ be generators of the $k[\mathcal{G}]^{\mathcal{N}}$-ideal I^{l+1} and

$$v_l : k\mathcal{G} \to (k\mathcal{G})^{q_l}, h \mapsto (f_1.h, \ldots, f_{q_l}.h).$$

By the proof of [51, 9.6], the map $u_l := v_l \otimes id_Z : N_n \to N_n^{q_l}$ is $\mathcal{N}$-linear and has kernel N_l.
For $1 \leq j \leq n$ we define the $\mathcal{N}$-linear maps $p_{l,j} := u_{l|N_j}$. Note that these maps depend on the choice of the generators $f_1, \ldots, f_{q_l}$. In the case that I is a principal ideal, we fix a generator f of I and will always choose f^{l+1} as the generator of I^{l+1}.

Proposition 4.2.2. *Assume that I is a principal ideal. Then the following hold:*

(a) u_l is an $\mathcal{N}$-linear endomorphism of N_n with $\ker u_l = N_l$*,*

(b) the dimension of N_l is equal to $(l+1)\dim_k Z$*,*

(c) $\operatorname{im} p_{l,j} = N_{j-l-1}$ *for $1 \leq l \leq j \leq n$, and*

(d) $p_{m,i} \circ p_{l,j} = p_{m+l+1,j}$ *for all* $1 \leq j \leq i \leq n$.

Proof. Since I is a principal ideal the same holds for I^{l+1}. Therefore u_l is an $\mathcal{N}$-linear endomorphism of N_n, so that (a) holds.
Due to 4.2.1, the image of the restriction $u_{l|N_{l+1}}$ must lie in $N_0 \cong Z$. Hence the $\mathcal{N}$-module N_{l+1}/N_l is isomorphic to a submodule of Z. But by 4.2.1 it is also isomorphic to a non-zero direct sum of copies of Z. Therefore it must be isomorphic to Z, which yields (b). For $l \leq j \leq n$ another application of 4.2.1 yields $N_j/N_l \cong \operatorname{im} p_{l,j} \subseteq N_{j-l-1}$, with equality due to dimension reasons.
To show that (d) holds, we first note that $p_{m,i} \circ p_{l,j} = p_{m,j} \circ p_{l,j}$. Now consider the map $v_l : k\mathcal{G} \to k\mathcal{G}, h \mapsto f^{l+1}.h$, where f is the generator of I. Then we obtain $v_m \circ v_l(h) = f^{m+l+2}.h = v_{m+l+1}(h)$ for all $h \in k\mathcal{G}$. This yields $p_{m,j} \circ p_{l,j} = p_{m+l+1,j}$. $\square$

Voigt also gave a generalized version of Clifford theory for the decomposition of an induced module ([51, 9.9]):

Remark 4.2.3. The stabilizer $\mathcal{G}_Z$ of Z (see [51, 1.3]) equals $\mathcal{G}$ if and only if for all $l \in \{0, \ldots, n\}$ the short exact sequence

$$0 \to N_{l-1} \longrightarrow N_l \longrightarrow N_l/N_{l-1} \to 0$$

splits.

The modules of our interest are in a somewhat opposite situation. We are interested in conditions, when none of these sequences split.
We say that for a k-algebra A an A-module M is a brick, if $\operatorname{End}_A(M) \cong k$.

Proposition 4.2.4. *Assume that the following conditions hold:*

(i) I *is a principal ideal,*

(ii) $\dim_k \operatorname{Ext}^1_{\mathcal{N}}(Z, Z) = 1$*, and*

(iii) $k\mathcal{G} \otimes_{k\mathcal{N}} Z$ *is a brick.*

Then for all $l \in \{1, \ldots, n\}$ *the short exact sequence*

$$0 \to N_{l-1} \longrightarrow N_l \longrightarrow Z \to 0$$

does not split.

Proof. Since $N_n = k\mathcal{G} \otimes_{k\mathcal{N}} Z$ is a brick, it is indecomposable and the sequence

$$0 \to N_{n-1} \longrightarrow N_n \xrightarrow{p_{n-1,n}} Z \to 0$$

cannot split. Hence there is a minimal $l \in \{1, \ldots, n\}$ such that the short exact sequence

$$0 \to N_{l-1} \longrightarrow N_l \xrightarrow{p_{l-1,l}} Z \to 0$$

does not split. Assume $l > 1$. Then the diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & N_{l-1} \arrow[r] \arrow[d, "p_{0,l-1}"] & N_l \arrow[r, "p_{l-1,l}"] \arrow[d, "p_{0,l}"] & Z \arrow[r] \arrow[d, "id"] & 0 \\ 0 \arrow[r] & N_{l-2} \arrow[r] & N_{l-1} \arrow[r, "p_{l-2,l-1}"'] & Z \arrow[r] & 0 \end{tikzcd}$$

is commutative. If we identify the rows with elements in $\mathrm{Ext}^1_{\mathcal{N}}(Z, N_{l-1})$ and $\mathrm{Ext}^1_{\mathcal{N}}(Z, N_{l-2})$, then the map $p^*_{0,l-1} : \mathrm{Ext}^1_{\mathcal{N}}(Z, N_{l-1}) \to \mathrm{Ext}^1_{\mathcal{N}}(Z, N_{l-2})$ sends the first row to the second row ([45, 7.2]).
By assumption (iii) and Frobenius reciprocity we have

$$1 \leq \dim_k \mathrm{Hom}_{\mathcal{N}}(Z, Z) \leq \dim_k \mathrm{Hom}_{\mathcal{N}}(Z, k\mathcal{G} \otimes_{k\mathcal{N}} Z) = \dim_k \mathrm{End}_{\mathcal{G}}(k\mathcal{G} \otimes_{k\mathcal{N}} Z) = 1.$$

As $\mathrm{Hom}_{\mathcal{N}}(Z, -)$ is left exact, the spaces $\mathrm{Hom}_{\mathcal{N}}(Z, N_{l-1})$ and $\mathrm{Hom}_{\mathcal{N}}(Z, N_{l-2})$ can be identified with subspaces of $\mathrm{Hom}_{\mathcal{N}}(Z, k\mathcal{G} \otimes_{k\mathcal{N}} Z)$. As $l > 1$ they are non-trivial and consequently also one-dimensional. By assumption (ii) we have $\dim_k \mathrm{Ext}^1_{\mathcal{N}}(Z, Z) = 1$. Therefore the short exact sequence

$$0 \to Z \longrightarrow N_{l-1} \overset{p_{0,l-1}}{\longrightarrow} N_{l-2} \to 0$$

induces the long exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{N}}(Z, Z) \overset{\sim}{\longrightarrow} \mathrm{Hom}_{\mathcal{N}}(Z, N_{l-1}) \overset{0}{\longrightarrow} \mathrm{Hom}_{\mathcal{N}}(Z, N_{l-2})$$

$$\overset{\sim}{\longrightarrow} \mathrm{Ext}^1_{\mathcal{N}}(Z, Z) \overset{0}{\longrightarrow} \mathrm{Ext}^1_{\mathcal{N}}(Z, N_{l-1}) \overset{p^*_{0,l-1}}{\longrightarrow} \mathrm{Ext}^1_{\mathcal{N}}(Z, N_{l-2}).$$

Hence $p^*_{0,l-1}$ is injective and sends non-split exact sequences to non-split exact sequences ([45, 7.2]). Thus the short exact sequence

$$0 \to N_{l-2} \longrightarrow N_{l-1} \overset{p_{l-2,l-1}}{\longrightarrow} Z \to 0$$

does not split, a contradiction. Consequently $l = 1$. $\square$

The following proposition gives us a tool for realizing modules belonging to homogeneous tubes. The assumptions are for example fulfilled for $SL(2)_1T_r$.

Proposition 4.2.5. *Assume that $\mathcal{N}$ is infinitesimal of height 1 and that the following conditions hold:*

(a) I is a principal ideal,

(b) $k\mathcal{G} \otimes_{k\mathcal{N}} Z$ is a brick, and

(c) Z belongs to a homogeneous tube Θ of the stable Auslander-Reiten quiver $\Gamma_s(\mathcal{N})$.

Then N_l is the indecomposable $\mathcal{N}$-module of quasi-length $l+1$ in Θ.

Proof. We first show, that Z is the quasi-simple module in Θ. Let

$$\mathcal{E} : 0 \to Z \xrightarrow{\alpha} E \xrightarrow{\beta} Z \to 0$$

be the almost split sequence ending in Z. By 2.2.8 we have a decomposition $E = \bigoplus_{i=1}^{n} E_i$ into non-projective indecomposable $\mathcal{N}$-modules. Applying $\operatorname{Hom}_{\mathcal{N}}(Z, -)$ to $\mathcal{E}$ yields the sequence

$$0 \to \operatorname{End}_{\mathcal{N}}(Z) \overset{\beta_*}{\to} \operatorname{Hom}_{\mathcal{N}}(Z, E) \overset{\alpha_*}{\to} \operatorname{End}_{\mathcal{N}}(Z)$$

As $\mathcal{E}$ does not split and $\operatorname{End}_{\mathcal{N}}(Z)$ is isomorphic to k we obtain $\alpha_* = 0$ and that β_* is an isomorphism. Since for each $i \in \{1, \dots, n\}$ there is an irreducible map $Z \to E_i$ we obtain $\operatorname{Hom}_{\mathcal{N}}(Z, E_i) \neq 0$. Consequently $n \leq \dim_k \operatorname{Hom}_{\mathcal{N}}(Z, E) = \dim_k \operatorname{End}_{\mathcal{N}}(Z) = 1$, so that $n = 1$. This is only possible if Z is quasi-simple.
Now define for all $l, j \in \{1, \dots, n\}$ with $j \geq l$ the maps $\delta_l := \sum_{i=0}^{l-1} p_{i,l} : N_l \to N_{l-1}$ and the injections $\iota_{l,j} : N_l \to N_j$. Then we get:

$$p_{0,l+1} \circ \delta_l = \sum_{i=0}^{l-1} p_{0,l+1} \circ p_{i,l} \underset{4.2.2(d)}{=} \sum_{i=0}^{l-1} p_{i+1,l} = \sum_{i=1}^{l} p_{i,l} = \sum_{i=1}^{l-1} p_{i,l} = \delta_l - p_{0,l}.$$

This gives us

$$\begin{pmatrix} \iota_{l-1,l} \; , \; p_{0,l+1} \end{pmatrix} \circ \begin{pmatrix} -\delta_l \\ \iota_{l,l+1} + \delta_l \end{pmatrix} = -\delta_l + p_{0,l} + p_{0,l+1} \circ \delta_l = 0.$$

Therefore we obtain a short exact sequence:

$$0 \longrightarrow N_l \xrightarrow{\begin{pmatrix} -\delta_l \\ \iota_{l,l+1}+\delta_l \end{pmatrix}} N_{l-1} \oplus N_{l+1} \xrightarrow{\begin{pmatrix} \iota_{l-1,l} \; , \; p_{0,l+1} \end{pmatrix}} N_l \longrightarrow 0.$$

Now we show by induction over l that N_l belongs to Θ and has quasi-length $l+1$. By 2.1.5, every exact sequence

$$0 \to Z \longrightarrow X \longrightarrow Z \to 0$$

is either split or almost split. Hence we have $\dim_k \operatorname{Ext}^1_{\mathcal{N}}(Z, Z) = 1$. Thanks to 4.2.4, the short exact sequence

$$0 \to N_{j-1} \longrightarrow N_j \xrightarrow{p_{j-1,j}} Z \to 0$$

does not split for all $j \in \{1, \dots, n\}$. Especially the exact sequence

$$0 \to Z \longrightarrow N_1 \longrightarrow Z \to 0$$

does not split and therefore is almost split. As Z is the quasi-simple module in Θ and since by 2.2.8 the middle term of the above sequence has no non-zero projective direct summand, it follows that N_1 is the indecomposable $\mathcal{N}$-module of quasi-length 2 in Θ.
Now let $l \geq 1$ and assume for all $j \leq l$ that N_j is a module of quasi-length $j+1$ in Θ. As N_l and N_{l-1} are indecomposable $\mathcal{N}$-modules which are not isomorphic to each other the exact sequence

$$0 \longrightarrow N_l \xrightarrow{\begin{pmatrix} -\delta_l \\ \iota_{l,l+1}+\delta_l \end{pmatrix}} N_{l-1} \oplus N_{l+1} \xrightarrow{\begin{pmatrix} \iota_{l-1,l} \ , \ p_{0,l+1} \end{pmatrix}} N_l \longrightarrow 0$$

cannot split. Applying standard properties of almost split sequences ([2, V.1]) we obtain $\dim_k \operatorname{Hom}_{\mathcal{N}}(N_l, N_l) = l+1$. As for all $-1 \leq i \leq l-1$ the map $\iota_{l-i-1,l} \circ p_{i,l}$ with image N_{l-i-1} belongs to $\operatorname{Hom}_{\mathcal{N}}(N_l, N_l)$, we get that these maps form a k-basis of $\operatorname{Hom}_{\mathcal{N}}(N_l, N_l)$. The only isomorphism of these maps is $\iota_{l,l} \circ p_{-1,l} = id_{N_l}$. Hence if $\varphi = \sum_{i=-1}^{l-1} \lambda_i \, \iota_{l-i-1,l} \circ p_{i,l} \in \operatorname{Hom}_{\mathcal{N}}(N_l, N_l)$ is not an isomorphism, then $\lambda_{-1} = 0$. Thus the image of φ must be a submodule of N_{l-1}. But then φ factors through $\begin{pmatrix} \iota_{l-1,l} \\ p_{0,l+1} \end{pmatrix}$ and by 2.1.4 the above exact sequence is almost split. Moreover, by 2.2.8 the middle term of this sequence has no non-zero projective direct summand. Therefore N_{l+1} must be a successor of N_l in Θ. As Θ is a homogeneous tube, the module N_l of quasi-length $l+1$ has exactly two successors, one of quasi-length l and one of quasi-length $l+2$. Since N_{l-1} has quasi-length l it follows that N_{l+1} must be the indecomposable $\mathcal{N}$-module of quasi-length $l+2$ in Θ. $\square$

4.3. Realizations of periodic $SL(2)_1T_r$-modules

Let k be an algebraically closed field of characteristic $p > 2$, $T \subseteq SL(2)$ be the torus of diagonal matrices and $B \subseteq SL(2)$ the standard Borel subgroup of upper triangular matrices. Let $C \subseteq SL(2,k)$ be a set of representatives for $SL(2,k)/B$ with $\{1, w_0\} \subseteq C$ and $g \in C \setminus \{1, w_0\}$. Set $\mathcal{G} := SL(2)_1T_r$ for $r \geq 1$ and $\mathcal{N} := SL(2)_1$. For $0 \leq a \leq p-2$ we consider the filtration by $\mathcal{N}$-modules

$$Z(a)^g \cong N_0 \subseteq N_2 \subseteq \ldots \subseteq N_{p^{r-1}-1} = k\mathcal{G} \otimes_{k\mathcal{N}} Z(a)^g$$

of the induced module $k\mathcal{G} \otimes_{k\mathcal{N}} Z(a)^g$.

Proposition 4.3.1. *For all $l \in \{0, \ldots, p^{r-1}-1\}$, the $\mathcal{N}$-module N_l is isomorphic to $g.W((l+1)p+a)$.*

Proof. The augmentation ideal of $k[\mathcal{G}/\mathcal{N}] \cong k[\mu_{(p^{r-1})}] = k[T]/(T^{p^{r-1}} - 1)$ is a principal ideal. By 4.1.2 the restriction of the induced $\mathcal{G}$-module $k\mathcal{G} \otimes_{k\mathcal{N}} Z(a)^g$ to $\mathcal{N}$ is isomorphic to $g.W(p^r + a)$. Therefore 2.2.5 and Frobenius reciprocity yield $\dim_k \operatorname{End}_{\mathcal{G}}(k\mathcal{G} \otimes_{k\mathcal{N}} Z(a)^g) = \dim_k \operatorname{Hom}_{\mathcal{N}}(Z(a)^g, g.W(p^r + a)) = 1$, so that $k\mathcal{G} \otimes_{k\mathcal{N}} Z(a)^g$ is a brick. By 4.1.1, the module $Z(a)^g$ is quasi-simple and belongs to a homogeneous tube Θ of the stable Auslander-Reiten quiver $\Gamma_s(\mathcal{N})$. Additionally, 4.1.1 yields that $g.W(lp+a)$ is the $\mathcal{N}$-module of quasi-length l in Θ. The assertion now follows by applying 4.2.5. $\square$

Remark 4.3.2. The above result can also be applied if $g \in \{1, w_0\}$. One only has to use another torus $\hat{T}$ such that the induction of $Z(a)$ respectively $Z(a)^{w_0}$ to $SL(2)_1\hat{T}_r$ is indecomposable.

Our next result will now use this new description of these $SL(2)_1$-modules and the filtration of induced modules to obtain a realization of the $SL(2)_1T_r$-modules which

belong to homogeneous tubes. Moreover, for a subgroup scheme H of $N_{SL(2)}(T) \cap B^g$ we are able to extend these modules to $SL(2)_1T_rH$, which later will be of interest for the classification of modules for domestic group schemes.
Denote by B^g the subgroup of $SL(2)$ which is obtained by conjugating all elements of B by g. Then $Z(a)^g \cong g.W(p+a)$ is stable under the action of B^g, so that $Z(a)^g$ is an $SL(2)_1B^g$-module.

Theorem 4.3.3. *Let $g \in C \setminus \{1, w_0\}$, $0 \leq a \leq p-2$ and H be a subgroup scheme of $N_{SL(2)}(T) \cap B^g$. For $n \geq 1$ let $\mathcal{H}_{(n)} := SL(2)_1T_nH$ and $\mathcal{N} := SL(2)_1$. Let $r, s \geq 1$, $Y := k\mathcal{H}_{(r)} \otimes_{k\mathcal{H}_{(1)}} Z(a)^g$ and denote the filtration by $\mathcal{H}_{(r)}$-modules of the induced module $N := k\mathcal{H}_{(r+s)} \otimes_{k\mathcal{H}_{(r)}} Y$ by*

$$k\mathcal{H}_{(r)} \otimes_{k\mathcal{H}_{(1)}} Z(a)^g \cong N_0 \subseteq N_1 \subseteq \ldots \subseteq N_{p^s-1} = N.$$

Then $\operatorname{res}_{\mathcal{N}}^{\mathcal{H}_{(r)}} N_{l-1} \cong g.W(lp^r + a)$ for all $1 \leq l \leq p^s$.

Proof. Set $\mathcal{G} := SL(2)_1T_{r+s}$. As $\mathcal{H}_{(r+s)}/\mathcal{H}_{(1)} \cong \mu_{(p^{r+s-1})} \cong \mathcal{G}/\mathcal{N}$ we obtain $k[\mathcal{H}_{(r+s)}]^{\mathcal{H}_{(1)}} \cong k[\mu_{(p^{r+s-1})}] \cong k[\mathcal{G}]^{\mathcal{N}}$. Denote the filtration by $\mathcal{H}_{(1)}$-modules of the induced module $N \cong k\mathcal{H}_{(r+s)} \otimes_{k\mathcal{H}_{(1)}} Z(a)^g$ by

$$Z(a)^g \cong M_0 \subseteq M_1 \subseteq \ldots \subseteq M_{p^{r+s-1}-1} = N.$$

The $\mathcal{H}_{(r+s)}$-module N is over $\mathcal{G}$ isomorphic to $k\mathcal{G} \otimes_{k\mathcal{N}} Z(a)^g$. These modules are also isomorphic over $k[\mu_{(p^{r+s-1})}]$ with respect to the action defined in section 4.2. Applying 4.2.1 yields that the restriction of the modules M_i to $\mathcal{N}$ is the filtration by $\mathcal{N}$-modules of the induced module $k\mathcal{G} \otimes_{k\mathcal{N}} Z(a)^g$. Let J be the kernel of the canonical projection $k[\mathcal{H}_{(r+s)}] \to k[\mathcal{H}_{(1)}]$. Then the ideal $J^{p^{r-1}}$ is the kernel of the canonical projection $k[\mathcal{H}_{(r+s)}] \to k[\mathcal{H}_{(r)}]$. By 4.2.1, we get the equality $M_{p^{r-1}l-1} = \operatorname{res}_{\mathcal{H}_{(1)}}^{\mathcal{H}_{(r)}} N_{l-1}$ for all $1 \leq l \leq p^s$.
By 4.1.2, there is for any $l \geq 1$ a unique $SL(2)_1T_r$-module $X(i, g, l)$ which is isomorphic to $g.W(lp^r + a)$ over $\mathcal{N}$. Thanks to 4.3.1, this module is isomorphic to $\operatorname{res}_{\mathcal{N}}^{\mathcal{H}_{(1)}} M_{p^{r-1}l-1} = \operatorname{res}_{\mathcal{N}}^{\mathcal{H}_{(r)}} N_{l-1}$. □

Remark 4.3.4. Thanks to this result we have realized the $SL(2)_1T_r$-modules $X(i, g, l)$. Consequently, we have completed the classification of the indecomposable $SL(2)_1T_r$-modules.

5. Modules for domestic finite group schemes

In this section we will develop the tools for the classification of the indecomposable modules of an amalgamated polyhedral group scheme. At first, we will investigate a group action on the rank variety of the Lie algebra associated to a finite group scheme. The stabilizers of this action are connected to the stabilizers of the corresponding modules. The main result will be that for a tame group scheme these stabilizers are cyclic groups. After that, we will consider the decomposition of an induced module. Under certain circumstances we are able to give a description of this decomposition. In the last subsection we will combine our results to obtain methods for describing the modules of an amalgamated polyhedral group scheme.

5.1. Actions on rank varieties and their stabilizers

Let k be an algebraically closed field. We say that X is a variety, if it is a separated reduced prevariety over k and we will identify it with its associated separated reduced k-scheme of finite type (c.f. [25],[23]). A point $x \in X$ is always supposed to be closed and therefore also to be k-rational, as k is algebraically closed. The structure sheaf of X will be denoted by $\mathcal{O}_X$. Let $x \in X$ be a point. The local ring of X at x will be denoted by $\mathcal{O}_{X,x}$ and its maximal ideal $\mathfrak{m}_{X,x}$. The tangent space $T_{X,x}$ of x is defined as the dual space $(\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2)^*$.

Definition 5.1.1. Let X be a variety. A point $x \in X$ is called simple, if $\mathcal{O}_{X,x}$ is a regular local ring.

The following result can be found in [40, Lemma 4] for $\operatorname{char} k = 0$, but the proof can easily be modified such that it applies to finite groups whose order are relatively prime to the characteristic of the field.

Lemma 5.1.2. *[40, Lemma 4] Let X be an irreducible variety and G be a finite group with $p \nmid |G|$ which acts faithfully on X. Let $x \in X$ be a fixed point of G. Then the induced action of G on $T_{X,x}$ is faithful.*

Remark 5.1.3. The result can also be generalized to finite linearly reductive group schemes acting on X. Consequently there are also generalizations of the following results to this situation.

Let k be a field of characteristic $p > 0$, $\mathcal{G}$ a finite group scheme and $\mathfrak{g} := \operatorname{Lie}(\mathcal{G})$ its Lie algebra. The nullcone $V_{\mathfrak{g}}$ is a cone, so that we can consider the projective variety $\mathbb{P}(V_{\mathfrak{g}})$. There is an action of the group-like elements of $k\mathcal{G}$ on its primitive elements and therefore we obtain an action of $\mathcal{G}(k)$ on $\mathfrak{g}$. Moreover, this action induces an action of $G := \mathcal{G}(k)$ on $\mathbb{P}(V_{\mathfrak{g}})$.
Now the rank variety of a twisted module M^g can be computed as $\mathbb{P}(V_{\mathfrak{g}}(M^g)) = g.\mathbb{P}(V_{\mathfrak{g}}(M))$. If $\mathbb{P}(V_{\mathfrak{g}}(M)) = \{x\}$, then it is easy to see that $G_M \subseteq G_x$, where G_x is the stabilizer of x.

Lemma 5.1.4. *Let $\mathcal{G}$ be a finite group scheme with Lie algebra $\mathfrak{g} := \mathrm{Lie}(\mathcal{G})$ such that the variety $\mathbb{P}(V_\mathfrak{g})$ is irreducible. Assume that the order of $G := \mathcal{G}(k)$ is relatively prime to p and that G acts faithfully on $\mathbb{P}(V_\mathfrak{g})$. Moreover, let $r := \dim \mathbb{P}(V_\mathfrak{g})$ and $x \in \mathbb{P}(V_\mathfrak{g})$ be a simple point. Then there is an injective homomorphism $G_x \to GL_r(k)$.*

Proof. Since x is a fixed point of G_x and x is a simple point, the action of G_x on $T_{\mathbb{P}(V_\mathfrak{g}),x}$ is faithful, by Lemma 5.1.2. As the point x is simple, we have $r = \dim_k T_{\mathbb{P}(V_\mathfrak{g}),x}$. So, there is an injective homomorphism $G_x \to GL(T_{\mathbb{P}(V_\mathfrak{g}),x}) \cong GL_r(k)$. □

Remark 5.1.5. Let $\mathcal{G} \subseteq SL(2)$ with $\mathcal{G}^0 \cong SL(2)_1$ and M be a $\mathcal{G}^0$-module which belongs to a homogeneous tube Θ. Then there are $g \in SL(2,k)$ and $d \in \mathbb{N}$ with $M \cong g.W(d)$ and $\mathbb{P}(V_{\mathfrak{sl}(2)}(g.W(d))) = \{g.[e]\}$. Let $h \in G_{g.[e]}$. Then $g^{-1}hg.[e] = [e]$ and hence $g^{-1}hg$ is an element of the standard Borel subgroup of upper triangular matrices B. From this follows that $hg.W(d) = g.W(d)$ and thus $h \in G_{g.W(d)}$. Therefore we obtain

$$G_{g.W(d)} = G_{g.[e]}.$$

Corollary 5.1.6. *Let $\mathcal{G}$ be a finite group scheme with Lie algebra $\mathfrak{g} = \mathrm{Lie}(\mathcal{G})$ such that $\mathbb{P}(V_\mathfrak{g})$ is smooth and irreducible. Assume that $\mathcal{G}/\mathcal{G}_1$ is linearly reductive and that $G = \mathcal{G}(k)$ acts faithfully on $\mathbb{P}(V_\mathfrak{g})$. Let $r := \dim \mathbb{P}(V_\mathfrak{g})$ and M be an indecomposable $\mathcal{G}^0$-module of complexity 1. Then there is an injective homomorphism $G_M \to GL_r(k)$. If additionally $r = 1$ (for example when $k\mathcal{G}^0$ is tame), then G_M is a cyclic group.*

Proof. By 1.4.18, the extension $k\mathcal{G}^0 : k\mathcal{G}_1$ is separable, so that there is an indecomposable direct summand N of $\mathrm{res}^{\mathcal{G}^0}_{\mathcal{G}_1} M$ such that M is a direct summand of $\mathrm{ind}^{\mathcal{G}^0}_{\mathcal{G}_1} N$. Thanks to 1.5.12 the module $\mathrm{res}^{\mathcal{G}^0}_{\mathcal{G}_1} M$ is indecomposable and therefore $N = \mathrm{res}^{\mathcal{G}^0}_{\mathcal{G}_1} M$. General properties of the complexity ([16, II.2]) yield

$$1 = \mathrm{cx}_{k\mathcal{G}^0}(M) \leq \mathrm{cx}_{k\mathcal{G}^0}(\mathrm{ind}^{\mathcal{G}^0}_{\mathcal{G}_1} N) \leq \mathrm{cx}_{k\mathcal{G}_1}(N) \leq \mathrm{cx}_{k\mathcal{G}_1}(M) \leq \mathrm{cx}_{k\mathcal{G}^0}(M) = 1.$$

Hence N has also complexity 1 and the variety $\mathbb{P}(V_{\mathfrak{sl}(2)}(M))$ has dimension 0. By 1.6.4, the indecomposability of M yields that the variety $\mathbb{P}(V_{\mathfrak{sl}(2)}(M))$ consists of only one point x. Thus $G_M \subseteq G_x$. The first assertion now follows from 5.1.4. If $r = \dim \mathbb{P}(V_\mathfrak{g}) = 1$, then G_M is isomorphic to a finite subgroup of $k^\times$ and therefore cyclic. □

Example 5.1.7. Let $\mathcal{H}$ be a finite reduced linearly reductive subgroup of GL_n. Then $\mathcal{H}$ acts naturally on the n-fold product $\mathbb{G}^n_{a(1)}$ of the first Frobenius kernel of the additive group $\mathbb{G}_a$. Hence we can form the semi-direct product $\mathcal{G} := \mathbb{G}^n_{a(1)} \rtimes \mathcal{H}$. Then $\mathcal{G}^0 = \mathbb{G}^n_{a(1)}$, $\mathcal{G}_{red} = \mathcal{H}$ and $\mathfrak{g} := \mathrm{Lie}(\mathcal{G}^0)$ is an n-dimensional abelian restricted Lie algebra with trivial p-map. The nullcone can be computed as $\mathbb{P}(V_\mathfrak{g}) \cong \mathbb{P}(\mathfrak{g}) \cong \mathbb{P}^{n-1}$ and $G := \mathcal{G}(k) = \mathcal{H}(k)$ acts faithfully on this variety. Hence we can apply 5.1.6, so that the stabilizer G_M of every periodic $\mathcal{G}^0$-module M is isomorphic to a finite subgroup of $GL_{n-1}(k)$.

5.2. Decomposition of induced modules

For strongly group graded algebras there are ways to describe the decomposition of an induced module ([26, 4.5]). This decomposition is connected to the ring structure of the endomorphism ring of this induced module via the following ring theoretic results:

Lemma 5.2.1 ([26, 4.5.11])**.** *Let R be a ring, N be a nil ideal of R and $\pi : R \to R/N$ be the canonical projection. Assume that R/N has a decomposition $\bigoplus_{i=1}^{n} J_i$ into finitely many indecomposable left ideals. Then R has a decomposition $\bigoplus_{i=1}^{n} I_i$ into indecomposable left ideals such that*

1. *$\pi(I_i) = J_i$ for all $i \in \{1, \dots, n\}$, and*

2. *$I_i \cong I_j$ as R-modules if and only if $J_i \cong J_j$ as R/N-modules.*

Lemma 5.2.2 ([26, 4.5.12])**.** *Let R be a ring, M be an R-module and assume that $E := \operatorname{End}_R(M)^{op}$ has a decomposition $\bigoplus_{i=1}^{n} I_i$ into finitely many indecomposable left ideals. Then the following statements hold:*

1. *$M \cong \bigoplus_{i=1}^{n} M \otimes_E I_i$ is a decomposition into indecomposable R-modules.*

2. *$M \otimes_E I_i \cong M \otimes_E I_j$ as R-modules if and only if $I_i \cong I_j$ as E-modules.*

We want to generalize the result for group graded algebras to the context of certain modules of a finite group scheme. In the situation of G-graded algebras, these decomposition results hold for G-invariant modules. In our situation this is not enough. There are already some generalized results in the context of H-Galois extensions and H-stable modules. Some of these results can be found in [51], [47] and [50]. In general, it is not possible for these induced modules to obtain a decomposition as in the group graded case. Therefore, we want to consider modules satisfying a stronger stability condition.
Let $\mathcal{M}$ be a multiplicative group scheme. Then we obtain for any $\mathcal{M}$-module M a weight space decomposition $M = \bigoplus_{\lambda \in X(\mathcal{M})} M_\lambda$ with

$$M_\lambda := \{m \in M \mid hm = \lambda(h)m \text{ for all } h \in k\mathcal{M}\}.$$

Let $\mathcal{G}$ be a finite group scheme and $\mathcal{N} \subseteq \mathcal{G}^0$ a normal subgroup scheme of $\mathcal{G}$ such that $\mathcal{G}^0/\mathcal{N}$ is linearly reductive. By Nagata's Theorem [7, IV,§3,3.6], an infinitesimal group scheme is linearly reductive if and only if it is multiplicative.
Let M be a $\mathcal{G}^0$-module. For any $\lambda \in X(\mathcal{G}^0/\mathcal{N})$ we obtain a $\mathcal{G}^0$-module $M \otimes_k k_\lambda$, the tensor product of M with the one-dimensional $\mathcal{G}^0/\mathcal{N}$-module defined by λ. This defines an action of $X(\mathcal{G}^0/\mathcal{N})$ (here we identify $X(\mathcal{G}^0/\mathcal{N})$ with a subgroup of $X(\mathcal{G}^0)$ via the canonical inclusion $X(\mathcal{G}^0/\mathcal{N}) \hookrightarrow X(\mathcal{G}^0)$ which is induced by the canonical projection $\mathcal{G}^0 \twoheadrightarrow \mathcal{G}^0/\mathcal{N}$) on the isomorphism classes of $\mathcal{G}^0$-modules and we define the stabilizer of a $\mathcal{G}^0$-module as $X(\mathcal{G}^0/\mathcal{N})_M := \{\lambda \in X(\mathcal{G}^0/\mathcal{N}) \mid M \otimes_k k_\lambda \cong M\}$. A $\mathcal{G}^0$-module M is called $\mathcal{G}^0/\mathcal{N}$-regular if its stabilizer is trivial.

Proposition 5.2.3. *Let $\mathcal{G}$ be a finite group scheme over an algebraically closed field k and $\mathcal{N}$ be an infinitesimal normal subgroup scheme such that $\mathcal{G}^0/\mathcal{N}$ is multiplicative. Let $k(\mathcal{G}/\mathcal{N}) = \bigoplus_{i=1}^{n} P_i$ be the decomposition into projective indecomposable $\mathcal{G}/\mathcal{N}$-modules. Let M be a $\mathcal{G}$-module such that $\operatorname{res}^{\mathcal{G}}_{\mathcal{G}^0} M$ is indecomposable and M is $\mathcal{G}^0/\mathcal{N}$-regular. Then $k\mathcal{G} \otimes_{k\mathcal{N}} M$ is isomorphic to the direct sum $\bigoplus_{i=1}^{n} M \otimes_k P_i$ of indecomposable $\mathcal{G}$-modules and $M \otimes_k P_i \cong M \otimes_k P_j$ as $\mathcal{G}$-modules if and only if $P_i \cong P_j$ as $\mathcal{G}/\mathcal{N}$-modules.*

Proof. Let $G = \mathcal{G}(k)$. Then $k\mathcal{G}^0 \subseteq k\mathcal{G}$ is a kG-Galois extension and $k\mathcal{N} \subseteq k\mathcal{G}^0$ is a $k(\mathcal{G}^0/\mathcal{N})$-Galois extension. We set $E := \mathrm{End}_{\mathcal{G}}(k\mathcal{G} \otimes_{k\mathcal{N}} M)$ and $E' := \mathrm{End}_{\mathcal{N}}(M)$. As M is a $\mathcal{G}$-module, we can apply 1.4.13 twice to get isomorphisms

$$E \cong \mathrm{End}_{\mathcal{G}^0}(k\mathcal{G}^0 \otimes_{k\mathcal{N}} M)\#kG \cong (E'\#k(\mathcal{G}^0/\mathcal{N}))\#kG.$$

Denote by $\pi : k\mathcal{G}^0 \to k(\mathcal{G}^0/\mathcal{N})$ the canonical projection. Then the comodule map $\rho : k\mathcal{G}^0 \to k\mathcal{G}^0 \otimes k(\mathcal{G}^0/\mathcal{N})$ is given by $\rho(a) = \sum_{(a)} a_{(1)} \otimes \pi(a_{(2)})$. This yields

$$\beta(\sum_{(a)} \eta(a_{(1)}) \otimes a_{(2)}) = 1 \otimes \pi(a).$$

Therefore the $k(\mathcal{G}^0/\mathcal{N})$-action on E' is given by

$$(h.f)(m) = \sum_{(a)} a_{(1)} f(\eta(a_{(2)})m)$$

for $h \in k(\mathcal{G}^0/\mathcal{N})$, $f \in E'$, $m \in M$ and $a \in k\mathcal{G}^0$ with $\pi(a) = h$. Hence we obtain $(E')^{\mathcal{G}^0/\mathcal{N}} = \mathrm{End}_{\mathcal{G}^0}(M)$.

Since $\mathcal{G}^0/\mathcal{N}$ is multiplicative, the space E' affords a decomposition into weight spaces $E' = \bigoplus_{\lambda \in X(\mathcal{G}^0/\mathcal{N})} E'_\lambda$ with $E'_0 = (E')^{\mathcal{G}^0/\mathcal{N}} = \mathrm{End}_{\mathcal{G}^0}(M)$. As M is regular we have: (c.f. for example the proof of [15, 3.1.4])

$$\mathrm{Rad}(E') = \mathrm{Rad}(\mathrm{End}_{\mathcal{G}^0}(M)) \oplus \bigoplus_{\substack{\lambda \in X(\mathcal{G}^0/\mathcal{N}) \\ \lambda \neq 0}} E'_\lambda.$$

Hence the Jacobson radical $\mathrm{Rad}(E')$ is stable under the action of $\mathcal{G}^0/\mathcal{N}$ so that $\mathrm{Rad}(E') = \mathrm{Rad}^{k(\mathcal{G}^0/\mathcal{N})}(E')$. By 1.4.15, we get

$$\mathrm{Rad}(E')\#k(\mathcal{G}^0/\mathcal{N}) \subseteq \mathrm{Rad}(E'\#k(\mathcal{G}^0/\mathcal{N})).$$

Moreover, by 1.4.15, we have

$$\mathrm{Rad}(E'\#k(\mathcal{G}^0/\mathcal{N}))\#kG \subseteq \mathrm{Rad}((E'\#k(\mathcal{G}^0/\mathcal{N}))\#kG).$$

Therefore, $(\mathrm{Rad}(E')\#k(\mathcal{G}^0/\mathcal{N}))\#kG$ is nilpotent. By 1.5.12, the indecomposability of $\mathrm{res}^{\mathcal{G}}_{\mathcal{G}^0} M$ yields that $\mathrm{res}^{\mathcal{G}}_{\mathcal{N}} M$ is indecomposable and therefore E' is local. As k is algebraically closed, we obtain isomorphisms

$$\begin{aligned} &(E'\#k(\mathcal{G}^0/\mathcal{N}))\#kG)/((\mathrm{Rad}(E')\#k(\mathcal{G}^0/\mathcal{N}))\#kG) \\ \cong&(k\#k(\mathcal{G}^0/\mathcal{N}))\#kG \cong k(\mathcal{G}^0/\mathcal{N})\#kG \\ \cong&k(\mathcal{G}/\mathcal{N}). \end{aligned}$$

Now let $p : E \to k(\mathcal{G}/\mathcal{N})$ be the surjection given by the above isomorphisms. Then the map $\psi : k\mathcal{G} \otimes_{k\mathcal{N}} M \otimes_{E^{op}} E^{op} \to M \otimes_k k(\mathcal{G}/\mathcal{N})$ with $\psi(a \otimes m \otimes \varphi) = \sum_{(a)} a_{(1)} m \otimes \pi(a_{(2)}) p(\varphi)$ is an isomorphism of $\mathcal{G}$-modules:

The $\mathcal{G}$-linearity follows directly from the definition of the operation on $M \otimes_k k(\mathcal{G}/\mathcal{N})$. Let $m \in M$ and $y \in k\mathcal{G}$. Then

$$\begin{aligned}\psi(\sum_{(y)} y_{(1)} \otimes \eta(y_{(2)})m \otimes 1) &= \sum_{(y)} y_{(1)}\eta(y_{(2)})m \otimes \pi(y_{(3)}) \\ =\sum_{(y)} \varepsilon(y_{(1)})m \otimes \pi(y_{(2)}) &= m \otimes \pi(y),\end{aligned}$$

so that ψ is surjective and therefore bijective for dimension reasons.
Hence we have a decomposition $k\mathcal{G} \otimes_{k\mathcal{N}} M \cong \bigoplus_{i=1}^n M \otimes_k P_i$ of $\mathcal{G}$-modules. Now let $e_1, \dots, e_n \in k(\mathcal{G}/\mathcal{N})$ be primitive idempotents with $P_i = k(\mathcal{G}/\mathcal{N})e_i$ and set $Q_i := e_i k(\mathcal{G}/\mathcal{N})$. Then $k(\mathcal{G}/\mathcal{N})^{op} = \bigoplus_{i=1}^n Q_i$ is a decomposition into indecomposable left ideals and by 5.2.1 this decomposition lifts to a decomposition $E^{op} = \bigoplus_{i=1}^n I_i$ of indecomposable left ideals of E^{op} such that $p(I_i) = Q_i$. Let $f_1, \dots, f_n \in E^{op}$ be primitive idempotents such that $p(f_i) = e_i$. Then

$$\psi(k\mathcal{G} \otimes_{k\mathcal{N}} M \otimes_{E^{op}} I_i) = M \otimes_k k(\mathcal{G}/\mathcal{N})p(f_i) = M \otimes_k k(\mathcal{G}/\mathcal{N})e_i = M \otimes_k P_i.$$

By 5.2.2 the $\mathcal{G}$-module $k\mathcal{G} \otimes_{k\mathcal{N}} M \otimes_{E^{op}} I_i$ is indecomposable so that $M \otimes_k P_i$ is also indecomposable. Moreover, we have $M \otimes_k P_i \cong M \otimes_k P_j$ as a $\mathcal{G}$-module if and only if $k\mathcal{G} \otimes_{k\mathcal{N}} M \otimes_{E^{op}} I_i$ is isomorphic to $k\mathcal{G} \otimes_{k\mathcal{N}} M \otimes_{E^{op}} I_j$ as a $\mathcal{G}$-module and by 5.2.2 these are isomorphic if and only if $I_i \cong I_j$ as left ideals of E^{op}. By 5.2.1 we have $I_i \cong I_j$ as left ideals of E^{op} if and only if $Q_i \cong Q_j$ as left ideals of $k(\mathcal{G}/\mathcal{N})^{op}$ and these are isomorphic if and only if $P_i \cong P_j$ as $\mathcal{G}/\mathcal{N}$-modules.

□

Example 5.2.4. There is a somewhat weaker result for non-regular M. Let $\mathcal{G}$ be infinitesimal and $M \cong M \otimes_k k_\lambda$ for all $\lambda \in X(\mathcal{G}/\mathcal{N})$. By [15, 2.1.5], the ring $\operatorname{End}_{\mathcal{N}}(M)$ is isomorphic to a crossed product $\operatorname{End}_{\mathcal{G}}(M)\#_\sigma kX(\mathcal{G}/\mathcal{N})$. Since M is a $\mathcal{G}$-module, we get as above an isomorphism $\operatorname{End}_{\mathcal{G}}(k\mathcal{G} \otimes_{k\mathcal{N}} M) \cong \operatorname{End}_{\mathcal{N}}(M)\#k(\mathcal{G}/\mathcal{N})$. Due to the isomorphism $(k(\mathcal{G}/\mathcal{N}))^* \cong kX(\mathcal{G}/\mathcal{N})$ and [33, 9.4.17], there is an isomorphism $E := \operatorname{End}_{\mathcal{G}}(k\mathcal{G} \otimes_{k\mathcal{N}} M) \cong \operatorname{End}_{\mathcal{G}}(M) \otimes M_n(k)$. Hence $\operatorname{Rad}(E) \cong \operatorname{Rad}(\operatorname{End}_{\mathcal{G}}(M)) \otimes_k M_n(k)$ and $E/\operatorname{Rad}(E) \cong M_n(k)$. Consequently

$$k\mathcal{G} \otimes_{k\mathcal{N}} M \cong M^n \cong \bigoplus_{\lambda \in X(\mathcal{G}/\mathcal{N})} M \otimes_k k_\lambda.$$

5.3. Modules of domestic finite group schemes

For any amalgamated polyhedral group scheme $\mathcal{G}$ there is an $r \geq 1$ such that $\mathcal{G}^0$ is isomorphic to $SL(2)_1T_r$. Recall that for $l \in \mathbb{N}$, $i \in \{0, \dots, p-2\}$ and $g \in SL(2) \setminus (B \cup w_0 B)$ there is a unique $SL(2)_1T_r$-module $X(i, g, l)$ such that $\operatorname{res}_{SL(2)_1} X(i, g, l) \cong g.W(lp^r + i)$. For $g \in B \cup w_0B$, the $SL(2)_1$-module $g.W(lp + i)$ is stable under the action of $SL(2)_1T_r$. By abuse of notation we write in this situation $X(i, g, l) = g.W(lp + i)$.

Let $\mathcal{Z}$ be the center of the group scheme $SL(2)$. As $p > 2$, the group scheme $\mathcal{Z}$ is reduced with $\mathcal{Z}(k) = \langle \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \rangle$. Let $H := N_{SL(2)}(T) \cap B^g$, $\mathcal{G} := SL(2)_1 T_r H/\mathcal{Z}$ and $\pi : SL(2)_1 T_r H \to \mathcal{G}$ be the canonical projection. For all $g \in SL(2)(k)$ the module $X(i,g,l)$ can be viewed as an $SL(2)_1 T_r H$-module as in 4.3.3 and $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ acts via $(-1)^i$. Therefore, for even i the action factors through π, so that $X(i,g,l)$ is a $\mathcal{G}$-module. For odd i we need to twist the action by a special character $\gamma : SL(2)_1 T_r H \to \mu_{(1)}$. To define this character consider the homomorphism

$$\varphi : SL(2)_1 T_r H \to SL(2)_1 T_r^{g^{-1}} H^{g^{-1}} \ , \ x \mapsto x^{g^{-1}}$$

of group schemes and the character

$$\tilde{\gamma} : SL(2)_1 T_r^{g^{-1}} H^{g^{-1}} \to \mu_{(1)} \ , \ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto d^{p^r}$$

(this is a character as $H^{g^{-1}} \subseteq B$ and as it is trivial on $SL(2)_1 T_r^{g^{-1}}$). Then $\gamma := \tilde{\gamma} \circ \varphi$. By definition $\gamma_{|SL(2)_1 T_r}$ is trivial and $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ acts trivially on $X(i,g,l) \otimes_k k_\gamma$. Consequently, the action of $SL(2)_1 T_r H$ on $X(i,g,l) \otimes_k k_\gamma$ factors through π. Therefore we can lift the $SL(2)_1 T_r H$-module $X(i,g,l)$ to a $\mathcal{G}$-module $Y(i,g,l)$ for all $i \in \{0, \ldots, p-2\}$.
In the same way we can twist the Weyl modules $V(d)$ and obtain a module $\tilde{V}(d)$ for any amalgamated polyhedral group scheme.

As in 4.1.3, let $C \subseteq SL(2,k)$ be a set of representatives for $SL(2,k)/B$ with $1, w_0 \in C$. Let $\mathcal{G}$ be a group scheme, $\mathcal{N}$ be a normal subgroup scheme of $\mathcal{G}$ with $\mathcal{G}^0 \subseteq \mathcal{N}$ and Y be an $\mathcal{N}$-module. As in 1.5.11, the stabilizer $\mathcal{G}_Y$ is uniquely determined by the stabilizer $\mathcal{G}(k)_Y$. In the same way, for $h \in \mathcal{G}(k)$, the conjugated subgroup scheme $\mathcal{G}_Y^h$ is determined by $\mathcal{G}(k)_Y^h$.
We denote by $h_4 \in GL(2)(k)$ the element which appears in the definition of the binary dihedral group scheme (c.f. 3.2).

Proposition 5.3.1. *Let $\mathcal{G}$ be an amalgamated polyhedral group scheme and r be the height of $\mathcal{G}^0$. Then the following statements hold:*

(a) For all $l \in \mathbb{N}$, $i \in \{0, \ldots, p-2\}$ and $g \in C$ there is an $h \in SL(2)(k)$ such that the group scheme $\mathcal{G}_{Y(i,g,l)}^h$ is either equal to $P\mathcal{SQ}_{(p^r)}$ or to $P\mathcal{SC}_{(np^r)}$ for one $n \in \mathbb{N}$ with $(n,p) = 1$.

(b) If $\mathcal{G}_{Y(i,g,l)}^h$ is equal to $P\mathcal{SQ}_{(p^r)}$, then $hg \in h_4 B \cup h_4 w_0 B$.

(c) If $\mathcal{G}_{Y(i,g,l)}^h$ is equal to $P\mathcal{SC}_{(np^r)}$ and $n > 1$, then $hg \in B \cup w_0 B$.

(d) $Y(i,g,l)$ is a $\mathcal{G}_{Y(i,g,l)}$-module.

Proof. (a) The variety $\mathbb{P}(V_{\mathfrak{sl}(2)}) \cong \mathbb{P}^1$ is smooth. Let $G := \mathcal{G}(k)$. Then $k\mathcal{G} \cong k\mathcal{G}^0 * G$ and $k\mathcal{G}_{Y(i,g,l)} \cong k\mathcal{G}^0 * G_{Y(i,g,l)}$. Thanks to 3.4.1 the group scheme $\mathcal{G}$ is domestic. By 3.1.4 this yields that $\mathcal{G}^0$ and $\mathcal{G}_{Y(i,g,l)}$ are domestic. Another application of 3.4.1 now shows that $\mathcal{G}_{Y(i,g,l)}$ is isomorphic to an amalgamated polyhedral group scheme.

In particular, by 3.3.2 this isomorphism is given via conjugation by an element $h \in SL(2)(k)$. Due to 5.1.6, the stabilizer $\mathcal{G}_{Y(i,g,l)}(k) = G_{Y(i,g,l)}$ is a cyclic group, so that we obtain (a) from 3.4.1 and the table in 3.3.2.

(b) A direct computation shows $\mathcal{G}^h_{Y(i,g,l)} = \mathcal{G}_{Y(i,hg,l)}$. Assume $\mathcal{G}_{Y(i,g,l)} = \mathcal{PSQ}_{(p^r)}$. Let $\pi : SL(2)(k) \to PSL(2)(k)$ be the canonical projection, $\hat{G} := \pi^{-1}(G)$ and $\hat{G}_{Y(i,g,l)} := \pi^{-1}(G_{Y(i,g,l)})$. By definition of $\mathcal{PSQ}_{(p^r)}$, the group $\hat{G}_{Y(i,g,l)}$ is a subgroup of T^{h_4}. Therefore $\hat{G}_{Y(i,g,l)}$ does only stabilize the points $[h_4.e]$ and $[h_4w_0.e]$ in $\mathbb{P}(V_{\mathfrak{sl}(2)})$. By 5.1.5, we have $\hat{G}_{Y(i,g,l)} = \hat{G}_{[g.e]}$. Therefore we obtain $[g.e] \in \{[h_4.e], [h_4w_0.e]\}$ which yields the assertion.

(c) Assume $\mathcal{G}_{Y(i,g,l)} = \mathcal{PSC}_{(np^r)}$ with $n > 1$ and $(n, p) = 1$. By definition of $\mathcal{PSC}_{(np^r)}$, the group $\hat{G}_{Y(i,g,l)}$ is a subgroup of T. The only points in $\mathbb{P}(V_{\mathfrak{sl}(2)})$ which are stabilized by T are $[e]$ and $[w_0.e]$. This yields as above $g \in B \cup w_0B$.

(d) First let $\mathcal{G}_{Y(i,hg,l)} = \mathcal{G}^h_{Y(i,g,l)} = \mathcal{PSC}_{(np^r)}$ with $n > 1$ and $(n, p) = 1$. By (c), we obtain $hg \in B \cup w_0B$. Therefore $T_{(2n)} \subseteq N_{SL(2)}(T) \cap B^{hg}$ and $Y(i, hg, l)$ is an $SL(2)_1T_{(2np^r)}$-module. As $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ acts trivially on $Y(i, hg, l)$, it is also a $\mathcal{PSC}_{(np^r)}$-module.
If $\mathcal{G}_{Y(i,hg,l)}$ is equal to $\mathcal{PSQ}_{(p^r)}$, we obtain $hg \in h_4B \cup h_4w_0B$ by (b). Hence $\mathcal{H}_4 \subseteq N_{SL(2)}(T) \cap B^{hg}$. As $(SL(2)_1T_r\mathcal{H}_4)/\mathcal{Z} = \mathcal{PSQ}_{(p^r)}$, we obtain that $Y(i, hg, l)$ is a $\mathcal{PSQ}_{(p^r)}$-module.
In both cases conjugation by h^{-1} yields that $Y(i, g, l)$ is a $\mathcal{G}_{Y(i,g,l)}$-module.

$\square$

Let $\mathcal{G}$ be an amalgamated polyhedral group scheme. Then $\mathcal{G}(k)$ acts on $X := SL(2)/B$ via left multiplication of the preimage of the projection $SL(2)(k) \to PSL(2)(k)$. We denote by $C_\mathcal{G}$ a set of representatives for $X/\mathcal{G}(k)$ with $1 \in C_\mathcal{G}$ and $w_0 \in C_\mathcal{G}$, if w_0 is not in the same orbit as 1.

Theorem 5.3.2. *Let $\mathcal{G}$ be an amalgamated polyhedral group scheme. Let M be an indecomposable $\mathcal{G}$-module of complexity 1. Then there are unique $l \in \mathbb{N}$, $i \in \{0, \dots, p-2\}$ and $g \in C_\mathcal{G}$ such that M is isomorphic to $\operatorname{ind}^{\mathcal{G}}_{\mathcal{G}_{Y(i,g,l)}}(Y(i, g, l) \otimes_k k_\alpha)$ for a character $\alpha \in X(\mathcal{G}_{Y(i,g,l)})$. Moreover, the character α can be chosen as a unique element of a subgroup of $X(\mathcal{G}_{Y(i,g,l)})$ determined by the following cases:*

(a) If $g \in \{1, w_0\}$, then $\alpha \in X(\mathcal{G}_{Y(i,g,l)}/\mathcal{G}_1)$.

(b) If $g \notin \{1, w_0\}$, then $\alpha \in X(\mathcal{G}_{Y(i,g,l)}/\mathcal{G}^0)$.

Proof. The group algebra $k\mathcal{G}$ is isomorphic to $k\mathcal{G}^0\#k\mathcal{G}(k)$ and thanks to 3.1.5 the group $\mathcal{G}(k)$ is linearly reductive. By 1.4.18, the extension $k\mathcal{G} : k\mathcal{G}^0$ is separable, so that there is an indecomposable direct summand N of $\operatorname{res}^{\mathcal{G}}_{\mathcal{G}^0} M$ such that M is a direct summand of $\operatorname{ind}^{\mathcal{G}}_{\mathcal{G}^0} N$. General properties of the complexity ([16, II.2]) yield

$$1 = \operatorname{cx}_{k\mathcal{G}}(M) \leq \operatorname{cx}_{k\mathcal{G}}(\operatorname{ind}^{\mathcal{G}}_{\mathcal{G}^0} N) \leq \operatorname{cx}_{k\mathcal{G}^0}(N) \leq \operatorname{cx}_{k\mathcal{G}^0}(M) \leq \operatorname{cx}_{k\mathcal{G}}(M) = 1.$$

Hence N has also complexity 1 and consequently there is $g \in C$ with $V_{\mathfrak{sl}(2)}(N) = \{g.[e]\}$. We first assume that $g \in \{1, w_0\}$. Then there are unique $l \in \mathbb{N}$ and $i \in \{0, \dots, p-2\}$ with $\operatorname{res}^{\mathcal{G}^0}_{\mathcal{G}_1} N \cong g.W(lp+i)$. Moreover, as $\mathcal{G}^0/\mathcal{G}_1$ is linearly reductive, 1.4.18 yields that $k\mathcal{G}^0 : k\mathcal{G}_1$ is separable. Therefore, N is isomorphic to a direct summand of $\operatorname{ind}^{\mathcal{G}^0}_{\mathcal{G}_1} g.W(lp+i)$. As p does not divide $\dim_k Y(i,g,l)$, [15, 2.1.4] shows that $Y(i,g,l)$ is $\mathcal{G}^0/\mathcal{G}_1$-regular. As $\mathcal{G}_{Y(i,g,l)} \cong \mathcal{PSC}_{(np^r)}$ we have that $k(\mathcal{G}_{Y(i,g,l)}/\mathcal{G}_1) = \bigoplus_{\beta \in X(\mathcal{G}_{Y(i,g,l)}/\mathcal{G}_1} k_\beta$ is a decomposition into simple $\mathcal{G}_{Y(i,g,l)}/\mathcal{G}_1$-modules. Hence an application of 5.2.3 yields a decomposition

$$\operatorname{ind}^{\mathcal{G}_{Y(i,g,l)}}_{\mathcal{G}_1} Y(i,g,l) \cong \bigoplus_{\beta \in X(\mathcal{G}_{Y(i,g,l)}/\mathcal{G}_1)} Y(i,g,l) \otimes_k k_\beta.$$

Since $\operatorname{res}^{\mathcal{G}_{Y(i,g,l)}}_{\mathcal{G}_1} Y(i,g,l)$ is isomorphic to $g.W(lp+i)$, the module $\operatorname{ind}^{\mathcal{G}_{Y(i,g,l)}}_{\mathcal{G}^0} N$ is a direct summand of $\operatorname{ind}^{\mathcal{G}_{Y(i,g,l)}}_{\mathcal{G}_1} Y(i,g,l)$. The group algebra $k\mathcal{G}$ is isomorphic to the skew group algebra $k\mathcal{G}^0 \# k\mathcal{G}(k)$. Therefore an application of 1.3.8 yields that the indecomposable direct summands of $\operatorname{ind}^{\mathcal{G}}_{\mathcal{G}^0} N$ are isomorphic to $\operatorname{ind}^{\mathcal{G}}_{\mathcal{G}_{Y(i,g,l)}}(Y(i,g,l) \otimes_k k_\alpha)$ with a unique $\alpha \in X(\mathcal{G}_{Y(i,g,l)}/\mathcal{G}_1)$.
If $g \notin \{1, w_0\}$, then there are unique $l \in \mathbb{N}$ and $i \in \{0, \dots, p-2\}$ with $N \cong X(i,g,l)$. Similar arguments as above now yield the second assertion.
Now let $h \in C$ with $M \cong \operatorname{ind}^{\mathcal{G}}_{\mathcal{G}_{Y(i,h,l)}}(Y(j,h,m) \otimes_k k_\beta)$ for β in $X(\mathcal{G}_{Y(j,h,m)}/\mathcal{G}_1)$ respectively $X(\mathcal{G}_{Y(j,h,m)}/\mathcal{G}^0)$, $m \in \mathbb{N}$ and $j \in \{0, \dots, p-2\}$. Then

$$\operatorname{res}^{\mathcal{G}}_{\mathcal{G}_1} M \cong \bigoplus_{u \in \mathcal{G}(k)} ug.W(lp+i) \cong \bigoplus_{v \in \mathcal{G}(k)} vh.W(mp+j).$$

Thanks to 4.1.2, we obtain $m = l$, $j = i$ and there is $v \in \mathcal{G}(k)$ such that vh is represented in C by g. Therefore h and g are in the same $\mathcal{G}(k)$-orbit and we obtain a unique element $\tilde{g} \in C_{\mathcal{G}}$ with $M \cong \operatorname{ind}^{\mathcal{G}}_{\mathcal{G}_{Y(i,\tilde{g},l)}}(Y(i,\tilde{g},l) \otimes_k k_\alpha)$. □

Theorem 5.3.3. *Let $\mathcal{G}$ be a finite subgroup scheme of $PSL(2)$ with $\mathcal{G}_1 \cong PSL(2)_1$ and tame principal block. Let M be an indecomposable $\mathcal{G}$-module of complexity 2. Then M is isomorphic to one of the following pairwise non-isomorphic $\mathcal{G}$-modules:*

1. *$\tilde{V}(d) \otimes_k S$ or $M \cong \tilde{V}(d)^* \otimes_k S$ for $d \geq p$ with $d \not\equiv -1 \pmod{p}$ and a simple $\mathcal{G}/\mathcal{G}_1$-module S*

2. *$\tilde{V}(i) \otimes_k S$ with $0 \leq i \leq p-1$ and a simple $\mathcal{G}/\mathcal{G}_1$-module S.*

Proof. Since $\mathcal{G}/\mathcal{G}_1$ is linearly reductive, 1.4.18 yields that the extension $k\mathcal{G} : k\mathcal{G}_1$ is separable, so that there is an indecomposable direct summand N of $\operatorname{res}^{\mathcal{G}}_{\mathcal{G}_1} M$ such that M is a direct summand of $\operatorname{ind}^{\mathcal{G}}_{\mathcal{G}_1} N$. As in the proof of 5.3.2, we obtain $\operatorname{cx}_{\mathcal{G}^0}(N) = \operatorname{cx}_{\mathcal{G}}(M) = 2$. Since $\mathcal{G}_1$ is isomorphic to $PSL(2)_1 \cong SL(2)_1$, we obtain that N is isomorphic to $V(d)$, $V(d)^*$ or $V(i)$ for a unique $d \geq p$ with $d \not\equiv -1 \pmod{p}$ or a unique $0 \leq i \leq p-1$. As noted at the beginning of this section these modules are the restrictions of the $\mathcal{G}$-modules $\tilde{V}(d)$, $\tilde{V}(d)^*$ and $\tilde{V}(i)$. Hence we can assume that N is a $\mathcal{G}$-module which is isomorphic to one of these modules. By [15, 2.1.4], the module N is $\mathcal{G}^0/\mathcal{G}_1$-regular and 5.2.3 yields that the indecomposable direct summands of $\operatorname{ind}_{\mathcal{G}_1}^{\mathcal{G}} N$ are of the form $N \otimes_k S$ for a unique $\mathcal{G}/\mathcal{G}_1$-module S. $\square$

Remark 5.3.4. The modules listed in (2) are exactly the simple $\mathcal{G}$-modules.

6. Induction of almost split sequences

Let k be an algebraically closed field. Given a finite group scheme $\mathcal{G}$ with normal subgroup scheme $\mathcal{N}$, we want to investigate conditions under which the induction functor $\operatorname{ind}_{\mathcal{N}}^{\mathcal{G}} : \operatorname{mod}\mathcal{N} \to \operatorname{mod}\mathcal{G}$ sends an almost split exact sequence to a direct sum of almost split exact sequences.

6.1. Almost split sequences for skew group algebras

In [42, 3.8] Riedtmann and Reiten used the functorial approach of almost split sequences to show the following:

Theorem 6.1.1 ([42, 3.8]). *Let k be a field, G be a finite group such that $|G|$ is invertible in k and be $A * G$ a skew group algebra. Then the induction functor $\operatorname{ind}_1^G : A \to A * G$ (or the restriction functor $\operatorname{res}_1^G : A * G \to A$) sends almost split sequences over A (or over $A * G$, respectively) to direct sums of almost split sequences over $A * G$ (or over A, respectively).*

The proof relies on the following properties of the involved functors:

(A) (i) There is a split monomorphism of functors $\operatorname{id}_{\operatorname{mod} A} \to \operatorname{res}_1^G \operatorname{ind}_1^G$.

(ii) There is a split epimorphism of functors $\operatorname{ind}_1^G \operatorname{res}_1^G \to \operatorname{id}_{\operatorname{mod} A*G}$.

(B) $(\operatorname{ind}_1^G, \operatorname{res}_1^G)$ and $(\operatorname{res}_1^G, \operatorname{ind}_1^G)$ are adjoint pairs of functors.

($\tilde{\text{C}}$) The finite group G is acting on $\operatorname{mod} A$ such that for every A-module M there is a decomposition $\operatorname{res}_1^G \operatorname{ind}_1^G M = \bigoplus_{g\in G} M^g$ and if $\varphi : M \to N$ is A-linear, then $\operatorname{res}_1^G \operatorname{ind}_1^G(\varphi) = (g.\varphi)_{g\in G} : \bigoplus_{g\in G} M^g \to \bigoplus_{g\in G} N^g$.

Recall from 1.3.5 that $M^g = Ag \otimes_A M$ and $N^g = Ag \otimes_A N$. The morphism $g.\varphi : M^g \to N^g$ is therefore given by $g.\varphi = Ag \otimes_A \varphi$.
It was shown that these properties also hold for the induced functors

$$\operatorname{ind}_1^G : \operatorname{mmod} A \to \operatorname{mmod} A * G \text{ and } \operatorname{res}_1^G : \operatorname{mmod} A * G \to \operatorname{mmod} A$$

and that they imply the following property:

(C) $\operatorname{ind}_1^G : \operatorname{mmod} A \to \operatorname{mmod} A * G$ and $\operatorname{res}_1^G : \operatorname{mmod} A * G \to \operatorname{mmod} A$ preserve semisimple objects and projective covers.

6.2. Induction of almost split sequences for finite group schemes

We want to apply the ideas of the proof of 6.1.1 in the context of group algebras of finite group schemes. In this situation we will not always have analogous results for the induction and restriction functor. For example, in [15, 3.1.4] it was already shown that for the restriction functor this is possible if and only if the ending term of the almost split sequence fulfills a certain regularity property.

Let $\mathcal{N}$ be a normal subgroup scheme of a finite group scheme $\mathcal{G}$. The $\mathcal{N}$-modules which will be of our interest are restrictions of $\mathcal{G}$-modules. To obtain an analogue of property ($\tilde{\mathrm{C}}$) we will use the following result:

Lemma 6.2.1. *Let $\mathcal{G}$ be a finite group scheme and $\mathcal{N}$ be a normal subgroup scheme of $\mathcal{G}$. Denote by $\mathcal{C}$ the full subcategory of* $\operatorname{mod}\mathcal{N}$ *consisting of direct sums of indecomposable $\mathcal{N}$-modules which are restrictions of $\mathcal{G}$-modules. Then for each $M \in \mathcal{C}$ we have a natural isomorphism $\operatorname{res}^{\mathcal{G}}_{\mathcal{N}} \operatorname{ind}^{\mathcal{G}}_{\mathcal{N}} M \cong M^n$ where $n = \dim_k k(\mathcal{G}/\mathcal{N})$. In particular, if $\iota : \mathcal{C} \to \operatorname{mod}\mathcal{N}$ denotes the canonical embedding, we have a split monomorphism $\mathrm{id}_{\operatorname{mod} A} \to \operatorname{res}^{\mathcal{G}}_{\mathcal{N}} \circ \operatorname{ind}^{\mathcal{G}}_{\mathcal{N}} \circ \iota$ of functors.*

Proof. Let M be a $\mathcal{G}$-module. Then there is a $\mathcal{G}$-linear isomorphism

$$\psi_M : \operatorname{ind}^{\mathcal{G}}_{\mathcal{N}} \operatorname{res}^{\mathcal{G}}_{\mathcal{N}} M \to M \otimes_k k(\mathcal{G}/\mathcal{N})$$

which is natural in M:
This follows directly from the tensor identity. Alternatively, exactly as in the proof of 5.2.3 one can show that the map

$$\psi_M : \operatorname{ind}^{\mathcal{G}}_{\mathcal{N}} M \to M \otimes_k k(\mathcal{G}/\mathcal{N}),\ a \otimes m \mapsto \sum_{(a)} a_{(1)} m \otimes \pi(a_{(2)})$$

is an isomorphism of $\mathcal{G}$-modules, where $\pi : k\mathcal{G} \to k(\mathcal{G}/\mathcal{N})$ is the canonical projection. A direct computation shows that it is natural in M. As a direct consequence we obtain the assertion. $\square$

Proposition 6.2.2. *Let $\mathcal{G}$ be a finite group scheme and $\mathcal{N}$ be a normal subgroup scheme such that $\mathcal{G}/\mathcal{N}$ is linearly reductive. Let X, Y and E be $\mathcal{N}$-modules such that*

1. *every indecomposable direct summand of the modules X, Y and E is the restriction of a $\mathcal{G}$-module, and*
2. *$\mathcal{E} : 0 \to X \xrightarrow{\varphi} E \xrightarrow{\psi} Y \to 0$ is an almost split exact sequence of $\mathcal{N}$-modules.*

Then $\operatorname{ind}^{\mathcal{G}}_{\mathcal{N}} \mathcal{E}$ is a direct sum of almost split exact sequences.

Proof. Denote by $\mathcal{C}$ the full subcategory of $\operatorname{mod}\mathcal{N}$ consisting of direct sums of indecomposable $\mathcal{N}$-modules which are restrictions of $\mathcal{G}$-modules. Denote by $\iota : \mathcal{C} \to \operatorname{mod}\mathcal{N}$ the canonical embedding. We first show that the following analogues of (A),(B) and ($\tilde{\mathrm{C}}$) hold:

($\hat{\mathrm{A}}$) (i) There is a split monomorphism of functors $\mathrm{id}_{\mathcal{C}} \to \operatorname{res}^{\mathcal{G}}_{\mathcal{N}} \circ \operatorname{ind}^{\mathcal{G}}_{\mathcal{N}} \circ \iota$.
(ii) There is a split epimorphism of functors $\operatorname{ind}^{\mathcal{G}}_{\mathcal{N}} \operatorname{res}^{\mathcal{G}}_{\mathcal{N}} \to \mathrm{id}_{\operatorname{mod}\mathcal{G}}$.

($\hat{\mathrm{B}}$) $(\operatorname{ind}^{\mathcal{G}}_{\mathcal{N}}, \operatorname{res}^{\mathcal{G}}_{\mathcal{N}})$ and $(\operatorname{res}^{\mathcal{G}}_{\mathcal{N}}, \operatorname{ind}^{\mathcal{G}}_{\mathcal{N}})$ are adjoint pairs of functors.

($\breve{\mathrm{C}}$) For every $M \in \mathcal{C}$ there is a decomposition $\operatorname{res}^{\mathcal{G}}_{\mathcal{N}} \operatorname{ind}^{\mathcal{G}}_{\mathcal{N}} M = M^n$ with $n = \dim_k k(\mathcal{G}/\mathcal{N})$ and if $\varphi : M \to N$ is $\mathcal{N}$-linear in $\mathcal{C}$, then $\operatorname{res}^{\mathcal{G}}_{\mathcal{N}} \operatorname{ind}^{\mathcal{G}}_{\mathcal{N}}(\varphi) = (\varphi, \varphi, \ldots, \varphi) : M^n \to N^n$.

In 6.2.1 we have already seen that the properties (Â)(i) and (Č) hold. As $\mathcal{G}/\mathcal{N}$ is linearly reductive, the $k(\mathcal{G}/\mathcal{N})$-Galois extension $k\mathcal{G} : k\mathcal{N}$ is separable by 1.4.18. Basic properties of separable extensions (c.f. [39, Proposition 10.8]) yield that property (Â)(ii) holds. Thanks to 1.4.21, the ring extension $k\mathcal{G} : k\mathcal{N}$ is a free Frobenius extension of first kind. Hence, by 1.4.20 the induction and coinduction functors are equivalent, so that property (B̂) holds. As in [42, 3.5] the properties (Â), (B̂) and (Č) also hold for the induced functors on the functor categories.
Since ψ is minimal right almost split, the exact sequence

$$\mathrm{Hom}_{\mathcal{N}}(-, E) \to \mathrm{Hom}_{\mathcal{N}}(-, Y) \to S^Y \to 0$$

is a minimal projective presentation of S^Y. From this point, we can proceed as in the proof of [42, 3.6] to show that the functor $\mathrm{ind}_{\mathcal{N}}^{\mathcal{G}}$ preserves semisimple objects and projective covers which are finitely presented over $\mathcal{C}$:
We start by showing that if the functor $\mathfrak{F}$ is indecomposable in $\mathrm{mmod}\,\mathcal{G}$ and $\mathrm{res}_{\mathcal{N}}^{\mathcal{G}} \mathfrak{F}$ is semisimple in $\mathrm{mmod}\,\mathcal{N}$, then $\mathfrak{F}$ is simple. If $\mathfrak{F}$ is not simple, then there is a non-split exact sequence

$$0 \to \mathfrak{H} \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{G} \to 0.$$

Since by (B̂), $\mathrm{res}_{\mathcal{N}}^{\mathcal{G}}$ and $\mathrm{ind}_{\mathcal{N}}^{\mathcal{G}}$ are both left and right adjoint functors, they are exact. By assumption, the functor $\mathrm{res}_{\mathcal{N}}^{\mathcal{G}} \mathfrak{F}$ is semisimple. Therefore, the sequence

$$0 \to \mathrm{res}_{\mathcal{N}}^{\mathcal{G}} \mathfrak{H} \longrightarrow \mathrm{res}_{\mathcal{N}}^{\mathcal{G}} \mathfrak{F} \longrightarrow \mathrm{res}_{\mathcal{N}}^{\mathcal{G}} \mathfrak{G} \to 0$$

splits. This yields a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{ind}_{\mathcal{N}}^{\mathcal{G}} \mathrm{res}_{\mathcal{N}}^{\mathcal{G}} \mathfrak{H} \arrow[r] & \mathrm{ind}_{\mathcal{N}}^{\mathcal{G}} \mathrm{res}_{\mathcal{N}}^{\mathcal{G}} \mathfrak{F} \arrow[r] \arrow[d] & \mathrm{ind}_{\mathcal{N}}^{\mathcal{G}} \mathrm{res}_{\mathcal{N}}^{\mathcal{G}} \mathfrak{G} \arrow[r] \arrow[d] & 0 \\
 & & \mathfrak{F} \arrow[r] & \mathfrak{G} \arrow[r] & 0
\end{tikzcd}$$

Due to (Â)(ii) the vertical arrows split. As the upper row splits, this yields the splitting of the lower row, a contradiction.
Now let $\mathfrak{F}$ be a simple functor that is finitely presented over $\mathcal{C}$. Then property (Č) yields, that $\mathrm{res}_{\mathcal{N}}^{\mathcal{G}} \mathrm{ind}_{\mathcal{N}}^{\mathcal{G}} \mathfrak{F} = \bigoplus_{i=1}^n \mathfrak{F}$ is semisimple. Thanks to the above, this yields that $\mathrm{ind}_{\mathcal{N}}^{\mathcal{G}} \mathfrak{F}$ is semisimple.
Next we show that $\mathrm{ind}_{\mathcal{N}}^{\mathcal{G}} S^Y = S^{\mathrm{ind}_{\mathcal{N}}^{\mathcal{G}} Y}$ for $Y \in \mathcal{C}$ (this shows that $\mathrm{ind}_{\mathcal{N}}^{\mathcal{G}}$ preserves projective covers). Applying $\mathrm{ind}_{\mathcal{N}}^{\mathcal{G}}$ to the exact sequence

$$\mathrm{Hom}_{\mathcal{N}}(-, Y) \to S^Y \to 0$$

yields the exact sequence

$$\mathrm{Hom}_{\mathcal{G}}(-, \mathrm{ind}_{\mathcal{N}}^{\mathcal{G}} Y) \to \mathrm{ind}_{\mathcal{N}}^{\mathcal{G}} S^Y \to 0.$$

Hence there is a direct summand X of $\operatorname{ind}_{\mathcal{N}}^{\mathcal{G}} Y$ such that

$$\operatorname{Hom}_{\mathcal{G}}(-, X) \to \operatorname{ind}_{\mathcal{N}}^{\mathcal{G}} S^Y \to 0$$

is a projective cover. Applying $\operatorname{res}_{\mathcal{N}}^{\mathcal{G}}$ to this sequence yields the exact sequence

$$\operatorname{Hom}_{\mathcal{N}}(-, \operatorname{res}_{\mathcal{N}}^{\mathcal{G}} X) \to \operatorname{res}_{\mathcal{N}}^{\mathcal{G}} \operatorname{ind}_{\mathcal{N}}^{\mathcal{G}} S^Y \to 0.$$

By (Č) we obtain $\operatorname{res}_{\mathcal{N}}^{\mathcal{G}} \operatorname{ind}_{\mathcal{N}}^{\mathcal{G}} S^Y = \bigoplus S^Y$. The projective cover of this functor is $\operatorname{Hom}_{\mathcal{N}}(-, \bigoplus Y)$. As $\operatorname{res}_{\mathcal{N}}^{\mathcal{G}} X$ is a direct summand of $\operatorname{res}_{\mathcal{N}}^{\mathcal{G}} \operatorname{ind}_{\mathcal{N}}^{\mathcal{G}} Y = \bigoplus Y$, this yields $\operatorname{res}_{\mathcal{N}}^{\mathcal{G}} X = \operatorname{res}_{\mathcal{N}}^{\mathcal{G}} \operatorname{ind}_{\mathcal{N}}^{\mathcal{G}} Y$ and hence $X = \operatorname{ind}_{\mathcal{N}}^{\mathcal{G}} Y$.

Therefore, the functor $\operatorname{ind}_{\mathcal{N}}^{\mathcal{G}} S^Y$ is semisimple and the exact sequence

$$\operatorname{ind}_{\mathcal{N}}^{\mathcal{G}} \operatorname{Hom}_{\mathcal{N}}(-, E) \to \operatorname{ind}_{\mathcal{N}}^{\mathcal{G}} \operatorname{Hom}_{\mathcal{N}}(-, Y) \to \operatorname{ind}_{\mathcal{N}}^{\mathcal{G}} S^Y \to 0$$

is a minimal projective presentation of $\operatorname{ind}_{\mathcal{N}}^{\mathcal{G}} S^Y \cong S^{\operatorname{ind}_{\mathcal{N}}^{\mathcal{G}} Y}$. Hence, $\operatorname{ind}_{\mathcal{N}}^{\mathcal{G}} \psi$ is a direct sum of minimal right almost split homomorphisms.
Dually, one can show that $\operatorname{ind}_{\mathcal{N}}^{\mathcal{G}} \varphi$ is a direct sum of minimal left almost split homomorphisms. □

Lemma 6.2.3. *Let $\mathcal{G}$ be a reduced group scheme, $\mathcal{N}$ be a normal subgroup scheme of $\mathcal{G}$ and X and M be $\mathcal{G}$-modules such that $X|_{\mathcal{N}}$ and $M|_{\mathcal{N}}$ are indecomposable. Then* $\operatorname{Rad}_{\mathcal{N}}(X, M)$ *and* $\operatorname{Rad}^2_{\mathcal{N}}(X, M)$ *are $\mathcal{G}$-submodules of* $\operatorname{Hom}_{\mathcal{N}}(X, M)$.

Proof. As $\mathcal{G}$ is reduced, we only need to prove that $\operatorname{Rad}_{\mathcal{N}}(X, M)$ and $\operatorname{Rad}^2_{\mathcal{N}}(X, M)$ are $\mathcal{G}(k)$-submodules of $\operatorname{Hom}_{\mathcal{N}}(X, M)$ (see [25, Remark after 2.8]).
Let $g \in \mathcal{G}(k)$ and $\varphi \in \operatorname{Rad}_{\mathcal{N}}(X, M)$. Assume that $g.\varphi$ is an isomorphism with inverse ψ. Then $g^{-1}.\psi$ is an inverse of φ, a contradiction.
Let $g \in \mathcal{G}(k)$ and $\varphi \in \operatorname{Rad}^2_{\mathcal{N}}(X, M)$. Then there is an $\mathcal{N}$-module Z, $\alpha \in \operatorname{Rad}_{\mathcal{N}}(X, Z)$ and $\beta \in \operatorname{Rad}_{\mathcal{N}}(Z, M)$ such that $\varphi = \beta \circ \alpha$. Denote by Z^g the $\mathcal{N}$-module Z with action twisted by g^{-1}, i.e. $h.z = h^{g^{-1}} z$ for $h \in k\mathcal{N}$ and $z \in Z^g$. As X and M are $\mathcal{G}$-modules, we can define the $\mathcal{N}$-linear maps

$$\tilde{\alpha} : X \to Z^g,\ x \mapsto \alpha(g^{-1}x) \text{ and } \tilde{\beta} : Z^g \to M,\ z \mapsto g\beta(z).$$

Then we obtain $g.\varphi = \tilde{\beta} \circ \tilde{\alpha}$. If $\tilde{\alpha}$ is an isomorphism with inverse $\gamma : Z^g \to X$, then the $\mathcal{N}$-linear map

$$\tilde{\gamma} : Z \to X,\ z \to g^{-1}\gamma(z)$$

is an inverse of α, a contradiction. In the same way, $\tilde{\beta}$ is not an isomorphism. Hence $g.\varphi = \tilde{\beta} \circ \tilde{\alpha} \in \operatorname{Rad}^2_{\mathcal{N}}(X, M)$. □

Proposition 6.2.4. *Let $\mathcal{G}$ be a finite subgroup scheme of a reduced group scheme $\mathcal{H}$ and $\mathcal{N} \subseteq \mathcal{G}$ a normal subgroup scheme of $\mathcal{H}$ such that $\mathcal{G}/\mathcal{N}$ is linearly reductive. Let X and M be $\mathcal{G}$-modules such that $X|_{\mathcal{N}}$ and $M|_{\mathcal{N}}$ are indecomposable and such that there is an almost split exact sequence*

$$\mathcal{E} : 0 \to \tau_{\mathcal{N}}(M) \longrightarrow X^n \longrightarrow M \to 0$$

of $\mathcal{N}$-modules. Then there is a short exact sequence

$$\tilde{\mathcal{E}} : 0 \to N \longrightarrow X \otimes_k \operatorname{Irr}_{\mathcal{N}}(X, M) \longrightarrow M \to 0$$

of $\mathcal{G}$-modules such that $\operatorname{res}^{\mathcal{G}}_{\mathcal{N}} \tilde{\mathcal{E}} = \mathcal{E}$.

Proof. Consider the map

$$\tilde{\psi} : X \otimes_k \operatorname{Hom}_{\mathcal{N}}(X, M) \to M,\ x \otimes f \mapsto f(x)$$

of $\mathcal{G}$-modules. As $\mathcal{G}/\mathcal{N}$ is linearly reductive, the short exact sequence

$$0 \to \operatorname{Rad}^2_{\mathcal{N}}(X, M) \longrightarrow \operatorname{Rad}_{\mathcal{N}}(X, M) \longrightarrow \operatorname{Irr}_{\mathcal{N}}(X, M) \to 0$$

of $\mathcal{G}$-modules splits. This yields a decomposition

$$\operatorname{Rad}_{\mathcal{N}}(X, M) \cong \operatorname{Rad}^2_{\mathcal{N}}(X, M) \oplus \operatorname{Irr}_{\mathcal{N}}(X, M).$$

Hence there results a $\mathcal{G}$-linear map $\psi : X \otimes_k \operatorname{Irr}_{\mathcal{N}}(X, M) \to M$ by restricting $\tilde{\psi}$ to $X \otimes_k \operatorname{Irr}_{\mathcal{N}}(X, M)$. Let $(f_i)_{1 \le i \le n} : X^n \to M$ be the surjection given by $\mathcal{E}$. Since $\mathcal{E}$ is almost split, the maps $(f_i)_{1 \le i \le n}$ form a k-basis of $\operatorname{Irr}_{\mathcal{N}}(X, M)$ (c.f. [1, IV.4.2]). Therefore the restriction $\operatorname{res}^{\mathcal{G}}_{\mathcal{N}}(\psi)$ equals $(f_i)_{1 \le i \le n}$. □

In 6.2.2 we have seen that if we apply the induction functor to certain almost split sequences, they will be the direct sum of almost split sequences. Our next result enables us to describe this decomposition under a certain regularity condition.

Proposition 6.2.5. *In the situation of 6.2.4 assume that M and N are $\mathcal{G}^0/\mathcal{N}$-regular. Let S be a simple $\mathcal{G}/\mathcal{N}$-module. Then the short exact sequence $\tilde{\mathcal{E}} \otimes_k S$ is almost split.*

Proof. Let $k(\mathcal{G}/\mathcal{N}) = \bigoplus_{i=1}^n S_i$ be the decomposition into simple $\mathcal{G}/\mathcal{N}$-modules. Due to 6.2.1, there is for any $\mathcal{G}$-module U a natural isomorphism $\psi_U : \operatorname{ind}^{\mathcal{G}}_{\mathcal{N}} U \to U \otimes_k k(\mathcal{G}/\mathcal{N})$. Therefore, we obtain the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{ind}^{\mathcal{G}}_{\mathcal{N}} N \arrow[r] \arrow[d, "\psi_N"] & \operatorname{ind}^{\mathcal{G}}_{\mathcal{N}}(X \otimes_k \operatorname{Irr}_{\mathcal{N}}(X, M)) \arrow[r] \arrow[d, "\psi_{X \otimes_k \operatorname{Irr}_{\mathcal{N}}(X,M)}"] & \operatorname{ind}^{\mathcal{G}}_{\mathcal{N}} M \arrow[r] \arrow[d, "\psi_M"] & 0 \\
0 \arrow[r] & N \otimes_k k(\mathcal{G}/\mathcal{N}) \arrow[r] & X \otimes_k \operatorname{Irr}_{\mathcal{N}}(X, M) \otimes_k k(\mathcal{G}/\mathcal{N}) \arrow[r] & M \otimes_k k(\mathcal{G}/\mathcal{N}) \arrow[r] & 0
\end{tikzcd}$$

As all vertical arrows are isomorphisms, the two exact sequences $\operatorname{ind}^{\mathcal{G}}_{\mathcal{N}} \mathcal{E}$ and $\bigoplus_{i=1}^n \tilde{\mathcal{E}} \otimes_k S_i$ are equivalent.
Thanks to 6.2.2 the exact sequence $\operatorname{ind}^{\mathcal{G}}_{\mathcal{N}} \mathcal{E}$ is a direct sum of almost split sequences. By 5.2.3, the $\mathcal{G}$-modules $M \otimes_k S$ and $N \otimes_k S$ are indecomposable. Moreover, the sequence $\tilde{\mathcal{E}} \otimes_k S$ does not split. Otherwise, the sequence $\operatorname{res}^{\mathcal{G}}_{\mathcal{N}}(\tilde{\mathcal{E}} \otimes_k S)$ would split and therefore also $\mathcal{E}$. Hence, the sequence $\tilde{\mathcal{E}} \otimes_k S$ is equivalent to an indecomposable direct summand of $\operatorname{ind}^{\mathcal{G}}_{\mathcal{N}} \mathcal{E}$. As the Krull-Schmidt theorem holds in the category of short exact sequences of finite-dimensional $\mathcal{G}$-modules (as the space of morphisms between those sequences is finite-dimensional, c.f. [29]), it follows that $\tilde{\mathcal{E}} \otimes_k S_i$ is almost split. □

7. The McKay and Auslander-Reiten quiver of domestic finite group schemes

By 1.5.9, the principal block of a domestic finite group scheme is isomorphic to the principal block of an amalgamated polyhedral group scheme. Hence, to determine the Auslander-Reiten quiver of this block, it is enough to understand the Auslander-Reiten quiver of the amalgamated polyhedral group schemes. Due to 3.4.3, the blocks are Morita-equivalent to a trivial extension of a radical square zero tame hereditary algebra and therefore the Auslander-Reiten quiver is known in an abstract way. In this chapter we will compute the Euclidean components of this quiver by giving a concrete connection to the McKay quiver of the associated binary polyhedral group scheme.

7.1. Euclidean AR-components of amalgamated polyhedral group schemes

In the following it will be convenient to have a common notation for the Weyl modules and their duals. We set

$$V(n,i) := \begin{cases} V(np+i) & \text{if } n \geq 0 \\ V(-np+i)^* & \text{if } n \leq 0 \end{cases}$$

for $n \in \mathbb{Z}$ and $0 \leq i \leq p-1$.

Lemma 7.1.1. *Let $\mathcal{G}$ be a finite subgroup scheme of $SL(2)$ with $SL(2)_1 \subseteq \mathcal{G}$ such that $\mathcal{G}/SL(2)_1$ is linearly reductive. Let $0 \leq i \leq p-2$ and $n \in \mathbb{Z}$. Then*

$$\Omega^{2n}_{\mathcal{G}}(L(i)) \cong V(2n,i)$$

and

$$\Omega^{2n+1}_{\mathcal{G}}(L(i)) \cong V(2n+1, p-2-i)$$

In particular, $\Theta(i) = \{V(n,i) \mid n \in \mathbb{Z}\}$.

Proof. By [54, 7.1.2], there is for $n \leq 0$ a short exact sequence

$$0 \to V(n,i) \longrightarrow P(i) \otimes_k (V(n)^*)^{[1]} \longrightarrow V(n-1, p-2-i) \to 0$$

of $SL(2)$-modules. Here $P(i)$ denotes the $SL(2)$-module such that $P(i)|_{SL(2)_1}$ is the projective cover of $L(i)$. The projectivity of $P(i)|_{SL(2)_1}$ and Frobenius reciprocity yield $\operatorname{Ext}^1_{\mathcal{G}}(\operatorname{ind}^{\mathcal{G}}_{SL(2)_1} P(i), -) \cong \operatorname{Ext}^1_{SL(2)_1}(P(i), -) \circ \operatorname{res}^{\mathcal{G}}_{SL(2)_1} = 0$. Consequently $\operatorname{ind}^{\mathcal{G}}_{SL(2)_1} P(i)$ is projective and as $\mathcal{G}/\mathcal{N}$ is linearly reductive the module $P(i)|_{\mathcal{G}}$ is a direct summand of $\operatorname{ind}^{\mathcal{G}}_{SL(2)_1} P(i)$. Hence $P(i)|_{\mathcal{G}}$ is projective and consequently the $\mathcal{G}$-module $P(i) \otimes_k (V(n)^*)^{[1]}$ is projective. Hence there is a projective $\mathcal{G}$-module P with $\Omega_{\mathcal{G}}(V(n-1, p-2-i)) \oplus P \cong V(n,i)$. Since $V(n,i)$ is a non-projective indecomposable $\mathcal{G}$-module, we obtain

$$\Omega_{\mathcal{G}}(V(n-1, p-2-i)) \cong V(n,i).$$

Dualizing the above sequence yields $\Omega_{\mathcal{G}}(V(n,i))) = V(n+1, p-2-i)$ for $n \geq 0$ in the same way. $\square$

Lemma 7.1.2. *For all $n \in \mathbb{Z}$ and $0 \leq i \leq p-2$ the $SL(2)$-module $L(1)^{[1]}$ is isomorphic to $\mathrm{Irr}_{SL(2)_1}(V(n+1,i), V(n,i))$.*

Proof. As in the proof of 7.1.1 there is for all $n \in \mathbb{Z}$ a short exact sequence

$$0 \to V(n+1,i) \longrightarrow P \longrightarrow V(n, p-2-i) \to 0$$

of $SL(2)$-modules with P being projective over $SL(2)_1$. Applying $\mathrm{Hom}_{SL(2)_1}(-, V(n,i))$ to this sequence yields the exact sequence

$$\begin{aligned} &\mathrm{Hom}_{SL(2)_1}(P, V(n,i)) \xrightarrow{\alpha} \mathrm{Hom}_{SL(2)_1}(V(n+1,i), V(n,i)) \\ &\xrightarrow{\beta} \mathrm{Ext}^1_{SL(2)_1}(V(n,p-2-i), V(n,i)) \to \mathrm{Ext}^1_{SL(2)_1}(P, V(n,i)) \end{aligned}$$

of $SL(2)$-modules. As P is projective, we obtain $\mathrm{Ext}^1_{SL(2)_1}(P, V(n,i)) = (0)$. Hence the map β is surjective. By [11, 2.4], the $SL(2)$-module $\mathrm{Ext}^1_{SL(2)_1}(V(n,p-2-i), V(n,i))$ is isomorphic to $L(1)^{[1]}$. Hence $L(1)^{[1]} \cong \mathrm{Hom}_{SL(2)_1}(V(n+1,i), V(n,i))/\ker\beta$. As $\mathrm{im}\,\alpha = \ker\beta$ is contained in $\mathrm{Rad}^2_{SL(2)_1}(V(n+1,i), V(n,i))$, we obtain a surjective morphism

$$L(1)^{[1]} \to \mathrm{Irr}_{SL(2)_1}(V(n+1,i), V(n,i))$$

of $SL(2)$-modules. Since both modules are 2-dimensional, this map is an isomorphism. $\square$

If $\mathcal{G} = (SL(2)_1\tilde{\mathcal{G}})/\mathcal{Z}$ is an amalgamated polyhedral group scheme we set $\hat{\mathcal{G}} := SL(2)_1\tilde{\mathcal{G}}$. For any Euclidean diagram $(\tilde{A}_n)_{n\in\mathbb{N}}$, $(\tilde{D}_n)_{n\geq 4}$ and $(\tilde{E}_n)_{6\leq n\leq 8}$ we will denote in the same way the quiver where each edge $\bullet - \bullet$ is replaced by a pair of arrows $\bullet \leftrightarrows \bullet$. As shown in the proof of [13, 7.2.3], the McKay quiver $\Upsilon_{L(1)^{[1]}}(\hat{\mathcal{G}}/\hat{\mathcal{G}}_1)$ is isomorphic to one of the quivers $\tilde{A}_{2np^{r-1}-1}, \tilde{D}_{np^{r-1}+2}, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$, where r is the height of $\hat{\mathcal{G}}^0$ and $(n,p) = 1$.
For any quiver Q we denote by Q_s its separated quiver. If $\{1, \ldots, n\}$ is the vertex set of Q, then Q_s has $2n$ vertices $\{1, \ldots, n, 1', \ldots, n'\}$ and arrows $i \to j'$ if and only if $i \to j$ is an arrow in Q. The separated quiver of one of the quivers $\tilde{A}_{2np^{r-1}-1}, \tilde{D}_{np^{r-1}+2}, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ is the union of 2 quivers with the same underlying graph as the original quiver and each vertex is either a source or a sink.

Theorem 7.1.3. *Let $\mathcal{G}$ be an amalgamated polyhedral group scheme and Θ a component of $\Gamma_s(\mathcal{G})$ containing a $\mathcal{G}$-module of complexity 2. Let Q be a connected component of $\Upsilon_{L(1)^{[1]}}(\hat{\mathcal{G}}/\hat{\mathcal{G}}_1)_s$. Then Θ is isomorphic to $\mathbb{Z}[Q]$.*

Proof. Thanks to 3.4.3, all the non-simple blocks of $k\mathcal{G}$ are Morita equivalent to the principal block $\mathcal{B}_0(\mathcal{G})$ of $k\mathcal{G}$. Additionally, by 1.5.9, the block $\mathcal{B}_0(\mathcal{G})$ is isomorphic to the block $\mathcal{B}_0(\hat{\mathcal{G}})$. Therefore it suffices to prove this result for $\mathcal{G} := \hat{\mathcal{G}}$.
In view of 1.6.4 and 2.2.6, all modules belonging to the component Θ have complexity 2. Let $S_1, \ldots, S_m$ be the simple $\mathcal{G}/\mathcal{G}_1$-modules and $M \in \Theta$. Due to 5.3.3, there are $0 \leq l \leq p-2$, $n \in \mathbb{N}$ and $1 \leq j \leq m$ such that $M \cong V(n,l) \otimes_k S_j$.

Thanks to 3.4.2, the group algebra $k\mathcal{G}$ is symmetric. Therefore the Auslander-Reiten translation $\tau_{\mathcal{G}}$ equals $\Omega^2_{\mathcal{G}}$ (see 2.1.3). Applying 6.2.4 and 6.2.5 to the almost split exact sequence

$$0 \to V(n+2,l) \longrightarrow V(n+1,l) \oplus V(n+1,l) \longrightarrow V(n,l) \to 0$$

of $SL(2)_1$-modules yields the almost split exact sequence

$$0 \to \tau_{\mathcal{G}}(V(n,l)) \longrightarrow V(n+1,l) \otimes_k \mathrm{Irr}_{SL(2)_1}(V(n+1,l), V(n,l)) \longrightarrow V(n,l) \to 0$$

of $\mathcal{G}$-modules. Due to 7.1.1, we have $\tau_{\mathcal{G}}(V(n,l)) \cong \Omega^2_{\mathcal{G}}(V(n,l)) \cong V(n+2,l)$. In conjunction with 7.1.2 and 6.2.2, we now obtain the almost split exact sequence

$$0 \to V(n+2,l) \otimes_k S_j \longrightarrow V(n+1,l) \otimes_k L(1)^{[1]} \otimes_k S_j \longrightarrow V(n,l) \otimes_k S_j \to 0.$$

Hence $\tau_{\mathcal{G}}(V(n,l) \otimes_k S_j) \cong V(n+2,l) \otimes_k S_j$. Moreover, due to the decomposition $L(1)^{[1]} \otimes_k S_j \cong \bigotimes_{i=1}^m a_{ij} S_i$, there are a_{ij} arrows $V(n+1,l) \otimes_k S_i \to V(n,l) \otimes_k S_j$ and a_{ij} arrows $V(n+2,l) \otimes_k S_j \to V(n+1,l) \otimes_k S_i$ in Θ. Without loss we can now assume that $n = 0$, so that $V(0,l) \otimes_k S_j$ belongs to Θ.
Denote by $\{1,\dots,m\}$ the vertex set of $\Upsilon_{L(1)^{[1]}}(\mathcal{G}/\mathcal{G}_1)$ and by $\{1,\dots,m,1',\dots,m'\}$ the vertex set of its separated quiver. Let Q be the connected component of $\Upsilon_{L(1)^{[1]}}(\mathcal{G}/\mathcal{G}_1)_s$ which contains j'. If N is another module in Θ, then there are $\mu \in \mathbb{Z}$, $0 \le \tilde{l} \le p-2$ and $t \in \{1,\dots,m\}$ with $N \cong V(\mu,\tilde{l}) \otimes_k S_t$. By the above, M and N can only lie in the same component if $l = \tilde{l}$. If $\mu = 2\nu$ is even, then $\tau_{\mathcal{G}}^{-\nu}(N) \cong V(0,l) \otimes_k S_t$. As M and N are in the same component, there is a path

$$V(0,l) \otimes_l S_j \leftarrow V(1,l) \otimes_l S_{i_1} \to V(0,l) \otimes_l S_{i_2} \leftarrow \dots \leftarrow V(1,l) \otimes_l S_{i_r} \to V(0,l) \otimes_l S_t$$

in Θ. This gives rise to a path

$$j' \leftarrow i_1 \to i_2' \leftarrow \dots \leftarrow i_r \to t'$$

in the separated quiver $\Upsilon_{L(1)^{[1]}}(\mathcal{G}/\mathcal{G}_1)_s$. Consequently, $t' \in Q$. Similarly, if μ is odd, we obtain $t \in Q$.
Moreover, for each arrow $i \to t'$ in Q and $\mu \in \mathbb{Z}$ we have arrows

$$\varphi_{i,t',\mu} : V(\mu+1,l) \otimes_k S_i \to V(\mu,l) \otimes_k S_t \text{ and}$$
$$\varphi_{t',i,\mu+1} : V(\mu+2,l) \otimes_k S_t \to V(\mu+1,l) \otimes_k S_i$$

in Θ.
Now let $\psi : \mathbb{Z}[Q] \to \Theta$ be the morphism of stable translation quivers given by

$$\psi(\nu,t) = V(2\nu+1,l) \otimes_k S_t \text{ for each } \nu \in \mathbb{Z} \text{ and } t \in \{1,\dots,m\},$$
$$\psi(\nu,t') = V(2\nu,l) \otimes_k S_t \text{ for each } \nu \in \mathbb{Z} \text{ and } t' \in \{1',\dots,m'\},$$
$$\psi((\nu,i) \to (\nu,t')) = \varphi_{i,t',2\nu},$$
$$\psi((\nu+1,t') \to (\nu,i)) = \varphi_{t',i,2\nu+1}.$$

One now easily checks that this is an isomorphism. □

8. Classification of modules for amalgamated polyhedral group schemes

Let k be an algebraically closed field of characteristic $p > 2$. In this section we will classify the modules of an amalgamated polyhedral group scheme. As for any domestic finite group scheme $\mathcal{G}$, the factor group $\mathcal{G}/\mathcal{G}_{lr}$ is isomorphic to an amalgamated polyhedral group scheme and the principal blocks of these group schemes are isomorphic, we obtain a classification of all indecomposable modules of the principal blocks of domestic finite group schemes.
Moreover, we are able to assign to each module its component in the Auslander-Reiten quiver by using the results in this work.

8.1. Amalgamated cyclic group schemes

Due to 3.4.3, the stable Auslander-Reiten quiver of the $\frac{p-1}{2}$ non-simple blocks of $kP\mathcal{SC}_{(m)}$ consists of two Euclidean components of type $\mathbb{Z}[\tilde{A}_{2np^{r-1}-1}]$, four exceptional tubes $\mathbb{Z}[A_\infty]/(\tau^{np^{r-1}})$ and infinitely many homogeneous tubes.
For $r = 1$ and $n = 1$ we obtain $P\mathcal{SC}_{(m)} \cong PSL(2)_1 \cong SL(2)_1$. Therefore, we do not have to consider this case.

Proposition 8.1.1. *Let $m \geq 1$ with $m = p^r n$, $(n,p) = 1$ and $r > 1$ or $n > 1$. Let $\mathcal{G} = P\mathcal{SC}_{(m)}$ and assume $\{1, w_0\} \subseteq C_{\mathcal{G}}$. Let M be an indecomposable non-projective $\mathcal{G}$-module. Then M is isomorphic to one of the modules belonging to the following list of pairwise non-isomorphic $\mathcal{G}$-modules:*

(i) $\tilde{V}(d) \otimes_k k_\lambda$, $\tilde{V}(d)^* \otimes_k k_\lambda$, $\tilde{V}(i) \otimes_k k\lambda$ *for* $d \geq p$, $d \not\equiv -1 \bmod p$, $\lambda \in X(\mu_{(np^{r-1})})$ *and* $0 \leq i \leq p-1$. *(Modules belonging to Euclidean components)*

(ii) $Y(i, w_0^j, l) \otimes_k k_\lambda$ *for* $i \in \{0, \dots, p-2\}$, $l \in \mathbb{N}$, $j \in \{0,1\}$ *and* $\lambda \in X(\mu_{(np^{r-1})})$. *(Modules belonging to exceptional tubes of rank np^{r-1})*

(iii) $\operatorname{ind}_{P\mathcal{SC}_{(p^r)}}^{\mathcal{G}} Y(i, g, l)$ *for* $g \in C_{\mathcal{G}} \setminus \{1, w_0\}$, $i \in \{0, \dots, p-2\}$ *and* $l \in \mathbb{N}$. *(Modules belonging to homogeneous tubes)*

Proof. This proof has two parts. The first part uses the results of chapter 5 to determine the modules as listed above. In the second part we will use the results about the Auslander-Reiten quiver and stabilizers to assign these modules to their components in the Auslander-Reiten quiver.

1. If M has complexity 2 it is by 5.3.3 isomorphic to one of the modules in (i). Hence we can assume that M has complexity 1. Then an application of 5.3.2 yields unique $l \in \mathbb{N}$, $i \in \{0, \dots, p-2\}$ and $g \in C_{\mathcal{G}}$ such that M is isomorphic to $\operatorname{ind}_{\mathcal{G}_{Y(i,g,l)}}^{\mathcal{G}}(Y(i,g,l) \otimes_k k_\alpha)$ for a unique character α.
Let $G := \mathcal{G}(k)$. By 5.1.5 we know $G_{Y(i,g,l)} = G_{[g.e]}$. By computing the stabilizers for the action of G on $\mathbb{P}(V_{\mathfrak{sl}(2)})$, we obtain the following cases:

a) If $g \in \{1, w_0\}$ then $\mathcal{G}_{Y(i,g,l)} = P\mathcal{SC}_{(m)}$ and $\alpha \in X(\mathcal{G}_{Y(i,g,l)}/\mathcal{G}_1) \cong X(\mu_{(np^{r-1})})$.

b) Otherwise $\mathcal{G}_{Y(i,g,l)} = P\mathcal{SC}_{(p^r)}$ and $\alpha \in X(\mathcal{G}_{Y(i,g,l)}/\mathcal{G}^0) \cong X(\mu_{(1)}) = \{1\}$.

These are exactly the cases (ii) and (iii).

2. If M has complexity 2 then by 2.2.7 the component of M is not a tube and therefore Euclidean.
Let M be of complexity 1 and $g \in C_{\mathcal{G}}$ be arbitrary. Thanks to 2.4.5, the component $\Theta \subseteq \Gamma_s(\mathcal{G}_{Y(i,g,l)})$ of the $\mathcal{G}_{Y(i,g,l)}$-module $Y(i,g,l) \otimes_k k_\alpha$ is isomorphic to the component of M. As M has complexity 1, an application of 2.2.7 yields that the component Θ is a tube. Let $\Xi \subseteq \Gamma_s(\mathcal{G}^0)$ be the component of $\operatorname{res}^{\mathcal{G}_{Y(i,g,l)}}_{\mathcal{G}^0} Y(i,g,l)$. Then Ξ is a tube of rank $s \in \{1, p^{r-1}\}$. Denote by q the rank of Θ. Due to 2.4.7 we obtain $q \leq |G_{Y(i,g,l)}|s$.
Assume that Θ is a tube of rank np^{r-1}. Then $np^{r-1} \leq |G_{Y(i,g,l)}|s \leq np^{r-1}$ and therefore $|G_{Y(i,g,l)}| = n$ and $s = p^{r-1}$. If $r > 1$, then $V_{\mathfrak{sl}(2)}(\Xi) \in \{ke, kf\}$ and therefore $g \in \{1, w_0\}$. If $n > 1$, the only points in $\mathbb{P}(V_{\mathfrak{sl}(2)})$ stabilized by $G_{Y(i,g,l)} = G_{[g.e]}$ are $[e]$ and $[w_0.e]$. Hence we have in both cases $g \in \{1, w_0\}$. The quasi-length of $\operatorname{res}^{\mathcal{G}_{Y(i,g,l)}}_{\mathcal{G}^0} Y(i,g,l)$ in Ξ is l. By 2.4.7 the morphism $\operatorname{res}^{\mathcal{G}_{Y(i,g,l)}}_{\mathcal{G}^0} : \Theta \to \Xi$ preserves the quasi-length, so that the module $Y(i,g,l) \otimes_k k_\beta$ has also quasi-length l. As the number of $\mathcal{G}$-modules of quasi-length l belonging to exceptional tubes of rank np^{r-1} is $(p-1)np^{r-1}$ and this number equals the number of modules in (ii) for a fixed l, all these modules belong to an exceptional tube of rank np^{r-1}.
Now the remaining modules in (iii) have to belong to homogeneous tubes.

□

8.2. Amalgamated non-reduced-dihedral group schemes

By [13, 7.4], the stable Auslander-Reiten quiver of the $\frac{p-1}{2}$ non-simple blocks of the group scheme $kP\mathcal{SQ}_{(m)}$ consists of two Euclidean components of type $\mathbb{Z}[\tilde{D}_{np^{r-1}+2}]$, two exceptional tubes $\mathbb{Z}[A_\infty]/(\tau^{np^{r-1}})$, four exceptional tubes $\mathbb{Z}[A_\infty]/(\tau^2)$ and infinitely many homogeneous tubes.
The representation theory of the amalgamated dihedral group schemes is the most complicated case. If $n = 2$ and $r = 1$ we are not able to distinguish the two cases of exceptional tubes and all exceptional tubes are of rank 2. For $r > 1$ we are in the non-reduced case.

Proposition 8.2.1. *Let $m \geq 2$ with $m = p^r n$ and $(n,p) = 1$. Assume $r > 1$ or $n > 2$. Let $\mathcal{G} = P\mathcal{SQ}_{(m)}$ and assume $\{1, h_4, h_4w_0\} \subseteq C_{\mathcal{G}}$. Let M be an indecomposable non-projective $\mathcal{G}$-module. Then M is isomorphic to one of the modules belonging to the following list of pairwise non-isomorphic $\mathcal{G}$-modules:*

(i) $\tilde{V}(d) \otimes_k S$, $\tilde{V}(d)^* \otimes_k S$, $\tilde{V}(i) \otimes_k S$ for $d \geq p$, $d \not\equiv -1 \bmod p$, S a simple $\mathcal{G}/\mathcal{G}_1$-module and $0 \leq i \leq p-1$. (Modules belonging to Euclidean components)

(ii) $\operatorname{ind}^{\mathcal{G}}_{\mathcal{PSC}_{(m)}}(Y(i,1,l) \otimes_k k_\lambda)$ *for* $i \in \{0,\dots,p-2\}$, $l \in \mathbb{N}$ *and* $\lambda \in X(\mu_{(np^{r-1})})$. *(Modules belonging to exceptional tubes of rank* np^{r-1}*)*

(iii) $\operatorname{ind}^{\mathcal{G}}_{\mathcal{PSQ}_{(p^r)}}(Y(i,h_4w_0^j,l) \otimes_k k_\lambda)$ *for* $j \in \{0,1\}$, $i \in \{0,\dots,p-2\}$, $l \in \mathbb{N}$ *and* $\lambda \in X(\mu_{(2)})$. *(Modules belonging to exceptional tubes of rank* 2*)*

(iv) $\operatorname{ind}^{\mathcal{G}}_{\mathcal{PSC}_{(p^r)}} Y(i,g,l)$ *for* $g \in C_{\mathcal{G}} \setminus \{1, h_4, h_4w_0\}$, $i \in \{0,\dots,p-2\}$ *and* $l \in \mathbb{N}$. *(Modules belonging to homogeneous tubes)*

Proof. This proof has two parts. The first part uses the results of chapter 5 to determine the modules as listed above. In the second part we will use the results about the Auslander-Reiten quiver and stabilizers to assign these modules to their components in the Auslander-Reiten quiver.

1. If M has complexity 2 it is by 5.3.3 isomorphic to one of the modules in (i). Hence we can assume that M has complexity 1. Then an application of 5.3.2 yields unique $l \in \mathbb{N}$, $i \in \{0,\dots,p-2\}$ and $g \in C_{\mathcal{G}}$ such that M is isomorphic to $\operatorname{ind}^{\mathcal{G}}_{\mathcal{G}_{Y(i,g,l)}}(Y(i,g,l) \otimes_k k_\alpha)$ for a unique character α.
   Let $G := \mathcal{G}(k)$. By 5.1.5 we know $G_{Y(i,g,l)} = G_{[g.e]}$. By computing the stabilizers for the action of G on $\mathbb{P}(V_{\mathfrak{sl}(2)})$, we obtain the following cases:
   a) If $g = 1$, then $\mathcal{G}_{Y(i,1,l)} = \mathcal{PSC}_{(m)}$ and $\alpha \in X(\mathcal{G}_{Y(i,1,l)}/\mathcal{G}_1) \cong X(\mu_{(np^{r-1})})$.
   b) If $g \in \{h_4, h_4w_0\}$, then $\mathcal{G}_{Y(i,g,l)} = \mathcal{PSQ}_{(p^r)}$ and $\alpha \in X(\mathcal{G}_{Y(i,g,l)}/\mathcal{G}^0) \cong X(\mu_{(2)})$.
   c) Otherwise, $\mathcal{G}_{Y(i,g,l)} = \mathcal{PSC}_{(p^r)}$ and $\alpha \in X(\mathcal{G}_{Y(i,g,l)}/\mathcal{G}^0) \cong X(\mu_{(1)}) = \{1\}$.

   These are exactly the cases (ii)-(iv).

2. If M has complexity 2 then by 2.2.7 the component of M is not a tube and therefore Euclidean.
   Let M be of complexity 1 and $g \in C_{\mathcal{G}}$ be arbitrary. Thanks to 2.4.5, the component $\Theta \subseteq \Gamma_s(\mathcal{G}_{Y(i,g,l)})$ of the $\mathcal{G}_{Y(i,g,l)}$-module $Y(i,g,l) \otimes_k k_\alpha$ is isomorphic to the component of M. As M has complexity 1, an application of 2.2.7 yields that the component Θ is a tube. Let $\Xi \subseteq \Gamma_s(\mathcal{G}^0)$ be the component of $\operatorname{res}^{\mathcal{G}_{Y(i,g,l)}}_{\mathcal{G}^0} Y(i,g,l)$. Then Ξ is a tube of rank $s \in \{1, p^{r-1}\}$. Denote by q the rank of Θ. Due to 2.4.7, we obtain $q \leq |G_{Y(i,g,l)}|s$. An application of 5.1.6 yields that $|G_{Y(i,g,l)}| = |G_{[g.e]}| \leq n$.
   Assume that Θ is a tube of rank np^{r-1}. Then $np^{r-1} \leq |G_{Y(i,g,l)}|s \leq np^{r-1}$ and therefore $|G_{Y(i,g,l)}| = n$ and $s = p^{r-1}$. If $r > 1$, then $V_{\mathfrak{sl}(2)}(\Xi) \in \{ke, kf\}$ and therefore $g \in \{1, w_0\}$. If $n > 2$, the only points in $\mathbb{P}(V_{\mathfrak{sl}(2)})$ stabilized by the group $G_{Y(i,g,l)} = G_{[g.e]}$ are $[e]$ and $[w_0.e]$. Hence we have in both cases $g = 1$, as w_0 is in the same G-orbit as 1. The quasi-length of $\operatorname{res}^{\mathcal{G}_{Y(i,g,l)}}_{\mathcal{G}^0} Y(i,g,l)$ in Ξ is l. By 2.4.7, the morphism $\operatorname{res}^{\mathcal{G}_{Y(i,g,l)}}_{\mathcal{G}^0} : \Theta \to \Xi$ preserves the quasi-length, so that the module $Y(i,g,l) \otimes_k k_\beta$ has also quasi-length l. As the number of $\mathcal{G}$-modules of quasi-length

l belonging to exceptional tubes of rank np^{r-1} is $(p-1)np^{r-1}$ and this number equals the number of modules in (ii) for a fixed l, all these modules belong to an exceptional tube of rank np^{r-1}.
Now assume that Θ is a tube of rank 2. Then $g \neq 1$. By 8.1.1, the equality $s = p^{r-1}$ is only given for $g \in \{1, w_0\}$. As w_0 is in the same orbit as 1 we therefore obtain $s = 1$. Moreover, we are now in the cases (b) and (c) so that $|G_{Y(i,g,l)}| \leq 2$. Since $2 \leq |G_{Y(i,g,l)}|s = |G_{Y(i,g,l)}|$ we obtain $|G_{Y(i,g,l)}| = 2$. Therefore M is isomorphic to a module in (iii). With the same arguments as above, all modules in (iii) belong to an exceptional tube of rank 2.
Now the remaining modules in (iv) have to belong to homogeneous tubes.

□

8.3. Amalgamated reduced-polyhedral group schemes

Let $\mathcal{G}$ be an amalgamated reduced-polyhedral group scheme so that $\mathcal{G}/\mathcal{G}_1$ is reduced and set $G := \mathcal{G}(k)$. Then G acts faithfully on $\mathbb{P}(\mathcal{V}_{\mathfrak{sl}(2)})$. The map

$$\zeta : \mathbb{P}^1 \to \mathbb{P}(V_{\mathfrak{sl}(2)}), (a:b) \mapsto \left[\begin{pmatrix} ab & a^2 \\ -b^2 & -ab \end{pmatrix}\right]$$

is an isomorphism of varieties and induces a faithful action of G on $\mathbb{P}^1$ such that ζ is G-equivariant. Up to an automorphism of $\mathbb{P}^1$, this action is given by the natural action of $PSL(2)(k)$ on $\mathbb{P}^1$. This action plays a role in the classification of the polyhedral groups. Let F be the set of points in $\mathbb{P}^1$ with non-trivial stabilizer. Then G acts on the finite set F. Let $p_1, \dots, p_d$ be a complete set of representatives of the orbits of this action. For each $i \in \{1, \dots, d\}$, there is $g_i \in SL(2)(k)$ such that $\zeta(p_i) = [g_i.e]$. Set $c_i := |G_{p_i}|$ and $c := |G|$. Thanks to [49, 4.4], we have $d = 2$ if G is cyclic and $d = 3$ otherwise. For $d = 3$ the tuple (c, c_1, c_2, c_3) belongs to the set $\{(2n, 2, 2, n), (12, 2, 3, 3), (24, 2, 3, 4), (60, 2, 3, 5)\}$. Define

$$Q := \begin{cases} \tilde{D}_{n+2} & \text{if } (c, c_1, c_2, c_3) = (2n, 2, 2, n) \\ \tilde{E}_6 & \text{if } (c, c_1, c_2, c_3) = (12, 2, 3, 3) \\ \tilde{E}_7 & \text{if } (c, c_1, c_2, c_3) = (24, 2, 3, 4) \\ \tilde{E}_8 & \text{if } (c, c_1, c_2, c_3) = (60, 2, 3, 5) \end{cases}$$

Note that these numbers also coincide with the tubular types of Q ([44, 3.6]). Moreover, this quiver coincides with the McKay quiver of G.
By 3.4.3, the stable Auslander-Reiten quiver of the $\frac{p-1}{2}$ non-simple blocks of $k\mathcal{G}$ consists of two Euclidean components of type $\mathbb{Z}[Q]$, two exceptional tubes $\mathbb{Z}[A_\infty]/(\tau^{c_1})$, two exceptional tubes $\mathbb{Z}[A_\infty]/(\tau^{c_2})$, two exceptional tubes $\mathbb{Z}[A_\infty]/(\tau^{c_3})$ and infinitely many homogeneous tubes.

Proposition 8.3.1. *Let $\mathcal{G}$ be an amalgamated reduced-polyhedral group scheme and set $G := \mathcal{G}(k)$. Let M be an indecomposable non-projective $\mathcal{G}$-module. Then M is isomorphic to one of the modules belonging to the following list of pairwise non-isomorphic $\mathcal{G}$-modules:*

(i) $\tilde{V}(d)\otimes_k S$, $\tilde{V}(d)^*\otimes_k S$, $\tilde{V}(i)\otimes_k S$ *for* $d \geq p$, $d \not\equiv -1 \bmod p$, S *a simple* $\mathcal{G}/\mathcal{G}_1$*-module and* $0 \leq i \leq p-1$*. (Modules belonging to Euclidean components)*

(ii) $\mathrm{ind}^{\mathcal{G}}_{P\mathcal{SC}^{g_1}_{(c_1)}}(Y(i,g_1,l)\otimes_k k_\lambda)$ *for* $i \in \{0,\dots,p-2\}$, $l \in \mathbb{N}$ *and* $\lambda \in X(\mu_{(c_1)})$*. (Modules belonging to exceptional tubes of rank* c_1*)*

(iii) $\mathrm{ind}^{\mathcal{G}}_{P\mathcal{SC}^{g_2}_{(c_2)}}(Y(i,g_2,l)\otimes_k k_\lambda)$ *for* $j \in \{0,1\}$, $i \in \{0,\dots,p-2\}$, $l \in \mathbb{N}$ *and* $\lambda \in X(\mu_{(c_2)})$*. (Modules belonging to exceptional tubes of rank* c_2*)*

(iv) $\mathrm{ind}^{\mathcal{G}}_{P\mathcal{SC}^{g_3}_{(c_3)}}(Y(i,g_3,l)\otimes_k k_\lambda)$ *for* $j \in \{0,1\}$, $i \in \{0,\dots,p-2\}$, $l \in \mathbb{N}$ *and* $\lambda \in X(\mu_{(c_3)})$*. (Modules belonging to exceptional tubes of rank* c_3*)*

(v) $\mathrm{ind}^{\mathcal{G}}_{SL(2)_1} Y(i,g,l)$ *for* $g \in C_{\mathcal{G}} \setminus \{g_1,g_2,g_3\}$, $i \in \{0,\dots,p-2\}$ *and* $l \in \mathbb{N}$*. (Modules belonging to homogeneous tubes)*

Proof. If M has complexity 2, it is by 5.3.3 isomorphic to one of the modules in (i). By 2.2.7, the component of M is not a tube and therefore Euclidean.
Hence we can assume that M has complexity 1. Then an application of 5.3.2 yields unique $l \in \mathbb{N}$, $i \in \{0,\dots,p-2\}$ and $g \in C_{\mathcal{G}}$ such that M is isomorphic to $\mathrm{ind}^{\mathcal{G}}_{\mathcal{G}_{Y(i,g,l)}}(Y(i,g,l)\otimes_k k_\alpha)$ for a unique character α.
As $\mathcal{G}/\mathcal{G}_1$ is reduced and the group $G_{Y(i,g,l)}$ is cyclic, we obtain $\mathcal{G}_{Y(i,g,l)} = P\mathcal{SC}^g_{(n)}$ for $n = |G_{Y(i,g,l)}| = |G_{[g.e]}|$. Let $\Theta \subseteq \Gamma_s(\mathcal{G})$ be the component of M and $\Xi \subseteq \Gamma_s(\mathcal{G}_{Y(i,g,l)})$ be the component $Y(i,g,l)\otimes_k k_\alpha$. As $\mathcal{G}/\mathcal{G}_1$ is reduced, 2.4.5 yields that $\mathrm{ind}^{\mathcal{G}}_{\mathcal{G}_{Y(i,g,l)}} : \Xi \to \Theta$ is an isomorphism of stable translation quivers. Moreover, $\Xi^{g^{-1}} \subseteq \Gamma_s(P\mathcal{SC}_{(n)})$ is the component containing $Y(i,1,l)$. By 8.1.1, the component Ξ is an exceptional tube of rank n. Hence, Θ also has rank n and all modules belonging to Θ are of the form $\mathrm{ind}^{\mathcal{G}}_{P\mathcal{SC}^g_{(n)}}(Y(i,g,s)\otimes_k k_\beta)$ with $s \in \mathbb{N}$ and $\beta \in X(\mu_{(n)})$.
Now the assertion follows from the description of the stabilizers given above. □

Remark 8.3.2. The problem, that we have to analyze the non-reduced case separately, is due to the fact that we do not have the isomorphism $\mathrm{ind}^{\mathcal{G}}_{\mathcal{G}_{Y(i,g,l)}} : \Xi \to \Theta$ (c.f. [15, 4.3]). Therefore, one has to find a functor that can substitute the induction functor in this context.
One way could be, to use an alternate approach to realize the modules $X(i,g,l)$. The filtration we used seems to be connected to a filtration in the context of Hopf-Galois extensions introduced in [47].

9. Quotients of support varieties and ramification

The goal of this chapter is to describe a geometric connection between the tubes in the Auslander-Reiten quiver of a finite group scheme $\mathcal{G}$ and the corresponding tubes in the Auslander-Reiten quiver of a normal subgroup scheme $\mathcal{N}$ of $\mathcal{G}$. We will see that the support variety of $\mathcal{N}$ is a geometric quotient of the support variety of $\mathcal{G}$. The geometric connection will then be given via the ramification indices of the quotient morphism.

9.1. Quotients of varieties and ramification

Let $x \in X$, R be a commutative k-algebra and $\iota_R : k \to R$ be the canonical inclusion. Then we denote by $x_R := X(\iota_R)(x)$ the image of x in $X(R)$. An action of a group scheme on a variety is always supposed to be an action via morphisms of schemes.

Definition 9.1.1. Let $\mathcal{H}$ be a group scheme acting on a variety X.

1. A pair (Y, q) consisting of a variety Y and an $\mathcal{H}$-invariant morphism $q : X \to Y$ is called categorical quotient of X by the action of $\mathcal{H}$, if for every $\mathcal{H}$-invariant morphism $q' : X \to Y'$ of varieties there is a unique morphism $\alpha : Y \to Y'$ such that $q' = \alpha \circ q$.

2. A pair (Y, q) consisting of a variety Y and an $\mathcal{H}$-invariant morphism $q : X \to Y$ of varieties is called geometric quotient of X by the action of $\mathcal{H}$ if the following holds:
   a) The underlying topological space of Y is the quotient of the underlying topological space of X by the action of the group $\mathcal{H}(k)$.
   b) $q : X \to Y$ is an $\mathcal{H}$-invariant morphism of schemes such that the induced homomorphism of sheafs $\mathcal{O}_Y \to q_*(\mathcal{O}_X)^{\mathcal{H}}$ is an isomorphism.

3. If $x \in X$ is a point, then the stabilizer $\mathcal{H}_x$ is the subgroup scheme of $\mathcal{H}$ given by $\mathcal{H}_x(R) = \{g \in \mathcal{H}(R) \mid g.x_R = x_R\}$ for every commutative k-algebra R.

Thanks to [37, 12.1], there is for any finite group scheme $\mathcal{H}$ and any quasi-projective variety X an up to isomorphism uniquely determined geometric quotient which will be denoted by $X/\mathcal{H}$. Moreover, the quotient morphism $q : X \to X/\mathcal{H}$ is a finite morphism, i.e. there exists an open affine covering $X/\mathcal{H} = \bigcup_{i \in I} V_i$ such that $q^{-1}(V_i)$ is affine and the homomorphism $k[V_i] \to k[q^{-1}(V_i)]$ of rings is finite for all $i \in I$.

Example 9.1.2. Let $\mathcal{H}$ be a finite group scheme and A be a finitely generated commutative k-algebra. Then the set $X := \operatorname{Maxspec} A$ of maximal ideals of A is an affine variety. An action of $\mathcal{H}$ on X gives A the structure of a $k\mathcal{H}$-module algebra. Then the set

$$A^{\mathcal{H}} := \{a \in A \mid h.a = \varepsilon(h)a \text{ for all } h \in k\mathcal{H}\}$$

of $\mathcal{H}$-invariants is a subalgebra of A. As $\mathcal{H}$ is finite, the subalgebra is also finitely generated and $X/\mathcal{H} = \operatorname{Maxspec} A^{\mathcal{H}}$.
Let $A = \bigoplus_{n \geq 0} A_n$ be additionally graded with $A_0 = k$ and $A_+ = \bigoplus_{n>0} A_n$ be its irrelevant ideal. Then the set $X = \operatorname{Proj} A$ of maximal homogeneous ideals which do not contain A_+ is a projective variety. If $\mathcal{H}$ acts on X, then $X/\mathcal{H} = \operatorname{Proj} A^{\mathcal{H}}$.

Let X be a variety with structure sheaf $\mathcal{O}_X$. For a point $x \in X$ we denote by $\mathcal{O}_{X,x}$ the local ring at the point x. We say that a point $x \in X$ is simple, if its local ring $\mathcal{O}_{X,x}$ is regular. If R is a commutative local ring, we will denote by $\hat{R}$ its completion at its unique maximal ideal. Let V be a k-vector space of dimension n with basis $b_1, \dots, b_n$. Let $t_1, \dots, t_n$ be the corresponding dual basis of V^*. We define the ring of polynomial functions of V as $k[V] := k[t_1, \dots, t_n]$. Its completion $k[[t_1, \dots, t_n]]$ at the maximal ideal $(t_1, \dots, t_n)$ will be denoted by $k[[V]]$.
For future reference we will recall the following facts:

Remark 9.1.3. Let X be an n-dimensional variety and $x \in X$.

(i) A point $x \in X$ is simple if and only if $\hat{\mathcal{O}}_{X,x} \cong k[[x_1, \dots, x_n]]$.

(ii) Let $\mathcal{H}$ be a finite group scheme acting on X and assume that there is a geometric quotient (Y, q) of this action. Then $\hat{\mathcal{O}}_{Y,q(x)} \cong (\hat{\mathcal{O}}_{X,x})^{\mathcal{H}_x}$ ([35, Exercise 4.5(ii)]).

(iii) Let $\mathcal{H}$ be a linearly reductive group scheme acting on $k[[x_1, \dots, x_n]]$ via algebra automorphisms. Then there is an n-dimensional $\mathcal{H}$-module V such that there is an $\mathcal{H}$-equivariant isomorphism $k[[V]] \cong k[[x_1, \dots, x_n]]$ of k-algebras. (c.f. [46, Proof of 1.8])

Definition 9.1.4. Let $f : X \to Y$ be a finite morphism, $x \in X$ and $y = f(x)$. Let $\mathfrak{m}_y$ be the maximal ideal of the local ring $\mathcal{O}_{Y,y}$. Then $e_x(f) := \dim_k \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is called the ramification index of f at x.

As the morphism f is finite, the induced homomorphism $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ endows $\mathcal{O}_{X,x}$ with the structure of a finitely generated $\mathcal{O}_{Y,y}$-module. Therefore the number $e_x(f)$ is finite.

Lemma 9.1.5. *Let $\mathcal{H}$ be a finite linearly reductive group scheme which acts faithfully on a 1-dimensional irreducible quasi-projective variety X. Let $q : X \to X/\mathcal{H}$ denote the quotient morphism. If $x \in X$ is a simple point, then $\mathcal{H}_x \cong \mu_{(n)}$ for some $n \in \mathbb{N}$ and $e_x(q) = |\mathcal{H}_x|$.*

Proof. Since x is a simple point and X is one-dimensional, 9.1.3 (i) yields a $\mathcal{H}_x$-equivariant isomorphism $\hat{\mathcal{O}}_{X,x} \cong k[[T]]$. By 9.1.3 (iii), we can assume that the one-dimensional k-vector space $\langle T \rangle_k$ is an $\mathcal{H}_x$-module. As X is irreducible, the field of fractions of $\mathcal{O}_{X,x}$ is the function field $k(X)$ of X. If $\mathcal{K}$ is the kernel of the action of $\mathcal{H}_x$ on $\mathcal{O}_{X,x}$, then it acts also trivially on $k(X)$ and therefore also on X. As $\mathcal{H}$ acts faithfully on X, it follows that $\mathcal{K}$ is trivial. Therefore, $\langle T \rangle_k$ is a faithful $\mathcal{H}_x$-module and we can assume $\mathcal{H}_x = \mu_{(n)} \subseteq GL_1$ where $n = |\mathcal{H}_x|$. As $\mathcal{H}_x$ acts via algebra automorphisms, this yields $k[[T]]^{\mathcal{H}_x} = k[[T^n]]$. By 9.1.3 (ii), we have $\hat{\mathcal{O}}_{X/\mathcal{H},q(x)} \cong \hat{\mathcal{O}}_{X,x}^{\mathcal{H}_x}$. Since $\hat{\mathcal{O}}_{X,x}$ is the completion of $\mathcal{O}_{X,x}$ at is maximal ideal $\mathfrak{m}_x$ and the ideal $\mathfrak{m}_{q(x)}\mathcal{O}_{X,x}$ is contained in $\mathfrak{m}_x$, we obtain an isomorphism $\hat{\mathcal{O}}_{X,x}/\mathfrak{m}_{q(x)}\hat{\mathcal{O}}_{X,x} \cong \mathcal{O}_{X,x}/\mathfrak{m}_{q(x)}\mathcal{O}_{X,x}$. As a result, we obtain

$$e_x(q) = \dim_k \hat{\mathcal{O}}_{X,x}/\mathfrak{m}_{q(x)}\hat{\mathcal{O}}_{X,x} = \dim_k k[T]/(T^n) = n = |\mathcal{H}_x|.$$

□

9.2. Ramification of the restriction morphism

If $\mathcal{N}$ is a normal subgroup scheme of $\mathcal{G}$ then $\mathcal{G}/\mathcal{N}$ acts via automorphisms of graded algebras on $H^\bullet(\mathcal{N},k)$ and therefore on $\mathbb{P}(\mathcal{V}_\mathcal{N})$.

Proposition 9.2.1. *Let $\mathcal{G}$ be a finite group scheme and $\mathcal{N}$ be a normal subgroup scheme of $\mathcal{G}$ such that $\mathcal{G}/\mathcal{N}$ is linearly reductive. Then $(\mathbb{P}(\mathcal{V}_\mathcal{G}), \iota_{*,\mathcal{N}})$ is a geometric quotient for the action of $\mathcal{G}/\mathcal{N}$ on $\mathbb{P}(\mathcal{V}_\mathcal{N})$.*

Proof. Since $\mathcal{G}/\mathcal{N}$ is linearly reductive, the Lyndon-Hochschild-Serre spectral sequence ([25, I.6.6(3)]) yields an isomorphism $\iota^\bullet : H^\bullet(\mathcal{G},k) \to H^\bullet(\mathcal{N},k)^{\mathcal{G}/\mathcal{N}}$. Therefore, $(\mathbb{P}(\mathcal{V}_\mathcal{G}), \iota_{*,\mathcal{N}})$ is a geometric quotient for the action of $\mathcal{G}/\mathcal{N}$ on $\mathbb{P}(\mathcal{V}_\mathcal{N})$. □

Let $\mathcal{G}$ be a finite group scheme and $\mathcal{N} \subseteq \mathcal{G}$ be a normal subgroup scheme with $\mathcal{G}^0 \subseteq \mathcal{N}$. Set $G := \mathcal{G}/\mathcal{N}(k)$. As in 2.4, the group G acts on the set of components of $\Gamma_s(\mathcal{N})$. As before, there is a unique subgroup scheme $\mathcal{G}_\Theta \subseteq \mathcal{G}$ with $k\mathcal{G}_\Theta = (k\mathcal{G})_{G_\Theta}$.
As in 2.4, let N be an indecomposable non-projective $\mathcal{N}$-module and Ξ the corresponding component in $\Gamma_s(\mathcal{N})$. Assume there is an indecomposable non-projective direct summand M of $\operatorname{ind}_\mathcal{N}^\mathcal{G} N$ and let Θ be the corresponding component in $\Gamma_s(\mathcal{G})$.
Now assume that Θ and Ξ are tubes. Then the varieties $\mathbb{P}(\mathcal{V}_\mathcal{G}(\Theta))$ and $\mathbb{P}(\mathcal{V}_\mathcal{N}(\Xi))$ consist of single points x_Θ and x_Ξ, respectively. Thanks to [20, 5.6], we have $\iota_{*,\mathcal{N}}^{-1}(\mathbb{P}(\mathcal{V}_\mathcal{G}(M))) = \mathbb{P}(\mathcal{V}_\mathcal{N}(\operatorname{res}_\mathcal{N}^\mathcal{G} M))$. By 2.4.4, the module $\operatorname{res}_\mathcal{N}^\mathcal{G} M$ has an indecomposable direct summand which belongs to Ξ. Hence,

$$x_\Xi \in \iota_{*,\mathcal{N}}^{-1}(\mathbb{P}(\mathcal{V}_\mathcal{G}(M))) = \iota_{*,\mathcal{N}}^{-1}(x_\Theta),$$

so that $\iota_{*,\mathcal{N}}(x_\Xi) = x_\Theta$.

Proposition 9.2.2. *Let Ξ be a tube of rank n and Θ be a tube of rank m. Assume that $\mathcal{G}^0 \subseteq \mathcal{N}$ and set $G := (\mathcal{G}/\mathcal{N})(k)$. Moreover, assume the following*

1. *$\mathcal{G}/\mathcal{N}$ is linearly reductive, i.e. p does not divide the order of G,*
2. *G acts faithfully on $\mathbb{P}(\mathcal{V}_\mathcal{N})$,*
3. *the variety $\mathbb{P}(\mathcal{V}_\mathcal{N})$ is one-dimensional and irreducible,*
4. *x_Ξ is a simple point of $\mathbb{P}(\mathcal{V}_\mathcal{N})$, and*
5. *all modules belonging to Ξ are $\mathcal{G}_\Xi$ stable.*

Then $m \leq e_{x_\Xi}(\iota_{*,\mathcal{N}})n$.

Proof. The group algebra $k\mathcal{G}$ is a strongly G-graded k-algebra with $(k\mathcal{G})_1 = k\mathcal{N}$. As G acts faithfully on the one-dimensional irreducible variety $\mathbb{P}(\mathcal{V}_\mathcal{N})$ with simple point x_Ξ, we obtain due to 9.1.5, that the stabilizer G_{x_Ξ} is a cyclic group and we have the equality $e_{x_\Xi}(\iota_{*,\mathcal{N}}) = |G_{x_\Xi}|$. Now the assertion follows directly from 2.4.7. □

In the case of amalgamated polyhedral group schemes we can show that the ramification index is actually equal to the ranks of the corresponding tubes.

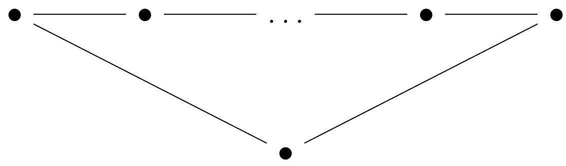
Proposition 9.2.3. *Let $\mathcal{G}$ be an amalgamated polyhedral group scheme, $\mathcal{N} := \mathcal{G}_1$ its first Frobenius kernel and Θ be a tube. Then Θ has rank $e_{x_\Xi}(\iota_{*,\mathcal{N}})$.*

Proof. If $\mathcal{G}$ is an amalgamated non-reduced-dihedral group scheme or an amalgamated cyclic group scheme, then one obtains this result directly from 8.2.1 and 8.1.1 by comparing the numbers. The other cases are either proved in the same way by using the corresponding classification or with the following arguments:
Assume that $\mathcal{G}$ is an amalgamated reduced-polyhedral group scheme. Then the group algebra $k\mathcal{G}$ is isomorphic to the skew group algebra $kSL(2)_1 * \mathcal{G}(k)$. Let M be an indecomposable $\mathcal{G}$-module which belongs to Θ and U be an indecomposable direct summand of $\mathrm{res}^{\mathcal{G}}_{\mathcal{G}_\Xi} M$ with $\mathrm{ind}^{\mathcal{G}}_{\mathcal{G}_\Xi} U = M$. Denote by Λ the Auslander-Reiten component of $\Gamma_s(\mathcal{G}_\Xi)$ which contains U. Then 2.4.5 yields, that $\mathrm{ind}^{\mathcal{G}}_{\mathcal{G}_\Xi} : \Lambda \to \Theta$ is an isomorphism of stable translation quivers. Now the assertion follows from the cyclic case, as $\mathcal{G}_\Xi(k)$ is a cyclic group. $\square$

A. Euclidean Diagrams

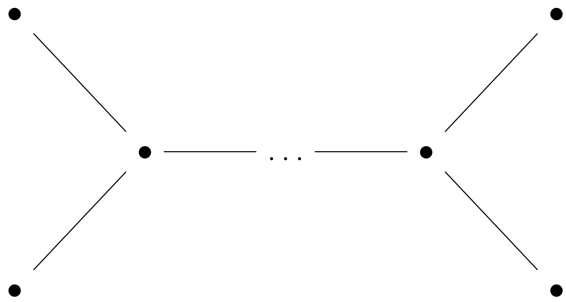
The following list shows the Euclidean diagrams which occur in this work:

$\tilde{A}_n$:



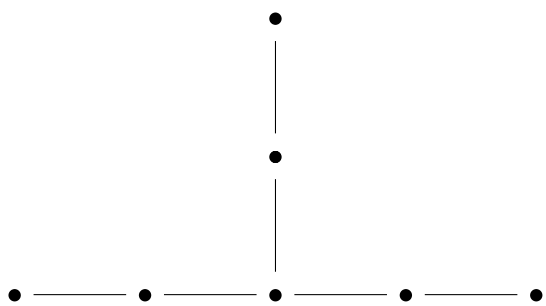
$n+1$ nodes

$\tilde{D}_n$:

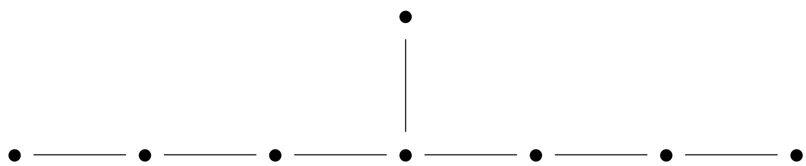


$n+1$ nodes

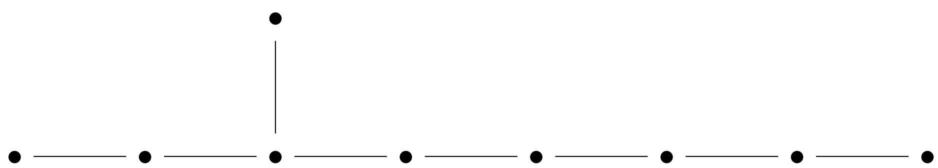
$\tilde{E}_6$:



$\tilde{E}_7$:

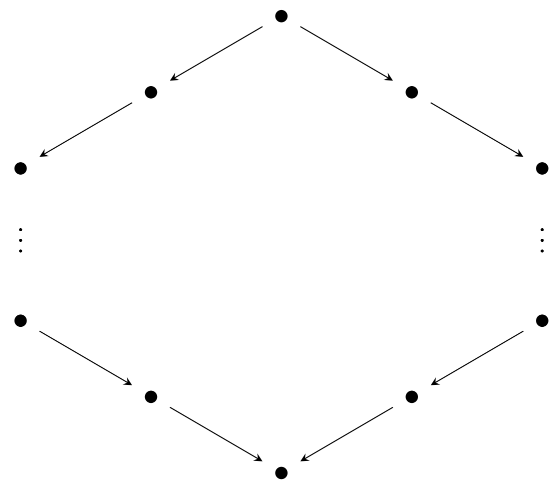


$\tilde{E}_8$:



The arrows of the directed graph $\tilde{A}_{n,n}$ with underlying diagram $\tilde{A}_{2n-1}$ are directed as follows:

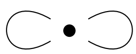
$\tilde{A}_{n,n}$:



$2n$ nodes

Another diagram which may come up at some point is the point with two loops:

$\tilde{L}_0$:



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Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit abgesehen von der Beratung durch den Betreuer meiner Promotion unter Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft selbstständig angefertigt habe und keine anderen als die angegebenen Hilfsmittel verwendet habe.

Einige Hauptergebnisse der Dissertation sind in folgenden Publikationen enthalten:

- Kirchhoff, Dirk: Classification of indecomposable modules for finite group schemes of domestic representation type. *Submitted for Publication to Transformation Groups (Preprint: arXiv:1509.05203)*, 2015.

- Kirchhoff, Dirk: AR-Components of domestic finite group schemes: McKay-Quivers and Ramification. *Submitted for Publication to The Quarterly Journal of Mathematics (Preprint: arXiv:1512.04821)*, 2015.

Kiel, den 15. März 2016

(Dirk Kirchhoff)