Games in Networks under Robustness, Locality, and Coloring Aspects

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Zusammenfassung

Dezentralisierung ist ein wichtiges Konzept für viele moderne Netzwerke, wie zum Beispiel das Internet, soziale Netzwerke, oder Netzwerke aus drahtlosen Telefonen oder Sensoren. Ein systematisches Studium von Netzwerken, die von einer Vielzahl an nicht-koorperativen oder schwach-kooperativen Spielern geformt werden, ist ein lebhaftes Thema in der Diskreten Mathematik, Informatik und den Wirtschaftswissenschaften. Im Jahr 2010 wurde ein neues Modell eingeführt, das es erlaubt, Robustheitsaspekte zu untersuchen. In diesem Modell, bekannt als *Gegnermodell* oder *Zerstörermodell*, wird eine zufällig ausgewählte Kante im Netzwerk zerstört, nachdem das Netzwerk aufgebaut wurde. Die Spieler rechnen mit diesem Eingriff und versuchen, das Netzwerk so zu bauen, dass es auch nach der Zerstörung der einen Kante noch gute Verbindungseigenschaften hat. Wie effizient ist dies in dezentraler Weise machbar?

Diese Dissertationsschrift bringt das Gegnermodell in zwei Richtungen voran: Zum einen verwenden wir zum ersten Mal in diesem Modell das moderne Equilibriumskonzept bekannt als *Swap-Equilibrium*, und zum anderen erweitern wir das Modell auf die Zerstörung eines *Knotens*. Dieser Eingriff wird generell schwerwiegender sein als wenn nur eine Kante zerstört wird. Wir charakterisieren verschiedene Szenarien, von denen einige nachweislich effiziente Netzwerke hervorbringen, während wir von anderen zeigen, dass die Netzwerke in einem gewissen Maße ineffizient sein können.

Abgesehen vom Gegnermodell betrachten wir ein Modell, in welchem jeder Spieler v versucht, die Anzahl anderer Spieler zu maximieren, die sich im Abstand höchstens k von v befinden für einen festen Parameter k. Wenn das Netzwerk zum Beispiel Freundschaften ausdrückt, dann würden im Falle $k = 2$ die Spieler versuchen, die Anzahl ihrer Freunde plus die Anzahl der Freunde ihrer Freunde zu maximieren, was in der Soziologie in interessantes Maß ist. Wir beweisen Struktur- und Effizienzeigenschaften solcher Netzwerke.

Im letzten Kapitel ist das Netzwerk fest, und jeder Spieler kann eine von k Farben wählen. Zum Beispiel könnte das Netzwerk die räumliche Anordnung der Spieler beschreiben, und die Farben korrespondieren zu Radiofrequenzen. Dann ist es das Ziel jedes Spielers, eine Frequenz für sich zu finden, die zu möglichst wenig Interferenz führt. Wir zeigen, dass unter moderaten Annahmen an die Art, wie Interferenz modelliert wird, die erzielten Färbungen erstaunlich effizient sind, nämlich ihre Güte ist nur einen kleinen konstanten Faktor vom Optimum entfernt.

Abstract

Decentralization is a key concept in modern networks, such as the Internet, social networks, or wireless phone or sensor networks. The systematic study of how networks are formed by a multitude of non-cooperative or only mildly cooperative players, is a vivid topic in Discrete Mathematics, Computer Science, and Economics. In 2010, a variant addressing *robustness aspects* was introduced. In this model, known as *adversary model* or *destruction model*, one link in the network is destroyed at random after the network has been formed. Players anticipate this disruption and try to build a network that gives them good connectivity even after the destruction. How efficiently can this be done in a decentralized setting?

This thesis advances our knowledge regarding the adversary model: we bring in the modern equilibrium concept of *swap equilibrium* and we extend to the destruction of one *vertex*. This disruption tends to be more severe compared to the case that just one link is destroyed. We characterize several settings, where for some, the formed networks are provably efficient, while for others, we show they can be inefficient up to a certain degree.

Apart from the adversary model, we study a model where each player v tries to maximize the number of players at distance at most k from v, for a fixed parameter k. For example, when the network models friendship, then for $k = 2$, players would try to maximize the number of friends plus friends of their friends, which is an interesting metric in Sociology. We prove results on the structure and efficiency of such networks.

In the final chapter, the network is fixed and each player chooses one of k colors. For example, the network might describe the spatial relations between the players and colors might correspond to radio frequencies, so each player's aim is to choose a frequency that causes as few interference as possible with the frequencies of her neighbors. We show that under mild assumptions on how interference between colors is modeled, the resulting colorings are surprisingly efficient, namely their performance is within a small constant factor of the optimum.

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Contents

Chapter 1

Introduction and Summary of Results

Decentralization is a key concept in many modern networks, such as the Internet, social networks, or wireless phone or sensor networks. The systematic study of how networks are formed by a multitude of non-cooperative or only mildly cooperative players, is a vivid topic in Economics (since around 1996, starting with [JW96]) and Discrete Mathematics and Computer Science (since around 2003, starting with [FLM+03]). Most of the studies concentrated on global, centrality-type metrics in order to evaluate the situation of a single player and the performance of the network as a whole. For example, a player might aim to reduce the sum of the distances between her and all the other players [FLM+03] or the maximum distance to any other player [DHM+07]. In 2010, a variant addressing *robustness aspects* was introduced [Kli10a; Kli11] and further studied since [**Kli16**; **CLMM16**; Kli15]. In this model, known as *adversary model* or *destruction model*, one link in the network is destroyed at random after the network has been formed. Players anticipate this disruption – although in general they cannot tell with certainty which link will be destroyed – and try to build a network that gives them good connectivity even after the destruction. The main question is how efficiently this can be done when players behave non-cooperatively or only show a small degree of cooperation.

In Chapter 3, this thesis advances our knowledge regarding the adversary model in two directions. On the one hand, we bring in the modern equilibrium concept of *swap equilibrium*, which was introduced in 2010 [ADH+10]. On the other hand, we extend from the destruction of one link to the destruction of one *vertex*, meaning that the destroyer choses one vertex randomly according to a particular probability distribution and then removes all links incident to this vertex. Clearly, the disruption will tend to be more severe compared to the case that just one link is destroyed.

In Chapter 4, we study a variant called *local network formation*. This introduces another alternative to the traditional global, centrality-type metrics. Namely, in local network formation, we allow players to evaluate the network by considering it only within a certain radius of themselves. For example, a player might aim for a position in the network that maximizes the sum of the degrees of her neighbors [NPR+13] or her clustering coefficient [BK11]. In this thesis, we consider that each player v tries to maximize the size of her *k-neighborhood* for a parameter $k \in \mathbb{N}$, that is, the number of players at distance at most k from v. For example, when the network models friendship, then for $k = 2$, players would try to maximize the number of friends plus friends of their friends, which is an interesting metric in sociology.

In Chapter 5, we switch perspective. Now the network is fixed and each player has to choose one of k colors. For example, the network might describe the spatial relations between the players and colors might correspond to radio frequencies, so each player's aim is to choose a frequency that does not interfere with the frequencies of her neighbors, or causes interference only up to a certain degree. Applications arise in wireless phone or sensor networks.

Results of This Thesis

Destruction Model (Chapter 3)

This chapter is based on **[KSS17a]**. It considers the destruction model under the *swap equilibrium* (SE) concept. A graph is called an SE if no player can improve her cost by removing one of her incident edges and creating a different new edge instead (or by just removing an incident edge). Players anticipate the destruction of one edge or one vertex in the graph. This destruction happens according to a probability measure on the edges or vertices, respectively, which is known to the players. Each player's cost function is the expected number of other players whom she will no longer be able to reach after the destruction took place. We speak of *edge destruction* or *vertex destruction* in order to indicate whether an edge or vertex will be destroyed. We consider the following probability measures:

- The *uniform destroyer* occurs in edge and vertex destruction. It chooses an edge or vertex to destroy uniformly at random from the set of all edges or vertices, respectively.
- The *extreme destroyer* also occurs in edge and vertex destruction. It chooses the edge or vertex to destroy uniformly at random from the set of max-sep edges or max-sep vertices, respectively, that is, where the destruction of each causes a maximum number of vertex pairs to be separated.
- The *uniform bridge destroyer* is only for edge destruction. It chooses an edge to destroy uniformly at random from the set of bridges in the graph.
- The *degree proportional destroyer* is only for vertex destruction. Here, the probability for destruction of a vertex is proportional to its degree.

For all three variants of edge destruction (that is, *uniform edge destruction*, *extreme edge destruction*, and *uniform bridge destruction*), we prove an upper bound of $\mathcal{O}(n)$ on the social cost of any SE. The essential proof idea is to show that SE are bridgeless or have a star-like structure. For vertex destruction, the overall picture is more diverse:

- For *uniform vertex destruction*, we prove an $\mathcal{O}(n)$ bound on the social cost of any SE. The proof works by showing that an SE is two-connected or a tree.
- For *degree-proportional vertex destruction*, we give a lower bound, namely we show that the star is an SE with social cost $\Omega(n^2)$.
- For *extreme vertex destruction*, we prove a super-linear lower bound of $\Omega(n^{3/2})$ on the social cost of SE by constructing a graph which contains a clique of certain size at the center such that paths of a certain length are attached. We also prove that if $n \geqslant 8$, there is no tree SE with only one max-sep vertex.

Local Network Formation (Chapter 4)

In this chapter, we study the k-neighborhood model, which belongs to *local network formation games*, where each player's utility function depends on a certain graph-theoretic neighborhood around her. We consider three equilibrium concepts for this model: *pairwise stability*, *pairwise Nash equilibrium*, and *swap equilibrium*, which was also studied in Chapter 3. A graph is called *pairwise stable* (PS) if both of the following two conditions hold:

(i) Removing an edge does not improve utility for any of its endpoints.

(ii) Adding an edge does not improve utility for any of its endpoints, or it impairs utility (i. e., strictly decrease utility) for at least one endpoint.

Pairwise Nash equilibrium (PNE) is similar to pairwise stability, but in condition (i), instead of considering the deletion of one edge, deletion of several edges is assessed. Hence in particular, a pairwise Nash equilibrium is also pairwise stable, but not vice versa in general. We prove for the k-neighborhood model:

- Pairwise stability and pairwise Nash equilibrium are equivalent.
- For $k = 2$, the diameter of a pairwise stable graph is upper bounded by 3.
- In pairwise stability, if $\alpha \leqslant \frac{n}{2}$, then a tree with diameter at most k is an optimum; if $\alpha > \frac{n}{2}$, the empty graph is an optimum.
- For $k = 2$, an upper bound of $2 - o(1)$ on the price of anarchy when restricting to trees. (The price of anarchy is the ratio of optimum social utility to worst-case social utility, taken over a class of equilibrium graphs – in this case the class of all pairwise stable trees.)
- An upper bound of k on the diameter of swap equilibrium trees.
- For $k = 2$, an upper bound of 4 on the diameter of general swap equilibrium graphs and an example of a swap equilibrium graph with diameter 4, proving the upper bound tight.

Coloring (Chapter 5)

This chapter is based on **[KSS17]**. Let $G = (V, E)$ be a fixed graph and $k \in \mathbb{N}_{\geqslant 2}$. The vertices in V correspond to players, and each player has to choose one of k colors. A function $c : V \longrightarrow [k]$, specifying the choice of each player, is called a *k-coloring* or *coloring*, where $[k] = \{1, \ldots, k\}$. The set of colors $[k]$ is called the *spectrum*. Let f be a concave function defined on $[0, k]$. The *payoff* or *utility* for player $v \in V$ is

$$U_v(c) := \sum_{w \in N(v)} f\Big(|c(v) - c(w)|\Big) \ .$$

Thus, a player's utility depends on the distances between her color and the colors of her neighbors. A coloring is called *stable* if no player can increase her payoff by changing her color. A coloring is called *optimal* if it has maximum social utility, where the social utility is the sum over the payoffs of all players. The *price of anarchy* is the social utility of an optimal coloring devided by the social utility of a worst-case stable coloring.

We denote $f^* := \max_{i \in \mathcal{D}} f(i)$ the maximum that f can attain on the possible distances $\mathcal{D} := \{0, \ldots, k-1\}$ between two colors, and $\mathcal{D}^*(f) := \{i \in \mathcal{D}\,;\ f(i) = f^*\}$. We prove several constant upper bounds on the price of anarchy depending on the shape of the concave function f, cf. Theorem 5.9.9:

- An upper bound of 2 for non-decreasing f. This contains the natural case of *distance payoff*, that is, $U_v(c) = \sum_{w \in N(v)} |c(v) - c(w)|$.
- An upper bound of 2, for all concave functions f for which $\mathcal{D}^*(f) \cap \{0, \ldots, \lfloor \frac{k}{2} \rfloor\} \neq \varnothing$, that is, which attain their maximum in the left part of the spectrum. This contains *cyclic payoff*, that is,

  $$U_v(c) = \sum_{w \in N(v)} \min\{|c(v) - c(w)|, k - |c(v) - c(w)|\} \ .$$

  We show that for this class of functions f, the upper bound of 2 is the best possible.
- An upper bound of 3 for all concave functions f.
- An upper bound of 2.5 for $k \geqslant 16$ and all concave functions f for which $\mathcal{D}^*(f) \cap \{\lfloor \frac{k}{2} \rfloor + 1, \ldots, k-1\} \neq \varnothing$, that is, which attain their

maximum in the right part of the spectrum. This resolves a conjecture from [**KSS17**].

Chapter 2

Preliminaries for Network Formation Games

In this chapter we briefly introduce some important game-theoretical definitions and notations that we require in Chapter 3 and Chapter 4 in our study of network formation games. *Network formation games*, also called *network creation games*, provide a framework for the game-theoretic analysis of how networks, modeled as graphs, may form. In a network creation game, n players attempt to create a network that should be efficient in a certain sense. These games have been studied for more than two decades by now, with the work of Jackson and Wolinsky in 1996 [JW96] marking a starting point.

We mostly use standard graph theory notation. All our graphs $G = (V, E)$ are finite, simple and undirected, that is, V is a finite set and $E \subseteq \binom{V}{2}$. If G is any graph, not necessarily introduced in the form $G = (V, E)$, then its vertex set is denoted by $V(G)$ and its edge set by $E(G)$. When A is a subgraph of G and P is a path in G, we denote the set of vertices in A and the set of vertices that are visited by P by $V(A)$ and $V(P)$, respectively.

We denote the *undirected link* (a.k.a. *link*, *undirected edge*, *edge*) between v and w by $\{v, w\}$. We write $G + \{v, w\}$ for the graph with edge set $E(G) \cup \{\{v, w\}\}$ and $G + \sum_{i \in [k]} \{v_i, w_i\}$ for the graph with edge set

$$E(G) \cup \{\{v_1, w_1\}, \ldots, \{v_k, w_k\}\} \ .$$

By $G - \{v, w\}$ we denote the graph with edge set $E(G) \backslash \{\{v, w\}\}$ and by $G - \sum_{i \in [k]} \{v_i, w_i\}$ the graph with edge set

$$E(G) \backslash \{\{v_1, w_1\}, \ldots, \{v_k, w_k\}\} \ .$$

This allows us to easily add or remove one or multiple links from our graphs. We denote by $\deg_G(v)$, or $\deg(v)$ if G is clear from context, the *degree* of v in G, i. e., the number of edges incident in v in G.

Let $n \geqslant 3$ and $V_n = [n] = \{1, \ldots, n\}$ be the set of players. We consider these players as vertices of a graph and use the terms "player" and "vertex" interchangeably. By $\mathcal{G}_n$, we denoted the set of all undirected simple graphs on V_n. For each player v, there is either a function $\mathcal{C}_v : \mathcal{G}_n \longrightarrow \mathbb{R}$ where $\mathcal{C}_v(G)$ is interpreted as the *cost* experienced by player v in G, or a function $\mathcal{U}_v : \mathcal{G}_n \longrightarrow \mathbb{R}$ where $\mathcal{U}_v(G)$ is interpreted as the *utility* or *payoff*[1] enjoyed by player v in G. The *social cost* or *social utility* of G is $\mathsf{SC}(G) := \sum_{v \in V} \mathcal{C}_v(G)$ or $\mathsf{SU}(G) := \sum_{v \in V} \mathcal{U}_v(G)$, respectively, which evaluates the graph from a global perspective. Cost and utility can have different structure, depending on the equilibrium concept.

Intuitively speaking, a graph is an *equilibrium* with respect to a particular cost or utility function if no player can improve her cost or utility by choosing from a set of possible changes to the graph. Since whether a graph is an equilibrium depends also on the cost or utility function, we should strictly say that $(G, \mathcal{C})$ or $(G, \mathcal{U})$ is an equilibrium. However, for the sake of a more concise notation, we will usually speak of a graph being in equilibrium. It will always be clear from context which cost or utility function is meant.

We consider three equilibrium concepts defined precisely in the following. All concepts have in common that there is, for each player v, a function $D_v : \mathcal{G}_n \longrightarrow \mathbb{R}_{\geqslant 0}$, called *disutility* function, which is part of the cost function $\mathcal{C}_v$; or, in case of utility, an *income* function $I_v : \mathcal{G}_n \longrightarrow \mathbb{R}_{\geqslant 0}$, which is part of the utility function $\mathcal{U}_v$. Disutility expresses that part of cost that is associated with *using* the graph for a task like routing, while income expresses a benefit derived from the graph. A classical disutility function for example is $D_v(G) := \sum_{w \in V} \operatorname{dist}_G(v, w)$, the sum-distance model (see [FLM+03; CP05; AEE+06; DHM+07; MS10]) or $D_v(G) := \max_{w \in V} \operatorname{dist}_G(v, w)$, the max-distance model (see [DHM+07]). In Chapter 3, we will consider disutility functions associated with the destruction model, which express the expected damage caused by the destroyer. In Chapter 4, we will use an income function, namely the size

[1] The term "utility" is more common with network formation games and "payoff" is more common with network coloring games. In economics, the two words are sometimes used with different meanings, whereas for us, they are synonymous.

of the k-neighborhood of the player.

2.1 Pairwise Stability, Pairwise Nash Equilibrium, Price of Anarchy

We explain the equilibrium concept of *pairwise stability* (PS), which was introduced by Jackson and Wolinsky [JW96] in 1996. It will be required in Chapter 4. In addition to the number n of players and the disutility or income function, we need the *link cost* parameter $\alpha > 0$. If a disutility function is given, the cost of a player v is defined as

$$\mathcal{C}_v(G) := \deg_G(v)\,\alpha + D_v(G) \ ,$$

or if an income function is given, the utility of a player is defined as

$$U_v(G) := I_v(G) - \deg_G(v)\,\alpha \ .$$

The term $\deg(v)\,\alpha$ is called *building cost*. So each player can be thought of maintaining her incident links to the price of α each. Commonly, but not always, disutility decreases (or income increases) when more links are built, so players will balance their expenses for links against disutility (or against income). We obtain expressions for social cost and social utility:

$$\mathsf{SC}(G) = 2|E(G)|\alpha + \sum_{v \in V} D_v(G) \quad \text{and} \quad \mathsf{SU}(G) = \sum_{v \in V} I_v(G) - 2|E(G)|\alpha \ .$$

2.1.1 Definition. A graph $G = (V, E) \in \mathcal{G}_n$ is called *pairwise stable* (PS) if the following two conditions hold:

(i) For any $\{v, w\} \in E$, we have

$$\mathcal{C}_v(G) \leqslant \mathcal{C}_v(G - \{v, w\}) \ .$$

(Which includes $\mathcal{C}_w(G) \leqslant \mathcal{C}_w(G - \{v, w\})$ due to symmetry.)

(ii) For any $\{v, w\} \notin E$,

$$\text{if} \quad \mathcal{C}_v(G) > \mathcal{C}_v(G + \{v, w\}) \quad \text{then} \quad \mathcal{C}_w(G) < \mathcal{C}_w(G + \{v, w\}) \ .$$

(Including $\mathcal{C}_w(G) > \mathcal{C}_w(G + \{v,w\}) \implies \mathcal{C}_v(G) < \mathcal{C}_v(G + \{v,w\})$, again due to symmetry.)

Or, in case of income and utility:

(i) For any $\{v,w\} \in E$, we have

$$U_v(G) \geqslant U_v(G - \{v,w\}) \ .$$

(ii) For any $\{v,w\} \notin E$,

$$\text{if} \quad U_v(G) < U_v(G + \{v,w\}) \quad \text{then} \quad U_w(G) > U_w(G + \{v,w\}) \ .$$

The first condition says that it is impossible to remove a link and by doing so to improve utility or cost for any of its endpoints. The second condition says that each absent link must be justified by the fact that adding it does not improve utility or cost for any of its endpoints, or adding it impairs utility or cost (i. e., strictly decreases utility or strictly increases cost) for at least one endpoint.

For the rest of this section, we will state everything only in terms of utility (since we will only require this in Chapter 4), but for cost the notions apply likewise.

2.1.2 Definition. A graph $G = (V,E) \in \mathcal{G}_n$ is a *pairwise Nash equilibrium* (PNE) if the following two conditions hold:

(i) For any $\{v,w_1\},\ldots,\{v,w_k\} \in E$, we have

$$U_v(G) \geqslant U_v(G - \{v,w_1\} \ldots - \{v,w_k\}) \ .$$

(ii) For any $\{v,w\} \notin E$, we have

$$\text{if} \quad U_v(G) < U_v(G + \{v,w\}) \quad \text{then} \quad U_w(G) > U_w(G + \{v,w\}) \ .$$

This is similar to PS, but players can also evaluate the effect of severing *any number* of their incident links.

2.1.3 Remark. It is obvious that a pairwise Nash equilibrium graph is pairwise stable. The converse holds if the cost is pseudo-convex in a certain graph-theoretic sense, shown by Corbo and Parkes [CP05].

To prove the converse we use a result due to Antoni Calvó-Armengol and Rahmi İlkiliç [Cİ05].

2.1.4 Definition. Let $v \in V$. The utility function U_v is called *pseudo-convex*[2] in G if for all $\{w_1, \ldots, w_k\} \subseteq V$ we have

$$\begin{aligned} & U_v(G - \{v, w_1\} - \ldots - \{v, w_k\}) - U_v(G) \\ \leqslant & \sum_{i=1}^{k} (U_v(G - \{v, w_i\}) - U_v(G)) \ . \end{aligned}$$

2.1.5 Lemma. *Let G be a pairwise stable graph and the cost be convex in G. Then G is a pairwise Nash equilibrium graph.*

Proof. We need to show that deletion can not improve the utility of a player $v \in V$. Let $\{w_1, \ldots, w_k\} \subseteq V$. By pairwise stability, for each $i \in [k]$, we have

$$U_v(G - \{v, w_i\}) - U_v(G) \leqslant 0 \ .$$

Since for any $v \in V$, utility function U_v is pseudo-convex, it follows

$$\begin{aligned} & U_v(G - \{v, w_1\} - \ldots - \{v, w_k\}) - U_v(G) \\ \leqslant & \sum_{i=1}^{k} (U_v(G - \{v, w_i\}) - U_v(G)) \\ \leqslant & \ 0. \end{aligned}$$

Thus v has no incentive to remove all edges $\{v, w_1\}, \ldots, \{v, w_k\}$. □

A brief discussion of the motivation of PNE is in order. PNE provides additional ways in which the stability of the graph can be challenged. In PS, only single-link deviations matter, whereas in PNE, also the effect of removing any number of incident links is considered. Likewise, we could consider the *creation* of any number of links. A possible definition for such a kind of equilibrium would be that for each player v and each set of new links $\{v, w_1\}, \ldots, \{v, w_k\}$, the addition of all these links would mean no improvement for any of the players $v, w_1, \ldots, w_k$ or an impairment for at least one of them. This is a more complicated extension than allowing

[2] "Convex" is also common in the literature, but we use "pseudo-convex" here in order to not cause confusion with convexity as in convex optimization, etc.

the removal of multiple links, and most of the literature is restricted to the latter. There is another reason why the extension to the removal of multiple links can be considered more natural than the extention to adding multiple links. Let each player v specify a function $S_v : V \backslash \{v\} \longrightarrow \{0, 1\}$. The function S_v is called the *strategy* of player v, and the family $S = (S_v)_{v \in V}$ of all such functions is called a *strategy profile*. Given a strategy profile S, we define a set of edges $E(S) := \{\{v, w\}\, ;\, S_v(w) = 1 \wedge S_w(v) = 1\}$. In other words: players will be connected by an edge if and only if they have named each other. If a player v specifies $S_v(w) = 1$, but player w specifies $S_w(v) = 0$, that is, if v names w but w does not name v, then no edge between v and w is created. In order to simplify matters, traditionally there has been a restriction to *essential* strategy profiles, namely where $S_v(w) = 1 \iff S_w(v) = 1$ for all players v, w. We call a strategy profile S a *Nash equilibrium* if no player can improve her cost or utility by changing her strategy in S. Now it is easy to see that $G = (V, E)$ is a PNE if and only if G is PS and there is an essential Nash equilibrium S such that $E = E(S)$. This is also the reason for the name "pairwise Nash equilibrium".

We return to our main topic, namely we are interested in how efficient PS graphs or PNE graphs are compared to when a central authority would enforce an optimal graph. The following notion, the *price of anarchy*, is a measure for inefficiency of PS or PNE graphs in this sense. It is due to Koutsoupias and Papadimitriou [KP99] and Papadimitriou [Pap01]. First, we need a notion for optimality:

2.1.6 Definition. For fixed parameters n, α, utility function, and equilibrium concept (PS or PNE), an undirected graph $G^* \in \mathcal{G}_n$ is called *optimal*, if for all $G \in \mathcal{G}_n$, we have $\mathsf{SU}(G^*) \geqslant \mathsf{SU}(G)$, that is, $G^* \in \arg\max_{G \in \mathcal{G}_n} \mathsf{SU}(G)$. The optimum utility is denoted OPT, and assumed to be positive.

2.1.7 Definition. Let $\mathcal{E} \subseteq \mathcal{G}_n$ be a set of equilibrium graphs, e. g., all PS graphs or all PNE graphs on n players, or all such graphs with an additional property, e. g., all PS graphs on n players that are trees. The *price of anarchy* with respect to $\mathcal{E}$ is denoted $\mathsf{PoA}(\mathcal{E})$ and defined as the ratio of optimum social utility to worst-case equilibrium social utility, i. e.,

$$\mathsf{PoA}(\mathcal{E}) := \max_{G \in \mathcal{E}} \frac{\mathsf{OPT}}{\mathsf{SU}(G)} \ .$$

Usually, we omit the $\mathcal{E}$ in the notation when it is clear from context. When no further restriction is given, $\mathcal{E}$ is assumed to be the set of all equilibrium graphs with respect to the equilibrium concept currently under study and a fixed number of player, e. g., all PS graphs on n players.

2.2 Swap Equilibrium

The concept of *swap equilibrium* (SE) was introduced by Alon, Demaine, Hawvaghayi and Leighton [ADH+10] in 2010. In this model, cost is equal to disutility and utility is equal to income, in symbols $\mathcal{C}_v(G) = D_v(G)$ and $U_v(G) = I_v(G)$, respectively. Often, when considering SE, we will only speak of "cost" or "utility" and not mention "disutility" or "income". There is no link cost α, which is considered one of the benefits of this model. We will use SE in Chapter 3 and in Chapter 4. All notions in this section will only be given for cost, but they can easily be translated to the case of utility.

2.2.1 Definition. A *swap* for a graph $G \in \mathcal{G}_n$ is a triple of players (u, v, w), such that $\{u, v\} \in E$ and $\{u, w\} \notin E$.

Denote $S(G) \subseteq V_n^3$ the set of all swaps of G. Denote $G + (u, v, w)$ the graph that is obtained from G by removing $\{u, v\}$ and inserting $\{u, w\}$; we say that player u swaps her edge $\{u, v\}$ for the new edge $\{u, w\}$.

2.2.2 Definition. A graph $G \in \mathcal{G}_n$ is called a *swap equilibrium* (SE) with respect to cost $\mathcal{C}$ if the following two conditions hold:

(i) $\mathcal{C}_u(G) \leqslant \mathcal{C}_u(G + (u, v, w))$ for all $(u, v, w) \in S(G)$,

(ii) $\mathcal{C}_u(G) \leqslant \mathcal{C}_u(G - \{u, v\})$ for all $\{u, v\} \in E$.

That is, if no player can improve by swapping one of her incident links for another (first condition) or by removing one of her incident links (second condition).

Since we have no link cost, the price of anarchy as defined before does not apply since it includes comparing graphs with different numbers of edges, which does not make sense when we have no edge costs. Instead, when working with SE, we derive statements that give upper bounds on the social cost of any SE or, for lower bounds, show existence of SE graphs with high social cost.

For example, in [ADH+10], they considered the sum-distance model and the max-distance model under SE. In the sum-distance model, they obtained an upper bound of $2^{\mathcal{O}(\sqrt{\ln n})}$ and a lower bound of 3 on the diameter of SE graphs and proved that the star is the only equilibrium tree. For the max-distance model, they proved a lower bound of $\Omega(\sqrt{n})$ on the diameter of an SE and a tight bound of 3 for trees. Those bounds on the diameter imply bounds on the social cost.

2.3 Asymptotic Notation

In order not having to introduce names for all occurring constants, we use "$\mathcal{O}(\ldots)$" and "$\Omega(\ldots)$" notation. For the results proved in this thesis, we use this notation in the following understanding: we write "$x = \mathcal{O}(y)$" if there exists a constant $c > 0$ such that $x \leqslant cy$. The constant may only depend on other constants and is in particular independent of the non-constant quantities that constitute x and y, e. g., parameters n and α. We do not implicitly require that some quantities, e. g., n, have to be large. Analogously, we write "$x = \Omega(y)$" if there exists a constant $c > 0$ such that $x \geqslant cy$. Note that "$\mathcal{O}(\ldots)$" indicates an upper bound, making no statement about a lower bound; while "$\Omega(\ldots)$" indicates a lower bound, making no statement about an upper bound. We write $x = \Theta(y)$ if $x = \mathcal{O}(y)$ and $x = \Omega(y)$; the constants used in the "$\mathcal{O}(\ldots)$" and the "$\Omega(\ldots)$" statement may be different, of course.

The "$o(\ldots)$" notation is only used in one form, namely $o(1)$ substituting a quantity that tends to 0 when n tends to infinity, regardless whether other parameters are fixed or not. Whenever we write "$o(1)$" in an expression, it is meant as an upper bound, making no statement about a lower bound.

Chapter 3

Swap Equilibria under Link and Vertex Destruction

Now we look at a particular class of network formation games, which consider robustness aspects of the graphs in the sense of the *destruction* or *adversary model* [Kli10b]. Our equilibrium concept is *swap equilibrium* (SE) as explained in Section 2.2. The destruction model is a network formation game incorporating the robustness of a network under a more or less targeted attack. In addition to bringing in the SE concept, we extend the model from an attack on the edges to an attack on the vertices of the network. We prove structural results and linear upper bounds or super-linear lower bounds on the social cost of SE under different attack scenarios. For the case that the vertex to be destroyed is chosen uniformly at random from the set of *max-sep* vertices, that is, where each causes a maximum number of separated player pairs, we show that there is no tree SE with only one max-sep vertex. On the other hand, we show that for the uniform measure, all SE are trees (unless two-connected). This chapter is based on [**KSS17a**].

3.1 Previous and Related Work

Recently, robustness aspects have been addressed in the form of the destruction model (also known as the *adversary model*) [**Kli16**; Kli10a; Kli11; Kli15]. In this model, players anticipate the destruction of exactly one edge in the graph, and the cost function for each player v gives the expected number of other players that v will no longer be able to reach after the destruction. We recall that social cost is the sum of all players' costs, which is equal to the expected number of separated vertex pairs, that is the expected number of all ordered pairs (v, w) such that there is no path anymore between v and w after the destruction has taken place. The model allows many variations, since the edge to be destroyed is determined randomly according to a probability measure that may even depend on the graph (this dependence is known to the players)[1].

The destruction model was introduced by Kliemann in 2010 [Kli10a] and subsequently studied in a series of publications [**Kli16**; Kli11; Kli15]. The focus was on the price of anarchy for Nash equilibrium (NE)[2] and pairwise stability (PS). We give a comparison between the NE and PS results for the destruction model with the results on SE in this work in Section 3.3. Earlier works on robustness in network formation include [JW96; CFS+04; BG00; SH03; HS05]. None of those earlier models allows a structure-dependent destruction probability as in the destruction model; for a detailed discussion, we refer to [Kli11, Section 4].

Computationally deciding whether a graph constitutes an NE is hindered by an exponential search space and can indeed be NP-hard [FLM+03]. This has raised concerns about the applicability of the model since we should not expect players to solve an NP-hard problem. Therefore, many variations have been introduced in order to make the model more tractable. Usually, the idea is to limit the choices of the players, for example to single-edge deviations [JW96; Len12].

The following three recent publications address robustness in a network formation framework similar to ours:

- ▻ Meirom et al. [**MMA15**] consider a cost function that uses a linear combination of the lengths of two short disjoint paths. The idea is that the players build a graph where, for each shortest path, there is a backup path of reasonable length.

[1] The name “adversary model” is the original one. However, “adversary” was found to be more suited to describe an entity that aims at maximizing cost under equilibrium. This is not necessarily the case in our model.

[2] Nash equilibrium is another equilibrium concept, different from pairwise Nash equilibrium. We only give an informal description here, since it is not needed outside of the discussion. For details, we refer to the literature, e. g., [Kli11]. When using NE, each edge is owned by exactly one of its endpoints. Each player can remove any of the links that she owns and build any number of additional links (which then will be owned by her).

- ▻ Goyal et al. [**GJK16**] consider a model where each player, in addition to building edges, can choose to immunize themselves in exchange for a fee. Then, an adversary selects a connected component of non-immunized vertices to destroy; an alternative description is that the adversary picks a vertex, and then, the destruction spreads from there, while immunized vertices act as firewalls. A player's utility is the expected size of his or her connected component after the destruction has taken place, which is zero if the player herself is destroyed. This utility is almost exactly the positive version of our cost: if $\mathcal{C}_v$ is the expected number of cut-off vertices, then $n - \mathcal{C}_v$ is the expected size of v's component after the attack (we differ in that if the player herself is attacked, we say that she is cut off from $n-1$ vertices, so her component size is one, not zero). However, the kind of destruction is very different from ours. Although it can be formulated as an attack on a vertex, the contagious properties of the attack move the focus to different connectivity properties of the attacked vertex in comparison to our model. For example, if a leaf (that is, a vertex of degree one) is attacked in our model, the overall damage is relatively small: namely, we have $2(n-1)$ separated vertex pairs. In their model, however, if the neighbor of the leaf is not immunized, the destruction will spread, and the overall damage can be much higher. For earlier work on network formation with contagious risk, see [**BEK13**].
- ▻ Chauhan et al. [**CLMM16**] extended the edge destruction model by incorporating distances: the cost for player v is the expected sum of distances to all other players after edge destruction.

In another line of research [**DG13**; **HJ16**; **Hal16**; **BCT17**], the robustness of networks is studied in a two-player game. The first player is the *designer* and can choose which edges to form in the network, while the second player is the *adversary* or *disruptor*, who can destroy vertices or edges in the network. Some of those models allow immunization of vertices or edges. All of those models are very different from ours since the network is formed by a *central authority*, the designer, and not in a decentralized manner as in our model. Moreover, the adversaries in those models work differently from ours: they typically act more deterministically and not randomized as our adversaries (or they randomize in simpler ways than ours), but on the other hand, they are more powerful, since they can destroy more than one edge or one vertex.

If, for our type of adversary, we had a network designer, then that designer would simply build an optimal network. Optimal networks for the case that we have link costs α and edge destruction have been identified as cycle and star, independent of the adversary and only depending on the range of α [**Kli16**; Kli11]. Therefore, the designer's job would be simple: look at α, and then, decide whether to build a cycle or a star. If there are no link costs, as in the SE model, the notion of optimality is not straightforward in the first place. However, given a fixed number of edges k, we can define optimal graphs as those with minimal cost among all graphs with k edges. If $k \geqslant n$, then any graph with a cycle traversing all of the vertices is optimal in this sense, for edge and for vertex destruction independent of the actual destroyer. If $k = n - 1$, then for any edge destroyer, the star is optimal (optimality for vertex destruction is not straightforward for $k = n - 1$).

3.2 Contribution

We prove quantitative and structural results for two types of destroyers under the stability concept of swap equilibrium (SE): the *uniform destroyer* picks an edge or vertex uniformly at random, while the *extreme destroyer* picks an edge or vertex uniformly at random from the set of edges or vertices, respectively, where the destruction of each causes a maximum number of vertex pairs to be separated. An edge or vertex that does the latter is called a *max-sep edge* or *max-sep vertex*, respectively. In addition, we consider some variations. For edge destruction, the most notable variation is a destroyer that chooses an edge from the set of *bridges* uniformly at random. For vertex destruction, we consider that the probability for destruction of a vertex v is proportional to her *degree* $\deg(v)$, which we call the degree-proportional destroyer.

We prove that for uniform and extreme edge destruction and uniform bridge destruction, an SE is bridgeless or has a star-like structure. A consequence of this is that in terms of social cost, those SE are very efficient, namely the social cost of any of those SE is $\mathcal{O}(n)$. For uniform vertex destruction, we prove that if an SE is not two-connected, then it is a tree. This again implies an $\mathcal{O}(n)$ bound on the social cost.

For the degree-proportional vertex destroyer, the situation is very different. Social cost of an SE can be as high as $\Omega(n^2)$, which is the highest order possible in this model. This lower bound is attained on a simple graph, namely a star.

For extreme vertex destruction, we also give a super-linear lower bound on the social cost of SE, namely $\Omega(n^{3/2})$. The construction is still roughly star-like, but more complicated: we need a clique of certain size at the center unto which paths of a certain length are attached.

Finally, we prove a structural result for extreme vertex destruction: if $n \geqslant 8$, there is no tree SE with only one max-sep vertex. This means that in a tree SE (with $n \geqslant 8$), the extreme destroyer will always have at least two vertices to choose from.

3.3 Discussion of Results and Comparison with Previous Work

What kind of (decentralized) network formation is most effective against a destroyer? This question cannot be fully answered yet. However, we can make the following observations.

Consider first that we face the uniform edge destroyer. All three models, namely Nash equilibrium (NE), pairwise stability (PS) and swap equilibrium (SE), provide relatively robust networks. For NE and PS, this has been shown in previous work [**Kli16**; Kli11] by proving a price of anarchy of $\mathcal{O}(1)$, meaning that the cost of an equilibrium network is at most a constant factor away from that of an optimal network. For SE, we prove here that the social cost of an equilibrium is upper-bounded by $2(n-1)$, which means that the destroyer can cut off at most one vertex from the rest of the graph (resulting in $2(n-1)$ separated vertex pairs). This equates to relatively small damage. It is only beaten by a graph with a cycle that traverses all vertices, which has zero social cost. Note that non-equilibrium graphs can have super-linear social cost, e.g., the path has social cost $\Omega(n^2)$, following from the computation in [Kli11, Proposition 8.1].

In [**CLMM16**], the uniform edge destroyer is combined with a different cost function, which is the expected sum of distances to other players after destruction (instead of considering the number of cut-off vertices). NE is used as the equilibrium concept. A bound of $\mathcal{O}(1 + \alpha/\sqrt{n})$ is proven on the price of anarchy, which is a good bound. Note in particular that it is constant for $\alpha \leqslant \sqrt{n}$.

Now, consider that we face the extreme edge destroyer. One of the most striking previous results is that for NE, the price of anarchy is still $\mathcal{O}(1)$ [Kli11, Theorem 9.8]. For PS, the situation flips, and the worst possible order is obtained for the price of anarchy **[Kli16]**, if $\alpha = \Omega(1)$. The lower-bound example used in the proof exploits the two properties of PS: we only consider single-edge deviations, and the agreement of both endpoints is required to build an edge. For SE, we prove here that the destroyer can cut off at most one vertex from the rest of the graph, just like for the uniform destroyer. Therefore, although NE and SE differ computationally (exponential versus quadratic search space), they both provide relatively robust networks when faced with the extreme edge destroyer.

For vertex destruction, NE and PS have not been studied previously, so we cannot make a comparison with those. The results that we prove here for SE resemble those for edge destruction and PS: relatively robust networks for the uniform destroyer and less robust ones for the extreme destroyer (linear social cost versus super-linear social cost). It should be emphasized that the degree-proportional destroyer, which arguably takes a simpler approach than the extreme destroyer, can have SE even with quadratic cost, which is the worst possible order. An explanation is that this destroyer combines the unpredictability of the uniform destroyer, on the one hand, with the focus on destruction expressed by the extreme destroyer, on the other hand (higher-degree vertices tend to cause more damage when destroyed). Indeed, the proof for the quadratic lower bound relies crucially on the way in which this destroyer randomizes.

A comparison between edge destruction and vertex destruction can be done only for SE currently. Removing an edge can leave us with at most two connected components. Removing a vertex v can leave us with $\deg(v)$ connected components (not counting v itself; cf. the exact definition in Section 3.4). This suggests that the model is theoretically harder to handle and that the destroyer can do more damage in terms of separated vertex pairs. However, for the uniform destroyer, the fundamental proof ideas turn out to be roughly similar (namely opening a cycle to capture more vertices and bounding the diameter of a tree SE), although technically much more involved for vertex destruction.

For the extreme destroyer, edge and vertex destruction are clearly more divergent from each other than for the uniform destroyer. This is reflected by our results: for edge destruction, we have a linear bound on the social cost of SE, while for vertex destruction, we have a super-linear lower bound and no upper bound at this time. The lower-bound example clearly does not work for extreme edge destruction, as explained in Remark 3.8.2. can have profound effects on the destroyer's probability measure. That is, if a leaf a swaps its one edge $\{a, b\}$ for an edge $\{a, c\}$, then the number of components when c is destroyed increases. This can lead to c becoming the only max-sep vertex in the new graph. This signifies a drastic change, provided that before the swap, there were many max-sep vertices in different parts of the graph. It should be noted that we have no lemma saying that max-sep vertices are distributed in a certain pattern in the graph, whereas for edge destruction, it is known that they form a star-like structure [Kli11, Proposition 9.1].

3.4 Model and Notation

Edge destruction

Let $G \in \mathcal{G}_n$ be connected. For $v, w \in V_n$, denote $\mathcal{R}_G(v, w)$ the set of all v-w paths in G.

3.4.1 Definition. The *relevance* of $e \in E(G)$ for $v \in V_n$ is:

$$\operatorname{rel}_G(e, v) := |\{w \in V_n \,;\ \forall P \in \mathcal{R}_G(v, w) : e \in E(P)\}| \ .$$

That is, the number of those vertices where for each of them holds: in order to reach it from v, we necessarily have to traverse edge e. When e is removed from the graph, then there will be exactly $\operatorname{rel}_G(e, v)$ vertices that v will no longer be able to reach; we also say that those vertices are *cut off* from v or that v is *cut off* from them.

3.4.2 Definition. An *edge destroyer* $\mathcal{D}$ is a map that associates with each $G \in \mathcal{G}_n$ a probability measure $\mathcal{D}_G$ on $E(G)$, that is $\mathcal{D}_G(e) \in [0,1]$ for each e, and $\sum_{e \in E(G)} \mathcal{D}_G(e) = 1$.

Given $\mathcal{D}$, we define the *cost* for player v in G as:

$$\mathcal{C}_v(G) := \sum_{e \in E} \mathrm{rel}_G(e,v)\, \mathcal{D}_G(e) \ ,$$

that is, the expected number of vertices from which v will be cut off after one edge is removed randomly according to the measure $\mathcal{D}_G$. If G is disconnected, cost is defined to be ∞.

3.4.3 Definition. The *separation* $\mathrm{sep}(e)$ of an edge $e \in E(G)$ is the number of ordered player pairs (v,w) such that the removal of e will destroy all v-w paths in G.

If e is a non-bridge, then clearly $\mathrm{sep}(e) = 0$. Otherwise, $G - e$ has exactly two components, say $K_1, K_2 \subseteq V_n$, and we call $\nu(v) := \min\{|K_1|, |K_2|\}$ the *minimal-component size* of e. Then, $\mathrm{sep}(e) = 2\nu(e)(n - \nu(e))$. The social cost is:

$$\begin{aligned}
\mathsf{SC}(G) &= \sum_{v \in V(G)} \mathcal{C}_v(G) = \sum_{v \in V(G)} \sum_{e \in E} \mathrm{rel}_G(e,v)\, \mathcal{D}_G(e) \\
&= \sum_{e \in E} \sum_{v \in V(G)} \mathrm{rel}_G(e,v)\, \mathcal{D}_G(e) = \sum_{e \in E(G)} \mathrm{sep}(e)\, \mathcal{D}_G(e)
\end{aligned}$$

Vertex destruction

For the vertex destruction model, the destroyer $\mathcal{D}$ associates each $G \in \mathcal{G}_n$ with a probability measure on the vertices of G, that is $\mathcal{D}_G(v) \in [0,1]$ for each $v \in V_n$, and $\sum_{v \in V_n} \mathcal{D}_G(v) = 1$. We define the destruction of a vertex not as its removal from the graph, but as the removal of all of its incident edges. This is reflected by the following definition of relevance and cost.

3.4.4 Definition. The *relevance* of $u \in V_n$ for $v \in V_n$ is:

$$\mathrm{rel}_G(u,v) := |\{w \in V_n\,;\ \forall P \in \mathcal{R}_G(v,w) :\ u \in V(P)\}| \ .$$

Note that since v is in every v-w path, we have $\mathrm{rel}_G(v,v) = n - 1$, which is exactly the number of vertices that will be cut off from v if all edges incident with v are removed. Given a vertex destroyer $\mathcal{D}$, we define the cost for player v in G as:

$$\mathcal{C}_v(G) := \sum_{u \in V} \mathrm{rel}_G(u, v)\, \mathcal{D}_G(u) \ .$$

Again, a disconnected graph induces infinite cost for each player.

3.4.5 Definition. The *separation* $\mathrm{sep}(u)$ of a vertex $u \in V_n$ is the number of ordered player pairs (v, w) such that the removal of u will destroy all v-w paths in G.

If removal of u creates k components (not counting u itself) of sizes $\beta_1, \ldots, \beta_k$, we have:

$$\begin{aligned}
\mathrm{sep}(u) &= n - 1 + \sum_{i=1}^{k} \beta_i (n - \beta_i) \\
&= n - 1 + n \sum_{i=1}^{k} \beta_i - \sum_{i=1}^{k} \beta_i^2 \\
&= n - 1 + n(n-1) - \sum_{i=1}^{k} \beta_i^2 \\
&= n^2 - 1 - \sum_{i=1}^{k} \beta_i^2 \ . \qquad (3.4.6)
\end{aligned}$$

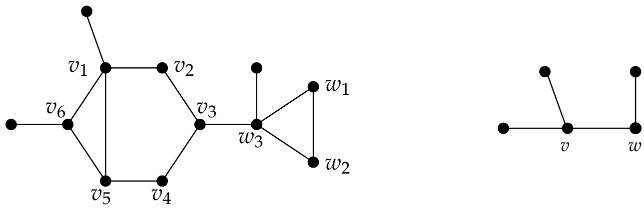
It is again easy to see that

$$\mathsf{SC}(G) = \sum_{u \in V_n} \mathrm{sep}(u)\, \mathcal{D}_G(u).$$

When the graph G is clear from context, we omit the G arguments. When a graph G' is defined, we write $\mathcal{C}'$, rel', SC', etc., instead of $\mathcal{C}(G')$, $\mathrm{rel}(G')$, $\mathsf{SC}(G')$, etc., respectively. The same goes for G''.

3.4.7 Definition. For any connected graph $G = (V, E)$, we call $I \subseteq V$ an *island* if it is inclusion-maximal under the condition that the induced subgraph $G[I]$ is bridge-free (it does not matter whether we mean bridges of G or bridges of $G[I]$).

3.4.8 Definition. The *bridge tree* $\widetilde{G}$ of G is obtained by collapsing each island I of G to a single vertex $\widetilde{I}$ and inserting an edge between $\widetilde{I}$ and $\widetilde{J}$ if



i) A graph G.

ii) The corresponding bridge tree $\widetilde{G}$

Figure 3.1. Players v and w consisting of vertices $v_1, v_2, \ldots, v_6$ and w_1, w_2, w_3 respectively.

and only if an edge runs in G between a vertex of I and a vertex of J. See Figure 3.1 for an example of the bridge tree.

Obviously, $IJ \mapsto \widetilde{I}\widetilde{J}$ is a bijection between the set of bridges of G and the set of bridges of $\widetilde{G}$, and we will often identify those two sets. We refer to [Kli11] for a more formal treatment of the bridge tree.

We will also use the better-known *block-cut-vertex* tree; see, e.g., the book by Diestel [Die05] for a definition.

3.5 Uniform Edge and Uniform Bridge Destruction

Denote $B(G) \subseteq E(G)$ the set of all of the bridges of $G \in \mathcal{G}_n$ and:

$$S^B(G) := \{(a,b,c) \in S(G)\,;\ \{a,b\} \in B(G) \wedge G + (a,b,c) \text{ is connected}\}$$

the set of *bridge swaps* (the case that $G + (a,b,c)$ is disconnected is not interesting since such a swap can never bring an improvement). Clearly, $\{a,c\} \in B(G + (a,b,c))$ for each $(a,b,c) \in S^B(G)$.

An edge destroyer $\mathcal{D}$ is called the *uniform edge destroyer* if $\mathcal{D}_G(e) = \frac{1}{|E(G)|}$ for each $e \in E(G)$ and each $G \in \mathcal{G}_n$. It is called the *uniform bridge destroyer* if $\mathcal{D}_G(e) = \frac{1}{|B(G)|}$ for each $e \in B(G)$ and $\mathcal{D}_G(e) = 0$ for each $e \in E(G) \setminus B(G)$, for each $G \in \mathcal{G}_n$ (if $B(G) = \emptyset$, the graph is two-edge-connected, and we can take any probability measure for $\mathcal{D}_G$ since the cost for each player is zero in any case).

In order to point out what are some of the essential properties of those destroyers, we look at more general destroyers first. Consider the following condition on a destroyer:

$$\begin{aligned}
&\forall G \in \mathcal{G}_n \ \forall s = (a, b, c) \in S^B(G) : \\
&\qquad \Big(\operatorname{sep}_G(\{a, b\}) = \operatorname{sep}_{G+s}(\{a, c\}) \implies \mathcal{D}_G(\{a, b\}) = \mathcal{D}_{G+s}(\{a, c\}) \Big) \\
&\wedge \quad \Big(\forall e \in E(G) \cap E(G+s) : \operatorname{sep}_G(e) = \operatorname{sep}_{G+s}(e) \\
&\qquad \implies \mathcal{D}_G(e) = \mathcal{D}_{G+s}(e) \Big)
\end{aligned} \tag{3.5.1}$$

This means that if after a bridge swap an edge maintains its separation, then it also maintains its probability. This clearly includes the uniform edge destroyer and the uniform bridge destroyer. The next proposition shows that (3.5.1) is equivalent to the following simpler condition:

$$\begin{aligned}
&\forall G \in \mathcal{G}_n \ \forall s = (a, b, c) \in S^B(G) : \\
&\qquad \mathcal{D}_G(\{a, b\}) = \mathcal{D}_{G+s}(\{a, c\}) \\
&\wedge \quad \forall e \in E(G) \cap E(G+s) : \mathcal{D}_G(e) = \mathcal{D}_{G+s}(e)
\end{aligned} \tag{3.5.2}$$

This means that each edge carries its fixed probability that, in the case of a bridge, sticks to it even when it is swapped for another edge.

3.5.3 Proposition. *(3.5.1) and (3.5.2) are equivalent.*

Proof. It is clear that (3.5.2) implies (3.5.1). Therefore, let $\mathcal{D}$ be a destroyer with property (3.5.1). Let $G \in \mathcal{G}_n$ and $s = (a, b, c) \in S^B(G)$. Since $\nu_G(\{a, b\}) = \nu_{G+s}(\{a, c\})$, we have $\operatorname{sep}_G(\{a, b\}) = \operatorname{sep}_{G+s}(\{a, c\})$; hence, $\mathcal{D}_G(\{a, b\}) = \mathcal{D}_{G+s}(\{a, c\})$. Now, consider the special case first that (a, b, c) is a path in the bridge tree. Then, the only separation that changes due to s is that of $\{b, c\}$. Since separations of all other edges are maintained, they also maintain their probabilities. Since all of the probabilities add up to one, the edge $\{b, c\}$ also maintains its probability. In the general case, we have a path $(v_0 = b, v_1, \ldots, v_k = c)$. Conducting the sequence of swaps $(a, v_0, v_1), (a, v_1, v_2), \ldots, (a, v_{k-1}, v_k)$ gives the graph $G + s$, and in each step, the edges maintain their probability. □

The following proof uses the basic idea from [ADH+10, Theorem 1].

3.5.4 Lemma. *Let $\mathcal{D}$ be a destroyer with property (3.5.2). Then, the bridge tree of an SE with respect to $\mathcal{D}$ has a diameter at most two.*

Proof. Let G be an SE, and for contradiction, assume that (a, b, c, d) is a path in its bridge tree. Denote n_a, n_b, n_c, n_d the number of vertices in the subtrees rooted at a, b, c and d, respectively; hence, $n = n_a + n_b + n_c + n_d$. Consider the swap $s = (a, b, c)$. By (3.5.2), we have $\mathcal{D}_G(\{a, b\}) = \mathcal{D}_{G+s}(\{a, c\})$, and all of the other edges maintain their probabilities, as well. We have $\mathrm{rel}_G(\{a, b\}, a) = \mathrm{rel}_{G+s}(\{a, c\}, a)$, and for all of the other edges, from the view of a, the only relevance that changes is that of $\{b, c\}$, namely from $n_c + n_d$ to n_b. Since G is an SE, this means $n_c + n_d \leqslant n_b$. Likewise, we consider the swap (d, c, b) and obtain $n_a + n_b \leqslant n_c$. Together, this implies $n_a \leqslant 0$, which is impossible. □

3.5.5 Theorem. *Let G be an SE for the uniform edge destroyer. Then, G is bridgeless or a star; hence, $\mathsf{SC}(G) \leqslant 2(n-1) = \mathcal{O}(n)$.*

Proof. If G is bridgeless, then $\mathsf{SC}(G) = 0$. Therefore, assume that G contains a bridge. We want to show that G is a tree, so for contradiction, assume that G is not a tree. Let I be an island containing a cycle, and let $\{b, c\}$ be a bridge with $b \in I$ (and $c \in I'$ for some island I'). Then, there is a cycle C in I that traverses b. Choose a so that $\{a, b\} \in E(C)$. Then, the swap (a, b, c) puts the bridge $\{b, c\}$ on a cycle and makes it part of the island, so its relevance for a drops from a positive value to zero. No new bridges are introduced, and the relevance of all other edges remains the same for player a. Hence, this is an improving swap, a contradiction to SE. The situation is depicted in Figure 3.2.

Since we know that G is a tree, G coincides with its bridge tree. By Lemma 3.5.4, $\mathrm{diam}(G) \leqslant 2$. Since $n \geqslant 3$, we conclude that G is a star. It follows that $\mathsf{SC}(G) = 2(n-1)$. □

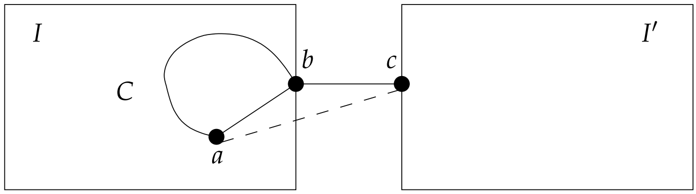


Figure 3.2. Proof of Theorem 3.5.5.

3.5.6 Theorem. *Let G be an SE for the uniform bridge destroyer. Then, G is bridgeless or $\widetilde{G}$ is a star, where each of the outer islands has exactly one vertex* [3]*. Hence,* $\mathsf{SC}(G) \leqslant 2(n-1) = \mathcal{O}(n)$.

Proof. If G is bridgeless, then $\mathsf{SC}(G) = 0$. Therefore, assume that G contains a bridge. By Lemma 3.5.4, $\widetilde{G}$ is a star. Let I be an island that is not the center of the star (i.e., it is an outer island) and that contains more than one vertex. Then, I contains a cycle. By a swap as in the proof of Theorem 3.5.5, the one bridge e between the center of the star and I can be put on a cycle. For the players in I, this is a strict improvement since for them, e had the strictly highest relevance of all bridges in G. The statement on the social cost follows since only one vertex can be separated from the rest of the graph by the removal of a bridge. □

3.6 Extreme Edge Destruction

For $G \in \mathcal{G}_n$, denote $\operatorname{sep}_{\max}(G) := \max_{e \in E(G)} \operatorname{sep}(e)$ and:

$$E_{\max}(G) := \{e \in E(G)\,;\ \operatorname{sep}(e) = \operatorname{sep}_{\max}(G)\}\ .$$

We call the edges in $E_{\max}(G)$ the *max-sep edges*. Recall $\operatorname{sep}(e) = 2\nu(e)(n - \nu(e))$ and note that $x \mapsto x(n-x)$ is strictly increasing on $[0, n/2]$; hence, $\nu(e) = \nu(e')$ for all $e, e' \in E_{\max}(G)$. Moreover, if $\nu(e) = \nu(e')$ for some $e \in E_{\max}(G)$ and $e' \in E(G)$, then $e' \in E_{\max}(G)$. In other words: exactly all of the edges with maximum $\nu(e)$ are max-sep.

[3] If the star has only one edge and thus there are exactly two islands, this statement means that one of the two islands has exactly one vertex.

An edge destroyer $\mathcal{D}$ is called the *extreme edge destroyer* if $\mathcal{D}_G(e) = \frac{1}{|E_{\max}(G)|}$ for each $e \in E_{\max}(G)$ and $\mathcal{D}_G(e) = 0$ for each $e \in E(G) \backslash E_{\max}(G)$.

3.6.1 Theorem. *Let G be an SE under the extreme edge destroyer. Then, G is bridgeless or $\widetilde{G}$ is a star, where each of the outer islands has exactly one vertex. Hence, $\mathsf{SC}(G) \leqslant 2(n-1) = \mathcal{O}(n)$.*

Proof. The expression for the social cost follows from the structural statement. Assume for contradiction that G contains bridges and is not of the stated form.

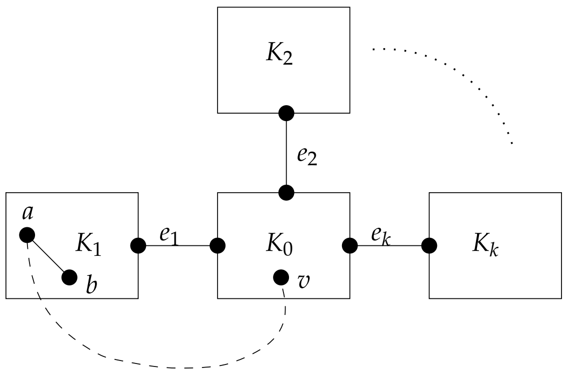


Figure 3.3. Proof of Theorem 3.6.1, Case 1.

Case 1: $E_{\max} = \{e_1, \ldots, e_k\}$ with $k \geqslant 2$. This situation is depicted in Figure 3.3. By ([Kli11] Proposition 9.1), the max-sep edges form a star in the bridge tree. For each $i \in [k]$, denote $K_i \subseteq V_n$ the (unique) minimal component of $G - e_i$, and denote K_0 the island at the center of the star formed by $E_{\max}$. Then, $V_n = \dot\bigcup_{i=0}^{k} K_i$ and $|K_i| = |K_j|$ for all $i, j \in [k]$. By assumption, $|K_i| \geqslant 2$ for each i.

Case 1.1: There is a leaf $a \in K_1$. Denote b the neighbor of a, and let $v \in K_0$. Define $G' := G + (a, b, v)$. When moving from G to G', all of the edges $e_2, \ldots, e_k$, and the edges in $G[K_2], \ldots, G[K_k]$ have their separation maintained. Edges in $G[K_1]$ have their separation maintained or reduced since their minimal-component size reduces. The edge e_1 has its separation reduced since its minimal-component size reduces. The new edge $\{a, v\}$

cannot become max-sep since $\nu(\{a, v\}) = 1$ while $\nu(e_2) \geqslant 2$. It follows that $E'_{\max} = \{e_2, \ldots, e_k\}$. We have $\mathrm{rel}'(e_i, a) = \mathrm{rel}(e_i, a) < \mathrm{rel}(e_1, a)$ for all $i \geqslant 2$. Hence, $\mathcal{C}'_a < \mathcal{C}_a$, a contradiction to SE.

Case 1.2: There is a cycle C in K_1. Let $\{a, b\} \in E(C)$ and $v \in K_0$. By essentially the same arguments as in Case 1.1, we show that player a improves by the swap (a, b, v).

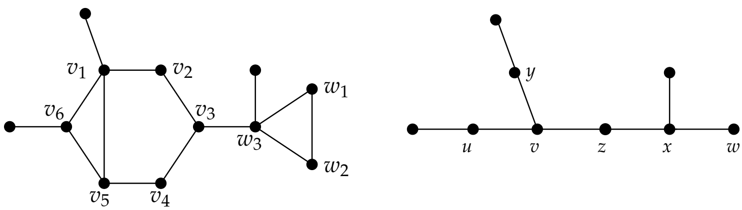
Case 2: $E_{\max} = \{e_1\}$. This case is more difficult since after reducing the separation of e_1, we have no other max-sep edges that could act as a reference. Denote K_1 a component of $G - e_1$ with minimum size (if both components of $G - e_1$ have the same size, then pick one arbitrarily), and denote K_0 the island containing the endpoint of e_1 that is not in K_1.

Case 2.1: There is a leaf $a \in K_1$. Denote b the neighbor of a, and let $v \in K_0$. Define $G' := G + (a, b, v)$. We have $\nu'(e_1) = \nu(e_1) - 1$, so $\mathrm{sep}'(e_1)$ has the next lower possible value below $\mathrm{sep}(e_1)$. Hence, $E'_{\max} = \{e_1, e_2, \ldots, e_k\}$ for zero or more additional edges $e_2, \ldots, e_k$. Since they all have the same minimum-component size, they cannot be in $G[K_1]$; hence, they form a star with K_0 as the center (this is the only way that they can form a star in the bridge tree). If $\{a, v\} \notin E'_{\max}$, then $\mathcal{C}'_a = \nu(e_1) - 1 < n - \nu(e_1) = \mathcal{C}_a$; hence, the swap is an improvement. If $\{a, v\} \in E'_{\max}$, then $\nu(e_1) - 1 = \nu'(\{a, v\}) = 1$; so $|K_1| = 2$ and $n \geqslant 4$. Moreover, $k \geqslant 2$. It follows that $\mathcal{C}'_a = \frac{1}{k}(n - 1 + (k-1)(\nu(e_1) - 1)) = \frac{n-2}{k} + 1$. On the other hand, $\mathcal{C}_a = n - 2$. If $n \geqslant 5$ or $k \geqslant 3$, this implies an improvement. The remaining case of $n = 4$ and $k = 2$ is impossible, since for such n, the graph G' is a star, and thus, $k = 3$.

Case 2.2: There is a cycle C in K_1. We consider a swap like the one in Case 1.2. It is not difficult to see that $\mathrm{rel}'(e, a) < \mathrm{rel}(e_1, a)$ for each $e \in E'_{\max}$; hence, the swap is an improvement. □

3.7 Uniform and Degree-Proportional Vertex Destruction

For any connected graph $G = (V, E)$, denote $\mathcal{B}(G)$ the set of its blocks (maximal biconnected subgraphs) and $A(G) \subseteq V$ the set of its cut-vertices (also known as *articulation points*). Denote $\widehat{G}$ the *block-cut-vertex tree* of G, that is $V(\widehat{G}) = \mathcal{B}(G) \dot\cup A(G)$, and each edge in $\widehat{G}$ runs between a block



i) A graph G.

ii) The corresponding block-cut-vertex tree $\widehat{G}$

Figure 3.4. Players v and w consisting vertices $v_1, v_2, ..., v_6$ and w_1, w_2, w_3 respectively. Vertices u, y, z and x in $\widehat{G}$ corresponds to cutvertices v_6, v_1, v_3 and w_3, respectively

and a cut-vertex, namely $\{B, v\} \in E(\widehat{G})$ if $B \in \mathcal{B}(G)$ and $v \in B \cap A(G)$. See Figure 3.4 for an example of the block-cut-vertex tree.

We have $|B| \geqslant 2$ for each $B \in \mathcal{B}(G)$. If $|B| \geqslant 3$, then $G[B]$ is two-connected; we also say that B is two-connected in G. Recall also that in a two-connected graph, for each vertex v, we can find a cycle that visits v. The following remark is proven by standard arguments, which are included here for completeness.

3.7.1 Remark. Let $G = (V, E)$ be any two-connected graph.

(i) Let $x, y, v \in V$ be three distinct vertices. Then, there exist paths $P = (v, \ldots, x)$ and $Q = (v, \ldots, y)$ with $V(P) \cap V(Q) = \{v\}$.

(ii) Let $W \subset V$ with $|W| \geqslant 2$ and $v \in V \backslash W$. Then, there are $x, y \in W$ with $x \neq y$ and paths $P = (v, \ldots, x)$ and $Q = (v, \ldots, y)$ with $V(P) \cap V(Q) = \{v\}$ and $V(P) \cap W = \{x\}$ and $V(Q) \cap W = \{y\}$.

Proof. (i) Add a new vertex z to the graph, and connect it with x and with y. The resulting graph is again two-connected. Using the global version of Menger's theorem (see, e.g., ([Die05] Theorem 3.3.6)), we find two independent (that is, internally vertex-disjoint) v-z paths. Taking subpaths yields the result.

(ii) Let $x', y' \in W$, $x' \neq y'$ be any two distinct vertices in W. By (i), we find $P' = (v, \ldots, x')$ and $Q' = (v, \ldots, y')$ with:

$$V(P') \cap V(Q') = \{v\} \ . \tag{3.7.2}$$

Let x be the first vertex on P' that is also in W. Define $P := (v, \ldots, x)$ as a subpath of P'. Likewise, let y be the first vertex on Q' that is also in W, and define $Q := (v, \ldots, y)$ as a subpath of Q'. By (3.7.2), we get $x \neq y$, and the other properties follow from the choice of x and y. □

3.7.3 Definition. Let $G = (V, E)$ be any graph and $(B_1, b_1, \ldots, B_k, b_k, B_{k+1})$, $k \geqslant 1$, be a path in its block-cut-vertex tree $\widehat{G}$. Assume $|B_1| \geqslant 3$, and let $C = (b_1, a, \ldots, b_1)$ be a cycle in B_1. Let $c \in B_{k+1} \backslash \{b_k\}$. Then, we call the swap (a, b_1, c) a *cycle extension* with respect to (B_1, B_{k+1}); note that $B_2, \ldots, B_k$ and $b_1, \ldots, b_k$ are uniquely determined by the pair (B_1, B_{k+1}) since $\widehat{G}$ is a tree.

The name "cycle extension" is chosen since the cycle C is extended into a larger cycle, thereby merging the blocks that are traversed by the new cycle. The merging property is proven in the next proposition.

3.7.4 Proposition. *With notation as in Definition 3.7.3, denote $G' := G + (a, b_1, c)$. Then, in G', all of the blocks $B_1, \ldots, B_{k+1}$ are merged into one block, and the remaining blocks are maintained; in particular, no new cut-vertices emerge in G', and the separation values of maintained cut-vertices do not increase.*

Proof. The only non-obvious part is that $B' := B_1 \cup \ldots \cup B_{k+1}$ is two-connected in G', which we will prove now. Denote $C = (b_1, a, a_1, \ldots, a_t, b_1)$ for some t. There is a cycle of the form $C' = (b_1, \ldots, b_k, \ldots, c, a, a_1, \ldots, a_t, u_1)$ that starts in B_1, runs through $B_2, \ldots, B_{k+1}$ and finally re-enters B_1.

Let $i \in [k+1]$ with $|B_i| \geqslant 3$ and $v \in B_i \backslash V(C')$. *Claim:* in G', there are paths $P = (v, \ldots, x)$ and $Q = (v, \ldots, y)$ with $x, y \in V(C')$ and $x \neq y$, such that $V(P) \cap V(Q) = \{v\}$ and $V(P) \cap V(C') = \{x\}$ and $V(Q) \cap V(C') = \{y\}$.

Proof of claim: Denote $W := V(C') \cap B_i$, then $|W| \geqslant 2$. By Remark 3.7.1(ii) applied with W as defined here, the claim is clear for $i \geqslant 2$, since such B_i is two-connected in G' (and in G). Hence, we consider $i = 1$. The difficulty is that B_1 may not be two-connected in G', since we removed the edge $a_1 u_1$. However, none of the paths P or Q guaranteed to exist by

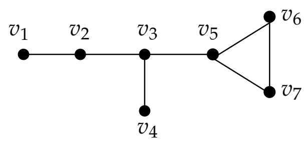


Figure 3.5. An example graph G

Remark 3.7.1(ii) in G can use a_1u_1 since then, that path would have more than one vertex in common with W. This concludes the proof of the claim.

Now, let $v, w \in B'$ with $v \neq w$. We show that in G', there are two independent v-w paths.

▻ If $v, w \in B_i$ for some $i \leqslant k$, then either B_i is two-connected and the statement is clear, or $|B_i| = 2$ in which case $v, w \in V(C')$, and the independent paths are given through C'.

▻ Let $v \in B_i$ and $w \in B_j$ for $i \neq j$. If v or w is located on C', then nothing has to be done for that vertex; otherwise, we connect it with C' via the paths guaranteed by the claim. It is easy to see that this gives two independent v-w paths.

▻ Let $v, w \in B_1$. In G, we find two independent v-w paths P and Q in B_1. At most, one of them, say P, uses $\{a, b_1\}$. Instead of using that edge, we can, starting at b_1, run along C' until we reach a. That b_1-a path runs outside of B_1 (except for a and b_1) and, thus, will not interfere with P or Q. □

A vertex destroyer $\mathcal{D}$ is called the *uniform vertex destroyer* if $\mathcal{D}_G(u) = \frac{1}{n}$ for each $u \in V_n$ and each $G \in \mathcal{G}_n$. In Figure 3.5, the cost of players in uniform vertex destruction model is

$$\mathcal{C}_{v_1}(G) = \frac{1}{n}(\sum_{i\in[7]} \operatorname{rel}(v_i, v_1)) = \frac{1}{7}(6+6+5+1+3+1+1) = \frac{23}{7}$$

$$\mathcal{C}_{v_2}(G) = \frac{1}{n}(\sum_{i\in[7]} \operatorname{rel}(v_i, v_2)) = \frac{1}{7}(1+6+5+1+3+1+1) = \frac{18}{7}$$

$$\mathcal{C}_{v_3}(G) = \frac{15}{7}, \qquad \mathcal{C}_{v_4}(G) = \frac{20}{7}, \qquad \mathcal{C}_{v_5}(G) = \frac{16}{7},$$

$$\mathcal{C}_{v_6}(G) = \mathcal{C}_{v_7}(G) = \frac{20}{7}.$$

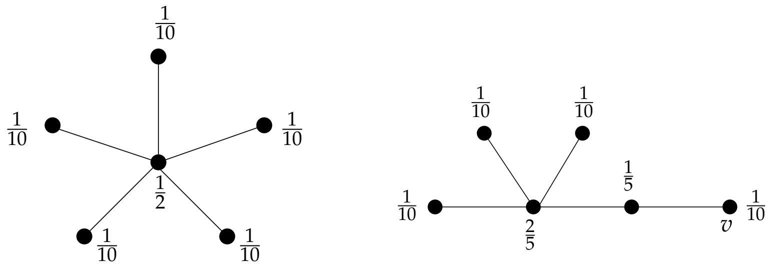
3.7.5 Theorem. *An SE for the uniform vertex destroyer is two-connected (that is, it has only one block) or it does not contain any cycle and, thus, is a tree.*

Proof. Let an SE graph $G \in \mathcal{G}_n$ be given, and assume it contains more than one block. We only need to prove that no block has a cycle, or, equivalently, that each block consists of only two vertices. Suppose for contradiction that B_1 is a block with $|B_1| \geqslant 3$, and let B_2 be another block. Let (a, b, c) be a cycle extension with respect to (B_1, B_2) and $G' := G + (a, b, c)$. By Proposition 3.7.4, it follows that $\mathrm{rel}'(b, a) < \mathrm{rel}(b, a)$, since in G', removal of b cannot cut a off the one or more vertices in $B_2 \backslash \{b_1\}$ anymore (recall that a block always contains at least two vertices). All other relevance for a is maintained or also reduced. Therefore, we have an improvement in the cost of player a, contradicting the stability of G. □

3.7.6 Corollary. *Let $G \in \mathcal{G}_n$ be an SE for the uniform vertex destroyer. Then, G is either two-connected or a star; hence, $\mathsf{SC}(G) = 2(n-1) = \mathcal{O}(n)$ or $\mathsf{SC}(G) = 3n - 5 + \frac{2}{n} = \mathcal{O}(n)$.*

Proof. Let G be non-two-connected. Then, by Theorem 3.7.5, G is a tree. By applying the same argument as in the proof of Lemma 3.5.4, we obtain that the diameter of G is at most two, i.e., G is a star. The social cost of star is easily computed to be $\frac{1}{n}\Big((n-1)(3n-4) + 2(n-1)\Big) = 3n - 5 + \frac{2}{n}$. □

A destroyer $\mathcal{D}$ is called the *degree-proportional vertex destroyer* if $\mathcal{D}_G(u) = \frac{\deg_G(u)}{2m}$ for each $u \in V_n$ and each $G \in \mathcal{G}_n$.



(a) Cost for an outer player is $4 \cdot \frac{1}{10} + 5 \cdot \frac{1}{2} + 5 \cdot \frac{1}{10} = \frac{9}{10} + \frac{5}{2} = \frac{17}{5}$.

(b) Cost for player v is $3 \cdot \frac{1}{10} + 4 \cdot \frac{2}{5} + 5 \cdot \frac{1}{5} + 5 \cdot \frac{1}{10} = \frac{4+8+5}{5} = \frac{17}{5}$.

Figure 3.6. Example for Proposition 3.7.7. Numbers give probabilities for the vertices to be picked for destruction. By computing cost, we see that an outer player cannot benefit from a swap.

3.7.7 Proposition. *The star is an SE for the degree-proportional vertex destroyer, and its social cost is* $\frac{1}{2}(n^2 + n) - 1 = \Omega(n^2)$.

Proof. Let $S \in \mathcal{G}_n$ be a star. First, we prove that S is an SE. Clearly, by just removing an edge (without creating a new one), no player can improve. It is also obvious that the only possibility for swapping is from one leaf to another leaf. Let $a, c \in V$ be leaves and b the center of the star. Denote $S' := S + (a, b, c)$. Although player a, by the swap (a, b, c), decreases the number of cut-off vertices in the case that the center b is destroyed, this does not improve her cost. This is because at the same time, the degree of c is increased, and destruction of c in G' will cut off a from all other vertices. The following calculations show that a indeed does not improve her cost by the swap. We have:

$$\begin{aligned}
\mathcal{C}_a(S) &= \frac{1}{2(n-1)} \sum_{w \in V} \mathrm{rel}(w, a) \cdot \deg(w) \\
&= \frac{1}{2(n-1)} \Big((n-1) + (n-2) + (n-1)(n-1) \Big) \\
&= \frac{1}{2} \Big(n + 1 - \frac{1}{n-1} \Big)
\end{aligned}$$

After swapping:

$$\begin{aligned}\mathcal{C}_a(S') &= \frac{1}{2(n-1)}\Big(3(n-1)+(n-2)^2+(n-3)\Big)\\ &= \frac{1}{2}\Big(n+1-\frac{1}{n-1}\Big)\end{aligned}$$

Hence, $\mathcal{C}_a(a) = \mathcal{C}_a(S')$. Therefore, the star is an SE (an example for $n = 6$ is given in Figure 3.6). Its social cost is:

$$\begin{aligned}\mathsf{SC}(S) &= \frac{1}{2(n-1)}\sum_{v\in V}\sum_{w\in V}\mathrm{rel}(w,v)\cdot\deg(w)\\ &= (n-1)\Big(\frac{1}{2}\Big(n+1-\frac{1}{n-1}\Big)\Big)+\frac{1}{2(n-1)}\cdot(n-1)\,n\\ &= \frac{1}{2}\Big((n^2-1)-1\Big)+\frac{n}{2} = \tfrac{1}{2}(n^2+n)-1 \qquad \square\end{aligned}$$

3.7.8 Corollary. *The social cost of SE for the degree-proportional vertex destroyer can be as high as $\Omega(n^2)$, which is the worst possible order in the destruction model.* $\square$

3.8 Extreme Vertex Destruction

This model is defined similarly to the extreme edge destruction model. Denote $V_{\max}$ the set of max-sep vertices and $n_{\max} := |V_{\max}|$. The extreme vertex destroyer picks the vertex to destroy uniformly at random from $V_{\max}$. In Figure 3.5, only player v_3 has maximum separation, $\mathrm{sep}_{\max} = \mathrm{sep}(v_3) = 34$. Then, $n_{\max} = 1$ and the cost for players in the extreme vertex destruction model is

$$\begin{aligned}&\mathcal{C}_{v_1}(G) = \mathrm{rel}(v_3,v_1) = 5,\;\; \mathcal{C}_{v_2}(G) = \mathrm{rel}(v_3,v_2) = 5,\\ &\mathcal{C}_{v_3}(G) = \mathrm{rel}(v_3,v_3) = 6,\;\; \mathcal{C}_{v_4}(G) = \mathrm{rel}(v_3,v_4) = 6,\\ &\mathcal{C}_{v_5}(G) = \mathcal{C}_{v_6}(G) = \mathcal{C}_{v_7}(G) = 4.\end{aligned}$$

We start with a first step toward understanding the worst-case order of social cost of an SE in this model, by giving an SE example with super-linear social cost, namely $\Omega(n^{3/2})$. The construction is shown in Figure 3.7. It is unknown at this time whether there is a matching upper bound. Before reading the proof of the following theorem, the reader is encouraged to take a look at Remark 3.8.3 in order to obtain an idea why we have to limit the length of the paths attached to the clique.

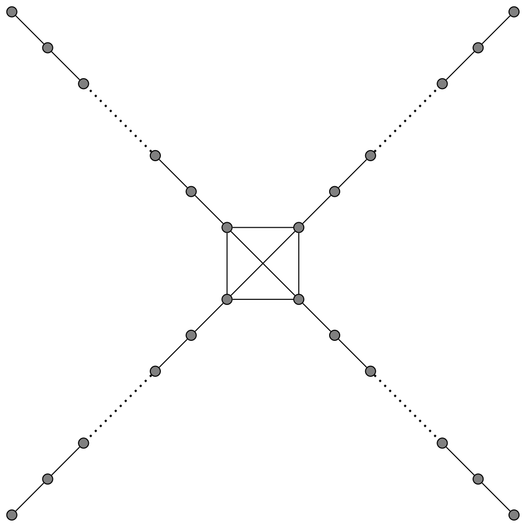


Figure 3.7. Construction from Theorem 3.8.1 for $t = 4$.

3.8.1 Theorem. *Let $t \geqslant 4$ and $0 \leqslant k \leqslant 4t - 5 = \Theta(t)$. Let $G = (V, E)$ be the graph consisting of a clique C on t vertices, and to each vertex of C, there is a path of length k attached (so, $n := |V| = t(k+1)$). Then, G is an SE with $\mathsf{SC}(G) = \Omega(n^{3/2})$.*

Proof. For each player v at distance $0 \leqslant i \leqslant k$ from C, we have:

$$\operatorname{sep}(v) = 2\Big((n-1) + (k-i)\,(n-1-(k-i))\Big)$$

Since $k - i \leqslant (n-1)/2$ and since the function $x \mapsto x(n-1-x)$ is strictly increasing on $[0, (n-1)/2]$, separation is strictly largest when $i = 0$, that is when $v \in C$. It follows $V_{\max} = C$ and:

$$\begin{aligned}\mathsf{SC}(G) &= 2((n-1) + k(n-1-k)) \\ &= 2\Big(t(k+1) - 1 + k(t(k+1) - 1 - k)\Big)\end{aligned}$$

$$
\begin{aligned}
&= 2(tk + t - 1 + k^2 t + kt - k - k^2) \\
&\geqslant 2\Big((2t-1)k + k^2(t-1)\Big) \\
&\geqslant 2k^2(t-1) \\
&\geqslant k^2 t
\end{aligned}
$$

If we choose k maximal, that is $k = 4t - 5$, we get:

$$
\begin{aligned}
\mathsf{SC}(G) \geqslant k^2 t &= (4t-5)^2 t \\
&= (16t^2 - 40t + 25)t \\
&\geqslant (6t^2 + t(10t - 40))t \\
&\geqslant 6t^3
\end{aligned}
$$

Now, for this k, we have $n = t(k+1) = t(4t-4) \leqslant 4t^2$; hence, $n^{3/2} \leqslant 8t^3$. It follows $\mathsf{SC}(G) \geqslant \frac{6}{8} n^{3/2} = \frac{3}{4} n^{3/2} = \Omega(n^{3/2})$ (in short: we have $\mathsf{SC}(G) = \Omega(tk^2)$; if we choose k maximal, then $k = \Theta(t)$; hence, $\mathsf{SC}(G) = \Omega(t^3) = \Omega(n^{3/2})$).

We prove the SE property. Just removing an edge (without building a new one) is clearly not an option for the players on the paths. For a player of the clique C, nothing changes when removing an edge, since because of $t \geqslant 4$, the set C remains two-connected.

For $u \in C$, denote V_u the k vertices on the path attached to u (note that $u \notin V_u$). In the following, whenever we consider a swap, by G', we refer to the graph we obtain from G by applying the swap.

We start with the different swaps available to a player $a \in C$. We have $\mathrm{rel}(u, a) = k + 1$ for each $u \in V_{\max} \setminus \{a\}$; hence:

$$
\mathcal{C}_a = \frac{(n-1) + (t-1)(k+1)}{t} \quad .
$$

(i) Consider a swap (a, b, c) with $b \in C \cap N(a)$ and $c \in V_u$ for $u \in C \setminus \{a\}$, that is player a swaps an edge from the clique to some path, but not the one connected to a itself. Since $t \geqslant 4$, the set C remains two-connected. Separation of u and of some vertices in V_u is reduced. All other separations are maintained. It follows:

$$
\mathcal{C}_a - \mathcal{C}'_a = \frac{(n-1) + (t-1)(k+1)}{t}
$$

$$
\begin{aligned}
&- \frac{(n-1)+(t-2)(k+1)}{t-1} \\
&= \frac{k-n+2}{t(t-1)} < 0
\end{aligned}
$$

Hence, the swap is no improvement for player a.

(ii) Consider a swap (a, b, c) with $b \in C \cap N(a)$ and $c \in V_a$, that is player a swaps an edge from the clique to its own path. Again, since $t \geqslant 4$, the set C remains two-connected. The separation values on some vertices in V_a decrease, but that does not change the set of max-sep vertices, nor their relevance for a. Hence, player a's cost is maintained.

(iii) Consider a swap (a, b, c) with $b \in V_a$; then, in order to keep the graph connected, we have $c \in V_a \backslash \{b\}$. That is, player a swaps the first edge on its path to some vertex on that path. We only have to exclude that c becomes max-sep; if we achieve that, then, we know that a's cost is maintained. Let $2 \leqslant l \leqslant k-1$ be the distance between a and c. Then, by (3.4.6):

$$
\begin{aligned}
& \operatorname{sep}'(c) < \operatorname{sep}_{\max} \\
\iff & n^2 - 1 - (l-1)^2 - (k-l)^2 - (n-k)^2 \\
< \; & n^2 - 1 - k^2 - (n-1-k)^2 \\
\iff & k^2 + (n-1-k)^2 < (l-1)^2 + (k-l)^2 + (n-k)^2 \\
\iff & (n-1-k)^2 < -2l(k+1-l) + 1 + (n-k)^2 \\
\iff & -2(n-k) < -2l(k+1-l) \\
\iff & n-k > l(k+1-l) \\
\Longleftarrow \; & n-k > \frac{(k+1)^2}{4} \\
\iff & t(k+1) > \frac{(k+1)^2}{4} + k \\
\iff & t > \frac{k+1}{4} + 1 - \frac{1}{k+1} \\
\Longleftarrow \; & 4t - 5 \geqslant k
\end{aligned}
$$

The latter is true by the restriction on k in the statement of the theorem. In this computation, we used again that the function $x \mapsto x(k+1-x)$ is increasing on $[0,(k+1)/2]$.

We continue with the swaps available to a player $a \in V_u$ for some $u \in C$.

(iv) Consider a swap (a,b,c) with $\operatorname{dist}(u,a) < \operatorname{dist}(u,b)$. Then, $c \in V_u$ with $\operatorname{dist}(u,b) < \operatorname{dist}(u,c)$; since otherwise, the graph would become disconnected. From a computation like in (iii), it follows that this swap cannot make c max-sep. Hence, the cost of player a does not change.

(v) Consider a swap (a,b,c) with $\operatorname{dist}(u,a) > \operatorname{dist}(u,b)$. If $c \in V_u$, then again, a's cost will not change. If $c \in C$, then c will become the only max-sep vertex in the new graph, clearly increasing a's cost.

Now, let $c \in V_w$ for some $w \neq u$, that is some vertices migrate from u's path to w's path. Separation of w and separation of the vertices $v \in V_w$ with $\operatorname{dist}(w,v) \leqslant \operatorname{dist}(w,c)$ will increase. All other separations are reduced or maintained, so we have $V'_{\max} \subseteq V_w \cup \{w\}$. In the best case, k vertices migrate to w's path; only w becomes max-sep, and $\operatorname{rel}'(w,a) = n-2k$. In this case:

$$\begin{aligned}
\mathcal{C}_a - \mathcal{C}'_a &= \frac{(t-1)(k+1)+(n-k)}{t} - (n-2k) \\
&= \frac{(t-1)(k+1)+(t(k+1)-k)-t(t(k+1)-2k)}{t} \\
&= \frac{(k+1)(2t-t^2-1)+(2t-1)k}{t} \\
&\leqslant (k+1)(2-t)+2k \\
&\leqslant -2(k+1)+2k \\
&< 0
\end{aligned}$$

Hence, the swap is no improvement for player a. □

3.8.2 Remark. For $k \geqslant 2$, the graph in Theorem 3.8.1 is no SE for extreme edge destruction.

Proof. The max-sep edges are the edges on the paths that have one endpoint in the clique. This will not change when a leaf swaps to the clique; hence, this leaf will experience an improvement. □

3.8.3 Remark. The graph in Theorem 3.8.1 is no SE if $k \geqslant 4t - 4 = 4(t-1)$, that is if k is larger than the upper bound in the theorem.

Proof. Let $a \in C$ and b be her neighbor in V_a. Let $c \in V_a \setminus \{b\}$ be at distance l from a, to be specified later. Denote $G' := G + (a, b, c)$. If c becomes max-sep in G', then player a's cost will decrease, since $\mathrm{rel}'(c, a) = k$, whereas the relevance of a max-sep vertex in G for a is $k+1$ or $n-1$. By the computation in Theorem 3.8.1(iii), we see that c becomes max-sep if $n - k \leqslant l(k+1-l)$. We have:

$$
\begin{aligned}
n - k \leqslant l(k+1-l) &\iff t \leqslant \frac{l(k+1-l)}{k+1} + 1 - \frac{1}{k+1} \\
&\Longleftarrow t \leqslant \frac{l(k+1-l)}{k+1} + \frac{4}{5}
\end{aligned}
$$

For odd $k \geqslant 4(t-1)$ and $l := \frac{k+1}{2}$, we have:

$$
\begin{aligned}
t \leqslant \frac{l(k+1-l)}{k+1} + \frac{4}{5} &\iff t \leqslant \frac{(k+1)^2/4}{k+1} + \frac{4}{5} \\
&\iff t \leqslant \frac{k+1}{4} + \frac{4}{5} \\
&\Longleftarrow t \leqslant \frac{4t-3}{4} + \frac{4}{5} \\
&\iff 0 \leqslant \frac{-3}{4} + \frac{4}{5} \\
&\iff 0 \leqslant \frac{1}{20}
\end{aligned}
$$

For even $k \geqslant 4(t-1)$ and $l := \frac{k}{2}$, we have:

$$
\begin{aligned}
t \leqslant \frac{l(k+1-l)}{k+1} + \frac{4}{5} &\iff t \leqslant \frac{\frac{k^2}{4} + \frac{k}{2}}{k+1} + \frac{4}{5} \\
&\iff t \leqslant \frac{k^2+k}{4(k+1)} + \frac{k}{4(k+1)} + \frac{4}{5} \\
&\Longleftarrow t \leqslant \frac{k}{4} + \frac{1}{4}\frac{4}{5} + \frac{4}{5} \\
&\iff t \leqslant \frac{k}{4} + 1 \\
&\Longleftarrow t \leqslant t - 1 + 1
\end{aligned}
$$

$\square$

In the remainder of this work, we provide more insight into the structure of tree SE graphs for the extreme vertex destroyer. The next theorem says that in a tree with only one max-sep vertex (and $n \geqslant 8$), there is always an improving swap, so it cannot be an SE. The main concepts are that having a single max-sep vertex makes the destroyer very predictable and that swapping to a leaf, that leaf (which will have degree two in the new graph) can become max-sep only under specific conditions.

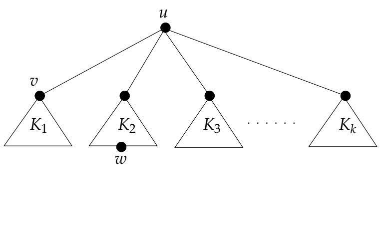


Figure 3.8. Proof of Theorem 3.8.4, Case 1.

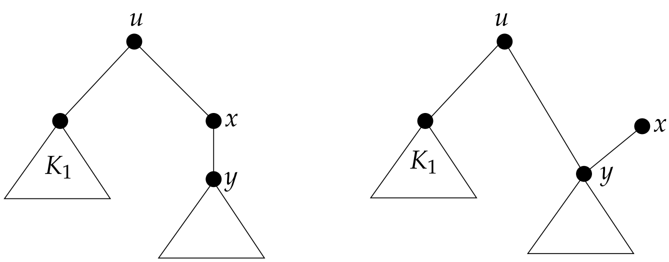


Figure 3.9. Proof of Theorem 3.8.4, Case 2.

3.8.4 Theorem. *There is no SE tree with $n_{\max} = 1$, provided that $n \geqslant 8$.*

Proof. Let $T = (V, E)$ be an SE tree with $n_{\max} = 1$ and $u \in V_{\max}$. It is clear that u is a cut-vertex, otherwise G is two-connected, and thus, $n_{\max} = n$. Denote $K_1, \ldots, K_k$, with $k \geqslant 2$, the components of $T - u$ ordered by non-decreasing sizes $|K_1| \leqslant |K_2| \leqslant \ldots \leqslant |K_k|$. For convenience, denote $n_i := |K_i|$ for each i.

Let $v \in K_1$ and $\{u, v\} \in E$ and $w \in K_2$ with $\deg(w) = 1$. We consider $T' := T + (v, u, w)$, that is we detach K_1 from u and re-attach it to a leaf of K_2. The situation is depicted in Figure 3.8.

Then, $\text{rel}'(w,v) = \text{rel}(u,v)$ and $\text{rel}'(w,v) > \text{rel}'(w',v)$ for all $w' \neq w$, and so, we have an improvement for v whenever $V'_{\max}$ contains at least one vertex distinct from w. The latter is the case if $\text{sep}'(w) \leqslant \text{sep}'(u)$. Using (3.4.6), we compute:

$$\begin{aligned}
\text{sep}'(w) \leqslant \text{sep}'(u) \iff & n^2 - 1 - n_1^2 - (n - n_1 - 1)^2 \\
\leqslant \quad & n^2 - 1 - (n_1 + n_2)^2 - \sum_{i=3}^{k} n_i^2 \\
\iff & n_1^2 + (n - n_1 - 1)^2 \geqslant (n_1 + n_2)^2 + \sum_{i=3}^{k} n_i^2 \\
\iff & (n - n_1 - 1)^2 \geqslant 2n_1 n_2 + \sum_{i=2}^{k} n_i^2 \\
\iff & \left(\sum_{i=2}^{k} n_i\right)^2 \geqslant 2n_1 n_2 + \sum_{i=2}^{k} n_i^2 \\
\iff & \sum_{2 \leqslant i < j \leqslant k} n_i n_j \geqslant n_1 n_2
\end{aligned} \tag{3.8.5}$$

Now, Condition (3.8.5) is true if $k \geqslant 3$, since $n_1 \leqslant n_3$.

Hence, we may assume that $k = 2$. Let x be the only vertex in $N(u) \cap K_2$. If $\deg(x) \geqslant 3$, then:

$$\begin{aligned}
\text{sep}(x) &\geqslant n^2 - 1 - 1 - (n_2 - 2)^2 - (n_1 + 1)^2 \\
&= n^2 - 1 - 1 - n_2^2 + 4n_2 - 4 - n_1^2 - 2n_1 - 1 \\
&= \text{sep}(u) - 1 + 4n_2 - 4 - 2n_1 - 1 \\
&= \text{sep}(u) + 2(2n_2 - n_1) - 6 \\
&\geqslant \text{sep}(u) + 2n_2 - 6 > \text{sep}(u).
\end{aligned}$$

The last step is true since $n_2 \geqslant (n-1)/2 > 3$. We conclude that $\deg(x) \leqslant 2$. Since $n \geqslant 8$, there is $y \in N(x) \setminus \{u\}$ with $\deg(y) \geqslant 2$. We consider $T' := T + (u, x, y)$, as shown in Figure 3.9. We have:

$$\begin{aligned}
\text{sep}'(u) < \text{sep}'(y) \iff & n_1^2 + n_2^2 > 1 + (n_2 - 2)^2 + (n_1 + 1)^2 \\
\iff & 0 > 1 - 4n_2 + 4 + 2n_1 + 1
\end{aligned}$$

$$\iff 0 > 2(n_1 - 2n_2) + 6$$
$$\iff 0 > n_1 - 2n_2 + 3 \impliedby n_2 > 3$$

Since the last statement is true, we know that $V'_{\max} = \{y\}$; hence, $\mathcal{C}'_u = n_2 < n - 1 = \mathcal{C}_u$. □

3.9 Open Problems

We have extensive experimental evidence and indications on the theory side that our structural result for the extreme vertex destroyer (that there is no SE tree with only one max-sep vertex for $n \geqslant 8$) can be extended in two ways. We conjecture for $n = \Omega(1)$:

(i) There is no SE graph under extreme vertex destruction with only one max-sep vertex (this extends our result for trees to general graphs).

(ii) There is no SE graph under extreme vertex destruction that is a tree (this extends our non-existence result for trees with one max-sep vertex to trees in general).

Recall that we proved that unless the graph is two-connected, an SE cannot contain cycles for the uniform vertex destroyer. On the other hand, for the extreme vertex destroyer, our Conjecture (ii) would imply that an SE is only possible if we have at least one cycle. It would then be interesting to find a family $(\mathcal{D}^{\varepsilon})_{\varepsilon \in [0,1]}$ of vertex destroyers, such that $\mathcal{D}^0$ is the uniform destroyer and $\mathcal{D}^1$ is the extreme destroyer, and the others are something suitable in between. When moving ε from zero to one, the probability measure should concentrate more and more on the max-sep vertices. It would be interesting to find out at which values of ε the situation switches from no non-two-connected SE having a cycle to each SE having a cycle. Our Conjecture (ii) would imply that it must switch an odd number of times.

Other interesting directions include:

- ▻ Closing the gap between our lower $\Omega(n^{3/2})$ bound and the trivial $\mathcal{O}(n^2)$ upper bound for social cost of extreme vertex destruction.
- ▻ Extension to edge ownerships and the asymmetric swap equilibrium.
- ▻ Study of the model from **[CLMM16]** under swap equilibrium or asymmetric swap equilibrium.
- ▻ Study of vertex destruction under Nash equilibrium or pairwise stability as the equilibrium concept.

Chapter 4

Local Network Formation

Let $G = (V, E)$ be an undirected graph with n vertices. For a vertex $v \in V$, we denote $N(v) := \{w \in V\,;\ \{v, w\} \in E\}$ the set of neighbors of v in G, where $\deg(v) = |N(v)|$.

4.0.1 Definition. For any $v \in V$ and $k \in \mathbb{N}$, the *k-neighborhood of vertex v*, which we denote $N_k(v)$, is the set of other players that v can reach through a path of length at most k, in other words $N_k(v) := \{w \in V\,;\ 1 \leqslant \operatorname{dist}(w, v) \leqslant k\}$ and $N(v) = N_1(v)$.

In this chapter, we study a type of network formation games called *local network formation games*, where the income $I_v(G)$ only depends on a certain graph-theoretic neighborhood around v. An example is taking the *clustering coefficient* as income, which was considered by Brautbar and Kearns [BK11], so their income is defined

$$I_v(G) = \frac{|\{\{u, w\} \in E(G)\,;\ u, w \in N(v)\}|}{\binom{\deg_G(v)}{2}}.$$

The *degree-sum* was considered by Nikoletseas et al. in 2013 [NPR+13] with income

$$I_v(G) = \sum_{w \in N_k(v)} \deg_G(w).$$

Both income functions given above only depend on the *local* structure around the respective player and are hence referred to as defining local network formation games.

Let $k \in \mathbb{N}$. We define another local network formation game, which is called *neighborhood-size* or *k-neighborhood* model, and is such that the income function of player v is defined

$$I_v(G) := |N_k(v)|.$$

Then player v's utility for pairwise stability with link cost α is

$$U_v(G) := |N_k(v)| - \deg(v)\alpha.$$

The social utility for graph G, also given for pairwise stability, is

$$\mathsf{SU}(G) = \sum_{v \in V} |N_k(v)| - 2|E|\alpha.$$

We consider two equilibrium concepts for the k-neighborhood model, namely pairwise stability and swap equilibrium (only in Section 4.3). We prove that due to pseudo-convexity of the income function, pairwise stability and pairwise Nash equilibrium are equivalent.

We say that player v can *see* player w if $w \in N_k(v)$, or, equivalently, if $v \in N_k(w)$. For a graph G and an edge $\{v, w\}$ denote $B^i_{v,w}(G)$ the players which are at distance exactly i from v, for $i \in \{2, ..., k\}$, but do not see v in $G - \{v, w\}$. With $L_j(G, v)$ we denote the set of players at exactly distance j from v, this is also called the layer j, since it is the jth BFS layer when doing a BFS starting from v.

4.0.2 Lemma. *The utility function U_v in the k-neighborhood model is pseudo-convex.*

Proof. Fix $v \in V$ and let $w_1, ..., w_{k'} \in N(v)$. We do induction on k'. The case $k' = 1$ is trivial, so let $k' > 1$ and $G' := G - \{v, w_1\} - ... - \{v, w_{k'-1}\}$. We assume

$$U_v(G') - U_v(G) \leqslant \sum_{i=1}^{k'-1} (U_v(G - \{v, w_i\}) - U_v(G)).$$

We show that for any $j \in \{2, ..., k\}$, $B^j_{v,w_{k'}}(G') \supseteq B^j_{v,w_{k'}}(G)$. Let j be such that $u \in B^j_{v,w_{k'}}(G)$, that is, $u \in L_j(G, v)$ and v cannot see u in $G - \{v, w_{k'}\}$. Then $u \in N_{j-1}(w_{k'})$. Since the $w_1, \ldots, w_{k'}$ are all distinct, the edge $\{v, w_{k'}\}$ is still present in G', so $u \in L_j(G', v)$. Clearly, since v cannot see u in $G - \{v, w_{k'}\}$, she can also not see u in $G' - \{v, w_{k'}\}$. This proves $u \in B^j_{v,w_{k'}}(G')$.

It follows that the loss in layer j for v when switching from G' to $G' - \{v, w_{k'}\}$ is at least as strong as when switching from G to $G - \{v, w_{k'}\}$. The same holds for layer 1: if v loses sight of $w_{k'}$ when switching from G to $G - \{v, w_{k'}\}$, she also loses sight when switching from G' to $G' - \{v, w_{k'}\}$. It follows

$$U_v(G' - \{v, w_{k'}\}) - U_v(G') \leqslant U_v(G - \{v, w_{k'}\}) - U_v(G).$$

By induction, we have:

$$\begin{aligned}
&U_v(G - \{v, w_1\} - ... - \{v, w_{k'}\}) - U_v(G) \\
&= U_v(G' - \{v, w_{k'}\}) - U_v(G) \\
&= U_v(G' - \{v, w_{k'}\}) - U_v(G) + U_v(G') - U_v(G') \\
&\leqslant U_v(G - \{v, w_{k'}\}) - U_v(G) + U_v(G') - U_v(G) \\
&\leqslant U_v(G - \{v, w_{k'}\}) - U_v(G) + \sum_{i=1}^{k'-1} (U_v(G - \{v, w_i\}) - U_v(G)) \\
&= \sum_{i=1}^{k'} (U_v(G - \{v, w_i\}) - U_v(G)).
\end{aligned}$$ □

From Lemma 4.0.2 and Lemma 2.1.5, we conclude that in the k-neighborhood model, PNE and PS coincide.

4.1 Diameter of PS Graph

In this section, we obtain an upper bound on the diameter of pairwise stable graphs in the 2-neighborhood model, and we show that for any k in the $k-$neighborhood model, the diameter of a PS tree is at most $k + 1$. Moreover, we see that a tree can only be pairwise stable if $\alpha \leqslant 1$.

4.1.1 Proposition. *Let $G = (V, E)$ be a PS graph in the $2-$neighborhood model (that is, $k = 2$). Then $\alpha \leqslant \deg(v)$ for each non-isolated vertex $v \in V$.*

Proof. Let $v \in V$, and $w \in N(v)$. After removing $\{w, v\}$, the loss in utility for w is at most $\deg(v)$. Then by PS, we have $\alpha \leqslant \deg(v)$. □

4.1.2 Lemma. *A PS graph in the $2-$neighborhood model has its diameter upper bounded by 3.*

Proof. We assume that there is a PS graph of diameter at least 4. Thus there are two vertices a_0, a_4 with $\text{dist}(a_0, a_4) = 4$. By adding the edge $\{a_0, a_4\}$, the gain in income for a_0 is $\deg(a_4) + 1$ and for a_4 it is $\deg(a_0) + 1$. This is because if a_0 could see any of the neighbors of a_4 in G, the distance would be less than 4; and the same argument holds for a_4 and the neighbors of a_0. Hence, since $\{a_0, a_4\}$ is not there, we have due to PS that $\alpha \geqslant \deg(a_0) + 1$ or $\alpha > \deg(a_4) + 1$. By Proposition 4.1.1, each of these two inequalities is impossible, so we obtain a contradiction. □

4.1.3 Lemma. *A PS tree in the $k-$neighborhood model has diameter at most $k+1$. A tree with diameter at most k can be PS only if $\alpha \leqslant 1$, and a tree with diameter $k+1$ can be PS only if $\alpha = 1$.*

Proof. Let there be a PS tree with diameter at least $k+2$. We already know from Proposition 4.1.1 that a tree can only be PS if $\alpha \leqslant 1$, since each tree contains a leaf. Let there be two vertices a_0, a_{k+2} with $\text{dist}(a_0, a_{k+2}) = k+2$. After creating the edge $\{a_0, a_{k+2}\}$, the gain for a_0 and for a_{k+2} is at least 2, which is a contradiction to $\alpha \leqslant 1$. So the diameter of PS trees is at most $k+1$.

We already know that a tree can only be PS if $\alpha \leqslant 1$, due to the existence of a leaf. If the diameter is $k+1$, then two players at distance $k+1$ of each other would both gain at least 1 in income if they put an edge between them. Thus $\alpha = 1$ due to PS. □

We conclude this section with two remarks that further restrict the class of PS graphs.

4.1.4 Remark. If a PS graph in the k-neighbourhood model is disconnected or a connected graph with diameter at least $k+1$, then $\alpha \geqslant 1$.

Proof. Let a PS graph $G = (V, E)$ be given. We assume that C and C' are two components. We build an edge from player $v \in V(C)$ to player $w \in V(C')$. By adding the edge $\{v, w\}$, player v gains at least one player (player w) and the player w also gains at least one player (player v). Hence, by pairwise stability we have $\alpha \geqslant 1$. We have the same argument for the connected graph with diameter $k+1$, but by choosing two players with distance $k+1$ and creating a link between them. □

4.1.5 Remark. In any PS graph, any player must lose sight of at least one player after removing any of its incident edges, since otherwise due to PS we get $\alpha \leqslant 0$, a contradiction. □

4.2 Optima, Equilibria and Price of Anarchy

We recall that the price of anarchy is the ratio of optimum social utility to worst-case equilibrium social utility.

In order to bound the price of anarchy, we would like to find the optimum. It is clear that among all connected graphs a tree with diameter at most k has optimum income and optimum number of edges. It is left to investigate in which cases it is better to omit some edges and get a disconnected graph.

Omitting edges will decrease building cost but will also decrease the income of each player, so the question is: When do the savings in building cost justify this decrease in income? Denote $\mathcal{T}$ a tree with diameter at most k. We compare $\mathcal{T}$ with a forest where each of its components has diameter at most k, e. g., where each component is a star. (Having diameter more than k in any component would strictly decrease social utility.)

Let $n \geqslant 3$ and $G' = (V', E')$ be a disconnected graph with $2 \leqslant k' \leqslant n$ components, such that each component is a tree with maximum diameter k. Denote n_i the number of vertices in component $i \in [k']$, thus $|V'| = \sum_{i=1}^{k'} n_i = n$. We have

$$\begin{aligned}
\mathsf{SU}(G') &= \sum_{i=1}^{k'} n_i(n_i - 1) - |E'|\alpha = \sum_{i=1}^{k'} n_i(n_i - 1) - \alpha \sum_{i=1}^{k'} (n_i - 1) \\
&= \sum_{i=1}^{k'} n_i^2 - \sum_{i=1}^{k'} n_i - \alpha(\sum_{i=1}^{k'} n_i - \sum_{i=1}^{k'} 1) \\
&= \sum_{i=1}^{k'} n_i^2 - n - \alpha(n - k') \\
&= (\sum_{i=1}^{k'} n_i)^2 - 2\sum_{i<j} n_i n_j - n - \alpha(n - k')
\end{aligned}$$

$$= n(n-1) - 2\sum_{i<j} n_i n_j - \alpha n + \alpha k'.$$

We need to show which characterization for $(n_i)_{i=1}^{k'}$ can make $\sum_{i<j} n_i n_j$ minimal.

4.2.1 Proposition. *Let $k' \in \mathbb{N}$ and $n_1, n_2, ..., n_{k'} \in \mathbb{N}_{\geqslant 1}$. Assume there are $s, t \in [k']$ such that $1 < n_s \leqslant n_t$. Define $n'_s := n_s - 1$ and $n'_t := n_t + 1$. For all $i \in [k'] \backslash \{s, t\}$ define $n'_i := n_i$. Then $\sum_{i<j} n'_i n'_j < \sum_{i<j} n_i n_j$.*

Proof. We have $\sum_{i<j} n'_i n'_j = \frac{1}{2}((\sum_{i=1}^{k'} n'_i)^2 - \sum_{i=1}^{k'} n'^2_i)$ and

$$\begin{aligned}
\sum_{i=1}^{k'} n'_i &= n'_s + n'_t + \sum_{i \in [k'] \backslash \{s,t\}} n'_i \\
&= n_s - 1 + n_t + 1 + \sum_{i \in [k'] \backslash \{s,t\}} n_i \\
&= \sum_{i=1}^{k'} n_i.
\end{aligned}$$

Then $(\sum_{i=1}^{k'} n'_i)^2 = (\sum_{i=1}^{k'} n_i)^2$. It follows

$$\begin{aligned}
\sum_{i=1}^{k'} n'^2_i &= \sum_{i \in [k'] \backslash \{s,t\}} n'^2_i + n'^2_s + n'^2_t \\
&= \sum_{i \in [k'] \backslash \{s,t\}} n'^2_i + (n_s - 1)^2 + (n_t + 1)^2 \\
&= \sum_{i=1}^{k'} n'^2_i + (-2n_s + 1) + (2n_t + 1) \\
&= \sum_{i=1}^{k'} n'^2_i + 2(n_t - n_s + 1) \\
&\geqslant \sum_{i=1}^{k'} n'^2_i + 2.
\end{aligned}$$

By replacing, it follows

$$\begin{aligned}
\sum_{i<j} n'_i n'_j &= \frac{1}{2}((\sum_{i=1}^{k'} n'_i)^2 - \sum_{i=1}^{k'} n'^2_i) \\
&\leqslant \frac{1}{2}((\sum_{i=1}^{k'} n_i)^2 - (\sum_{i=1}^{k'} n_i{}^2 + 2)) \\
&= \frac{1}{2}((\sum_{i=1}^{k'} n_i)^2 - \sum_{i=1}^{k'} n_i{}^2) - 2)) \\
&= \sum_{i<j} n_i n_j - 1 \\
&< \sum_{i<j} n_i n_j. \qquad \square
\end{aligned}$$

Proposition 4.2.1 shows that the minimum of $\sum_{i<j} n_i n_j$ is attained if we have a characterization for components such that for $2 \leqslant k' \leqslant n$ there exists t_0 such that $n_{t_0} = n - (k' - 1)$ and $n_t = 1$ for all $t \neq t_0$. In terms of graphs, this means $k' - 1$ isolated vertices and one component being a tree with maximum diameter k on $n - (k' - 1)$ vertices. Denote this graph by $\mathsf{k' - kdiam}$.

4.2.2 Proposition.

(i) If $\alpha \leqslant \frac{n}{2}$, $\mathcal{T}$ is an optimum; it has social utility $(n-1)(n-2\alpha)$.

(ii) If $\alpha > \frac{n}{2}$, the empty graph is an optimum; it has social utility zero.

Proof. Let $\alpha \leqslant \frac{n}{2}$. For any $2 \leqslant k' \leqslant n$ we prove that $\mathsf{SU}(\mathcal{T}) \geqslant \mathsf{SU}(\mathsf{k' - kdiam})$. With obtained characterization from Proposition 4.2.1 for $\mathsf{k' - kdiam}$, we have

$$\begin{aligned}
\mathsf{SU}(\mathcal{T}) &= n(n-1) - 2|E|\alpha = n(n-1) - 2\alpha(n-1) \\
&= (n-1)(n-2\alpha). \\
\mathsf{SU}(\mathsf{k' - kdiam}) &= (n - (k'-1))(n - (k'-1) - 1) - 2|E|\alpha \\
&= (n - k' + 1)(n - k') - 2(n - k')\alpha \\
&= n^2 - k'n - k'n + k'^2 + n - k' - 2\alpha n + 2k'\alpha \\
&= n^2 - 2k'n + k'^2 + n - k' - 2\alpha n + 2k'\alpha.
\end{aligned}$$

$$\begin{aligned}\mathsf{SU}(\mathcal{T}) - \mathsf{SU}(\mathsf{k}' - \mathsf{kdiam}) &= n^2 - n - 2\alpha n + 2\alpha - (n^2 - 2k'n + k'^2 \\ &\quad + n - k' - 2\alpha n + 2k'\alpha) \\ &= n^2 - n - 2\alpha n + 2\alpha - n^2 + 2k'n - k'^2 - n \\ &\quad + k' + 2\alpha n - 2k'\alpha \\ &= -2n + 2\alpha + 2kn - k'^2 + k' - 2k'\alpha \\ &= -2n + 2k'n - k'^2 + k' - 2(k'-1)\alpha \\ &= (2n - k')(k'-1) - 2(k'-1)\alpha \\ &= (k'-1)(2n - k' - 2\alpha).\end{aligned}$$

Since $\alpha \leqslant \frac{n}{2}$ and $k' \leqslant n$, we can get easily that $2n - k' - 2\alpha \geqslant 0$. Therefore $\mathsf{SU}(\mathcal{T}) \geqslant \mathsf{SU}(\mathsf{k}' - \mathsf{kdiam})$.

Now, let $\alpha > \frac{n}{2}$. We need to prove that the empty graph, i. e., $\mathsf{k}' - \mathsf{kdiam}$ with $k' = n$, has maximum social utility. We only need to prove that for any $2 \leqslant k' \leqslant n-1$, we have $\mathsf{SU}(\mathsf{n} - \mathsf{kdiam}) > \mathsf{SU}(\mathsf{k}' - \mathsf{kdiam})$. Since $\mathsf{SU}(\mathsf{n} - \mathsf{kdiam}) = 0$, it is enough to show that $\mathsf{SU}(\mathsf{k}' - \mathsf{kdiam}) < 0$ for $2 \leqslant k' \leqslant n-1$. We have

$$\begin{aligned}\mathsf{SU}(\mathsf{k}' - \mathsf{kdiam}) &= n^2 - 2k'n + k'^2 + n - k' - 2\alpha n + 2k'\alpha \\ &= n^2 - 2k'n + k'^2 + n - k' - 2\alpha(n - k') \\ &= (n-k')(n-k'+1) - 2\alpha(n-k') \\ &= (n-k')(n-k'+1-2\alpha).\end{aligned}$$

Since $k' \leqslant n-1$, we have $n - k' \geqslant 1$. Because of $k' \geqslant 2$, we have $\frac{k'-1}{2} \geqslant 0$. Moreover, we have $\alpha > \frac{n}{2}$, then $\alpha + \frac{k'-1}{2} \geqslant \alpha > \frac{n}{2}$, therefore $\alpha + \frac{k'-1}{2} > \frac{n}{2}$, it follows $n - k' + 1 - 2\alpha < 0$. Since $\mathsf{SU}(\mathcal{T}) < 0$ for $\alpha > \frac{n}{2}$, the empty graph is the optimum with social utility of zero. □

4.2.3 Definition. The *eccentricity* of a vertex v in a connected graph G is the maximum graph distance between v and any other vertex w of G. Let A be a subgraph of G; for any $i \in [k]$, we denote $N_i(A, v) = N_i(v) \cap V(A)$.

4.2.4 Proposition. *When restricting to trees in the* $2-$*neighborhood model, the price of anarchy is upper bounded by* $2 - \frac{2}{n}$.

Proof. From Lemma 4.1.3 we know the diameter of a PS tree is at most 3. If the diameter of each PS tree is 2, then obviously $\mathsf{PoA} = 1$. Then for finding a pairwise stable tree with a worst-case social utility, it is sufficient to consider a tree with diameter of 3, and by Lemma 4.1.3, $\alpha = 1$.

Let T be a tree of diameter 3. Then there are two players $a, b \in V$ and $\{a, b\} \in E$. Denote T_a, T_b the subtrees rooted at a and b, respectively, and $n_a := |V(T_a) \backslash \{a\}|$ and $n_b := |V(T_b) \backslash \{b\}|$. Hence, $n_a + n_b = n - 2$. Each player in $T_a \backslash \{a\}$ has sight of $n_a - 1$ players as well as players a and b, and each player in $T_b \backslash \{b\}$ has sight of $n_b - 1$ players as well as players a and b. Due to eccentricity of 2 of players a and b, we have $U_a(T) = U_b(T) = n - 1$.

Then, the social utility of T is

$$
\begin{aligned}
\mathsf{SU}(T) &= U_a(T) + U_b(T) + U_{v \in V(T_a) \backslash \{a\}}(T) + U_{v \in V(T_b) \backslash \{b\}}(T) \\
&= 2(n-1) + n_a(n_a+1) + n_b(n_b+1) - 2\alpha(n-1) \\
&= 2(n-1) + {n_a}^2 + n_a + {n_b}^2 + n_b - 2\alpha(n-1) \\
&= 2(n-1) + n - 2 + {n_a}^2 + {n_b}^2 - 2\alpha(n-a) \\
&= 3n - 4 + (n_a + n_b)^2 - 2n_1 n_2 - 2\alpha(n-1) \\
&= n^2 - n - 2n_a n_b - 2\alpha(n-1).
\end{aligned}
$$

It is clear that $\mathsf{SU}(T)$ is minimum where $n_a = \lfloor \frac{n}{2} - 1 \rfloor$ and $n_b = \lceil \frac{n}{2} - 1 \rceil$ i. e., where those two quantities are as equal as possible. We consider two cases, even and odd n.

Let n be even. Then, $n_a = n_b = \frac{n}{2} - 1$. Therefore, we have

$$
\begin{aligned}
\frac{\mathsf{OPT}}{\mathsf{SU}(T)} &= \frac{n(n-1) - 2(n-1)\alpha}{\sum_{v \in V} |N_2(v)| - 2|E|\alpha} \\
&= \frac{n(n-1) - 2(n-1)\alpha}{n^2 - n - 2n_a n_b - 2\alpha(n-1)} \\
&\underset{\alpha=1}{=} \frac{(n-1)(n-2)}{n^2 - n - 2(\frac{n}{2} - 1)^2 - 2(n-1)} \\
&= \frac{(n-1)(n-2)}{n(\frac{n}{2} - 1)} \\
&= 2 - \frac{2}{n}.
\end{aligned}
$$

Let n be odd. Without loss of generality, we assume $n_a = \frac{n}{2} - \frac{1}{2}$ and

$n_b = \frac{n}{2} - \frac{3}{2}$. Hence, we have

$$\begin{aligned}
\frac{\mathsf{OPT}}{\mathsf{SU}(T)} &= \frac{n(n-1) - 2(n-1)\alpha}{\sum_{v \in V} |N_2(v)| - 2|E|\alpha} \\
&= \frac{(n-1)(n-2)}{n^2 - n - 2n_a n_b - 2\alpha(n-1)} \\
&\underset{\alpha=1}{=} \frac{(n-1)(n-2)}{n^2 - n - 2(\frac{n}{2} - \frac{1}{2})(\frac{n}{2} - \frac{3}{2}) - 2(n-1)} \\
&= \frac{2(n-2)}{n-1} \\
&= 2 - \frac{2}{n-1} \\
&< 2 - \frac{2}{n}.
\end{aligned}$$

□

4.3 Swap Equilibrium

We analyze swap equilibrium trees for the k-neighborhood model and obtain an upper bound of k on the diameter of them and upper bound of 4 on the diameter for general SE graphs in the $2-$neighborhood model. We also show an example of a swap equilibrium graph with diameter of 4 for 2-neighborhood.

4.3.1 Theorem. *If a swap equilibrium graph is a tree in the $k-$neighbourhood model, then it has diameter at most k.*

Proof. Assume the equilibrium tree $T = (V, E)$ has diameter at least $k+1$ and we consider a path $(v_1, v_2, ..., v_{k+1}, v_{k+2})$ and $T_{v_1}, T_{v_2}, ..., T_{v_{k+1}}, T_{v_{k+2}}$ as their subtrees rooted at $v_1, v_2, ..., v_{k+1}, v_{k+2}$, respectively.

The utility for player v_1 is

$$U_{v_1}(T) = N_k(T_{v_1}, v_1) + N_{k-1}(T_{v_2}, v_1) + N_{k-2}(T_{v_3}, v_1) + ... + N_1(T_{v_k}, v_1).$$

Consider the swap (v_1, v_2, v_{k+1}). Then, we have

$$\begin{aligned}
U'_{v_1}(T) = 1 &+ N_k(T_{v_1}, v_1) + N_{k-2}(T_{v_{k+2}}, v_1) + N_{k-1}(T_{v_{k+1}}, v_1) \\
&+ N_{k-2}(T_{v_k}, v_1) + ... + N_1(T_{v_3}, v_1).
\end{aligned}$$

By stability $U'_{v_1}(T) \leqslant U_{v_1}(T)$, then we have

$$\begin{aligned} 1 &+ N_{k-2}(T_{v_{k+2}}, v_1) + N_{k-1}(T_{v_{k+1}}, v_1) + N_{k-2}(T_{v_k}, v_1) + \dots + N_1(T_{v_3}, v_1) \\ &\leqslant N_{k-1}(T_{v_2}, v_1) + N_{k-2}(T_{v_3}, v_1) + \dots + N_1(T_{v_k}, v_1). \end{aligned} \tag{4.3.2}$$

Moreover, the utility for player v_{k+1} is

$$\begin{aligned} U_{v_{k+2}}(T) = N_k(T_{v_{k+2}}, v_{k+2}) + N_{k-1}(T_{v_{k+1}}, v_{k+2}) + N_{k-2}(T_{v_k}, v_{k+2}) \\ + \dots + N_1(T_{v_3}, v_{k+2}). \end{aligned}$$

Consider the swap (v_{k+2}, v_{k+1}, v_2). Then, we have

$$\begin{aligned} U'_{v_{k+2}}(T) = 1 + N_k(T_{v_{k+2}}, v_{k+2}) + N_{k-1}(T_{v_2}, v_{k+2}) + N_{k-2}(T_{v_1}, v_{k+2}) \\ + N_{k-2}(T_{v_3}, v_{k+2}) + \dots + N_1(T_{v_k}, v_{k+2}) \end{aligned}$$

By stability $U'_{v_{k+2}}(T) \leqslant U_{v_{k+2}}(T)$, then we have

$$\begin{aligned} 1 + N_{k-1}(T_{v_2}, v_{k+2}) + N_{k-2}(T_{v_1}, v_{k+2}) + N_{k-2}(T_{v_3}, v_{k+2}) + \dots \\ + N_1(T_{v_k}, v_{k+2}) \leqslant N_{k-1}(T_{v_{k+1}}, v_{k+2}) + N_{k-2}(T_{v_k}, v_{k+2}) + \dots + N_1(T_{v_3}, v_{k+2}). \end{aligned} \tag{4.3.3}$$

From equations (4.3.2) and (4.3.3), we obtain

$$2 + N_{k-2}(T_{v_{k+2}}, v_{k+2}) + N_{k-2}(T_{v_1}, v_{k+2}) \leqslant 0 \ ,$$

a contradiction. □

4.3.4 Proposition. *Let a swap equilibrium graph $G = (V, E)$ in the $2-$neighborhood model be given and $a, c \in V$ with* $\mathrm{dist}(a, c) \geqslant 4$. *Then for all $b \in N(a)$ we have* $\deg(b) > \deg(c)$.

Proof. Let $b \in N(a)$. We analyze what happens if we consider the swap (a, b, c). When removing the edge $\{a, b\}$, the loss in utility for a is at most $\deg(b)$, since b contributes at most itself plus $\deg(b) - 1$ of its neighbors to $N_2(a)$. On the other hand, when subsequently adding $\{a, c\}$, the gain in utility is exactly $\deg(c) + 1$, since none of c's neighbors is in $N_2(a)$ due to the distance between a and c. Due to the SE condition, it follows that $\deg(c) + 1 \leqslant \deg(b)$. □

4.3.5 Theorem. *An SE graph in the $2-$neighborhood model has its diameter upper-bounded by 4.*

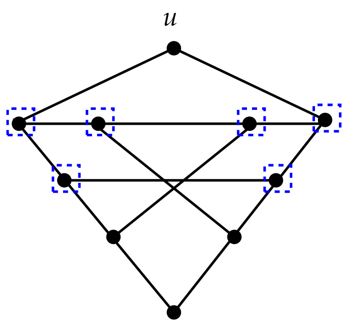


Figure 4.1. A 4−diameter swap equilibrium graph in the 2−neighborhood model.

Proof. For contradiction, assume there exists a shortest path $(v_0, v_1, ..., v_{k'})$ with $k' \geqslant 5$. Denote $a := v_1$ and $b := v_{k'}$. Then $\operatorname{dist}(a, b) = k' - 1 \geqslant 5 - 1 = 4$ and $v_0 \in N(a)$, so by Proposition 4.3.4 we have $\deg(v_0) > \deg(b) = \deg(v_{k'})$. The same argument with $a := v_{k'-1}$ and $b := v_0$ yields $\deg(v_{k'}) > \deg(v_0)$, a contradiction. $\square$

4.3.6 Theorem. *In the* $2-$*neighborhood model, there is a swap equilibrium graph with diameter 4.*

Proof. Figure 4.1 is an illustration for a swap equilibrium graph with a diameter of 4, which also can be drawn like Figure 4.2. We prove the swap equilibrium properties. It is clear that removing an incident edge for any player cannot improve the utility of them. We consider all possible edge swaps around each player:

(i) Players c, g, i and j have eccentricity of 2, hence they have maximum utility and no incentive to swap.

(ii) Player a; due to symmetry, we may restrict to one of the edges $\{a, h\}$ or $\{a, b\}$. We look at edge $\{a, h\}$. The swaps (a, h, j), (a, h, g), (a, h, f), (a, h, d) and (a, h, e) do not change the utility of player a. The swaps (a, h, i) and (a, h, c) reduce the utility by 1. Thus no swap can improve the utility of player a.

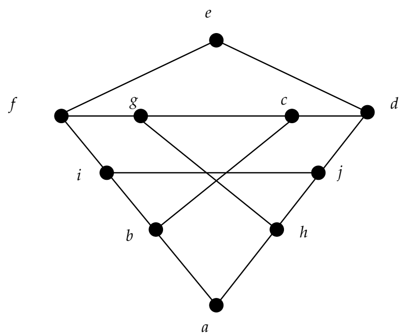


Figure 4.2. Different drawing of the graph from Figure 4.1.

(iii) Player b; we have three possibilities of swap for b:

(a) Swapping the incident edge $\{b, a\}$. The swaps (b, a, d), (b, a, e), (b, a, f), (b, a, g) and (b, a, j) reduce the utility of b by 1. The swap (b, a, h) does not change the utility of vertex b.

(b) Swapping the incident edge $\{b, i\}$. The swap (b, i, f) does not change the utility. The swaps (b, i, j), (b, i, d) and (b, i, e) reduce the utility of b by 1. The swaps (b, i, g) and (b, i, h) decrease the utility by 2.

(c) Swapping the incident edge $\{b, c\}$. The swap (b, c, d) does not change the utility of vertex b. The swaps (b, c, e), (b, c, g) and (b, c, f) reduce the utility of b by 1. The swaps (b, c, j) and (b, c, g) decrease the utility by 2.

(iv) Player e; by considering Figure 4.2, we have an argument similar to item (ii).

(v) By considering Figure 4.2, we have an argument similar to item (iii) for players d, f and h. □

Chapter 5

Network Coloring Games with Concave Payoff

Unlike in the previous chapters, now the network is fixed (in the form of an undirected graph). Players still correspond to vertices. But instead of building or removing links, each player can choose a *color* from a given set of possible colors. Utility, or *payoff*, as we will call it here, depends on the colors chosen by the player and her neighbors. This chapter is in part based on [**KSS17**]. Notably, however, here we give the resolution for a conjecture posed in [**KSS17**]; see Section 5.4 for a summary of our results.

5.1 Model, Notation, Basic Notions

Let $G = (V, E)$ be an undirected, simple graph without any isolated vertices ($n := |V|$ and $m := |E|$) and $k \in \mathbb{N}_{\geqslant 2}$. A function $c : V \longrightarrow [k]$ is called a *k-coloring* or *coloring*, where $[k] = \{1, \dots, k\}$. Vertices of the graph represent players, and each player is to choose exactly one color from the set $[k]$. We sometimes call this set $[k]$ the *spectrum*. Clearly, a k-coloring is used to collect the choices of all the players.[1] Given a coloring c, define the *payoff* or *utility* for player v as

$$U_v(c) := \sum_{w \in N(v)} f\Big(|c(v) - c(w)|\Big) \ , \tag{5.1.1}$$

where f is a non-negative, real-valued function defined on $[0, k]$ (choosing this domain instead of $\{0, \dots, k-1\}$ is technically easier). Given such f,

[1] The term "graph coloring game" in the literature also names a class of maker-breaker style games, which we are not referring to here.

we denote $f^* := \max_{i \in \mathcal{D}} f(i)$ the maximum value that f attains on the possible distances $\mathcal{D} := \{0, \ldots, k-1\}$ between two colors, and

$$\mathcal{D}^*(f) := \{i \in \mathcal{D}\,;\ f(i) = f^*\}.$$

We call $f(|c(v) - c(w)|)$ the *contribution* of edge $\{v, w\}$. So f^* is an upper bound on the contribution of any edge, and it is attained if the two players v and w manage to put a distance between each other which is in the set $\mathcal{D}^*(f)$. We assume $f^* > 0$, since otherwise the situation is uninteresting. If f is concave, this implies

$$f(i) > 0 \quad \forall i \in \mathcal{D}^+ := \{1, \ldots, k-1\}\ , \tag{5.1.2}$$

that is, f is positive for all the positive distances.

Let c be a k-coloring. For a player $v \in V$ and a color $t \in [k]$ we write the coloring where v *changes* to color t as $c[v \leftarrow t]$, so

$$c[v \leftarrow t](w) = \begin{cases} t & \text{if } w = v \\ c(w) & \text{otherwise} \end{cases} \qquad \text{for } w \in V.$$

A coloring c is called *stable* if

$$U_v(c[v \leftarrow t]) \leqslant U_v(c) \quad \forall t \in [k] \quad \forall v \in V\ . \tag{5.1.3}$$

This is our equilibrium concept. The *social utility* of c is $\mathsf{SU}(c) := \sum_{v \in V} U_v(c)$. We denote the optimum social utility as $\mathsf{SU}_{\mathsf{OPT}} := \max_c \mathsf{SU}(c)$, where c runs over all k-colorings; optimum colorings exist since there are only finitely many colorings. The price of anarchy is:

$$\mathsf{PoA}(G, f, k) := \max_{c \text{ is stable } k\text{-coloring}} \frac{\mathsf{SU}_{\mathsf{OPT}}}{\mathsf{SU}(c)}$$

The idea is the same as with network formation games: the price of anarchy measures the worst-case performance-loss due to non-cooperative behavior.

It is easy to see that stable colorings always exist, by a potential function argument. This has been observed before in the broader class of polymatrix common-payoff games [CD11],[2] and we repeat that simple argument here

[2] In [CD11], the term "coordination" is used instead of "common-payoff".

for completeness. Note that we can write social utility as follows:

$$\mathsf{SU}(c) = 2 \sum_{\{u,w\} \in E} f\Big(|c(u) - c(w)|\Big) \tag{5.1.4}$$

If a player v makes an *improvement step* resulting in coloring c' (i. e., there is $t \in [k]$ such that $c' = c[v \leftarrow t]$ and $U_v(c) < U_v(c')$) then

$$\sum_{\{v,w\} \in E} f\Big(|c(v) - c(w)|\Big) < \sum_{\{v,w\} \in E} f\Big(|c'(v) - c'(w)|\Big) ,$$

that is, the part of the sum in (5.1.4) with edges incident in v strictly increases. All other terms in the sum are maintained. Hence $\mathsf{SU}(c) < \mathsf{SU}(c')$. So in particular any optimal coloring is stable. Since existence of optimal colorings is guaranteed, we also have stable colorings. Note however that there can be sub-optimal stable colorings, which motivates the study of the price of anarchy.

5.2 Previous and Related Work

Graph coloring has been a theme in Combinatorics and Combinatorial Optimization for many decades. A coloring c for a graph $G = (V, E)$ is called *proper* or *legal* if $c(v) \neq c(w)$ for all $\{v, w\} \in E$, and given $v \in V$ we call a neighbor $w \in N(v)$ *properly colored* if $c(v) \neq c(w)$. Determining the minimal k such that a given graph admits a proper k-coloring, i. e., the *chromatic number* $\chi(G)$, is NP-hard [Kar72]. One of the classical highlights is Brooks' theorem [Bro41; Lov75], stating that for each graph G which is neither a complete graph nor a cycle of odd length, a proper Δ-coloring can be constructed by a combinatorial algorithm in polynomial time, where Δ is the maximum vertex degree in G. Another important algorithmic technique for finding proper colorings with provable worst-case performance is semi-definite programming with randomized hyperplane rounding [KMS98]. In addition, for more than three decades, distributed algorithms for proper colorings have been studied, see [BE13] for a recent survey. In the distributed model, each vertex is a processor and can communicate with its neighbors, where communication is done in rounds. For example, in one round, each vertex could communicate its color to all of its neighbors. In 2006, a study with human subjects was conducted by Kearns et al. [KSM06], where each subject controlled the color of one vertex, to be selected from a set of $\chi(G)$ colors, and each subject could see the colors of her neighbors. The goal was to construct a proper coloring. The study used graphs with $n = 38$ vertices and focused on certain classes of graphs, like cycles, small world graphs, and random graphs from the preferential attachment model. It is reported how the time required to reach a proper coloring is influenced by the structure of the graph.

A game-theoretic view on proper colorings was given in 2008 by Panagopoulou and Spirakis [PS08]. In their model, payoff for a player v is 0 whenever there exists a non-properly colored neighbor of v, otherwise payoff is the total number of players with color $c(v)$ in the graph. The idea is to incentivize players to create a proper coloring with few different colors. Indeed, two of the main results in [PS08] are that equilibria can be constructed by improvement steps in polynomial time, and that they are proper colorings using a total number of different colors that is upper-bounded by several known upper bounds on $\chi(G)$. So [PS08] yields a constructive proof for all of those bounds. For the same model, improved bounds and an extension to coalitions were later given by Escoffier et al. [EL10]; and Chatzigiannakis et al. [CKP+10] gave algorithmic improvements including experimental studies. Also in 2008, Chaudhuri et al. [CCJ08] considered the simpler payoff function that is 1 for player v if all of v's neighbors are properly colored and 0 otherwise. They showed that if the total number of available colors is $\Delta + 1$ and if players do improvement steps in a randomized manner, then an equilibrium (which also is a proper coloring) is reached in $\mathcal{O}(\log(n))$ steps with high probability.

A generalization of proper colorings are *distance-constrained labelings*, which are connected to the frequency assignment problem, where colors correspond to frequencies [Jan02]. Given integers $p_1, \ldots, p_l$, a coloring c is called an $L(p_1, \ldots, p_l)$-labeling if for all $i \in [l]$ and all $v, w \in V$ we have $|c(v) - c(w)| \geqslant p_i$ whenever $\text{dist}_G(v, w) = i$, where $\text{dist}_G(v, w)$ is the graph-theoretical distance between v and w, i. e., the length of a shortest v-w path in G. The notion of $L(1)$-labeling coincides with that of proper coloring. The case of $L(p, q)$-labelings, that is, where $l = 2$, has received special attention, in particular in connection with frequency assignment, see, e. g., [Yeh06; Cal11; HIO+14]. As a variation of this, not the distance

$|c(v) - c(w)|$ between colors is considered, but instead it is required that

$$\min\Big\{|c(v) - c(w)|, k - |c(v) - c(w)|\Big\} \geqslant p_i \ ,$$

whenever $\mathrm{dist}_G(v, w) = i$, as considered in [HLS98]. Another variation are T-colorings [Hal80; Rob91]: given a set of integers T, a T-coloring is one where $|c(v) - c(w)| \notin T$ whenever $\{v, w\} \in E$. This problem also arises in frequency assignment where T is the set of forbidden spectral distances between neighboring senders, which are known to cause interference.

In our model, we allow colorings being evaluated by the payoff function and not only ask whether they have a certain property (e. g., being proper, a T-coloring, etc.) or not. For example, using

$$f : [0, k] \longrightarrow \mathbb{R}_{\geqslant 0}, \ x \mapsto \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{otherwise} \end{cases} \tag{5.2.1}$$

as the function f, payoff for each player v is the number of her properly colored neighbors. We call this *basic payoff*. Note that this is a concave function. With this payoff, the game is also known as a *max-k-cut game* (in the unweighted version), since we partition the vertices in k clusters and payoff for v is the total number of cut edges incident to v, i. e., edges incident to v and running between clusters. These games were studied by Hoefer [Hoe07] in 2007, and a tight bound of $\frac{k}{k-1}$ on the price of anarchy was proved, where tightness is already attained on bipartite graphs. The upper bound works by a mean-value argument: for each coloring and for each player v, there is a color t that occurs on at most $\frac{\deg(v)}{k}$ neighbors of v. In a stable coloring, v chooses this t, or even better if possible, hence obtaining a payoff of at least $\frac{k-1}{k} \deg(v)$. Since optimum social utility is upper-bounded by $2m$, the theorem follows from this (using the handshaking lemma for the sum of degrees). In Section 5.11.1 we provide indication that already for $f(x) = x$, a straightforward generalization of this technique cannot yield a constant upper bound.

A weighted version of max-k-cut games is obtained by assigning a weight to each edge and defining payoff for v as the sum of the weights of the incident cut edges. This version under the aspect of coalitions among players was studied by Gourvès and Monnot [GM09] in 2009.

In 2013, Kun, Powers, and Reyzin [KPR13] considered complexity issues for basic payoff, i. e., payoff as per (5.2.1), combined with variations of the game. It is clear that since social utility for basic payoff can only change in integer steps and can never be more than $2m$, we will reach a stable coloring after at most $\mathcal{O}(m)$ improvement steps (m is the number of edges in the graph). On the other hand, Kun et al. show that for basic payoff, it is NP-hard to decide whether a graph admits a *strictly stable* coloring, where the latter notion is defined by replacing $\leqslant$ for $<$ in (5.1.3). They also show that for basic payoff, it is NP-hard to decide whether a *directed* graph has a stable coloring, where for directed graphs, payoff is defined by having the sum in (5.1.1) only range over the out-neighbors of v.

In 2014, Apt et al. [ARS+14] used the function

$$f : [0, k] \longrightarrow \mathbb{R}_{\geqslant 0}, \ x \mapsto \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}, \tag{5.2.2}$$

which counts the neighbors of the *same* color, together with the extension that each player has a set of colors to choose from (whereas we in our model always allow all colors for all players). They study this game under different aspects, including coalitions, price of anarchy, and complexity. As for the price of anarchy without coalitions, it is easy to see that it can be unbounded. The lower-bound construction depends on the fact that we can forbid certain colors for certain players: for example, for each player v let there be a distinct "private" color s_v, and in addition there is a "common" color t. Player v can choose her color from the set $\{s_v, t\}$. The social optimum, namely $2m$, is obtained when each player chooses t, whereas a worst-case stable coloring, with social utility 0, is obtained when each player v chooses s_v. Without the ability to restrict players to certain colors, i. e., if we use our framework for the function f in (5.2.2), a tight bound on the price of anarchy of k is easy to see (Section 5.11.2).

The graph coloring games studied in our work belong to the class of *polymatrix games* [Yan68; CD11]. In such a game, we have a graph $G = (V, E)$, for each player a set of strategies, and for each edge $\{v, w\} \in E$ a two-player matrix game $\Gamma_{\{v,w\}}$. Each player v chooses one strategy and has to play this same strategy in all the games $\{\Gamma_{\{v,w\}}\}_{w \in N(v)}$ corresponding to incident edges. Payoff for v is the sum of the payoffs over all those two-player games. A special case is that of *polymatrix common-payoff games*, which means that each $\Gamma_{\{v,w\}}$ is a *common-payoff game*, i. e., it always yields the same payoff for v as for w.[3] Thus our graph coloring games belong to this class, since each edge $\{v, w\}$ contributes the same value $f(|c(v) - c(w)|)$ to the payoffs of v and of w. Recently, in 2015, Rahn and Schäfer [RS15] studied polymatrix common-payoff games with coalitions. They consider (α, l)-equilibria, that are α-approximate equilibria under coalitions of size l. For the corresponding price of anarchy, they give a lower bound of $2\alpha(n-1)/(l-1) + 1 - 2\alpha$ and an upper bound of $2\alpha(n-1)/(l-1)$. Note that in our work, we have $l = 1$ since we do not consider coalitions, and for this case their bounds are ∞. This follows already from the example given in [ARS+14], which is based on restricting certain players to certain colors (the "private" and "public" colors) and is explained in the previous paragraph.

5.3 Our Direction and More Related Work

In 2013, Kun et al. [KPR13] named as open problems the study of the price of anarchy for payoff as per (5.1.1) and induced by f being the identity, i. e., the contribution of edge $\{v, w\}$ is the distance $|c(v) - c(w)|$. We call this *distance payoff*. A variation, which Kun et al. also refer to, is the notion studied by van den Heuvel et al. [HLS98] in the context of frequency assignment, namely the contribution of $\{v, w\}$ is $\min\{|c(v) - c(w)|, k - |c(v) - c(w)|\}$, which in our notation means $f(x) = \min\{x, k - x\}$. We call this *cyclic payoff*. It rewards players for keeping a "medium" distance from others. This has an additional interpretation in connection with an example often given in the context of proper colorings, namely where colors correspond to skills and people inside of an organization try to develop skills that are complementary to nearby colleagues (see, e. g., [KSM06; CCJ08]). Cyclic payoff refines that idea: subjects are incentivized to develop different *but still related (that is, not too far away)* skills. Distance and cyclic payoff both

[3] Such are called "coordination games" sometimes, but the use of this term is not consistent in the literature. We stick to the terminology from [LS08, Sec. 1.3.2], which is "common-payoff game".

result from concave functions f, so they are both special cases of the payoff functions studied in our work.

Spectrum sharing and frequency (or channel) assignment problems, that is, when a multitude of participants compete for using the same or similar frequencies, has received much attention lately, see, e. g., [PZZ06; FK07; HHL+10; DA12]. Many works in that field use some form of graph coloring (colors corresponding to frequencies) in a distributed or game-theoretic setting. It is common in the frequency assignment literature to consider not only feasible versus infeasible colorings but instead to quantify the "degree of interference". To this end, the spectral distance $|c(v) - c(w)|$ between players v and w is often used as a measure or as a substantial ingredient to a measure. In addition to the references we give above for distance-constrained labeling and for T-coloring, this is documented for example in [Gam86, Sec. 2], [ASH02, Sec. 2.1], [SP02, Sec. II.B], [AHK+07, Sec. 3], and [BEG+10, Sec. 11.6]. So our game can be used to model the case where each participant of a wireless network, e. g., a mobile telephone network or wireless sensor network, chooses her frequency non-cooperatively and with respect to a measure of interference expressed by a concave function applied to spectral distances (where higher values of that function mean less interference).

A first game-theoretic study of distance and cyclic payoff in our framework was conducted by Schink [Sch14] in 2014. He observed that a stable k-coloring for distance payoff can be constructed from a stable 2-coloring for basic payoff by replacing color 2 with k. A similar approach works for cyclic payoff if k is even, replacing color 2 with $\frac{k}{2} + 1$. Since a stable coloring for basic payoff can be constructed in $\mathcal{O}(m)$ improvement steps, this gives a runtime guarantee independent of k. Since social utility can reach up to $\Omega(mk)$ for distance and cyclic payoff, such a k-independent guarantee is not easily possible by a direct argument using improvement steps. Interestingly, for cyclic payoff and *odd* k, we have no k-independent runtime guarantee for the construction of stable colorings at this time. For price of anarchy, Schink proved an upper bound of $\Delta(G)$, the maximum vertex degree in G, for cyclic payoff. Apart from that, we are not aware of any previous bounds on the price of anarchy for graph coloring games with concave payoff, not even conjectures. The work by Rahn and Schäfer [RS15], for general polymatrix common-payoff games, can be considered orthogonal to ours since they do not consider the effects of restricting the two-player games $\Gamma_{\{v,w\}}$ to certain classes, whereas we restrict to such games arising from applying a concave function to the color distance $|c(v) - c(w)|$, resulting in small constant bounds on the price of anarchy. Moreover, [RS15] allows restricting players to certain sets of colors (making finite bounds on the price of anarchy impossible without coalitions), whereas in our model, each player has the same set of colors to choose from.

5.4 Our Contribution and Techniques

We prove constant upper bounds on the price of anarchy for several classes of concave functions f. We prove a bound of 2 for all concave functions f which assume f^* at a distance on or below $\lfloor \frac{k}{2} \rfloor$, that is, for which $\mathcal{D}^*(f) \cap \{0, \ldots, \lfloor \frac{k}{2} \rfloor\} \neq \emptyset$. This includes cyclic payoff. We show that for this class of functions, this bound is the best possible, since for cyclic payoff with even k, the price of anarchy is exactly 2. For non-decreasing concave functions, we also show an upper bound of 2. This includes distance payoff. Again, we show that for this class of functions, this bound is the best possible. The remaining concave functions are those which have $\mathcal{D}^*(f) \cap \{\lfloor \frac{k}{2} \rfloor + 1, \ldots, k-2\} \neq \emptyset$. For those, an upper bound of 3 on the price of anarchy was proved in [**KSS17**], and it was conjectured – based on computer experiments – that a bound of 2.5 should be possible. We resolve this conjecture here for $k \geqslant 16$.

It may be surprising at first that the situation is not symmetric: functions with their maximum left of the middle of the spectrum behave differently from functions with their maximum right of the middle. But in fact this is to be expected since for example, a player can always force all the distances to her neighbors to be on or below $\lfloor \frac{k}{2} \rfloor$ by choosing her own color as $\lfloor \frac{k}{2} \rfloor + 1$, but it is not always possible to force all distances beyond 1. So there is an asymmetry between short and long distances.

All our proofs work by *local* arguments. That is, if we are to prove an upper bound of $\lambda \geqslant 1$ on the price of anarchy, we show the following: for each player v, given the colors $(c(w))_{w \in N(v)}$ of her neighbors, there is a color $t \in [k]$ such that

$$\sum_{w \in N(v)} f\Big(|c(w) - t|\Big) \geqslant \frac{\deg(v) \cdot f^*}{\lambda} \quad .$$

Clearly, in a stable coloring, each player chooses such a color t, or better. Hence $\mathsf{SU}(c) \geqslant \frac{2mf^*}{\lambda}$ for a stable c. Since $\mathsf{SU}_{\mathsf{OPT}} \leqslant 2mf^*$, the bound λ follows.

In the next section, we provide a framework for the type of local arguments described in the above paragraph. To this end, we introduce the *local parameter* $\lambda(f,k)$ of a function f and show $\mathsf{PoA}(G,f,k) \leqslant \lambda(f,k)$. In Section 5.6, we show how to obtain an upper bound of 4 on the local parameter, and thus on the price of anarchy, for any concave function f by a simple technique. In Section 5.7, a more elaborate technique is introduced. It says that if we can find a representation of a number $\lambda \geqslant 1$ as a sum $\lambda_1 + \ldots + \lambda_k = \lambda$ with $\sum_{s=1}^{k} \lambda_s \cdot f(|s-p|) \geqslant f^*$ for all $p \in [k]$, then we have $\lambda(f,k) \leqslant \lambda$, and thus a bound of λ on the price of anarchy. Using this *splitting technique*, we prove our main results in Section 5.7 and Section 5.9. For example, the upper bound of 2 for cyclic payoff can be proven using the splitting defined by $\lambda_1 := 1$ and $\lambda_{\lfloor \frac{k}{2} \rfloor + 1} := 1$ and $\lambda_s := 0$ for all remaining indices $s \in [k] \backslash \{1, \lfloor \frac{k}{2} \rfloor + 1\}$.

Appropriate splittings up to a certain granularity for moderate values of k and given function f can be found by computer, doing a simple enumeration. We fix $\delta = (\delta_1, \ldots, \delta_r)$ such that $\lambda = \sum_{i=1}^{r} \delta_i$, for example $\delta = (1, 1, \frac{1}{4}, \frac{1}{4})$ for $\lambda = 2.5$. For each $v = (v_1, \ldots, v_r) \in [k]^r$ define a splitting $\lambda(v)$ by $\lambda_s(v) := \sum_{\substack{i \in [r] \\ v_i = s}} \delta_i$ for each $s \in [k]$. Then we let the computer enumerate all $v \in [k]^r$ and output those corresponding splittings $\lambda(v)$ that satisfy the necessary conditions. Although this is for a concrete k and not general, it can give valuable hints on how to do a general proof. We used this to obtain essential ideas for many of our proofs.

5.5 The Local Parameter

For a function $f : [0, k] \longrightarrow \mathbb{R}_{\geqslant 0}$ we define the *local parameter* of f as

$$\lambda(f, k) := \max_{\nu \in \mathbb{N}} \max_{c_1, \ldots, c_\nu \in [k]} \frac{\nu f^*}{\max_{t \in [k]} \sum_{i=1}^{\nu} f(|c_i - t|)} \quad . \tag{5.5.1}$$

By (5.1.2), the denominator is never zero. Clearly, the denominator is never greater than νf^*, so $\lambda(f, k) \geqslant 1$. The intuition is that we capture the best way a player can react to a given set of colors $c_1, \ldots, c_\nu$ (which will be the colors of her neighbors when applying this to the graph coloring game) relative to the maximum conceivable payoff νf^*.

5.5.2 Remark. We have $\mathsf{PoA}(G, f, k) \leqslant \lambda(f, k)$.

Proof. Let c be stable and $v \in V$. By definition of the local parameter, we have

$$\lambda(f, k) \geqslant \frac{\deg(v) \cdot f^*}{\max_{t \in [k]} \sum_{w \in N(v)} f(|c(w) - t|)}$$

hence,

$$\max_{t \in [k]} \sum_{w \in N(v)} f(|c(w) - t|) \geqslant \frac{\deg(v) \cdot f^*}{\lambda(f, k)} \quad .$$

That is, there is a color t which v can choose to obtain payoff at least that much. Since c is stable, player v chooses such a color, or better, so $U_v(c) \geqslant \frac{\deg(v) \cdot f^*}{\lambda(f,k)}$. Choosing c as a worst-case stable coloring, and using the trivial upper bound $\mathsf{SU}_{\mathsf{OPT}} \leqslant 2mf^*$ and the handshaking lemma, we obtain:

$$\mathsf{PoA}(G, f, k) = \frac{\mathsf{SU}_{\mathsf{OPT}}}{\mathsf{SU}(c)} \leqslant \frac{2mf^*}{\sum_{v \in V} U_v(c)} \leqslant \frac{2mf^*}{\frac{f^*}{\lambda(f,k)} \sum_{v \in V} \deg(v)} = \lambda(f, k)$$

□

The local parameter has the following simple properties:

5.5.3 Remark.

- The local parameter is invariant against scaling, that is, $\lambda(f,k) = \lambda(\gamma f,k)$ for all $\gamma \in \mathbb{R}_{>0}$.
- The local parameter is monotone in the following way. Let $f,g : [0,k] \longrightarrow \mathbb{R}_{\geqslant 0}$ such that $f(i) \geqslant g(i)$ for all $i \in \mathcal{D}$, and $f^* = g^*$. Then $\lambda(f,k) \leqslant \lambda(g,k)$.

Proof. Follows directly from the definition in (5.5.1). □

5.6 A First Upper Bound

As a start, we provide a lose bound of 4 on the local parameter by a simple argument. This bound is later improved, depending on where f assumes its maximum.

5.6.1 Theorem. *Let f be concave. Then $\lambda(f,k) \leqslant 4$.*

Proof. Let $\nu \in \mathbb{N}$ and $c_1,\ldots,c_\nu \in [k]$ and define the function $\phi(t) := \sum_{i=1}^{\nu} f(|c_i - t|)$. We are done if we can prove that there is $t \in [k]$ with $\phi(t) \geqslant \frac{\nu f^*}{4}$. Let $k^* \in \mathcal{D}^*(f)$, i. e., $k^* \in \{0,\ldots,k-1\}$ with $f(k^*) = f^*$. By concavity, f assumes a value of at least $\frac{f^*}{2}$ between k^* and the half-way point to 0 as well as to the half-way point to k, i. e., for all $x \in H := [\frac{k^*}{2}, k^* + \frac{k-k^*}{2}] = [\frac{k^*}{2}, \frac{k+k^*}{2}]$ we have $f(x) \geqslant \frac{f^*}{2}$. One of the following two cases is given: $c_i \leqslant \lfloor \frac{k}{2} \rfloor$ for at least $\frac{\nu}{2}$ distinct $i \in [\nu]$, or $c_i \geqslant \lfloor \frac{k}{2} \rfloor + 1$ for at least $\frac{\nu}{2}$ distinct $i \in [\nu]$.

Assume the first case. We choose $t := \lceil \frac{k+k^*}{2} \rceil \leqslant \lceil \frac{2k}{2} \rceil = k$. Let $i \in [\nu]$ be an index with $c_i \leqslant \lfloor \frac{k}{2} \rfloor$ (of which we have at least $\frac{\nu}{2}$ ones in this case). Then

$$|c_i - t| = t - c_i \leqslant t - 1 = \left\lceil \frac{k+k^*}{2} \right\rceil - 1 \leqslant \frac{k+k^*}{2} \quad ,$$

and

$$t - c_i \geqslant \left\lceil \frac{k+k^*}{2} \right\rceil - \left\lfloor \frac{k}{2} \right\rfloor \geqslant \frac{k+k^*}{2} - \frac{k}{2} = \frac{k^*}{2} \quad .$$

Hence $|c_i - t| \in H$, which means that $f(|c_i - t|) \geqslant \frac{f^*}{2}$. Since we have at least $\frac{\nu}{2}$ of those indices, it follows $\phi(t) \geqslant \frac{\nu}{2}\frac{f^*}{2} = \frac{\nu f^*}{4}$.

The other case can be treated likewise; define $t := \lceil \frac{k-k^*}{2} \rceil$. Let $i \in [\nu]$ be an index with $c_i \geqslant \lfloor \frac{k}{2} \rfloor + 1$. Then

$$|c_i - t| = c_i - t \leqslant k - \lceil \tfrac{k-k^*}{2} \rceil \leqslant k - \tfrac{k-k^*}{2} = \tfrac{k+k^*}{2} \ ,$$

and

$$c_i - t \geqslant \lfloor \tfrac{k}{2} \rfloor + 1 - \lceil \tfrac{k-k^*}{2} \rceil \geqslant \tfrac{k}{2} - \tfrac{1}{2} + 1 - \left(\tfrac{k-k^*}{2} + \tfrac{1}{2}\right) = \tfrac{k^*}{2} \ .$$

So also in the case, $|c_i - t| \in H$. □

5.7 The Splitting Technique

We give a simple but powerful technique to prove upper bounds on the local parameter of a given function. Given a number $\lambda \in \mathbb{R}_{\geqslant 1}$, we call a family of numbers $\lambda_1, \dots, \lambda_k \in \mathbb{R}_{\geqslant 0}$ a *splitting of* λ, provided that $\lambda = \sum_{s=1}^{k} \lambda_s$.

5.7.1 Lemma. *Let $\lambda \geqslant 1$ and $\lambda_1, \dots, \lambda_k \in \mathbb{R}_{\geqslant 0}$ be a splitting of λ. Let $f : [0,k] \longrightarrow \mathbb{R}_{\geqslant 0}$. Assume that the following condition is given:*

$$\forall p \in [k] : \ \sum_{s=1}^{k} \lambda_s \cdot f(|s-p|) \geqslant f^* \tag{$*$}$$

Then $\lambda(f,k) \leqslant \lambda$.

Proof. Let $\nu \in \mathbb{N}$ and $c_1, \dots, c_\nu \in [k]$. We prove that

$$\lambda \cdot \max_{t \in [k]} \sum_{i=1}^{\nu} f(|c_i - t|) \geqslant \nu f^* \ .$$

For each $p \in [k]$ denote $\nu_p := |\{i \in [\nu] \,;\ c_i = p\}|$, that is, how many times the number p occurs in the family $c_1, \dots, c_\nu$. We have:

$$\begin{aligned} &\lambda \cdot \max_{t \in [k]} \sum_{i=1}^{\nu} f(|c_i - t|) \\ = &\sum_{s=1}^{k} \lambda_s \cdot \max_{t \in [k]} \sum_{i=1}^{\nu} f(|c_i - t|) \qquad \text{def. of splitting} \end{aligned}$$

$$
\begin{aligned}
&\geqslant \sum_{s=1}^{k} \lambda_s \cdot \sum_{i=1}^{\nu} f(|c_i - s|) && \text{maximum} \\
&= \sum_{s=1}^{k} \lambda_s \cdot \sum_{p=1}^{k} \nu_p \cdot f(|p - s|) && \text{subsume same values} \\
&= \sum_{p=1}^{k} \nu_p \cdot \sum_{s=1}^{k} \lambda_s \cdot f(|p - s|) && \text{exchange summation} \\
&\geqslant \sum_{p=1}^{k} \nu_p f^* && \text{by } (*) \\
&= \nu f^* && \square
\end{aligned}
$$

We demonstrate the use of the splitting technique by a couple of simple proofs.

5.7.2 Proposition. *Let $a \in \mathbb{R}_{>0}$ and $b \in \mathbb{R}_{\geqslant 0}$ and $f(x) = ax + b$. Then $\lambda(f,k) \leqslant \rho(a,b,k) := 2\frac{a(k-1)+b}{a(k-1)+2b} \leqslant 2$.*

Proof. All we have to do is check $(*)$ for this function f and an appropriate splitting of $\lambda = \rho(a,b,k)$. We have $f^* = a\,(k-1) + b$. Define $\lambda_1 := \lambda_k := \frac{\rho(a,b,k)}{2} = \frac{f^*}{a\,(k-1)+2b}$ and $\lambda_s := 0$ for all other s, that is, all $s \in [k]\backslash\{1,k\}$. We have to check

$$\forall p \in [k]: \quad \frac{f^*}{a\,(k-1)+2b} \cdot \Big(f(p-1) + f(k-p)\Big) \geqslant f^* \quad ,$$

that is,
$$\forall p \in [k]: \;\; f(p-1) + f(k-p) \geqslant a\,(k-1) + 2b \quad .$$

The latter is clearly true due to the definition of f. $\square$

5.7.3 Corollary. *Let f be concave, non-constant, and non-decreasing. Then $\lambda(f,k) \leqslant \rho(a,b,k) \leqslant 2$, where $a := \frac{f(k-1)-f(0)}{k-1}$ and $b := f(0)$. (Since f is non-constant, $a > 0$. The constant case is trivial and needs no attention.)*

Proof. Define the function $g: [0,k] \longrightarrow \mathbb{R}_{\geqslant 0}$, $x \mapsto ax + b$. By concavity of f, we have $g(x) \leqslant f(x)$ for all $x \in [0, k-1]$. By monotonicity of f, we have $f^* = g^*$. The corollary follows from Remark 5.5.3. $\square$

We also treat the case of a decreasing affine and then a non-increasing concave function. Here it makes sense to allow f to assume negative values in $(k-1, k]$.

5.7.4 Proposition. *Let $a \in \mathbb{R}_{>0}$ and $b \in \mathbb{R}_{\geqslant 0}$ and $f(x) = b - ax$, such that $f(k-1) \geqslant 0$. Then $\lambda(f,k) \leqslant \rho'(a,b,k) := \frac{2b}{2b-a(k-1)}$, which is 2 for the case of $f(k-1) = 0$.*

Proof. We have $f^* = b$. Define $\lambda_1 := \lambda_k := \frac{\rho'(a,b,k)}{2} = \frac{f^*}{2b-a(k-1)}$ and $\lambda_s := 0$ for all other s. Now $(*)$ follows from a simple calculation as in the proof of Proposition 5.7.2. □

5.7.5 Corollary. *Let f be concave, non-constant, and non-increasing. Then $\lambda(f,k) \leqslant \rho'(a,b,k) \leqslant 2$, where $a := \frac{f(0)-f(k-1)}{k-1}$ and $b := f(0)$.*

Proof. Like Corollary 5.7.3. □

5.7.6 Proposition. *Let $f(x) = \min\{x, k-x\}$, that is, cyclic payoff. Then the price of anarchy is upper-bounded by 2.*

Proof. All we have to do is check $(*)$ for this function f and an appropriate splitting of $\lambda = 2$. For $i \in \mathbb{N}$ denote $k_i := \lfloor \frac{k}{2} \rfloor + i$. Then $f^* = k_0$ and $f(x) = x$ if $x \leqslant k_0$ and $f(x) = k - x$ if $x \geqslant k_1$. Define $\lambda_1 := 1$ and $\lambda_{k_1} := 1$ and $\lambda_s := 0$ for all other s. Condition $(*)$ reduces to:

$$\forall p \in [k]: \ f(p-1) + f(|k_1 - p|) \geqslant k_0 \tag{5.7.7}$$

To show (5.7.7), let $p \in [k]$. If $1 \leqslant p \leqslant k_1$, then $p - 1 \leqslant k_0$ and $k_1 - p \leqslant k_0$, so we have:

$$f(p-1) + f(|k_1 - p|) = f(p-1) + f(k_1 - p) = p - 1 + k_1 - p = k_0$$

If $k_2 \leqslant p \leqslant k$, then $p - 1 \geqslant k_2 - 1 = k_1$ and $p - k_1 \leqslant k - k_1 = \lceil \frac{k}{2} \rceil - 1 \leqslant k_0$, so we have:

$$\begin{aligned} f(p-1) + f(|k_1 - p|) &= f(p-1) + f(p - k_1) \\ &= k - (p-1) + p - k_1 = k - k_0 \geqslant k_0 \end{aligned}$$

This concludes the proof. □

5.8 Lower Bounds

5.8.1 Proposition. *The bound of $\rho(a,b,k)$ implied by Proposition 5.7.2 on the price of anarchy for affine functions f is the best possible, and the worst case is assumed already on bipartite graphs.*

Proof. We give an instance of the graph coloring game with function $f(x) = ax + b$ that has price of anarchy $\rho(a,b,k)$. Consider the complete bipartite graph $K_{2,2}$ and denote $\{u_1, u_2\}$ the vertices of one partition and $\{w_1, w_2\}$ those of the other (so edges are all $\{u_i, w_j\}$ with $i, j \in \{1,2\}$). Define coloring c by:

$$c(u_1) := 1 \quad c(u_2) := k \quad c(w_1) := \lfloor \tfrac{k+1}{2} \rfloor \quad c(w_2) := \lceil \tfrac{k+1}{2} \rceil$$

It is easy to see that c is stable: players w_1 and w_2 have payoff $a\,(k-1) + 2b$ each, no matter which color they choose. Players u_1 and u_2 also have payoff $a\,(k-1) + 2b$ each, but only for colors 1 and k; for all other colors they get less. An optimal coloring is obtained by $c(u_i) := 1$ and $c(w_i) := k$ for $i \in \{1,2\}$, with each edge giving contribution $a\,(k-1) + b$. We have, using the number $m = 4$ of edges,

$$\frac{\mathsf{SU}_{\mathsf{OPT}}}{\mathsf{SU}(c)} = \frac{2 \cdot 4 \cdot (a\,(k-1) + b)}{4 \cdot (a\,(k-1) + 2b)} = \rho(a,b,k) \ . \qquad \square$$

5.8.2 Proposition. *The bound of $\rho'(a,b,k)$ implied by Proposition 5.7.4 on the price of anarchy for affine decreasing functions f is the best possible, and the worst case is assumed already on bipartite graphs.*

Proof. We use the same graph as in the proof of Proposition 5.8.1. However, we define coloring c by:

$$c(u_1) := 1 \quad c(u_2) := k \quad c(w_1) := 1 \quad c(w_2) := k$$

It is easy to see that this is stable with $\mathsf{SU}(c) = 4\,(2b - a(k-1))$. Since the optimum is $8b$, when all players choose the same color, a price of anarchy of $\frac{2b}{2b-a(k-1)} = \rho'(a,b,k)$ follows. $\square$

5.8.3 Proposition. *The bound of 2 implied by Proposition 5.7.6 on the price of anarchy for cyclic payoff is the best possible for even k, and the worst case is* *assumed already on a cycle of even length. For odd k, we have a lower bound of* $\frac{3}{2}(1 - \frac{1}{k})$.

Proof. Again for each $i \in \mathbb{N}_0$ denote $k_i := \lfloor \frac{k}{2} \rfloor + i$. First let k be even. Consider a cycle of length $4n$ for some $n \in \mathbb{N}_{\geqslant 1}$ and color like so:

$$1, 1, k_1, k_1, 1, 1, k_1, k_1, \ldots$$

Then half of the edges have contribution 0, namely between players of the same color, and the other half has contribution k_0 each, so the welfare is nk_0. We prove that this coloring is stable. Let v be a player with $c(v) = 1$. Her payoff is k_0. If she changes to a color $2 \leqslant t \leqslant k_1$, her new payoff will be $(t-1) + (k_1 - t) = k_1 - 1 = k_0$, so no improvement. If she changes to a color $k_1 + 1 \leqslant t \leqslant k$, her new payoff will be $(t - k_1) + k - (t-1) = k - k_1 + 1 = k_0 - 1 + 1 = k_0$, so also no improvement. The case $c(v) = k_1$ is treated likewise. An optimal coloring uses 1 and k_1 alternately and yields welfare $2nk_0$. This proves the claim.[4]

For odd k, we take a cycle of length $6n$ for some $n \in \mathbb{N}_{\geqslant 1}$ and color like so: $1, k_1, k_2, 1, k_1, k_2, \ldots$. The pattern $1, k_1, k_2$ can be repeated an integral number of times since the number of vertices is a multiple of 3. This yields welfare $2n(k_0 + 1 + k_0) = 4nk_0 + 2n$, so in comparison with the optimum (still attained by using 1 and k_1 alternately, since number of vertices is even) we have

$$\frac{6nk_0}{4nk_0 + 2n} = \frac{3}{2}\left(1 - \frac{1}{2k_0 + 1}\right) = \frac{3}{2}\left(1 - \frac{1}{k}\right) .$$

We prove that this coloring is stable. Let v be a player with $c(v) = 1$. Her payoff is $2k_0$. If she changes to color t with $2 \leqslant t \leqslant k_1$, her new payoff will be $(k_1 - t) + (k_2 - t) = 2k_0 + 3 - 2t \leqslant 2k_0 - 1$, so no improvement. If she changes to color t with $k_2 \leqslant t \leqslant k = 2k_0 + 1$, her new payoff will be $(t - k_1) + (t - k_2) = 2t - 2k_0 - 3 \leqslant 2(2k_0 + 1) - 2k_0 - 3 = 2k_0 - 1$, so also no improvement.

Now let $c(v) = k_1$. Her payoff is $k_0 + 1 = k_1$. If she changes to color t with $2 \leqslant t \leqslant k_1 - 1 = k_0$, her new payoff will be $(t-1) + (k_2 - t) =$

[4] The above construction is not stable for odd k, since then for example a player with color k_1 could change to k_2: this would not change the contribution of the edge to the 1-colored neighbor (it remains k_0) but would increase distance from 0 to 1 regarding the k_1-colored neighbor, hence increasing payoff by 1.

$k_2 - 1 = k_1$, so it is no improvement. If she changes color to 1, then her new payoff will be $k - (k_2 - 1) = k - k_1 = k - k_0 - 1 = k_1 - 1 = k_0$, so no improvement; note that $k - k_0 = k_1$. If she changes to color t with $k_1 + 1 \leqslant t \leqslant k$, her new payoff will be $k - (t-1) + (t - k_2) = k + 1 - k_0 - 2 = k_1 - 1$, so also no improvement. The case $c(v) = k_2$ can be treated likewise and is omitted here. $\square$

5.9 General Concave f

We define a family of "prototype" concave functions. For each $\ell \in \mathbb{N}$ with $1 \leqslant \ell < k - 1$ define:

$$f_\ell : [0,k] \longrightarrow \mathbb{R}_{\geqslant 0}, \ x \mapsto \begin{cases} \frac{x}{\ell} & \text{if } x \leqslant \ell \\ \frac{k-x}{k-\ell} & \text{if } x \geqslant \ell \end{cases} \tag{5.9.1}$$

So this function rises in a linear fashion from 0 until it reaches value 1 in ℓ, and then it drops in a linear fashion until it reaches value 0 in k. Clearly, $f_\ell^* = 1$. For even k and $\ell = \frac{k}{2}$, this function is that for cyclic payoff. The following remark shows how to transfer bounds on the local parameter on members of this familiy to general concave functions. Note that the cases $0 \in \mathcal{D}^*(f)$ and $k - 1 \in \mathcal{D}^*(f)$ describe monotone functions f and have been covered in Section 5.7 already (monotone on $[0, k-1]$, which is sufficient).

5.9.2 Remark. Let $f : [0,k] \longrightarrow \mathbb{R}_{\geqslant 0}$ be concave. Then $\lambda(f,k) \leqslant \lambda(f_\ell, k)$ for all $\ell \in \mathcal{D}^*(f) \setminus \{0, k-1\}$.

Proof. Let $\ell \in \mathcal{D}^*(f) \setminus \{0, k-1\}$. Define $g := \frac{f}{f^*}$, then $g^* = 1$ and $g(\ell) = 1 = f_\ell(\ell)$, so $g^* = f_\ell^*$. By concavity,[5] $f_\ell(x) \leqslant g(x)$ for all $x \in [0,k]$. Applying Remark 5.5.3 two times yields: $\lambda(f,k) = \lambda(\frac{f}{f^*}, k) = \lambda(g,k) \leqslant \lambda(f_\ell, k)$. $\square$

[5]Due to the particular shape of f_ℓ, in order for this concavity argument to work, it is important that f is concave on $[0,k]$ and not only on $[0,k-1]$.

5.9.3 Remark. Let $k \in \mathbb{N}_{\geqslant 2}$ and $f : [0,k] \longrightarrow \mathbb{R}_{\geqslant 0}$ be any function. Assume we have $\alpha \in \mathbb{R}_{>0}$ and $s,t \in [k]$ such that:

$$\forall p \in [k] : f(|s-p|) + f(|t-p|) \geqslant \alpha f^* \tag{5.9.4}$$

Then $\lambda(f,k) \leqslant \frac{2}{\alpha}$.

Proof. Then this is an application of the splitting technique with $\lambda_s = \lambda_t = \frac{1}{\alpha}$ and $\lambda_r = 0$ for all $r \in [k]\backslash\{s,t\}$. □

5.9.1 Maximum on the Left

5.9.5 Theorem. *For $1 \leqslant \ell < \lceil \frac{k}{2} \rceil$ we have $\lambda(f_\ell, k) \leqslant 2$.*

Proof. For each $i \in \mathbb{N}_0$ denote $k_i := \lfloor \frac{k}{2} \rfloor + i$ and also $k' := \lceil \frac{k}{2} \rceil$. Define $s := k'$ and $t := k' - \ell$. Let $p \in [k]$. By Remark 5.9.3 (note that $f_\ell^* = 1$) applied with $\alpha = 1$, it remains to show:

$$\phi(p) := f_\ell(|k'-p|) + f_\ell(|k'-\ell-p|) \geqslant 1$$

The following observations help to make the necessary case distinction:

$$|k'-p| \leqslant \ell \iff k'-\ell \leqslant p \leqslant k'+\ell$$
$$|k'-\ell-p| \leqslant \ell \iff k'-2\ell \leqslant p \leqslant k'$$

Case $1 \leqslant p < k' - 2\ell$:

$$\begin{aligned}\phi(p) &= \frac{k-(k'-p)+k-(k'-\ell-p)}{k-\ell} = \frac{2(k-k'+p)+\ell}{k-\ell} \\ &\geqslant \frac{2(k_0+1)+\ell}{k-\ell} \geqslant \frac{k+\ell}{k-\ell} > 1\end{aligned}$$

Case $k'-2\ell \leqslant p < k'-\ell$:

$$\phi(p) = \frac{k-(k'-p)}{k-\ell} + \frac{k'-\ell-p}{\ell} \geqslant \frac{k-(k'-p)+k'-\ell-p}{k-\ell} = 1$$

Case $k'-\ell \leqslant p \leqslant k'$:

$$\phi(p) = \frac{k'-p+p-k'+\ell}{\ell} = 1$$

Case $k' < p \leqslant k' + \ell$:

$$\phi(p) = \frac{p-k'}{\ell} + \frac{k-(p-k'+\ell)}{k-\ell} \geqslant \frac{p-k'+k-(p-k'+\ell)}{k-\ell} = 1$$

Case $k' + \ell < p \leqslant k$:

$$\begin{aligned}\phi(p) &= \frac{k-(p-k')+k-(p-k'+\ell)}{k-\ell} = \frac{2(k+k'-p)-\ell}{k-\ell} \\ &\geqslant \frac{2k'-\ell}{k-\ell} \geqslant \frac{k-\ell}{k-\ell} = 1\end{aligned}$$

□

5.9.2 Maximum on the Right

Now we will treat the case of $\lceil \frac{k}{2} \rceil \leqslant \ell < k-1$. For this case, we will ultimately give an upper bound in the form of a function of k. That function yields values at most 3 for all relevant k and values at most 2.5 for $k \geqslant 16$.

5.9.6 Theorem. *Let $k \in \mathbb{N}_{\geqslant 2}$ and $\ell \in \mathbb{N}$ with $\lceil \frac{k}{2} \rceil \leqslant \ell < k-1$. Then:*

$$\lambda(f_\ell, k) \leqslant 2 \cdot \frac{(k-\ell)\,\ell + \ell^2}{(k-\ell)\,(k-\ell-1) + \ell^2} = 2 \cdot \frac{k\ell}{2\ell^2 - \ell\,(2k-1) + k\,(k-1)}$$

Proof. We can verify by a routine calculation that the two expressions on the right-hand side are the same. Aiming for an application of Remark 5.9.3, define:

$$\alpha := \frac{(k-\ell)\,(k-\ell-1) + \ell^2}{(k-\ell)\,\ell + \ell^2} = \frac{2\ell^2 - \ell\,(2k-1) + k\,(k-1)}{k\ell}$$

Define $s := \lfloor (1-\alpha)\,\ell + 1 \rfloor \geqslant (1-\alpha)\,\ell$. Since $\frac{k}{2} \leqslant \ell$, we have $k - \ell - 1 < \ell$, hence $\alpha < 1$, hence $s \geqslant 1$. Finally, define $t := \ell + 1$. For this t, we know $f_\ell(|t-p|) = f_\ell(|\ell+1-p|) = \frac{|\ell+1-p|}{\ell}$ for each p. What we need to show thus reduces to:

$$\forall p \in [k] : \; f_\ell(|s-p|) + \tfrac{|(\ell+1)-p|}{\ell} \geqslant \alpha$$

Let $p \in [k]$. We distinguish four cases. Note that by definition, $(\ell+1) - s \geqslant \alpha\ell$.

Case $1 \leqslant p < s$*:* Since $s \leqslant \ell + 1$, we have $s - p \leqslant \ell$. Hence we are in the first case of the definition of f_ℓ:

$$f_\ell(|s-p|) + \tfrac{|(\ell+1)-p|}{\ell} = \tfrac{s-p+(\ell+1)-p}{\ell} \overset{p<s}{>} \tfrac{s+(\ell+1)-2s}{\ell} = \tfrac{(\ell+1)-s}{\ell} \geqslant \alpha$$

Case $s \leqslant p \leqslant \ell + 1$*:* Again, we are in the first case of the definition of f_ℓ, since $p - s \leqslant \ell$. Hence:

$$f_\ell(|s-p|) + \tfrac{|(\ell+1)-p|}{\ell} = \tfrac{p-s+(\ell+1)-p}{\ell} = \tfrac{(\ell+1)-s}{\ell} \geqslant \alpha$$

Case: $\ell + 1 < p \leqslant \ell + s$*:* We are still in the first case of the definition of f_ℓ. Hence:

$$f_\ell(|s-p|) + \tfrac{|(\ell+1)-p|}{\ell} = \tfrac{p-s+p-(\ell+1)}{\ell} \overset{\ell+1<p}{\geqslant} \tfrac{2(\ell+2)-s-(\ell+1)}{\ell} = \tfrac{\ell+3-s}{\ell} > \alpha$$

Case: $\ell + s < p \leqslant k$*:* This is the only case where we are in the second case of the definition of f_ℓ, since $p - s \geqslant \ell$. Hence:

$$\begin{aligned}
& f_\ell(|s-p|) + \tfrac{|(\ell+1)-p|}{\ell} = \tfrac{k-(p-s)}{k-\ell} + \tfrac{p-(\ell+1)}{\ell} \geqslant \alpha \\
\iff & p\,(k-2\ell) - k + \ell\,(s+\ell+1) \geqslant \alpha\,(k-\ell)\,\ell \\
\Longleftarrow & k\,(k-2\ell) - k + \ell\,(s+\ell+1) \geqslant \alpha\,(k-\ell)\,\ell && \text{by } k-2\ell \leqslant 0 \\
\Longleftarrow & k\,(k-2\ell) - k + \ell\,((1-\alpha)\,\ell+\ell+1) \geqslant \alpha\,(k-\ell)\,\ell && \text{by } s \geqslant (1-\alpha)\,\ell \\
\iff & k\,(k-2\ell) - k + \ell\,(\ell+\ell+1) \geqslant \alpha\,((k-\ell)\,\ell+\ell^2) \\
\iff & k\,(k-2\ell) - k + 2\ell^2 + \ell \geqslant \alpha k\ell \\
\iff & k\,(k-1) + \ell\,(2k-1) + 2\ell^2 + \ell \geqslant \alpha k\ell
\end{aligned}$$

The last condition is true by definition of α. □

5.9.7 Corollary. *Let* $k \in \mathbb{N}_{\geqslant 2}$ *and* $\ell \in \mathbb{N}$ *with* $\lceil \frac{k}{2} \rceil \leqslant \ell < k-1$*. Then*

$$\lambda(f_\ell, k) \leqslant \frac{k}{\sqrt{2k(k-1)} - k + 0.5} =: \psi_0(k) \ ,$$

where $\psi_0(k) \to \frac{1}{\sqrt{2}-1} \approx 2.414$ *for* $k \to \infty$*.*

Proof. The asymptotics follow easily since

$$\psi_0(k) = \frac{1}{\sqrt{2}\sqrt{1-\frac{1}{k}} - 1 + \frac{1}{2k}} \ .$$

Define, for fixed k,

$$\psi(\ell) := 2 \cdot \frac{k\ell}{2\ell^2 - \ell\,(2k-1) + k\,(k-1)} \ ,$$

which is the bound from the previous theorem. Treating ℓ as a real variable, we have:

$$\psi'(\ell) = 2 \cdot \frac{k\,(2\ell^2 - \ell\,(2k-1) + k\,(k-1)) - k\ell\,(4\ell - (2k-1))}{\left(2\ell^2 - \ell\,(2k-1) + k\,(k-1)\right)^2} > 0$$

$$\iff k(k-1) - 2\ell^2 > 0$$

$$\iff \ell < \sqrt{\tfrac{k(k-1)}{2}} =: \ell_0$$

Hence ψ attains its maximum in ℓ_0. Writing

$$\psi(\ell) = 2 \cdot \frac{k}{2\ell - (2k-1) + \frac{k\,(k-1)}{\ell}} \ ,$$

a routine calculation shows $\psi(\ell_0) = \psi_0(k)$. □

5.9.8 Corollary. *Let $k \in \mathbb{N}_{\geqslant 2}$ and $\ell \in \mathbb{N}$ with $\lceil \frac{k}{2} \rceil \leqslant \ell < k-1$. Then $\lambda(f_\ell, k) \leqslant 3$ and $\lambda(f_\ell, k) \leqslant 2.5$ for $k \geqslant 16$.*

Proof. Due to $\lceil \frac{k}{2} \rceil \leqslant \ell < k-1$, we have $k \geqslant 4$. Using the function ψ_0 from the previous corollary, we compute $\psi_0(4) < 2.86$ and $\psi_0(16) < 2.5$. Since ψ_0 is non-increasing, the claim follows. □

We summarize the upper-bound results in the following theorem:

5.9.9 Theorem. *Let $f : [0,k] \longrightarrow \mathbb{R}_{\geqslant 0}$ be a concave function.*

(i) If $\mathcal{D}^*(f) \cap \{0, \ldots, \lfloor \frac{k}{2} \rfloor\} \neq \emptyset$, then the price of anarchy is upper-bounded by 2.

(ii) If $\mathcal{D}^*(f) \cap \{\lfloor \frac{k}{2} \rfloor + 1, \ldots, k-1\} \neq \emptyset$, then the price of anarchy is upper-bounded by 3. If additionally, $k \geqslant 16$, then it is upper-bounded by 2.5.

Proof. If $0 \in \mathcal{D}^*(f)$, we may assume that f is non-increasing; an upper bound of 2 follows from Corollary 5.7.5. If $k-1 \in \mathcal{D}^*(f)$, we may assume that f is non-decreasing; an upper bound of 2 follows from Corollary 5.7.3. For the remaining cases, we are allowed to use Remark 5.9.2.

(i) Let $\ell \in \mathcal{D}^*(f) \cap \{1, \ldots, \lfloor \frac{k}{2} \rfloor\} \neq \emptyset$. For $1 \leqslant \ell < \lceil \frac{k}{2} \rceil$, an upper bound of 2 follows from Theorem 5.9.5. For odd k, we have $\lfloor \frac{k}{2} \rfloor < \lceil \frac{k}{2} \rceil$, so all cases are covered. For even k, the only remaining case is $\ell = \lfloor \frac{k}{2} \rfloor = \lceil \frac{k}{2} \rceil$, which corresponds to cyclic payoff, and an upper bound of 2 follows from Proposition 5.7.6.

(ii) Let $\ell \in \mathcal{D}^*(f) \cap \{\lfloor \frac{k}{2} \rfloor + 1, \ldots, k-2\} \neq \emptyset$. For $\lceil \frac{k}{2} \rceil \leqslant \ell < k-1$, the claim follows from Corollary 5.9.8. This covers all cases since $\lceil \frac{k}{2} \rceil \leqslant \lfloor \frac{k}{2} \rfloor + 1$. □

5.10 Open Problems

- Computer experiments suggest that for functions assuming their maximum left of the middle of the spectrum, better bounds than 2 on the local parameter should possible, depending on the exact location of the maximum. However, the evidential basis for this is thin at this time, since enumerative computer experiments are hindered by the exponential search space (cf. the definition in (5.5.1)).

- Apart from the splitting technique, are there further theoretical methods to compute or to approximate the local parameter? Are there faster practical methods for this task than simple enumeration?

- For which functions f and which ranges for parameter k can we find graphs G such that $\mathsf{PoA}(G, f, k) = \lambda(f, k)$? In this work in Section 5.8, we construct such graphs for $f(x) = \min\{x, k-x\}$ and even k (cyclic payoff) and for f of the form $f(x) = ax + b$.

- Computational issues for finding stable colorings should be addressed, in particular the construction of stable colorings for cyclic payoff and odd k in a number of steps being independent of k.

5.11 Appendix

5.11.1 Upper Bound by Mean-Value Argument

We prove a rough bound on the price of anarchy for distance payoff (that is, $f(x) = x$) using a straightforward generalization of a mean-value argument from the proof of [KPR13, Prop. 2]. We believe that not much better bounds than this can be obtained without extending the technique.

Proposition. *The price of anarchy for distance payoff is upper-bounded by* $2k$*.*

Proof. Let c be a stable k-coloring for G and fix a player v. Denote

$$\ell := \max_{t \in [k]} |\{w \in N(v)\,;\; c(w) = t\}| \tag{5.11.1}$$

the cardinality of the largest color class in v's neighborhood. Then clearly $\ell \geqslant \lceil \frac{\deg(v)}{k} \rceil$. Let t be a color where the maximum in (5.11.1) is attained, so there are ℓ neighbors of v with color t. At least one of the two numbers, $t + \lfloor \frac{k}{2} \rfloor$ or $t - \lfloor \frac{k}{2} \rfloor$ is in $[k]$. By choosing an appropriate one of them, v puts distance $\lfloor \frac{k}{2} \rfloor$ between herself and those ℓ neighbors, so each of them will contribute $\lfloor \frac{k}{2} \rfloor$ to v's payoff. In a stable coloring, such as c, player v chooses such color or better, hence

$$\mathsf{SU}(c) \geqslant \sum_{v \in V} \tfrac{\deg(v)}{k} \lfloor \tfrac{k}{2} \rfloor = \tfrac{2m}{k} \lfloor \tfrac{k}{2} \rfloor \geqslant \tfrac{2m}{k} \tfrac{k-1}{2} \ .$$

Here, m is the number of edges in G. Using the trivial upper bound $\mathsf{SU}_{\mathsf{OPT}} \leqslant 2m\,(k-1)$, we obtain:

$$\frac{\mathsf{SU}_{\mathsf{OPT}}}{\mathsf{SU}(c)} \leqslant \frac{2m\,(k-1) \cdot 2k}{2m\,(k-1)} = 2k \qquad \square$$

5.11.2 Counting Neighbors with Same Color

Proposition. *Define f as counting the neighbors with same color, that is,*

$$f : [0, k] \longrightarrow \mathbb{R}_{\geqslant 0},\ x \mapsto \begin{cases} 1 & \textit{if } x = 0 \\ 0 & \textit{otherwise} \end{cases} \ .$$

Then the price of anarchy with respect to f is upper-bounded by k, and this bound is tight.

Proof. The upper bound follows from a mean-value argument like in the proof of the proposition in Section 5.11.1: for each player v, there is one color with which at least $\lceil \frac{\deg(v)}{k} \rceil$ of her neighbors are colored, so choosing this color will yield at least that much payoff for v. Hence $\mathsf{SU}(c) \geqslant \sum_{v \in V} \frac{\deg(v)}{k} = \frac{2m}{k}$ for each stable coloring c. Using the trivial upper bound $\mathsf{SU}_{\mathsf{OPT}} \leqslant 2m$ yields the claim.

For the lower bound, consider the complete bipartite graph $K_{k,k}$. Clearly, $\mathsf{SU}_{\mathsf{OPT}} = 2k^2$, which is attained if all players choose the same color, for example color 1. Enumerate vertices in one partition $\{v_1, \ldots, v_k\}$ and in the other $\{w_1, \ldots, w_k\}$ and define $c(v_i) := c(w_i) := i$ for each $i \in [k]$. Then c is stable since whatever color a player chooses, her payoff is always 1. We have $\mathsf{SU}(c) = 2k$, and the price of anarchy is at least $\frac{2k^2}{2k} = k$. $\square$

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Declaration

I hereby declare that,

1. apart from the guidance of Prof. Dr. Anand Srivastav, the content and design of this thesis is all my own work,
2. parts of this thesis have already been published, in detail:
   (a) "Price of anarchy for graph coloring games with concave payoff". Published in: *AIMS Journal of Dynamics and Games4.1 (2017). Conference version at UECE Lisbon Meetings in Game Theory and Applications 2015.*, pp. 41–58. `doi:10.3934/jdg.2017003`.
   (b) "Swap equilibria under link and vertex destruction". Published in: *Games 8.1 (2017), 18 pages.* `doi:10.3390/g8010014`.
3. this thesis has been prepared subject to the Rules of Good Scientific Practice of the German Research Foundation,
4. a written statement from Prof. Dr. Anand Srivastav identifies my contribution to the above publications.

Elmira Shirazi Sheykhdar Abadi
Kiel, 18.09.2017