Lebesgue Measure of Escaping Sets of Entire Functions in the Eremenko-Lyubich Class

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Abstract

The main aim of this thesis is to try to understand dynamical behaviours of transcendental entire functions from a measure-theoretic point of view. More precisely, we concentrate on a large family of entire functions, which is the Eremenko-Lyubich class $\mathcal{B}$, and study the measure properties of the Julia and escaping sets.

For a transcendental entire function, the complex plane $\mathbb{C}$ is partitioned into two completely invariant subsets – the *Fatou set* and the *Julia set* – based on the theory of normal families in the sense of Montel. Various properties of the Fatou and Julia sets have been studied for a long time. Another interesting set which is of equal importance is the *escaping set*, consisting of points in $\mathbb{C}$ tending to infinity under iterates. For entire functions in class $\mathcal{B}$, i.e., those functions with a bounded set of singular values, the escaping set turns out to be a subset of the Julia set. Thus this gives us a way to estimate the size of Julia sets from below.

In 1987, McMullen showed that the Julia set of λe^z has Hausdorff dimension 2 and the area of Julia set of $\sin(\alpha z + \beta)$ is positive, where $\lambda \in \mathbb{C} \setminus \{0\}$, $\alpha, \beta \in \mathbb{C}$ and $\alpha \neq 0$. The result on the Hausdorff dimension has been extended to more general transcendental entire functions by various authors, while the generalization of the result on the area of Julia set to larger class of functions was not undertook until recent work of Aspenberg and Bergweiler. They gave a condition that is satisfied by many functions, in particular the ones considered by McMullen. We continue their work on this respect and show that their condition is essentially sharp by constructing an entire function for which the escaping set has zero area.

In 1992, on the other hand, Eremenko and Lyubich gave a condition under which the area of escaping set of entire functions in the class $\mathcal{B}$ is zero. The condition is quite general and in particular applies to finite order entire functions in class $\mathcal{B}$ whose inverse has a finite logarithmic singularity. We shall generalize this result to certain functions of infinite order, by adapting the method we use above.

Zusammenfassung

Das Hauptziel dieser Arbeit ist das Verständnis des dynamischen Verhaltens ganz transzendenter Funktionen unter einem maßtheoretischen Gesichtspunkt. Genauer konzentrieren wir uns auf eine große Familie von ganzen Funktionen - die Eremenko-Lyubich-Klasse - und untersuchen Eigenschaften des Maßes der Julia- und entkommenden Menge.

Für eine ganz transzendente Funktion lässt sich die komplexe Ebene im Montelschen Sinne in zwei invariante Teilmengen zerlegen, die sogenannte Fatou- und Juliamenge. Diverse Eigenschaften dieser Mengen wurden lange Zeit untersucht. Eine weitere Menge mit ebenso großer Bedeutung ist die entkommende Menge, welche aus allen Punkten in $\mathbb{C}$ besteht, die unter Iteration gegen ∞ streben. Man kann zeigen, dass für jede ganze Funktion in der Klasse $\mathcal{B}$, d.h. Funktionen, deren singuläre Menge beschränkt ist, die entkommende Menge in der Juliamenge enthalten ist. Dies gibt uns eine Möglichkeit, die Größe der Juliamenge nach unten abzuschätzen.

Im Jahr 1987 hat McMullen gezeigt, dass die Juliamenge von $\lambda \exp z$ Hausdorffdimension 2 hat und die Fläche der Juliamenge von $\sin(\alpha z + \beta)$ positiv ist, wobei $\lambda, \alpha \in \mathbb{C} \setminus \{0\}$ und $\beta \in \mathbb{C}$ sind. Das Resultat über die Hausdorffdimension wurde von verschiedenen Autoren auf größere Klassen ganz transzendenter Funktionen verallgemeinert, während das Ergebnis über die Fläche der Juliamenge für größere Funktionenklassen erst in einer neueren Arbeit von Aspenberg und Bergweiler betrachtet wurde. Ihre Verallgemeinerung liefert eine Bedingung, die von vielen Funktionen erfüllt wird. Insbesondere wird sie von den Funktionen erfüllt, die McMullen betrachtet hat. Wir setzen ihre Arbeit fort und zeigen, dass ihre Bedingung scharf ist, indem wir eine ganze Funktion konstruieren, deren entkommende Menge Fläche 0 hat.

Andererseits wurde im Jahr 1992 von Eremenko und Lyubich eine Bedingung gegeben, unter der die entkommende Menge ganzer Funktionen aus der Klasse $\mathcal{B}$ Fläche 0 hat. Diese Bedingung ist recht allgemein und lässt sich insbesondere auf ganze Funktionen endlicher Ordnung in der Klasse $\mathcal{B}$ anwenden, deren Inverse eine endliche logarithmische Singularität besitzen. Wir werden dieses Resultat auf gewisse Funktionen unendlicher Ordnung verallgemeinern, indem wir die obige Methode verwenden.

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Contents

1 Introduction **1**

2 Preliminaries **5**
2.1 Function Theory . . . 5
2.2 Quasiconformal Mappings . . . 12
2.3 Transcendental Dynamics . . . 14
2.3.1 Dynamics of entire functions . . . 14
2.3.2 Eremenko-Lyubich class . . . 18
2.4 Miscellaneous . . . 22

3 Aspenberg-Bergweiler Condition and Sharpness **23**
3.1 A Reference Function . . . 24
3.2 The Aspenberg-Bergweiler Condition . . . 29
3.3 Sharpness . . . 30

4 Eremenko-Lyubich Condition and Generalization **47**
4.1 The Eremenko-Lyubich Condition . . . 48
4.2 Generalization . . . 49

Bibliography **55**

Chapter 1

Introduction

In this thesis we only consider transcendental dynamics, i.e., iteration of transcendental entire functions. Much of the theory in early times, as with polynomials and rational functions, is based on Montel's theory of *normal families*. For a transcendental entire function f, the *Fatou set* $\mathcal{F}(f)$ is then defined as the set of points where $\{f^n\}$ forms a normal family. The *Julia set* $\mathcal{J}(f)$ is the complement of $\mathcal{F}(f)$ in $\mathbb{C}$. A complete classification of Fatou components, i.e., connected components of the Fatou set, is known, while the dynamics on the Julia set is in general far more complicated.

As in the case of polynomials and rational functions, where dynamical behaviours are often dominated by critical values, the dynamics of transcendental entire functions has close relations with singular values, that is, critical and asymptotic values. By definition, w is a *critical value* of f if there exists $z \in \mathbb{C}$ with $f'(z) = 0$ and $f(z) = w$. A point w is called an *asymptotic value* of f if there exists a curve $\gamma : (0, \infty) \to \mathbb{C}$ with $\gamma(t) \to \infty$ as $t \to \infty$ but $f(\gamma(t)) \to w$ as $t \to \infty$. By a *singular value* of f we always mean a critical value or an asymptotic value of f. The *Eremenko-Lyubich class* $\mathcal{B}$ consists of those entire functions with a bounded set of singular values. Many entire functions, which have similar dynamical behaviours to those of polynomials, are contained in this class, for instance, the exponential map $\exp(z)$ and the sine map $\sin(z)$.

For a transcendental entire function f, a singularity z of f^{-1} is called a *direct singularity* if there exists a disk $D_\chi(z, r)$ with respect to the spherical metric such that f omits the value z in a component V_r of $f^{-1}(D_\chi(z, r))$ for some $r > 0$. In particular, a direct singularity is called *logarithmic* if the restriction $f : V_r \to D_\chi(z, r) \setminus \{z\}$ is a universal covering for some $r > 0$. The domain V_r is called a *direct tract* or *logarithmic tract* over z, respectively. It follows that for every entire function in class $\mathcal{B}$, ∞ is a logarithmic singularity.

The *escaping set* $\mathcal{I}(f)$ of an entire function f consists of points in $\mathbb{C}$ tending to infinity under iteration. For a transcendental entire function f, $\mathcal{I}(f)$ is non-empty and intersects with $\mathcal{J}(f)$ [Ere89]. Moreover, every connected component of $\overline{\mathcal{I}(f)}$ is unbounded [Ere89]. The escaping set has attracted a lot of interest recently, mainly due to a conjecture proposed by Eremenko: every component of $\mathcal{I}(f)$ is unbounded. If, in addition, $f \in \mathcal{B}$, then $\mathcal{I}(f)$ is a subset of $\mathcal{J}(f)$ [EL92]. Thus for functions in class $\mathcal{B}$ we can estimate the size of Julia sets by considering the escaping sets.

Various structures of escaping and Julia sets have been investigated, from either topological, geometrical or measure-and-dimension-theoretical points of view. Our aim here is to study escaping and Julia sets of transcendental entire functions in view of their two-dimensional Lebesgue measure.

McMullen [McM87] proved that the Lebesgue measure of the escaping set of $\sin(\alpha z + \beta)$ is always positive for $\alpha \neq 0$ and $\alpha, \beta \in \mathbb{C}$. This result is generalized substantially by Aspenberg and Bergweiler [AB12] to a large class of entire functions. More precisely, if an entire function f is in class $\mathcal{B}$ with a finite number, say N, logarithmic tracts over infinity, and if

$$\log\log M(r,f) \leq \left(\frac{N}{2} + \frac{1}{\Phi(r)}\right)\log r$$

for large r, then the escaping set (and hence the Julia set) has positive area. Here, the function $\Phi : (0,\infty) \to \mathbb{R}$ is increasing, tending to ∞ but slower than any of the functions $\log^m r := \log(\log^{m-1} r)$ for $m \in \mathbb{N}$; see Section 3.1 for the exact definition of Φ. In [AB12] the authors give an example to show that $1/\Phi(r)$ can not be replaced by a positive constant. However, it remained open to what extent the condition they give is best possible. By constructing an example, we can show that their condition is essentially sharp. More precisely, we have the following result.

Theorem 1.1. *There exists an entire function f in the class $\mathcal{B}$ with*

$$\log\log M(r,f) \leq \left(\frac{1}{2} + \frac{1}{\log\Phi(r)}\right)\log r + \mathcal{O}(1),$$

for which the escaping set has Lebesgue measure zero.

Our next intention is to extend a result of Eremenko and Lyubich also concerning the Lebesgue measure of escaping and Julia sets. Recall that the order $\rho(f)$ of an entire function f is defined as $\rho(f) := \limsup_{r\to\infty} \log\log M(r,f)/\log r$, where $M(r,f) := \max_{|z|=r}|f(z)|$ is the maximum modulus of f. In [EL92], the authors gave a condition, which ensures that an entire function in class $\mathcal{B}$ satisfying this condition has zero area Julia set. In particular, if an entire function f is in class $\mathcal{B}$ and of finite order, and the inverse has a finite logarithmic singularity, then the area of escaping set is zero. We generalize this result to entire functions of "small" infinite order in the following sense:

Theorem 1.2. *Let $f \in \mathcal{B}$ be a transcendental entire function and $r' > 0$. Suppose that the inverse of f has a direct singularity $a \in \mathbb{C}$. Suppose furthermore that f satisfies*

$$\log \log M(r, f) \leq A(r) \log r$$

for $r \geq r'$, for some continuous and increasing function $A : [r', \infty) \to \mathbb{R}$ satisfying $A(r) < \log r$ for large r and

$$\sum_{k=1}^{\infty} \frac{1}{A\left(E^k\left(0\right)\right)} = \infty.$$

Then $\operatorname{area} \mathcal{I}(f) = 0$.

Note that if $A(r)$ is bounded above, then f is of finite order and the divergence of sum is automatically satisfied. In this case, our theorem is exactly that of Eremenko and Lyubich. In this sense, we can view our theorem as a generalization of Eremenko and Lyubich's result mentioned above.

The structure of this thesis is as follows: In Chapter 2 we give preliminaries that will be needed in our proofs of the theorems. This includes some notations, definitions, and results from function theory, quasiconformal mappings and others. In particular, we present a brief introduction to transcendental dynamics. Chapter 3 is devoted to the proof of Theorem 1 above. To this aim, we first construct an entire function using the Weierstrass canonical product and show that it is in class $\mathcal{B}$. Then we prove that this function indeed shows the essential sharpness of the condition given by Aspenberg and Bergweiler. Finally in Chapter 4, we give a proof of Theorem 2.

Chapter 2

Preliminaries

In this thesis, we need some facts from function theory, quasiconformal mappings, complex dynamics and others. Therefore, this chapter will be concentrated on presenting some preliminaries from those theories. We begin with some notations that will be used throughout.

Notations. Throughout the complex plane is denoted by $\mathbb{C}$ and the Riemann sphere by $\widehat{\mathbb{C}}$. We also denote by $\mathbb{R}$ the set of real numbers and by $\mathbb{R}^n$ the n-dimensional Euclidean space. We define by $D(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}$ the Euclidean disk for $z_0 \in \mathbb{C}$ and $r > 0$, that is, a disk with respect to Euclidean metric and in case of the unit disk we use $\mathbb{D}$ instead. We will also use disks with respect to the spherical metric χ and in this case the disk is denoted by $D_\chi(z_0, r)$. The set of rational numbers and natural numbers are denoted by $\mathbb{Q}$ and $\mathbb{N}$, respectively.

Let A, $B \subset \mathbb{C}$ be measurable. The *area*, i.e. 2-dimensional Lebesgue measure, of A is denoted by $\operatorname{area}(A)$. The *density* of A in B is defined to be

$$\operatorname{dens}(A, B) = \frac{\operatorname{area}(A \cap B)}{\operatorname{area} B}.$$

2.1 Function Theory

By an entire function f we always mean a holomorphic function in the whole complex plane $\mathbb{C}$. If, in addition, f is not a polynomial, then f is a transcendental entire function. Aside from the function-theoretic aspects of transcendental entire functions, the study of these functions from a dynamical point of view also attracts many interests recently. The interplay between the function properties and dynamical properties of transcendental entire functions can be seen from many works, see, for instance, [Ber93].

From either aspect of entire functions, the set of singularities of the inverse function plays an important role, as in the case of rational or polynomial dynamics. For this reason, we first give the definition of singularities.

A point $a \in \mathbb{C}$ is called a *critical point* of f if $f'(a) = 0$, and then $f(a)$ is called a *critical value* of f. We say that $b \in \widehat{\mathbb{C}}$ is an *asymptotic value* of f, if there exists a curve $\gamma : (0, \infty) \to \mathbb{C}$ with $\gamma(t)$ tending to ∞ as $t \to \infty$ but $f(\gamma(t))$ tends to b as $t \to \infty$. A point $z \in \widehat{\mathbb{C}}$ is called a *singularity* of the inverse of a transcendental entire function f, if it is an asymptotic value or a critical value of f. By Iversen's theorem, ∞ is an asymptotic value for every entire function, see [GO08, Chapter 5] and [Nev70, Chapter XI]. The closure of the set of singularities of f^{-1} in $\mathbb{C}$ of an entire function f is denoted by $\operatorname{Sing}(f^{-1})$. A classification of singularities is given in [BE95]. For our purpose, the following definition is sufficient.

Definition 2.1 (Direct/logarithmic singularity)**.** *For a transcendental entire function f, a singularity $z \in \widehat{\mathbb{C}}$ of f^{-1} is called a direct singularity if there exists a disk $D_\chi(z, r)$ with respect to the spherical metric such that f omits the value z in a component V_r of $f^{-1}(D_\chi(z, r))$ for some $r > 0$. In particular, a direct singularity is called logarithmic if the restriction $f : V_r \to D_\chi(z, r) \setminus \{z\}$ is a universal covering for some $r > 0$. The domain V_r is called a direct tract or logarithmic tract over z, respectively.*

Now we can define entire functions in the Eremenko-Lyubich class, which is the main object we will study.

Definition 2.2 (Eremenko-Lyubich class)**.** *An entire function f is in the Eremenko-Lyubich class $\mathcal{B}$, if the singular set* $\operatorname{Sing}(f^{-1})$ *is bounded. If, in particular,* $\operatorname{Sing}(f^{-1})$ *is finite, then f is in the Speiser class $\mathcal{S}$.*

Remark 2.1.1. Functions in class $\mathcal{B}$ are also said to be of *bounded type*, while functions in class $\mathcal{S}$ are said to be of *finite type*.

Remark 2.1.2. Many familiar entire functions belong to the class $\mathcal{B}$. This includes, for instance, the exponential map $\exp(z)$ which has 0 as the only singular value and the sine function $\sin(\alpha z + \beta)$ for $\alpha \in \mathbb{C} \setminus \{0\}, \beta \in \mathbb{C}$, which have only the two critical values ± 1 and no asymptotic values. In particular, these functions all belong to class $\mathcal{S}$. An entire function, which is in class $\mathcal{B}$ but not in class $\mathcal{S}$, is $\sin(z)/z$.

Later in our construction of entire functions, we need the following class of functions, which is exactly the closure of real polynomials with real zeros. Therefore, we give the following definition.

Definition 2.3 (Laguerre-Pólya class)**.** *An entire function f is in the Laguerre-Pólya class $\mathcal{LP}$, if f is the locally uniform limit of real polynomials with only real zeros.*

See [Obr63, Section II.9] for a discussion of this class. As an example, we mention that the function $\exp(-z^2) \in \mathcal{LP}$, while $\exp(z^2) \notin \mathcal{LP}$. We will need some basic properties of entire functions in class $\mathcal{LP}$.

Proposition 2.1. *Let $f \in \mathcal{LP}$ be an entire function. Then*

(i) f is a real entire function (i.e., $f(\mathbb{R}) \subset \mathbb{R}$), and if $f \not\equiv 0$ then f has only real zeros;

(ii) $f' \in \mathcal{LP}$;

(iii) if f is transcendental then $f^{(k)}$ has only real zeros;

(iv) f can be represented in the following form:

$$f(z) = e^{-az^2+bz+c} \prod_{k=1}^{\infty} \left(1 - \frac{z}{x_k}\right) e^{z/x_k}, \quad \text{for} \;\; a, b, c, x_k \in \mathbb{R}, \;\; a \geq 0.$$

Remark 2.1.3. A conjecture of Wiman states that if f is a real entire function and f and f'' have only real zeros then $f \in \mathcal{LP}$. The conjecture was finally confirmed in 2003 by Bergweiler, Eremenko and Langley [BEL03].

Nevanlinna theory. For an entire function f, the *maximum modulus* is defined as

$$M(r, f) := \max_{|z|=r} |f(z)| \quad \text{for} \quad r \geq 0.$$

Definition 2.4 (Order of growth)**.** *Let f be an entire function. The lower order $\lambda(f)$ and the order $\rho(f)$ of f are defined respectively as*

$$\lambda(f) := \liminf_{r\to\infty} \frac{\log\log M(r, f)}{\log r},$$
$$\rho(f) := \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r}.$$

The above notion of order is not sufficient if we consider meromorphic functions. On the other hand, we have another quantity to characterize the growth scale of a meromorphic function. To this aim, we need Nevanlinna's theory on value distribution of meromorphic functions. A thorough introduction to this theory can be found in [Hay64], [GO08] and [Nev70].

For $x > 0$, we denote

$$\log^+ x := \max\{0, \log x\}.$$

For a function f meromorphic in the complex plane $\mathbb{C}$, the *proximity function* of f is defined as

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left|f(re^{i\theta})\right| d\theta.$$

We use $n(t,f)$ to denote the number of poles of f in $|z| \leq t$, counting multiplicities, and define

$$N(r,f) = \int_0^r \frac{n(t,f) - n(0,f)}{t} dt + n(0,f).$$

Definition 2.5 (Nevanlinna characteristic)**.** *The Nevanlinna characteristic function $T(r,f)$ of a meromorphic function f is defined to be*

$$T(r,f) = m(r,f) + N(r,f).$$

Now we can give a definition of order and lower order of a meromorphic function.

Definition 2.6 (Order of growth)**.** *The lower order $\lambda(f)$ and the order $\rho(f)$ of a meromorphic function f in $\mathbb{C}$ are defined as*

$$\lambda(f) := \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r},$$
$$\rho(f) := \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

The following result is commonly called *the first fundamental theorem of Nevanlinna theory.*

Theorem 2.1. *For $a \in \mathbb{C}$,*

$$T(r,f) = T\left(r, \frac{1}{f-a}\right) + \mathcal{O}(1). \tag{2.1}$$

When f is a transcendental entire function, the theorem below roughly says that instead of $\log T(r,f)$ one can use $M(r,f)$ in the definition of order of growth. Compare Definition 2.4 with Definition 2.6.

Theorem 2.2. *Let f be an entire function, and let $0 < r < R$. Then*

$$T(r,f) \leq \log M(r,f) \leq \frac{R+r}{R-r} T(R,f). \tag{2.2}$$

For a meromorphic function f with a logarithmic tract G over ∞, if we define

$$M_G(r,f) := \max_{z\in G, |z|=r} |f(z)|,$$

and replace $M(r,f)$ with $M_G(r,f)$, the second inequality in (2.2) is still true.

Theorem 2.3. *Let f be a meromorphic function with a logarithmic tract G over ∞ and let $0 < r < R$. Then*

$$\log M_G(r,f) \leq \frac{R+r}{R-r} \left(T(R,f) + \mathcal{O}(1)\right). \tag{2.3}$$

Proof. See [Nev70, Chapter XI, Section 4.3] for a proof. □

Remark 2.1.4. From a dynamical point of view, meromorphic functions in $\mathbb{C}$ with direct or logarithmic singularities over ∞ share many properties with those of entire functions. More generally, one can study dynamics of functions defined only in these tracts. We refer to [BRS08] for detailed discussions.

The following well-known theorem gives us a way to estimate the number of logarithmic tracts in terms of the order of growth, see [GO08, Chapter 5, Theorem 1.4]. In particular, an entire function in class $\mathcal{B}$ of finite order has only finitely many logarithmic tracts over ∞.

Theorem 2.4 (Denjoy-Carleman-Ahlfors Theorem)**.** *If an entire function f has n distinct asymptotic values in $\mathbb{C}$, then*

$$\liminf_{r\to\infty} \frac{T(r,f)}{r^{n/2}} > 0.$$

Remark 2.1.5. Under the conditions of the above theorem, it is clear that the lower order of the function is at least $n/2$.

Remark 2.1.6. It follows from the Denjoy-Carleman-Ahlfors theorem that if $f \in \mathcal{B}$, then $\rho(f) \geq 1/2$ ([BE95], [AB12]).

We recall the following result due to Tsuji [Tsu75], which is connected to the proof of the Denjoy-Carleman-Ahlfors theorem. See, for instance, [GO08, Chapter 5]. To formulate it precisely, we need some notations. For an unbounded domain G with boundary Γ and $r > 0$ such that $\{z : |z| = r\} \cap \Gamma \neq \emptyset$, put

$$\beta(r) = \operatorname{meas}\left\{\theta \in [0, 2\pi] : re^{i\theta} \in G\right\}.$$

If $\{z : |z| = r\} \cap \Gamma = \emptyset$ then we define $\beta(r) = \infty$.

Theorem 2.5 (Tsuji's inequality)**.** *Let G be an unbounded domain and Γ its boundary. Let f be continuous in $G \cup \Gamma$ and holomorphic in G. Suppose that f is bounded on Γ but unbounded in G. Let $0 < \alpha < 1$ and $r_1 > 0$. Then*

$$\log\log M_G(r,f) \geq \pi \int_{r_1}^{\alpha r} \frac{dt}{t\beta(t)} + \mathcal{O}(1).$$

Tsuji's inequality implies, in particular, that one can estimate the size of a tract over infinity for an entire function from the growth of the function. And conversely, one can also estimate the order of growth from the size of the tract.

Canonical product. Our main aim in this thesis is to construct an entire function with certain dynamical properties. In general, there are many methods, including, for instance, Cauchy integrals ([RRRS11, Section 7], [RG14], [Sta91]), approximation ([EL87]) and more recently quasiconformal folding ([Bis15]). However, we shall adopt the method using infinite products, which is classical in the theory of meromorphic functions ([GO08], [Hay64]). One important aspect of this method is that we can assign zeros or poles for our desired functions. For example, we prescribe the zeros along the real axis. In this case, the order can be estimated according to the distribution of zeros. We first give the following definition.

Definition 2.7 (Exponent of convergence)**.** *Let* $\{a_n\}_{n\in\mathbb{N}}$ *be a sequence of non-zero complex numbers with* $\lim_{n\to\infty}|a_n| = \infty$*. The exponent of convergence of the sequence is defined by*

$$\lambda := \inf\left\{\mu > 0 : \sum_{k=0}^{\infty}\frac{1}{|a_k|^{\mu}} < \infty\right\}.$$

The exponent of convergence of a sequence can be used to determine whether a canonical product converges to an entire function. More precisely, recall that for an nonnegative integer p, the *Weierstrass primary factors* are defined as

$$E(z,p) = \begin{cases} 1-z & \text{if } p = 0,\\ (1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right), & \text{if } p \geq 1.\end{cases}$$

Theorem 2.6. *Let* $\{a_n\}_{n\in\mathbb{N}}$ *be a sequence of non-zero complex numbers have a finite exponent of convergence* λ *and let* p *be a nonnegative integer with* $p > \lambda - 1$*. Then the canonical product*

$$\prod_{n=1}^{\infty} E\left(\frac{z}{a_k}, p\right) \tag{2.4}$$

defines an entire function. Here multiple zeros correspond to points which occur in the sequence $\{a_n\}$ *repeatedly.*

The entire function defined in (2.4) is called the *Weierstrass canonical product of genus* p. Now let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence as in Definition 2.7. Denote by $n(r)$ the number of the a_n lying in the disk $\{z : |z| \leq r\}$, which is called the *counting function of the sequence* and set

$$N(r) = \int_0^r n(t)\frac{dt}{t}.$$

Then

$$\sum_{k=0}^{\infty}\frac{1}{|a_k|^{\mu}} \quad \text{and} \quad \int_0^{\infty} n(t)\frac{dt}{t^{\mu+1}}$$

converge simultaneously, see, for instance, [Hay64, Lemma 1.4]. Therefore, the exponent of convergence of $\{a_n\}_{n\in\mathbb{N}}$ can also be defined as

$$\lambda := \inf\left\{\mu > 0 : \int_0^\infty n(t)\frac{dt}{t^{\mu+1}} < \infty\right\}.$$

Moreover, one can prove by the above discussion the following lemma.

Lemma 2.1. *We have*

$$\lambda = \limsup_{r\to\infty} \frac{\log n(r)}{\log r}.$$

Proximate order. The notion of proximate order, defined below, can help us to estimate asymptotic behaviours of a Weierstrass canonical product outside of a sector or a curvilinear sector and will be crucial for us.

Definition 2.8 (Proximate order)**.** *A function $\rho(r)$ defined on $[r_0, \infty)$, where $r_0 > 0$, is called a proximate order if it satisfies the following conditions:*

(1) $\rho(r) \geq 0$;

(2) $\lim_{r\to\infty} \rho(r) = \rho$;

(3) $\rho(r)$ *is continuously differentiable on* $[r_0, \infty)$;

(4) $\lim_{r\to\infty} r\rho'(r)\log r = 0$.

See [GO08, Section 2] for a complete discussion of proximate orders. We only state here a few properties of proximate orders.

Proposition 2.2 (Slowly varying)**.** *If $\rho(r)$ is a proximate order, then $L(r) := r^{\rho(r)-\rho}$ is a slowly varying function. That is,*

$$\lim_{r\to\infty} \frac{L(kr)}{L(r)} = 1$$

uniformly on each interval $0 < a \leq k \leq b < \infty$.

Proposition 2.3. *If $\rho(r)$ is a proximate order and $\rho > 0$, then the function $V(r) := r^{\rho(r)}$ is increasing for sufficiently large r.*

Proposition 2.4. *Let $\rho(r)$ be a proximate order and $\rho(r) \to \rho$ as $r \to \infty$. Then*

$$\int_1^r t^{\rho(t)-q}dt \sim \frac{r^{\rho(r)+1-q}}{\rho+1-q}, \qquad \text{if} \quad q < \rho + 1,$$

and

$$\int_r^\infty t^{\rho(t)-q}dt \sim \frac{r^{\rho(r)+1-q}}{q-\rho-1}, \qquad \text{if} \quad q > \rho + 1.$$

2.2 Quasiconformal Mappings

We recall the definition and some basic results from the theory of quasiconformal mappings. For a good account, we suggest [Ahl06], [LV73] and [FM07] and for applications of quasiconformal mappings we refer to [Hub06]. Note that there are several different definitions of quasiconformal mappings, which are equivalent to each other. Each definition has its own advantages over the others.

Geometric definition. Let Γ be a family of rectifiable curves in a domain Ω. A function $\rho : \Omega \to \mathbb{R}$ is called *admissible* if ρ is measurable, $\rho(z) \geq 0$ and $A(\rho) \neq 0$ or ∞, where

$$A(\rho) = \iint_{\Omega} \rho(z)^2 dxdy.$$

The length of $\gamma \in \Gamma$ with respect to ρ is

$$L_\gamma(\rho) = \int_\gamma \rho(z)|dz|.$$

Then the *extremal length* of Γ is defined as

$$\lambda(\Gamma) = \sup_\rho \frac{(\inf_{\gamma\in\Gamma} L_\gamma(\rho))^2}{A(\rho)},$$

where the supremum ranges over all admissible metrics ρ of Ω. Note that the extremal length is a conformal invariant, which means it is invariant under the action of conformal mappings.

A (topological) *quadrilateral* is a Jordan domain with four marked points on the boundary. As a consequence of the Riemann mapping theorem and the Carathéodory theorem, we can find a map f, which maps a quadrilateral Q conformally onto a rectangle with sides of length a and b. If Γ is the family of curves connecting the b-sides, then the *module* $\mathrm{mod}(Q)$ of Q is

$$\mathrm{mod}(Q) = \frac{1}{\lambda(\Gamma)} = \frac{b}{a}.$$

Definition 2.9 (Geometric definition)**.** *Let $f : \Omega \to f(\Omega)$ be a homeomorphism. We say that f is K-quasiconformal if for every quadrilateral $Q \subset \Omega$,*

$$\frac{\mathrm{mod}(Q)}{K} \leq \mathrm{mod}(f(Q)) \leq K\,\mathrm{mod}(Q).$$

The advantage of the geometric definition is that, one can easily deduce that if f is K-quasiconformal, then so is f^{-1}. Moreover, if f is K_1-quasiconformal and g

is K_2-quasiconformal, then both $f \circ g$ and $g \circ f$ are K_1K_2-quasiconformal (if the domains and ranges of f and g are such that the compositions are defined). Also we have that every 1-quasiconformal mapping is conformal.

Analytic definition. Let $\Omega \subset \mathbb{C}$ be a domain, $f : \Omega \to f(\Omega) \subset \mathbb{C}$ and let a rectangle $R \subset \Omega$ have sides parallel to the x and y axes. Then f is *absolutely continuous on lines* (ACL) on R if f is absolutely continuous on almost every horizontal and vertical line in R. The map f is ACL on Ω if f is ACL on every rectangle $R \subset \Omega$.

Definition 2.10 (Analytic definition)**.** *A homeomorphism $f : \Omega \to f(\Omega)$ is K-quasiconformal if the following holds:*

(a) f is ACL on Ω,

(b) $|f_{\overline{z}}| \leq k|f_z|$ almost everywhere, where $k = \frac{K-1}{K+1}$.

The analytic definition can be used to show that a K-quasiconformal mapping is locally Hölder continuous with exponent $1/K$. This in turn can be used to show that certain families of quasiconformal mappings are compact. For instance, the family of all K-quasiconformal mappings $f : \mathbb{D} \to \mathbb{D}$ which are surjective and satisfy $f(0) = 0$, is compact.

Metric definition. Let $f : \Omega \to \Omega'$ be an orientation-preserving homeomorphism. The *circle dilatation* $H_f(z)$ of f at $z \in \Omega$ is defined as

$$H_f(z) = \limsup_{r \to 0+} \frac{\sup_{|\xi - z| = r} |f(\xi) - f(z)|}{\inf_{|\xi - z| = r} |f(\xi) - f(z)|}$$

Then clearly $1 \leq H_f(z) \leq \infty$ for every $z \in \Omega$.

Definition 2.11 (Metric definition)**.** *Let $f : \Omega \to \Omega'$ be an orientation-preserving homeomorphism. Then f is K-quasiconformal if and only if the circle dilatation $H_f \leq K$ a.e. in Ω.*

It follows from this definition that every bi-Lipschitz map is quasiconformal. More importantly, this definition allows for generalizations to general metric space with approximate geometry [HK98].

Definition 2.12 (Complex dilatation)**.** *For a quasiconformal mapping f, the complex dilatation μ_f of f is defined as*

$$\mu_f = \frac{f_{\overline{z}}}{f_z}.$$

One of the most important results in the theory of quasiconformal mappings is the *measurable Riemann mapping theorem*, which says that for every measurable μ with $||\mu||_\infty \leq k < 1$ there exists a quasiconformal mapping f such that

$$f_{\bar{z}}(z) = \mu(z) f_z(z)$$

holds. Moreover, if μ depends analytically on some parameter t, the solution also depends analytically on t.

Remark 2.2.1. Except for the importance of itself, the theory of quasiconformal mappings plays an extremely important role in complex dynamics. It was introduced into complex dynamics as an essential tool by Sullivan [Sul85], Douady and Hubbard [DH85] in 1980s. Its applications in complex dynamics are everywhere, notably Sullivan's proof of no wandering domain for rational functions [Sul85], investigation of parameter spaces by Douady and Hubbard [DH85], etc.

2.3 Transcendental Dynamics

The iteration of transcendental entire functions was started by Fatou [Fat26], who extended some results from iteration of polynomials and rational maps to this transcendental setting. This extension works well in some cases, while in general the iteration of transcendental entire functions exhibits more complicated dynamical behaviours. We will mainly concentrate on giving a very brief introduction to the subject here. For background on complex dynamics, we refer to [Bea91], [CG93], [Mil06] and [Ste93]. For transcendental dynamics, we refer to survey papers [Ber93] and [Sch09].

2.3.1 Dynamics of entire functions

We will always assume that f is a transcendental entire function unless otherwise stated, and denote by f^n the n-th iterate of f. To give definitions of Fatou and Julia sets, we need the notion of normal family. Let $U \subset \mathbb{C}$ be a domain and $\mathscr{F} := \{f : U \to \widehat{\mathbb{C}}\}$ a family of meromorphic functions in U. Then $\mathscr{F}$ is a *normal family* if every sequence in $\mathscr{F}$ has a locally uniformly convergent subsequence with respect to the spherical metric. One of the most important results in the theory of normal family is the following:

Theorem 2.7 (Montel's theorem)**.** *The family of all meromorphic maps of an arbitrary domain U into the three-punctured sphere $\widehat{\mathbb{C}} \setminus \{a, b, c\}$ is normal.*

Based on the definition of normality, we have the following fundamental concepts in complex dynamics.

Definition 2.13 (Fatou set and Julia set)**.** *The Fatou set $\mathcal{F}(f)$ of f is defined to be the set of points $z \in \mathbb{C}$ which have a neighbourhood such that the family of iterates f^n forms a normal family in the sense of Montel. The Julia set $\mathcal{J}(f)$ is the complement of the Fatou set.*

Remark 2.3.1*.* It follows from the definition that $\mathcal{F}(f)$ is open and $\mathcal{J}(f)$ is closed. Moreover, both sets are completely invariant, that is, $z \in \mathcal{F}(f)$ if and only if $f(z) \in \mathcal{F}(f)$ and $z \in \mathcal{J}(f)$ if and only if $f(z) \in \mathcal{J}(f)$. In addition, $\mathcal{F}(f^p) = \mathcal{F}(f)$ and $\mathcal{J}(f^p) = \mathcal{J}(f)$ for all $p \in \mathbb{N}$.

For an entire function f, a point $z \in \mathbb{C}$ is called a *periodic point* of f, if there exists $n \in \mathbb{N}$ such that $f^n(z) = z$. The smallest such n is called the *period* of z. In case that z is a periodic point of period 1, we say that it is a fixed point of f. By a *periodic cycle* of f we mean a set of points $\{z, f(z), \ldots, f^{n-1}(z)\}$ if z is a periodic point of period n. For such a periodic point, the *multiplier* of z is defined as

$$\lambda := (f^n)'(z).$$

And a periodic point z is said to be *attracting*, *indifferent* or *repelling* if $|\lambda|$ is less than, equal to or greater than 1 respectively. In case $\lambda = 0$ we say z is *superattracting*. If $|\lambda| = 1$, then there exists $\alpha \in [0, 1)$ such that $\lambda = e^{2\pi i\alpha}$. An indifferent periodic point is *rational indifferent* (or *parabolic*) if $\alpha \in \mathbb{Q}$, and *irrational indifferent* if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. A point z is *preperiodic* if there exists $m \in \mathbb{N}$ such that $f^m(z)$ is a periodic point of f.

Remark 2.3.2*.* We need to mention the existence of periodic points. Some transcendental entire functions may not have fixed points in $\mathbb{C}$. For instance, $e^z + z$ has no fixed points in $\mathbb{C}$. On the other hand, Bergweiler showed that, except possibly for period 1, every entire function has repelling periodic points of all periods [Ber91].

Fatou set. A connected component of the Fatou set is called a *Fatou component*. The classification of Fatou components for entire or rational functions is connected to periodic cycles of the function. As above, a Fatou component U of f is *periodic* if there exists $p \in \mathbb{N}$ such that $f^p(U) \subset U$. The smallest p with this property is called the period of U. In case $p = 1$, we say that U is *invariant*. U is a *preperiodic* Fatou component, if there exists $q \geq 0$ such that $f^q(U) \subset V$ and V is a periodic Fatou component. On the other hand, U is called a *wandering domain* if U is not preperiodic.

For an entire function f, let U be an invariant Fatou component of f, then U can be only one of the following types:

(i) *Böttcher domain*, if U contains a superattracting fixed point;

(ii) *Schröder domain*, if U contains an attracting but not superattracting fixed point;

(iii) *Leau domain* (or parabolic basin), if there exists a fixed point $z_0 \in \partial U$ with multiplier 1, such that $f^n|_U \to z_0$ as $n \to \infty$;

(iv) *Baker domain*, if f is transcendental and $f^n|_U \to \infty$ as $n \to \infty$;

(v) *Siegel disk*, if $f|_U : U \to U$ is bijective and there exists a subsequence $(n_k) \in \mathbb{N}$ such that $f^{n_k}|_U \to id_U$.

If U is of type (i) or (ii), we say that U is an *attracting basin*. More generally, if U is a periodic Fatou component of period p, then $f^p(U)$ is contained in a Fatou component, which is invariant under f^p.

Rational functions do not have wandering domains and Baker domains. For transcendental entire functions, all these types can occur except for Herman rings. The nonexistence of Herman rings follows from the maximum principle. The relation between dynamical behaviours inside periodic Fatou components and various functional equations is omitted here. We refer to the above-mentioned literature for a detailed explanation.

Remark 2.3.3. Every multiply connected Fatou component of a transcendental entire function is a wandering domain and bounded [Bak75, Bak84]. While the understanding of dynamical behaviours in any periodic Fatou component is already clear, a detailed study of dynamics inside a multiply connected Fatou component is only recently given by Bergweiler, Rippon and Stallard [BRS13].

Remark 2.3.4. While Sullivan proved that rational functions do not have wandering domains [Sul85], the first example of wandering domains was given even earlier by Baker [Bak63, Bak76]. The connectivity problem of wandering domains was studied by Kisaka and Shishikura [KS08]. In particular, they use quasiconformal surgery to construct entire functions with multiply connected wandering domains of finite connectivity.

Julia set. The Julia set is the set where chaotic behaviours occur and is often a fractal. In the following we collect some properties of Julia sets.

Theorem 2.8. *For an entire function f,*

(i) the Julia set $\mathcal{J}(f)$ is non-empty and perfect (i.e., no isolated points);

(ii) repelling cycles belong to $\mathcal{J}(f)$;

(iii) for any $z \in \mathcal{J}(f)$, the backward orbit is dense in $\mathcal{J}(f)$, that is,

$$\mathcal{J}(f) = \overline{\bigcup_{n \geq 0} f^{-n}(\{z\})};$$

(iv) the repelling cycles are dense in $\mathcal{J}(f)$.

Remark 2.3.5. All the properties (i)-(iv) hold also for rational functions. The proof of (iv) for transcendental entire functions is due to Baker [Bak68], who used Ahlfors's theory of covering surfaces (in particular, the Ahlfors five islands theorem) as an essential ingredient. The relation between dynamics and Ahlfors's theory of covering surfaces is addressed in [Ber00].

Escaping set. Another concept which has attracted a lot of attention in transcendental dynamics is the escaping set. It is also an important set in polynomial dynamics since this set is the attracting basin of ∞. Building on Böttcher coordinates, one can define external rays and study landing properties of them. The local connectivity of Julia sets is closely connected with this landing property. However, in the transcendental setting, the escaping set itself becomes a more interesting object. Many topological, geometric and measure-dimension theoretic properties of escaping sets are studied for some specific and even more general entire functions.

Definition 2.14 (Escaping set). *The escaping set $\mathcal{I}(f)$ is the set of points z with $f^n(z) \to \infty$ as $n \to \infty$.*

The first fundamental result concerning this set is the following:

Theorem 2.9. *The escaping set $\mathcal{I}(f)$ is non-empty.*

Remark 2.3.6. The proof of this theorem is due to Eremenko [Ere89]. The original proof by Eremenko uses the Wiman-Valiron theory from entire function theory, while Domínguez gave a more topological proof [Dom98]. As a matter of fact, they proved the theorem for the *fast escaping set* in the sense of [BH99].

Theorem 2.10 (Properties of escaping sets). *For every transcendental entire function f, we have*

(i) $\mathcal{I}(f) \cap \mathcal{J}(f) \neq \emptyset$;

(ii) $\mathcal{J}(f) = \partial \mathcal{I}(f)$;

(iii) every component of $\overline{\mathcal{I}(f)}$ *is unbounded.*

Remark 2.3.7. The theorem is also due to Eremenko [Ere89]. The property (ii) is also valid for polynomials. Under this circumstance, the complement of the escaping set in $\widehat{\mathbb{C}}$ is called the *filled Julia set*, the boundary of which is the Julia set.

Remark 2.3.8. Many sets concerning the escaping speed are defined, most notably the aforementioned fast escaping set, which is defined by Bergweiler and Hinkkanen in [BH99] and studied by Rippon, Stalllard and others in details, see [RS12], for instance.

Remark 2.3.9. The topology of escaping sets is often difficult to understand. It is conjectured by Eremenko that each component of the escaping set is unbounded [Ere89]. This is now known as *Eremenko's conjecture* and is still open. A strong form of this conjecture asks if every point in $\mathcal{I}(f)$ can be connected to infinity by a curve in $\mathcal{I}(f)$. Even though the strong form of Eremenko's conjecture is true for the exponential family [SZ03], the sine family [Sch07] and more generally entire functions in class $\mathcal{B}$ with finite order [RRRS11], it is shown in [RRRS11] that there exist entire functions in class $\mathcal{B}$ such that every path connected component of $\mathcal{J}(f)$ is bounded.

2.3.2 Eremenko-Lyubich class

The investigation of transcendental entire functions from a dynamical point of view is more complicated than that of polynomials, mainly due to the wide function-theoretic variety of transcendental entire functions. The theory of transcendental dynamics is, therefore, focused on either specific functions, such as exponential maps and sine maps, or studying general properties which hold for all transcendental entire functions. Recently, many works are concentrated on a subfamily of entire functions, which is the Eremenko-Lyubich class $\mathcal{B}$. On one hand, this class contains many interesting and elementary functions, such as the above mentioned exponential family and sine family. On the other hand, some concepts from dynamical systems make sense only in this class, for instance, the notion of hyperbolicity [RGS].

Recall that from Definition 2.2, class $\mathcal{S}$ is also contained in this class $\mathcal{B}$. The dynamical behaviours of entire functions in class $\mathcal{S}$ are similar to those of polynomials. For instance, they do not have wandering domains [EL92, GK86], as rational functions [Sul85]. However, there are entire functions in class $\mathcal{B}$ with wandering domains [Bis15]. Note that there are functions in class $\mathcal{B}$ but not in class $\mathcal{S}$.

Logarithmic change of variables. To study dynamics of functions in class $\mathcal{B}$, Eremenko and Lyubich introduced *a logarithmic change of variables* as a main tool [EL92, Section 2]. To describe this technique, first recall the definition of entire functions in class $\mathcal{B}$.

Let $f \in \mathcal{B}$. Then by definition we can find a constant, say $r_0 > 0$, such that all the singularities of f lie in $\{z : |z| \leq r_0\}$. This also implies that all components of $f^{-1}(\{z : |z| \leq r_0\})$ are logarithmic tracts over ∞. Without loss of generality, by choosing a suitable large constant r_0, we may assume that $|f(0)| \leq r_0$. Now we put

$$\begin{aligned} A &= \{z \in \mathbb{C} : \ |z| > r_0\}, \\ U &= f^{-1}(A), \\ H &= \{z \in \mathbb{C} : \operatorname{Re} z > \log r_0\}. \end{aligned} \tag{2.5}$$

Each component U_j of U is a simply connected domain whose boundary is an analytic curve tending ∞ on both sides. Then $f : U_j \to A$ and $\exp : H \to A$ are universal coverings. Thus from each component U_j of U there exists a biholomorphic map $G_j : U_j \to H$ such that

$$f|_{U_j} = \exp \circ G_j.$$

Therefore, in this way we obtain a map $G : U \to H$ with $G|_{U_j} = G_j$. Put

$$W = \exp^{-1}(U). \tag{2.6}$$

We can define

$$F : W \to H, \quad F(z) = G(\exp(z)). \tag{2.7}$$

So we have the following commutative diagram:

$$\begin{array}{ccc} W & \xrightarrow{F} & H \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ U & \xrightarrow[f]{} & A \end{array}$$

Moreover, by construction F maps every component of W biholomorphically onto H. In this way, we say that F is obtained from f by a logarithmic change of variable.

Expanding property. By using a logarithmic change of variable to $f \in \mathcal{B}$, Eremenko and Lyubich obtain the following estimate, see [EL92, Lemma 1].

Lemma 2.2 (Expanding property)**.** *Let $f \in \mathcal{B}$ and suppose F is obtained through the logarithmic change of variable from f, then*

$$|F'(z)| \geq \frac{\operatorname{Re} F(z) - \log r_0}{4\pi} \tag{2.8}$$

for $z \in W$.

Eremenko and Lyubich used the expanding property to prove the following result [EL92, Theorem 1].

Theorem 2.11. *Let $f \in \mathcal{B}$. Then $\mathcal{I}(f) \subset \mathcal{J}(f)$.*

From this we can see the striking difference between transcendental entire functions and polynomials. As mentioned, for the latter the escaping set is an attracting basin of the Fatou set, while here it is contained in the Julia set. Combined with Theorem 2.10 (ii), we have

Corollary 2.1. *Let $f \in \mathcal{B}$. Then $\overline{\mathcal{I}(f)} = \mathcal{J}(f)$.*

The result is useful in the sense that we can estimate the size of Julia set from below by considering the escaping set.

Hyperbolicity. The notion of hyperbolicity is one of the most important concepts in dynamical systems. In rational dynamics, a map is hyperbolic if it is expanding with respect to a smooth Riemannian metric in a neighborhood of the Julia set. The dynamical behaviours of hyperbolic rational maps are understood quite clearly. For transcendental entire functions, the following definition is often adopted, see [BFRG15, Definition 1.1] and [RG16, Section 2].

Definition 2.15 (Hyperbolicity and disjoint type)**.** *An entire function f is said to be hyperbolic if the singular set* $\operatorname{Sing}(f^{-1})$ *is bounded and every point in* $\operatorname{Sing}(f^{-1})$ *tends to an attracting periodic cycle of f under iteration. If f is hyperbolic and furthermore the Fatou set $\mathcal{F}(f)$ is connected, then we say that f is of disjoint type.*

Remark 2.3.10. According to the definition, all hyperbolic entire functions are in class $\mathcal{B}$. For a discussion of the notion of hyperbolicity in transcendental dynamics, see [RGS]. The escaping set of a hyperbolic entire function is always disconnected (this fact is established for a larger family of entire functions in [MB12, Corollary 1.4], which includes, in particular, hyperbolic entire functions).

Remark 2.3.11. All Fatou components of hyperbolic entire functions are simply connected, as is shown [EL92, Proposition 3]. Hence the Fatou set of a disjoint type entire function is simply connected.

For disjoint type entire functions, the following result can be viewed as an alternative definition; see [BJR12, Lemma 3.1], [MB12, Proposition 2.8] and also [RG16, Proposition 2.1].

Proposition 2.5. *A transcendental entire function f is of disjoint type if and only if there exists a bounded Jordan domain D with* $\operatorname{Sing}(f^{-1}) \subset D$ *and* $f(\overline{D}) \subset D$.

For any given entire function $f \in \mathcal{B}$, the function

$$f_\lambda : \mathbb{C} \to \mathbb{C}, \qquad z \mapsto \lambda f(z) \tag{2.9}$$

is of disjoint type for sufficiently small λ, see [Rem09]. For disjoint type entire functions, we have the following useful characterization of their Julia sets [RG16, Proposition 2.2].

Proposition 2.6. *If f is of disjoint type and D is as in Proposition 2.5, then*

$$\mathcal{J}(f) = \left\{ z \in \mathbb{C} : \; f^n(z) \notin \overline{D} \;\; \textit{for all} \;\; n \geq 0 \right\}.$$

Remark 2.3.12. For a disjoint type function f, take $A := \mathbb{C} \setminus \overline{D}$, where D is as in Proposition 2.5. Then for $U := f^{-1}(A)$, $f : U \to A$ is a covering map since

$\operatorname{Sing}(f^{-1}) \subset D$. Every component of U is called a *tract* of f. Then Proposition 2.6 means that the Julia set of f is exactly the set of points which stays in the tracts of f under any iterates. More precisely,

$$\mathcal{J}(f) = \bigcap_{n \geq 0} f^{-n}(U).$$

Escaping rigidity. A remarkable result of Rempe [Rem09, Theorem 1.1] says that for $f \in \mathcal{B}$ and f_λ defined in (2.9), f and f_λ have the same dynamics near infinity. To state his result more precisely, we need the following notions.

Definition 2.16 (Equivalence)**.** *Let f, g be entire functions. We say f and g are topologically equivalent if there are planar homeomorphisms φ and ψ such that*

$$\psi \circ f = g \circ \varphi. \tag{2.10}$$

In particular, if we can choose φ and ψ to be quasiconformal/conformal homeomorphisms of the plane such that (2.10) *holds, then we say that they are quasiconformally/conformally equivalent.*

Remark 2.3.13*.* The above definition can be found in [EL92, Section 3].

Remark 2.3.14*.* Following Epstein and Rempe-Gillen [ERG15], we say that φ and ψ in the above definition are *witnessing homeomorphisms.* In particular, if f and g are topologically equivalent entire functions in class $\mathcal{S}$, then they are quasiconformally equivalent, see [ERG15, Proposition 2.3].

The following notion was used by Rempe to formulate his rigidity result.

Definition 2.17 (Quasiconformal equivalence near infinity)**.** *Let f and g be entire functions in class $\mathcal{B}$. We say that f and g are quasiconformally equivalent near ∞ if there exist quasiconformal homeomorphisms $\varphi, \psi : \mathbb{C} \to \mathbb{C}$ such that*

$$\psi \circ f = g \circ \varphi$$

whenever $|f(z)|$ or $|g(\varphi(z))|$ is large enough.

Remark 2.3.15*.* Let $f \in \mathcal{B}$ and $g := \lambda f$ for $\lambda \neq 0$. Then f and g are conformally equivalent. In particular, if we choose λ to be sufficiently small, this would imply the conformal equivalence between any entire function in class $\mathcal{B}$ and a particularly simple (i.e., disjoint type) function.

Now we can state the above-mentioned result by Rempe [Rem09, Theorem 1.1].

Theorem 2.12 (Rigidity near infinity)**.** *Let $f, g \in \mathcal{B}$ be quasiconformally equivalent near infinity. Then there exist $R > 0$ and a quasiconformal map $\theta : \mathbb{C} \to \mathbb{C}$ such that*

$$\theta \circ f = g \circ \theta \quad on \quad \mathcal{J}_R(f) := \{z \in \mathbb{C} : |f^n(z)| \geq R \ \ for\ all \ \ n \geq 1\}.$$

2.4 Miscellaneous

The following Vitali type lemma is standard and can be found in [Fal03, Lemma 4.8]. It holds for any bounded set in $\mathbb{R}^n$, but we only use it for sets in the complex plane $\mathbb{C}$.

Lemma 2.3. *Let $Q \subset \mathbb{C}$ be a bounded set, $R > 0$ and $r : Q \to (0, R]$ be a real positive function. Then there exists an at most countable subset L of Q such that*

$$D\left(x, r(x)\right) \cap D\left(y, r(y)\right) = \emptyset \quad \textit{for} \quad x, y \in L,\ x \neq y,$$

and

$$\bigcup_{x \in Q} D\left(x, r(x)\right) \subset \bigcup_{x \in L} D\left(x, 4r(x)\right).$$

We shall also use the following theorem, which can be found in [Pom92, Section 1.3].

Theorem 2.13 (Koebe's Distortion Theorem)**.** *Let f be a univalent function in $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$. Then*

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}$$

and

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}.$$

Moreover,

$$f(\mathbb{D}) \supset D\left(0, 1/4\right).$$

An immediate consequence of the above theorem is the following corollary.

Corollary 2.2. *Let $z_0 \in \mathbb{C}$, $r > 0$ and let f be a univalent function in $D(z_0, r)$ and let $0 < \lambda < 1$. Then*

$$\frac{\lambda}{(1+\lambda)^2} |f'(z_0)| \leq \left| \frac{f(z) - f(z_0)}{z - z_0} \right| \leq \frac{\lambda}{(1-\lambda)^2} |f'(z_0)|$$

and

$$\frac{1-\lambda}{(1+\lambda)^3} |f'(z_0)| \leq |f'(z)| \leq \frac{1+\lambda}{(1-\lambda)^3} |f'(z_0)|$$

for $|z - z_0| \leq \lambda r$. Moreover,

$$f\left(D(z_0, r)\right) \supset D\left(f(z_0), \frac{1}{4} |f'(z_0)|\, r\right).$$

Chapter 3

Aspenberg-Bergweiler Condition and Sharpness

From now on we are mainly concerned with the area of escaping sets of entire functions in the Eremenko-Lyubich class $\mathcal{B}$. In this chapter, we present a condition given by Aspenberg and Bergweiler in [AB12], which ensures that functions in this class satisfying the condition always have positive area escaping sets and hence Julia sets. Then we show that their condition is essentially sharp by constructing an entire function.

To formulate the condition given by Aspenberg and Bergweiler, we shall need a function, which roughly means that it is increasing but slower than any iterates of the logarithmic function. We call this function a *reference function.* This function will also appear in our construction.

Now let Φ be the reference function, whose definition will be given later. Then our main result is as follows.

Theorem 3.1. *There exists an entire function f in class $\mathcal{B}$ with*

$$\log\log M(r,f) \leq \left(\frac{1}{2} + \frac{1}{\log \Phi(r)}\right) \log r + \mathcal{O}(1), \tag{3.1}$$

for which the escaping set and the Julia set have Lebesgue measure zero.

To construct such an entire function, we consider a Weierstrass canonical product with zeros distributed along the positive real axis and then control the asymptotic behaviour of f outside of a small curvilinear sector containing the positive real axis. By applying the Denjoy-Carleman-Ahlfors theorem, we show that the function is in fact bounded in this sector. It is clear from our construction that the function f belongs to *Laguerre-Pólya class* $\mathcal{LP}$. Using basic properties of functions in this class we see that all the critical points of f are contained in the above sector and hence the set of all critical values is bounded. For this function the only possible asymptotic value is 0, so it is in the Eremenko-Lyubich class $\mathcal{B}$. The fact that the area of the escaping set of this function is zero will follow from a result given after the construction.

The chapter is thus divided into three sections. In the first section, we introduce the reference function and give some of its properties. In the second section, we present the Aspenberg-Bergweiler condition. Then we construct an entire function satisfying conditions given in Theorem 3.1. Finally we show that the escaping set of the constructed function has zero area.

3.1 A Reference Function

To begin with, we consider the function

$$E_\beta(z) = e^{\beta z} \quad \text{for} \quad \beta \in (0, 1/e).$$

The function E_β is clearly a real entire function, i.e., it maps the real line into itself. It has a repelling fixed point ξ on $\mathbb{R}$ with multiplier $\lambda = \beta\xi > 1$. Around this repelling fixed point, Schröder's functional equation

$$\Phi\left(E_\beta(z)\right) = \lambda \Phi(z) \tag{3.2}$$

has a unique local holomorphic solution Φ normalized by $\Phi(\xi) = 0$ and $\Phi'(\xi) = 1$, see, for instance, [Mil06, Section 8]. We call this solution Φ the *reference function*[1].

The reference function Φ has a continuation to $[\alpha, \infty)$, where α is the attracting fixed point of E_β. This is because Φ is defined in some interval $[x_1, x_2]$ around ξ, then (3.2) gives a continuation to the larger interval $[E_\beta(x_1), E_\beta(x_2)]$. Repeating this process one gets a continuation to $[E_\beta^n(x_1), E_\beta^n(x_2)]$ for every $n \in \mathbb{C}$ and thus to (α, ∞). The diagram (3.2) still holds for $z \in [\xi, \infty)$. Note that Φ is increasing on the real axis and $\lim_{x\to\infty} \Phi(x) = \infty$. Moreover, the function Φ tends to ∞ but slower than any iterate of the logarithm, in other words, for all $m \in \mathbb{N}$ we have

$$\lim_{x\to\infty} \frac{\Phi(x)}{\log^m x} = 0. \tag{3.3}$$

Here $\log^m$ denotes the m-th iterate of the logarithm. For a proof of (3.3), we refer to [Pet08, Section 5.2]. Therefore, if we consider the function $\varepsilon_0 : (\xi, \infty) \to (0, \infty)$ defined by

$$\varepsilon_0(x) = \frac{1}{\Phi(x)}, \tag{3.4}$$

then this function tends to zero but slower than $1/\log^m$ for any $m \in \mathbb{N}$. For this function, the following result holds.

[1] In some literature, the function Φ is called Kœnigs' function. And the inverse of Φ is called a linearizing map which can be continued analytically to an entire function.

Lemma 3.1. *For $x \geq \xi$,*

$$\sum_{k=1}^{\infty} \varepsilon_0 \left(E_\beta^k(x)\right) < \infty. \tag{3.5}$$

Proof. It follows from (3.2) and $\lambda > 1$ that

$$\sum_{k=1}^{\infty} \varepsilon_0 \left(E_\beta^k(x)\right) = \sum_{k=1}^{\infty} \frac{1}{\Phi\left(E_\beta^k(x)\right)} = \sum_{k=1}^{\infty} \frac{1}{\lambda^k \Phi(x)} < \infty.$$

□

For later purpose, we also define $\varepsilon : (\xi, \infty) \to (0, \infty)$ by

$$\varepsilon(x) = \frac{1}{\log \Phi(x)}. \tag{3.6}$$

Then this function also tends to zero slower than any of the functions $1/\log^m$ where $m \in \mathbb{N}$. However, similar result as in Lemma 3.1 does not hold for $\varepsilon(x)$. Instead, we have the following.

Lemma 3.2. *For $\varepsilon(x)$ defined in (3.6) and $x > 0$, we have*

$$\sum_{k=1}^{\infty} \varepsilon\left(E^k(x)\right) = \infty.$$

Proof. Recall that $E_\beta(x) = e^{\beta x}$, $E(x) = e^x$. We shall show first that for any given x_1 there exists $x_2 > x_1$ such that for any $k \in \mathbb{N}$,

$$E_\beta^k(x_2) \geq E^k(x_1).$$

To see this we first consider the following real function defined as

$$F_\beta(x) = \beta e^x.$$

In particular, $F_1(x) = E(x)$. An easy computation shows that $F_\beta(\beta x) = \beta E_\beta(x)$. Let $c > \log(2/\beta)$ and $x > \log c$. We have

$$F_\beta(x + c) = \beta e^c e^x > 2e^x > e^x + c = F_1(x) + c > F_1(x).$$

Since for any $k \in \mathbb{N}$, $E^k(x) > c$ since $x > \log c$, by induction we have

$$F_\beta^k(x + c) \geq F_1^k(x) + c > F_1^k(x). \tag{3.7}$$

If we take $x = x_1$ and $x_2 = (x + c)/\beta$, then

$$F_\beta^k(x + c) = F_\beta^k(\beta x_2) = \beta E_\beta^k(x_2).$$

Together with (3.7) we see that

$$E_\beta^k(x_2) > \beta E_\beta^k(x_2) = F_\beta^k(x+c) > F_1^k(x_1) = E^k(x_1).$$

Our assertion now follows. So for $x > 0$ there exists $x' > x$ with $E_\beta^k(x') > E^k(x)$. Thus

$$\begin{aligned}\sum_{k=1}^{\infty} \varepsilon(E^k(x)) &= \sum_{k=1}^{\infty} \frac{1}{\log \Phi\left(E^k(x)\right)} \\ &\geq \sum_{k=1}^{\infty} \frac{1}{\log \Phi\left(E_\beta^k(x')\right)} \\ &= \sum_{k=1}^{\infty} \frac{1}{\log\left(\lambda^k \Phi(x')\right)} \\ &= \sum_{k=1}^{\infty} \frac{1}{k \log \lambda + \log \Phi(x')} \\ &= \infty.\end{aligned}$$

□

Building on same ideas as in the proof of Lemma 3.2, we can rewrite our Lemma 3.1 as follows:

Lemma 3.3.

$$\sum_{k=1}^{\infty} \varepsilon_0\left(E^k(0)\right) < \infty. \tag{3.8}$$

The following estimate for $\varepsilon(x)$ would be useful in the construction of our function.

Lemma 3.4. *For $\varepsilon(x)$ defined in (3.6) and for $N \in \mathbb{N}_0$, we have the following estimate:*

$$\varepsilon'(x) \prod_{j=0}^{N} \log^j x \leq \varepsilon(x)^3 \tag{3.9}$$

for large x.

Proof. We recall how the function Φ in (3.2) is constructed; see, for instance, [Mil06, Section 8]. First let $T_\xi(x) = x + \xi$ and $L_\beta(x) = E_\beta^{-1}(x)$. Define

$$L = T_\xi^{-1} \circ L_\beta \circ T_\xi. \tag{3.10}$$

Thus,

$$L(x) = \frac{\log(x+\xi)}{\beta} - \xi.$$

The function L satisfies that $L(0) = 0$, $L'(0) = 1/\lambda$ (recall that $\lambda = \beta\xi$). Since λ is greater than one, 0 is an attracting fixed point of L. Schröder's functional equation has a unique local holomorphic solution $\Psi(z)$ normalised by $\Psi(0) = 0$ and $\Psi'(0) = 1$, such that

$$\Psi(L(x)) = \frac{1}{\lambda}\Psi(x). \tag{3.11}$$

Then $\Psi(x) = \Phi(x + \xi)$. Define

$$\Psi_n(x) = \lambda^n L^n(x),$$

then $\Psi(x) = \lim_{n\to\infty} \Psi_n(x)$. We can now compute explicitly the first derivative of Ψ_n:

$$\begin{aligned}\Psi_n'(x) &= \lambda^n \left(L^n\right)'(x) = \lambda^n L'\left(L^{n-1}\right) \cdot \left(L^{n-1}\right)'(x)\\ &= \frac{\lambda^n}{\beta\left(L^{n-1}(x) + \xi\right)} \cdot \left(L^{n-1}\right)'(x) = \lambda^n \prod_{j=0}^{n-1} \frac{1}{\beta\left(L^j(x) + \xi\right)}\\ &= \left(\frac{\lambda}{\beta\xi}\right)^n \cdot \prod_{j=0}^{n-1} \frac{1}{1 + L^j(x)/\xi} = \prod_{j=0}^{n-1} \frac{1}{1 + L^j(x)/\xi}.\end{aligned}$$

Now we see that, for any $N \in \mathbb{N}_0$,

$$\begin{aligned}\Psi_n'(x) \prod_{j=0}^{N} \log^j x &= \prod_{j=0}^{n-1} \frac{1}{1 + L^j(x)/\xi} \prod_{j=0}^{N} \log^j x\\ &= \prod_{j=0}^{N} \frac{\log^j x}{1 + L^j(x)/\xi} \prod_{j=N+1}^{n-1} \frac{1}{1 + L^j(x)/\xi}.\end{aligned}$$

For any fixed finite integer N (for our following applications, $N \leq 5$ is enough), the definition of $L(x)$ in (3.10) can be put into the above equality. An easy computation shows that

$$\Psi_n'(x) \prod_{j=0}^{N} \log^j x \leq \frac{1}{2}, \tag{3.12}$$

for sufficiently large x. If we define

$$\begin{aligned}\eta(x) &= \frac{1}{\log \Psi(x)},\\ \eta_n(x) &= \frac{1}{\log \Psi_n(x)},\end{aligned} \tag{3.13}$$

then $\eta(x) = \lim_{n\to\infty} \eta_n(x)$ and also $\eta'(x) = \lim_{n\to\infty} \eta_n'(x)$. Then, by applying (3.12) and the definition of $\eta_n(x)$ in (3.13) we have

$$\eta_n'(x)\prod_{j=0}^{N}\log^j x = \frac{1}{[\log\Psi_n(x)]^2}\frac{1}{\Psi_n(x)}\Psi_n'(x)\prod_{j=0}^{N}\log^j x \le \frac{1}{2\Psi_n(x)\,[\log\Psi_n(x)]^2}$$
$$\le \frac{1}{2\,[\log\Psi_n(x)]^3} = \frac{1}{2}\eta_n(x)^3.$$

Now we see that

$$\eta'(x)\prod_{j=0}^{N}\log^j x \le \frac{3}{4}\eta(x)^3$$

for large x. Recall the definition of $\varepsilon(x)$ and $\Phi(x)$. We thus have $\eta(x) = \varepsilon(x+\xi)$. For large $x \in (\xi, \infty)$, we see that

$$\begin{aligned}\varepsilon'(x+\xi)\prod_{j=0}^{N}\log^j(x+\xi) &= \left(\eta'(x)\prod_{j=0}^{N}\log^j x\right)\frac{\prod_{j=0}^{N}\log^j(x+\xi)}{\prod_{j=0}^{N}\log^j x}\\ &\le \frac{3}{4}\eta(x)^3\frac{\prod_{j=0}^{N}\log^j(x+\xi)}{\prod_{j=0}^{N}\log^j x}\\ &= \frac{3}{4}\varepsilon(x+\xi)^3\frac{\prod_{j=0}^{N}\log^j(x+\xi)}{\prod_{j=0}^{N}\log^j x}\\ &\le \varepsilon(x+\xi)^3.\end{aligned}$$

This finishes our proof. □

Now we set

$$\rho(r) = \frac{1}{2} + \varepsilon(r). \tag{3.14}$$

An immediate consequence of the above result is the following

Lemma 3.5. *$\rho(r)$ is a proximate order.*

Proof. First recall the definition of proximate order given in Definition 2.8. To show that the statement is true we only need to check whether $r\rho'(r)\log r \to 0$ as $r \to \infty$. This follows easily from the above lemma. Taking $N = 1$ in (3.9), we see that

$$|r\rho'(r)\log r| = |\varepsilon'(r)r\log r| \le \varepsilon(r)^3 \to 0 \quad \text{as} \quad r \to \infty.$$

The other conditions are easy to verify. We omit it here. □

3.2 The Aspenberg-Bergweiler Condition

The first result concerning the Lebesgue measure of Julia sets of transcendental entire functions is due to McMullen [McM87].

Theorem 3.2. *For $\alpha, \beta \in \mathbb{C}$ and $\alpha \neq 0$, we have*

$$\operatorname{area} \mathcal{I}(\sin(\alpha z + \beta)) > 0.$$

A substantial generalization of the above McMullen's result is due to Aspenberg and Bergweiler [AB12].

Theorem 3.3. *Let f be an entire function in class $\mathcal{B}$ which has N logarithmic tracts over ∞, and suppose that*

$$\log\log M(r, f) \leq \left(\frac{N}{2} + \frac{1}{\Phi(r)}\right) \log r \tag{3.15}$$

for large r. Then $\operatorname{area} \mathcal{I}(f) > 0$.

We call condition given in (3.15) the *Aspenberg-Bergweiler condition.*

Remark 3.2.1. The proof shows that (3.15) may be relaxed to

$$\log\log M(r, f) \leq \left(\frac{N}{2} + \frac{1}{(\log \Phi(r))^{1+\delta}}\right) \log r \tag{3.16}$$

for large r and some $\delta > 0$; see Remark 3.3.3 at the end of this chapter. Theorem 3.1 shows that this does not hold for $\delta = 0$, so this condition is essentially sharp.

Remark 3.2.2. Note that, for an entire function in class $\mathcal{B}$ with N logarithmic tracts over ∞, the Denjoy-Carleman-Ahlfors theorem says that the order of such a function is at least $N/2$. Therefore, functions satisfying conditions in the above theorem only grow slightly faster than guaranteed by the Denjoy-Carleman-Ahlfors theorem.

As we mentioned in the introduction, the function $\varepsilon_0(r)$ in the Aspenberg-Bergweiler condition (3.15) cannot be replaced by a constant, as is shown by the following example, which is given in [AB12].

Example 3.1. *Consider the Mittag-Leffler function*

$$F_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}$$

for $\alpha \in (0, 2)$.

This function satisfies the following properties:

(i) The order $\rho(F_\alpha) = \frac{1}{\alpha}$;

(ii) F_α is bounded in the sector $\{re^{it} : r > 0, |t - \pi| \leq (1 - \frac{\alpha}{2})\pi\}$;

(iii) F_α belongs to the class $\mathcal{B}$.

For a proof of (i) and (ii), see [GO08, pp. 83-86] and for the property (iii) see [AB12, Section 4]. Now it follows from a result of Eremenko and Lyubich [EL92, Theorem 7] that the area of escaping set of the Mittag-Leffler function is zero.

3.3 Sharpness

In this section we shall construct an entire function in class $\mathcal{B}$, with which we can show the essential sharpness of the Aspenberg-Bergweiler condition (3.15) as claimed.

First recall that $\varepsilon(r)$ is defined in (3.6) and $\rho(r)$ in (3.14). To construct an entire function as required, let $\{a_n\}_{n\in\mathbb{N}}$ be a positive sequence tending to infinity such that

$$1 \leq a_0 \leq a_1 \leq \ldots,$$

and

$$n(r) = r^{\rho(r)} + \mathcal{O}\left(1\right). \tag{3.17}$$

Here $n(t)$ is the number of elements of the sequence $\{a_k\}$ which are contained in $\{z : |z| \leq r\}$. Then, by the Lemma 2.1 we see that the exponent of convergence of the sequence $\{a_n\}$ is $1/2$. Then Theorem 2.6 implies that the infinite product

$$f(z) = \prod_{n=0}^{\infty} \left(1 - \frac{z}{a_n}\right) \tag{3.18}$$

converges locally uniformly and hence it is an entire function. Note that $\varepsilon(r)^3 r^{\rho(r)} \to \infty$ as $r \to \infty$. Instead of (3.17) it suffices to assume that

$$n(r) = r^{\rho(r)} + \mathcal{O}\left(\varepsilon(r)^3 r^{\rho(r)}\right), \tag{3.19}$$

and the infinite product defined in above way is still an entire function.

For a Weistrass canonical product g, one can find an asymptotic formula for $\log g(z)$ in the angle $\delta < \arg z < 2\pi - \delta$ for the branch of $\log g(z)$ for which $\log g(0) = 0$. For a detailed explanation of this estimate, see [GO08, Chapter 2, Section 5]. Using the notion of proximate orders, we can modify this to control the asymptotic behaviour for $\log f(z)$ outside of a curvilinear sector, where f is defined in (3.18). For a similar result, we refer to [BC16, Theorem 1.5].

Lemma 3.6 (Asymptotic representation)**.** *For f, ε and ρ defined above, we have*

$$\log|f(re^{i\theta})| = \frac{\pi\cos((\theta-\pi)\rho(r))}{\sin(\pi\rho(r))} r^{\rho(r)} + \mathcal{O}\left(\varepsilon(r)^2 r^{\rho(r)}\right), \tag{3.20}$$

for

$$\varepsilon(r) \le \theta \le 2\pi - \varepsilon(r) \tag{3.21}$$

as $r \to \infty$.

Proof. Following the standard argument using Riemann-Stieltjes integral (see [GO08, Chapter 2, Section 5]), we have

$$\log f(z) = -z\int_0^\infty \frac{n(t)}{t(t-z)}dt$$

and

$$I(z) := \int_0^\infty \frac{t^{\rho(r)}}{t(t-z)}dt = -\frac{\pi e^{-i\pi\rho(r)}}{\sin(\pi\rho(r))} z^{\rho(r)-1}, \tag{3.22}$$

defined for $0 < \arg z < 2\pi$ and $z = re^{i\theta}$. Here $n(t)$ denotes the number of zeros of f in the disk $\{|z| < t\}$ which satisfies (3.19). For $z = re^{i\theta}$,

$$\operatorname{Re}(zI(z)) = -\frac{\pi\cos(\rho(r)(\pi-\theta))}{\sin(\pi(\rho(r))} r^{\rho(r)}. \tag{3.23}$$

By (3.19), (3.22) and (3.23) we have

$$\begin{aligned} \left|\log|f(z)| + \operatorname{Re}(zI(z))\right| &= \left|\operatorname{Re}\left(z\int_0^\infty \frac{t^{\rho(r)} - n(t)}{t(t-z)}dt\right)\right| \\ &\le r\int_0^\infty \frac{\left|n(t) - t^{\rho(r)}\right|}{t\,|t-z|}dt. \end{aligned} \tag{3.24}$$

To estimate the integral on the right-hand side of (3.24), we consider

$$b(r) = \frac{1}{a(r)} = \exp\left\{\varepsilon(r)^3 \log\log r\right\}. \tag{3.25}$$

We claim two properties of $a(r)$ and $b(r)$. First, $b(r) \to \infty$ as $r \to \infty$. This follows easily from (3.6) and (3.3), from which we see that $\Phi(r) \le \log^k(r)$ for large r and for any $k \in \mathbb{N}$. Hence, $\varepsilon(r) \ge 1/\log^k(r)$ for any $k \ge 1$. Now the first claim follows. The second one is that, for any positive constant $\delta < 1$,

$$a(r)^\delta = \frac{1}{b(r)^\delta} = o\left(\varepsilon(r)^2\right). \tag{3.26}$$

In fact it can easily be seen that, this is true if the following holds:

$$\frac{\varepsilon(r)^3 \log\log r}{\log \varepsilon(r)} \to -\infty.$$

Again, the same argument as above implies this. We omit details here.

Now following the argument in the proof of Theorem 2.2 in [GO08, Chapter 2, Section 2], for every $k \in [a(r), b(r)]$ there exists $k^* \in [a(r), b(r)]$ such that

$$\begin{aligned} |\rho(kr) - \rho(r)| &= k^* r\, |\rho'(k^* r)|\, |\log k| \\ &= k^* r\, |\rho'(k^* r)|\, |\log(k^* r)| \frac{|\log k|}{|\log(k^* r)|}. \end{aligned} \tag{3.27}$$

By Definition 2.8, $\rho(r)$ is a proximate order. The definition of proximate orders and Lemma 3.4 then imply that

$$k^* r\, |\rho'(k^* r)|\, |\log(k^* r)| = k^* r\, |\varepsilon'(k^* r)|\, |\log(k^* r)| \le \frac{1}{\log\log(k^* r)} \tag{3.28}$$

for large r. Recall that $a(r)$ and $b(r)$ are defined in (3.25). Since $k \le b(r)$ and $k^* \ge a(r)$, we see that

$$\frac{|\log k|}{|\log(k^* r)|} \le \frac{\log b(r)}{\log(ra(r))} = \frac{\log b(r)}{\log r - \log b(r)} = \frac{\varepsilon(r)^3 \log\log r}{\log r - \varepsilon(r)^3 \log\log r}, \tag{3.29}$$

which, by a simple computation, (3.27) and (3.28), yields

$$|\rho(kr) - \rho(r)| \le (1 + o(1)) \frac{\varepsilon(r)^3}{\log r} \quad \text{as} \quad r \to \infty.$$

It follows from the definition (3.14) of $\rho(r)$ that the above estimate also holds with ρ replaced by ε, that is,

$$|\varepsilon(kr) - \varepsilon(r)| \le (1 + o(1)) \frac{\varepsilon(r)^3}{\log r} \quad \text{as} \quad r \to \infty. \tag{3.30}$$

Therefore we see that, for all such k,

$$\begin{aligned} r^{\rho(kr)-\rho(r)} &= \exp\{(\rho(kr) - \rho(r)) \log r\} \\ &= \exp\left\{(1 + o(1))\varepsilon(r)^3\right\} \text{ as } r \to \infty. \end{aligned} \tag{3.31}$$

Take $\delta = 1/4$, then $\delta < \min\{\rho(t), 1 - \rho(t)\}$ for large t since $\rho(t) \to 1/2$ by definition. Moreover, note that for such δ, we have the estimate (3.26). Now we separate the integral on the right-hand side of (3.24) into three parts as follows:

$$\begin{aligned} \int_0^\infty \frac{\left|n(t) - t^{\rho(r)}\right|}{t\,|t - z|} dt = \int_0^{a(r)r} \frac{\left|n(t) - t^{\rho(r)}\right|}{t\,|t - z|} dt + \int_{b(r)r}^\infty \frac{\left|n(t) - t^{\rho(r)}\right|}{t\,|t - z|} dt \\ + \int_{a(r)r}^{b(r)r} \frac{\left|n(t) - t^{\rho(r)}\right|}{t\,|t - z|} dt. \end{aligned}$$

For the first integral, by using standard properties of the proximate order, (3.26), (3.31), and the fact that for $t \in [0, a(r)r]$, we have $|t-z| \geq r(1-a(r)) \geq r/2$, we find that there exists a constant C such that

$$\begin{aligned}\int_0^{a(r)r} \frac{\left|n(t)-t^{\rho(r)}\right|}{t\,|t-z|}dt &\leq \frac{C}{r}\left\{\int_1^{a(r)r} t^{\rho(t)-1}dt + \int_1^{a(r)r} t^{\rho(r)-1}dt\right\} \\ &\leq Ca(r)^{\delta} r^{\rho(r)-1} \\ &= o\left(\varepsilon(r)^2 r^{\rho(r)-1}\right).\end{aligned} \tag{3.32}$$

For the second integral, we have $|t-z| \geq t/2$ for $t \geq b(r)r$ and similarly we obtain the following estimates:

$$\begin{aligned}\int_{b(r)r}^{\infty} \frac{\left|n(t)-t^{\rho(r)}\right|}{t\,|t-z|}dt &\leq C\left\{\int_{b(r)r}^{\infty} t^{\rho(t)-2}dt + \int_{b(r)r}^{\infty} t^{\rho(r)-2}dt\right\} \\ &\leq C\frac{1}{b(r)^{\delta}} r^{\rho(r)-1} \\ &= Ca(r)^{\delta} r^{\rho(r)-1} \\ &= o\left(\varepsilon(r)^2 r^{\rho(r)-1}\right).\end{aligned} \tag{3.33}$$

For $t \in [a(r)r, b(r)r]$ and $z = re^{i\theta}$, we have $|t-z| \geq (t+r)\sin\frac{\varepsilon(r)}{2}$. Combining this with (3.19), (3.30) and (3.31), we can obtain an estimate of the last integral as follows:

$$\begin{aligned}\int_{a(r)r}^{b(r)r} \frac{\left|n(t)-t^{\rho(r)}\right|}{t\,|t-z|}dt &\leq \int_{a(r)r}^{b(r)r} \frac{\left|n(t)-t^{\rho(t)}\right|}{t\,|t-z|}dt \\ &\quad + \int_{a(r)r}^{b(r)r} \frac{\left|t^{\rho(t)}-t^{\rho(r)}\right|}{t\,|t-z|}dt \\ &\leq \frac{C}{\sin\frac{\varepsilon(r)}{2}} \int_{a(r)r}^{b(r)r} \frac{\varepsilon(t)^3\, t^{\rho(t)}}{t\,(t+r)}dt \\ &= \mathcal{O}\left(\frac{\varepsilon(r)^2}{r}\int_{a(r)}^{b(r)} \frac{(\tau r)^{\rho(\tau r)}}{\tau\,(1+\tau)}d\tau\right) \\ &= \mathcal{O}\left(\varepsilon(r)^2\, r^{\rho(r)-1}\int_0^{\infty} \frac{\tau^{\rho(r)}}{\tau\,(1+\tau)}d\tau\right) \\ &= \mathcal{O}\left(\varepsilon(r)^2\, r^{\rho(r)-1}\right).\end{aligned} \tag{3.34}$$

Putting all the estimates (3.24), (3.32), (3.33) and (3.34) together, we see that for $z = re^{i\theta}$ with θ satisfying (3.21), $\log|f(z)|$ has the following asymptotic representation:

$$|\log|f(z)| + \operatorname{Re}(zI(z))| = \mathcal{O}\left(\varepsilon(r)^2 r^{\rho(r)}\right).$$

□

Now, for some $r_0 > 0$ we define

$$\gamma^+ = \left\{ re^{i\varepsilon(r)} : r \geq r_0 \right\},$$

$$\gamma^- = \left\{ re^{-i\varepsilon(r)} : r \geq r_0 \right\},$$

and

$$G(\gamma) = \mathbb{C} \setminus \left\{ re^{i\theta} \in \mathbb{C} : \varepsilon(r) \leq \theta \leq 2\pi - \varepsilon(r),\ r \geq r_0 \right\}.$$

A consequence of the above lemma is the following fact.

Lemma 3.7. *The function f is bounded on γ^+ and γ^-.*

Proof. This follows from the asymptotic representation of f given in Lemma 3.6. For sufficiently large r,

$$\rho(r)(\pi - \varepsilon(r)) = \frac{\pi}{2} + \varepsilon(r)\left(\pi - \frac{1}{2} - \varepsilon(r)\right) > \frac{\pi}{2}.$$

So we have

$$\begin{aligned} \cos\left(\rho(r)(\pi - \varepsilon(r))\right) &= -\sin\left(\varepsilon(r)\left(\pi - \frac{1}{2} - \varepsilon(r)\right)\right) \\ &= -(1 + o(1))\left(\pi - \frac{1}{2}\right)\varepsilon(r). \end{aligned}$$

Since $\varepsilon(r)^2 = o\left(\varepsilon(r)\right)$, it follows from the asymptotic representation of $\log|f|$ in Lemma 3.6 that

$$\log\left|f(re^{i\theta})\right| < 0,$$

for $|\theta| = \varepsilon(r)$ and for r sufficiently large. □

Lemma 3.8. *$f(z)$ is bounded in $G(\gamma)$.*

Proof. By the Denjoy-Carleman-Ahlfors Theorem, if g is entire then the number of components of $\{z \in \mathbb{C} : |g(z)| > R\}$ for $R > 0$ is less than or equal to $\max\{1, 2\rho(g)\}$, where $\rho(g)$ is the order of g. Therefore, our function f, which is constructed in (3.18) and has order of growth $1/2$, has at most one tract. However, the asymptotic formula (3.20) for f in Lemma 3.6 implies that $f(z)$ is unbounded when z goes to infinity along the negative real axis. So this means that f has a tract containing the negative real axis. By Lemma 3.7 f is bounded on γ^+ and γ^-. Therefore, the only tract of f should be contained in $\mathbb{C} \setminus G(\gamma)$, which means that f is bounded in $G(\gamma)$. □

To show that the entire function f we constructed in (3.18) is in the Eremenko-Lyubich class $\mathcal{B}$, we need to prove, by Definition 2.2, that the set of all critical values and asymptotic values of f is bounded. To this aim, we define

$$P_n(z) = \prod_{k=1}^{n} \left(1 - \frac{z}{a_k}\right).$$

Then

$$f(z) = \lim_{n\to\infty} P_n(z).$$

Clearly f is the locally uniform limit of real polynomials P_n with only real zeros. By Definition 2.3, the function f belongs to the Laguerre-Pólya class $\mathcal{LP}$. By using Proposition 2.1, we will deduce the following result.

Lemma 3.9. *The function f constructed in* (3.18) *is in class $\mathcal{B}$.*

Proof. As noted above, $P_n \to f$ locally uniformly. Thus also $P_n' \to f'$. Since all the polynomials P_n' have only real positive zeros, Hurwitz's theorem implies that all zeros of f' are also real and positive. So f has only real positive critical points. As $[r_0, \infty)$ lies in the unbounded domain $G(\gamma)$ defined above, Lemma 3.8 means that the set of critical values of f is bounded. Moreover, f at most one asymptotic value by the Denjoy-Carleman-Ahlfors theorem. Therefore, f belongs to the Eremenko-Lyubich class $\mathcal{B}$. □

Lemma 3.10. *The function f constructed in* (3.18) *satisfies the condition* (3.1).

Proof. This follows from our Lemma 3.6, which yields

$$\begin{aligned}\log M(r,f) &\le \frac{\pi}{\sin(\pi\rho(r))} r^{\rho(r)} + \mathcal{O}\left(\varepsilon(r)^2 r^{\rho(r)}\right)\\ &= (1+o(1))\,\pi r^{\rho(r)}.\end{aligned}$$

Thus, we obtain that

$$\begin{aligned}\log\log M(r,f) &\le \log r^{\rho(r)} + \mathcal{O}(1)\\ &= \rho(r)\log r + \mathcal{O}(1)\\ &= \left(\frac{1}{2} + \varepsilon(r)\right)\log r + \mathcal{O}(1).\end{aligned}$$

□

Until now we have constructed an entire function in class $\mathcal{B}$ which satisfies all the conditions in Theorem 3.1 except that the escaping set of this function has zero area. To show this, we will prove the following theorem, which may be of independent interests. Given an entire function $f \in \mathcal{B}$, recall notations in (2.5) and (2.6), which are connected to the logarithmic change of variable. Moreover,

$$V = \mathbb{C} \setminus U, \tag{3.35}$$

and

$$\theta(r) = \operatorname{meas}\left\{t \in [0, 2\pi] : re^{it} \in V\right\}. \tag{3.36}$$

Here meas denotes the one-dimensional Lebesgue measure. Now our result can be stated as follows.

Theorem 3.4. *Let $f \in \mathcal{B}$ be of finite order. Let V and θ be as above. Suppose that $\theta(r) \geq \theta_0(r)$ for large $r > 0$, where $\theta_0(r)$ is decreasing and satisfies*

$$\sum_{k=1}^{\infty} \theta_0 \left(E^k(0) \right) = \infty. \tag{3.37}$$

Then $\operatorname{area} \mathcal{I}(f) = 0$. *If, in addition, f is of disjoint type, then* $\operatorname{area} \mathcal{J}(f) = 0$.

We shall prove this theorem first for certain disjoint type entire functions (the following Theorem 3.5), which are obtained by considering $\lambda f(z)$ for f satisfying conditions in the theorem (see the discussion after Proposition 2.5). Then we can apply Theorem 2.12 to transfer to entire functions not necessarily being of disjoint type.

Theorem 3.5. *Let $f \in \mathcal{B}$ be of finite order. Let $r_0 > |f(0)|$ be such that the singularities of f^{-1} are contained in $\{z : |z| < r_0\}$. Suppose that there exists R_1 with $R_1 > \max\{2 \log r_0, \log r_0 + 64\pi\}$ such that $|z| > e^{R_1}$ if $|f(z)| > r_0$. Suppose that $\theta(r) \geq \theta_0(r)$ for large $r > 0$, where $\theta_0(r)$ is decreasing and satisfies* (3.37). *Then* $\operatorname{area} \mathcal{J}(f) = 0$.

Proof. First we apply a logarithmic change of variable to f and thus obtain a map $F : W \to H$ as in (2.7). Under the assumption in the theorem, we know that by Definition 2.15 the function f is of disjoint type. We also say in this case that F is of disjoint type. See [Rem09, Definition 2.2] for a discussion of such functions. More precisely, we can deduce from the conditions in the theorem that

$$W \subset \{z : \operatorname{Re} z > R_1\}. \tag{3.38}$$

Now we consider the following set

$$T = \{z : F^n(z) \in W, \text{ for all } n \geq 0\}.$$

Clearly the function satisfying (3.38) is of disjoint type, and thus the Fatou set of f consists of a single immediate attracting basin, and

$$\mathcal{J}(f) = \exp(T).$$

Moreover, the assumption that $f \in \mathcal{B}$ implies that $\mathcal{I}(f) \subset \mathcal{J}(f)$. And since the exponential map preserves sets of zero Lebesgue measure, to show that $\operatorname{area} \mathcal{J}(f) = 0$ (and hence $\operatorname{area} \mathcal{I}(f) = 0$) it suffices to show that $\operatorname{area} T = 0$. In the following discussion, we will mainly concentrate on this set and prove that the Lebesgue measure of T is zero.

Define

$$T_n = \left\{ z \in \mathbb{C} : F^k(z) \in W, \text{ for } k = 0, \ldots, n \right\}$$

and

$$S = \mathbb{C} \setminus T_0 = \mathbb{C} \setminus W. \tag{3.39}$$

By definition of T and T_n, we have

$$T = \bigcap_{n=0}^{\infty} T_n.$$

Put

$$c_n = \sum_{j=1}^{n} \frac{\tau R_1}{K^j}, \qquad K = \frac{R_1 - R}{4\pi} > 1, \tag{3.40}$$

where $\tau = 1/16$.

For a point $w \in W$, we consider a sequence of squares centred at w as follows:

$$P_n(w) = \left\{ z \in \mathbb{C} : |\mathrm{Re}(z-w)| \le \frac{\mathrm{Re}\, w}{64} - c_n, |\mathrm{Im}(z-w)| \le \frac{\mathrm{Re}\, w}{64} - c_n \right\}. \tag{3.41}$$

We write P_n instead of $P_n(w)$ for simplicity. For $z \in T_n$, we define

$$r_n(z) = \frac{\mathrm{Re}\, F^n(z)}{|(F^n)'(z)|}. \tag{3.42}$$

We show that there exists a countable subset $L_n \subset T_n \cap P_n$ satisfying the following conditions:

(i) $\bigcup_{z \in T_n \cap P_n} D(z, \tau r_n(z)) \subset \bigcup_{z \in L_n} D(z, 4\tau r_n(z))$;

(ii) $D(z_1, \tau r_n(z_1)) \cap D(z_2, \tau r_n(z_2)) = \emptyset$ for distinct $z_1, z_2 \in L_n$;

(iii) $D(z, \tau r_n(z)) \subset P_{n-1}$, for $z \in L_n$;

(iv) for each $z \in L_n$, the disk $D(z, \tau r_n(z))$ contains a compact subset $A_n(z)$ such that F^n maps $A_n(z)$ bijectively onto a square $Q(z_n)$ centred at $z_n := F^n(z)$ with sidelength $\mathrm{Re}\, z_n/32$, that is,

$$Q(z_n) = \left\{ z \in \mathbb{C} : |\mathrm{Re}(z - z_n)| \le \frac{1}{64} \mathrm{Re}\, z_n,\ |\mathrm{Im}(z - z_n)| \le \frac{1}{64} \mathrm{Re}\, z_n \right\}; \tag{3.43}$$

(v) $D(z, \tau r_n(z)) \subset T_{n-1}$, for $z \in L_n$.

The existence of L_n satisfying (i) and (ii) follows from Lemma 2.3. To see that the conclusion (iii) holds, note that $z \in T_n$ and hence $F^k(z) \in W$ for $0 \le k \le n$, which in particular means that $\operatorname{Re} F^n(z) > R_1$. Moreover, it follows from Lemma 2.2 and (3.40) that

$$|F'(z)| \geq \frac{\operatorname{Re} F(z) - R}{4\pi} \geq \frac{R_1 - R}{4\pi} = K.$$

Thus

$$\begin{aligned} |(F^n)'(z)| &= \left|F'(F^{n-1}(z))\right| \cdot \left|(F^{n-1})'(z)\right| \\ &\geq \frac{\operatorname{Re} F^n(z) - R}{4\pi} \cdot \prod_{j=0}^{n-2} \left|F'(F^j(z))\right| \\ &\geq \frac{\operatorname{Re} F^n(z) - R}{4\pi} \cdot \left(\frac{R_1 - R}{4\pi}\right)^{n-1}. \end{aligned}$$

Therefore, for $z \in T_n$ we have

$$\begin{aligned} r_n(z) = \frac{\operatorname{Re} F^n(z)}{|(F^n)'(z)|} &\leq \frac{\operatorname{Re} F^n(z)}{\operatorname{Re} F^n(z) - R} \cdot 4\pi \cdot \left(\frac{4\pi}{R_1 - R}\right)^{n-1} \\ &\leq \frac{R_1}{R_1 - R} \cdot 4\pi \cdot \left(\frac{4\pi}{R_1 - R}\right)^{n-1} \\ &= \left(\frac{4\pi}{R_1 - R}\right)^{n} R_1 \\ &= \frac{R_1}{K^n}. \end{aligned}$$

This implies (iii). Essentially (iv) follows from the above Corollary 2.2, which is a consequence of Koebe's theorem. Since $z \in L_n \subset T_n$, by definition of T_n we have $\operatorname{Re} z_k > R_1$ for $0 \leq k \leq n$. Now since

$$\frac{63}{64} \operatorname{Re} z_n > \frac{63}{64} R_1 > \frac{63}{64} \cdot 2R > R,$$

we obtain that $Q(z_n)$ is contained in H. If ϕ is the inverse branch of F which maps z_n to z_{n-1}, then $\phi(Q(z_n))$ is contained in W and hence the preimage of $Q(z_n)$ under the pullback of the inverse branch of F^k which maps z_n to z_{n-k} is contained in W for each $k = 1, \ldots, n$. If we denote by ϕ_n the branch of the inverse of F^n which maps z_n to z, then ϕ_n extends to a univalent map on $D(z_n, \frac{1}{2} \operatorname{Re} z_n)$ since $\frac{1}{2} \operatorname{Re} z_n > \frac{1}{2} R_1 > R$. By using Corollary 2.2 and by taking $\sigma = 1/256$, we have

$$D\left(z, \sigma r_n(z)\right) \subset \phi_n\left(Q(z_n)\right) \subset D\left(z, \tau r_n(z)\right). \tag{3.44}$$

The conclusion (iv) follows if we take

$$A_n(z) = \phi_n\left(Q(z_n)\right).$$

The last conclusion (v) follows if we consider the following square centred at z_n

$$Q'(z_n) = \left\{ z \in \mathbb{C} : |\mathrm{Re}(z - z_n)| \le \frac{1}{4}\,\mathrm{Re}\, z_n,\ |\mathrm{Im}(z - z_n)| \le \frac{1}{4}\,\mathrm{Re}\, z_n \right\}.$$

Similar arguments as above show that

$$\phi_n\left(Q'(z_n)\right) \subset T_{n-1}$$

and

$$D\left(z, \tau r_n(z)\right) \subset \phi_n\left(Q'(z_n)\right).$$

Therefore, $D(z, \tau r_n(z)) \subset T_{n-1}$ for $z \in L_n$. The conclusion (v) follows.

Now we split the rest of the proof into two steps. First we estimate the area of $T_{n-1} \setminus T_n$ in $D(z, \tau r_n(z))$ for $z \in T_n$, which we call *local estimate*. Then we spread the local estimate to a *global estimate*, which is the area of $T_{n-1} \setminus T_n$ in $P_n(w)$, by using the above (i), (ii) and (iii).

First we note that

$$T_{n-1} \setminus T_n = F^{-n}(S) \cap T_{n-1}.$$

Together with (v) above we have

$$\begin{aligned} &\mathrm{area}\left((T_{n-1} \setminus T_n) \cap D(z, \tau r_n(z))\right) \\ &= \mathrm{area}\left(F^{-n}(S) \cap T_{n-1} \cap D(z, \tau r_n(z))\right) \\ &= \mathrm{area}\left(F^{-n}(S) \cap D(z, \tau r_n(z))\right). \end{aligned} \tag{3.45}$$

Recall our definition of S in (3.39) and $\theta(r)$ in (3.36). We define

$$\varphi(x) = \mathrm{meas}\,\{\ y \in [0, 2\pi] : x + iy \in S\ \},$$

and

$$\varphi_0(x) = \theta_0(e^x).$$

Then $\varphi(x) = \theta(e^x)$. Since $\theta(x) \ge \theta_0(x)$ for large x, we see that $\varphi(x) \ge \varphi_0(x)$ for large x. Now we can give a lower bound for the area of S in the square $Q(z_n)$ given in (3.43). For simplicity we put $Q = Q(z_n)$. We use the fact that θ_0 is a continuous and decreasing function. Since the square Q contains at least $[\frac{\mathrm{Re}\, z_n}{64\pi}]$ horizontal strips of width 2π and $\mathrm{Re}\, z_n > R_1 > 64\pi$, we obtain

$$\begin{aligned} \mathrm{area}\,(S \cap Q) &\ge \left[\frac{\mathrm{Re}\, z_n}{64\pi}\right] \int_{\frac{63}{64}\mathrm{Re}\, z_n}^{\frac{65}{64}\mathrm{Re}\, z_n} \varphi(t)dt \\ &\ge \left[\frac{\mathrm{Re}\, z_n}{64\pi}\right] \int_{\frac{63}{64}\mathrm{Re}\, z_n}^{\frac{65}{64}\mathrm{Re}\, z_n} \varphi_0(t)dt \\ &\ge \left[\frac{\mathrm{Re}\, z_n}{64\pi}\right] \varphi_0\left(\frac{65}{64}\,\mathrm{Re}\, z_n\right) \frac{1}{32}\,\mathrm{Re}\, z_n. \end{aligned} \tag{3.46}$$

Here $[\,\cdot\,]$ denotes the integer part. Therefore,

$$\begin{aligned}
\operatorname{dens}(S,Q) &\geq \left[\frac{\operatorname{Re} z_n}{64\pi}\right] \frac{\varphi_0\left(\frac{65}{64}\operatorname{Re} z_n\right)\frac{1}{32}\operatorname{Re} z_n}{\left(\frac{1}{32}\operatorname{Re} z_n\right)^2} \\
&\geq \frac{1}{2}\frac{\operatorname{Re} z_n}{64\pi}\frac{\varphi_0\left(\frac{65}{64}\operatorname{Re} z_n\right)}{\frac{1}{32}\operatorname{Re} z_n} \\
&= \frac{1}{4\pi}\varphi_0\left(\frac{65}{64}\operatorname{Re} z_n\right) \\
&\geq \frac{1}{4\pi}\varphi_0\left(2\operatorname{Re} z_n\right) \\
&=: \varphi_1\left(\operatorname{Re} z_n\right).
\end{aligned} \tag{3.47}$$

As we mentioned above, ϕ_n, which is the inverse branch of F^n which maps z_n to z, extends to a univalent map on $D(z_n, \frac{1}{2}\operatorname{Re} z_n)$. Thus by Koebe's theorem, there exist positive constants K_1 and K_2 such that

$$K_1 \operatorname{dens}(S,Q) \leq \operatorname{dens}(\phi_n(S), \phi_n(Q)) \leq K_2 \operatorname{dens}(S,Q)$$

for $n \in \mathbb{N}$. Then by using (3.44) and (3.47) we have

$$\begin{aligned}
\operatorname{dens}\big(F^{-n}(S), D(z,\tau r_n(z))\big) &= \frac{\operatorname{area}\left(F^{-n}(S)\cap D(z,\tau r_n(z))\right)}{\operatorname{area} D(z,\tau r_n(z))} \\
&\geq \frac{\operatorname{area}\left(\phi_n(S)\cap \phi_n(Q)\right)}{\operatorname{area} D(z,\tau r_n(z))} \\
&\geq \frac{K_1 \operatorname{dens}(S,Q)\cdot \operatorname{area}\phi_n(Q)}{\operatorname{area} D(z,\tau r_n(z))} \\
&\geq \frac{K_1 \operatorname{dens}(S,Q)\cdot \operatorname{area} D(z,\sigma r_n(z))}{\operatorname{area} D(z,\tau r_n(z))} \\
&= K_1\left(\frac{\sigma}{\tau}\right)^2 \operatorname{dens}(S,Q) \\
&\geq K_1\left(\frac{\sigma}{\tau}\right)^2 \varphi_1(\operatorname{Re} z_n).
\end{aligned}$$

So by (3.45), for $z \in T_n$ we have, with $\varphi_2(x) = K_1(\sigma/\tau)^2\varphi_1(x)$,

$$\begin{aligned}
&\operatorname{area}\left((T_{n-1}\setminus T_n)\cap D\left(z,\tau r_n(z)\right)\right) \\
&\quad\geq K_1\left(\frac{\sigma}{\tau}\right)^2 \varphi_1(\operatorname{Re} z_n)\cdot \operatorname{area} D\left(z,\tau r_n(z)\right) \\
&\quad= \varphi_2(\operatorname{Re} z_n)\cdot \operatorname{area} D(z,\tau r_n(z))).
\end{aligned} \tag{3.48}$$

Since f is of finite order, there exists some constant $\rho' < \infty$ such that

$$\log\log M(r,f) \leq \rho' \log r \quad \text{for large} \quad r.$$

Now, we see that for any point z with large real part,

$$\operatorname{Re} F(z) \leq \exp(\rho' \operatorname{Re} z) = E(\rho' \operatorname{Re} z) \leq E^2(\operatorname{Re} z), \tag{3.49}$$

and hence

$$\operatorname{Re} z_k = \operatorname{Re} F^k(z) \leq E^{2k}(\operatorname{Re} z). \tag{3.50}$$

Now we can deduce our global estimate from the conclusions (i), (ii), (iii), (3.48) and (3.50) as follows:

$$\begin{aligned}
&\operatorname{area}\left((T_{n-1} \setminus T_n) \cap P_{n-1}\right) \\
&\geq \operatorname{area}\left((T_{n-1} \setminus T_n) \cap \bigcup_{z \in L_n} D(z, \tau r_n(z))\right) \\
&= \sum_{z \in L_n} \operatorname{area}\left((T_{n-1} \setminus T_n) \cap D(z, \tau r_n(z))\right) \\
&\geq \sum_{z \in L_n} \varphi_2\left(\operatorname{Re} z_n\right) \cdot \operatorname{area} D(z, \tau r_n(z)) \\
&\geq \sum_{z \in L_n} \varphi_2\left(E^{2n}(\operatorname{Re} z)\right) \cdot \operatorname{area} D(z, \tau r_n(z)) \\
&\geq \varphi_2\left(E^{2n}\left(\frac{65}{64}\operatorname{Re} w\right)\right) \sum_{z \in L_n} \operatorname{area} D(z, \tau r_n(z)) \\
&\geq \frac{1}{16}\varphi_2\left(E^{2n}\left(\frac{65}{64}\operatorname{Re} w\right)\right) \cdot \operatorname{area}\left(\bigcup_{z \in L_n} D(z, 4\tau r_n(z))\right) \\
&\geq \frac{1}{16}\varphi_2\left(E^{2n}\left(\frac{65}{64}\operatorname{Re} w\right)\right) \cdot \operatorname{area}\left(T_n \cap P_n\right).
\end{aligned} \tag{3.51}$$

Since

$$\begin{aligned}
\operatorname{area}\left((T_{n-1} \setminus T_n) \cap P_{n-1}\right) &= \operatorname{area}\left(T_{n-1} \cap P_{n-1}\right) - \operatorname{area}\left(T_n \cap P_{n-1}\right) \\
&\leq \operatorname{area}\left(T_{n-1} \cap P_{n-1}\right) - \operatorname{area}\left(T_n \cap P_n\right),
\end{aligned}$$

we obtain

$$\operatorname{area}\left(T_{n-1} \cap P_{n-1}\right) \geq \left[1 + \frac{1}{16}\varphi_2\left(E^{2n}\left(\frac{65}{64}\operatorname{Re} w\right)\right)\right] \cdot \operatorname{area}\left(T_n \cap P_n\right).$$

Let

$$\begin{aligned}
P_\infty &= \bigcap_{n \geq 1} P_n \\
&= \left\{z : \ |\operatorname{Re}(z-w)| \leq \frac{\operatorname{Re} w}{M} - c, |\operatorname{Im}(z-w)| \leq \frac{\operatorname{Re} w}{M} - c\right\},
\end{aligned}$$

where

$$c = \sum_{j=1}^{\infty} \frac{\tau R_1}{K^j} = \frac{\tau R_1}{K-1}.$$

We have

$$\begin{aligned}\operatorname{area}(T_n \cap P_\infty) &\le \operatorname{area}(T_n \cap P_n) \\ &\le \prod_{k=1}^{n} \frac{1}{1+\frac{1}{16}\varphi_2\left(E^{2k}\left(\frac{65}{64}\operatorname{Re} w\right)\right)} \cdot \operatorname{area}(T_1 \cap P_1),\end{aligned}$$

which means that

$$\operatorname{area}(T \cap P_\infty) \le \prod_{k=1}^{\infty} \frac{1}{1+\frac{1}{16}\varphi_2\left(E^{2k}\left(\frac{65}{64}\operatorname{Re} w\right)\right)} \cdot \operatorname{area}(T_1 \cap P_1). \tag{3.52}$$

Note that

$$\begin{aligned}\frac{1}{16}\varphi_2\left(E^{2k}\left(\frac{65}{64}\operatorname{Re} w\right)\right) &= \alpha \cdot \varphi_0\left(2E^{k}\left(\frac{65}{64}\operatorname{Re} w\right)\right) \\ &\ge \alpha \cdot \varphi_0\left(E^{2k+2}(\operatorname{Re} w)\right) \\ &= \alpha \cdot \theta_0\left(E^{2k+3}(\operatorname{Re} w)\right) \\ &\ge \alpha \cdot \theta_0\left(E^{2k+4}(\operatorname{Re} w)\right),\end{aligned}$$

where $\alpha = \frac{K_1}{64\pi}(\sigma/\tau)^2$. Since

$$\begin{aligned}\sum_{k=1}^{\infty} \theta_0\left(E^{k}(x)\right) &= \sum_{k=1}^{\infty} \theta_0\left(E^{2k}(x)\right) + \sum_{k=1}^{\infty} \theta_0\left(E^{2k-1}(x)\right) \\ &\le \sum_{k=1}^{\infty} \theta_0\left(E^{2k}(x)\right) + \sum_{k=1}^{\infty} \theta_0\left(E^{2k-2}(x)\right) \\ &= 2\sum_{k=1}^{\infty} \theta_0\left(E^{2k}(x)\right) + \theta_0(x),\end{aligned} \tag{3.53}$$

we deduce from (3.37) that

$$\sum_{k=1}^{\infty} \frac{1}{16}\varphi_2\left(E^{2k}\left(\frac{65}{64}\operatorname{Re} w\right)\right) = \infty.$$

This implies that

$$\prod_{k=1}^{\infty} \frac{1}{1+\frac{1}{16}\varphi_2\left(E^{2k}\left(\frac{65}{64}\operatorname{Re} w\right)\right)} = 0,$$

which, together with (3.52), finishes the proof:

$$\text{area}(T \cap P_\infty) = 0.$$

Since the point $w \in W$ is chosen arbitrarily, we have in particular that $\text{area}\, T = 0$. By the discussion at the beginning of this section we finally have $\text{area}\, \mathcal{J}(f) = 0$ and in particular $\text{area}\, \mathcal{I}(f) = 0$. □

Proof of Theorem 3.4. Let $f \in \mathcal{B}$ be as in Theorem 3.4. Now we consider

$$f_\lambda : \mathbb{C} \to \mathbb{C}; \quad z \mapsto f(\lambda z).$$

By choosing λ to be sufficiently small, the function f_λ will satisfy all conditions in Theorem 3.5. Therefore, we see that $\text{area}\, \mathcal{J}(f_\lambda) = 0$. By definition, f and f_λ are equivalent near infinity. By Theorem 2.12, there exists $R > 0$ and a quasiconformal mapping $\theta : \mathbb{C} \to \mathbb{C}$ such that

$$\theta \circ f_\lambda = f \circ \theta$$

on the set

$$\mathcal{J}_R(f_\lambda) := \{z \in \mathbb{C} : |f_\lambda^n(z)| \geq R \ \text{ for all } \ n \geq 1\}.$$

Now we put

$$\mathcal{I}_R(f_\lambda) := \mathcal{I}(f_\lambda) \cap \mathcal{J}_R(f_\lambda).$$

Then $\text{area}\, \mathcal{I}_R(f_\lambda) = 0$. So we see that there exists a constant $R' > 0$ such that

$$\mathcal{I}_{R'}(f) \subset \theta(\mathcal{I}_R(f_\lambda)).$$

Recall that

$$\mathcal{I}(f) = \bigcup_{n \geq 0} f^{-n}\left(\mathcal{I}_{R'}(f)\right).$$

Therefore, it is clear that $\text{area}\, \mathcal{I}(f) = 0$. □

Remark 3.3.1. The hypothesis that f has finite order was used only in (3.49) to conclude that $\text{Re}\, F(z) \leq E^2(\text{Re}\, z)$ if $\text{Re}\, z$ is large enough. The proof goes through with only minor modifications if instead we only have $\text{Re}\, F(z) \leq E^N(\text{Re}\, z)$ for some $N \in \mathbb{N}$ and $\text{Re}\, z$ sufficiently large. This implies that the condition in Theorem 3.4 and Theorem 3.5 that f has finite order can be replaced by the condition that

$$\log^N M(r, f) \leq r$$

for some $N \in \mathbb{N}$ and large r.

Now we will finish the proof of Theorem 3.1, by showing that the escaping set of the functions f constructed in (3.18) which satisfies the condition (3.1) has zero area. We only need to check that our function f satisfies all the conditions in Theorem 3.4. By considering $f(\lambda z)$ for a sufficiently small constant λ if necessary, we can see that the condition (3.38) is satisfied. Moreover, since f is bounded in $G(\gamma)$ and the opening angle of $G(\gamma)$ at radius r is equal to $2\varepsilon(r)$. Hence, we need to check that

$$2\sum_{k=1}^{\infty}\varepsilon\left(E^{k}\left(x\right)\right)=\infty.$$

This follows from our Lemma 3.2. Now it follows from Theorem 3.4 that area $\mathcal{J}(f) = 0$ and in particular area $\mathcal{I}(f) = 0$.

Remark 3.3.2. The function we constructed above shows the sharpness of the Aspenberg-Bergweiler condition (3.15) in the case of only one logarithmic tract over infinity. To obtain an entire function with any finite number, say N, logarithmic tracts over infinity, we let f be as in Theorem 3.1 and let $g(z) = f(z)^N$ and $h(z) = f(z^N)$. Then h is in class $\mathcal{B}$ and has N logarithmic tracts over infinity. Moreover, the function h satisfies that

$$\log\log M(r,h)\leq\left(\frac{N}{2}+\frac{N}{\log\Phi(r)}\right)\log r+\mathcal{O}(1).$$

Now we need to show that for such function h we have area $\mathcal{I}(h) = 0$. This follows since $z \in \mathcal{I}(h)$ if and only if $z^N \in \mathcal{I}(g)$. Thus area $\mathcal{I}(h) = 0$ if and only if area $\mathcal{I}(g) = 0$. Now area $\mathcal{I}(g) = 0$ by the same argument that gives area $\mathcal{I}(f) = 0$. Hence area $\mathcal{I}(h) = 0$.

Remark 3.3.3. We constructed an entire function f in class $\mathcal{B}$ which satisfies the condition (3.1). Now we compare this condition with the Aspenberg-Bergweiler condition (3.15). Recall that $\varepsilon_0(r)$ is defined in (3.4). Lemma 3.3 says that

$$\sum_{k=1}^{\infty}\varepsilon_0(E^k(0))<\infty. \tag{3.54}$$

While for the function $1/\log\Phi(r)$ in (3.1), by Lemma 3.2 we have

$$\sum_{k=1}^{\infty}\frac{1}{\log\Phi(E^k(0))}=\infty. \tag{3.55}$$

Instead of $E^k(0)$ we could take $E^k(x)$ for some $x \in \mathbb{R}$ in (3.54). Since there always exists $N \in \mathbb{N}$ such that $E^N(x) \geq 0$, we have

$$\begin{aligned}\sum_{k=1}^{\infty} \varepsilon_0\left(E^k\left(x\right)\right) &= \sum_{k=1}^{N} \varepsilon_0\left(E^k\left(x\right)\right) + \sum_{k=1}^{\infty} \varepsilon_0\left(E^{k+N}\left(x\right)\right) \\ &\leq \sum_{k=1}^{N} \varepsilon_0\left(E^k\left(x\right)\right) + \sum_{k=1}^{\infty} \varepsilon_0(E^k(0)) \\ &< \infty.\end{aligned}$$

As a matter of fact, the convergence or divergence of the above sums is crucial in both the work of Aspenberg and Bergweiler and our proof of Theorem 3.1. The series (3.54) converges also for $\varepsilon(x) = 1/(\log \Phi(x))^{1+\delta}$ if $\delta > 0$, and as mentioned after Theorem 3.3, the argument of Aspenberg and Bergweiler still works in this case and shows that the conclusion of their theorem also holds in this case. Theorem 3.1 shows that this does not hold for $\delta = 0$.

Chapter 4

Eremenko-Lyubich Condition and Generalization

This chapter is also concerned with the area of escaping set of entire functions in class $\mathcal{B}$. In [EL92], Eremenko and Lyubich gave a condition under which the escaping set of a transcendental entire function in class $\mathcal{B}$ has zero Lebesgue measure. Their condition was formulated in terms of the quantity

$$\theta(r) := \left\{t \in [0, 2\pi] : \left|f(re^{it})\right| < R\right\}, \tag{4.1}$$

which also appeared in the last chapter (compare (3.36)). We call the condition given by them the *Eremenko-Lyubich condition*. In particular, if an entire function is of finite order for which the inverse of f has a finite logarithmic singularity, then the Eremenko-Lyubich condition is satisfied. Hence, the result of Eremenko and Lyubich can be formulated as: if an entire function in class $\mathcal{B}$ is of finite order and the inverse has a finite logarithmic singularity $a \in \mathbb{C}$, then the area of the escaping set is zero. The main intention of this chapter is devoted to generalize this result. Roughly speaking, our result says that one can relax the function to be of small infinite order and in this case the same conclusion holds. More precisely, we have the following result.

Theorem 4.1. *Let $f \in \mathcal{B}$ be a transcendental entire function and $r' > 0$. Suppose that the inverse of f has a direct singularity $a \in \mathbb{C}$. Suppose furthermore that f satisfies*

$$\log\log M(r, f) \leq A(r) \log r \tag{4.2}$$

for $r \geq r'$, for some continuous and increasing function $A : [r', \infty) \to \mathbb{R}$ satisfying $A(r) < \log r$ for large r and

$$\sum_{k=1}^{\infty} \frac{1}{A\left(E^k(0)\right)} = \infty. \tag{4.3}$$

Then $\operatorname{area}\mathcal{I}(f) = 0$.

The chapter is divided into two parts. In the first part we present the Eremenko-Lyubich condition mentioned above. Then in the second part we prove our theorem.

4.1 The Eremenko-Lyubich Condition

Recall that the quantity $\theta(r)$ for a transcendental entire function f is defined in (4.1). We use the notations given in (2.5) and (2.6), and put

$$V = \mathbb{C} \setminus U.$$

In terms of the tracts over infinity, we can also define $\theta(r)$ as follows:

$$\theta(r) = \operatorname{meas}\left\{t \in [0, 2\pi] : re^{it} \in V\right\}. \tag{4.4}$$

Now we state the result by Eremenko and Lyubich, see [EL92, Theorem 7].

Theorem 4.2. *Let* $f \in \mathcal{B}$ *be a transcendental entire function satisfying*

$$\liminf_{r\to\infty} \frac{1}{\log r}\int_1^r \theta(t)\frac{dt}{t} > 0. \tag{4.5}$$

Then $\operatorname{area}\mathcal{I}(f) = 0$.

We call (4.5) the *Eremenko-Lyubich condition.* It is clear that if an entire function $f \in \mathcal{B}$ is bounded in a sector, or in other words if the set V contains a sector of a definite opening, then the area of escaping set of f is zero (this also follows from our Theorem 3.4). In particular, the Mittag-Leffler function discussed in Example 3.1 satisfies this condition and has zero area escaping set.

In [EL92, Proposition 4], Eremenko and Lyubich showed the following

Proposition 4.1. *If the order of an entire function* f *is finite and the inverse* f^{-1} *has a logarithmic singularity* $a \in \mathbb{C}$, *then the Eremenko-Lyubich condition is satisfied.*

Combining the above two results, we have the following

Theorem 4.3. *Let* $f \in \mathcal{B}$ *be a transcendental entire function of finite order such that the inverse of* f *has a finite logarithmic singularity. Then* $\operatorname{area}\mathcal{I}(f) = 0$.

Our main aim in this chapter is to generalize this result to "small" infinite order entire functions.

4.2 Generalization

To prove Theorem 4.1, we shall first prove the following

Theorem 4.4. *Let $f \in \mathcal{B}$ be a transcendental entire function and $r' > 0$. Suppose that f satisfies* (4.2) *and*

$$\frac{1}{\log r}\int_{r^c}^{r}\theta(t)\frac{dt}{t} \geq \frac{1}{A(r)} \tag{4.6}$$

for a constant c with $63/65 \leq c < 1$, and for some continuous and increasing function $A : [r', \infty) \to \mathbb{R}$ satisfying $A(r) < \log r$ for large r and (4.3). *Then* $\operatorname{area}\mathcal{I}(f) = 0$.

Proof. For the function f given in the theorem, we define

$$g(z) := \lambda f(z),$$

where $\lambda > 0$ is a constant to be determined later. Then g belongs to class $\mathcal{B}$ and is conformally equivalent to f. So we can apply a logarithmic change of variables to g and hence we obtain a function G, corresponding to g, such that G is a biholomophic map from every component of W onto H', where

$$H' = \{\ z \in \mathbb{C} :\ \operatorname{Re} z > \log r_0 + \log\lambda\ \},$$

here r_0 is as in Section 2.3.2. Now the constant λ is chosen sufficiently small such that

$$\inf\{\operatorname{Re} z :\ z \in W\} > 2(\log r_0 + \log\lambda)$$

and

$$\inf\{\operatorname{Re} z :\ z \in W\} - (\log r_0 + \log\lambda) > 64\pi.$$

Therefore, the function g defined above satisfies the conditions in Theorem 3.5 by taking

$$R_1 := \inf\{\operatorname{Re} z :\ z \in W\}.$$

Now the proof of the theorem is similar to that of Theorem 3.5. We choose a point w with large real part and a sequence of squares $P_n(w)$ exactly the same as those in (3.41). In the same way we define the sets T_n and S. Now there exists a countable subset L_n of $T_n \cap P_n$ which satisfies conditions (i)-(v) as before. Now the proof follows if we can obtain appropriate estimates: the local estimate and the global estimate.

The local estimate is achieved by using the inequality (4.6). In terms of φ, where $\varphi(x) = \theta(e^x)$, the condition (4.6) can be written as

$$\int_{cx}^{x}\varphi(s)ds = \int_{e^{cx}}^{e^x}\varphi(\log t)\frac{dt}{t} = \int_{e^{cx}}^{e^x}\theta(t)\frac{dt}{t} \geq \frac{x}{A(e^x)} = \frac{x}{A(E(x))}. \tag{4.7}$$

Then since $63/65 \le c < 1$, by letting $z_n := G^n(z)$ we obtain, instead of (3.46),

$$\begin{aligned} \operatorname{area}(S \cap Q) &\ge \left[\frac{\operatorname{Re} z_n}{64\pi}\right] \int_{\frac{63}{64}\operatorname{Re} z_n}^{\frac{65}{64}\operatorname{Re} z_n} \varphi(t)dt \\ &\ge \left[\frac{\operatorname{Re} z_n}{64\pi}\right] \int_{c\frac{65}{64}\operatorname{Re} z_n}^{\frac{65}{64}\operatorname{Re} z_n} \varphi(t)dt \\ &\ge \left[\frac{\operatorname{Re} z_n}{64\pi}\right] \frac{\frac{65}{64}\operatorname{Re} z_n}{A(E(\frac{65}{64}\operatorname{Re} z_n))}, \end{aligned}$$

and

$$\operatorname{dens}(S, Q) \ge \frac{C_0}{A\left(E^2(\operatorname{Re} z_n)\right)}$$

for some constant $C_0 > 0$. Therefore, we obtain the following local estimate as in (3.48) by using Koebe's theorem as before:

$$\begin{aligned} &\operatorname{area}\left((T_{n-1} \setminus T_n) \cap D\left(z, \tau r_n(z)\right)\right) \\ &\quad \ge \frac{C_1}{A\left(E^2(\operatorname{Re} z_n)\right)} \cdot \operatorname{area} D\left(z, \tau r_n(z)\right), \end{aligned} \tag{4.8}$$

where C_1 is some positive constant.

To spread this to the global estimate, we need the assumption that $A(r) < \log r$ for large r. So in this way we have an estimate for $\operatorname{Re} G(z)$:

$$\begin{aligned} \operatorname{Re} G(z) &\le \log M(e^{\operatorname{Re} z}, g) \\ &\le \exp\left(A(e^{\operatorname{Re} z}) \cdot \operatorname{Re} z\right) \\ &\le \exp^2(\operatorname{Re} z), \end{aligned} \tag{4.9}$$

which implies that

$$\operatorname{Re} z_n = \operatorname{Re} G^n(z) \le \exp^{2n}(\operatorname{Re} z) = E^{2n}(\operatorname{Re} z).$$

Now by using (4.8) the global estimate, which is similar to (3.51), is given as follows:

$$\begin{aligned} \operatorname{area}\left((T_{n-1} \setminus T_n) \cap P_{n-1}\right) &\ge \sum_{z \in L_n} \operatorname{area}\left((T_{n-1} \setminus T_n) \cap D(z, \tau r_n(z))\right) \\ &\ge \sum_{z \in L_n} \frac{C_1}{A\left(E^2(\operatorname{Re} z_n)\right)} \cdot \operatorname{area} D(z, \tau r_n(z)) \\ &\ge \frac{C_1}{A\left(E^{2n+2}\left(\frac{65}{64}\operatorname{Re} w\right)\right)} \sum_{z \in L_n} \operatorname{area} D(z, \tau r_n(z)) \\ &\ge \frac{1}{16} \frac{C_1}{A\left(E^{2n+2}\left(\frac{65}{64}\operatorname{Re} w\right)\right)} \cdot \operatorname{area}\left(T_n \cap P_n\right). \end{aligned}$$

This, as in (3.52), implies that

$$\operatorname{area}(T \cap P_\infty) \le \prod_{n=1}^{\infty} \frac{1}{1 + \dfrac{1}{16} \dfrac{C_1}{A\left(E^{2n+2}\left(\frac{65}{64}\operatorname{Re} w\right)\right)}} \cdot \operatorname{area}(T_1 \cap P_1).$$

Since the function $A(r)$ is increasing, we can use similar argument as in (3.53) to deduce from condition (4.3) that

$$\sum_{n=1}^{\infty} \frac{1}{A\left(E^{2n+2}\left(\frac{65}{64}\operatorname{Re} w\right)\right)} = \infty.$$

The rest is the same as before and we omit details here. Therefore, we see that area $\mathcal{J}(g) = 0$ for the disjoint type function g.

We still need to transfer this result to our original function f. This is done as in the proof of Theorem 3.4, using Theorem 3.5.

□

Proof of Theorem 4.1. We define

$$g(z) = \frac{1}{f(z) - a}.$$

Since f has a finite direct singular value $a \in \mathbb{C}$, we find that g has a direct tract over infinity. Choose $\varepsilon > 0$ small enough, and denote

$$\alpha(r) := \operatorname{meas}\left\{t \in [0, 2\pi] : \left|f(re^{it}) - a\right| < \varepsilon\right\}.$$

We will follow the notations given in the introduction with respect to f. Recall that $\theta(r) = \operatorname{meas}\left\{t \in [0, 2\pi] : |f(re^{it})| < e^R\right\}$, which is defined in (3.36). By choosing $R > 0$ large enough we can have $e^R > |a| + \varepsilon$. Therefore, we see that

$$\theta(r) \ge \alpha(r) \tag{4.10}$$

holds for large r. On the other hand, $\alpha(r)$ also denotes the linear measure of the set $U_0 := \{t \in [0, 2\pi] : |g(re^{it})| > 1/\varepsilon\}$. Applying Theorem 2.5 to the function g, with $G = U_0$ and $\beta(r) = \alpha(r)$ we see that

$$\log\log M_{U_0}(r, g) \ge \pi \int_{r_1}^{\alpha_0 r} \frac{dt}{t \cdot \alpha(t)} + \mathcal{O}(1), \tag{4.11}$$

where $r_1 > 0$ and $0 < \alpha_0 < 1$. This, together with (4.10), implies that

$$\begin{aligned} \log\log M_{U_0}(r, g) &\ge \pi \int_{r_1}^{\alpha_0 r} \frac{dt}{t \cdot \theta(t)} + \mathcal{O}(1) \\ &\ge \pi \int_{(\alpha_0 r)^c}^{\alpha_0 r} \frac{dt}{t \cdot \theta(t)} + \mathcal{O}(1), \end{aligned} \tag{4.12}$$

here c satisfies the condition from Theorem 4.4.

Then by applying Theorem 2.2, Theorem 2.1 and Theorem 2.3, and by taking $R = 2r$ we have

$$\begin{aligned}\log M_{U_0}(r,g) &\le 3T(2r,g) + \mathcal{O}(1) \\ &= 3T(2r,f) + \mathcal{O}(1) \\ &\le 3\log M(2r,f) + \mathcal{O}(1).\end{aligned}$$

Therefore, we get

$$\begin{aligned}\pi \int_{(\alpha_0 r)^c}^{\alpha_0 r} \frac{dt}{t\cdot\theta(t)} &\le \log\log M(2r,f) + \mathcal{O}(1) \\ &\le A(2r)\log(2r) + \mathcal{O}(1).\end{aligned} \tag{4.13}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}((1-c)\log(\alpha_0 r))^2 = \left(\int_{(\alpha_0 r)^c}^{\alpha_0 r} \frac{dt}{t}\right)^2 &= \left(\int_{(\alpha_0 r)^c}^{\alpha_0 r} \sqrt{\frac{\theta(t)}{t}}\frac{1}{\sqrt{\theta(t)t}}dt\right)^2 \\ &\le \int_{(\alpha_0 r)^c}^{\alpha_0 r} \theta(t)\frac{dt}{t} \cdot \int_{(\alpha_0 r)^c}^{\alpha_0 r} \frac{dt}{\theta(t)\cdot t}.\end{aligned}$$

Together with (4.13) we see that

$$\frac{1}{\log r}\int_{(\alpha_0 r)^c}^{\alpha_0 r} \theta(t)\frac{dt}{t} \ge \frac{C_0}{A(2r)} \tag{4.14}$$

for large r, where $C_0 > 0$ is some constant. Thus, we obtain from (4.14) that

$$\begin{aligned}\frac{1}{\log r}\int_{r^c}^{r} \theta(t)\frac{dt}{t} &\ge \frac{1}{\log(r/\alpha_0)}\int_{r^c}^{r} \theta(t)\frac{dt}{t} \\ &\ge \frac{C_0}{A(2r/\alpha_0)} := \frac{1}{A_0(r)}.\end{aligned} \tag{4.15}$$

Thus, (4.6) is satisfied with $A(r)$ replaced by $A_0(r)$. Moreover, by using (4.3) it is easy to show that

$$\sum_{k=1}^{\infty} \frac{1}{A_0(E^k(0))} = \infty. \tag{4.16}$$

Therefore, we can apply Theorem 4.4 to obtain that area $\mathcal{I}(f) = 0$.

□

Remark 4.2.1. Compare our Theorem 4.1 with Theorem 4.3 by Eremenko and Lyubich. If the function $A(r)$ in (4.2) is bounded above by some constant, that is, f is of finite order, then the condition (4.3) is automatically satisfied. The statement of our theorem is exactly that of Theorem 4.3. In this sense, we can view Theorem 4.1 as a generalization of Eremenko and Lyubich's result.

Remark 4.2.2. The condition (4.3) implies that $A(r) < \log r$ is only a mild restriction. While our argument requires some upper bound for $A(r)$, a much weaker bound like $A(r) \leq \exp^N r$ for some $N \in \mathbb{N}$ would suffice. Only (4.9) and the subsequent estimates have to be changed slightly.

Remark 4.2.3. Even though the Hausdorff dimension of escaping and Julia sets is not considered in this thesis, we mention that for entire functions satisfying all conditions in Theorem 4.1, the escaping set has full Hausdorff dimension 2. This is a consequence of a result by Bergweiler, Karpińska and Stallard [BKS09, Theorem 1.1].

Bibliography

[AB12] M. Aspenberg and W. Bergweiler, *Entire functions with Julia sets of positive measure*, Math. Ann. **352** (2012), no. 1, 27–54.

[Ahl06] L. V. Ahlfors, *Lectures on quasiconformal mappings*, second ed., University Lecture Series, vol. 38, American Mathematical Society, Providence, RI, 2006.

[Bak63] I. N. Baker, *Multiply connected domains of normality in iteration theory*, Math. Z. **81** (1963), 206–214.

[Bak68] ———, *Repulsive fixpoints of entire functions*, Math. Z. **104** (1968), 252–256.

[Bak75] ———, *The domains of normality of an entire function*, Ann. Acad. Sci. Fenn., Ser. A. I. Math. **1** (1975), 277–283.

[Bak76] ———, *An entire function which has wandering domains*, J. Aust. Math. Soc. Ser. A **22** (1976), no. 2, 173–176.

[Bak84] ———, *Wandering domains in the iteration of entire functions*, Proc. London Math. Soc. (3) **49** (1984), no. 3, 563–576.

[BC16] W. Bergweiler and I. Chyzhykov, *Lebesgue measure of escaping sets of entire functions of completely regular growth*, J. London Math. Soc. (2) **94** (2016), no. 2, 639–661.

[BE95] W. Bergweiler and A. Eremenko, *On the singularities of the inverse to a meromorphic function of finite order*, Rev. Mat. Iberoam **11** (1995), no. 2, 355–373.

[Bea91] A. F. Beardon, *Iteration of rational functions: Complex analytic dynamical systems*, Graduate Texts in Mathematics, vol. 132, Springer-Verlag, New York, 1991.

[BEL03] W. Bergweiler, A. Eremenko, and J. Langley, *Real entire functions of infinite order and a conjecture of Wiman*, Geom. Funct. Anal. **13** (2003), no. 5, 975–991.

[Ber91] W. Bergweiler, *Periodic points of entire functions: proof of a conjecture of Baker*, Complex Var. Theory Appl. **17** (1991), no. 1-2, 57–72.

[Ber93] ______, *Iteration of meromorphic functions*, Bull. Amer. Math. Soc. **29** (1993), no. 2, 151–188.

[Ber00] ______, *The role of the Ahlfors five islands theorem in complex dynamics*, Conform. Geom. Dyn. **4** (2000), no. 2, 22–34.

[BFRG15] W. Bergweiler, N. Fagella, and L. Rempe-Gillen, *Hyperbolic entire functions with bounded Fatou components*, Comment. Math. Helv. **90** (2015), no. 4, 799–829.

[BH99] W. Bergweiler and A. Hinkkanen, *On semiconjugation of entire functions*, Math. Proc. Cambridge Philo. Soc. **126** (1999), no. 3, 565–574.

[Bis15] C. J. Bishop, *Constructing entire functions by quasiconformal folding*, Acta Math. **214** (2015), no. 1, 1–60.

[BJR12] K. Barański, X. Jarque, and L. Rempe, *Brushing the hairs of transcendental entire functions*, Topology Appl. **159** (2012), no. 8, 2102–2114.

[BKS09] W. Bergweiler, B. Karpińska, and G. M. Stallard, *The growth rate of an entire function and the Hausdorff dimension of its Julia set*, J. Lond. Math. Soc. (2) **80** (2009), no. 3, 680–698.

[BRS08] W. Bergweiler, P. J. Rippon, and G. M. Stallard, *Dynamics of meromorphic functions with direct or logarithmic singularities*, Proc. London Math. Soc. (3) **97** (2008), no. 2, 368–400.

[BRS13] ______, *Multiply connected wandering domains of entire functions*, Proc. London Math. Soc. (3) **107** (2013), no. 6, 1261–1301.

[CG93] L. Carleson and T. W. Gamelin, *Complex dynamics*, Universitext: Tracts in Mathematics, Springer-Verlag, New York, 1993.

[DH85] A. Douady and J. H. Hubbard, *Étude dynamique des polynômes complexes. I, II*, Publications Mathématiques d'Orsay, vol. 84–85, Université de Paris-Sud, Département de Mathématiques, Orsay, 1984–1985.

[Dom98] P. Domínguez, *Dynamics of transcendental meromorphic functions*, Ann. Acad. Sci. Fen. Math. **23** (1998), no. 2, 225–250.

[EL87] A. Eremenko and M. Lyubich, *Examples of entire functions with pathological dynamics*, J. London Math. Soc. (2) **36** (1987), no. 3, 458–468.

[EL92] ______, *Dynamical properties of some classes of entire functions*, Ann. Inst. Fourier **42** (1992), no. 4, 989–1020.

[Ere89] A. Eremenko, *On the iteration of entire functions*, Dynamical systems and ergodic theory (Warsaw, 1986), vol. 23, PWN, Warsaw, 1989, pp. 339–345.

[ERG15] A. L. Epstein and L. Rempe-Gillen, *On invariance of order and the area property for finite-type entire functions*, Ann. Acad. Sci. Fenn. Math. **40** (2015), no. 2, 573–599.

[Fal03] K. J. Falconer, *Fractal geometry: Mathematical foundations and applications*, second ed., John Wiley & Sons, Inc., Hoboken, NJ, 2003.

[Fat26] P. Fatou, *Sur l'itération des fonctions transcendantes entières*, Acta Math. **47** (1926), no. 4, 337–370.

[FM07] A. Fletcher and V. Markovic, *Quasiconformal maps and Teichmüller theory*, Oxford Graduate Texts in Mathematics, vol. 11, Oxford University Press, Oxford, 2007.

[GK86] L. R. Goldberg and L. Keen, *A finiteness theorem for a dynamical class of entire functions*, Ergodic Theory Dynam. Systems **6** (1986), no. 2, 183–192.

[GO08] A. A. Goldberg and I. V. Ostrovskii, *Value distribution of meromorphic functions*, Translations of Mathematical Monographs, vol. 236, American Mathematical Society, Providence, RI, 2008.

[Hay64] W. K. Hayman, *Meromorphic functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.

[HK98] J. Heinonen and P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry*, Acta. Math. **181** (1998), no. 1, 1–61.

[Hub06] J. H. Hubbard, *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 1*, Matrix Editions, Ithaca, NY, 2006.

[KS08] M. Kisaka and M. Shishikura, *On multiply connected wandering domains of entire functions*, Transcendental dynamics and complex analysis, London Math. Soc. Lecture Note Ser., vol. 348, Cambridge University Press, Cambridge, 2008, pp. 217–250.

[LV73] O. Lehto and K. I. Virtanen, *Quasiconformal mappings in the plane*, second ed., vol. 126, Springer-Verlag, New York-Heidelberg, 1973.

[MB12] H. Milhaljević-Brandt, *Semiconjugacies, pinched Cantor bouquets and hyperbolic orbifolds*, Trans. Amer. Math. Soc. **364** (2012), no. 8, 4053–4083.

[McM87] C. T. McMullen, *Area and Hausdorff dimension of Julia sets of entire functions*, Trans. Amer. Math. Soc. **300** (1987), no. 1, 329–342.

[Mil06] J. Milnor, *Dynamics in one complex variable*, Annals of Mathemtaics Studies, vol. 160, Princeton University Press, Princeton, NJ, 2006.

[Nev70] R. Nevanlinna, *Analytic functions*, Translated from the second German edition by Phillip Emig. Die Grundlehren der mathematischen Wissenschaften, Band 162, Springer-Verlag, New York-Berlin, 1970.

[Obr63] N. Obreschkoff, *Verteilung und Berechnung der Nullstellen reeller Polynome*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1963.

[Pet08] J. Peter, *Hausdorff measure of escaping and Julia sets of exponential maps*, Ph.D. thesis, University of Kiel, 2008.

[Pom92] Ch. Pommerenke, *Boundary behaviour of conformal maps*, Grundlehren der mathematischen Wissenschaften, vol. 299, Springer-Verlag, Berlin, 1992.

[Rem09] L. Rempe, *Rigidity of escaping dynamics for transcendental entire functions*, Acta Math. **203** (2009), no. 2, 235–267.

[RG14] L. Rempe-Gillen, *Hyperbolic entire functions with full hyperbolic dimension and approximation by Eremenko–Lyubich functions*, Proc. Lond. Math. Soc. (3) **108** (2014), no. 5, 1193–1225.

[RG16] ______, *Arc-like continua, Julia sets of entire functions, and Eremenko's Conjecture*, arXiv:1610.06278 (2016).

[RGS] L. Rempe-Gillen and D. J. Sixsmith, *Hyperbolic entire functions and the Eremenko-Lyubich class: Class $\mathcal{B}$ or not class $\mathcal{B}$?*, Math. Z., to appear.

[RRRS11] G. Rottenfusser, J. Rückert, L. Rempe, and D. Schleicher, *Dynamic rays of bounded-type entire functions*, Ann. of Math. (2) **173** (2011), no. 1, 77–125.

[RS12] P. J. Rippon and G. M. Stallard, *Fast escaping points of entire functions*, Proc. London Math. Soc. (3) **105** (2012), no. 4, 787–820.

[Sch07] D. Schleicher, *The dynamical fine structure of iterated cosine maps and a dimension paradox*, Duke Math. J. **136** (2007), no. 2, 343–356.

[Sch09] ______, *Dynamics of entire functions*, Complex Dynamics–Families and Friends (D. Schleicher, ed.), A K Peters, Ltd., 2009, pp. 295–339.

[Sta91] G. M. Stallard, *The Hausdorff dimension of Julia sets of entire functions*, Ergodic Theory Dynam. Systems **11** (1991), no. 4, 769–777.

[Ste93] N. Steinmetz, *Rational iteration: Complex analytic dynamical systems*, De Gruyter Studies in Mathematics, vol. 16, Walter de Gruyter & Co., Berlin, 1993.

[Sul85] D. P. Sullivan, *Quasiconformal homeomorphisms and dynamics. I. Solution of the Fatou-Julia problem on wandering domains*, Ann. of Math. (2) **122** (1985), no. 3, 401–418.

[SZ03] D. Schleicher and J. Zimmer, *Escaping points of exponential maps*, J. London Math. Soc. (2) **67** (2003), no. 2, 380–400.

[Tsu75] M. Tsuji, *Potential theory in modern function theory*, Chelsea Publishing Co., New York, 1975.

Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit, abgesehen von der Beratung durch den Betreuer meiner Promotion, unter Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft selbstsändig angefertigt habe und keine anderen als die angegebenen Hilfsmittel verwendet habe. Die Arbeit oder Auszüge wurden bislang noch bei keiner anderen Stelle im Rahmen eines Prüfungsverfahrens vorgelegt oder veröffentlicht. Einige Hauptergebnisse der Dissertation sind in der folgenden Publikation enthalten:

Cui, Weiwei. *Lebesgue measure of escaping sets of transcendental entire functions in the Eremenko-Lyubich class*. Preprint, arXiv: 1608.04600.

Kiel, den 10. 07. 2017

Weiwei Cui