Approximate pricing of barrier options in Lévy models

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Zusammenfassung

In dieser Arbeit betrachten wir einen bestimmten Typ von Derivaten, nämlich Europäische Barriereoptionen. Hierbei fokussieren wir uns auf die Frage der Optionspreisbewertung in geometrischen Lévy-Modellen, welche im Gegensatz zu dem wohlbekannten Black-Scholes-Modell Sprünge im Aktienkursprozess zulassen. Die Bepreisung des obigen Derivats wird dabei dadurch enorm erschwert, dass der Aktienkurs die Barriere nicht notwendigerweise stetig überschreitet, sondern jederzeit über die Barriere springen kann aus verschiedenen Positionen.

Obwohl semi-explizite Lösungen für das Preisproblem in obigen Modellen existieren, sind diese häufig schwer zu berechnen, da diese von den Wiener-Hopf-Faktoren des treibenden Prozesses abhängen, welche in den meisten Modellen von praktischer Relevanz unbekannt sind. Dieser Mangel an qualitativen, einfachenen Preismethoden führt natürlicherweise zu Approximationen. Unsere Approximationsidee interpretiert das Sprungmodell als perturbiertes Black-Scholes-Modell, in dem wir einen Korrekturterm erster Ordnung berechnen.

Während der Berechnung der Approximation ist es notwendig die Option aufzuteilen, wobei die zweite Option eine polynomielle Auszahlung des *overshoots* des zugrundeliegenden Lévy-Prozesses besitzt. Die Approximation für die erste Option besteht aus Momenten des Aktienkursprozesses sowie Sensitivitäten (*cash greeks*) des perturbierten Black-Scholes Derivatepreises.

Die Approximation für die *overshoot*-option besteht aus einem eigenständigen Resultat zur Approximation von Momenten des *overshoot*, welches auf der Fluktuationstheorie für Lévyprozesse gründet. Der Korrekturterm besteht aus einem zweidimensionalen komplexen Integral, das lediglich vom charakteristischen Exponenten des Prozesses abhängt, welches sich effizient numerisch berechnen lässt.

Wir zeigen in einer numerischen Fallstudie in verschiedenen parametrischen Modellen, dass unsere Approximation zufriedenstellende Resultate liefert, falls unser Sprungmodell nahe dem Black-Scholes-Modell ist, in dem Sinne dass die vierten Kumulanten des treibenden Lévy Prozesses nicht zu groß sind, sowie robust und einfach zu berechnen ist.

Abstract

This thesis deals with pricing of a certain type of derivatives, namely European barrier options. We consider the question of pricing this option in geometric Lévy models, which in contrast to the famous Black-Scholes model allow jumps in the stock price. This increases the difficulty of computing an option price enormously due to the fact that the stock price does not necessarily cross the barrier continuously, but is able to jump over it from different space points.

Although there exist semi explicit solutions to the pricing problem in these models, those are often hard to evaluate in practice, as they depend on the so called *'Wiener-Hopf-factors'* of the underlying process, which are unknown for most models of practical relevance. We will give a comparative overview about the different pricing methods. The lack of satisfactory, easy to evaluate pricing concepts in this framework leads naturally to approximative ideas. The main idea of our approach will be the interpretian of the jump model as a perturbed Black-Scholes model, where we compute a first order correction term.

In the process of evaluating the approximation it will be necessary to split up our option into two, one of them paying a polynomial of the overshoot of the underlying Lévy process. The approximation for the first option will consist of moments of the stock price as well as sensitivities (so called *'greeks'*) of the Black-Scholes derivative price.

The approximation for the overshoot option will consist of an independent result on the approximation of overshoot moments using Lévy process fluctuation theory. The correction term consists of a 2-dimensional complex integral formula depending only on the characteristic exponent of the underlying Lévy process, which may be efficiently evaluated numerically.

We show in a numerical illustration for several parametric models used in practice, that our approximation yields good results if the Lévy model is reasonably close to a Black-Scholes model with same volatility in the sense that the fourth order cumulant of the Lévy process should not be too large, yet arguably being robust and simple to evaluate.

Contents

List of Tables

1. Introduction

1.1. Geometric Lévy models and Barrier options

We consider an arbitrage free market (B, S) with a fixed maturity $T > 0$ and two assets, a bank account B which we set constantly equal to 1 and a non dividend paying stock S whose price process is given by

$$S(t) = S(0)e^{X(t)},\ t \in [0, T], \tag{1.1.1}$$

for a Lévy process X living on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, P)$, for which we assume $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ to be the naturally generated filtration of X and $S(0) > 0$. Furthermore, we consider a European up-and-out-barrier option on S, which has the following payoff C:

$$C = h(S(T))1_{\{\sup\limits_{0\leq t\leq T} S(t)<B\}}, \tag{1.1.2}$$

for a function $h : \mathbb{R}^+ \to \mathbb{R}$ and a barrier $S(0) \neq B > 0$.

According to the first fundamental theorem of asset pricing [16], the only reasonable arbitrage free price is given by

$$V(0) = E\left(h(S(T))1_{\{\sup\limits_{0\leq t\leq T} S(t)<B\}}\right), \tag{1.1.3}$$

where the expectation is taken under an equivalent martingale measure for S. To avoid technicalities regarding measure changes, we assume P to be an equivalent martingale measure for S, hence we assume S to be a martingale.

As an example, consider a European up-and-out put option, where we set $h(x) = (K - x)^+$ for a strike $K > 0$. If we take $X = -\frac{\sigma^2}{2}I + \sigma W$, with $\sigma > 0$, I being the identity process and W a standard Brownian motion, we have the special case of the Black-Scholes model, for which the fair option price is well known:

Example 1.1.1 (Up-and-out put in the Black-Scholes model)**.** *The fair price of a European up-and-out put in the Black-Scholes model defined as above with interest rate $r = 0$ is given by:*

$$\begin{aligned} V(0) = K\Phi(-\frac{\ln(\frac{S(0)}{K}) + \frac{\sigma^2}{2}}{\sigma\sqrt{T}}) - S(0)\Phi(-\frac{\ln(\frac{S(0)}{K}) - \frac{\sigma^2}{2}}{\sigma\sqrt{T}}) \\ - \frac{S(0)K}{B}(\Phi(-\frac{\ln(\frac{B^2}{S(0)K}) - \frac{\sigma^2}{2}}{\sigma\sqrt{T}}) - B\Phi(-\frac{\ln(\frac{B^2}{S(0)K}) - \frac{\sigma^2}{2}}{\sigma\sqrt{T}})) \end{aligned} \tag{1.1.4}$$

where Φ is the cumulative distribution function of the standard normal distribution.

This can be shown in various ways, for example via the reflection principle of the Brownian motion [25] or direct integration [45]. However, if we move on from the Black-Scholes framework to geometric Lévy models, things get more complicated mainly due to two factors:

- The price process incorporates jumps, hence it is possible to cross the barrier not only continuously, but from a whole area of space points. This implies that there is no reflection principle at the barrier to use.
- The density of the stock S or the log price X is generally unknown as well as the density of the corresponding running supremum processes $\overline{X}$.

This implies that we need to look for different pricing methods, as we will do in the subsequent chapters.

1.2. Overview on pricing methods

The pricing methods for barrier options in geometric Lévy models can roughly be split up in three different approaches:

1. Integral transforms
2. Monte-Carlo simulation
3. Partial-integro-differential equations

We will discuss each approach and the main advantages and disadvantages separately.

1.2.1. Integral transform approach

For illustrational sake, let us first discuss the case of a European option with payoff $f(X(T))$, $f : \mathbb{R}^+ \to \mathbb{R}$ with respect to the log price $X(T)$. Define the bilateral Laplace transform of f at the point $z \in \mathbb{C}$ to be

$$\widetilde{f}(z) = \int_{\mathbb{R}} e^{-zx} f(x) dx, \tag{1.2.5}$$

assuming $\int_{\mathbb{R}} |e^{-zx} f(x)| dx < \infty$ holds, which implies the existence of (1.2.5) in the Lebesgue-sense. Under certain regularity conditions, e.g. if $\widetilde{f} \in \mathcal{L}^1(\mathbb{R})$, f allows for the representation

$$f(x) = \int_{R+i\mathbb{R}} e^{zx} \widetilde{f}(z) dz, x \in \mathbb{R} \tag{1.2.6}$$

for a suitable $R \in \mathbb{R}$. If we now assume that we are allowed to interchange integration and expectation, this leads to

$$V_0 = E\left(f(X(T)\right) = E\left(\int_{R+i\mathbb{R}} e^{zX(T)} \widetilde{f}(z) dz\right) = \int_{R+i\mathbb{R}} E\left(e^{zX(T)}\right) \widetilde{f}(z) dz \tag{1.2.7}$$

where $E\left(e^{zX(T)}\right) = \kappa^{X(T)}(-z)$ is the analytically extended Laplace exponent of $X(T)$ defined in remark 2.0.2. The main advantage is now that in contrast to the density, this characteristic function is well known for Lévy processes due to the Lévy Khintchine formula 2.0.1 and hence the integral (1.2.7) may be evaluated through numerical integration.

So far so good, but when we get to barrier options, things naturally get more complicated. If we try to mirror this approach, we would like to know whether the characteristic function of the random vector $(X, \overline{X})$ is easily computable. This corresponds to the question whether the Wiener-Hopf factors from theorem 2.0.13 are explicitly known. Let us suppose for the moment that this is the case.

As an example consider again an up-and-out-put option and set $S(0) = 1$ for simplicity. We write its fair price as a function of maturity T, log strike $k := \ln(K)$ and log barrier $b := \ln(B)$:

$$V_0(T,k,b) := E\left((e^k - e^{X(T)})^+ 1_{\left\{\overline{X}(T)<b\right\}}\right) = \int_{\mathbb{R}^2} (e^k - e^x)^+ 1_{\{y<b\}} f_T(x,y) d(x,y), \tag{1.2.8}$$

where $f_T(x,y)$ is the joint density of $(X(T), \overline{X}(T))$. Under regularity conditions we may interchange integrals to calculate the generalized Fourier transform in log strike and log barrier as shown in [13], Chapter 11:

$$\int_{\mathbb{R}^2} e^{iuk+ivb} V_0(T,k,b) d(b,k) = \frac{\psi(u-i,v)}{uv(1+iu)}, \tag{1.2.9}$$

where ψ here denotes the joint characteristic function of $(X(T), \overline{X}(T))$, $\mathrm{Im}\,(u) > 1$, $\mathrm{Im}\,(v) > 0$. Computing the Laplace transform in maturity T using corollary 2.0.14 we conclude that

$$q \int_0^\infty \int_{\mathbb{R}^2} e^{iuk+ivb-qT} V_0(T,k,b) d(b,k) dT = \frac{\psi_q^+(v+u-i)\psi_q^-(u-i)}{uv(1+iu)}. \tag{1.2.10}$$

If we now assume enough regularity for the Lévy process such as that the Fourier- and Laplace inversion exist, we may apply them to compute the option price. However, besides that numerical inversion of a Laplace transform is known to be unstable, this method heavily relies on the fact that the Wiener-Hopf factors have to be known explicitly. A sample case where closed form solutions may be obtained is the Kou jump diffusion model (see [32] for the model and [43] for the barrier option price). However, these closed form solutions still involve some series expansions of functions, which have to be approximated numerically.

1.2.2. Monte-Carlo simulation

The basic idea of the Monte-Carlo simulation for option pricing is to simulate the payoff of the option and use the empirical mean due to the law of large numbers as an approximation for the option price. In our context of barrier options in Lévy models, the efficiency of the method heavily depends on the jump activity of the Lévy process.

If we consider a process with finite jump activity there exists an efficient algorithm to simulate the payoff. We follow [13], chapter 6 here. Consider a jump diffusion model, where our Lévy process has the representation

$$X(t) = \gamma t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i \tag{1.2.11}$$

so that N is a Poisson process with some jump activity α and Y_i i.i.d. jumps with some distribution. If we stick to our example of an up-and-out put, again putting $S(0) = 1$, we want to compute

$$V_0 = E\left((K - e^{X(T)})^+ 1_{\{\overline{X}(T)<b\}}\right). \tag{1.2.12}$$

The main idea of the method is to simulate the jump times of the Poisson process first, and then use it as a stochastic grid for the simulation of the process. Denote by

$$\mathcal{F}^* := \sigma\left\{N(t), 0 \le t \le T, Y_i, W(\tau_i), \tau_i, i = 0, \ldots, n\right\} \tag{1.2.13}$$

where $\tau_i, i \le n, n \in \mathbb{N}$ are the jump times of N where we set $\tau_0 = 0, \tau_n = T$. Because $X(T)$ is $\mathcal{F}^*$-measurable, we may rewrite (1.2.12) as

$$V_0 = E\left((K - e^{X(T)})^+ E\left(1_{\{\overline{X}(T)<b\}} | \mathcal{F}^*\right)\right). \tag{1.2.14}$$

The conditional expectation may be rewritten as

$$E\left(1_{\{\overline{X}(T)<b\}} | \mathcal{F}^*\right) = \prod_{i=1}^{n} P(\overline{X}_i \le b | X(\tau_{i-1}), X(\tau_i -)) \tag{1.2.15}$$

where we set $\overline{X}_i := \sup_{\tau_{i-1} \le t < \tau_i} X(t)$. Now we use the fact that X does not jump in (τ_{i-1}, τ_i) and hence its trajectory in this interval is nothing else than a Brownian bridge. Therefore, using the strong Markov property and the independence of increments, we get

$$\begin{aligned} &P(\overline{X}_i \le b | X(\tau_{i-1}), X(\tau_i -), \tau_{i-1}, \tau_i) \\ &= P(\sup_{0<t<\tau_i - \tau_{i-1}} \sigma W(t) + \gamma t \le b - X(\tau_i -) | A), \end{aligned} \tag{1.2.16}$$

where we define $A := \{\sigma W(\tau_i - \tau_{i-1}) + \gamma(\tau_i - \tau_{i-1}) = X(\tau_i -) - X(\tau_{i-1})\}$. For this expression there exists an analytical solution (e.g. [8]):

$$P_x(\sup_{0 \le s < t} \sigma W(t) + \gamma s < b | \sigma W(t) + \gamma t = z) = 1 - \exp(-\frac{2(z-b)(x-b)}{t\sigma^2}), \tag{1.2.17}$$

where P_x denotes the probability measure with $P(W(0) = x) = 1$ in the Markovian sense. Substituting into (1.2.14) leads to

$$V_0 = E\left((K - e^{X(T)})^+ \prod_{i=1}^{n} 1_{\{X(\tau_{i-}), X(\tau_i) < b\}} (1 - \exp(-\frac{2(X(\tau_{i-} - b)(X(\tau_{i-1}) - b)}{(\tau_i - \tau_{i-1})\sigma^2}))\right) \tag{1.2.18}$$

Thus we simplified the path dependent payoff to a payoff just depending on the jump times of N, which can be evaluated via usual Monte-Carlo simulation. The computational complexity of the algorithm is proportional to the number of jumps n. We will apply this algorithm to generate reference values in jump-diffusion models, see chapter 5.

However, when we consider Lévy processes with infinite jump activity, the above algorithm cannot be generalized directly. For some Lévy process, one is able to at least simulate increments when the process has a representation as a subordinated Brownian motion, e.g. for variance gamma (VG) or normal inverse Gaussian (NIG) processes (e.g. [13], chapter 4). We can then simulate the trajectory on a fixed time grid. However, we pay the price of an additional source of error because we only do a discrete barrier check. When there is no direct simulation method available one can approximate the process by a jump diffusion process putting the small jumps in the Brownian motion part. Here we of course make an additional approximation error. This method has some similarities to our perturbation approach, except that we do not need to do Monte-Carlo simulation for pricing. See [1] for details. If the Wiener-Hopf factors of the process are known, there exists a Monte-Carlo method simulating X on a stochastic grid determined by those factors, see [33].

All in all we can conclude that the Monte-Carlo method is efficient only for jump-diffusion-models.

1.2.3. PIDE-method

The basic idea of this approach is to use the Markovian structure of the stock price process. Let us again consider a European option with payoff $f(S(T))$ first. Because of the Markovian property we may write the (discounted) option price at time $t \in [0,T]$ as a function of the current stock price and time:

$$V(t) = E\left(f(S(T))|\mathcal{F}_t\right) = E\left(f(S(T))|S(t)\right) = v(t, S(t)),\ S(t) > 0,\ t \in [0,T]. \tag{1.2.19}$$

If the function v is smooth enough to apply Ito's formula, e.g. $v \in C^{1,2}$ we may write

$$dV(t) = a(v)(t, S(t))dt + dM(t) \tag{1.2.20}$$

for some drift $a(v)(S(t),t))$, where a is some integro-differential-operator and a martingale term $M(t)$. As the (discounted) option price has to be a martingale under an equivalent martingale measure, the drift part has to vanish and hence $a(v(S(t),t)) = 0$ has to be fulfilled, with the boundary condition $v(S(T),T) = f(S(T))$.

For barrier options, define $H(x) := h(x)1_{\{x<B\}}$ and furthermore the stopping time

$$\tau_B := \inf\{s \leq T : X(s) \geq B\}. \tag{1.2.21}$$

Using the strong Markov property, we see that for $\{t \in [0, \tau_B]\}$ the following holds:

$$\begin{aligned} V(t) &= E\left(h(S(T))1_{\{\sup\limits_{0\leq s\leq T} S(s)<B\}}|\mathcal{F}_t\right) = E\left(H(S(T\wedge\tau_B))|\mathcal{F}_t\right) \\ &= E\left(H(S(T\wedge\tau_B))|\mathcal{F}_t\right) = E\left(H(S(T)\wedge\tau_B)|S(t)\right) =: v(t, S(t)). \end{aligned} \tag{1.2.22}$$

Hence,

$$V(t \wedge \tau_B) = v(t \wedge \tau_B, S(t \wedge \tau_B)). \tag{1.2.23}$$

If we want to apply the same mechanism as in the European case, we have to show that $v(t,x) \in C^{1,2}$, especially for $x = B$. Here one needs certain regularity assumptions on the Lévy process, for example a nonzero diffusion coefficient. If this holds, one gets to the following partial differential equation:

Theorem 1.2.1 (PIDE for up-and-out put)**.** *Let (b, c, K) be the characteristic triple of X as in Theorem 2.0.1, with $c > 0$. Then the pricing functional of an up-and-out-put is the unique function $v : ([0,T] \times \mathbb{R}^+) \to \mathbb{R}^+$ that is continuous on $[0,T] \times \mathbb{R}^+$, $v \in C^{1,2}$ on $(0,T] \times \mathbb{R}^+$ and satisfies*

$$\begin{aligned}
&\forall (t,x) \in [0,T) \times (0,B) : \frac{\partial v}{\partial t}(t,x) + \frac{cx^2}{2}\frac{\partial^2 v}{\partial x^2}(t,x) \\
&+ \int (v(t, xe^y) - v(t,x) - x(e^y - 1)\frac{\partial v}{\partial x}(t,x))K(dy) = 0. \\
&\forall x \in (0,B) : v(T,x) = (K-x)^+. \\
&\forall x \geq B, \forall t \in [0,T] : v(t,x) = 0.
\end{aligned} \tag{1.2.24}$$

Proof. See [5] Chapter 3. □

Remark 1.2.2. *1. In the case $c = 0$, smoothness on the barrier can fail to hold. Solutions exist usually only in the viscosity sense, see [14] for more details.*

2. If one admits a change of variables $u(\tau, x) = v(T - \tau, S(0)e^x)$ with $\tau = T - t$, the PIDE may be rewritten in a more Markovian style equation:

$$\begin{aligned}
&\forall (\tau, x) \in [0,T) \times (0, \log(\frac{B}{S(0)})) : \frac{\partial u}{\partial \tau}(\tau, x) = \mathcal{L}(u)(\tau, x). \\
&\forall x \in (0, \log(\frac{B}{S(0)})) : u(0,x) = (K - S(0)e^x)^+. \\
&\forall x \geq \log(\frac{B}{S(0)}), \forall t \in [0,T] : u(\tau, x) = 0.
\end{aligned} \tag{1.2.25}$$

where

$$\mathcal{L}(f)(\tau, x) := \lim_{t \to 0} \frac{E\left(f(\tau, x + X(t))\right) - f(\tau, x)}{t} \tag{1.2.26}$$

is the infinitesimal generator of X defined for all functions f so that the above limes is well defined (see [20] for details). As e^X is a martingale, it can be shown (e.g [20], chapter 8) that

$$\begin{aligned}
\mathcal{L}(f)(\tau, x) = &\frac{c}{2}\big(\frac{\partial^2 f}{\partial x^2}(\tau, x) - \frac{\partial f}{\partial x}(\tau, x)\big) \\
&+ \int_{\mathbb{R}} (f(\tau, x+y) - f(\tau, x) - (e^y - 1)\frac{\partial f}{\partial x})(\tau, x)K(dy).
\end{aligned}$$

Analytical solutions for this PIDE are usually hard to come by. In practice one uses numerical schemes like finite differences to solve these equations. In comparison to the computation of PDE's from diffusion models, the integro term provides an additional challenge because of its non locality. Hence, we have to split up the integral operator which leads to an additional source of instability. See [13] chapter 12 and [24] for a comprehensive overview over the different numerical methods available to solve these equations and its advantages and drawbacks.

This short overview shows that the Holy Grail for pricing barrier options in geometric Lévy models has yet to be found, hence an approximation approach makes a lot of sense.

1.3. Approximations in Mathematical Finance

"In regione caecorum rex est luscus"(Adagia, Desiderius Erasmus (1500)).[1]

The first natural question that arises when talking about approximations to a problem is why would you even consider an approximation in the first place. There seem to be several reasons to do so.

- **Complexity**: Sometimes the original problem just might be too difficult to solve, hence we use an approximation as a way out.
- **Computational speed**: Even if the problem is solvable, the computation of the solution might involve complex numerical algorithms, which are too time consuming. In practice, one often prefers simple and easy to implement approximations.
- **Structural insight**: Sometimes, even when we solve a problem analytically, the obtained formulas do not offer much insight in the structurally most important factors of the equation. Approximations typically offer more insight in the relevant structure of the problem and reveal the main driving factors.

In the context of Mathematical Finance, the most used way to apply an approximation seems to be a perturbation approach. We consider a complex problem as a perturbation of a simpler problem, usually in the Black-Scholes model:

- **Portfolio optimization and utility indifference pricing and hedging under transaction costs**
  The idea of utility indifference pricing is to set the initial option price in such a way that optimal trading with or without the option should lead to the same expected utility of terminal wealth. The corresponding optimal trading strategy with the obligation to deliver the option payoff at maturity may be used as a hedging strategy. Some problems as well as the choice of an optimal portfolio under transaction costs lead usually to free boundary value problems, which are

[1]In the valley of the blind, the one-eyed man is king

hard to solve even numerically. Therefore, asymptotic expansions with respect to the size ϵ of the transaction costs were considered in the Black-Scholes model by [48] for utility indifference pricing and [31] for portfolio optimization. For a recent overview over proportional transaction costs consider [29], whereas [21] deals with fixed transaction costs.

- **Hedging errors**
  In incomplete market models, for an option with payoff H there is usually no perfect replicating portfolio ϕ so that its value process V_ϕ equals H at maturity T. Therefore it is of interest to study the hedging error $H - V_\phi(T)$ and especially the *mean squared hedging error* $E\left((H - V_\phi(T))^2\right)$. The main focus in the literature lies on the effect of hedging on discrete time steps of small step size Δ_t. We refer to [46] for a first order expansion of delta-hedging in the Black-Scholes model and to [47] for mean squared hedging errors in geometric Lévy models.

- **No arbitrage option pricing**
  For no arbitrage option prices in stochastic volatility models, expansions with respect to the local volatility process are considered in the literature, e.g. [23] and [39].

1.4. Perturbation approach for Lévy models

The basic idea of perturbation is to consider a difficult problem as a small deviation from a simpler one, which is easier to calculate. An approximation is then typically obtained by the solution of the simpler problem plus a correction term, where the amount of deviation is quantified by a small parameter, let us say ϵ. The simple problem is usually stated in the Black-Scholes world. The question that arises is how to quantify the deviation ϵ? Sometimes the problem itself naturally inhabits such a parameter as was shown in the previous chapter.

In our case of no arbitrage option pricing, the choice of such a natural parameter is not obvious, as we do not want to restrict ourselves to a certain parametric class of Lévy processes. Thus we introduce an artificial parameter $\lambda \in [0, 1]$, where the closest interpretation for λ that comes to mind would be the size of jumps. This allows us to connect our original stock price process S through a curve S^λ with a Black-Scholes model, where:

- $\lambda = 1$ corresponds to our original stock price.
- $\lambda = 0$ corresponds to a Black-Scholes price process with the same volatility as the original proces.
- $\lambda \in (0, 1)$ corresponds to some kind of interpolation, which will be specified in the next chapter.

This family of stock processes S^λ now leads to a new family of quantities $q(\lambda)$, $\lambda \in [0, 1]$, which are in our case barrier option prices. If $q(\lambda)$ is differentiable in λ, we consider a first order expansion of q as an approximation for our barrier option price V:

$$V = q(1) \approx q(0) + q'(0). \tag{1.4.28}$$

We call $\mathcal{A} := q(0) + q'(0)$ the first order approximation of V.

1.5. Related literature and discussion of the approach

Although perturbation ideas to the Black-Scholes world are well documented in the literature, in this specific context they are relatively new. The idea to connect the price processes through a curve using an artificial parameter was first used in the series of papers [2],[3],[4], where approximations for options in local volatility models were considered. For geometric Lévy models, [11] used this approach first to approximate hedging problems. [17] considered no arbitrage option pricing for European options in a more general setup of time changed Lévy models, including the class of geometric Lévy models. [38] determined approximations for utility indifference pricing in geometric Lévy models, whereas [28] considered volatility option pricing and [22] deals with approximate option pricing in the Lévy Libor model.

The structure of the papers is generally the following.

1. Find some suitable representation of $q(\lambda)$ and show smoothness in λ.
2. Calculate the corresponding derivatives $q^{(n)}(0), n \in \mathbb{N}$.
3. Show via numerical illustration that the approximation error is reasonably small for practical purposes.

The first step will usually be carried out through some integral transform method, leading to some representation of $q(\lambda)$ in the Fourier-space. The increments of the approximation usually consist of moments of the driving Lévy process and some derivatives of Black-Scholes functionals, for which analytical formulas exist. The third step will be necessary as explicit and reasonably tight error bounds are hard to come by. [35] gives some error bounds for European option prices depending on the cumulants of the underlying Lévy process, but they lack tightness for practical purposes.

In our case of barrier options, it turns out that Step 1 is already quite challenging due to the structure of the barrier payoff, which does not combine well with Fourier methods. As the option prices lack smoothness in λ, it will be necessary to split up our payoff into two separate options

$$\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 \tag{1.5.29}$$

where $\mathcal{C}_1$ corresponds to a slightly modified option, which we can treat with similar methods as in [38]. The price we have to pay is an option $\mathcal{C}_2$ with a polynomial payoff of the *overshoot* of S over the barrier B, where a first order approximation will be obtained with Lévy process fluctuation theory methods.

There arise the natural questions in which order the approximation should be applied and whether the function $q(\lambda)$ might be developed into a power series $q(\lambda) = \sum_{k=1}^{\infty} \frac{q^k(\lambda)}{k!}$. The general consent in the papers above seems to be that an approximation of order two offers a good balance between accuracy and computability. In our scenario the lack of expliciteness in the representation of $q(\lambda)$ shifts this balance to a first-order approximation. The second question was discussed in [35] and [28], where it turned out that the necessary conditions for such a result to hold are way too strict to be fulfilled by any Lévy model of practical relevance.

2. Preliminaries on Lévy process fluctuation theory

In this chapter we will do a brief recapitulation on some theory about Lévy processes and develop the notation which we will use later in this thesis. A special focus is laid on the fluctuation theory involving Wiener-Hopf factorization. For further reading we refer to the monographs of Sato for Lévy processes [44] and Kypreanou for fluctuation theory [34]. For general background on stochastic processes, we refer to the monograph of Jacod and Shiryaev [26].

Let ψ be the characteristic exponent of a one-dimensional Lévy process X defined by $E\left(e^{iuX(t)}\right) = e^{t\psi(u)},\ t \in \mathbb{R}^+$. The well known Lévy-Khintschin-formula yields the following representation for the exponent.

Theorem 2.0.1 (Lévy-Khintschin-formula). *Let X be a Lévy process, $h : \mathbb{R} \to \mathbb{R}$ be a truncation function, i.e. a bounded measurable function with $h(x) = x$, $x \in U$, U a neighbourhood of 0. Then $\psi : \mathbb{R} \to \mathbb{C}$ admits the representation*

$$\psi(u) := iub(h) - \frac{1}{2}u^2c + \int_{\mathbb{R}} (e^{iux} - 1 - iuh(x))K(dx) \tag{2.0.1}$$

with $b(h) \in \mathbb{R}$, $c \geq 0$ and K a so-called Lévy-measure, i.e. a measure on $\mathbb{R}$ which satisfies $K(\{0\}) = 0$ and $\int_{\mathbb{R}}(1 \wedge |x^2|)K(dx) < \infty$. We call $(b(h), c, K)$ the characteristic triple of X, it is unique modulo the choice of the truncation function h. The standard truncation function is $h(x) := x1_{\{|x|\leq 1\}}$.

Proof. [44], Theorem 8.1 $\square$

Remark 2.0.2. *1. It is well known (cf. [44], Theorem 25.3) that*

$$\int_{\mathbb{R}} (x1_{\{|x|>1\}})K(dx) < \infty \Leftrightarrow E\left(|X(1)|\right) < \infty.$$

As this will be the case throughout our thesis, we may use $h = id$ as the truncation function and $b := b(id) = E\left(X(1)\right)$. Hence, by characteristic triple we will mean the triple (b, c, K).

2. The characteristic function ψ may be extended to

$$D_X := \left\{ z \in \mathbb{C}, \int e^{\Re(z)x} K(dx) < \infty \right\},$$

(cf. [44], Proof of Theorem 8.1). Sometimes it will be more convenient for us to work with the Laplace exponent of X. We define $\kappa(z) = \psi(iz)$, $-z \in D_X$.

Next we will gather some facts about subordinators.

Definition 2.0.3. *A subordinator Y is an almost surely non decreasing Lévy process. Let Y be a subordinator and T_η be an independent, exponentially distributed random variable with parameter $\eta \geq 0$, (for $\eta = 0$ set $T_\eta = \infty$). Then a killed subordinator with killing rate η is the process*

$$X(t) := \begin{cases} Y(t) \text{ if } t < T_\eta \\ \delta \text{ if } t \geq T_\eta, \end{cases} \tag{2.0.2}$$

where δ is a 'graveyard state'.

The Laplace exponent κ of a killed subordinator X has the representation

$$\kappa(z) = \eta - dz + \int (1 - e^{zx}) K(dx) \tag{2.0.3}$$

for $z \in \left\{z \in \mathbb{C},\ E\left(e^{-zX(1)}\right) < \infty\right\}$with $d \geq 0$ and $(\int (1 \wedge x) K(dx)) < \infty$ (cf. [34], Chapter 5).

Definition 2.0.4. *Let X be a subordinator, $q \geq 0$. Then we define the q-potential measure of X as*

$$\mathcal{U}^q(A) := \int_0^\infty P(X(t) \in A) dt = \int_0^\infty e^{-qt} P(X(t) \in A) dt, \tag{2.0.4}$$

for Borel sets A. We set $\mathcal{U} := \mathcal{U}^0$.

Note that, for a killed subordinator with killing rate η the q-potential measure is equal to the $(q + \eta)$-potential measure of the corresponding non killed subordinator. This can be shown by computing the Laplace transform of both measures (see [34] Lemma 5.2). The potential measure measures the average time a process spends at a certain level and will be an important tool in the analysis of undershoots and overshoots. The next corollary concerns the limiting behaviour of potential measures. Therefore we have to exclude the case that the potential measure lives on a lattice grid: We say that the support of a real-valued measure Π is lattice, if there exists a discrete set of points $A = \{a + h_i n_i\}$, where $a \in \mathbb{R},\ h_i > 0, n_i \in \mathbb{Z}, i \in \mathbb{Z}$, such that $\Pi(\mathbb{R}) = \Pi(A)$. It can be shown that, if the Lévy measure K has no lattice support, the potential measures $\mathcal{U}^q$ also do not have a lattice support for $q \geq 0$ (cf. [34], Theorem 5.4).

Theorem 2.0.5. *Let X be a subordinator (no killing) with $\mu := E(X(1)) < \infty$.*

1. *If $\mathcal{U}$ has no lattice support, then for $y > 0$,*

$$\lim_{x \uparrow \infty} (\mathcal{U}([0, x + y]) - \mathcal{U}([0, x])) = \frac{y}{\mu}. \tag{2.0.5}$$

2.

$$\lim_{x \uparrow \infty} \frac{\mathcal{U}([0, x])}{x} = \frac{1}{\mu}. \tag{2.0.6}$$

Proof. This follows from renewal theory for random walks, (cf. [6], pp. 38 and 74). □

Define

$$\tau_x^+ := \inf\{t > 0, X(t) > x\}. \tag{2.0.7}$$

In the following we denote by $O(x) := X(\tau_x^+) - x$ the overshoot of X over a fixed level $x > 0$ and $U(x) := x - X(\tau_x -)$ the undershoot of X over the same level. For a killed subordinator the joint distribution is given by the following theorem:

Theorem 2.0.6. *Let X be a killed subordinator. Then for $A \in \mathcal{B}(\mathbb{R}^+)$ and $B \in \mathcal{B}([0, x])$:*

$$P(O(x) \in A, U(x) \in B) = \mathcal{U}(x - B)K(A + B) \tag{2.0.8}$$

Proof. [34], Theorem 5.6. □

The intuition behind this is, that in order to achieve an undershoot of size y and an overshoot of size u over the level x, the process has to be at level $x - y$ which is measured by the potential measure, and perform a jump of size $y + u$. This result will later be used to determine the asymptotic overshoot distribution in our approximation setting.

The next object to discuss for us will be the running supremum process

$$\overline{X}(t) := \sup_{0 \le s \le t} X(s), t \ge 0 \tag{2.0.9}$$

and the running infimum

$$\underline{X}(t) := \inf_{0 \le s \le t} X(s), t \ge 0. \tag{2.0.10}$$

Note that these processes are no longer Lévy processes (but still Markovian). However, there exists a time change ν, so that the time changed running supremum $\overline{X}_\nu$ is a (possibly killed) subordinator. The structure of the time change depends on the short time behaviour of the Lévy process. We say that 0 is *regular* for an open or closed set B, if $P(\tau^B = 0) = 1$, for $\tau^B := \inf\{t > 0 : X(t) \in B\}$.

Definition 2.0.7 (Local time at the maximum (continuous version)). *A continuous, non decreasing, $[0, \infty)$-valued, $(\mathcal{F}_t)_{t \in \mathbb{R}^+}$-adapted process $L = \{L(t) \, : \, t \ge 0\} \neq 0$ is called a continuous local time at the maximum (or just local time for short) if the following hold.*

1. *The support of the Stiltjes measure $dL(t)$ is the closure of the (random) set of times $\left\{t \ge 0 \, : \, \overline{X}(t) = X(t)\right\}$.*

2. *For every $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$-stopping time τ such as that $\overline{X}(\tau) = X(\tau)$ on $\{\tau < \infty\}$ almost surely, the shifted trivariate process*

$$\left\{\left(X(\tau+t) - X(\tau), \overline{X}(\tau+t) - X(\tau+t), L(\tau+t) - L(\tau)\right) : t \geq 0\right\}$$

is independent of $\mathcal{F}_\tau$ on $\{\tau < \infty\}$ and has the same law as $\left(X, \overline{X} - X, L\right)$ under P.

Definition 2.0.8 (Local time at the maximum (right-continuous version))**.** *Let $N = N(t), t \in \mathbb{R}^+$ be the counting process defined via*

$$N(t) = |\left\{0 \leq s \leq t : \overline{X}(s) = X(s)\right\}|. \tag{2.0.11}$$

Let $\{T^i_\lambda, i \in \mathbb{N}\}$ be independent exponential distributed random times with some parameter $\lambda > 0$. Define the right continuous version of the local time L via

$$L(t) = \sum_{i=1}^{N(t)} T^i_\lambda, t \in \mathbb{R}^+. \tag{2.0.12}$$

To ensure that L is adapted, define $(\mathcal{G}_t)_{t\in\mathbb{R}^+} ;= (\mathcal{F}_t)_{t\in\mathbb{R}^+} \cup \sigma(\{T^i_\lambda, i \in \mathbb{N}\})$ and use this as our new filtration from now on.

Remark 2.0.9.

- *Local times may be defined only up to a multiplicative constant. A common normalization is to take the local time so that*

$$E\left(\int_0^\infty e^{-t} dL(t)\right) = 1, \tag{2.0.13}$$

(cf. [12]). We will use this normalization from now on.

- *Note that we have a finite limit $L(\infty)$, if and only if the process drifts to $-\infty$.*

Intuitively speaking, a local time is a process where something happens only on $\left\{t \geq 0 \ : \ \overline{X}_t = X_t\right\}$. The following theorem provides the connection between regularity in 0 and local times.

Theorem 2.0.10. *Let X be any Lévy process.*

1. *There exists a continuous local time at the maximum if and only if 0 is regular for $[0, \infty)$.*
2. *If 0 is not regular for $[0, \infty)$, the set $\left\{0 < s \leq t : \overline{X}_t = X_t\right\}$ is finite for all $t > 0$ and there exists a right continuous version of the local time as in* (2.0.12)

Proof. For the first part see [7] and for the second part [34] Chapter 6.1. □

Definition 2.0.11 (Ladder process)**.** *Define the inverse local time process* $L^{-1} = \{L(t)^{-1} \ : \ t \geq 0\}$ *where* $L(t)^{-1} := \inf\{s > 0 \ : \ L(s) > t\}$*. The process* $H = \{H(t) \ : \ t \geq 0\}$ *is defined via*

$$H(t) := \begin{cases} X(L(t)^{-1}) \text{ for } t < L(\infty) \\ \infty \text{ otherwise} \end{cases} . \tag{2.0.14}$$

The (ascending) ladder process is defined as the bivariate process (L^{-1}, H)*.*

Similarly one may define the *(descending) ladder process* constructed from $-X$. One can show that both $L(t)^{-1}$ and $L(t-)^{-1}$ are stopping times (cf. [34] Lemma 6.9). The important link between ladder process and subordinators is given in the following theorem.

Theorem 2.0.12. *Let* X *be a Lévy process and* T_q *an independent and exponentially distributed random variable with parameter* $q \geq 0$*. Then*

$$P\left(\limsup_{t\uparrow\infty} X_t < \infty\right) \in \{0, 1\} \tag{2.0.15}$$

and the process (L^{-1}, H) *satisfies the following properties*

1. *If* $P\left(\limsup_{t\uparrow\infty} X(t) < \infty\right) = 0$*, then* (L^{-1}, H) *has the law of a bivariate subordinator.*
2. *If* $P\left(\limsup_{t\uparrow\infty} X(t) < \infty\right) = 1$*, then for some* $q > 0$*,* $L(\infty) = T_q$ *in distribution and the process* (L^{-1}, H) *has the law of a killed bivariate subordinator with killing rate* q*.*

Proof. [34], Theorem 6.10. □

It follows from the law of large numbers that the limiting behaviour of X depends on the sign of the expectation $E(X(1))$, if it is well defined. We can now define the (analytically extended) Laplace exponent function for $\alpha, \beta \in \mathbb{C}^+$:

$$e^{-\kappa(\alpha,\beta)} := E\left(e^{-\alpha L(1)^{-1} - \beta H(1)} \mathbf{1}_{\{1 < L(\infty)\}}\right) \tag{2.0.16}$$

Using a multidimensional version of formula (2.0.3) we get the representation

$$\kappa(\alpha, \beta) = q + \alpha a + \beta b + \int_{(0,\infty)^2} (1 - e^{-\alpha x - \beta y}) \Gamma(dx, dy), \tag{2.0.17}$$

for $q \geq 0$ as in Theorem 2.0.12, a being the drift of the inverse local time, b being the drift of the ladder process and $\Gamma(dx, dy)$ the Levy measure of (L^{-1}, H).

Similarly, one defines the (analytically extended) Laplace exponent $\widetilde{\kappa}(\alpha,\beta)$ for the descending ladder process. Furthermore we set

$$\overline{G}(t) := \sup\left\{s < t : \overline{X}(s) = X(s)\right\} \tag{2.0.18}$$

and

$$\underline{G}(t) := \sup\left\{s < t : \underline{X}(s) = X(s)\right\}. \tag{2.0.19}$$

Note that for technical reasons we exclude compound Poisson processes from the following formula, as we do not want our Lévy process to reach the same maximum at two distinct ladder times. Now we may present the key formula of this chapter:

Theorem 2.0.13 (Wiener-Hopf-factorization). *Let X be a Lévy process which is not a compound Poisson process and $\boldsymbol{e}_q$ an independent and exponentially distributed random variable with parameter $q \geq 0$.*

1. *The vectors $(\overline{G}_{\boldsymbol{e}_q}, \overline{X}_{\boldsymbol{e}_q})$and $(\boldsymbol{e}_q - \overline{G}_{\boldsymbol{e}_q}, \overline{X}_{\boldsymbol{e}_q} - X_{\boldsymbol{e}_q})$ are independent and infinitely divisible, yielding the factorisation*

$$\frac{q}{q - i\nu + \psi(\theta)} = \psi_q^+(\nu,\theta)\psi_q^-(\nu,\theta)\ \nu,\theta \in \mathbb{R}, \tag{2.0.20}$$

where ψ is the characteristic exponent of X,

$$\psi_q^+(\nu,\theta) = E\left(e^{i\nu\overline{G}_{e_q} + i\theta\overline{X}_{e_q}}\right) \tag{2.0.21}$$

$$\psi_q^-(\nu,\theta) = E\left(e^{i\nu\underline{G}_{e_q} + i\theta\underline{X}_{e_q}}\right). \tag{2.0.22}$$

$\psi_q^+(\nu,\theta)$ and $\psi_q^-(\nu,\theta)$ are called Wiener-Hopf factors.

2. *The Wiener-Hopf-factors may be obtained from the Laplace exponents of the ascending and descending ladder processes for $\alpha,\beta \in \mathbb{C}^+$:*

$$E\left(e^{-\alpha\overline{G}_{e_q} - \beta\overline{X}_{e_q}}\right) = \frac{\kappa(q,0)}{\kappa(q+\alpha,\beta)} \tag{2.0.23}$$

$$E\left(e^{-\alpha\underline{G}_{e_q} - \beta\underline{X}_{e_q}}\right) = \frac{\widetilde{\kappa}(q,0)}{\widetilde{\kappa}(q+\alpha,\beta)} \tag{2.0.24}$$

3. *The Laplace exponents $\kappa(\alpha,\beta)$ and $\widetilde{\kappa}(p+\alpha,\beta)$ may also be obtained from the law of X due to*

$$\kappa(\alpha,\beta) = k\exp\left(\int_0^\infty \int_{(0,\infty)} (e^{-t} - e^{-\alpha t - \beta x})\frac{1}{t}P^{X(t)}(dx)dt\right) \tag{2.0.25}$$

and

$$\widetilde{\kappa}(\alpha,\beta) = \widetilde{k}\exp\left(\int_0^\infty \int_{(-\infty,0)} (e^{-t} - e^{-\alpha t - \beta x})\frac{1}{t}P^{X(t)}(dx)dt\right), \tag{2.0.26}$$

4. *By taking limits as q tends to zero and using all previous equations, the characteristic exponent factorizes into the Laplace exponents:*

$$\psi(\theta) = \kappa(0, -i\theta)\widetilde{\kappa}(0, i\theta) \tag{2.0.27}$$

Proof. [34] Theorem 6.16 □

We finish this section with two corollaries, one concerning the joint Laplace transform of the Wiener-Hopf factors and the other one the joint Laplace transform of the overshoot and first hitting time of a Lévy process, which will be especially useful in Chapter 4.

Corollary 2.0.14. *The Laplace transform in time t of the joint characteristic function of $(\overline{X}, \overline{X} - X)$ is given by*

$$p \int_0^\infty e^{-pt} E\left(e^{ix\overline{X}(t)+iy(\overline{X}(t)-X(t))}\right) dt = \psi_p^+(0, x)\psi_p^-(0, y), \tag{2.0.28}$$

for any $q > 0, x, y \in \mathbb{R}$.

Proof. [44] Theorem 45.7 □

Corollary 2.0.15. *For a Lévy process other than a subordinator it holds that*

$$\int_0^\infty e^{-qx} E\left(e^{-\alpha\tau_x^+ - \beta(X(\tau_x^+)-x)} 1_{\{\tau_x^+<\infty\}}\right) dx = \frac{\kappa(\alpha, q) - \kappa(\alpha, \beta)}{(q - \beta)\kappa(\alpha, q)}. \tag{2.0.29}$$

Proof. [34], Exercise 6.8. □

3. Approximation for barrier options

In this chapter, we will introduce the mathematical setup we are working with, specify our approximation curve for the perturbation approach and derive our first-order approximation.

3.1. Mathematical Setup

3.1.1. Market model

We consider a market (S^0, S) consisting of two traded assets, a non-dividend paying stock and a bond, where we set $S^0 \equiv 1$. The stock price process S will be given as

$$S(t) = S(0)e^{X(t)},\ t \in \mathbb{R}^+, \tag{3.1.1}$$

where X is a Lévy process with càdlàg paths with $X(0) = 0$, living on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, P)$, where we assume $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ to be the completed naturally generated filtration of X and $S(0) > 0$.

Assumption 3.1.1 (Moment conditions on the Lévy process)**.** *We assume that*

1. $E\left(e^{X(1)}\right) = 1$.
2. $\operatorname{Var}(X(1)) > 0$.
3. $E\left(e^{6X(1)}\right), E\left(e^{-6X(1)}\right) < \infty$.

As we want to apply martingale modelling, the first assumption ensures that the discounted stock price process S is a martingale relative to P. The second assumption avoids the degenerate case. The third assumption provides us with enough integrability for the forthcoming analysis.

3.1.2. Barrier option pricing

We consider a fixed European up-and-out barrier option on S:

Definition 3.1.2 (Up-and-out barrier option)**.** *A European up-and-out barrier option is defined by a barrier $B > S(0)$, maturity $T > 0$ and a measurable payoff function $h : \mathbb{R}^+ \to \mathbb{R}$ through its combined payoff*

$$\mathcal{C} = h(S(T))1_{\{\sup\limits_{0\le t\le T} S(t)<B\}}. \tag{3.1.2}$$

As S is a martingale relative to P, the first fundamental theorem of asset pricing implies that a reasonable no-arbitrage price is given by

$$V = E\left(h(S(T))1_{\{\sup\limits_{0\leq t\leq T} S(t)<B\}}\right). \tag{3.1.3}$$

Remark 3.1.3.
- *As the precise definition of the term "no arbitrage" is quite delicate in continuous time models, we do not want to enter this minefield in this thesis and just refer to [16] for further details.*
- *As markets driven by a geometric Lévy process are generally incomplete, they allow for infinitely many different martingale measures. We will assume that the right choice of martingale measure P has been made beforehand. See [13], Chapter 10 for concepts on choosing the martingale measure P.*

We further note that, if we consider an up-and-out option, it is possible to define the corresponding up-and-in barrier options through

$$\widetilde{\mathcal{C}} = h(S(T))1_{\{\sup\limits_{0\leq t\leq T} S(t)\geq B\}}. \tag{3.1.4}$$

Using the identity

$$\widetilde{\mathcal{C}} = h(S(T)) - \mathcal{C}, \tag{3.1.5}$$

this option splits into a European option and an up-and-out-option. Hence if we are able to price the corresponding European option e.g. via integral transform method as in section 1.2.1 , this naturally leads to a price of up-and-in options.

Assumption 3.1.4 (Smoothness condition on the payoff function)**.** *We assume that*

1. *h is in $C^6(\mathbb{R}^+,\mathbb{R})$ and all six derivatives of h are bounded.*
2. *$h^{(n)}(B) = 0$ for $n = 0, \dots, 6$.*

Remark 3.1.5. *Note that this condition on h corresponds to the smoothness condition for European payoff functions in [17]. In [38], Theorem 2, there was an alternative assumption that either $c > 0$ or there exists $\beta \in (0,2)$ so that $\liminf_{r\downarrow 0} \frac{\int_{[-r,r]} x^2 K(dx)}{r^{2-\beta}} > 0$ plus weaker regularity on the payoff function. In our case, this regularity was difficult to achieve due to a lack of a good representation of the barrier option price in the Fourier space. See [38], Lemma 5 for details.*

3.2. Perturbation curve

Now we want to connect our geometric Lévy process with a geometric Brownian motion in a reasonable manner. We will stick to the choice of [17] here. The connection will be performed via a suitable time change, shift and rescaling of the original process. For $\lambda \in (0,1]$, we define a family of Lévy processes X^λ via

$$X^\lambda(t) := \lambda X(\frac{t}{\lambda^2}) - \frac{1}{\lambda^2}\kappa(-\lambda)t \tag{3.2.6}$$

with κ being the Laplace exponent of $X(1)$. We obtain the following properties for the family $(X^\lambda)_{\lambda\in(0,1]}$:

Lemma 3.2.1. *For all $\lambda \in (0,1]$ the following holds:*

1. *X^λ is a Lévy process with càdlàg paths such as that $E\left(e^{X^\lambda(1)}\right) = 1$ and* $\operatorname{Var}\left(X^\lambda(1)\right) > 0$.
2. *For all $t \in \mathbb{R}^+$: $\operatorname{Var}\left(X^\lambda(t)\right) = \operatorname{Var}(X(t)) = t\operatorname{Var}(X(1))$.*
3. *It holds, that*

$$\begin{aligned} D^\lambda &:= \left\{z \in \mathbb{C} : E\left(e^{-\Re(z)X^\lambda(1)}\right) < \infty\right\} \\ &\subset D := \left\{z \in \mathbb{C} : E\left(e^{-\Re(z)X(1)}\right) < \infty\right\}. \end{aligned}$$

4. *The characteristic triple $(b^\lambda, c^\lambda, K^\lambda)$ of X^λ (with respect to the truncation function id) is given by:*

$$b^\lambda = -\frac{c}{2} - \int (e^x - 1 - x)K^\lambda(dx) \tag{3.2.7}$$

$$c^\lambda = c \tag{3.2.8}$$

$$K^\lambda(dx) = \frac{1}{\lambda^2}K(d\lambda x), x \in \mathbb{R}. \tag{3.2.9}$$

Hence, the Laplace exponent $\kappa^\lambda : D^\lambda \to \mathbb{C}$ of X^λ is given by

$$\kappa^\lambda(z) = -\frac{1}{\lambda^2}\kappa(\lambda)z - \frac{1}{\lambda^2}\kappa(\lambda z). \tag{3.2.10}$$

Proof. The fourth statement follows directly from [44], Corollary 8.3 and Proposition 11.10, where the formulas of the characteristic triples of time-changed and linear transformed Lévy processes are stated. The other properties can be deduced of from the characteristic triple. □

For $\lambda = 0$ the expression 3.2.6 does not make much sense, but we obtain a Brownian motion with the correct drift in the limit (cf. [17], Lemma 5.2.7):

Lemma 3.2.2. *For $\lambda \to 0$ the family $(X^\lambda)_{\lambda\in(0,1]}$ of Lévy processes converges in law with respect to the Skorokhod topology (cf. [26], Section VI.1 for details) to a Brownian motion:*

$$X^\lambda \to -\frac{1}{2}\operatorname{Var}(X(1))\, I + \sqrt{\operatorname{Var}(X(1))}W \text{ for } \lambda \to 0, \tag{3.2.11}$$

where I denotes the identity process I(t)=t and W is a standard Brownian motion.

Proof. Lemma 3.2.1 yields directly that

$$\lim_{\lambda\to 0} e^{\kappa^\lambda(z)} = e^{-\frac{1}{2}\mathrm{Var}(X(1))z+\frac{1}{2}\mathrm{Var}(X(1))z^2}, z \in i\mathbb{R}. \tag{3.2.12}$$

Lévy's continuity theorem (cf.[44], Proposition 2.5 .vii) implies that the univariate marginals of X^λ converge to the univariate marginals of

$$-\frac{1}{2}\mathrm{Var}\left(X(1)\right) I + \sqrt{\mathrm{Var}\left(X(1)\right)} W$$

for a standard Brownian motion W. As all processes are Lévy processes, [26], Corollary VII.3.6 implies convergence of the whole process with respect to the Skorokhod topology. □

Hence, we denote the limiting process by

$$X^0 := -\frac{1}{2}\mathrm{Var}\left(X(1)\right) I + \sqrt{\mathrm{Var}\left(X(1)\right)} W, \tag{3.2.13}$$

where, without loss of generality, W is supposed to live on the same probability space as X. This implies that the Laplace exponent $\kappa^0 : \mathbb{C} \longrightarrow \mathbb{C}$ is given by

$$\kappa^0(z) = \frac{1}{2}\mathrm{Var}\left(X(1)\right) z + \frac{1}{2}\mathrm{Var}\left(X(1)\right) z^2, z \in \mathbb{C}. \tag{3.2.14}$$

This naturally leads to a curve of stock processes

$$S^\lambda = S(0)e^{X^\lambda}, \lambda \in [0,1]. \tag{3.2.15}$$

Remark 3.2.3.
- *The choice of curve is a natural one in the sense that for all $\lambda \in [0,1]$, X^λ will be an exponential martingale with the same variance for all $\lambda \in [0,1]$. As $\mathrm{Var}\left(X(1)\right) = c + \int x^2 K(dx)$, it can be seen that the volatility implied by the jumps shifts into the Brownian motion part for $\lambda = 0$.*
- *However, 3.2.6 is not the only way to construct such a curve. In [38], another curve is constructed via*

$$\widetilde{X}^\lambda = \lambda \widetilde{X}(\frac{t}{\lambda^2}),\ \lambda \in (0,1], \tag{3.2.16}$$

where $\widetilde{X}$ is the unique process so that $e^X = \mathcal{E}(\widetilde{X})$, with $\mathcal{E}(\widetilde{X})$ denoting the stochastic exponential:

$$\mathcal{E}(\widetilde{X})(t) = e^{\widetilde{X}(t)-\frac{c}{2}t} \prod_{0\le s\le t} (1+\Delta\widetilde{X}(s))e^{-\Delta\widetilde{X}(s)}. \tag{3.2.17}$$

Note that although both curves have similar properties, they are not the same due to the different rescaling of the jumps. Hence, the approximation formula one gets using this curve also differ. For our purposes, we stick to the curve of [17] because the dependence of the overshoot of S^λ on λ is more obvious, which will come in handy for the later analysis.

3.3. Decomposition of the barrier option

For the upcoming discussion it is more convenient to use a representation involving crossing times.

Definition 3.3.1 (Crossing time of the barrier). *For a barrier B and $\lambda \in [0,1]$ we define via*

$$\tau_B^\lambda := \inf\left\{t \in [0,T], S^\lambda(t) \geq B\right\} \tag{3.3.18}$$

the crossing time of the barrier of the process S^λ before maturity.

Using the Markovian property of S, we define the pricing functional $v : [0,1] \times [0,T] \times [0,B] \to \mathbb{R}_+$ of an up-and-out barrier option:

Definition 3.3.2 (Pricing functional of an up-and-out option). *Let $\mathcal{C}$ be the payoff of an up-and-out barrier option defined as in definition 3.1.2. The function $v : [0,1] \times [0,T] \times [0,B] \to \mathbb{R}_+$ defined via*

$$v(\lambda, t, x) := E\left(h(S^\lambda(T))1_{\left\{\tau_B^\lambda \leq T\right\}^c} | S^\lambda(t) = x, t < \tau_B^\lambda\right) \tag{3.3.19}$$

is called the pricing functional of the up-and-out barrier-option.

From the discussion in Chapter 1.2.2, we know that $v(\lambda, t, x) = 0$ for $x \geq B$ and that the pricing functional is not infinitely often differentiable in $x = B$. As we will see in the next part, a certain amount of smoothness in the third coordinate of the pricing functional will be required to make our approach work. Therefore we will modify the original payoff of the up-and-out option in such a way that we include a suitable payoff behind the barrier at the moment τ_B^λ when the process jumps above the barrier. As our representation for the pricing functional will require to apply Ito's formula to the functional $v(0, T, S^\lambda(T))$, it will turn out that we need a smooth continuation of that pricing functional behind the barrier.

Lemma 3.3.3. *The functional $v(0,t,x)$ is six times partially differentiable in $(0,B)$ with respect to x and six times left-side partially differentiable in B with respect to x.*

Proof. As the price process S^0 is a geometric Brownian motion, this is the special case of $\lambda = 0$ in Lemma 3.4.9 □

Hence, we may choose a smooth continuation behind the barrier:

Definition 3.3.4. *Denote by $P : [0,T] \times \mathbb{R} \to \mathbb{R}$ the time-dependent polynomial function of sixth order such as that*

$$\frac{\partial^n}{\partial x^n} P(t,B) = \lim_{x \uparrow B} \frac{\partial^n}{\partial x^n} v(0,t,x), \; n = 0, \dots, 6. \tag{3.3.20}$$

Definition 3.3.5 (Decomposition of the barrier option payoff). *We define*

$$\mathcal{C}_1 = h(S(T))1_{\{\sup\limits_{0\le t\le T} S(t)<B\}} + P(\tau_B, S(\tau_B))1_{\{\sup\limits_{0\le t\le T} S(t)\ge B\}} \tag{3.3.21}$$

$$\mathcal{C}_2 = -P(\tau_B, S_{\tau_B})1_{\{\sup\limits_{0\le t\le T} S(t)\ge B\}} \tag{3.3.22}$$

with P as in Definition 3.3.4.

This will allow us to treat the option with payoff $\mathcal{C}_1$ in the same spirit as European style options using a representation of the pricing functional similar to [38]. But we have to pay the price of having to deal with a new option with payoff $\mathcal{C}_2$, leading to a payoff of the overshoot of X, which will be treated in chapter 4. It is per se not obvious why we choose this polynomial continuation. However, we will see in chapter 4 that this eases the valuation of $E(\mathcal{C}_2)$, as we are able to compute an approximation for moments of overshoots of Lévy processes more conveniently than for general payoff functions.

3.4. First order approximation for $\mathcal{C}_1$

Applying the ideas of section 1.4, we define our pricing curve q via

Definition 3.4.1 (Curve of option prices).

$$q^{\mathcal{C}_1}(\lambda) := E\left(h(S^\lambda(T))1_{\{\sup\limits_{0\le t\le T} S^\lambda(t)<B\}} + P(\tau_B^\lambda, S^\lambda(\tau_B^\lambda))1_{\{\sup\limits_{0\le t\le T} S(t)\ge B\}}\right), \lambda \in [0,1]. \tag{3.4.23}$$

Lemma 3.4.2. $q^{\mathcal{C}_1} : [0,1] \to \mathbb{R}^+$ *is a two times differentiable function.*

Proof. See the proof of Theorem 3.4.6. □

Definition 3.4.3 (First order approximation).

$$\mathcal{A}^{q^{\mathcal{C}_1}} := q^{q^{\mathcal{C}_1}}(0) + q^{q^{\mathcal{C}_1}\prime}(0). \tag{3.4.24}$$

For the representation of our approximation, we need two ingredients, namely the cumulants of our Lévy process X and certain derivatives of the pricing functional $v(0,\cdot,\cdot)$, so called 'cash-greeks'.

Definition 3.4.4 (Cumulants of X). *Let κ be the Laplace exponent of X as in 2.0.2 and $n \in \mathbb{N}$. We define the n-th cumulant κ_n of X via*

$$\kappa_n = (-1)^n \frac{\partial^n}{\partial x^n}\kappa(z)|_{z=0}. \tag{3.4.25}$$

Definition 3.4.5 (Black-Scholes-Cash-greeks)**.** *Let $v_{BS}:[0,T]\times[0,B]\to\mathbb{R}^+$ be defined via $v_{BS}(t,x)=v(0,t,x)$. For $n\in\mathbb{N}$ we define the n-th 'Black-Scholes-cash-greek' $D_n:[0,T]\times[0,B]\to\mathbb{R}$ via*

$$D_n(t,x)=x^n\frac{\partial^n}{\partial x^n}v_{BS}(t,x). \tag{3.4.26}$$

One easily sees that the coefficients of $P(t,B)$ have to be linear combinations of $\frac{\partial^k}{\partial x^k}v_{BS}(t,B),\ k=1,\dots,6$, as $v_{BS}(t,B)=0, t\in[0,T]$. The exact representation will be calculated in chapter 5, where we do the numerical implementation of our approximation formula. Thus, we are ready to state the main result of this section:

Theorem 3.4.6 (First order approximation $\mathcal{A}^{C_1}$)**.** *Let $\mathcal{C}_1$ be the payoff of a modified up-and-out barrier option defined as in definition 3.3.5. The first order approximation $\mathcal{A}^{C_1}$ has the following representation:*

$$\mathcal{A}^{C_1}=v_{BS}(0,S(0))+\frac{1}{6}T\kappa_3(3D_2(0,S(0))+D_3(0,S(0))). \tag{3.4.27}$$

Before we start proving this, let us shortly discuss the formula.

Remark 3.4.7.

- *The cumulant κ_3 is typically well known because the characteristic function is known explicitly for most Lévy processes.*
- *The cash greeks D_n also have an analytical expression as v_{BS} can be calculated directly, see e.g. example 1.1.1 for an up-and-out-put option.*
- *For the payoff $\mathcal{C}_1$ higher order approximations might be considered and proven in similar fashion as in the above theorem. However, as higher order approximations for $\mathcal{C}_2$ are not as simple, we decide to restrict ourselves to the first order. See [28] for a general formula for n-th order approximations.*

Proof of Theorem 3.4.6. For simplicity we will choose $S(0)=1$ in the following. The difficult part of the proof is to find a suitable representation of q. We adapt an idea from [38], Lemma 5. We denote by

$$\widetilde{v}_{BS}(t,x):=\begin{cases}v_{BS}(t,x) \text{ for } x<B\\ P(t,x) \text{ for } x\geq B,\ t\in[0,T]\end{cases}, \tag{3.4.28}$$

the continuation of v_{BS} behind the barrier. Furthermore, for $t\in[0,T], x>0$ we define

$$f_\lambda(t,x):=E\left(1_{\{\tau_B^\lambda>t\}}\widetilde{v}_{BS}(t,xS^\lambda(t))\right). \tag{3.4.29}$$

Lemma 3.4.8. *For $\lambda\in[0,1]$:*

$$q^{\mathcal{C}_1}(\lambda)=E\left(\widetilde{v}_{BS}(T\wedge\tau_B^\lambda)\,,S^\lambda(T\wedge\tau_B^\lambda)\right). \tag{3.4.30}$$

Proof. Using the representations of $q^{\mathcal{C}_1}$ from Definition 3.4.1 and v_{BS}, we compute

$$\begin{aligned}
&E\left(\widetilde{v}_{BS}(T\wedge\tau_B^\lambda, S^\lambda(T\wedge\tau_B^\lambda))\right)\\
&= E\left(h(S^\lambda(T))1_{\{\sup\limits_{0\le t\le T} S^\lambda(t)<B\}} + P(\tau_B^\lambda, S^\lambda(\tau_B^\lambda))1_{\{\sup\limits_{0\le t\le T} S^\lambda(t)\ge B\}}\right) = q^{\mathcal{C}_1}(\lambda).
\end{aligned} \tag{3.4.31}$$

□

We now adapt a similar approach as to the PIDE method in Remark 1.2.2. As $\widetilde{v}_{BS}$ is by construction in $C^{(1,2)}$ and $S^\lambda = e^{X^\lambda}$ is a semimartingale, we apply Ito's formula:

$$\begin{aligned}
&\widetilde{v}_{BS}(T, S^\lambda(T)) = \widetilde{v}_{BS}(0,1) + \int_0^T \frac{\partial\widetilde{v}_{BS}}{\partial t}(t, S^\lambda(t))dt + \int_0^T \frac{\partial\widetilde{v}_{BS}}{\partial x}(t, S^\lambda(t))dS^\lambda(t)\\
&+ \frac{1}{2}\int_0^T \frac{\partial\widetilde{v}_{BS}^2}{\partial x^2}(t, S^\lambda(t))d[S^\lambda, S^\lambda]^c(t)\\
&+ \sum_{\Delta X\neq 0} \widetilde{v}_{BS}(t, S^\lambda(t)) - \widetilde{v}_{BS}(t, S^\lambda(t-)) - \Delta S^\lambda(t)\frac{\partial\widetilde{v}_{BS}}{\partial x}(t, S^\lambda(t-)).
\end{aligned} \tag{3.4.32}$$

Note that $[S^\lambda, S^\lambda]^c(t) = (S^\lambda)^2 ct$ and using the fact that S^λ is a martingale, we get:

$$\begin{aligned}
&E\left(\widetilde{v}_{BS}(T\wedge\tau_B^\lambda, S^\lambda(T\wedge\tau_B^\lambda))\right) - \widetilde{v}_{BS}(0,1)\\
&= E\left(\int_0^{\tau_B^\lambda} \frac{\partial\widetilde{v}_{BS}}{\partial t}(t, S^\lambda(t))dt + \frac{1}{2}\int_0^{\tau_B^\lambda} c(S^\lambda)^2\frac{\partial\widetilde{v}_{BS}^2}{\partial x^2}(t, S^\lambda(t))dt\right)\\
&+ E\left(\int_0^{\tau_B^\lambda}\int_{\mathbb{R}} (\widetilde{v}_{BS}(t, e^y S^\lambda(t-))K^\lambda(dy)dt\right)\\
&- E\left(\int_0^{\tau_B^\lambda} \widetilde{v}_{BS}(t, S^\lambda(t-)) - (e^y-1)S^\lambda(t-)\frac{\partial\widetilde{v}_{BS}}{\partial x}(t, S^\lambda(t-)))K^\lambda(dy)dt\right).
\end{aligned} \tag{3.4.33}$$

Now we use the fact that $\widetilde{v}_{BS}$ solves the Black-Scholes-PDE for S^0, which means that

$$\frac{\partial\widetilde{v}_{BS}}{\partial t}(t,x) + \frac{1}{2}\mathrm{Var}\,(X(1))\,x^2\frac{\partial\widetilde{v}_{BS}^2}{\partial x^2}(t,x) = 0,\ t\in[0,T],\ x\in[0,B], \tag{3.4.34}$$

where $\mathrm{Var}\,(X(1)) = c + \int y^2K(dy)$. As for $t\in[0,\tau_B^\lambda]$ it holds that $S^\lambda(t)\in[0,B]$, we compute:

$$
\begin{aligned}
&E\left(\widetilde{v}_{BS}(T\wedge\tau_B^\lambda, S^\lambda(T\wedge\tau_B^\lambda))\right)-\widetilde{v}_{BS}(0,1)\\
&=E\left(\int_0^{\tau_B^\lambda}\int_{\mathbb{R}}\widetilde{v}_{BS}(t,e^yS^\lambda(t))-\widetilde{v}_{BS}(t,S^\lambda(t))K^\lambda(dy)dt\right)\\
&-E\left(\int_0^{\tau_B^\lambda}\int_{\mathbb{R}}(e^y-1)S^\lambda(t)\frac{\partial\widetilde{v}_{BS}}{\partial x}(t,S^\lambda(t))K^\lambda(dy)dt\right)\\
&-E\left(\int_0^{\tau_B^\lambda}\int_{\mathbb{R}}\frac{y^2}{2}(S^\lambda(t))^2\frac{\partial\widetilde{v}_{BS}^2}{\partial x^2}(t,S^\lambda(t))K^\lambda(dy)dt\right),
\end{aligned}
\tag{3.4.35}
$$

using the fact that $\int_{\mathbb{R}}y^2K(dy)=\int_{\mathbb{R}}y^2K^\lambda(dy)$ due to Lemma 3.2.1. Now we apply Fubini's theorem, which is ensured by Lemma 3.4.9:

$$
\begin{aligned}
&E\left(\int_0^{T}\int_{\mathbb{R}}1_{\{\tau_B^\lambda>t\}}\widetilde{v}_{BS}(t,e^yS^\lambda(t))-\widetilde{v}_{BS}(t,S^\lambda(t))K^\lambda(dy)dt\right)\\
&-E\left(\int_0^{T}\int_{\mathbb{R}}(e^y-1)S^\lambda(t)\frac{\partial\widetilde{v}_{BS}}{\partial x}(t,S^\lambda(t))K^\lambda(dy)dt\right)\\
&-E\left(\int_0^{T}\int_{\mathbb{R}}1_{\{\tau_B^\lambda>t\}}(\frac{y^2}{2}S^\lambda(t)\frac{\partial\widetilde{v}_{BS}^2}{\partial x^2}(t,S^\lambda(t)))K^\lambda(dy)dt\right)\\
&=\int_0^{T}\int_{\mathbb{R}}(f_\lambda(t,e^y)-f_\lambda(t,1)-(e^y-1)\frac{\partial}{\partial x}f_\lambda(t,1)-\frac{y^2}{2}\frac{\partial^2}{\partial x^2}f_\lambda(t,1))K^\lambda(dy)dt.
\end{aligned}
\tag{3.4.36}
$$

Applying a Taylor series expansion with integral remainder term, using the fact that f_λ is six times differentiable in the second coordinate due to Lemma 3.4.9

$$
\begin{aligned}
&f_\lambda(t,e^y)-f_\lambda(t,1)-(e^y-1)\frac{\partial}{\partial x}f_\lambda(t,1)\\
&=\frac{(e^y-1)^2}{2}\frac{\partial^2}{\partial x^2}f_\lambda(t,1)+\frac{(e^y-1)^3}{6}\frac{\partial^3}{\partial x^3}f_\lambda(t,1)\\
&+\int_0^1\frac{(e^y-1)^4}{6}(1-s)^3\frac{\partial^4}{\partial x^4}f_\lambda(t,1+s(e^y-1))ds,
\end{aligned}
\tag{3.4.37}
$$

leads to

$$
\begin{aligned}
&\int_0^{T}\int_{\mathbb{R}}(f_\lambda(t,e^y)-f_\lambda(t,1)-(e^y-1)\frac{\partial}{\partial x}f_\lambda(t,1)-\frac{y^2}{2}\frac{\partial^2}{\partial x^2}f_\lambda(t,1))K^\lambda(dy)dt\\
&=\int_0^{T}\int_{\mathbb{R}}(\frac{(e^y-1)^2}{2}\frac{\partial^2}{\partial x^2}f_\lambda(t,1)-\frac{y^2}{2}\frac{\partial^2}{\partial x^2}f_\lambda(t,1)+\frac{(e^y-1)^3}{6}\frac{\partial^3}{\partial x^3}f_\lambda(t,1)\\
&+\int_0^1\frac{(e^y-1)^4}{6}(1-s)^3\frac{\partial^4}{\partial x^4}f_\lambda(t,1+s(e^y-1))ds)K^\lambda(dy)dt.
\end{aligned}
\tag{3.4.38}
$$

Using another Taylor series expansion with integral remainder term of appropiate order for e^y:

$$
\begin{aligned}
e^y &= 1 + \int_0^1 (ye^{ry})dr = 1 + y + \int_0^1 (\frac{y^2}{2}(1-r)e^{ry})dr \\
&= 1 + y + \frac{y^2}{2} + \int_0^1 (\frac{y^3}{6}(1-r)^2 e^{ry})dr,
\end{aligned}
\tag{3.4.39}
$$

we compute

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}} (\frac{(e^y-1)^2}{2}\frac{\partial^2}{\partial x^2} f_\lambda(t,1) - \frac{y^2}{2}\frac{\partial^2}{\partial x^2} f_\lambda(t,1) + \frac{(e^y-1)^3}{6}\frac{\partial^3}{\partial x^3} f_\lambda(t,1) \\
&+ \int_0^1 (\frac{(e^y-1)^4}{6}(1-s)^3 \frac{\partial^4}{\partial x^4} f_\lambda(t, 1 + s(e^y-1))ds))K^\lambda(dy)dt \\
&= \int_0^T \int_{\mathbb{R}} \Bigg(\frac{y^3}{6}(3\frac{\partial^2}{\partial x^2} f_\lambda(t,1) + \frac{\partial^3}{\partial x^3} f_\lambda(t,1)) \\
&+ ((\frac{y^4}{8} + (2y + y^2 + \int_0^1 \frac{y^3}{6}(1-r)^2 e^{ry} dr) \\
&\int_0^1 \frac{y^3}{6}(1-r)^2 e^{ry} dr)\frac{\partial^2}{\partial x^2} f_\lambda(t,1)) \\
&+ ((3y^2 + 3y\int_0^1 \frac{y^2}{2}(1-r)e^{ry}dr + (\int_0^1 \frac{y^2}{2}(1-r)e^{ry}dr)^2) \\
&\int_0^1 \frac{y^2}{2}(1-s)e^{ry}dr)\frac{\partial^3}{\partial x^3} f_\lambda(t,1))) \\
&+ \frac{(\int_0^1 (ye^{ry})dr)^4}{6}(1-s)^3 \frac{\partial^4}{\partial x^4} f_\lambda(t, 1 + s(e^y-1))ds) \Bigg) K^\lambda(dy)dt.
\end{aligned}
\tag{3.4.40}
$$

We now define for $(t,x) \in [0,T] \times \mathbb{R}$:

$$
\widetilde{f}_\lambda(t,x) := E\left(1_{\{\tau_B^\lambda < t\}^c}\left(3\frac{\partial^2}{\partial x^2}\widetilde{v}_{BS}(t, xS(t)) + \frac{\partial^3}{\partial x^3}\widetilde{v}_{BS}(t, xS(t))\right)\right) \tag{3.4.41}
$$

and

$$
\widetilde{w}_{BS}(t,x) := 3x^2 \frac{\partial^2}{\partial x^2}\widetilde{v}_{BS}(t,x) + x^3 \frac{\partial^3}{\partial x^3}\widetilde{v}_{BS}(t,x). \tag{3.4.42}
$$

As $\widetilde{v}_{BS} \in C^{1,6}$ by construction, $\widetilde{w}_{BS} \in C^{1,2}$ and we may use the same mechanism we applied to $\widetilde{v}_{BS}$ in (3.4.36), using Ito's formula, expectation, Black-Scholes-PDE and Fubini's theorem, justified again by Lemma 3.4.9, to see that

$$
\begin{aligned}
&\widetilde{f}_\lambda(t,1) = \widetilde{f}_\lambda(0,1) \\
&+ \int_0^t \int_{\mathbb{R}} \widetilde{f}_\lambda(s, e^y) - \widetilde{f}_\lambda(s,1) - (e^y - 1)\frac{\partial}{\partial x}\widetilde{f}_\lambda(s,1) - \frac{y^2}{2}\frac{\partial^2}{\partial x^2}\widetilde{f}_\lambda(s,1)K^\lambda(dy)ds,
\end{aligned}
\tag{3.4.43}
$$

and afterwards apply the same steps as from (3.4.36) to (3.4.40):

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}} \widetilde{f}_\lambda(s,e^y) - \widetilde{f}_\lambda(s,1) - (e^y-1)\frac{\partial}{\partial x}\widetilde{f}_\lambda(s,1) - \frac{y^2}{2}\frac{\partial^2}{\partial x^2}\widetilde{f}_\lambda(s,1)K^\lambda(dy)ds \\
&= \int_0^t \int_{\mathbb{R}} (\frac{(e^y-1)^2}{2}\frac{\partial^2}{\partial x^2}\widetilde{f}_\lambda(t,1) - \frac{y^2}{2}\frac{\partial^2}{\partial x^2}\widetilde{f}_\lambda(t,1) + \frac{(e^y-1)^3}{6}\frac{\partial^3}{\partial x^3}\widetilde{f}_\lambda(t,1) \\
&+ \int_0^1 (\frac{(e^y-1)^4}{6}(1-s)^3\frac{\partial^4}{\partial x^4}\widetilde{f}_\lambda(t,1+s(e^y-1))ds))K^\lambda(dy)dt \\
&= \int_0^t \int_{\mathbb{R}} \Bigg(\frac{y^3}{6}(3\frac{\partial^2}{\partial x^2}\widetilde{f}_\lambda(u,1) + \frac{\partial^3}{\partial x^3}\widetilde{f}_\lambda(u,1)) \\
&+ ((\frac{y^4}{8} + (2y+y^2 + \int_0^1 \frac{y^3}{6}(1-r)^2 e^{ry}dr) \\
&\int_0^1 \frac{y^3}{6}(1-r)^2 e^{ry}dr)\frac{\partial^2}{\partial x^2}\widetilde{f}_\lambda(u,1)) \\
&+ ((3y^2 + 3y\int_0^1 \frac{y^2}{2}(1-r)e^{ry}dr + (\int_0^1 \frac{y^2}{2}(1-r)e^{ry}dr)^2) \\
&\int_0^1 \frac{y^2}{2}(1-s)e^{ry}dr)\frac{\partial^3}{\partial x^3}\widetilde{f}_\lambda(t,1))) \\
&+ \frac{(\int_0^1 (ye^{ry})dr)^4}{6}(1-s)^3\frac{\partial^4}{\partial x^4}\widetilde{f}_\lambda(t,1+s(e^y-1))ds)\Bigg)K^\lambda(dy)du.
\end{aligned}
\tag{3.4.44}
$$

Inserting this into formula (3.4.40) and using the fact that

$$
\widetilde{f}_\lambda(0,1) = \widetilde{w}_{BS}(0,1) = (3\frac{\partial^2}{\partial x^2}v_{BS}(0,1) + \frac{\partial^3}{\partial x^3}v_{BS}(0,1)) \tag{3.4.45}
$$

this leads to

$$
\begin{aligned}
(3.4.40) &= \int_0^T \int_{\mathbb{R}} \frac{y^3}{6}(3\frac{\partial^2}{\partial x^2}v_{BS}(0,1) + \frac{\partial^3}{\partial x^3}v_{BS}(0,1))K^\lambda(dy)dt \\
&+ \int_0^T \int_0^t \int_{\mathbb{R}} \Bigg(\frac{y^3}{6}(3\frac{\partial^2}{\partial x^2}\widetilde{f}_\lambda(u,1) + \frac{\partial^3}{\partial x^3}\widetilde{f}_\lambda(u,1)) \\
&+ ((\frac{y^4}{8} + (2y+y^2 + \int_0^1 \frac{y^3}{6}(1-r)^2 e^{ry}dr) \\
&\int_0^1 \frac{y^3}{6}(1-r)^2 e^{ry}dr)\frac{\partial^2}{\partial x^2}\widetilde{f}_\lambda(u,1))
\end{aligned}
\tag{3.4.46}
$$

$$
\begin{aligned}
&+((3y^2+3y\int_0^1\frac{y^2}{2}(1-r)e^{ry}dr+(\int_0^1\frac{y^2}{2}(1-r)e^{ry}dr)^2)\\
&\int_0^1\frac{y^2}{2}(1-s)e^{ry}dr)\frac{\partial^3}{\partial x^3}\widetilde{f}_\lambda(u,1)))\\
&+\frac{(\int_0^1(ye^{ry})dr)^4}{6}(1-s)^3\frac{\partial^4}{\partial x^4}\widetilde{f}_\lambda(u,1+s(e^y-1))ds)\frac{y^3}{6}\Big)K^\lambda(dy)dudt\\
&+\int_0^T\int_{\mathbb{R}}\Big(\frac{y^3}{6}(3\frac{\partial^2}{\partial x^2}f_\lambda(t,1)+\frac{\partial^3}{\partial x^3}f_\lambda(t,1))\\
&+((\frac{y^4}{8}+(2y+y^2+\int_0^1\frac{y^3}{6}(1-r)^2e^{ry}dr)\\
&\int_0^1\frac{y^3}{6}(1-r)^2e^{ry}dr)\frac{\partial^2}{\partial x^2}f_\lambda(t,1))\\
&+((3y^2+3y\int_0^1\frac{y^2}{2}(1-r)e^{ry}dr+(\int_0^1\frac{y^2}{2}(1-r)e^{ry}dr)^2)\\
&\int_0^1\frac{y^2}{2}(1-s)e^{ry}dr)\frac{\partial^3}{\partial x^3}f_\lambda(t,1)))\\
&+\frac{(\int_0^1(ye^{ry})dr)^4}{6}(1-s)^3\frac{\partial^4}{\partial x^4}f_\lambda(t,1+s(e^y-1))ds)\Big)K^\lambda(dy)dt\\
&=:\int_0^T\int_{\mathbb{R}}\frac{y^3}{6}(3\frac{\partial^2}{\partial x^2}v_{BS}(0,1)+\frac{\partial^3}{\partial x^3}v_{BS}(0,1))K^\lambda(dy)dt+\mathcal{R}
\end{aligned}
$$

where $\mathcal{R}$ denotes the remaining integrals from (3.4.46). Plugging this into formula (3.4.32) from the beginning of the proof and using $\kappa_3=\int_{\mathbb{R}}y^3K(dy)$ and $K^\lambda(A)=\frac{1}{\lambda^2}K(\lambda A), A\in\mathcal{B}(\mathbb{R})$ due to Lemma 3.2.1, we conclude that

$$
q^{\mathcal{C}_1}=v_{BS}(0,1)+\lambda T\frac{\kappa_3}{6}(3\frac{\partial^2}{\partial x^2}v_{BS}(0,1)+\frac{\partial^3}{\partial x^3}v_{BS}(0,1))+\mathcal{R}. \tag{3.4.47}
$$

It remains to show that we are able to control the derivatives of f_λ such that $\frac{\mathcal{R}}{\lambda}\to 0$ to finish the proof. This will be ensured by the following technical lemmas: □

Lemma 3.4.9. *Under the assumption of Theorem 3.4.6, for all* $0\le k\le 6, (t,x)\in[0,T]\times\mathbb{R}^+$ *the following holds:*

1. $\frac{\partial^k}{\partial x^k}f_\lambda(t,x)$ *exists, is continuous in* x *and satisfies*

$$
\left|\frac{\partial^k}{\partial x^k}f_\lambda(t,x)\right|\le C(1+x^n+\frac{1}{x^n}) \tag{3.4.48}
$$

   for some $n\ge 0$ *and some constant* C *independent of* t *and* λ.

2. $f_\lambda(t,x)\to f_0(t,x)$ *pointwise, as* $\lambda\to 0$.

Proof. 1. Recall that

$$f_\lambda(t,x) := E\left(1_{\{\tau_B^\lambda > t\}} \widetilde{v}_{BS}(t, xS^\lambda(t))\right). \tag{3.4.49}$$

Consider $0 \le k \le 6$. We observe that $v_{BS}(t,x) = E\left(h(S^0(T))1_{\{\tau_B^0 \le T\}^c} | S^0(t) = x\right)$, hence, we may write

$$v_{BS}(t, xS^\lambda(t)) = \int h(e^{\widetilde{\sigma} z} xS^\lambda(t)) p(\frac{\log(\frac{B}{xS^\lambda(t)})}{\widetilde{\sigma}}, z, T-t) dz, \tag{3.4.50}$$

where $p(b,z,t)$ is the transition density of the process $-\frac{\widetilde{\sigma}}{2} I + W$, I the identity process, W the standard Brownian motion absorbed at $b > 0$. From [8], page 299, we see that

$$p(b,z,t) = 1_{\{z \le b\}} (\varphi(z, -\frac{\widetilde{\sigma}}{2}, t) - e^{\frac{b}{\widetilde{\sigma}}} \varphi(z, -\frac{\widetilde{\sigma}}{2} + 2b, t)), \tag{3.4.51}$$

with

$$\varphi(z, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(z-\mu)^2}{2\sigma^2}) \tag{3.4.52}$$

being the density of the $N(\mu, \sigma^2)$ distribution. W.l.o.g. we take h so that $h(x) = 0$ for $x > B$, so that we can forget about the indicator in $p(b,z,t)$. Inserting this yields:

$$\begin{aligned} &\int h(e^{\widetilde{\sigma} z} xS^\lambda(t)) p(\frac{\log(\frac{B}{xS^\lambda(t)})}{\widetilde{\sigma}}, z, T-t) dz \\ &= \int h(e^{\widetilde{\sigma} z} xS^\lambda(t)) \varphi(z, -\frac{\widetilde{\sigma}}{2}, T-t) dz \\ &- \int h(e^{\widetilde{\sigma} z} xS^\lambda(t)) \frac{B}{xS^\lambda(t)} \varphi(z, -\frac{\widetilde{\sigma}}{2} + 2\frac{\log(\frac{B}{xS^\lambda(t)})}{\widetilde{\sigma}}, T-t) dz. \end{aligned} \tag{3.4.53}$$

For the first integral observe that, since the first k derivatives of h are bounded,

$$|\frac{\partial^k}{\partial x^k} h(e^{\widetilde{\sigma} z} xS^\lambda(t)) \varphi(z, -\frac{\widetilde{\sigma}}{2}, T-t)| \le M e^{kz} S^\lambda(t)^k \varphi(z, -\frac{\widetilde{\sigma}}{2}, T-t) \tag{3.4.54}$$

for some positive constant M which is an integrable function in z as the normal distribution has all exponential moments. Hence, we may apply [19] Theorem 5.7 to interchange differentiation and integration to see that

$$\begin{aligned} &\frac{\partial^k}{\partial x^k} \int h(e^{\widetilde{\sigma} z} xS^\lambda(t)) \varphi(z, -\frac{\widetilde{\sigma}}{2}, T-t) dz \\ &= \int \frac{\partial^k}{\partial x^k} h(e^{\widetilde{\sigma} z} xS^\lambda(t)) \varphi(z, -\frac{\widetilde{\sigma}}{2}, T-t) dz \\ &= \int h(e^{\widetilde{\sigma} z} xS^\lambda(t)) e^{\widetilde{\sigma} zk} S^\lambda(t)^k \varphi(z, -\frac{\widetilde{\sigma}}{2}, T-t) dz \\ &\le | \max_{x \in (0,B)} (\frac{\partial^k}{\partial x^k} h(x)) \int e^{\widetilde{\sigma} zk} S^\lambda(t)^k \varphi(z, -\frac{\widetilde{\sigma}}{2}, T-t) dz | \le |M_1 S^\lambda(t)^k| \end{aligned} \tag{3.4.55}$$

for some constant $M_1 > 0$. For the second integral, substituting $\widetilde{z} := z - 2\frac{\log(\frac{B}{xS^\lambda(t)})}{\widetilde{\sigma}}$ yields

$$\begin{aligned}&\int h(e^{\widetilde{\sigma}z}xS^\lambda(t))\frac{B}{xS(t)}\varphi(z,-\frac{\widetilde{\sigma}}{2}+2\frac{\log(\frac{B}{xS^\lambda(t)})}{\widetilde{\sigma}},T-t)dz\\ &=\int h(e^{\widetilde{\sigma}\widetilde{z}}\frac{B^2}{xS^\lambda(t)})\frac{B}{xS^\lambda(t)}\varphi(\widetilde{z},-\frac{\widetilde{\sigma}}{2},T-t)d\widetilde{z},\end{aligned} \tag{3.4.56}$$

and applying the same arguments as for the first integral and using Faà di Bruno's formula [15] we find a constant $M_2 > 0$ so that

$$\begin{aligned}&\frac{\partial^k}{\partial x^k}\int h(e^{\widetilde{\sigma}\widetilde{z}}\frac{B^2}{xS^\lambda(t)})\frac{B}{xS^\lambda(t)}\varphi(\widetilde{z},-\frac{\widetilde{\sigma}}{2},T-t)d\widetilde{z}\\ &=\int\frac{\partial^k}{\partial x^k}h(e^{\widetilde{\sigma}\widetilde{z}}\frac{B^2}{xS^\lambda(t)})\frac{B}{xS^\lambda(t)}\varphi(\widetilde{z},-\frac{\widetilde{\sigma}}{2},T-t)d\widetilde{z}\\ &\le M_2\frac{1}{S^\lambda(t)^k}.\end{aligned} \tag{3.4.57}$$

Finally, we conclude with [19] Theorem 5.7 that

$$\begin{aligned}&|\frac{\partial^k}{\partial x^k}f_\lambda(t,x)| = |\frac{\partial^k}{\partial x^k}E\left(1_{\{\tau_B^\lambda\le t\}}\widetilde{v}_{BS}(\tau_B^\lambda,S(\tau_B^\lambda))+1_{\{\tau_B^\lambda>t\}}\widetilde{v}_{BS}(t,xS^\lambda(t))\right)|\\ &\le E\left(|\frac{\partial^k}{\partial x^k}1_{\{\tau_B^\lambda>t\}}\widetilde{v}_{BS}(t,xS^\lambda(t))|\right)\\ &\le E\left(|\frac{\partial^k}{\partial x^k}\widetilde{v}_{BS}(t,xS^\lambda(t))|\right)\\ &\le E\left(M_1S^\lambda(t)^k+C_2\frac{1}{S^\lambda(t)^kx^k}\right)\le M_3(1+\frac{1}{x^k}),\end{aligned} \tag{3.4.58}$$

where due to Assumption 4.1.2 all moments are finite and we may choose $M_3 > 0$ independent of (t,λ) due to Lemma 3.2.1.

2. As we know from Lemma 3.2.2, $S^\lambda \to S^0$ for $\lambda \to 0$ w.r.t the Shorokhod topology. Let $D(\mathbb{R})$ be the Skorokhod space, $\alpha \in D(\mathbb{R})$. [26], VII 2.11 implies that the operation $\alpha \to \alpha^{\tau_x}$ is Shorokhod-continuous for all $\alpha \in D(\mathbb{R})/(J_1 \cup J_2)$, $J_1 := \{x \in \mathbb{R} : \tau_x(\alpha) < \tau_{x+}(\alpha)\}$, $J_2 := \{x \in \mathbb{R} : \exists t \in \mathbb{R}^+ :\ \alpha(t-) = x < \alpha(t)\}$. As S^0 is a continuous process, J_1 and J_2 are null sets (cf. Lemma 4.5.6.3), hence [26], VII 3 implies that

$$E\left(1_{\{\tau_B^\lambda\le t\}}\right)\to E\left(1_{\{\tau_B^0\le t\}}\right), \tag{3.4.59}$$

$$E\left(1_{\{\tau_B^\lambda\le t\}^c}\widetilde{v}_{BS}(t,S^\lambda(t))\right)\to E\left(1_{\{\tau_B^0\le t\}^c}\widetilde{v}_{BS}(t,S^0(t))\right), \tag{3.4.60}$$

as $\widetilde{v}_{BS}(t,x)$ is bounded and continuous in $x \in [0,B]$, hence $f_\lambda(t,x) \to f_0(t,x)$ for $\lambda \to 0$.

□

Lemma 3.4.10. *For the integral remainder term $\mathcal{R}$ defined in* (3.4.46) *the following holds:*

$$\frac{\mathcal{R}}{\lambda} \to 0, \textit{ for } \lambda \to 0. \tag{3.4.61}$$

Proof. Remember that (3.4.46) implies that

$$\begin{aligned}
\mathcal{R} = &\int_0^T \int_0^t \int_{\mathbb{R}} \Bigg(\frac{y^3}{6} (3 \frac{\partial^2}{\partial x^2} \widetilde{f}_\lambda(u,1) + \frac{\partial^3}{\partial x^3} \widetilde{f}_\lambda(u,1)) \\
&+ ((\frac{y^4}{8} + (2y + y^2 + \int_0^1 \frac{y^3}{6}(1-r)^2 e^{ry} dr) \\
&\int_0^1 \frac{y^3}{6}(1-r)^2 e^{ry} dr) \frac{\partial^2}{\partial x^2} \widetilde{f}_\lambda(u,1)) \\
&+ ((3y^2 + 3y \int_0^1 \frac{y^2}{2}(1-r) e^{ry} dr + (\int_0^1 \frac{y^2}{2}(1-r)e^{ry} dr)^2) \\
&\int_0^1 \frac{y^2}{2}(1-s)e^{ry} dr) \frac{\partial^3}{\partial x^3} \widetilde{f}_\lambda(u,1))) \\
&+ \frac{(\int_0^1 (ye^{ry})dr)^4}{6} (1-s)^3 \frac{\partial^4}{\partial x^4} \widetilde{f}_\lambda(u, 1 + s(e^y - 1))ds) \frac{y^3}{6} \Bigg) K^\lambda(dy) du dt \\
&+ \int_0^T \int_{\mathbb{R}} \Bigg(\frac{y^3}{6} (3 \frac{\partial^2}{\partial x^2} f_\lambda(t,1) + \frac{\partial^3}{\partial x^3} f_\lambda(t,1)) \\
&+ ((\frac{y^4}{8} + (2y + y^2 + \int_0^1 \frac{y^3}{6}(1-r)^2 e^{ry} dr) \\
&\int_0^1 \frac{y^3}{6}(1-r)^2 e^{ry} dr) \frac{\partial^2}{\partial x^2} f_\lambda(t,1)) \\
&+ ((3y^2 + 3y \int_0^1 \frac{y^2}{2}(1-r) e^{ry} dr + (\int_0^1 \frac{y^2}{2}(1-r)e^{ry} dr)^2) \\
&\int_0^1 \frac{y^2}{2}(1-s)e^{ry} dr) \frac{\partial^3}{\partial x^3} f_\lambda(t,1))) \\
&+ \frac{(\int_0^1 (ye^{ry})dr)^4}{6} (1-s)^3 \frac{\partial^4}{\partial x^4} f_\lambda(t, 1 + s(e^y - 1))ds) \Bigg) K^\lambda(dy) dt.
\end{aligned} \tag{3.4.62}$$

Using the rule $K^\lambda(A) = \frac{1}{\lambda^2} K(\lambda A), A \in \mathcal{B}(\mathbb{R})$ due to Lemma 3.2.1 and Jensen's inequality, we compute

$$
\begin{aligned}
|\mathcal{R}| \le |&\frac{1}{\lambda^2}\int_0^T\int_0^t\int_{\mathbb{R}}\bigg(\frac{(y\lambda)^3}{6}(3\frac{\partial^2}{\partial x^2}\widetilde{f}_\lambda(u,1)+\frac{\partial^3}{\partial x^3}\widetilde{f}_\lambda(u,1))\\
&+((\frac{(y\lambda)^4}{8}+(2(y\lambda)+(y\lambda)^2+\int_0^1\frac{(y\lambda)^3}{6}(1-r)^2e^{ry\lambda}dr)\\
&\int_0^1\frac{(y\lambda)^3}{6}(1-r)^2e^{ry\lambda}dr)\frac{\partial^2}{\partial x^2}\widetilde{f}_\lambda(u,1))\\
&+((3(y\lambda)^2+3(y\lambda)\int_0^1\frac{y^2}{2}(1-r)e^{ry\lambda}dr+\int_0^1(\frac{(y\lambda)^2}{2}(1-r)e^{ry\lambda})^2dr)\\
&\int_0^1\frac{(y\lambda)^2}{2}(1-s)e^{ry\lambda}dr)\frac{\partial^3}{\partial x^3}\widetilde{f}_\lambda(u,1))\\
&+\frac{\int_0^1(((y\lambda)e^{ry\lambda})^4)dr}{6}(1-s)^3\frac{\partial^4}{\partial x^4}\widetilde{f}_\lambda(u,1+s(e^y-1))ds)\frac{(y\lambda)^3}{6}\bigg)K(dy)dudt\\
&+\frac{1}{\lambda^2}\int_0^T\int_{\mathbb{R}}\bigg(\frac{(y\lambda)^3}{6}(3\frac{\partial^2}{\partial x^2}f_\lambda(t,1)+\frac{\partial^3}{\partial x^3}f_\lambda(t,1))\\
&+((\frac{(y\lambda)^4}{8}+(2(y\lambda)+(y\lambda)^2+\int_0^1\frac{(y\lambda)^3}{6}(1-r)^2e^{ry\lambda}dr)\\
&\int_0^1\frac{(y\lambda)^3}{6}(1-r)^2e^{ry\lambda}dr)\frac{\partial^2}{\partial x^2}f_\lambda(t,1))\\
&+((3(y\lambda)^2+3(y\lambda)\int_0^1\frac{(y\lambda)^2}{2}(1-r)e^{ry\lambda}dr+\int_0^1(\frac{(y\lambda)}{2}(1-r)e^{ry\lambda})^2dr)\\
&\int_0^1\frac{(y\lambda)^2}{2}(1-s)e^{ry\lambda}dr)\frac{\partial^3}{\partial x^3}f_\lambda(t,1))\\
&+\frac{\int_0^1(y\lambda e^{ry\lambda})^4dr}{6}(1-s)^3\frac{\partial^4}{\partial x^4}f_\lambda(t,1+s(e^{y\lambda}-1))ds)\bigg)K(dy)dt|.
\end{aligned}
\tag{3.4.63}
$$

Factoring out the λ coefficients, we see immediately that

$$
\begin{aligned}
|\mathcal{R}| \le |\lambda^2&\int_0^T\int_0^t\int_{\mathbb{R}}\bigg(\frac{y^3}{6}(3\frac{\partial^2}{\partial x^2}\widetilde{f}_\lambda(u,1)+\frac{\partial^3}{\partial x^3}\widetilde{f}_\lambda(u,1))\\
&+((\frac{y^4}{8}+(2y+y^2+\int_0^1\frac{y^3}{6}(1-r)^2e^{ry\lambda}dr)\\
&\int_0^1\frac{y^3}{6}(1-r)^2e^{ry\lambda}dr)\frac{\partial^2}{\partial x^2}\widetilde{f}_\lambda(u,1))
\end{aligned}
\tag{3.4.64}
$$

$$
\begin{aligned}
&+ ((3y^2 + 3y\int_0^1 \frac{y^2}{2}(1-r)e^{ry\lambda}dr + (\int_0^1 \frac{y^2}{2}(1-r)e^{ry\lambda}dr)^2) \\
&\int_0^1 \frac{y^2}{2}(1-s)e^{ry\lambda}dr)\frac{\partial^3}{\partial x^3}\widetilde{f}_\lambda(u,1))) \\
&+ \frac{(\int_0^1(ye^{ry\lambda})dr)^4}{6}(1-s)^3\frac{\partial^4}{\partial x^4}\widetilde{f}_\lambda(u, 1+s(e^y-1))ds)\frac{y^3}{6}\Bigg)K(dy)dudt \\
&+ \int_0^T \int_{\mathbb{R}} \Bigg(\frac{y^3}{6}(3\frac{\partial^2}{\partial x^2}f_\lambda(t,1) + \frac{\partial^3}{\partial x^3}f_\lambda(t,1)) \\
&+ ((\frac{y^4}{8} + (2y + y^2 + \int_0^1 \frac{y^3}{6}(1-r)^2e^{ry\lambda}dr) \\
&\int_0^1 \frac{y^3}{6}(1-r)^2e^{ry\lambda}dr)\frac{\partial^2}{\partial x^2}f_\lambda(t,1)) \\
&+ ((3y^2 + 3y\int_0^1 \frac{y^2}{2}(1-r)e^{ry\lambda}dr + (\int_0^1 \frac{y^2}{2}(1-r)e^{ry\lambda}dr)^2) \\
&\int_0^1 \frac{y^2}{2}(1-s)e^{ry\lambda}dr)\frac{\partial^3}{\partial x^3}f_\lambda(t,1))) \\
&+ \frac{(\int_0^1(ye^{ry\lambda})dr)^4}{6}(1-s)^3\frac{\partial^4}{\partial x^4}f_\lambda(t, 1+s(e^{y\lambda}-1))ds)\Bigg)K(dy)dt|
\end{aligned}
$$

Due to Lemma 3.4.9.(2) and moment condition 3.1.1 on the Lévy measure K we may use dominated convergence to find a boundary $M > 0$ so that

$$\mathcal{R} \leq \lambda^2 M, \tag{3.4.65}$$

which concludes the proof. □

4. Approximation for overshoot moments

4.1. First order approximation for $\mathcal{C}_2$

So far, we have handled the option with payoff $\mathcal{C}_1$. This leaves us with the second option with payoff

$$\mathcal{C}_2 = -P(\tau_B, S_{\tau_B}), \tag{4.1.1}$$

with P as in definition 3.3.4. In the same spirit as in section 1.4, we set up our curve $q^{\mathcal{C}_2}$ via:

Definition 4.1.1.

$$q^{\mathcal{C}_2}(\lambda) := -E\left(P(\tau_B^\lambda, S^\lambda_{\tau_B^\lambda})1_{\left\{\tau_B^\lambda \in [0,T]\right\}}\right). \tag{4.1.2}$$

We pose the following assumptions on our Lévy process X and its characteristic triplet (b, c, K):

Assumption 4.1.2. *1. There exists $\delta > 0$ such that $\Delta X \leq \delta$.*

2. The diffusion part c is greater than zero.

3. The Lévy measure of X has non lattice support.

Remark 4.1.3. *The first assumption ensures that we can find global boundaries for the jump measure of the ladder height processes in our curve, which eases moment-convergence arguments and also implies that*

$$O(X^\lambda, b) \leq \lambda\delta, \ \lambda \in [0,1], \tag{4.1.3}$$

where $O(X^\lambda, b) := X^\lambda_{\tau_b^\lambda} - b$, is the overshoot over the log barrier $b = \log(B)$. (4.1.3) tends to be fulfilled if one can control the expected overshoot over a Lévy martingale, if the barrier goes to infinity. We refer to [18] for a discussion on that subject. The non-zero diffusion part is necessary because otherwise we lack integrability in the Bromwich-inversion of the Wiener-Hopf factors of X, used in the proof of the asymptotical independence of overshoot and hitting time. The difference in the numerical part is negligible as adding a very small ϵ of diffusion to any pure jump model does not make a huge difference quantitatively. The third assumption excludes the case in which the process jumps on a discrete space grid.

The problem we have is that we lack a good representation of τ_B^λ as a function of λ, which makes it difficult to obtain the same smoothness as for q^{C_1}. Fortunately, one is able to obtain exactly enough smoothness for our first order approximation to make sense.

Lemma 4.1.4. *$q^{\mathcal{C}_2} : [0,1] \to \mathbb{R}^+$ is a continuous function and differentiable in $\lambda = 0$.*

Proof. The continuity follows with the same argument as in the proof of lemma 3.4.9, part 2, whereas for the differentiability in 0 see the proof of Proposition 4.1.6 together with the proof of Theorem 4.5.13. $\square$

Definition 4.1.5 (First order approximation)**.**

$$\mathcal{A}^{\mathcal{C}_2} := q^{\mathcal{C}_2}(0) + q^{\mathcal{C}_2'}(0). \tag{4.1.4}$$

Now, the main observation in this section is now that we can simplify this problem to the derivation of a simpler functional, which depends on the overshoot $O(X^\lambda, b)$.

Proposition 4.1.6.

$$q^{\mathcal{C}_2}(\lambda) = -BE\left(v'_{BS}(\tau_B^\lambda, B)O(X^\lambda, b)\right) + o(\lambda), \tag{4.1.5}$$

where we set $v'_{BS}(t, B) = \lim_{x\uparrow B} \frac{\partial}{\partial x} v_{BS}(t, x)$.

Proof. As $\frac{\partial^n}{\partial x^n} P(t, B) = \lim_{x\uparrow B} \frac{\partial^n}{\partial x^n} v_{BS}(t, x),\ n = 0, \dots, 6$, we see that $P(t, B) = 0, t \in [0, T]$. Using a Taylor expansion argument, we write

$$\begin{aligned} &P(\tau_B^\lambda, S^\lambda_{\tau_B^\lambda}) = P(\tau_B^0, B) - 0 = P(\tau_B^\lambda, S^\lambda_{\tau_B^\lambda}) - P(\tau_B^\lambda, B) \\ &= P'(\tau_B^\lambda, B)(S^\lambda_{\tau_B^\lambda} - B) + \int_0^1 \frac{(S^\lambda_{\tau_B^\lambda} - B)^2}{2}(1-s)P''(\tau_B^\lambda, s(S^\lambda_{\tau_B^\lambda} - B))ds. \end{aligned} \tag{4.1.6}$$

Using the fact that $P'(\tau_B^\lambda, B) = v'_{BS}(\tau_B^\lambda, B)$ via construction of P, and $(S^\lambda_{\tau_B^\lambda} - B) = B(e^{O(X^\lambda, b)} - 1)$, another Taylor expansion together with the fact that $O(X^\lambda, b) \leq \lambda\delta$ from assumption 4.1.2 yields in the same spirit as in the proof of theorem 3.4.6:

$$\begin{aligned} &|P'(\tau_B^\lambda, B)(S^\lambda_{\tau_B^\lambda} - B) + \int_0^1 \frac{(S^\lambda_{\tau_B^\lambda} - B)^2}{2}(1-s)P''(\tau_B^\lambda, s(S^\lambda_{\tau_B^\lambda} - B))ds| \\ &= |v'_{BS}(\tau_B^\lambda, B)O(X^\lambda, b) \\ &+ \int_0^1 (1-s)(\frac{(S^\lambda_{\tau_B^\lambda} - B)^2}{2}P''(\tau_B^\lambda, s(S^\lambda_{\tau_B^\lambda} - B)) + \frac{O(X^\lambda, b)^2}{2}e^{O(X^\lambda, b)s})ds| \\ &\leq |v'_{BS}(\tau_B^\lambda, B)O(X^\lambda, b)| + \lambda^2 M \end{aligned} \tag{4.1.7}$$

for some constant $M > 0$. $\square$

4.2. Approximation idea

Hence, we have reduced the problem to evaluating a functional of the type

$$E\left(g(\tau_B^\lambda)O(X^\lambda, b)\right) \tag{4.2.8}$$

for a bounded function $g : [0, T] \to \mathbb{R}$. The strategy for the evaluation of $q^{C_2'}(0)$ involves the following steps:

1. Show weak convergence $\lim_{\lambda\to 0} P^{\frac{O}{\lambda}} = P^{O^*}$ for an asymptotic overshoot distribution P^{O^*}.
2. Calculate moments of O^*, which is defined as a random variable which has distribution P^{O^*}.
3. Show asymptotic independence: $P^{(\frac{O}{\lambda},\tau^\lambda)} \to_w P^{O^*} \otimes P^{\tau^0}$.
4. Evaluate the functional via numerical integration.

4.3. Asymptotic distribution of O^*

Our next concern is the object $\frac{O(X^\lambda,b)}{\lambda}$. The main idea is that instead of studying the overshoot of a curve of processes where the jumps tend to zero over a fixed barrier, we transform the problem into studying the overshoot of a curve of processes $\widetilde{X}^\lambda_{\lambda\in[0,1]}$, where the jump size stays similar, but the barrier tends to infinity. This will be realized in the following way: First, observe that

$$\frac{O(X^\lambda, b)}{\lambda} = \frac{X^\lambda(\tau_b^{X^\lambda}) - b}{\lambda} = \frac{X^\lambda}{\lambda}(\tau_{\frac{b}{\lambda}}^{\frac{X^\lambda}{\lambda}}) - \frac{b}{\lambda} = O(\frac{X^\lambda}{\lambda}, \frac{b}{\lambda}). \tag{4.3.9}$$

Note that $\frac{X^\lambda}{\lambda}(t) = -\frac{1}{\lambda^3}\kappa(\lambda)t + X(\frac{t}{\lambda^2})$. A time change $s = \frac{t}{\lambda^2}$ yields

$$\frac{X^\lambda}{\lambda}(s\lambda^2) = X(s) - \frac{1}{\lambda}\kappa(\lambda)s. \tag{4.3.10}$$

For $\lambda \in (0, 1]$ we define the process $\widetilde{X}^\lambda$ via

$$\widetilde{X}^\lambda(t) = X(t) - \frac{1}{\lambda}\kappa(\lambda)t \tag{4.3.11}$$

and for $\lambda = 0$, observing that $\frac{\kappa(\lambda)}{\lambda} \to E(X(1))$ for $\lambda \to 0$ due to lemma 4.3.1, we define

$$\widetilde{X}^0(t) = X(t) - E(X(1))\, t \tag{4.3.12}$$

hence

$$O(\frac{X^\lambda}{\lambda}, \frac{b}{\lambda}) = O(\widetilde{X}^\lambda, \frac{b}{\lambda}). \tag{4.3.13}$$

To determine the asymptotic distribution P^{O^*}, we want to use the fluctuation theory from chapter 2 to connect the Lévy curve $(X^\lambda)_{\lambda\in[0,1]}$ to its corresponding ladder height processes. Since for a Lévy process the overshoot coincides with the overshoot of its ladder height process, we will study the latter object.

As in definition 2.0.11, for $\lambda \in [0,1]$ let H^λ be the ladder height process of $\widetilde{X}^\lambda$.

Lemma 4.3.1. *For $\lambda \in (0,1]$, $\widetilde{X}^\lambda$ is a supermartingale, and for $\lambda = 0$ a martingale.*

Proof. Using as usual the truncation function id, we get from the Lévy-Khintchin-formula for $\lambda \in [0,1]$:

$$\begin{aligned} &E\left(\widetilde{X}^\lambda(1)\right) \\ &= E\left(X(1)\right) - \frac{\kappa(\lambda)}{\lambda} \\ &= -\frac{\lambda}{2}c - \frac{1}{\lambda}\int (e^{\lambda x} - 1 - \lambda x) K(dx) \\ &\leq 0, \end{aligned} \tag{4.3.14}$$

as $c \geq 0$ and $x \mapsto e^x - 1 - x$ is a non negative function for $x \in \mathbb{R}$. As

$$\lim_{\lambda \to 0} \frac{\kappa(\lambda)}{\lambda} = E\left(X(1)\right),$$

we get

$$E\left(\widetilde{X}^0\right) = 0.$$

□

From Theorem 2.0.12 and because of the previous lemma we know that $H^\lambda_{\lambda \in [0,1]}$ is a family of killed subordinators, which differ only by their killing rate $\eta(\lambda)$, which tends to zero as $\lambda \to 0$.

Theorem 4.3.2. *Let $(H^\lambda)_{\lambda \in [0,1]}$ be the ladder height processes of $\widetilde{X}^\lambda$, $\eta(\lambda) > 0$ their respective killing rate. Then for $x > 0$ we have weak convergence*

$$\lim_{\lambda \to 0} P(O^\lambda(\frac{1}{\lambda}x) > u, U^\lambda(\frac{1}{\lambda}x) > v) = \frac{1}{\mu}\int_{u+v}^\infty K^{H^0}(z,\infty)dz, \tag{4.3.15}$$

where K^{H^0} is the jump measure of H^0, $\mu = E\left(H^0(1)\right)$, $O^\lambda(x)$ the overshoot of H^λ w.r.t x, $U^\lambda(x)$ the undershoot of H^λ w.r.t x.

Proof. Using theorem 2.0.6 we know that

$$P(O^\lambda(x) > u, U^\lambda(x) > v) = \int_{(0,\infty)}\int_{(0,x)} 1_{\{z > u+x-y\}} 1_{\{0 \leq y < x-v\}} \mathcal{U}^\lambda(dy) K^{H^0}(dz) \tag{4.3.16}$$

with

$$\mathcal{U}^\lambda(A) := \int_0^\infty P(H^\lambda(t) \in A)dt = \int_0^\infty e^{-\eta(\lambda)t} P(H^0(t) \in A)dt, \ A \in \mathcal{B}(\mathbb{R}) \tag{4.3.17}$$

being the potential measure of H^λ. Hence, using the notation $\mathcal{U}^\lambda(x) = \mathcal{U}^\lambda([0,x])$,

$$\begin{aligned}
&\int_{(0,\infty)} \int_{(0,x)} 1_{\{z>u+x-y\}} 1_{\{0\le y<x-v\}} \mathcal{U}^\lambda(dy) K^{H^0}(dz) \\
&= \int_{(0,\infty)} \int_{(0,x)} 1_{\{y>u+x-z\}} 1_{\{0\le y<x-v\}} \mathcal{U}^\lambda(dy) K^{H^0}(dz) \\
&= \int_{(u+v,\infty)} 1_{\{z<u+x\}} (\mathcal{U}^\lambda(x-v) - \mathcal{U}^\lambda(u+x-z)) K^{H^0}(dz) \\
&+ \int_{(x+u,\infty)} \mathcal{U}^\lambda(x-v) K^{H^0}(dz).
\end{aligned} \tag{4.3.18}$$

For the second integral, observe that $\mathcal{U}^\lambda(y) < \mathcal{U}^0(y)$, hence, using theorem 2.0.5(2.), we find an $\epsilon > 0$ for λ sufficiently small such that

$$\begin{aligned}
&\int_{(x+u,\infty)} \mathcal{U}^\lambda(x-v) K^{H^0}(dz) \le \int_{(x+u,\infty)} \mathcal{U}^0(x-v) K^{H^0}(dz) \\
&\le \frac{1+\epsilon}{\mu} \int_{(x+u,\infty)} z K^{H^0}(dz) \to 0,
\end{aligned} \tag{4.3.19}$$

for $x \to \infty$ as $\mu = \int_{(0,\infty)} z K^{H^0}(dz)$. To solve the first integral, we note that $\mathcal{U}^\lambda(x)$ converges uniformly in x against $\mathcal{U}^0(x)$. Hence, using an $\frac{\epsilon}{3}$-argument, dominated convergence and Theorem 2.0.5(1.), we conclude that

$$\begin{aligned}
&\int_{(u+v,\infty)} 1_{\{z<u+x\}} (\mathcal{U}^\lambda(x-v) - \mathcal{U}^\lambda(u+x-z)) - (z-u-v) K^{H^0}(dz) \\
&\le \int_{u+v}^{x+u} |\mathcal{U}(x-v) - \mathcal{U}(u+x-z) - \frac{(z-u-v)}{\mu}| K^{H^0}(dz) \\
&+ |\mathcal{U}^\lambda(x-v) - \mathcal{U}^\lambda(u+x-z) - \mathcal{U}^0(x-v) - \mathcal{U}^0(u+x-z)| K^{H^0}(dz) \to 0
\end{aligned} \tag{4.3.20}$$

for $x \to \infty$, therefore,

$$\begin{aligned}
&\lim_{\lambda\to 0} P(O^\lambda(\frac{1}{\lambda}x) > u, U^\lambda(\frac{1}{\lambda}x) > v) \\
&= \frac{1}{\mu} \int_{u+v}^\infty (z-u-v) K^{H^0}(dz) \\
&= \frac{1}{\mu} \int_{u+v}^\infty K^{H^0}(z,\infty) dz
\end{aligned} \tag{4.3.21}$$

□

This result leads to the limiting distribution of P^{O^*} via the following corollary:

Corollary 4.3.3. *The asymptotic overshoot distribution has the law of $O^* := UZ$, where U is uniform-[0,1] distributed and independent of Z, and Z has the cumulative distribution function*

$$P(Z \leq x) = \frac{d^{H^0}}{\mu} + \frac{1}{\mu}\int_{[0,x]} zK^{H^0}(dz), \ \ x \geq 0, \tag{4.3.22}$$

where d^{H^0} is the drift of H^0 (w.r.t the truncation function 0), $\mu = E\left(H^0(1)\right)$.

Proof. Using the notation from the previous theorem, we see that

$$\begin{aligned} 1 - \lim_{\lambda\to 0} P(O^\lambda(\frac{1}{\lambda}x) = 0) &= \lim_{\lambda\to 0} P(O^\lambda(\frac{1}{\lambda}x) > 0, U^\lambda(\frac{1}{\lambda}x) > 0) \\ &= \frac{1}{\mu}\int_0^\infty K^{H^0}(z,\infty)dz. \end{aligned} \tag{4.3.23}$$

As $\mu = d^{H^0} + \int_0^\infty zK^{H^0}(dz) = d^{H^0} + \int_0^\infty K^{H^0}(z,\infty)dz$ using partial integration, we get $P(O^\lambda(\frac{1}{\lambda}x) = 0) = \frac{d^{H^0}}{\mu}$. Furthermore, applying Theorem 2.0.5 ,Theorem 2.0.6 and Theorem 4.3.2, we observe for $z > 0$:

$$\begin{aligned} &\lim_{\lambda\to 0} P(O^\lambda(\frac{1}{\lambda}x) + U^\lambda(\frac{1}{\lambda}x) \in [0,z]) \\ &= \lim_{\lambda\to 0}\int_0^z\int_0^y 1_{[0,y]}(\frac{1}{\lambda}x - u)\mathcal{U}^\lambda(du)K^{H^0}(dy) \\ &= \lim_{\lambda\to 0}\int_0^z (\mathcal{U}^\lambda([0,\frac{1}{\lambda}x]) - \mathcal{U}^\lambda([0,\frac{1}{\lambda}x - y]))K^{H^0}(dy) \\ &= \int_0^z \frac{1}{\mu}yK^{H^0}(dy). \end{aligned} \tag{4.3.24}$$

Now, denoting $Z := O^* + U^*$ with $U^* := \lim_{\lambda\to 0} U^\lambda(\frac{1}{\lambda}x)$ and using Theorem 4.3.2 and a result in [49], one sees that

$$\begin{aligned} &P((1-U)Z > u, UZ > y) = P(Z > u + y, \frac{y}{Z} < U < 1 - \frac{z}{Z}) \\ &= \int\int (1_{(z>y+u)}1_{(\theta\in(y/z,1-u/z))}1_{(\theta\in(0,1))}\frac{1}{\mu}z)d\theta K^{H^0}(dz) \\ &= \int 1_{(z>y+u)}\frac{z-u-y}{z}\frac{z}{y}K^{H^0}(dz) \\ &= \frac{1}{\mu}\int 1_{(z>y+u)}(\int 1_{(y+u<s<z)(ds)})K^{H^0}(dz) \\ &= \frac{1}{\mu}\int\int 1_{(z>y+u)}(\int 1_{(y+u<s)})K^{H^0}(dz)(ds) \\ &= \frac{1}{\mu}\int K^{H^0}(s,\infty)ds \\ &= \lim_{\lambda\to 0} P(O^\lambda(\frac{x}{\lambda}) > u, U^\lambda(\frac{x}{\lambda}) > y) \\ &= P(O^* > u, U^* > y). \end{aligned} \tag{4.3.25}$$

□

The previous two results imply convergence in distribution for the overshoots: $\lim_{\lambda \to 0} O(H^\lambda, \frac{b}{\lambda}) = O^*$.

4.4. Moments of O^*

For the further progress, we need convergence of moments of the overshoots and ladder processes, which is ensured by the following lemma:

Lemma 4.4.1. *For $\lambda \to 0$*

1.
$$E\left(O(H^\lambda, \frac{b}{\lambda})\right) \to E\left(O^*\right). \tag{4.4.26}$$

2.
$$E\left(H^\lambda(1)\right) \to E\left(H^0(1)\right). \tag{4.4.27}$$

3.
$$\operatorname{Var}\left(H^\lambda(1)\right) \to \operatorname{Var}\left(H^0(1)\right). \tag{4.4.28}$$

Proof. As the jumps of X are bounded by $\delta > 0$ due to condition 4.1.2, using the representation of $\widetilde{X}^\lambda$ in (4.3.11), we see that $O(H^\lambda, \frac{b}{\lambda}) \leq \delta, \lambda \in (0,1]$ and $O^* < \delta$ has to hold. Thus, the first part follows by convergence in law. Furthermore, we know that the jumps of H^0 are bounded by δ, hence $\int_{(0,\infty)} y^{2+\epsilon} \mathbb{K}^{H^0}(dy) < \infty$ for $\epsilon > 0$, as H is a subordinator, which implies $E\left((H^0(1))^{2+\epsilon}\right) < \infty$ due to the Lévy-Khintchin formula. But as H^λ is the subordinator H^0 with killing rate $\eta(\lambda)$, we know that

$$E\left((H^0(1))^{2+\epsilon}\right) = \sup_{\lambda \in [0,1]} E\left((H^\lambda(1))^{2+\epsilon}\right), \tag{4.4.29}$$

hence, using [27] Theorem 27.2, this implies $H^\lambda(1)_{\lambda \in [0,1]}$ and $(H^\lambda(1))^2_{\lambda \in [0,1]}$ are uniformly integrable. Hence

$$\lim_{\lambda \to 0} E\left((H^\lambda(1))\right) = \lim_{\lambda \to 0} \lim_{c \to \infty} (E\left((H^\lambda(1)) 1_{\left\{H^\lambda(1) \leq c\right\}}\right) + E\left((H^\lambda(1)) 1_{\left\{H^\lambda(1) > c\right\}}\right)), \tag{4.4.30}$$

where the first expectation converges to $E\left((H^0(1))\right)$ due to weak convergence and the second one to zero due to uniform integrability. The same argument holds true for $(H^\lambda(1))^2_{\lambda \in [0,1]}$. □

Corollary 4.4.2.

$$E\left(O^*\right) = \frac{\operatorname{Var}\left(H^0(1)\right)}{2E\left(H^0(1)\right)} \tag{4.4.31}$$

Proof. Corollary 4.3.3 directly implies

$$E(O^*) = \frac{E(U)\int y^2 K^{H^0}(dy)}{E(H^0(1))} \tag{4.4.32}$$

for U uniform distributed on $[0,1]$. But as H^0 is a subordinator, $\int y^2 K^{H^0}(dy) = \mathrm{Var}(H^0(1))$ and $E(U) = \frac{1}{2}$, hence the result follows. □

This means that for the computation of $E(O^*)$ we need moments of the ladder height process H^0. Unfortunately, due to the fact that $\widetilde{X}$ is a martingale, we have not found a way to compute these directly. On the other hand for $\lambda \in [0,1)$, we are able to compute moments $E\left(H^\lambda(1)\right)$ and $\mathrm{Var}\left(H^\lambda(1)\right)$:

Theorem 4.4.3 (Cumulants of H^λ)**.** *For $\lambda \in [0,1)$, the first two cumulants of H^λ are given by*

$$E\left(H^\lambda(1)\right) = \eta(\lambda)\frac{1}{\pi}\int_0^\infty \Re\left(\frac{1}{(R+iu)^2}\log(-\kappa_{\widetilde{X}^\lambda}(R+iu))\right)du \tag{4.4.33}$$

$$\begin{aligned} \mathrm{Var}\left(H^\lambda(1)\right) &= \eta(\lambda)\Big(\frac{1}{\pi}\int_0^\infty \Re\left(\frac{2}{(R+iu)^3}\log(-\kappa_{\widetilde{X}^\lambda}(R+iu))\right)du \\ &+ \Big(\frac{1}{\pi}\int_0^\infty \Re\left(\frac{1}{(R+iu)^2}\log(-\kappa_{\widetilde{X}^\lambda}(R+iu))\right)du\Big)^2\Big) \end{aligned} \tag{4.4.34}$$

with $R \in (0, \gamma^\lambda(0))$, where $\gamma^\lambda(0)$ is the unique positive zero so that $\kappa_{\widetilde{X}^\lambda}(\gamma^\lambda(0)) = 0$, $\kappa_{\widetilde{X}^\lambda}$ being the characteristic exponent of $\widetilde{X}^\lambda$, $\eta(\lambda)$ being the killing rate of H^λ.

Remark 4.4.4. *Note that $\gamma^\lambda(0)$ is unique because, due to Lemma 4.3.1, $\widetilde{X}^\lambda$ is a supermartingale and hence the first derivative of $\kappa_{\widetilde{X}^\lambda}$ has a negative slope, therefore the claim follows from the convexity of the characteristic exponent. Note that for $\lambda \to 0$ both the nominator and the denominator in* (4.4.31) *do not make any sense because $\gamma^\lambda(0) \to 0$. This is why we are not allowed to use the Laplace inversion techniques applied in the proof. But as corollary 4.4.1 ensures the convergence of $E\left(H^\lambda(1)\right) \to E(H^0(1))$ as well as $\mathrm{Var}\left(H^\lambda(1)\right) \to \mathrm{Var}(H^0(1))$, we may use an adaptive algorithm to obtain an approximation for $E(O^*)$.*

Proof. W.l.o.g. let $\lambda = 1$ (in this case $X = \widetilde{X}$). Using the Wiener-Hopf-factorization from Theorem 2.0.13 for $\widetilde{X}$, one knows that on the one hand,

$$e^{-\kappa(\alpha,\beta)} = E\left(e^{-\alpha L^{-1}(1)-\beta H(1)}1_{\{1<L(\infty)\}}\right), \tag{4.4.35}$$

with $\kappa(\alpha,\beta)$ being the Wiener-Hopf factor of $\widetilde{X}$ having the expression

$$\kappa(\alpha,\beta) = \eta(1) + \alpha a + \beta b + \int_{(0,\infty)^2}(1 - e^{-\alpha x - \beta y})\Gamma(dx,dy), \tag{4.4.36}$$

and on the other hand,

$$\kappa(\alpha,\beta) = \exp\left(\int_0^\infty \int_{(0,\infty)} (e^{-t} - e^{-\alpha t - \beta x}) \frac{1}{t} P^{\widetilde{X}(t)}(dx)dt\right), \tag{4.4.37}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial\beta}\kappa(0,\beta)|_{\beta=0} \\
&= \kappa(0,0)\frac{\partial}{\partial\beta}|_{\beta=0} \int_0^\infty \int_{(0,\infty)} (e^{-t} - e^{-\alpha t-\beta x})\frac{1}{t}P^{\widetilde{X}(t)}(dx)dt \\
&= \eta(1)\int_0^\infty \int_{(0,\infty)} \frac{\partial}{\partial\beta}\kappa(0,\beta)|_{\beta=0}(e^{-t} - e^{-\alpha t-\beta x})\frac{1}{t}P^{\widetilde{X}(t)}(dx)dt \\
&= \eta(1)\int_0^\infty \int_{(0,\infty)} x\frac{1}{t}P^{\widetilde{X}(t)}(dx)dt,
\end{aligned} \tag{4.4.38}$$

where interchanging differentiating and integration works because of Lemma 4.4.5 and [19], Satz 5.7.

Since Lemma 4.4.5 ensures the application of Fubini's theorem, we now want to apply inverse Laplace transformation for x^+, which is applicable due to Lemma 4.4.6. Hence, using [42], Theorem 9.11 we compute:

$$\int_0^\infty \int_{(0,\infty)} x\frac{1}{t}P^{\widetilde{X}(t)}(dx)dt = \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{R+i\mathbb{R}}\int_{(0,\infty)} \frac{1}{z^2}\frac{1}{t^{1-\epsilon}}e^{t\kappa_{\widetilde{X}}(z)}dtdz. \tag{4.4.39}$$

Substituting $-s = t\kappa_X(z)$ ($Re(\kappa_X(z)) < 0$ for $R \in (0, \gamma^\lambda(0))$), using the integral representation of the gamma function and a power series representation leads to

$$\begin{aligned}
&\lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{R+i\mathbb{R}}\int_{(0,\infty)} \frac{1}{z^2}\frac{1}{t^{1-\epsilon}}e^{t\kappa_{\widetilde{X}}(z)}dtdz \\
&= \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{R+i\mathbb{R}}\frac{1}{z^2}\int_{(0,\infty)}\frac{1}{s^{1-\epsilon}}e^{-s}(-\kappa_{\widetilde{X}}(z))^{-\epsilon}dsdz \\
&= \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{R+i\mathbb{R}}\frac{1}{z^2}\Gamma(\epsilon)(\kappa_{\widetilde{X}}(z))^{-\epsilon}dz \\
&= \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{R+i\mathbb{R}}\frac{1}{z^2}\Gamma(\epsilon)\sum_{n=0}^\infty \frac{-\epsilon^n(\log(-\kappa_{X_1}(z)))^n}{n!}dz \\
&= \lim_{\epsilon\to 0}\frac{1}{2\pi i}\sum_{n=0}^\infty\int_{R+i\mathbb{R}}\frac{1}{z^2}\Gamma(\epsilon)\frac{-\epsilon^n(\log(-\kappa_{X_1}(z)))^n}{n!}dz,
\end{aligned} \tag{4.4.40}$$

where we are allowed to interchange sum and integral as we have a geometric series for ϵ small enough. Furthermore, as $\lim\limits_{\epsilon\to 0}\epsilon\Gamma(\epsilon) = 1$, $z \to \frac{1}{z^2}$ is a holomorph function on $z \neq 0$ and applying monotone convergence, one gets

$$\begin{aligned}
&\lim_{\epsilon \to 0} \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{R+i\mathbb{R}} \frac{1}{z^2} \Gamma(\epsilon) \frac{-\epsilon^n (\log(-\kappa_{X_1}(z)) =^n}{n!} dz \\
&= \frac{1}{2\pi i} \int_{R+i\mathbb{R}} \frac{1}{z^2} \log(-\kappa_{X_1}(z)) dz \\
&= \frac{1}{\pi} \int_0^{\infty} \Re(\frac{1}{(R+iu)^2} \log(-\kappa_{\widetilde{X}}((R+iu)))) du,
\end{aligned} \tag{4.4.41}$$

where the last equality follows as the integrand is an even function.

Applying the chain rule for differentiation and a similar argument as in Lemma 4.4.6, one gets

$$\begin{aligned}
&- \frac{\partial^2}{\partial \beta^2} \kappa(0,\beta)|_{\beta=0} \\
&= \kappa(0,0)(\int_0^{\infty} \int_{(0,\infty)} x^2 \frac{1}{t} \mathbb{P}(X_t \in dx) dt + (\int_0^{\infty} \int_{(0,\infty)} x \frac{1}{t} \mathbb{P}(X_t \in dx) dt)^2)
\end{aligned} \tag{4.4.42}$$

and with the same reasoning as before

$$\begin{aligned}
&\int_0^{\infty} \int_{(0,\infty)} x^2 \frac{1}{t} \mathbb{P}(X_t \in dx) dt \\
&= \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{R+i\mathbb{R}} \frac{1}{z^3} \int_{(0,\infty)} \frac{1}{s^{1-\epsilon}} e^{-s} (-\kappa_{X_1}(z))^{-\epsilon} ds dz \\
&= \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{R+i\mathbb{R}} \frac{1}{z^3} \Gamma(\epsilon) (\kappa_{X_1}(z))^{-\epsilon} dz \\
&= \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{R+i\mathbb{R}} \frac{1}{z^3} \Gamma(\epsilon) \sum_{n=0}^{\infty} \frac{-\epsilon^n (\log(-\kappa_{X_1}(z)))^n}{n!} dz \\
&= \lim_{\epsilon \to 0} \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{R+i\mathbb{R}} \frac{1}{z^3} \Gamma(\epsilon) \frac{-\epsilon^n (\log(-\kappa_{X_1}(z)))^n}{n!} dz \\
&= \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{R+i\mathbb{R}} \frac{1}{z^3} \Gamma(\epsilon) \epsilon \log(-\kappa_{X_1}(z)) dz \\
&= \frac{1}{2\pi i} \int_{R+i\mathbb{R}} \frac{1}{z^3} \log(-\kappa_{X_1}(z)) dz \\
&= \frac{2}{\pi} \int_0^{\infty} \Re(\frac{1}{(R+iu)^3} \log(-\kappa_{\widetilde{X}}((R+iu)))) du.
\end{aligned} \tag{4.4.43}$$

This concludes the proof, as $E\left(H^1(1)\right) = \frac{\partial}{\partial \beta} \kappa(0,\beta)|_{\beta=0}$ and $\mathrm{Var}\left(H^1(1)\right) = -\frac{\partial^2}{\partial \beta^2} \kappa(0,\beta)|_{\beta=0}$. □

Note that the proof may be generalized for higher overshoot moments in the same manner leading to an approximation formula for overshoot moments.

Lemma 4.4.5. *Under the assumptions of the previous theorem, the following holds:*

1. *For $\beta \geq 0$:*
$$\int_0^\infty \int_{(0,\infty)} e^{-\beta x} x \frac{1}{t} \mathbb{P}(\widetilde{X}(t) \in dx) dt < \infty. \tag{4.4.44}$$

2.
$$\int_0^\infty \int_{(0,\infty)} x \frac{1}{t} \mathbb{P}(\widetilde{X}(t) \in dx) dt = \lim_{\epsilon \to 0} \int_0^\infty \int_{(0,\infty)} x \frac{1}{t^{1-\epsilon}} P^{\widetilde{X}(t)}(dx) dt. \tag{4.4.45}$$

Proof. First note that for $\beta \geq 0$

$$\begin{aligned} &\int_0^\infty \int_{(0,\infty)} e^{-\beta x} x \frac{1}{t} P^{\widetilde{X}(t)}(dx) dt \leq \int_0^\infty \int_{(0,\infty)} x \frac{1}{t} P^{\widetilde{X}(t)}(dx) dt \\ &= \int_0^1 \frac{E\left(X(t)^+\right)}{t} dt + \int_1^\infty \frac{E\left(X(t)^+\right)}{t} dt \end{aligned} \tag{4.4.46}$$

We treat both integrals separately. Denote $\mu := E(X(1))$ and $M(t) := X(t) - \mu t$. This leads to $E\left(X(t)^+\right) = E\left(f(M(t), t)\right)$ with $f(x,t) := (x + \mu t)^+, \ x \in \mathbb{R}, t \geq 1$. Because e^X is a martingale, Jensen's inequality leads to $\mu < 0$. For each $t \geq 1$ we find a fourth order monomial dominating and being tangent to $f(.,t)$. Define $c := -\frac{27}{256\mu^3}$, then
$f(M(t), t) \leq c \frac{M(t)^4}{t^3}$, hence
$E\left(X(t)^+\right) \leq c \frac{E\left(M(t)^4\right)}{t^3} = O(t^{-1})$ and this implies
$\int_1^\infty \frac{E\left(X(t)^+\right)}{t} dt < \infty$.
For the second integral observe

$$E\left(X(t)^+\right) \leq E\left(|X(t)|\right) \leq ||X(t)||_{L^2} \leq ||X(t)||_{H^2},$$

with the H_2-Norm defined as in [40], and $||.||_{L^2}$ denoting the L^2-Norm. Furthermore the moment conditions imply

$$||X_t||_{H^2} = E\left([X,X]_t\right)^{\frac{1}{2}} + ||bt||_{L^2} = ct^{\frac{1}{2}} + bt \leq Ct^{\frac{1}{2}} \text{ for } t \in [0,1], \tag{4.4.47}$$

for some $C > 0$, hence $\int_0^1 \frac{E\left(X(t)^+\right)}{t} dt \leq \infty$. The second assertion follows with monotone convergence. □

Lemma 4.4.6. *For $R \in (0, \gamma^\lambda(0))$:*

$$\int_{R+i\mathbb{R}} \int_{(0,\infty)} |\frac{1}{z^2} \frac{1}{t^{1-\epsilon}} e^{t\kappa_{\widetilde{X}}(z)}| dt dz < \infty \tag{4.4.48}$$

Proof. Observe that $\Re(\kappa_{\widetilde{X}}(z)) < 0$, hence

$$\int_{(0,\infty)} |\frac{1}{t^{1-\epsilon}} e^{t\kappa_{\widetilde{X}}(z)}| dt = -\Re(\kappa_{\widetilde{X}}(z))^{\epsilon}\Gamma(\epsilon). \tag{4.4.49}$$

Furthermore for $z \in R + i\mathbb{R}$, $|z|$ is bounded away from zero, hence $|\frac{-\Re(\kappa_{\widetilde{X}}(z))^{\epsilon}}{z^2}|$ is integrable on $R + i\mathbb{R}$. $\square$

4.5. Asymptotic independence of the overshoot and the first barrier crossing time

The previous section gave us the distribution of $\lim_{\lambda\to 0}(\frac{O}{\lambda}) = O^*$. We now want to show the weak convergence for the first crossing times and asymptotic independence:

$$\lim_{\lambda\to 0} P^{(\frac{O}{\lambda},\tau^\lambda)} = P^{O^*}\otimes P^{\tau^0} \tag{4.5.50}$$

The strategy for the proof is using the representation from [41]. They showed the following theorem:

Theorem 4.5.1. *Let X be a Levy process with $E\left(X(1)\right) = 0$ and $c > 0$ in the characteristic triple (b, c, K) such that*

1.

$$\int_{-\infty}^{-1} e^{sy}K(dy) < \infty, \textit{for all } s \in (-\infty, 0).$$

2.

$$\int_{1}^{\infty} e^{sy}K(dy) < \infty, \textit{for some } s \in (0, \infty).$$

3.

$$\kappa(-r_K) \in (0, \infty],$$

with $r_K := \sup\left\{s \geq 0; \int_1^\infty e^{sy}K(dy) < \infty\right\}$, *$\kappa$ the Laplace exponent of X.*

4.

$$\forall B \in (0, B_K)\ \exists C, R_0 : |\int_1^\infty e^{qy}K(dy)| \leq C|q|$$

for $q \in \{z \in \mathbb{C}, \mathrm{Re}\,(z) \in [-B, 0], |\mathrm{Im}\,(z)\,| \geq R_0\}$ *and*

$$B_K := \sup\left\{b > 0; q \mapsto \int_1^\infty e^{qy}K(dy)\right.$$
$$\left.\textit{admits a meromorphic extension to} \{q \in \mathbb{C}, \mathrm{Re}\,(q) > -b\}\right\}.$$

Then

$$\lim_{x\to\infty} P^{(\frac{\tau^x}{x^2},O(x))} = P^\nu \otimes P^\omega, \tag{4.5.51}$$

where ν is the hitting time of level $\sqrt{\frac{1}{\mathrm{Var}(X(1))}}$ of a standard Brownian motion, ω the distribution of $O^*(X)$ and the limes means weak convergence.

Their proof relies on finding an asymptotic expansion of the joint Laplace transform

$$F(\alpha,\beta,x) := E\left(\exp(-\alpha\tau_x - \beta O(x)1_{\{\tau_x<\infty\}})\right) :$$

$$F(\alpha,\beta,x) \sim C_0(\alpha,\beta)e^{-\gamma_0(\alpha)x}\text{for } x \to \infty,$$

where $\gamma_0(\alpha)$ is the left zero of $\kappa(.) - \alpha$, with κ being the Laplace exponent of $X(1)$. This is done via finding a representation of the Laplace transformation of F

$$\widetilde{F}(\alpha,\beta,q) := \int_0^\infty e^{-qx}F(\alpha,\beta,x)dx$$

and then using inverse Laplace transform and residual calculus.

In our setting is $E\left(X(1)\right) < 0$ but when we use the space and time transformed process $\widetilde{X}^\lambda = X - \frac{\kappa(\lambda)}{\lambda}I$ from the previous chapter we get a centered process in the limit. For this chapter let κ^λ be the Laplace exponent of $\widetilde{X}^\lambda$. The strategy will be to make everything above dependent on λ and show that by taking the limes $\lambda \to 0$ everything still works fine.

For simplicity, we assume $c = 1$ in the following.

Lemma 4.5.2. *For $\lambda \in [0,1], \widetilde{X}^\lambda$ fulfills all conditions of Theorem 4.5.1 except $E\left(\widetilde{X}^\lambda(1)\right) = 0$ for $\lambda \in (0,1]$.*

Proof. This follows from the conditions on X^1 in Assumption 4.1.2 $\square$

Lemma 4.5.3. *There exists $C > 0$ such that for all $\lambda \in [0,1], \alpha \in [0,C]$:*

- *There exists a unique $\gamma_0^{\lambda *}(\alpha) \geq 0$ such that $\kappa^\lambda(\gamma_0^{\lambda *}(\alpha)) = \alpha$.*
- *There exists a unique $\gamma_0^{\lambda}(\alpha) \geq 0$ such that $\kappa^\lambda(-\gamma_0^{\lambda}(\alpha)) = \alpha$.*

Proof. This follows from the convexity of $\kappa^\lambda(z)$ and the fact that $\kappa^\lambda(z) = \kappa(z) - \frac{\kappa(\lambda)}{\lambda}z$ for $z \in \mathbb{R}$. $\square$

Lemma 4.5.4. *For any $\alpha \in [0,C]$, $B \in (0,B_K), \lambda \in [0,1]$, $\kappa^\lambda(\cdot) - \alpha$ admits a finite number of conjugated zeros in the strip $\mathcal{D}_{-B,0} := \{q \in \mathbb{C}, -B \leq \Re(q) \leq 0\}$. This set of zeros is equal to:*

1. *$\left\{-\gamma_0^\lambda(\alpha), -\gamma_1^\lambda(\alpha), \bar{\gamma}_1^\lambda(\alpha), \ldots - \gamma_l^\lambda(\alpha), \bar{\gamma}_l^\lambda(\alpha)\right\}$ for $\alpha > 0$.*
2. *$\left\{0, -\gamma_0^\lambda(\alpha), -\gamma_1^\lambda(\alpha), \bar{\gamma}_1^\lambda(\alpha), \ldots - \gamma_l^\lambda(\alpha), \bar{\gamma}_l^\lambda(\alpha)\right\}$ for $\alpha = 0$.*
3. *$\left\{0 = -\gamma_0^\lambda(\alpha), -\gamma_1^\lambda(\alpha), \bar{\gamma}_1^\lambda(\alpha), \ldots - \gamma_l^\lambda(\alpha), \bar{\gamma}_l^\lambda(\alpha)\right\}$ for $\alpha = 0$ and $\lambda = 0$, in this case $-\gamma_0$ is a double zero.*

Proof. In the previous lemma we showed that the only real zeros of $\kappa^\lambda(\cdot) - \alpha$ are $\gamma_0^{\lambda *}(\alpha)$ and $-\gamma_0^\lambda(\alpha)$. We first show that
there exist no complex zeros in the strip $\left\{q \in \mathbb{C}, -\gamma_0^\lambda(\alpha) \leq \Re(q) \leq \gamma_0^{\lambda *}(\alpha)\right\}$:

- If $-\gamma_0^\lambda(\alpha) < \Re(q) < \gamma_0^{\lambda *}(\alpha)$:

$$|e^{\kappa^\lambda(q)-\alpha}| = |E\left(e^{-q\widetilde{X}^\lambda(1)-\alpha}\right)| \leq E\left(-\Re(q)\widetilde{X}^\lambda(1) - \alpha\right) = e^{\kappa^\lambda(\Re(q))-\alpha} < 1. \tag{4.5.52}$$

– If $q = -\gamma_0^\lambda(\alpha) + iu, u \in \mathbb{R}$, we calculate $\kappa^\lambda(q) - \alpha$:

$$\begin{aligned}\kappa^\lambda(q) - \alpha = \kappa^\lambda(-\gamma_0^\lambda(\alpha)) - \alpha - \frac{q^2}{2} + \int e^{\gamma_0^\lambda(\alpha)x}(\cos(bx) - 1)K(dx) \\ + i\left(-u\gamma_0^\lambda(\alpha) + bu - \int (e^{\gamma_0^\lambda(\alpha)x}\sin(ux) - uh(x))K(dx)\right).\end{aligned} \tag{4.5.53}$$

As $\kappa^\lambda(-\gamma_0^\lambda(\alpha)) - \alpha = 0$, we know that $\Re(\kappa^\lambda(q) - \alpha) \leq -\frac{u^2}{2}$ holds and this implies $\Re(\kappa^\lambda(q) - \alpha) = 0 \Leftrightarrow u = 0$. The same argument applies to $q = \gamma_0^{\lambda *}(\alpha) + iu$

Furthermore, according to assumption 4.1.2, there exists $R > R_0 > 0, k > 0$ such that

$$|\kappa^\lambda(z)| \geq k|q^2| \tag{4.5.54}$$

for $z \in \{z \in \mathbb{C} : B \leq \Re(z) \leq 0, |\Im(z)| \leq R\}$. Hence, as $\kappa^\lambda(\cdot) - \alpha$ is meromorphic in $\{z \in \mathbb{C}, \Re(z) > -B\}$, it admits a finite number of zeros in the above compact domain.

□

We now have to define several quantities depending on λ which will be needed later:

Definition 4.5.5. *For $\lambda \in [0, 1]$, $\alpha \geq 0$, $\beta \geq 0$ define:*

1. *$\kappa^\lambda(z)$ the Laplace exponent of $\widetilde{X}^\lambda$ for $\mathrm{Re}\,(z) \in (-r_K, \infty)$.*
2. *$\gamma_i(\lambda, \alpha)$ the solutions of $\kappa^\lambda - \alpha = 0$ in the left half plane.*
3. *$F(\lambda, \alpha, \beta, x) := E\left(\exp(-\alpha\tau^\lambda - \beta O(\widetilde{X}^\lambda, x))1_{(\tau^\lambda < \infty)}\right)$, if $\sigma > 0$ or the Lévy measure has no atoms.*
4. *$\bar{F}(\lambda, \alpha, \beta, x) := F(\lambda, \alpha, \beta, x)1_{[0,\infty)}(x) + (1 + x)1_{[-1,0)}(x), \forall x \in \mathbb{R}$.*
5. *$\widetilde{\bar{F}}(\lambda, \alpha, \beta, q) := \int_0^\infty e^{-qx}\bar{F}(\lambda, \alpha, \beta, x)dx$, for all $q \in \mathbb{C}$ such that $\int_0^\infty |e^{-qx}\bar{F}(\lambda, \alpha, \beta, x)|dx < \infty$.*
6. *$\widetilde{F}(\lambda, \alpha, \beta, q) := \int_0^\infty e^{-qx}F(\lambda, \alpha, \beta, x)dx$, for all $q \in \mathbb{C}$ such that $\int_0^\infty |e^{-qx}F(\lambda, \alpha, \beta, x)|dx < \infty$.*
7. *$C_0(\lambda, \alpha, \beta) := Res(\widetilde{\bar{F}}(\lambda, \alpha, \beta, q), \gamma_0(\lambda, \alpha))$ with $Res(f(x), y)$ being the residuum of the function f at the point y.*
8. *$C_i(\lambda, \alpha, \beta) := e^{\gamma_i(\lambda,\alpha)x}Res(e^{zx}\widetilde{\bar{F}}(\lambda, \alpha, \beta, q), \gamma_i(\lambda, \alpha))$.*

Lemma 4.5.6. *The above functions are continuous in all parameters.*

Proof. 1. $\widetilde{X}^\lambda = X - \frac{\kappa(\lambda)}{\lambda}I$ implies $\varphi^\lambda(z) = \kappa^1(z) - \frac{\kappa(\lambda)}{\lambda}z$ which is continuously partially differentiable in λ due to the Lévy Khintchine formula.

2. As $\kappa^\lambda(z)-\alpha$ is continuous in z and differentiable in λ and α, we may apply the implicit function theorem to show that the zeros $\gamma_i(\lambda,\alpha)$ are continuous in λ and α.
3. Let $D(\mathbb{R})$ be the Skorokhod space. Due to [26] Chapter VII 2.11 the operation $\tau_x : D(\mathbb{R}) \to \mathbb{R}$ is continuous on all $\alpha \in D(\mathbb{R})/J_1$ where $J_1 := \{x \in \mathbb{R} : \tau_x(\alpha) < \tau_{x+}(\alpha)\}$. Furthermore, the function $\alpha \to \alpha^{\tau_x}$ is continuous in Skorokhod in all $\alpha \in D(\mathbb{R})/(J_1 \cup J_2)$, $J_2 := \{x \in \mathbb{R} : \exists t \in \mathbb{R}^+ \alpha(t-) = x < \alpha(t)\}$. We want to show that J_1 and J_2 are nullsets for all $P^\lambda, \lambda \in [0,1]$.

   We start with J_2. Consider $\widetilde{\tau}_x := \inf\{t \in \mathbb{R}^+, X_-(t) \geq x\}$. As X_- is an optional process, $\widetilde{\tau}_x$ is a stopping time. Hence, using the strong Markov property, $P(X \in J_2) = P(\Delta X(0) \neq 0) = 0$.

   Regarding J_1, first note that a point x is said to be *regular* for a Borel set B, if $P_x(\tau^B = 0) = 1$, i.e. if the process hits the set immediately. Due to the strong Markov property, regularity of the point 0 for $(0,\infty)$ implies $P(J_1) = 0$. [34], Theorem 6.5 tells us that this is the case if the Lévy process has unbounded variation or positive drift. Consider now the case that X has a non positive drift and bounded variation. We show that such a process cannot hit positive points, i.e. $P(\inf\{t > 0 : X(t) = x\} < \infty) = 0$. Because the process is of bounded variation, we may write it as the difference of two pure jump subordinators and a possible negative drift, $X = X^u - X^d - a, a \geq 0$. As they are independent, we may condition on the path of one of them and note that

$$\begin{aligned} 0 &= E\left(P(\inf\left\{t > 0 : X^u(t) = x + at + X^d(t)\right\} < \infty | X^d)\right) \\ &= P(\inf\{t > 0 : X(t) = x\} < \infty). \end{aligned} \tag{4.5.55}$$

   Hence, it remains to show that $P(\inf\{t > 0 : X^u(t) = x + dt\} < \infty) = 0$. Let us write $\Phi(t,y,\omega) := 1_{\{x+at-X^u(t-)\}}(y)$. As Φ is previsible, we may use [34], Theorem 4.4, to see that

$$P(\inf\{t > 0 : X^u(t) = x + at\} < \infty) \leq E\left(\int_{[0,\infty]} \int_{\mathbb{R}} \Phi(t,y) K^u(dy)dt\right) \tag{4.5.56}$$

   and this equals 0 because $K^u\{y : y = x + dt - X^u(t-)\} = 0$ if the Lévy measure has no atoms.

   As $(x,y) \to e^{-\alpha x - \beta y} 1_{\{x<\infty\}}$ is almost surely a bounded, continuous function for $\alpha, \beta \geq 0$ and

$$X^{\widetilde{\lambda}} \to X^\lambda \text{ for } \widetilde{\lambda} \to \lambda,$$

   [26], VII 3.8, shows

$$\lim_{\widetilde{\lambda} \to \lambda} E\left(-\alpha \tau^{\widetilde{\lambda}} - \beta O(X^{\widetilde{\lambda}})_x 1_{\{\tau^{\widetilde{\lambda}} < \infty\}}\right) = E\left(-\alpha \tau^\lambda - \beta O(X^\lambda)_x 1_{\{\tau^\lambda < \infty\}}\right), \tag{4.5.57}$$

hence continuity follows.

4. This is immediate as $1_{[0,\infty)}(x) + (1+x)1_{[-1,0)}(x)$ is independent of λ.
5. This follows from 4. as continuity extends to Laplace transforms.
6. This follows from 3. as continuity extends to Laplace transforms.
7. This follows from $Res(\widetilde{\widetilde{F}}(\lambda,\alpha,\beta,q), \gamma_0(\lambda,\alpha)) = \frac{1}{2\pi i}\int_{\partial U_{\gamma_0(\lambda,\alpha)}} \widetilde{\widetilde{F}}(\lambda,\alpha,\beta,q)dq$, 2. and 4.
8. Similar to 6.

□

We now start investigating the object $\widetilde{F}(\lambda,\alpha,\beta,q)$. There are two different representations one can work with. In Corollary 2.0.15 we got one representation. Furthermore, there is a second representation in [41] as a solution of an integral equation:

Theorem 4.5.7. *Under the conditions of the first theorem, for $\alpha,\beta \geq 0$, $q \in \mathbb{C}, \Re(q) > 0$ we get:*

$$\begin{aligned} &\widetilde{F}(\lambda,\alpha,\beta,q) \\ &= \frac{1}{\kappa^\lambda(q)-\alpha}\Bigg(\frac{q-\gamma_0^{\lambda *}(\alpha)}{2} + \int_0^\infty \left[\frac{e^{-qy}-e^{-\beta y}}{q-\beta} - \frac{e^{-\gamma_0^{\lambda *}(\alpha) y}-e^{-\beta y}}{\gamma_0^{\lambda *}(\alpha)-\beta}\right] K(dy) \\ &+ R(F(\lambda,\alpha,\beta,\cdot))(q) - R(F(\lambda,\alpha,\beta,\cdot))(\gamma_0^{\lambda *}(\alpha))\Bigg), \end{aligned} \tag{4.5.58}$$

where $\gamma_0^{\lambda *}(\alpha)$ is the unique positive root of $\kappa^\lambda(z) - \alpha$ and

$$R(h)(q) := \int_{-\infty}^0 \int_0^{-y} (e^{-q(b+y)} - 1)h(b)db$$

for $q \in \mathbb{C}, q > 0$ and $h \in \mathcal{L}^\infty(\mathbb{R}^+)$.

Proof. See [41], Theorem 2.5. □

The other representation follows from corollary 2.0.15:

For $\alpha,\beta \geq 0$, $q \in \mathbb{C}, \Re(q) > 0$ we get:

$$\widetilde{F}(\lambda,\alpha,\beta,q) = \frac{\kappa^\lambda(\alpha,q) - \kappa^\lambda(\alpha,\beta)}{(q-\beta)\kappa^\lambda(\alpha,q)} \tag{4.5.59}$$

Note that we use the second representation mainly to ensure integrability conditions, which are hard to come by for the first, arguably nicer representation.

Lemma 4.5.8. *The poles of $\widetilde{F}(\lambda,\alpha,\beta,q)$ correspond to $\gamma_i(\lambda,\alpha)$.*

Proof. First note that representation 2 implies that

$$\widetilde{F}(\lambda, \alpha, \beta, q) = \infty \Leftrightarrow \kappa^{\lambda}(\alpha, q) = 0 \text{ and } \Re(q) \leq 0, \tag{4.5.60}$$

because when $q = \beta$, we may use the rule of L'Hôpital to show

$$\widetilde{F}(\lambda, \alpha, \beta, \beta) = \frac{\kappa'^{\lambda}(\alpha, \beta)}{\kappa^{\lambda}(\alpha, \beta)}. \tag{4.5.61}$$

The Wiener-Hopf representation from Theorem 2.0.13 implies
$q - \kappa^{\lambda}(\alpha) = \kappa^{\lambda}(\alpha, q)\widetilde{\kappa}^{\lambda}(\alpha, q)$. For $\Re(q) > 0$ we obtain using the same theorem:

$$\left|\frac{\kappa^{\lambda}(\alpha, 0)}{\kappa^{\lambda}(\alpha, q)}\right| = \left|E\left(e^{-q\bar{X}_{e_{\alpha}}}\right)\right| < E\left(\left|e^{-q\bar{X}_{e_{\alpha}}}\right|\right) \leq 1. \tag{4.5.62}$$

The same argument holds for $\widetilde{\kappa}^{\lambda}(\alpha, q)$ and $\Re(q) < 0$, hence, using Definition 4.5.5 for $\gamma_i(\lambda, \alpha)$, one gets the desired correspondence. □

We now want to show the following asymptotic expansion:

Theorem 4.5.9. *For any $\lambda \in [0, 1]$, $\alpha \in [0, C]$, $\beta \geq 0$, there exists a positive number $C_0^{\lambda}(\alpha, \beta) > 0$ and complex functions $C_1^{\lambda}(\alpha, \beta, x), \ldots, C_l^{\lambda}(\alpha, \beta, x)$, which are polynomials in x, such that $F(\lambda, \alpha, \beta, x)$ has the following asymptotic expansion as $x \to \infty$:*

$$\begin{aligned} F(\lambda, \alpha, \beta, x) = {} & C_0^{\lambda}(\alpha, \beta)e^{-\gamma_0(\lambda,\alpha)x} \\ & + \sum_{i=1}^{l} \frac{1}{2}(C_i^{\lambda}(\alpha, \beta, x)e^{-\gamma_i(\lambda,\alpha)x} + \bar{C}_i^{\lambda}(\alpha, \beta, x)e^{-\bar{\gamma}_i(\lambda,\alpha)x}) \\ & + O(e^{-Bx}), \end{aligned} \tag{4.5.63}$$

for a boundary $B > 0$ which does not depend on λ, α and β.

Proof. We will prove this theorem in three steps:

1. Replace $F(\lambda, \alpha, \beta, x)$ by $\bar{F}(\lambda, \alpha, \beta, x)$.
2. Show $u \mapsto \widetilde{\bar{F}}(\lambda, \alpha, \beta, q_1 + iu) \in \mathcal{L}(\mathbb{R})$ for $q_1 > 0$.
3. Prove the asymptotic expansion using the Bromwich inversion theorem and residual calculus.

First step:
We extend $F(\lambda, \alpha, \beta, x)$ continuously as in Definition 4.5.5:

$$\bar{F}(\lambda, \alpha, \beta, x) = F(\lambda, \alpha, \beta, x)1_{[0,\infty)}(x) + (1 + x)1_{[-1,0)}(x), \forall x \in \mathbb{R}. \tag{4.5.64}$$

Let $\widetilde{\bar{F}}(\lambda, \alpha, \beta, \cdot)$ be the Laplace transform of $\bar{F}(\lambda, \alpha, \beta, x)$:

$$\widetilde{\bar{F}}(\lambda, \alpha, \beta, q) := \int e^{-qx}\bar{F}(\lambda, \alpha, \beta, x)dx. \tag{4.5.65}$$

We are doing this because we can show in step 2 that $u \mapsto \widetilde{\widetilde{F}}(\lambda, \alpha, \beta, q_1 + iu)$ is an integrable function on $\mathbb{R}$ for $q_1 > 0$. As $\int_{-1}^{0}(1+x)e^{-qx} = \frac{e^{-q}-1-q}{q^2}$, the first representation of $\widetilde{F}(\lambda, \alpha, \beta, q)$ implies:

$$\begin{aligned}\widetilde{\widetilde{F}}(\lambda, \alpha, \beta, q) = \frac{e^{-q}-1-q}{q^2} + \frac{1}{\kappa^\lambda(q)-\alpha}\Bigg(&\frac{q-\gamma_0^{\lambda *}(\alpha)}{2} \\ &+ \int_0^\infty \left[\frac{e^{-qy}-e^{-\beta y}}{q-\beta} - \frac{e^{-\gamma_0^{\lambda *}(\alpha)y}-e^{-\beta y}}{\gamma_0^{\lambda *}(\alpha)-\beta}\right] K(dy) \\ &+ R(F(\lambda, \alpha, \beta, \cdot))(q) - R(F(\lambda, \alpha, \beta, \cdot))(\gamma_0^{\lambda *}(\alpha))\Bigg).\end{aligned} \tag{4.5.66}$$

We see that

$$\begin{aligned}&\frac{\kappa^\lambda(q)-\alpha}{q+\gamma_0^{\lambda *}(\alpha)} = \frac{\kappa^\lambda(q)-\kappa^\lambda(\gamma_0^{\lambda *}(\alpha))}{q+\gamma_0^{\lambda *}(\alpha)} \\ &= \frac{q-\gamma_0^{\lambda *}(\alpha)}{2} + b(\lambda)\frac{q-\gamma_0^{\lambda *}(\alpha)}{q+\gamma_0^{\lambda *}(\alpha)} + \int \frac{e^{-qy}-e^{-\gamma_0^{\lambda *}(\alpha)y}+(q-\gamma_0^{\lambda *}(\alpha))1_{\{|y|<1\}}}{q+\gamma_0^{\lambda *}} K(dy).\end{aligned} \tag{4.5.67}$$

Hence, for all $q \in \mathbb{C}^+$:

$$\begin{aligned}\widetilde{\widetilde{F}}(\lambda, \alpha, \beta, q) = &\frac{e^q-1}{q^2} - \frac{\gamma_0^*(\lambda, \alpha)}{q(q+\gamma_0^*(\lambda, \alpha)} \\ &+ \frac{1}{\kappa(q)-\alpha}(-b(\lambda)\frac{q-\gamma_0^*(\lambda, \alpha)}{q+\gamma_0^*(\lambda, \alpha)} \\ &- \int \frac{e^{-qy}-e^{-y\gamma_0^*(\lambda, \alpha)}+(q-y\gamma_0^*(\lambda, \alpha))y1_{\{|y|<1\}}}{q+\gamma_0^*(\lambda, \alpha)} K(dy) \\ &+ \int_0^\infty (\frac{e^{-qy}-e^{-\alpha y}}{q-\alpha} K(dy) - \frac{\gamma_0^*(\lambda, \alpha)y-e^{-\alpha y}}{\gamma_0^*(\lambda, \alpha)-\alpha}) K(dy) \\ &+ R(F(\lambda, \alpha, \beta, \cdot))(q) - R(F(\lambda, \alpha, \beta, \cdot))(\gamma_0^{\lambda *}(\alpha))).\end{aligned} \tag{4.5.68}$$

4.1.2 implies that $R(F(\lambda, \alpha, \beta, \cdot))$ is an entire function on $\mathbb{C}$ and all integrals are holomorphic in D_{-B_K}. Hence, the right hand side is a meromorpic extension of $\widetilde{\widetilde{F}}(\lambda, \alpha, \beta, \cdot)$ to D_{-B_K}.

Second Step:
The next step is to show that the function $u \mapsto \widetilde{\widetilde{F}}(\lambda, \alpha, \beta, q_1 + iu)$ belongs to $\mathcal{L}^1(\mathbb{R})$ for all $\lambda \in [0, 1]$. Fix $\alpha \in [0, C]$ and $q_1 > 0$. We show the following technical lemma:

Lemma 4.5.10. *Let $\alpha_1 < \alpha_2$. Then:*

1. *$\exists k > 0$ such that for any $q \in \{z \in \mathbb{C} : \alpha_1 \le \Re(q) \le \alpha_2\}$:*

$$\left|\int_{-1}^{1}(e^{-qy}-1+qy1_{\{|y|<1\}})K(dy)\right| \le k|q|, \tag{4.5.69}$$

$$\left|\int_{-1}^{0}(e^{-qy}-1+qy1_{\{|y|<1\}})K(dy)\right|\leq k|q|, \tag{4.5.70}$$

$$\left|\int_{0}^{1}(e^{-qy}-1+qy1_{\{|y|<1\}})K(dy)\right|\leq k|q|, \tag{4.5.71}$$

2. *$\forall d>0,\exists k_0>0$ such that* $\sup_{\Re(q)\leq d}\left|\int_{-\infty}^{-1}(e^{-qy})K(dy)\right|\leq k_0,$

3. *$\forall A>0,\exists k_1>0$ such that $\forall q\in\{z\in\mathbb{C}:-B\leq\Re(q)\leq A,|\Im(q)\geq R_0|\}$:*

$$\left|\int_{-\infty}^{\infty}(e^{-qy}-1+qy1_{\{|y|<1\}})K(dy)\right|\leq k_1(1+|q|), \tag{4.5.72}$$

$$\frac{1}{k_1}|q|^2\leq|\kappa(q)|\leq k_1|q|^2, \tag{4.5.73}$$

4. $\forall h\in\mathbb{R},\ \sup_{\Re(q)\leq h}|R(F(\lambda,\alpha,\beta,q))|<\infty.$

Proof. This is immediate from assumption 4.1.2. □

As 4.5.8 tells us that $\widetilde{F}(\lambda,\alpha,\beta,q)$ has no poles in $\mathbb{C}^+$, $u\mapsto\widetilde{\widetilde{F}}(\lambda,\alpha,\beta,q_1+iu)$ is a continuous function. Now the Lemma above tells us that all the numerators in 4.5.68 are bounded on the line $\{q_1+iu,u\in\mathbb{R}\}$ and the denominators are smaller than $M|q^2|$ for some $M>0$ and $|q|$ large. Hence, $u\mapsto\widetilde{\widetilde{F}}(\lambda,\alpha,\beta,q_1+iu)\in\mathcal{L}^1(\mathbb{R})$

Third Step:

Fix $q_1>0$. Since $u\mapsto\widetilde{\widetilde{F}}(\lambda,\alpha,\beta,q_1+iu)\in\mathcal{L}^1(\mathbb{R})$, we may use the Bromwich inversion formula for Laplace transforms (e.g. [42], Theorem 9.11). As $\bar{F}(\lambda,\alpha,\beta,x)=F(\lambda,\alpha,\beta,x)$ for $x\in\mathbb{R}^+$:

$$e^{q_1x}\bar{F}(\lambda,\alpha,\beta,x)=e^{q_1x}F(\lambda,\alpha,\beta,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{iux}\widetilde{\widetilde{F}}(\lambda,\alpha,\beta,q_1+iu)du, \tag{4.5.74}$$

hence

$$F(\lambda,\alpha,\beta,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{q_1+iux}\widetilde{\widetilde{F}}(\lambda,\alpha,\beta,q_1+iu)du=-i\frac{1}{2\pi}\int_{\Gamma_A}e^{zx}\widetilde{\widetilde{F}}(\lambda,\alpha,\beta,z)dz \tag{4.5.75}$$

with Γ_A being the path

$$\Gamma_A:=\{z=q_1+iu,u\in\mathbb{R},u\text{ increasing}\}. \tag{4.5.76}$$

In Lemma 4.5.8 it has been proven that there exists $R_1>R_0$ such that $\widetilde{\widetilde{F}}(\lambda,\alpha,\beta,z)$ has no poles in the two half-strips $\{z\in\mathbb{C}:-B\leq\Re(z)\leq q_1,|\Im(z)|>R_1\}$. Therefore, $\widetilde{\widetilde{F}}(\lambda,\alpha,\beta,z)$ is holomorphic in this domain. Let $\Gamma_{ABCD,R}$ be the following rectangular path:

$$\Gamma_{ABCD,R}:=\Gamma_{A,R}\cup\Gamma_{B,R}\cup\Gamma_{C,R}\cup\Gamma_{D,R}, \tag{4.5.77}$$

with:

$$\Gamma_{A,R} := \{z = q_1 + iu, |u| < R, u \text{ increasing}\} \tag{4.5.78}$$

$$\Gamma_{B,R} := \{z = u + iR, -B \leq u \leq q_1, u \text{ decreasing}\} \tag{4.5.79}$$

$$\Gamma_{C,R} := \{z = -B + iu, |u| < R, u \text{ decreasing}\} \tag{4.5.80}$$

$$\Gamma_{D,R} := \{z = u - iR, -B \leq u \leq q_1, u \text{ increasing}\}. \tag{4.5.81}$$

Now we apply the residual theorem to the meromorphic extension of $z \mapsto e^{zx}\tilde{\tilde{F}}(\lambda, \alpha, \beta, z)$ in D_{B_K} and we get for any $R > R_1$:

$$\int_{\Gamma_{ABCD,R}} e^{zx}\tilde{\tilde{F}}(\lambda, \alpha, \beta, z)dz$$

$$= 2i\pi \left(C_0^\lambda(\alpha, \beta)e^{-\gamma_0(\lambda,\alpha)x} + \sum_{i=1}^{l} \frac{1}{2}(C_i^\lambda(\alpha, \beta, x)e^{-\gamma_i(\lambda,\alpha)x} + \bar{C_i}^\lambda(\alpha, \beta, x)e^{-\bar{\gamma}_i(\lambda,\alpha)x}) \right) \tag{4.5.82}$$

with

$$C_0^\lambda(\alpha, \beta) := Res(\tilde{\tilde{F}}(\lambda, \alpha, \beta, q), \gamma_0(\lambda, \alpha)) \tag{4.5.83}$$

$$C_i^\lambda(\alpha, \beta, x) := e^{\gamma_i(\lambda,\alpha)x} Res(e^{zx}\tilde{\tilde{F}}(\lambda, \alpha, \beta, q), \gamma_i(\lambda, \alpha)). \tag{4.5.84}$$

$C_0^\lambda(\alpha, \beta)$ does not depend on x because $\gamma_0(\lambda, \alpha)$ is a simple pole of $\tilde{\tilde{F}}(\lambda, \alpha, \beta, q)$. Since $z \mapsto \frac{e^z-1-z}{z^2}$ has an holomorphic extension to $\mathbb{C}$, we conclude that

$$C_0^\lambda(\alpha, \beta) = Res(\tilde{F}(\lambda, \alpha, \beta, q), \gamma_0(\lambda, \alpha)) \tag{4.5.85}$$

$$C_i^\lambda(\alpha, \beta, x) = e^{\gamma_i(\lambda,\alpha)x} Res(e^{zx}\tilde{F}(\lambda, \alpha, \beta, q), \gamma_i(\lambda, \alpha)), \ i \geq 1. \tag{4.5.86}$$

Moreover, note that $\bar{C_i}^\lambda(\alpha, \beta) = e^{\bar{\gamma}_i(\lambda,\alpha)x} Res(e^{zx}\tilde{F}(\lambda, \alpha, \beta, q), \gamma_i(\lambda, \alpha))$. Note that it can be shown that $x \to C_i^\lambda(\alpha, \beta, x)$ is a polynomial function (see [41], section 4.5). Since $u \mapsto \tilde{\tilde{F}}(\lambda, \alpha, \beta, q_1 + iu) \in \mathcal{L}^1(\mathbb{R})$:

$$\begin{aligned} F(\lambda, \alpha, \beta, x) &= -i\frac{1}{2\pi} \lim_{R\to\infty} \int_{\Gamma_{A,R}} e^{zx}\tilde{\tilde{F}}(\lambda, \alpha, \beta, z)dz \\ =& -i\frac{1}{2\pi} \lim_{R\to\infty} (\int_{\Gamma_{ABCD,R}} e^{zx}\tilde{\tilde{F}}(\lambda, \alpha, \beta, z)dz - \int_{\Gamma_{B,R}} e^{zx}\tilde{\tilde{F}}(\lambda, \alpha, \beta, z)dz \\ &- \int_{\Gamma_{C,R}} e^{zx}\tilde{\tilde{F}}(\lambda, \alpha, \beta, z)dz - \int_{\Gamma_{D,R}} e^{zx}\tilde{\tilde{F}}(\lambda, \alpha, \beta, z)dz). \end{aligned} \tag{4.5.87}$$

We claim that the limits of the second and fourth integral are zero and for the second limit

$$\lim_{R\to\infty} \int_{\Gamma_{C,R}} e^{zx}\tilde{\tilde{F}}(\lambda, \alpha, \beta, z)dz = O(e^{-Bx}) \tag{4.5.88}$$

holds uniformly for $\alpha \in [0, C], \beta > 0, \lambda \in [0, 1]$. Hence, as $x \to \infty$:

$$\begin{aligned} F(\lambda, \alpha, \beta, x) = & C_0^\lambda(\alpha, \beta)e^{-\gamma_0(\lambda,\alpha)x} \\ & + \sum_{i=1}^{l} \frac{1}{2}(C_i^\lambda(\alpha, \beta, x)e^{-\gamma_i(\lambda,\alpha)x} + \bar{C_i}^\lambda(\alpha, \beta, x)e^{-\bar{\gamma}_i(\lambda,\alpha)x}) \\ & + O(e^{-Bx}). \end{aligned} \tag{4.5.89}$$

$\square$

Lemma 4.5.11.

$$\lim_{\lambda\to 0} e^{-\gamma_0^\lambda(\lambda^2\alpha)\frac{b}{\lambda}} = E\left(e^{-\alpha\tau_b^{X^0}}\right) \tag{4.5.90}$$

Proof. First note that the definition of γ_0^λ tells us that

$$\kappa^{\widetilde{X}^\lambda}(\gamma_0^\lambda(\lambda^2\alpha)) = \lambda^2\alpha \Longleftrightarrow \kappa^{X^\lambda}(\frac{\gamma_0^\lambda(\lambda^2\alpha)}{\lambda}) = \alpha. \tag{4.5.91}$$

Using the implicit function theorem,

$$\lim_{\lambda\to 0}\frac{\lambda^2\alpha}{\lambda} = \kappa_{X^0}^{-1}(\alpha) \tag{4.5.92}$$

holds. As

$$X^0 = -\frac{1}{2}\mathrm{Var}\left(X(1)\right) I + \sqrt{\mathrm{Var}\left(X(1)\right) W}, \tag{4.5.93}$$

we conclude

$$\kappa_{X^0}^{-1}(\alpha) = \frac{1}{2}\mathrm{Var}\left(X(1)\right) - \sqrt{2\alpha + \frac{1}{4}\mathrm{Var}\left(X(1)\right)}. \tag{4.5.94}$$

As $X^0 = -\frac{1}{2}\mathrm{Var}\left(X(1)\right) I + \sqrt{\mathrm{Var}\left(X(1)\right)}W$ is a Brownian motion with drift, from [8], page 274 it follows that

$$E\left(e^{-\alpha\tau_b^{X^0}}\right) = e^{-b(\frac{1}{2}\mathrm{Var}(X(1)) - \sqrt{2\alpha+\frac{1}{4}\mathrm{Var}(X(1))})}. \tag{4.5.95}$$

$\square$

Lemma 4.5.12.

$$C_0^0(0,\beta) = E\left(e^{-\beta O^*}\right) \tag{4.5.96}$$

Proof. On the one hand, using Corollary 4.3.3, one gets

$$\begin{aligned} E\left(e^{-\beta O^*}\right) &= \int e^{-\beta x} P^{O^*}(dx) = \frac{1}{\mu_{H(1)}}(\int b e^{-\beta x}\delta_0 + \int_{[0,1]}\int_{[0,\infty]}(e^{-\beta yz}y)K_H(dy)dz) \\ &= \frac{b\beta + \int_{[0,\infty]}(e^{-\beta y}-1)K_H(dy))}{\beta\mu_{H(1)}} = \frac{\kappa^{\widetilde{X}}(0,\beta)}{\beta\kappa^{\widetilde{X}\prime}(0,0)}, \end{aligned} \tag{4.5.97}$$

where H is the ladder height process of $\widetilde{X}^0$, b its drift w.r.t. the truncation function id, $\kappa^{\widetilde{X}\prime}(0,\cdot)$ its Laplace exponent.

On the other hand, using representation 1 for $\widetilde{F}(\lambda,\alpha,\beta,q)$ and using the fact that 0 is a simple pole for $\widetilde{F}(0,0,\beta,q)$, we see that

$$\begin{aligned} C(0,\beta) &= Res(\widetilde{F}(0,0,\beta,q),0) = Res(\frac{\kappa^0(0,q)-\kappa^0(0,\beta)}{(q-\beta)\kappa^0(0,q)},0) \\ &= \frac{\kappa^0(0,0)-\kappa^0(0,\beta)}{-\beta(\kappa^0(0,0)) + (q-\beta)\kappa^{0\prime}(0,0)} = \frac{\kappa^0(0,\beta)}{\beta\kappa^{0\prime}(0,0)}. \end{aligned} \tag{4.5.98}$$

$\square$

Now we have got our tools to show our result:

Theorem 4.5.13. *Let X be a Lévy process with assumption 4.1.2. Then*

$$\lim_{\lambda \to 0} P^{(\frac{O}{\lambda}, \tau^\lambda)} = P^{O^*} \otimes P^{\tau^0} \tag{4.5.99}$$

holds.

Proof. First we note that 4.3.13 tells us $\frac{O}{\lambda} = O(\widetilde{X}^\lambda, \frac{b}{\lambda})$, hence for $\alpha \in [0, C]$, $\beta \geq 0$:

$$E\left(e^{-\alpha\tau_b^\lambda - \beta\frac{O}{\lambda}}\right) = F(\lambda, \lambda^2\alpha, \beta, \frac{b}{\lambda}). \tag{4.5.100}$$

Applying Theorem 4.5.9 for $\widetilde{X}^\lambda$ we get:

$$\begin{aligned} &F(\lambda, \lambda^2\alpha, \beta, \frac{b}{\lambda}) = C_0^\lambda(\lambda^2\alpha, \beta)e^{-\gamma_0(\lambda,\lambda^2\alpha)\frac{b}{\lambda}} \\ &+ \sum_{i=1}^{l} \frac{1}{2}(C_i^\lambda(\lambda^2\alpha, \beta, \frac{b}{\lambda})e^{-\gamma_i(\lambda,\lambda^2\alpha)\frac{b}{\lambda}} + \bar{C}_i^\lambda(\lambda^2\alpha, \beta, \frac{b}{\lambda})e^{-\bar{\gamma}_i(\lambda,\lambda^2\alpha)\frac{b}{\lambda}}) + O(e^{-B\frac{b}{\lambda}}). \end{aligned} \tag{4.5.101}$$

As $B < \Re(-\gamma_i(\lambda, \lambda^2\alpha)) < 0$ for all $i = 1, .., l$, $\lambda \in [0, 1]$ and using Lemma 4.5.5, Lemma 4.5.11 and Lemma 4.5.12, one concludes:

$$\begin{aligned} \lim_{\lambda \to 0} F(\lambda, \lambda^2\alpha, \beta, \frac{b}{\lambda}) &= \lim_{\lambda \to 0} C_0^\lambda(\lambda^2\alpha, \beta)e^{-\gamma_0(\lambda,\lambda^2\alpha)\frac{b}{\lambda}} \\ &= C_0^0(0, \beta)e^{-b(\frac{1}{2}\mathrm{Var}(X(1)) - \sqrt{2\alpha + \frac{1}{4}\mathrm{Var}(X(1))})} \\ &= E\left(e^{-\alpha\tau_b^{X^0}}\right) E\left(e^{-\beta O^*}\right). \end{aligned} \tag{4.5.102}$$

Therefore, we got convergence in distribution to two independent random variables due to Lévys continuity theorem (e.g. [30], Satz 15.23). □

4.6. Approximation formula

As we now have got all our ingredients for our approximation for $\mathcal{C}_2$ together, we are finally able to state our approximation theorem.

Theorem 4.6.1. *Let $\mathcal{C}_2$ be the payoff defined in definition 3.3.5. The first order approximation $\mathcal{A}^{\mathcal{C}_2}$ defined in 4.1.5 has the representation*

$$\mathcal{A}^{\mathcal{C}_2} = -E\left(O^*\right) \int_0^T D_1(t, B) f_{\tau_b^0}(t) dt, \tag{4.6.103}$$

where $E\left(O^*\right)$ has the representation

$$\begin{aligned} E\left(O^*\right) = \lim_{\lambda \to 0} \Big(&\frac{\frac{1}{\pi}\int_0^\infty \Re(\frac{2}{(R+iu)^3}\log(-\kappa_{\widetilde{X}^\lambda}(R+iu)))du}{\frac{2}{\pi}\int_0^\infty \Re(\frac{1}{(R+iu)^2}\log(-\kappa_{\widetilde{X}^\lambda}(R+iu)))du} \\ &+ \frac{(\frac{1}{\pi}\int_0^\infty \Re(\frac{1}{(R+iu)^2}\log(-\kappa_{\widetilde{X}^\lambda}(R+iu)))du)^2}{\frac{2}{\pi}\int_0^\infty \Re(\frac{1}{(R+iu)^2}\log(-\kappa_{\widetilde{X}^\lambda}(R+iu)))du} \Big), \end{aligned} \tag{4.6.104}$$

$R \in (0, \lambda)$, $f_{\tau_b^0} : \mathbb{R}^+ \to \mathbb{R}^+$, $f_{\tau_b^0}(t) = \frac{b}{\sqrt{2\pi}t^{\frac{3}{2}}} \exp(-\frac{(b+\frac{\sigma^2}{2}t)^2}{2t})$ is the density of τ_b^0 (see [8], page 356), $D_1(t, B)$ is the first cash greek of v_{BS} defined in definition 3.4.5.

Proof. We basically put all the results from the previous chapters together. First we note that Proposition 4.1.6 ensures

$$\frac{q^{\mathcal{C}_2}(\lambda)}{\lambda} = \frac{E\left(D_1(\tau_b^\lambda, B)O(X^\lambda, b)\right)}{\lambda} + o(1). \tag{4.6.105}$$

The proof of Theorem 3.4.6 shows that $D_1(\cdot, B)$ is bounded, hence Theorem 4.5.13 together with Lemma 4.4.1 ensure that

$$\lim_{\lambda \to 0} \frac{q^{\mathcal{C}_2}(\lambda)}{\lambda} = E\left(D_1(\tau_b^0, B)\right) E\left(O^*\right). \tag{4.6.106}$$

Finally, Theorem 4.4.3 and Corollary 4.4 provide us with the representation for $E\left(O^*\right)$, noting that

$$\gamma^\lambda(0) = \lambda.$$

$\square$

5. Numerical Illustration

5.1. Generalities

We will consider European up-and-out put options from example 1.1.1 in two parametric models from the literature, namely the Merton model with normal jumps [37] as a model of jump-diffusion type and the variance-gamma model [36] as a model with infinite jump activity. Note that, strictly speaking, the European up-and-out put does not meet the regularity condition 3.1.4, as the payoff function is non smooth, and additionally, the models do not meet the regularity condition 4.1.2, as the jumps are not bounded and in the case of the variance-gamma model, we do not have a diffusion part in the driving Lévy process. Nevertheless, the formulas 3.4.6 and 4.6.1 still make sense for these models and payoff. The cash-greeks needed for the approximation 3.4.6 will be calculated in the appendix.

5.1.1. Option prices and benchmarks

We use European-up-and-out-put options with maturity

$$T \in \left\{ \frac{1}{12}, \frac{1}{4}, 1 \right\} \text{years}.$$

For each maturity, we will consider three different strikes K_j and three different barriers B_i. The barriers B_i are chosen in such a way that

$$P(\sup_{0 \le t \le T} S(t) < B_i) \approx \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, i = 1, 2, 3.$$

The Strikes K will be chosen so that at each maturity

$$K_j \approx B_1, S(0), \frac{S(0)^2}{B_1}, j = 1, 2, 3$$

holds. This will be done to get one strike in the money, at the money and out of the money with different distances to the barrier in a symmetrical setting. In this context $\approx$ means suitable rounded to a number divisible by 5. For reference prices we choose a numerical method to solve the PIDE in theorem 1.2.1, namely an explicit implicit finite difference algorithm according to [14] section 5, implemented in the software package MATLAB. As this algorithm prices European options in the same way as barrier options, we calibrate the grid of the algorithm

in such a way that the relative error of the European option price of a put option at the money is less than 0.05% compared to the European put price evaluated via integral transform method in [9], using the *quadgk* function in MATLAB for the numerical integration.
For every model and every parameter choice, we provide a table at each maturity. For every strike and barrier choice, we will evaluate the reference option price (FiDi), the option price of the naive Black-Scholes approximation (BS) and our approximation (J). In round brackets, we report the relative error of the approximation w.r.t the benchmark price in percent, i.e.

$$\frac{\text{Approximate price} - \text{Reference price}}{\text{Reference price}} * 100.$$

The round brackets in the head of the table correspond to the mean relative error over all strikes and barriers at the same maturity, i.e.

$$\frac{1}{9}\sum_{(i,j)} \frac{\text{Approximate price}(B_i, K_j) - \text{Reference price}(B_i, K_j)}{\text{Reference price}(B_i, K_j)} * 100.$$

5.1.2. Choice of model parameters

We will fit our parameter sets in such a way that theoretical and empirical moments coincide. The choice of parameters resembles [17] and lies in the range of empirical plausible values (cf. e.g. [[10] , Table 4]), at least if we agree that risk neutral parameters should not deviate too much from statistical ones:

$$E\left(\exp\left(X_1\right)\right) = 1$$

$$\begin{aligned} \mathrm{Var}\left(X(1)\right) &= 0.4^2, \\ \mathrm{Skew}(X(1)) &= \frac{0.1}{\sqrt{250}}, \\ \mathrm{ExKurt}(X(1)) &= \frac{5}{250}. \end{aligned} \tag{5.1.1}$$

For the Merton model, as we have five model parameters, we eliminate the additional degree of freedom by setting the variance arising from the jump component as 49% of the overall variance of X , following the choice in [11] and [17]. Additionally, we will provide a table in the Merton model with the exact and the approximation price of an up-and-out-put option fitted to different values of ExKurt$(X(1))$.

5.2. Merton model

5.2.1. Model specification

The Merton model is a jump diffusion model with normal distributed jumps:

$$X_t = -\gamma t + \sigma W_t + \sum_{k=1}^{N_t} J_k,\ t \in \mathbb{R}^+, \tag{5.2.2}$$

where $\sigma > 0$, W is a standard Brownian motion, $J_1, J_2, ..$ are independent and identically $N(\nu, \tau^2)-$distributed random variables, N is a Poisson process with intensity $\alpha > 0$ such that $W, J_1, J_2, ..., N$ are all independent. According to the martingale condition in (5.1.1), $\gamma = \frac{\sigma^2}{2} + \alpha(e^{\nu + \frac{\tau^2}{2}} - 1)$ holds.

5.2.2. Characteristic exponent

Using the Lévy Khintchin formula from theorem 2.0.1 we get for $u \in \mathbb{R}$:

$$\varphi(u) = -iu\gamma - \frac{\sigma^2 u^2}{2} + \alpha(e^{iu\nu - \frac{\tau^2 u^2}{2}} - 1). \tag{5.2.3}$$

5.2.3. Moments

The required moments in the approximation and for the calibration of the parameters are given by differentiating the characteristic exponent (e.g. [13], Table 4.3):

$$\begin{aligned} E\left(X(1)\right) &= \gamma + \alpha\nu \\ \operatorname{Var}\left(X(1)\right) &= \sigma^2 + \alpha(\nu^2 + \tau^2) \\ \operatorname{Skew}(X(1)) &= \frac{\alpha(\nu^3 + 3\tau^2\nu)}{\operatorname{Var}\left(X(1)\right)^{\frac{3}{2}}} \\ \operatorname{ExKurt}(X(1)) &= \frac{\alpha(\nu^4 + 6\tau^2\nu^2 + 3\tau^4)}{\operatorname{Var}\left(X(1)\right)^2}. \end{aligned} \tag{5.2.4}$$

Equating the theoretical moments in 5.2.4 with the empirical values in 5.1.1, one gets the following model parameters:

γ	σ	α	ν	τ
-0.0548	0.280	39.0	-0.00165	0.0457

Table 5.1.: Merton model parameters

5.2.4. Numerical comparison

In the Merton model, we see that our approximation J works rather well for maturity $T = 1$ with an average relative error of 0.0007. The error stays almost constant over all strikes and barriers, whereas the accuracy of BS decreases rapidly when the barrier B is close to the start price $S(0)$. The average relative error of J increases for shorter maturities up to 0.0108, which has to be expected due to the fact that the rescaling mechanism of the central limit theorem for the marginal distributions of X works better for larger maturities. The quality increases the further the strike K is away from the barrier. In table 5.5 we can see that the approximation quality depends roughly linear on the $\mathrm{ExKurt}(X(1))$, at leasts for values of $\mathrm{ExKurt}(X(1)) \geq \frac{10}{250}$. For the lower values, there might be effects due to the approximation of the exact price with the **FiDi** scheme, as we calibrated the scheme to an error tolerance of 0.0005.

T=1 B	K	FiDi (0.00)	BS (2.21)	J (0.07)
	110	10.4220 (0.00)	10.4308 (0.08)	10.4318 (0.09)
150	100	15.5455 (0.00)	15.5413 (-0.02)	15.5591 (0.09)
	90	21.5296 (0.00)	21.5018 (-0.13)	21.5459 (0.08)
	110	9.2219 (0.00)	9.1349 (-0.94)	9.2291 (0.08)
125	100	13.3890 (0.00)	13.2261 (-1.22)	13.3962 (0.05)
	90	18.0330 (0.00)	17.7708 (-1.45)	18.0388 (0.03)
	110	5.8565 (0.00)	5.5647 (-4.98)	5.8539 (-0.05)
110	100	8.1670 (0.00)	7.7288 (-5.37)	8.1623 (-0.06)
	90	10.6001 (0.00)	10.0000 (-5.66)	10.5937 (-0.06)

Table 5.2.: Option prices for maturity 1 and different barriers and strikes in the Merton model using the parameters in Table 5.1: **FiDi** refers to the option prices using the explicit-implicit finite difference scheme in [14] with $\delta_x = 0.005$ and $\delta_t = 0.00001$, *BS* refers to the Black-Scholes price relative to the volatility 0.4, *J* refers to the approximation derived in this thesis. Values in brackets in the body refer to the relative error w.r.t the **FiDi** price in percent, values in brackets in the head refer to the mean relative error over barriers and strikes.

T=1/4 B	K	FiDi (0.00)	BS (3.57)	J (0.35)
	105	5.4599 (0.00)	5.4792 (0.35)	5.4786 (0.34)
125	100	7.8586 (0.00)	7.8766 (0.23)	7.8876 (0.37)
	95	10.7221 (0.00)	10.7264 (0.04)	10.7554 (0.31)
	105	5.1546 (0.00)	5.1227 (-0.62)	5.1764 (0.42)
115	100	7.3051 (0.00)	7.2310 (-1.01)	7.3334 (0.39)
	95	9.7897 (0.00)	9.6464 (-1.46)	9.8166 (0.28)
	105	3.2299 (0.00)	2.9451 (-8.82)	3.2431 (0.41)
105	100	4.3588 (0.00)	3.9447 (-9.50)	4.3747 (0.37)
	95	5.5619 (0.00)	5.0000 (-10.10)	5.5783 (0.30)

Table 5.3.: Option prices for maturity 1/4 and different barriers and strikes in the Merton model using the parameters in Table 5.1: **FiDi** refers to the option prices using the explicit-implicit finite difference scheme in [14] with $\delta_x = 0.005$ and $\delta_t = 0.00001$, BS refers to the Black-Scholes price relative to the volatility 0.4, J refers to the approximation derived in this thesis. Values in brackets in the body refer to the relative error w.r.t the **FiDi** price in percent, values in brackets in the head refer to the mean relative error over barriers and strikes.

T=1/12 B	K	FiDi (0.00)	BS (2.64)	J (1.08)
	105	2.3864 (0.00)	2.4172 (1.29)	2.4112 (1.04)
125	100	4.5296 (0.00)	4.5746 (0.99)	4.5810 (1.13)
	95	7.5126 (0.00)	7.5329 (1.29)	7.5653 (0.70)
	105	2.3319 (0.00)	2.3504 (0.79)	2.3637 (1.36)
115	100	4.3710 (0.00)	4.3665 (-0.10)	4.4272 (1.29)
	95	7.1148 (0.00)	7.0082 (-1.50)	7.1612 (0.65)
	105	1.9852 (0.00)	1.9005 (-4.27)	2.0196 (1.73)
105	100	3.5481 (0.00)	3,3242 (-6.31)	3.5928 (1.26)
	95	5.4516 (0.00)	5.0000 (-8.28)	5.4821 (0.56)

Table 5.4.: Option prices for maturity 1/12 and different barriers and strikes in the Model model using the parameters in Table 5.1: **FiDi** refers to the option prices using the explicit-implicit finite difference scheme in [14] with $\delta_x = 0.005$ and $\delta_t = 0.00001$, *BS* refers to the Black-Scholes price relative to the volatility 0.4, *J* refers to the approximation derived in this thesis. Values in brackets in the body refer to the relative error w.r.t the **FiDi** price in percent, values in brackets in the head refer to the mean relative error over barriers and strikes.

ExKurt	α	τ	FiDi	J
5	39.0	0.0457	13.3890	13.3962 (0.05)
10	19.5075	0.0647	13.4598	13.4495 (-0.08)
15	13.0050	0.0792	13.5107	13.4838 (-0.20)
20	9.7537	0.0915	13.5528	13.5081 (-0.33)
25	7.8030	0.1022	13.5895	13.5259 (-0.47)
30	6.5025	0.1120	13.6225	13.5390 (-0.61)
35	5.5736	0.1210	13.6524	13.5485 (-0.76)
40	4.8769	0.1293	13.6797	13.5551 (-0.91)
45	4.3350	0.1372	13.7046	13.5592 (-1.06)
50	3.9015	0.1446	13.7274	13.5613 (-1.21)

Table 5.5.: Option prices for maturity $T = 1$, barrier $B = 125$ and strike $K = 100$ in the Merton model calibrated to different values of ExKurt normalized by $\frac{1}{250}$. α and τ denote the flexible Merton parameters corresponding to 5.2.1, the parameters $\sigma = 0.28$ and $\nu = -0.00165$ are left constant during the calibration. **FiDi** refers to the option prices using the explicit-implicit finite difference scheme in [14] with $\delta_x = 0.005$ and $\delta_t = 0.00001$, J refers to the approximation derived in this thesis. Values in brackets in the body refer to the relative error w.r.t the **FiDi** price in percent.

5.3. Variance gamma model

5.3.1. Model specification

The variance gamma model (VG) can be seen as a Brownian motion with drift $\theta \in \mathbb{R}$ and volatility $\sigma > 0$, subordinated by a gamma-process with variance $\nu > 0$:

$$X(t) = \mu t + \theta\Gamma(t, 1, \nu) + \sigma W(\Gamma(t, 1, \nu)), \tag{5.3.5}$$

where $\Gamma(t, 1, \nu)$ is a gamma process with mean 1 and variance ν and a drift $\mu \in \mathbb{R}$ is added. It is an infinite activity jump process without Brownian motion part, but with relatively low activity of small jumps. As it is a finite variation process, it may also be written as the difference of two gamma processes.

5.3.2. Characteristic exponent

The characteristic exponent of the variance gamma process is given by (e.g. [13], Table 4.5):

$$\varphi(u) = iu\mu - \frac{1}{\kappa}\log(1 + \frac{u^2\sigma^2\kappa}{2} - i\theta\kappa u), \tag{5.3.6}$$

where the branch of the complex logarithm is chosen such that the right-hand side is continuous and vanishes in 0.

5.3.3. Moments

The required moments are again given by differentiating the characteristic exponent, (e.g. [13], Table 4.5):

$$\begin{aligned} E\left(X(1)\right) &= \mu + \theta \\ \operatorname{Var}\left(X(1)\right) &= \sigma^2 + \theta^2\nu \\ \operatorname{Skew}(X(1)) &= \frac{3\sigma^2\theta\nu + 2\theta^3\nu^2}{\operatorname{Var}\left(X(1)\right)^{\frac{3}{2}}} \\ \operatorname{ExKurt}(X(1)) &= \frac{3\sigma^4\nu + 6\theta^4\nu^3 + 12\sigma^2\theta^2\nu^2}{\operatorname{Var}\left(X(1)\right)^2}. \end{aligned} \tag{5.3.7}$$

Once again equating the empirical moments in 5.1.1 with the theoretical moments in 5.3.7 leads to the model parameters 5.6.

5.3.4. Numerical comparison

In the VG model we see similar effects as in the Merton model. The average relative error of J for maturity $T = 1$ is 0.0023, increasing to 0.0185 for maturity $T = 1/12$, outperforming BS for every maturity. In addition, we see that the accuracy of J decreases the closer the barrier B gets to the start price $S(0)$, which

μ	θ	σ	ν
-0.2067	0.1267	0.3999	0.0067

Table 5.6.: VG model parameters

is a different behaviour than for the Merton model. This may be explained by the fact that the VG is in contrast to the Merton model a pure jump model, but with finite variation, hence it has relatively low jump activity. Consequently, the way the VG model crosses the barrier is way different from the Black-Scholes model which might lead to an error in the time integral in formula (4.6.103).

T=1 B	 K	FiDi (0.00)	BS (4.14)	J (0.23)
	110	10.4456 (0.00)	10.4308 (-0.14)	10.4364 (-0.09)
150	100	15.5895 (0.00)	15.5413 (-0.31)	15.5805 (-0.06)
	90	21.5983 (0.00)	21.5018 (-0.45)	21.5934 (-0.02)
	110	9.324 (0.00)	9.1349 (-2.03)	9.3293 (0.06)
125	100	13.5585 (0.00)	13.2261 (-2.45)	13.5730 (0.11)
	90	18.2847 (0.00)	17.7708 (-2.81)	18.3140 (0.16)
	110	6.1254 (0.00)	5.5647 (-9.15)	6.1519 (0.43)
110	100	8.5614 (0.00)	7.7288 (-9.72)	8.6057 (0.52)
	90	11.1327 (0.00)	10.0000 (-10.17)	11.1982 (0.59)

Table 5.7.: Option prices for maturity 1 and different barriers and strikes in the Variance-Gamma model using the parameters in Table 5.6: **FiDi** refers to the option prices using the explicit-implicit finite difference scheme in [14] with $\delta_x = 0.001$, $\delta_t = 0.000001$, $\epsilon = 0.0005$, BS refers to the Black-Scholes price relative to the volatility 0.4, J refers to the approximation derived in this thesis. Values in brackets in the body refer to the relative error w.r.t the **FiDi** price in percent, values in brackets in the head refer to the mean relative error over barriers and strikes.

T=1/4 B	K	FiDi (0.00)	BS (6.35)	J (0.83)
	105	5.4708 (0.00)	5.4792 (-0.15)	5.4805 (0.18)
125	100	7.8849 (0.00)	7.8766 (-0.11)	7.9006 (0.20)
	95	10.7661 (0.00)	10.7264 (-0.37)	10.7862 (0.19)
	105	5.2086 (0.00)	5.1227 (-1.65)	5.2347 (0.50)
115	100	7.4016 (0.00)	7.2310 (-2.31)	7.4407 (0.53)
	95	9.9408 (0.00)	9.6464 (-2.96)	9.9920 (0.50)
	105	3.4867 (0.00)	2.9451 (-15.53)	3.5500 (1.82)
105	100	4.7293 (0.00)	3.9447 (-16.59)	4.8146 (1.81)
	95	6.0610 (0.00)	5.0000 (-17.50)	6.1673 (1.75)

Table 5.8.: Option prices for maturity 1/4 and different barriers and strikes in the Variance-Gamma model using the parameters in Table 5.6: **FiDi** refers to the option prices using the explicit-implicit finite difference scheme in [14] with $\delta_x = 0.001$, $\delta_t = 0.000001$, $\epsilon = 0.0005$, BS refers to the Black-Scholes price relative to the volatility 0.4, J refers to the approximation derived in this thesis. Values in brackets in the body refer to the relative error w.r.t the **FiDi** price in percent, values in brackets in the head refer to the mean relative error over barriers and strikes.

T=1/12 B	K	FiDi (0.00)	BS (4.51)	J (1.85)
	105	2.3825 (0.00)	2.4172 (1.46)	2.4075 (1.05)
125	100	4.5455 (0.00)	4.5746 (0.64)	4.5887 (0.95)
	95	7.5522 (0.00)	7.5329 (-0.26)	7.5984 (0.61)
	105	2.3413 (0.00)	2.3504 (0.39)	2.3803 (1.66)
115	100	4.4205 (0.00)	4.3665 (-1.22)	4.4914 (1.60)
	95	7.2282 (0.00)	7.0082 (-3.04)	7.3181 (1.24)
	105	2.0684 (0.00)	1.9005 (-8.12)	2.1451 (3.71)
105	100	3.7472 (0.00)	3.3242 (-11.29)	3.8691 (3.25)
	95	5.8234 (0.00)	5.0000 (-14.14)	5.9730 (2.57)

Table 5.9.: Option prices for maturity 1/12 and different barriers and strikes in the Variance-Gamma model using the parameters in Table 5.6: **FiDi** refers to the option prices using the explicit-implicit finite difference scheme in [14] with $\delta_x = 0.001$, $\delta_t = 0.000001$, $\epsilon = 0.0005$, BS refers to the Black-Scholes price relative to the volatility 0.4, J refers to the approximation derived in this thesis. Values in brackets in the body refer to the relative error w.r.t the **FiDi** price in percent, values in brackets in the head refer to the mean relative error over barriers and strikes.

6. Conclusion

We have provided a first-order approximation for the price of barrier options in the framework of geometric Lévy models using a perturbation approach, viewing the Lévy model as an perturbed Black-Scholes model and connecting the logarithmic stock price processes according to section 1.4. Furthermore, we have provided an approximation formula for moments of the overshoot over a barrier for Lévy processes.

On a qualitative level, we have seen that difference between barrier option prices in jump models and the Black-Scholes models are essentially determined by the third moment of the Lévy process, its average overshoot and the first three Black-Scholes cash greeks of the option. The average overshoot can be expressed in terms of the characteristic exponent of the Lévy process.

On a quantitative level, we have shown in one model with finite jump activity and another one with infinite jump activity that our approximation leads to reasonable practical results, especially due to its low computational costs compared to other methods.

A. Black-Scholes up-and-out put cash greeks

Consider the risk-neutral price process of the stock in a Black-Scholes model with interest rate $r = 0$ and volatility $\sigma > 0$ given by

$$dS(t) = \sigma S(t) dW(t),\ S(0) > 0,$$

fo a standard Brownian motion W on some probability space. Let v be the pricing functional of an up-and-out put option with barrier $B > S(0)$, strike $0 \leq K \leq B$ and maturity $T > 0$, i.e. $v : [0,T] \times [0,B] \longrightarrow \mathbb{R}^+$ with:

$$v(t,x) := E\left((K - S(T))^+ 1_{\{\tau_B \leq T\}^c} | S(t) = x, t < \tau_B\right).$$

According to Example 1.1.1, the pricing functional is given by

$$\begin{aligned} v(t,x) = {} & K\Phi(-\frac{\ln(\frac{x}{K}) + \frac{\sigma^2}{2}}{\sigma\sqrt{T-t}}) - x\Phi(-\frac{\ln(\frac{x}{K}) - \frac{\sigma^2}{2}}{\sigma\sqrt{T-t}}) \\ & - \frac{xK}{B}(\Phi(-\frac{\ln(\frac{B^2}{xK}) - \frac{\sigma^2}{2}}{\sigma\sqrt{T-t}}) - B\Phi(-\frac{\ln(\frac{B^2}{xK}) - \frac{\sigma^2}{2}}{\sigma\sqrt{T-t}})), \end{aligned}$$

with Φ being the cumulative distribution function of the standard normal distribution. According to Lemma 3.3.3, the function v is three times differentiable with respect to x and we set as in Definition 3.4.5 for $n \in \{1, 2, 3\}$ the n-th cash greek D_n via:

$$D_n(t,x) = x^n \frac{\partial^n}{\partial x^n} v_{BS}(t,x).$$

Denoting by φ the density function of the standard normal distribution, steady but straightforward calculations yield:

$$
\begin{aligned}
D_1(t,x) &= x\Bigg(\Phi(d_1(t,\frac{S}{K})) + \frac{B}{x}\Phi(-d_1(t,\frac{B^2}{xK})) \\
&\quad - \frac{K}{B}\Phi(-d_2(t,\frac{B^2}{xK})) + (\Phi(d_1(t,\frac{B^2}{xK})) - 1)\Bigg) \\
D_2(t,x) &= x^2\left(\frac{\varphi(d_1(t,\frac{x}{K}))}{x\sigma\sqrt{T-t}} - \frac{B}{x^2}\frac{\varphi(d_1(t,\frac{B^2}{xK}))}{\sigma\sqrt{T-t}}\right) \\
D_3(t,x) &= x^3\Bigg(-\frac{\varphi(d_1(t,\frac{x}{K}))}{x^2\sigma\sqrt{T-t}}(d_1(t,\frac{x}{K})+1) \\
&\quad - B\frac{\varphi(d_1(t,\frac{B^2}{xK}))}{x^3\sigma^2\sqrt{T-t}^2}d_1(t,\frac{B^2}{xK}) \\
&\quad + 2\frac{B}{x^3}\frac{\varphi(d_1(t,\frac{B^2}{xK}))}{\sigma\sqrt{T-t}}\Bigg),
\end{aligned}
$$

with

$$
\begin{aligned}
d_1(t,x) &:= \frac{\log x + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}, \\
d_2(t,x) &:= d_1(t,x) - \sigma\sqrt{T-t}.
\end{aligned}
$$

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Erklärung

Hiermit erkläre ich, dass ich die vorliegende Dissertation - abgesehen von der Beratung durch meinen Betreuer Herrn Prof. Dr. Jan Kallsen - nach Inhalt und Form eigenständig angefertigt habe. Dabei habe ich die Regeln guter wissenschaftlicher Praxis der *Deutschen Forschungsgemeinschaft* eingehalten. Die Arbeit hat weder ganz noch zum Teil einer anderen Stelle im Rahmen eines Prüfungsverfahrens vorgelegen und ist weder ganz noch zum Teil veröffentlicht oder zur Veröffentlichung eingereicht worden.

Kiel, den 18. Juli 2017