Christian-Albrechts-Universität zu Kiel

Portfolio optimization in arbitrary dimensions in the presence of small bid-ask spreads

Dissertation

zur Erlangung des Doktorgrades

der Mathematisch-Naturwissenschaftlichen Fakultät

der Christian-Albrechts-Universität zu Kiel

vorgelegt von

Sergej Mikheev

Kiel, 2017

Erstgutachter:	Prof. Dr. Jan Kallsen
Zweitgutachter:	Prof. Dr. Sören Christensen

Tag der mündlichen Prüfung:	14.03.2018
Zum Druck genehmigt:	

gez. Prof. Dr. Natascha Oppelt, Dekanin

Declaration of Authorship

I, Sergej Mikheev, declare that this thesis titled, 'Portfolio optimization in arbitrary dimensions in the presence of small bid-ask spreads' and the work presented in it are my own. I confirm that:

- This work was done wholly while in candidature for a research degree at this University.
- No part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations and apart from the supervisor's guidance, this thesis is entirely my own work.
- The thesis has been prepared subject to the Rules of Good Scientific Practice of the German Research Foundation.

Signed: ______________________________

Date: ______________________________

CHRISTIAN-ALBRECHTS-UNIVERSITÄT ZU KIEL

Mathematisches Seminar

Abstract

Dissertation

Portfolio optimization in arbitrary dimensions in the presence of small bid-ask spreads

by Sergej Mikheev

This thesis deals with the problem of maximizing the expected utility of terminal wealth in financial markets with an arbitrary number of risky assets in the presence of small bid-ask spreads. The goal is to determine an asymptotically optimal trading strategy and to quantify the asymptotic welfare impact of small proportional fees levied on investor's transactions.

The approach taken in this study relies on the concept of a shadow price transforming the problem of portfolio optimization with proportional costs into a frictionless one. With the help of the shadow price, an asymptotically optimal trading strategy is shown to be a solution to a reflecting stochastic differential equation. The (stochastic) reflecting boundary is characterized as solution to a free-boundary problem. The boundary constrains the motion of the trading strategy to a domain known as the no-trade region. Instead of attempting to find exact solutions, we propose several simple domains as candidates for the no-trade region. Trading strategy to each of the domains is defined as solution to a stochastic Skorohod problem. By adapting the notion of the shadow price, we establish a duality relation between trading strategies and martingale measures for shadow-price processes. This allows us to derive an upper bound on the expected utility generated by each candidate strategy, which provides an estimate of the expected utility of the exact asymptotic optimizer.

Expected utility of each trading strategy together with the associated upper bound are evaluated by means of numerical simulations. The simulations are run on the Black-Scholes model for portfolios of up to 30 risky assets.

Acknowledgements

First and foremost, I would like to express my sincere gratitude to my project adviser Prof. Dr. Jan Kallsen for his continuous support, his guidance and his patience.
I also thank my family and my friends for their moral support in all the time of research and writing of this thesis.
Financial support through DFG-Sachbeihilfe KA 1682/4 – 1 is gratefully acknowledged.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

This thesis is concerned with the problem of maximizing the expected utility of terminal wealth in financial markets with an arbitrary number of risky assets in the presence of small bid-ask spreads. The general structure of the problem can be described as follows. Let u be the utility function of an economic agent investing in $n \geq 1$ risky assets $S^1, \ldots, S^n$ available on the financial market, for a certain time $T > 0$. The investor's trading strategy is modelled by an $\mathbb{R}^n$-valued stochastic process $(\varphi_t)_{t \in [0,T]}$. For $t \in [0,T]$, the random variable $\varphi_t = (\varphi_t^1, \ldots, \varphi_t^n)^\top$ describes the number of shares held by the investor in each of the risky assets. Let $X(\varphi)_t$ denote the investor's payoff if they were to liquidate the portfolio at the time $t \in [0,T]$. Assuming that there is no consumption, find a solution to the maximization problem

$$\max\colon \varphi \longmapsto E[u(X(\varphi)_T)] \ . \tag{1.1}$$

Suppose that we know the optimal solution if the market has no frictions (meaning that there are no fees on transactions) and denote this solution by φ^*. Now suppose that, for each transaction, the investor has to pay a fee proportional to the current value of the stock. In such a market, each stock has a *bid price*, $\underline{S}^j = S^j - \varepsilon_j S^j$, and an *ask price*, $\overline{S}^j = S^j + \varepsilon_j S^j$, with ε_j being a constant proportionality factor for the stock S^j, $j = 1, \ldots, n$. Markets with such proportional fees are referred to as *bid-ask spreads*.

We are interested in the *asymptotic effect* of proportional transaction costs. Put differently, we analyse the situation in which the proportionality factors are assumed to satisfy

$$\varepsilon_j = \mathcal{O}(\varepsilon), \quad j = 1, \ldots, n \ ,$$

with $\varepsilon > 0$ being a small parameter. In this asymptotic setting, our first goal is to find a trading strategy which is optimal *at the leading order* as ε becomes small. Based on existing results (e. g.,[SS94, Rog04]), asymptotic optimality of a trading strategy φ can be characterized as

$$E[u(X(\varphi)_T)] \geq E[u(X(\psi)_T)] + o(\varepsilon^{2/3}) \ ,$$

with ψ denoting any other strategy. Secondly, given that φ is asymptotically optimal, we want to quantify the expected utility loss

$$L = \left| E[u(X(\varphi^*)_T)] - E[u(X(\varphi)_T)] \right| \ . \tag{1.2}$$

Bid-ask spreads naturally reduce the set of (reasonable) trading strategies to the subset of finite-variation processes, since strategies having infinite variation generate infinite transaction costs and lead to bankruptcy. Due to the presence of transaction costs, trades result in money loss and thereby reduce the investor's expected utility. Hence, one aspect of optimality is to avoid unnecessary transactions. However, waiting too long results in large deviations from the frictionless optimizer, which also generates utility loss. Therefore, optimality of a trading strategy can be understood as a trade-off between the two opposite effects. This trade-off manifests itself in the form of a (stochastic) boundary around the frictionless optimizer. The region inside the boundary is known as the *no-trade region*. An optimal trading strategy remains constant inside the no-trade region. When the boundary is crossed, the strategy changes (meaning that trades are carried out by the investor) so as to return to the no-trade region. Finding an optimal boundary and determining the no-trade region is an essential part of the optimization procedure. To solve the problem, one still needs to determine the optimal behaviour of the strategy at the boundary.

To approach the optimization problem, we pass to a frictionless market extension, known as the *shadow-price market*. A shadow price is a fictitious price process

$$\widetilde{S} = (\widetilde{S}^1, \ldots, \widetilde{S}^n)^\top \in \prod_{j=1}^{n} [\underline{S}^j, \overline{S}^j] \ ,$$

evolving inside the bid-ask spread, whose defining property is that the frictionless optimizer with respect to $\widetilde{S}$ coincides with the optimizer with respect to the bid-ask spread. We assume that the shadow price can be represented as a sufficiently well-behaved function of the trading strategy. By using predominantly heuristic arguments, we show that, for small transaction costs, this function and the boundary of the (asymptotically) optimal no-trade region constitute a solution to a free-boundary problem. An optimal trading strategy can be obtained as solution to a reflecting stochastic differential equation with respect to the free boundary. Boundary conditions determining the optimal behaviour of the trading strategy at the boundary of the no-trade region follow from the properties of the shadow price. We are mainly interested in solutions in high dimensions, $n \gg 2$. Solving a free-boundary problem together with a system of reflecting stochastic differential equations with respect to a stochastic boundary[1] in high dimensions is a very challenging task. Moreover, in order to quantify the loss (1.2), the knowledge of the distribution of the trading strategy inside the no-trade region is required. Instead of attempting to solve the problem exactly, we propose several *candidate domains* as possible no-trade regions and estimate the loss generated when trading according to a strategy reflecting at the boundary of the domains. To assess the quality of the approximations, we propose an upper bound on the expected utility the candidates generate, which we derive using basic methods of the convex duality theory. The main idea is to construct a frictionless market, similar to the shadow-price market, and show that, under certain conditions, this market admits an *equivalent martingale measure*. The measure then serves as a *dual variable* generating the upper bound. The conjugate functions through which a duality relation is established are the utility function and the so-called *convex conjugate* of it, the latter being essentially the *Fenchel-Legendre transform* of the former. In short, the general structure of the results can be described as follows. Let φ be a candidate strategy (i. e., trading strategy corresponding to a candidate domain approximating the no-trade region), u the utility function and $\tilde{u}$ its convex conjugate. Let Z_T be the density of an equivalent martingale measure for an appropriately constructed frictionless market extension, as described above. Then, the duality relation is of the form

$$E[u(X(\varphi)_T)] \leq const + E[\tilde{u}(const' Z_T)] \ . \tag{1.3}$$

The left-hand side and the right-hand side of (1.3) are referred to as the *primal functional* and the *dual functional*, respectively. The functionals are evaluated using numerical simulations. Since the interval defined by the duality relation (1.3) necessarily contains the value generated by the exact asymptotic optimizer, simulated values of the primal and dual functionals provide an estimate for the asymptotically optimal value of the optimization problem.

This thesis is organised as follows. Chapter 2 introduces basic definitions and general notational conventions which will be used throughout this work. In Chapter 3, portfolio optimization in markets without frictions is discussed. The chapter aims at reviewing the main results of the frictionless theory and introducing the methodology of the convex-duality approach. Chapter 4 discusses theoretical aspects of optimizing portfolios in the presence bid-ask spreads. Apart from well-known general aspects, the chapter introduces the method of asymptotic expansions of the primal and dual functionals for small transaction costs, an essential tool for obtaining the main results of this thesis. Moreover, it is shown that, asymptotically, the optimization problem is equivalent to a free-boundary problem and the problem of finding a solution to a stochastic differential

[1] The situation is additionally complicated by the fact that the no-trade region turns out to be non-convex and its boundary non-smooth.

equation with reflecting boundary conditions. The chapter is concluded by presenting a few explicitly solvable special cases. In Chapter 5, a simple method for constructing *linear* candidate domains is presented and the notion of a trading strategy with respect to a domain is introduced. The method is then used to select three particular domains based on different heuristic arguments. Furthermore, a construction scheme for associating a dual variable with an arbitrary linear domain is proposed. In Chapter 6, the approximations are implemented to simulate large portfolios within the framework of the Black-Scholes model. First, a large portfolio is selected based on an implicit parameter-estimation scheme using real market data. Subsequently, the discretization method and the implementation algorithm are introduced. The chapter is concluded by presenting and discussing the results of the computer simulations.

Chapter 2

Preliminaries

This chapter contains the most basic definitions including the notational conventions that will be used throughout this thesis. Moreover, the notion of utility functions and their convex conjugates is introduced and their importance is explained.

2.1 Notation

This thesis will deal with continuous multivariate stochastic processes, i. e. continuous processes taking on their values in $\mathbb{R}^n$ for some $n \geq 1$. All vectors are assumed to be column vectors. To denote the components of a multivariate stochastic process, we will use superscript numbers $j \in \{0, \dots, n\}$, whereas subscripts will be used to indicate the time dependence of the process. Thus, e. g., the j-th component of a multivariate process $X = (X^1, \dots, X^n)^\top$ at the time t will be written as X_t^j. The superscript rule applies only to stochastic processes; the dimensional components of all non-stochastic objects (deterministic functions or constants) will be enumerated by using subscripts.
We now give a definition of multi-dimensional stochastic integrals. For the one-dimensional theory, the reader is referred, e.g., to [Pro04] Chapter 4, [JS13] Chapter I.4.

Definition 2.1. Let $X = (X^1, \dots, X^n)^\top$ be a semimartingale and $H = (H^1, \dots, H^n)^\top$ a predictable process. We call H integrable with respect to X and write $H \in L(X)$ if, for each $j = 1, \dots, n$, H^j is integrable with respect to X^j in the sense of the one-dimensional integration theory.

We will use the following notation. Given two *real-valued* processes X and H, $H \in L(X)$, the stochastic Itô integral will be written as

$$H \bullet X = \int H \, dX$$

when we mean the process in general. For a $t \geq 0$, we write

$$H \bullet X_t = \int_0^t H_s \, dX_s \ .$$

Let $H = (H^1, \ldots, H^n)^\top$ and $X = (X^n, \ldots, X^n)^\top$ be two multivariate stochastic processes such that $H \in L(X)$ in the sense of Definition 2.1. The multi-dimensional Itô integral of H with respect to X will be written as

$$X \bullet Y = \sum_{j=1}^{n} X^j \bullet Y^j = \int X \, dY = \sum_{j=1}^{n} \int X^j \, dY^j \ .$$

We remark that this definition of multi-dimensional Itô integrals is not the most general one. For a discussion of this subject, the reader is referred to [JS13] Chapter III.4a, Theorem 4.5 and Example 4.10.
An important role will be played by a particular class of semimartingales, the Itô processes.

Definition 2.2. A stochastic process $X = (X^1, \ldots, X^n)^\top$ is called Itô process if it is of the form

$$X_t^j = X_0^j + \int_0^t b_u^{X,j} du + \sum_{k=1}^{m} \int_0^t \sigma_u^{X,jk} dW_u^k \ ,$$

where $W^1, \ldots, W^m$, $m \geq n$, are standard Brownian motions, and the (predictable) processes b^X and σ^X satisfy

$$\int_0^t \left(\left| b_u^{X,j} \right| + \left(\sigma_u^{X,jk} \right)^2 \right) du < \infty \quad \text{a.s. for all } t \geq 0, \ (j,k) \in \{1, \ldots, n\} \times \{1, \ldots, m\} \ .$$

The matrix σ^X is referred to as the diffusion matrix, and we assume it to have full rank for each (t, ω). The vector-valued process b^X is called the drift process of X.
Let X, Y be two Itô processes. From the definition we see that the quadratic covariation of X and Y, and the quadratic variation of X can be represented as

$$d[X^j, Y^k]_t = c_t^{X,Y;jk} dt \quad \text{and} \quad d[X^j, X^k]_t = c_t^{X,jk} dt \ ,$$

where

$$c^{X,Y} = \sigma^X (\sigma^Y)^\top, \qquad c^X = \sigma^X (\sigma^X)^\top$$

are matrix-valued processes that will be referred to as the *local quadratic covariation* of X and Y, and the *local quadratic variation of* X, respectively.

Remark 2.3. Notice that our Definition 2.2 implies that an Itô process is continuous. This follows from the fact that integrals with respect to continuous local martingales (in our case Brownian motion) are again continuous local martingales for all predictable integrands (cf. [Pro04] Theorem IV.30).

2.2 Market model

Let $\mathcal{X} = (\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space with the filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$, $T \in (0, \infty)$, satisfying the usual conditions of completeness and right-continuity. For $n \in \mathbb{N}$, let $S^0, \dots, S^n \colon [0,T] \times \Omega \to \mathbb{R}$ be Itô processes on $\mathcal{X}$. We let the process S^0 describe a riskless cash account referred to as the *bond*, and $S = (S^1, \dots, S^n)^\top$ model n risky assets also called *stocks*. For $j \in \{0, \dots, n\}$, the process S^j will be referred to as the *price process* of the underlying asset. For notational convenience, the price process of the bond will be normalized to unity, $S_t^0 \equiv 1$. There is no loss of generality through this normalization since the bond can always be chosen as numéraire[1]. Thus, in the following, the bond price will not be mentioned explicitly, and the process $S = (S^1, \dots, S^n)^\top$ will be interpreted as modelling the discounted prices of n risky assets.
The stocks are assumed to be *non-dividend paying*.

2.3 Utility functions

As the name suggests, portfolio optimization is about maximizing the investor's wealth over a certain time interval. If we denote the investor's wealth at the endpoint, T, by X_T, the most naive approach to the optimization problem would therefore probably be to declare $E[X_T]$ to be the objective function. But maximizing the expected wealth directly, in general, leads to risky portfolios being selected as optimal, disregarding the fact that such investments involve a high chance of losing money. In order to account for a certain (investor-dependent) degree of risk aversion, the concept of utility functions is used.
In this work, only a specific class of utility functions, the so-called exponential utility, will be used. Nonetheless, we first provide a general definition of utility functions and collect some of their most important properties, as it will help present some of the conclusions we wish to make in the following in a more compact form. For a detailed discussion of utility functions and their properties in connection with portfolio optimization, the reader is referred to [Sch01].

Definition 2.4. A utility function is a mapping $u \colon \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ with the following properties:

[1] meaning that the price processes of the stocks can be viewed in units of the bond.

(i) u is increasing on $\mathbb{R}$ and continuous on $\{u > -\infty\}$;

(ii) u is differentiable and strictly concave on the interior of $\{u > -\infty\}$, and it satisfies

$$\lim_{x\to\infty} u'(x) = 0, \qquad \lim_{x\to-\infty} u'(x) = \infty \ . \tag{2.1}$$

Remark 2.5. The assumption of a utility function being an increasing, concave function makes good sense from an economic point of view. Whereas the property of being an increasing function is rather obvious (it means that the investor prefers more to less), it is the concavity of u which guarantees that the investor's negative reaction to the risk of losing money, *risk aversion*, is accounted for.

Given the concept of utility functions, the portfolio-optimization problem amounts to maximizing the expected utility of the investor's terminal wealth, $E[u(X_T)]$, over all "allowed" investment strategies. This subject and especially the notion of "allowed" strategies is discussed in a more rigorous way in Chapter 3.

Example 2.1 (Exponential utility)**.** *Since the main part of this thesis involves only one special type of utility functions, we pay special attention to it in this example.*
For a $p > 0$, the function

$$u(x) = -e^{-px}$$

is a utility function in the sense of Definition 2.4 and is referred to as the exponential utility function.

2.3.1 Convex conjugate

The theory of convex optimization or convex duality theory is an important tool for studying stochastic optimization problems (especially in a non-Markovian setting). Given the *primal* maximization problem, the idea is to construct a *dual* minimization problem whose optimal value coincides with (or is sufficiently close to) the optimal value of the primal problem one is actually attempting to solve. The objective functions of the primal and the dual optimization problem are connected via a transformation known as the *convex conjugation.* Sometimes this operation is also referred to as the *Fenchel-Legendre transformation.* The concept of approaching optimization problems by passing to convex duality theory plays an important role for the major part of this work. In the remainder of this subsection, we provide the definition as well as the most important properties of the convex conjugate only in connection with utility functions. For a general treatment of the theory of convex optimization refer, e.g., to [BP10] by V. Barbu and T. Precupanu or to [Roc70] by R. T. Rockafellar.

Definition 2.6. Let u be a utility function. The function $\tilde{u}\colon (0,\infty) \to \mathbb{R}$ defined by

$$\tilde{u}(y) = \sup_{x\in\mathbb{R}}\{u(x) - xy\} \tag{2.2}$$

is called the convex conjugate of u.

The following two propositions follow directly from the properties of the utility functions listed in Definition 2.4 and the definition of the convex conjugate.

Proposition 2.7. *Let u be a utility function and $\tilde{u}$ its convex conjugate. Then,*

$$\begin{gathered}\lim_{y\to 0+} \tilde{u}'(y) = -\infty, \qquad \lim_{y\to 0+} \tilde{u}(y) = \lim_{x\to\infty} u(x)\ , \\ \lim_{y\to\infty} \tilde{u}(y) = \lim_{y\to\infty} \tilde{u}'(y) = \infty\ .\end{gathered} \tag{2.3}$$

Proposition 2.8. *Let u be a utility function and $\tilde{u}$ its convex conjugate. Then,*

$$\tilde{u}(y) = u(x) - xy \ \Leftrightarrow\ x = (u')^{-1}(y),\ y > 0\ , \tag{2.4}$$

$$u(x) = \inf_{y>0}\{\tilde{u}(y) + xy\}\ , \tag{2.5}$$

$$\tilde{u}' = -(u')^{-1}\ . \tag{2.6}$$

Example 2.2. *Let $u(x) = -e^{-px}, p > 0$. Elementary maximization procedure then yields*

$$\tilde{u}(y) = \frac{y}{p}\left(\ln\frac{y}{p} - 1\right)\ . \tag{2.7}$$

Chapter 3

Portfolio Optimization: The Frictionless Case

In this chapter, a detailed discussion of the portfolio-optimization problem without any frictions (transaction costs or otherwise) is presented.

There are two methods that are used to approach stochastic optimization problems. The first one is known as *dynamic programming* and is based on the *Bellman's principle of optimality*. Its adaptation to the setting in which functionals of continuous Markov processes are considered is better known as *stochastic control theory*. This method allows to formulate a non-linear partial differential equation, known as the *Hamilton-Jacobi-Bellman equation (HJB equation)*, for the maximum expected utility being a function of the investor's initial endowment. In the case of a few tractable models, the problem can be solved explicitly for the optimal value as well as the optimizing strategy (optimal control). A short introduction with applications can be found in [Øk92], and for a detailed discussion of the control theory of diffusion processes the reader is referred to [ABG12]. The first explicit solution within the framework of the *Black-Scholes model*, a continuous-time financial model that uses geometric Brownian motion to represent the asset dynamics, was obtained by R. Merton in 1969 [Mer69, Mer71]. The multidimensional Black-Scholes model will be discussed in detail in Example 3.2.4 at the end of this chapter. There are very few tractable models for which an explicit solution can be found. Although potentially very well suited for obtaining explicit solutions, one should keep in mind that the stochastic-control method is only applicable to models having Markovian structure.

Another way to approach a portfolio-optimization problem is to apply martingale techniques relying on the theory of *convex optimization*. This approach amounts to establishing a duality relation between utility maximization and minimization of a convex

functional with respect to an appropriately chosen set of probability measures. It applies in great generality and does not require any specific structure of the underlying model. Important pioneering studies applying the convex duality theory to utility maximization in incomplete markets and continuous-time models are [HP91, KLSX91, KS99, Sch01, DGR+02].
The following discussion of the frictionless problem follows the convex duality approach. Apart from introducing the basic notions of the theory of portfolio optimization, the aim of this chapter is also to provide a compact illustration of the dual problem and its relation to exponential-utility maximization and to put the duality results into a form considered to be more convenient for further applications.

Remark 3.1. All results that are presented in the remainder of this chapter assume the price process to be only a locally bounded semimartingale. This is a more general market model than that introduced in Section 2.2 in which the stock prices are represented by Itô processes. Notice that a continuous semimartingale X with bounded X_0 is always locally bounded. In fact, this is true for all adapted and left-continuous processes. Recall that a process X with bounded X_0 is locally bounded if there exists a sequence of stopping times increasing to infinity, $(\tau_n)_{n\in\mathbb{N}}$, $\tau_n \nearrow \infty$ a.s., and a sequence $(C_n)_{n\in\mathbb{N}}$ in $(0,\infty)$ such that $|X_t^{\tau_n}| \leq C_n$ uniformly in (t,ω) for each n. For adapted, left-contiuunous (and thus predictable) processes, this can be realized by taking $\tau_n = \inf\{t \geq 0\colon |X_t| \geq n\}$ and $C_n = n$.

3.1 Basic notions from portfolio theory

In this section, the essentials of the general portfolio theory are presented. Self-financing and admissible trading strategies are introduced, and the notion of arbitrage absence is discussed.

Definition 3.2. For a given (discounted) price process $S = (S^1, \ldots, S^n)^\top$, a *trading strategy* (also referred to as *portfolio* or *policy*) is an $\mathbb{F}$-predictable $\mathbb{R}^{n+1}$-valued process $(\varphi^0, \varphi)^\top \colon [0,T] \times \Omega \to \mathbb{R} \times \mathbb{R}^n$ with $\varphi = (\varphi^1, \ldots, \varphi^n)^\top \in L(S)$.

For $t \geq 0$, the random variable φ_t^j, $j \in \{0, \ldots, n\}$, describes the *number of shares* of the security described by S^j held at time t.

Definition 3.3. For a price process S and a strategy $(\varphi^0, \varphi)^\top$, the stochastic process $(V_t)_{0\leq t\leq T}$,

$$V_t = \varphi_t^0 + \sum_{j=1}^{n} \varphi_t^j S_t^j \ , \tag{3.1}$$

is called the *wealth process*.

The random variable V_t then is the monetary value of the portfolio at time $t \geq 0$.

Definition 3.4. A portfolio $(\varphi^0, \varphi)^\top$ is called *self-financing* if

$$V_t = V_0 + \int_0^t \varphi_u \, dS_u = V_0 + \sum_{j=1}^n \int_0^t \varphi_u^j \, dS_u^j \ . \tag{3.2}$$

Thus, when there is no income from outside, the portfolio is self-financing if and only if the change of the investor's wealth is solely due to the change in asset prices. Note that in this case, buying new assets is financed only by selling assets (including the bond) from the portfolio.

Remark 3.5. A self-financing portfolio $(\varphi^0, \varphi)^\top$ can be identified with its part φ describing only the stock investment. More precisely, given the initial capital V_0 and an S-integrable, $\mathbb{R}^n$-valued process φ, there is a unique real-valued process φ^0,

$$\varphi_t^0 = V_0 + \int_0^t \varphi_u \, dS_u - \varphi_t^\top S_t \ ,$$

such that $(\varphi^0, \varphi)^\top$ is a self-financing strategy defining a wealth process whose initial value equals V_0.

Assumption A1. Unless otherwise stated, all trading strategies are assumed to be self-financing. Moreover, by Remark 3.5, we will often drop φ^0.

3.2 Portfolio optimization

This section gives a detailed introduction to the frictionless portfolio-optimization theory. For the special case of exponential utility, an appropriate dual minimization problem is formulated and used to establish an optimality criterion for the primal problem.

Definition 3.6 (Portfolio-optimization problem)**.** Let u be a utility function. Let S be the price process and $x = \sum_{j=0}^n x_j$ be the investor's initial endowment with x_0 and $x_1, \dots, x_n$ corresponding to their holdings in bond and stocks $1, \dots, n$, respectively. Define $\mathcal{A}(x) \subset L(S)$ to be a subset of predictable and S-integrable processes φ whose wealth process satisfies $V(\varphi)_0 = V_0 = x$. Note that this implies that $x_0 = \varphi_0^0$ and $x_j = \varphi_0^j S_0^j$, $j = 1, \dots, n$, with φ^0 being the unique process such that (φ^0, φ) is self-financing. In this setting, we first define the so-called *value function*, v,

$$v(x) = \sup_{\varphi \in \mathcal{A}(x)} E\left[u\left(V(\varphi)_T\right)\right] \ . \tag{3.3}$$

The value function thus describes the *maximum expected utility of terminal wealth* when optimizing over the set $\mathcal{A}(x)$.
The *portfolio-optimization problem* then amounts to finding the value function and determining a strategy φ^* such that

$$v(x) = E\left[u\left(V(\varphi^*)_T\right)\right] \ .$$

Put differently, one seeks an optimizer φ^* with respect to the set $\mathcal{A}(x)$ as well as the expected utility associated with it.

The necessity of restricting the set of strategies as well as the properties of $\mathcal{A}(x)$ are discussed in more detail in the following subsection.

3.2.1 Arbitrage and admissibility

The set $\mathcal{A}(x)$ appearing in the above definition can be characterized in a rather obvious, intuitive way: it must not allow for arbitrage opportunities and must be large enough to contain the optimizer. Trading strategies in $\mathcal{A}(x)$ are then called *admissible*. It was first noticed by Harrison et al. (cf. [HK79, HP81]) that by taking $\mathcal{A}(x)$ to be the whole set $L(S)$ one already fails to satisfy the first requirement: there exist arbitrage opportunities such as doubling strategies and alike investment schemes. A way out is to consider the set of those strategies in $L(S)$ whose wealth process is uniformly bounded from below. More precisely, one considers trading strategies $\varphi \in L(S)$ that, in addition to the initial condition $x = V(\varphi)_0$, satisfy

$$\forall t \geq 0\, \exists K > 0 \colon V(\varphi)_t \geq -K \quad \text{a.s.} \ . \tag{3.4}$$

If (3.4) is taken as the definition of admissibility, it can be easily seen that a sufficient condition for the absence of arbitrage opportunities among the strategies fulfilling the above boundedness condition is the existence of an *equivalent local martingale measure (ELMM)* for the price process S. To see this, note that an arbitrage strategy φ can be characterized as

$$(i)\ V(\varphi)_0 = 0, \quad (ii)\ P\left(V(\varphi)_T \geq 0\right) = 1, \quad (iii)\ P\left(V(\varphi)_T > 0\right) > 0 \ . \tag{3.5}$$

Now let $Q \sim P$ be such that the price process S is a Q-local martingale. Then, $\varphi \bullet S$ is a continuous Q-local martingale for each $\varphi \in L(S)$ since S is continuous (cf. [Pro04]

Theorem IV.30)[1]. Using (3.4) and applying Fatou's lemma yields that $\varphi \bullet S$ is a Q-supermartingale. Now let φ satisfy (ii) and (iii) of (3.5). Then, setting $V(\varphi) = \varphi \bullet S$, we obtain

$$V(\varphi)_0 \geq E_Q\left[V(\varphi)_T\right] = E_Q\left[V(\varphi)_T I_{\{V(\varphi)_T > 0\}}\right] > 0 \tag{3.6}$$

since the equivalence $Q \sim P$ implies that (3.5) holds for Q as well. Note that the first inequality is the supermartingale property of $\varphi \bullet S$.

Remark 3.7. There is a deep connection between equivalent martingale measures and arbitrage. It turns out that by relaxing the definition of arbitrage (3.5), one can obtain not only a necessary but also a sufficient condition for the existence of an ELMM for a price process S being a locally bounded semimartingale. The proper notion of arbitrage is called *"No Free Lunch with Vanishing Risk" (NFLVR)*. The reader is referred to the work by F. Delbaen and W. Schachermayer [DS94] (Theorem 1.1 and Corollary 1.2) in which the following remarkable result, known as *The Fundamental Theorem of Asset Pricing (FTAP)*, is proved: *For a price process S being a locally bounded semimartingale there exists an ELMM if and only if the NFLVR condition is fulfilled.*
For an extension to the case of unbounded asset prices see [DS96].

Recall the second requirement that the set of admissible strategies must be large enough to contain the optimizer. It turns out that defining admissibility by (3.4) works well only in the case of utility functions defined on $[0, \infty)$. This situation is analysed in detail, e. g., in [KS99]. However, if negative wealth is allowed, and this will be the only case of interest, one is forced to work with utility functions taking on finite values on the whole real line, $\mathbb{R}$. As pointed out, e.g., in [Sch01], for this class of utility functions, the condition of uniform boundedness (3.4) is too restrictive since already in the case of the Black-Scholes model with exponential utility, the optimizer in the sense of (3.3) fails to fulfil this condition. Although in [Sch01] the authors find a way to resolve this issue, we will take a different approach and introduce a different notion of admissibility as it was done in [DGR$^+$02, KS02]. We start by formulating a sufficient optimality condition which suggests how the set $\mathcal{A}$ could look like.

Proposition 3.8. *Let x be the investor's initial endowment, S the price process, and u a utility function. Assume that there exists a trading strategy φ such that*

$$\frac{dQ}{dP} = \frac{u'(x + \varphi \bullet S_T)}{E\left[u'(x + \varphi \bullet S_T)\right]}$$

[1]The integral $\varphi \bullet S$ can also be shown to be a local martingale for a local martingale S which is not necessarily continuous, and all strategies satisfying (3.4), cf. [Ans94], Proposition 3.3 .

is the density of a probability measure $Q \sim P$ with respect to which $\varphi \bullet S$ becomes a martingale. Then, the strategy φ is optimal with respect to the set

$$\mathcal{A}(x) = \{\psi \in L(S) \colon V(\psi) = x + \psi \bullet S \text{ is a } Q\text{-martingale}\} \ . \tag{3.7}$$

Proof. Since u is concave and differentiable, we have $u(y) \leq u(x) + u'(x)(y-x)$. Thus, for a strategy $\psi \in \mathcal{A}$,

$$\begin{aligned} E\left[u(x+\varphi \bullet S_T)\right] &\geq E\left[u(x+\psi \bullet S_T)\right] + E\left[u'(x+\varphi \bullet S_T)((\varphi-\psi)\bullet S_T)\right] \\ &= E\left[u(x+\psi \bullet S_T)\right] + E\left[u'(x+\varphi \bullet S_T)\right] E_Q\left[\varphi \bullet S_T - \psi \bullet S_T\right] \\ &= E\left[u(x+\psi \bullet S_T)\right] \ . \end{aligned}$$

The last term in the second line vanishes due to the martingale property demanded in the definition of $\mathcal{A}$. □

Remark 3.9. Defining the set of admissible strategies by (3.7) will, of course, exclude all arbitrage opportunities. This can be easily seen from (3.6) where the first relation then holds as an equality due to the martingale property.
Proposition 3.21 will show that the above condition is satisfied in the case of the exponential utility. It can then even be shown to be necessary if the measure Q happens to have some additional properties (cf. Remark 3.22).

In order to show that, in the case of exponential utility, the above condition holds, the problem of utility maximization will be linked to a (dual) minimization problem which, in the special case of interest, can be shown to have a unique solution.

3.2.2 The dual problem

A key role in establishing a duality result for the (primal) optimization problem (Definition 3.6) is played by the convex conjugate and a trivial consequence of its definition. From Definition 2.2 we immediately see that the convex conjugate is an upper-bound for the utility function,

$$u(x) \leq xy + \tilde{u}(y), \quad x \in \mathbb{R},\ y > 0 \ , \tag{3.8}$$

with (3.8) holding as an equality iff $x = (u')^{-1}(y)$, as already stated in Proposition 2.4. Some preparation needs to be done before the dual problem can be introduced.

Definition 3.10. By

$$\mathcal{M}_{loc} = \{Q \ll P \colon S \text{ is a } Q\text{-local martingale}\} \ , \qquad \mathcal{M}^e_{loc} = \{Q \sim P \colon S \text{ is a } Q\text{-local martingale}\} \ , \tag{3.9}$$

we denote the sets of all local martingale measures for the price process S that are absolutely continuous and equivalent with respect to the real-world measure P, respectively.

Let $x \in \mathbb{R}$ denote the investor's initial endowment, $Q \in \mathcal{M}_{loc}$ and $L^1(Q)$ be the set of all Q-integrable random variables. Define

$$U_Q(x) = \sup_{\eta \in M^x_Q} E\left[u(\eta)\right], \qquad M^x_Q = \left\{\eta \in L^1(Q) \colon E_Q[\eta] \le x,\ E[u(\eta)^-] < \infty\right\} \ . \tag{3.10}$$

Remark 3.11. Given the interpretation of $E_Q[\eta]$ as the price of the random payoff η with respect to the measure Q, the quantity $U_Q(x)$ can be understood as the maximum expected utility of those random payoffs satisfying the budget constraint $E_Q[\eta] \le x$, that is all affordable payoffs in the sense that their Q-price is at most equal to the investor's starting capital.

The next lemma provides a useful representation of the supremum utility U_Q.

Lemma 3.12 ([GR01], Lemma 4.1)**.** *Let $Q \in \mathcal{M}_{loc}$ and*

$$\forall \lambda > 0 \colon E_Q\left[(u')^{-1}\left(\lambda \frac{dQ}{dP}\right)\right] < \infty \ .$$

Then

$$U_Q(x) = \min_{\lambda > 0}\left\{\lambda x + E\left[\tilde{u}\left(\lambda \frac{dQ}{dP}\right)\right]\right\} \ . \tag{3.11}$$

Remark 3.13. The minimum in (3.11) is attained. To see this, note that, by Proposition 2.4,

$$\frac{d}{d\lambda}\left\{\lambda x + E\left[\tilde{u}\left(\lambda \frac{dQ}{dP}\right)\right]\right\} = 0$$

is equivalent to $x = E_Q\left[(u')^{-1}\left(\lambda \frac{dQ}{dP}\right)\right]$. It is assumed that this expectation is finite. It then follows from the properties of the utility function and of the convex conjugate that $E_Q\left[(u')^{-1}\left(\lambda \frac{dQ}{dP}\right)\right]$ is a continuous and monotonically decreasing function of λ. Moreover, by Proposition 2.4, $\tilde{u}'\left((0,\infty)\right) = \mathbb{R}$. Thus, there is a unique λ solving the above equation and minimizing the right-hand side of (3.11).

Definition 3.14 (The dual problem)**.** Let U_Q be as in the equation (3.10) above. The problem of finding

$$U(x) = \inf_{Q \in \mathcal{M}_{loc}} U_Q(x) \ . \tag{3.12}$$

will be referred to as the *dual problem*.

3.2.3 Duality and optimality

The goal of this subsection is to demonstrate that the minimization problem (3.12) is dual to the utility-maximization problem formulated in Definition 3.6 and to provide necessary conditions to ensure that there is no duality gap.
In the following, we consider only the exponential utility function, $u(x) = -e^{-px},\ p > 0$. As already mentioned in Example 2.2, the convex conjugate of the exponential utility reads $\tilde{u}(y) = \frac{y}{p}\left(\ln\frac{y}{p} - 1\right)$.

Definition 3.15 (Relative entropy)**.** For a probability measure $Q \ll P$, the expectation

$$H(Q,P) = E\left[\frac{dQ}{dP}\ln\frac{dQ}{dP}\right] \tag{3.13}$$

is referred to as the *relative entropy of Q with respect to P*.

Remark 3.16. By applying Jensen's inequality to the convex function $f(x) = x\ln x$, one can easily show that the relative entropy is always non-negative:

$$H(Q,P) = E\left[f\left(\frac{dQ}{dP}\right)\right] \geq f\left(E\left[\frac{dQ}{dP}\right]\right) = \ln(1) = 0\ .$$

Proposition 3.17. *Let $Q \in \mathcal{M}_{loc}$ be such that $H(Q,P) < \infty$. Then, in the case of the exponential utility, U_Q has the following representation:*

$$U_Q(x) = -e^{-px-H(Q,P)}\ . \tag{3.14}$$

Proof. We first note that $\tilde{u}'(y) = \frac{1}{p}\ln\frac{y}{p}$ and

$$\begin{aligned} E_Q\left[(u')^{-1}\left(\lambda\frac{dQ}{dP}\right)\right] &= -E_Q\left[\tilde{u}'\left(\lambda\frac{dQ}{dP}\right)\right] = -\frac{1}{p}E_Q\left[\ln\left(\frac{\lambda}{p}\frac{dQ}{dP}\right)\right] \\ &= -\frac{1}{p}\ln\frac{\lambda}{p} - \frac{1}{p}H(Q,P)\ . \end{aligned}$$

Since we assume that $H(Q,P) < \infty$, the requirements of Lemma 3.12 are met, and we obtain $U_Q(x)$ by minimizing over λ.

$$\frac{d}{d\lambda}\left\{\lambda x + E\left[\tilde{u}\left(\lambda\frac{dQ}{dP}\right)\right]\right\} = x + E_Q\left[\tilde{u}'\left(\lambda\frac{dQ}{dP}\right)\right] = x + \frac{1}{p}\ln\frac{\lambda}{p} + \frac{1}{p}H(Q,P)\ .$$

Equating this to zero yields the minimizer

$$\lambda_* = pe^{-px-H(Q,P)}\ . \tag{3.15}$$

By substituting λ_*, we obtain

$$\lambda_* x + E\left[\tilde{u}\left(\lambda_* \frac{dQ}{dP}\right)\right] = pxe^{-px-H(Q,P)} + e^{-px-H(Q,P)} E_Q\left[\ln \frac{dQ}{dP} - px - H(Q,P) - 1\right],$$

and the equation (3.14) follows. □

The next proposition due to Frittelli (2000) basically shows that the dual problem (3.12) for the exponential utility admits a unique solution. A similar result can also be found in [GR01, GR02].

Proposition 3.18 ([Fri00], Theorem 1 and Theorem 2)**.** *Let $Q \in \mathcal{M}_{loc}$ be a local martingale measure for the locally bounded price process S and $H(Q,P) < \infty$. Then, there exists a unique $Q_* \in \mathcal{M}_{loc}$ such that $H(Q_*, P) = \min_{Q \in \mathcal{M}_{loc}} H(Q,P)$.*
If $H(Q,P) < \infty$ holds for a $Q \in \mathcal{M}^e_{loc}$, then $Q_* \sim P$.

The minimizer Q_* is called the *Minimal Entropy Martingale Measure (MEMM).*
In the following, a standing assumption will be

Assumption A2.

$$\exists\, Q \in \mathcal{M}^e_{loc} \colon H(Q,P) < \infty \ .$$

An application of the above results to the dual problem will now be summarized in the following

Proposition 3.19. *Under the Assumption A2, there exists a unique MEMM $Q_* \sim P$ and, for u being the exponential utility function and $\tilde{u}$ its convex conjugate, the dual problem stated in Definition 3.14 admits a unique solution given through*

$$\begin{aligned} U(x) &= \min_{Q \in \mathcal{M}^e_{loc}} \min_{\lambda > 0} \left\{ \lambda x + E\left[\tilde{u}\left(\lambda \frac{dQ}{dP}\right)\right]\right\} \\ &= \min_{Q \in \mathcal{M}^e_{loc}} \left\{-e^{-px-H(Q,P)}\right\} = -e^{-px-H(Q_*,P)} \ . \end{aligned} \tag{3.16}$$

The next result shows that the density of the MEMM has a stochastic integral representation. This will be sufficient to establish a connection to the utility maximization problem.

Proposition 3.20. *Let $Q_* \sim P$ be the MEMM. Then, there exists a process $\overline{\varphi} \in L(S)$ such that $\overline{\varphi} \bullet S$ is a Q_*-martingale and the density of Q_* with respect to P has the representation*

$$\frac{dQ_*}{dP} = \exp\left(H(Q_*, P) - \int_0^T \overline{\varphi}_t \, dS_t\right) \tag{3.17}$$

Proof. Consider the function $f(x) = x \ln x$. Then, $f'(x) = 1 + \ln x$, and $-f'\left(\frac{dQ_*}{dP}\right)$ is bounded from below. The assertion then follows from [GR01] Theorem 2.2 combined with [Jac92] Theorem 3.4 since these theorems guarantee the existence of a $c \in \mathbb{R}$ and a strategy $\overline{\varphi} \in L(S)$ such that

$$f'\left(\frac{dQ_*}{dP}\right) = c - \overline{\varphi} \bullet S_T$$

with $\overline{\varphi} \bullet S$ being a Q_*-martingale. Since $f'\left(\frac{dQ_*}{dP}\right) = 1 + \ln \frac{dQ_*}{dP}$, taking the expectation with respect to the MEMM Q_* on both sides of the equation determines the constant and yields the result. □

The following proposition summarizes all results and shows the existence of an optimal solution to the utility maximization problem.

Proposition 3.21. *Assume there exists a measure $Q \in \mathcal{M}^e_{loc}$ such that $H(Q, P) < \infty$. Let $Q_* \sim P$ be the unique solution to the dual problem* (3.12)*, whose existence is ensured by Proposition 3.19. Then, there exists a strategy $\varphi^* \in L(S)$ maximizing the expected exponential utility of terminal wealth over the set*

$$\mathcal{A}(x) = \{\varphi \in L(S) \colon V(\varphi) = x + \varphi \bullet S \textit{ is a } Q_*\textit{-martingale}\} \ , \tag{3.18}$$

i. e.,

$$v(x) = \max_{\varphi \in \mathcal{A}(x)} E\left[u(V(\varphi)_T)\right] = E\left[-e^{-px - p\int_0^T \varphi_t^* \, dS_t}\right] \ , \tag{3.19}$$

with x being the investor's initial endowment. Moreover, the following relation between the primal maximizer φ^* and the dual minimizer Q_* holds:

$$\frac{dQ_*}{dP} = \frac{u'(x + \varphi^* \bullet S_T)}{E\left[u'(x + \varphi^* \bullet S_T)\right]} \ . \tag{3.20}$$

Proof. Let $\varphi \in L(S)$, $Q \sim P$ and $\lambda > 0$. From the properties of the utility function and its convex conjugate, one obtains (see Definition 2.2 and Proposition 2.8)

$$u(x + \varphi \bullet S_T) \leq \lambda \frac{dQ}{dP}(x + \varphi \bullet S_T) + \tilde{u}\left(\lambda \frac{dQ}{dP}\right) \ . \tag{3.21}$$

For a measure $Q \sim P$, define the set

$$\mathcal{A}_Q(x) = \{\varphi \in L(S) \colon x + \varphi \bullet S \text{ is a } Q\text{-martingale}\} \ .$$

Then, for each $\varphi \in \mathcal{A}_Q(x)$, the inequality (3.21) implies

$$E[u(x + \varphi \bullet S_T)] \leq \lambda x + E\left[\tilde{u}\left(\lambda \frac{dQ}{dP}\right)\right] \ . \tag{3.22}$$

Under the assumption

$$\mathcal{M}^e_{loc}(S) \cap \{Q \sim P \colon H(Q,P) < \infty\} \neq \emptyset \ ,$$

the right-hand side of (3.22) satisfies

$$\min_{Q\in\mathcal{M}^e_{loc}(S)} \min_{\lambda>0} \left\{ \lambda x + E\left[\tilde{u}\left(\lambda \frac{dQ}{dP}\right)\right]\right\} = \lambda_* x + E\left[\tilde{u}\left(\lambda_* \frac{dQ_*}{dP}\right)\right] \tag{3.23}$$

$$= -e^{-px-H(Q_*,P)} \tag{3.24}$$

by Proposition 3.17, 3.19, with the unique minimizers

$$\begin{aligned} Q_* &= \operatorname*{arg\,min}_{Q\in\mathcal{M}^e_{loc}(S)} H(Q,P) \ , \\ \lambda_* &= pe^{-px-H(Q_*,P)} \ . \end{aligned} \tag{3.25}$$

The integral representation of the density $\frac{dQ_*}{dP}$, which is presented in Proposition 3.20, ensures the existence of a strategy $\overline{\varphi} \in \mathcal{A}_{Q_*}(x)$ such that

$$\lambda_* \frac{dQ_*}{dP} = p\exp\left(-px - \overline{\varphi} \bullet S_T\right) = u'(x + \varphi^* \bullet S_T), \quad \varphi^* = \frac{\overline{\varphi}}{p} \ .$$

This is the relation between the the dual variable $\lambda_* \frac{dQ_*}{dP}$ and the primal variable $x + \varphi^* \bullet S_T$ that, by Proposition 2.8, guarantees their optimality and uniqueness and ensures that there is no duality gap.

We define

$$\mathcal{A}(x) = \mathcal{A}_{Q_*}(x) \ .$$

The value function in (3.19) reads as

$$v(x) = \max_{\varphi\in\mathcal{A}(x)} E\left[-e^{-px-p\int_0^T \varphi_t\, dS_t}\right] = -e^{-px-H(Q_*,P)} \ .$$

By using

$$e^{-H(Q_*,P)} = E\left[e^{-p\int_0^T \varphi^*_t\, dS_t}\right] \ ,$$

one obtains the right-hand side of the last equality in (3.19). Finally, Equation (3.20) follows from $\lambda_* = E\left[u'(x + \varphi^* \bullet S_T)\right]$. □

Remark 3.22. Proposition 3.21 shows that the sufficient condition 3.8 can be extended to give a necessary and sufficient optimality condition. Let u be the exponential utility function and x the investor's initial wealth. Assume there exists a measure $Q \in \mathcal{M}^e_{loc}$ such that $H(Q,P) < \infty$, and let $Q_* \sim P$ be the (unique) MEMM. Define the set $\mathcal{A}(x)$ of admissible strategies by (3.18). Then, a trading strategy φ maximizes the expected *exponential* utility of terminal wealth over the set $\mathcal{A}(x)$ if and only if

$$\frac{dQ_*}{dP} = \frac{u'(x + \varphi \bullet S_T)}{E\left[u'(x + \varphi \bullet S_T)\right]} \ .$$

The sufficiency of this condition has already been shown in Proposition 3.8. As for the necessity, if we assume a strategy φ to be optimal, it must already satisfy

$$\lambda_*(Q_*)\frac{dQ_*}{dP} = u'(x + \varphi \bullet S_T) \ .$$

If it was not the case, then, by Proposition 3.21, we could find a strategy which would satisfy this relation and whose expected utility would be equal to the minimum dual upper bound which, in turn, would dominate the expected utility of φ and thus contradict the optimality assumption.

Delbaen et al. 2002, Kabanov and Stricker 2002, Stricker 2002 solve the problem of maximizing exponential utility using a similar duality approach and show that the optimal wealth process, $V(\varphi^*) = \varphi^* \bullet S$, has further important properties which are summarized in the next proposition.

Proposition 3.23. *Let φ^* be the maximizer of the expected exponential utility. Then,*

(a) ([DGR+02] Theorem 3; [KS02] Theorem 2.1.(b)) *the process $(\varphi^* \bullet S_t)_{t\in[0,T]}$ is a Q-martingale for all $Q \in \mathcal{M}_{loc}$ such that $H(Q,P) < \infty$;*

(b) ([Str02] Theorem 5) *there exists a sequence of bounded simple trading strategies, $(\varphi_k)_{k\in\mathbb{N}}$, such that*

$$E\left[u(x + \varphi_k \bullet S_T)\right] \longrightarrow E\left[u(x + \varphi^* \bullet S_T)\right] \ . \tag{3.26}$$

A simple strategy has the form $X = \sum_{i=0}^{N} h_i I_{(\tau_i,\tau_{i+1}]}$ with $N \geq 1$, $0 = \tau_0 \leq \ldots \leq \tau_{N+1} \leq T$ stopping times and an $\mathbb{R}^n$-valued and $\mathcal{F}_{\tau_i}$-measurable random variables h_i, $0 \leq i \leq N$. In practice, be it actual trading or computer simulations, only such strategies are of relevance. Their financial meaning is that they represent linear combinations of buy-and-hold strategies: at the time τ_i the investor buys h_i units of some asset and holds this amount until the time τ_{i+1}. This fact emphasizes the importance of item (b) of the above theorem since it ensures that the optimizer can be attained on a sequence of such buy-and-hold strategies.

Remark 3.24. As pointed out in [Sch03], the class of martingales does not suffice to cover the case of more general utility functions. Instead, the class of *supermartingales* turns out to be the right choice for which the optimality can be proven even for price processes being general semimartingales. These results can be found, e.g., in [Sch03, DS06, BF07, BF08, Be11]

3.2.4 Example: Black-Scholes model

The Black-Scholes model is a special case of the market model introduced in Section 2.2 of Chapter 2, in which the bond is modelled by the deterministic function

$$S_t^0 = S_0^0 e^{\rho t}, \quad t \in [0,T] \ ,$$

with $\rho \geq 0$ being the constant interest rate, and the n risky assets $S^1, \ldots, S^n$ are represented by Itô processes with the dynamics

$$dS_t^j = S_t^j \Big(b_j dt + \sum_{k=1}^{n} \sigma_{jk} dW_t^k \Big), \quad t \in [0,T], \quad j = 1, \ldots, n \ , \tag{3.27}$$

with $W = (W^1, \ldots, W^n)^\top$ being a standard n-dimensional Brownian motion, $b = (b^1, \ldots, b^n)^\top \in \mathbb{R}^n$ and $(\sigma_{jk})_{jk=1,\ldots,n}$ an invertible matrix with positive entries. Put differently, $S = (S^1, \ldots, S^n)^\top$ is an n-dimensional Itô process with the drift vector $b^S = \operatorname{diag}(S^1, \ldots, S^n) b$ and the diffusion matrix $\sigma^S = \operatorname{diag}(S^1, \ldots, S^n)\sigma$ (cf. Definition 2.2). By applying Itô's formula to $\ln S_t^j$, one can easily obtain a solution to the stochastic differential equations (3.27), and it reads as

$$S_t^j = S_0^j \exp\Big((b_j - \frac{\sigma_j^2}{2}) t + \sum_{k=1}^{n} \sigma_{jk} W_t^k \Big), \quad t \in [0,T], \quad j = 1, \ldots, n \ , \tag{3.28}$$

with $\sigma_j = \sqrt{c_{jj}}$ and $c = \sigma\sigma^\top$ being the volatility of the j-th stock and the covariance matrix of the log-price $\ln S_t^j$, respectively. It is common practice to choose the cash account as the numéraire. In the Black-Scholes setting, the discounted price processes of the assets are then obtained simply by changing

$$S_0^j \mapsto \frac{S_0^j}{S_0^0}, \qquad b_j \mapsto b_j - \rho, \quad j = 1, \ldots, n \ . \tag{3.29}$$

In the following, for ease of notation, we take $S_t^0 \equiv 1$ and assume the discounting transformation (3.29) to have already been performed. We now turn to discussing the optimal trading strategy for maximizing the expected exponential utility of terminal wealth. The Black-Scholes model is one of the very few market models in which the problem of maximizing the expected utility can actually be solved explicitly using the techniques of the stochastic control theory. To derive the optimal policy, one follows a rather standard procedure of solving the HJB equation and proving that the solution is indeed an optimal control. We will forgo presenting this derivation here since it is purely technical and can be found in standard literature; see, e. g., the original papers by R. C. Merton [Mer69, Mer71] or Example 11.2.5 of Chapter 11 in [Øk92]. Instead, we take the Merton portfolio as a candidate and discuss its optimality using the duality result of Proposition 3.21 in the form as it is presented in Remark 3.22. This approach is much more instructive as it helps gain insight into the structure of the model.

Let the price process $S = (S^1, \dots, S^n)^\top$ be as defined in (3.28) and let $u(x) = -e^{-px}$, $p > 0$, be the utility function. Then, the strategy

$$\varphi_t^j = \frac{(c^{-1}b)_j}{pS_t^j} \tag{3.30}$$

maximizes the expected utility of terminal wealth over the set $\mathcal{A}$ of admissible strategies defined in (3.18). To see this, we compute all necessary ingredients for the construction of an equivalent measure, and we start with the wealth process of the candidate strategy. Define $r = \sigma^{-1}b$. Then,

$$\varphi \bullet S_t = \frac{1}{p}\sum_{j=1}^{n}(c^{-1}b)_j \int_0^t b_j du + \sum_{k=1}^{n}\sigma_{jk} dW_u^k = \frac{1}{p}\Big((r^\top r)t + r^\top W_t\Big) \ . \tag{3.31}$$

With $u'(x) = pe^{-px}$, we obtain

$$\begin{aligned} u'(x + \varphi \bullet S_T) &= pe^{-px}\exp\Big(-Tr^\top r - r^\top W_T\Big) = pe^{-px - \frac{T}{2}r^\top r}\exp\Big(-\frac{T}{2}r^\top r - r^\top W_T\Big) \\ &= pe^{-px - \frac{T}{2}r^\top r}\mathcal{E}\big(-r^\top W\big)_T \ , \end{aligned}$$

with $\mathcal{E}(X) = e^{X - \frac{1}{2}[X]}$ being the stochastic exponential. Since $r^\top W$ is a continuous local martingale, $\mathcal{E}\big(-r^\top W\big)$ is a uniformly integrable martingale by Novikov's Criterion (cf. Theorem 41 in [Pro04]). Hence, we define our candidate measure $dQ = Z_T dP$ by

$$Z_T = \frac{u'(x + \varphi \bullet S_T)}{E\left[u'(x + \varphi \bullet S_T)\right]} = \mathcal{E}\big(-r^\top W\big)_T \ . \tag{3.32}$$

Since $Z_T > 0$, we have $Q \sim P$. Now note that, for each $j = 1, \dots, n$, the process S^j is a P-semimartingale with the decomposition $S^j = A^j + M^j$ into a finite-variation process A^j and a local martingale M^j that are given through

$$A^j = b^{S,j} \bullet I, \qquad M^j = \sum_{k=1}^{n}\sigma^{S,jk} \bullet W^k \ .$$

By the Girsanov-Meyer Theorem (cf. Theorem 35 in [Pro04]), S^j also a Q-semimartingale and can be decomposed as $S^j = C^j + L^j$ into a Q-finite-variation process C^j and a Q-local martingale L^j. In order to conclude that S^j is a Q-local martingale, we need to show that $C^j = 0$. The same theorem tells us that

$$L^j = M^j - \frac{1}{Z} \bullet [Z, M^j] \ .$$

The stochastic exponential is known to satisfy $\mathcal{E}(X) = 1 + \mathcal{E}(X) \bullet X$. Thus, defining $N = -r^\top W$, we obtain

$$\begin{aligned} C^j &= S^j - L^j = A^j + \frac{1}{Z} \bullet [Z \bullet N, M^j] = A^j + [N, M^j] \\ &= A^j - \sum_{kl} r_k \sigma^{S,jl} \bullet [W^k, W^l] = \left(b^{S,j} - \left(\sigma^S r \right)^j \right) \bullet I \\ &= \left(b^{S,j} - \left(\operatorname{diag}(S^1, \ldots, S^n) \sigma \sigma^{-1} b \right)^j \right) \bullet I = 0 \ . \end{aligned}$$

The Q-finite-variation process vanishes, and S^j thus is a Q-local martingale. In order to show that the wealth process $\varphi \bullet W$ is a Q-martingale, we again apply the Girsanov-Meyer Theorem to conclude that

$$W^{Q,j} = W^j - [N, W^j] = W^j + r_j I \tag{3.33}$$

is a Brownian motion with respect to Q (and thus a Q-martingale). Therefore, by Equation (3.31), the wealth process of φ,

$$\varphi \bullet S = \frac{1}{p} \left((r^\top r) I + r^\top W^Q - (r^\top r) I \right) = \frac{1}{p} r^\top W^Q \ ,$$

is a Q-martingale. Altogether, the sufficient condition is satisfied, and φ is therefore optimal with respect to all admissible strategies.

We conclude this example by providing an explicit formula for the expected utility of terminal wealth. Notice that the utility function satisfies $u'(x) = -pu(x)$. Thus,

$$\begin{aligned} E\left[u(x + \varphi \bullet S_T)\right] &= -\frac{1}{p} E\left[u'(x + \varphi \bullet S_T)\right] = -\frac{1}{p} p e^{-px - \frac{T}{2} r^\top r} E\left[\mathcal{E}\left(-r^\top W\right)_T\right] \\ &= -e^{-px - \frac{T}{2} r^\top r} = -e^{-px - \frac{T}{2} b^\top c^{-1} b} \ . \end{aligned} \tag{3.34}$$

Chapter 4

Portfolio Optimization with Proportional Transaction Costs

The main differences between the frictionless problem and the one with proportional transaction costs can already be understood (on a qualitative level and only to a certain extent, of course) by analyzing the situation heuristically. Proportional transaction costs refer to a fee levied on each transaction, which is equal to a certain fixed percentage of the amount transacted. When there are no frictions in the market, an optimizing strategy is, in general, a process of unbounded variation. A prominent example is the Black-Scholes model, which is discussed in Section 3.2.4, where the optimizer follows a geometric Brownian motion. Obviously, adopting such a strategy for trading under transaction costs results in an immediate bankruptcy. Thus, it appears sensible to assume that when transaction fees are present, a good strategy selects an optimal initial portfolio (meaning that it starts with the frictionless optimizer) and keeps it unchanged for a certain time. But waiting too long, on the other hand, leads to utility loss due to a deviation from the optimal allocation which is described by a permanently changing stochastic process. There must therefore be a trade-off between doing nothing and trading, which can be thought of as being determined by a boundary around the frictionless optimum. When this boundary is crossed for the first time, the portfolio should be adjusted by buying or selling a minimum amount of stock (since the transaction fee is proportional, and we want to pay as little as possible) sufficient to get inside the boundary again. We conclude that finding an optimal strategy for trading under proportional transaction costs is equivalent to determining an optimal boundary around the frictionless optimizer. The region inside the boundary is referred to as the *no-transaction* or *no-trade region*.

The fact that an optimal trading strategy in the presence of proportional transaction costs is necessarily associated with a *certain region* about the frictionless optimizer was first discussed in the pioneering paper by Constantinides and Magill [MC76]. By means of heuristic arguments, the authors come to the conclusion that in the case of the Black-Scholes model it is optimal to start at the Merton point and to do nothing as long as the portfolio stays inside the buffer region. Once the boundary is crossed, the investor should transact so as to bring the portfolio back to the boundary of the region. However, the work contains no information about how to calculate the location of the boundaries and no precise description of the behaviour of the optimizer at these boundaries. A rigorous mathematical treatment of the problem was first presented by Davis and Norman [DN90]. The situation Davis and Norman considered was that of a Black-Scholes market with a single risky asset. They showed that the location of the boundaries of the no-trade region can be obtained by solving a free-boundary problem, and that the optimal buying and selling strategies are the local times of the stochastic process describing the monetary value of the bank account and the stock. The paper was given a detailed review by Shreve and Soner [SS94]. The authors remove many restrictive assumptions made in [DN90] and provide a very detailed analysis of the value function of the problem. Moreover, Shreve and Soner investigate the asymptotic impact of proportional transaction costs and show that the leading-order effect on the expected utility is $\mathcal{O}\left(\varepsilon^{2/3}\right)$ as $\varepsilon \to 0$ with ε being the proportionality factor. They also conclude that, asymptotically, the width of the no-trade region is of order $\mathcal{O}\left(\varepsilon^{1/3}\right)$. These asymptotic results will play a crucial role in the remainder of this thesis. The case of a Black-Scholes market with multiple risky assets has been approached in many studies by analysing the value function of the problem. In [AMS96], the problem is formulated for $n \geq 1$ uncorrelated risky assets, and numerical results are provided for the cases $n = 1$ and $n = 2$. Liu [Liu04] considers multiple risky assets and both proportional and fixed transaction costs. In the uncorrelated case, he obtains an expression for the optimal consumption and concludes that the optimal no-trade region is a rectangular box. In [LT10, MK06], asset correlation is incorporated. The authors show that non-zero correlation results in a distortion of the rectangular no-trade region which becomes parallelogram-like. In [GO10] the asymptotic effect of small proportional transaction costs is studied, and the probability density of the optimal portfolio is analysed. Atkinson and Ingpochai [AI12] study specifically the effect of correlation in the multidimensional Black-Scholes model with small proportional transaction costs by applying a perturbation technique for small correlation. The authors obtain numerical results for two and three correlated risky assets and observe that the no-trade region is a non-convex domain with a non-smooth around the frictionless optimizer. This important observation is also made by Altarovici et al. [ARS17]. In this study both fixed and proportional costs are considered, no restrictions are imposed on asset correlations, and algorithms are provided to determine the no-trade region as well as the optimal strategies at the leading order as the costs become small. Numerical computations are performed in the two-dimensional case. A different approach is taken by Kallsen and Muhle-Karbe in [KMK15]. The authors consider a general, non-Markovian model of a market with a single risky asset in the case that the investor's preference is described by the exponential utility. Instead of applying the dynamic-programming principle and analysing the value function of the problem, they use the shadow-price method (see also [KMK10] by the same authors) and calculate the leading-order asymptotic impact of small proportional transaction costs on the investor's expected utility.

We begin this chapter by giving a brief review of the main aspects of the theory of portfolio optimization in the presence of proportional transaction costs, we introduce the notion of a shadow price and discuss its importance in the context of constrained portfolio optimization. Subsequently, when focusing on the asymptotics for small transaction costs, we generalize the approach taken by Kallsen and Muhle-Karbe to markets with multiple risky assets.

The main goal of this thesis is to formulate a method for constructing a good upper bound on the utility loss of an arbitrary candidate policy in high dimensions. The construction scheme will be presented in Chapter 5. In Chapter 6, the quality of the upper bound will be assessed with the help of several heuristically constructed candidate strategies by means of simulations and numerical methods in up to 30 dimensions and for arbitrary asset correlations. Apart from analysing the main effects of proportional transaction costs and discussing the asymptotic approximation for small proportionality factors, this chapter mainly aims at reformulating the optimization problem. We show that the no-trade region can be described as a solution to a free-boundary problem, which then allows us to characterize the asymptotically optimal trading strategy as a solution to a multidimensional stochastic differential equation with reflections at boundary of the no-trade region. The results are summarized in Proposition 4.13. This and many of the intermediate results obtained in Section 4.3 will be important for the construction of our approximation scheme in Chapter 5.

Here and in the following chapters, only the exponential utility function will be used. Thus, unless otherwise stated, when a function u is used to denote a utility function, it always implies that

$$u(x) = -e^{-px}, \quad p > 0 .$$

4.1 Portfolio theory in the presence of proportional costs

Let $S\colon [0,T] \times \Omega \to \mathbb{R}^n$ and $(\varphi^0, \varphi)^\top\colon [0,T] \times \Omega \to \mathbb{R}^{n+1}$ be the price process (denominated in bond units) and the portfolio process, respectively. Assume that for each transaction the investor has to pay a fee which is proportional to the current value of the stock. We let the proportionality factor be the same for both selling and purchasing stocks, but we allow it to depend on which stock is traded. The presence of such *proportional transaction costs* causes each asset to have a purchase price and a selling price.

Definition 4.1. Let S^i be the price process of the i-th stock. The processes

$$\underline{S}^i = (1-\varepsilon_i)S^i \quad \text{and} \quad \overline{S}^i = (1+\varepsilon_i)S^i \tag{4.1}$$

are referred to as the *bid price* and *ask price*, respectively.

Hence, the stock can be purchased at a higher ask price and sold at a lower bid price. Transaction costs reduce the class of trading strategies in the sense that each portfolio process $\varphi^i,\ i \geq 1$, must be of finite variation because otherwise infinite transaction costs would be the result. It follows that for each φ^i there exist two increasing processes, $\varphi^{+,i}$ and $\varphi^{-,i}$, which do not increase at the same time, such that

$$\varphi^i = \varphi^{+,i} - \varphi^{-,i} \ . \tag{4.2}$$

Note that the requirement that $\varphi^{+,i}$ and $\varphi^{-,i}$ do not increase at the same time is quite natural since buying and selling the same amount of stock at the same time results in pure money loss due to transaction costs and can therefore never be a part of an optimal trading strategy.
The definition of a self-financing strategy and the dynamics of the wealth process must be generalized to account for the effect of the transaction costs. A trading strategy is called *self-financing* if stock purchases are financed from selling an amount of bonds which precisely covers the expenses including the transaction costs, and, similarly, money gained from selling stocks is immediately invested into the bond. The following definition is a formalization of this requirement.

Definition 4.2. A trading strategy $(\varphi^0, \varphi)^\top$ is called *self-financing* if it satisfies

$$d\varphi^0 = \underline{S} d\varphi^- - \overline{S} d\varphi^+ \ . \tag{4.3}$$

The dynamics of the wealth process of a self-financing trading strategy $(\varphi^0, \varphi)^\top$ must therefore have the form

$$\begin{aligned} dV &= d\varphi^0 + Sd\varphi^+ - Sd\varphi^- + \varphi dS \\ &= \sum_{j=1}^{n} \left((1-\varepsilon_j)S^j d\varphi^{-,j} - (1+\varepsilon_j)S^j d\varphi^{+,j}\right) + Sd\varphi^+ - Sd\varphi^- + \varphi dS \\ &= \varphi dS - \sum_{j=1}^{n} \varepsilon_j S^j d|D\varphi^j| \ , \end{aligned}$$

where $|D\varphi^j|_t$ is the total-variation process of φ^j at time t satisfying

$$d|D\varphi^j|_t = d\varphi_t^{+,j} + d\varphi_t^{-,j}, \quad j = 1, \ldots, n \ ,$$

To simplify notation, we set

$$d\bar{\varphi}^j = \varepsilon_j d|D\varphi^j| \ ,$$

which allows us to write the wealth process as

$$\begin{aligned} V_t &= V_0 + \int_0^t \varphi_u dS_u - \int_0^t S_u d\bar{\varphi}_u \ , \\ V_0 &= \varphi_0^0 + \sum_{j=1}^{n} \varphi_0^j S_0^j \ . \end{aligned} \tag{4.4}$$

If we compare the above expression with Equation (3.1) describing the investor's wealth in a market without transaction costs, we see that the right-hand side of (4.4) is just the expression known from the frictionless case reduced by $S \bullet \bar{\varphi}_t$ which equals the cumulative costs from all transactions up to time t.
The portfolio-optimization problem under proportional transaction costs amounts to finding an investment strategy such that trading according to this strategy maximizes the investor's expected utility of the payoff on the assets when the portfolio is liquidated the time $T \in (0, \infty)$. The payoff the investor seeks to maximize will be referred to as the *liquidation wealth.*

Definition 4.3. Let $(\varphi^0, \varphi)^\top$ be a self-financing trading strategy. The process

$$X(\varphi) = \varphi^0 + \sum_{j=1}^{n} \left(I_{\{\varphi^j \geq 0\}} \varphi^j \underline{S}^j + I_{\{\varphi^j < 0\}} \varphi^j \overline{S}^j \right) \tag{4.5}$$

is called the *liquidation-wealth process* of φ.

Remark 4.4. Notice that the liquidation wealth, $X(\varphi)_t$, can be obtain from the usual wealth process, $V(\varphi)_t$, if we assume that the portfolio is liquidated at the time $t > 0$ and thus subtract the liquidation costs, $\sum \varepsilon_j S_t^j |\varphi_t^j|$, from $V(\varphi)_t$. More precisely, let

$\varphi \in L(S)$ be of finite variation and $(\varphi^0, \varphi)^\top$ self-financing. Then,

$$\begin{aligned}
X(\varphi) &= \varphi^0 + \varphi^\top S - \sum_{j=1}^n \varepsilon_j S^j \left(\varphi^j I_{\{\varphi^j \geq 0\}} + \varphi^j I_{\{\varphi^j < 0\}}\right) \\
&= \varphi_0^0 + \underline{S} \bullet \varphi^- + \overline{S} \bullet \varphi^+ + \varphi_0^\top S_0 + \varphi \bullet S + S \bullet \varphi - \sum_{j=1}^n \varepsilon_j S^j |\varphi^j| \\
&= \varphi_0^0 + \varphi_0^\top S_0 + \varphi \bullet S - \sum_{j=1}^n \varepsilon_j S^j \bullet |D\varphi^j| - \sum_{j=1}^n \varepsilon_j S^j |\varphi^j| \\
&= V(\varphi) - \sum_{j=1}^n \varepsilon_j S^j |\varphi^j| \ .
\end{aligned}$$

In the second line we used the self-financing condition (3.2) and applied integration by parts to $\varphi^\top S$.

We can now obtain a precise statement of the optimization problem under proportional transaction costs directly from the statement made in Chapter 3, Definition 3.6 for markets without transaction costs simply by replacing the frictionless wealth process by the liquidation wealth from the above definition. Let $X(\psi)$ denote the liquidation-wealth process of a trading strategy ψ. Moreover, let $\mathcal{A}(x) \subset L(S)$ be a subset of S-integrable processes of finite variation such that $(\varphi^0, \varphi)^\top$ is self-financing, and $\varphi_0^0 + \varphi_0^\top S_0 = x$ is the initial wealth of the investor. Find a function v and a strategy φ such that

$$v(x) = \sup_{\psi \in \mathcal{A}(x)} E\left[u(X(\psi)_T)\right] = E\left[u(X(\varphi)_T)\right] \ . \tag{4.6}$$

As in the frictionless case, v will be called the *value function* of the problem.
The properties of the set $\mathcal{A}(x)$ are specified in the following subsection.

4.1.1 Arbitrage and admissibility

As already mentioned in Subsection 3.2.1, the set of admissible strategies $\mathcal{A}(x)$ must exclude arbitrage opportunities and contain the optimizing trading strategy. In the frictionless case, a sufficient criterion for arbitrage absence is the existence of a (local) martingale measure for the price process. This result does not extend to markets with bid-ask spreads. Under proportional transaction costs, the counterpart of (local) martingale measures ensuring arbitrage absence are the so-called *consistent price systems (CPS)*. A CPS is essentially a pair $(\widetilde{S}, \widetilde{Q})$ consisting of a process $\widetilde{S}$ evolving within the bid-ask spread, and an equivalent probability measure $\widetilde{Q}$ making $\widetilde{S}$ a martingale. Starting with the seminal paper [JK95], much research has been devoted to understanding the connection between CPS and arbitrage under proportional transaction costs, especially to the question of the existence of such price systems. We name only a few important contributions: [Sch04] proves a necessary and sufficient condition in finite and discrete time, [GRS08] provides an extension to continuous-time models with only one risky asset, and [GK12] generalizes the latter result to multidimensional models; [Gua06, IP08] also prove a sufficient condition for arbitrage absence for a large class of one-dimensional market models in terms of only the price process and without explicitly mentioning CPS or martingale measures. The reader is also referred to the detailed treatment of arbitrage theory under transaction costs in [KS10] Chapter 3.

For the purpose of this chapter, it will be sufficient to base the definition of admissibility on the following observation. Assume there exists a process $\widetilde{S}$ evolving inside the bid-ask spread, i. e. $\underline{S}^j \le \widetilde{S}^j \le \overline{S}^j$, $j = 1, \dots, n$. Then, implementing a given strategy φ in the market with the price process $\widetilde{S}$ without transaction costs will always yield a higher payoff than trading according to φ in the market with the bid-ask spread $(\underline{S}, \overline{S})$ since trades can be carried out at more favourable prices. More precisely, let $\widetilde{S}$, $\widetilde{S}^j \in [\underline{S}^j, \overline{S}^j]$, be a price process varying inside the bid-ask spread and let $\varphi = \varphi^+ - \varphi^-$ be of finite variation such that $(\varphi^0, \varphi)^\top$ is self-financing. Let $X = X(\varphi)$ denote the liquidation-wealth process generated by trading φ in the bid-ask spread and $V^0 = V^0(\varphi; \widetilde{S})$ denote the frictionless wealth of φ in the market with the price process $\widetilde{S}$. Then,

$$\widetilde{S} \bullet \varphi^+ \le \overline{S} \bullet \varphi^+ \quad \text{and} \quad \widetilde{S} \bullet \varphi^- \ge \underline{S} \bullet \varphi^- \ ,$$

and we obtain

$$\begin{aligned} X &= \varphi_0^0 + \underline{S} \bullet \varphi^- + \overline{S} \bullet \varphi^+ + \sum_{j=1}^n \left(I_{\{\varphi^j \ge 0\}} \varphi^j \underline{S}^j + I_{\{\varphi^j < 0\}} \varphi^j \overline{S}^j \right) \\ &\le \varphi_0^0 - \widetilde{S} \bullet \varphi + \sum_{j=1}^n \left(I_{\{\varphi^j \ge 0\}} \varphi^j \underline{S}^j + I_{\{\varphi^j < 0\}} \varphi^j \overline{S}^j \right) \\ &= \underbrace{\varphi_0^0 + \varphi_0^\top \widetilde{S}_0 + \varphi \bullet \widetilde{S}}_{=V^0(\varphi; \widetilde{S})} - \varphi^\top \widetilde{S} + \sum_{j=1}^n \left(I_{\{\varphi^j \ge 0\}} \varphi^j \underline{S}^j + I_{\{\varphi^j < 0\}} \varphi^j \overline{S}^j \right) \\ &= V^0 - \sum_{j=1}^n \Big(I_{\{\varphi^j \ge 0\}} \varphi^j \underbrace{(\widetilde{S}^j - \underline{S}^j)}_{\ge 0} + I_{\{\varphi^j < 0\}} \varphi^j \underbrace{(\widetilde{S}^j - \overline{S}^j)}_{\le 0} \Big) \le V^0 \ . \end{aligned} \tag{4.7}$$

This observation allows us to state a sufficient no-arbitrage condition for the subset of trading strategies which we introduce in the following

Definition 4.5. Let $x = \sum_{j=0}^n x_j$ denote the investor's initial wealth, where x_0 and $x_1, \dots, x_n$ correspond to their holdings in bond and stocks $1, \dots, n$, respectively. A self-financing policy $(\varphi^0, \varphi)^\top \colon [0, T] \times \Omega \to \mathbb{R} \times \mathbb{R}^n$ satisfying

$$x_0 = \varphi_0^0, \quad x_j = \varphi_0^j S_0^j, \quad j = 1, \dots, n \ ,$$

will be called *admissible* if φ is S-integrable and of finite-variation, such that

$$\exists K > 0 \, \forall t \geq 0 \colon X(\varphi)_t \geq -K \ .$$

The set of all admissible strategies to the initial wealth x will be denoted by $\mathcal{A}(x)$.

If we now assume the existence of a frictionless market evolving in the bid-ask spread, it can be easily seen that any no-arbitrage condition that holds in this frictionless market naturally extends to the market with proportional transaction costs. To see this, recall that a market is free of arbitrage if

$$P(X(\varphi)_T \geq 0) = 1 \quad \Longrightarrow \quad P(X(\varphi)_T = 0) = 1$$

for all admissible trading strategies. Let $\widetilde{S}$ be a frictionless market evolving in the bid-ask spread and being free of arbitrage. Then, for any non-zero trading strategy φ, there exists an event $E \subset \Omega$ with $P(E) > 0$ such that

$$V^0(\varphi; \widetilde{S})_T < 0 \quad \text{on the event } E \ .$$

Since V^0 is an upper bound to the liquidation wealth $X(\varphi)$, the above statement immediately implies that φ cannot be an arbitrage in the market with the bid-ask spread. Apart from being crucial to ensuring the arbitrage freedom of the market, a frictionless market extension with the price process evolving inside the bid-ask spread is also the main ingredient for our asymptotic analysis of the optimality under transaction costs. In the latter context we will speak of a *shadow-price process.* The existence of such a frictionless market is far from obvious. However, when the transaction costs become small, a shadow price can be characterized in terms of a free-boundary problem, as we shall see in Subsection 4.3.1.

4.2 Shadow price

The concept of a shadow price is essential for our approach to utility maximization under proportional transaction costs. The idea behind this concept is to construct a fictitious *frictionless* market equivalent to the market $(\underline{S}, \overline{S})$ in the sense that optimizing in the frictionless market yields the same maximum value of the expected utility of terminal wealth as it does in the market with transaction costs. The notion of a shadow price is made precise in the following

Definition 4.6. Let $\underline{S}$ and $\overline{S}$ be the bid and ask price as defined in (4.1), and let x denote the investor's initial endowment. Moreover, let the process $X(\psi)$ describe the investor's liquidation wealth from trading with proportional transaction costs according to a strategy ψ, as introduced in Definition 4.3. A *shadow price* is a process $\widetilde{S}$ satisfying

(i) $\underline{S}^j \leq \widetilde{S}^j \leq \overline{S}^j, \quad j = 1, \ldots, n$;

(ii) $\sup_{\psi \in \mathcal{A}(x)} E\left[u(x + \psi \bullet \widetilde{S}_T)\right] = \sup_{\psi \in \mathcal{A}(x)} E\left[u(X(\psi)_T)\right]$.

Remark 4.7. Recall that if $\widetilde{S}$ is an arbitrary frictionless price process evolving within the bid-ask spread, i.e. satisfying Definition 4.6 (i), and ψ is any non-zero trading strategy, we have

$$X(\psi)_T \leq x + \psi \bullet \widetilde{S}_T$$

since in the market with the price process $\widetilde{S}$ and no transaction costs shares can be bought at lower and sold at higher prices, and a higher wealth can therefore be generated (cf. Equation (4.7)). Thus, in order for $\widetilde{S}$ to be a shadow price in the sense of Definition 4.6, the optimizer φ in the frictionless market with the price process $\widetilde{S}$ must be such that φ^j increases on the set $\left\{\widetilde{S}^j = (1+\varepsilon_j)S^j\right\}$, decreases on the set $\left\{\widetilde{S}^j = (1-\varepsilon_j)S^j\right\}$ and is constant otherwise. This immediately implies that the frictionless optimizer for the shadow price automatically maximizes the expected utility in the market $(\underline{S}, \overline{S})$. Put differently, not only the values of the two optimization problems but also the corresponding optimizing strategies coincide.

At the beginning of this chapter, the most obvious properties of an optimizer φ in the presence of proportional transaction costs were discussed in a rather heuristic way. In particular, we noticed that a no-trade region around the frictionless optimizer should appear. This no-trade region would then completely determine the behaviour of φ by forcing it to remain constant in the interior and perform a minimal trade at the boundary. Taking into account the observations made in Remark 4.7, we define

$$\mathcal{C}^{+,j} = \left\{\widetilde{S}^j = (1+\varepsilon_j)S^j\right\}, \quad \mathcal{C}^{-,j} = \left\{\widetilde{S}^j = (1-\varepsilon_j)S^j\right\} \tag{4.8}$$

as subsets of $[0,T] \times \Omega$. These sets implicitly determine the no-trade region in the sense that the processes $\varphi^{+,j}$ and $\varphi^{-,j}$, $j = 1, \ldots, n$, which describe the optimal buying and selling policies, increase only on $\mathcal{C}^{+,j}$ and $\mathcal{C}^{-,j}$, respectively.

Note that, in general, the shadow price is not known. Thus, if one chooses to use the shadow-price method to solve a constrained utility maximization problem, finding a shadow-price process becomes a part of the optimization procedure.

4.3 Asymptotics for small transaction costs

In this section, the asymptotic effect of small proportional transaction costs will be discussed. The treatment of the problem will be mainly heuristic, meaning that some arguments will lack sufficient mathematical rigour. This will concern two crucial aspects. Firstly, discussing asymptotic behaviour of objects amounts to analysing their convergence properties. When dealing with the asymptotics of stochastic processes, we will not specify the type of convergence, tacitly assuming that it happens in some appropriate sense. Secondly, we will not distinguish between martingales and local martingales. In particular, this implies that a sufficient condition for arbitrage absence will be the existence of an equivalent probability measure making the price process a true martingale, and that integrals with respect to martingales are again true martingales.

We are interested in asymptotic results for small transaction costs $\varepsilon_1, \dots, \varepsilon_n$. To be more precise, when a transaction is made to purchase or sell a particular stock, the transaction costs paid are assumed to be small compared with the size of the transaction. The transaction costs levied on trading the individual assets are assumed to be small of the same order. We indicate this by introducing a real parameter $\varepsilon > 0$, which will be taken to be small throughout this thesis, and demanding

$$\varepsilon_j = \mathcal{O}(\varepsilon), \quad j = 1, \dots, n \; . \tag{4.9}$$

Put differently, for each $j \in \{1, \dots, n\}$, the proportionality factor ε_j can be written as $\varepsilon_j = \varepsilon \lambda_j$ for an appropriately chosen $\lambda_j > 0$ independent of ε.

As already mentioned at the beginning of this chapter, the effect of small proportional transaction costs is of order $\mathcal{O}\left(\varepsilon^{2/3}\right)$, as observed by Shreve and Soner [SS94], Rogers [Rog04] (see also [WW97, HMS13, DP13, Bic14]). Motivated by this fact and to clarify the meaning of "asymptotic results" in the above introduction, we formulate the following

Definition 4.8. A strategy φ is called *asymptotically optimal* if, for any other strategy ψ, it satisfies

$$E\left[u(X(\varphi)_T)\right] \geq E\left[u(X(\psi)_T)\right] + o\left(\varepsilon^{2/3}\right) \; . \tag{4.10}$$

In other words, we are looking for a strategy which is *optimal at the leading order* as ε tends to zero.

Remark 4.9. Notice that the liquidation costs do not contribute to the utility loss at the leading order since $\varepsilon_j S^j |\varphi^j| = \mathcal{O}\left(\varepsilon\right)$, $j = 1, \dots, n$. Thus, asymptotically, expected utility of the terminal wealth $V(\varphi)_T$ and that of the liquidation wealth $X(\varphi)_T$ are equal.

Our goal now is to establish a dual characterization of asymptotic optimality by using the shadow-price method. As already mentioned in the previous section, the fictitious market with a shadow-price process can be viewed as the least favourable frictionless market extension allowing us to reduce the constrained optimization problem to a well-known problem of utility maximization without transaction costs. Once we are in the frictionless setting, we can apply the results obtained in Section 3.2.3 where we saw how the convex-duality approach can be used to formulate a sufficient condition for a strategy φ to be optimal in the frictionless case (cf. Proposition 3.8). As we shall see, it is possible to obtain an analogous condition for asymptotic optimality by weakening the requirements of Proposition 3.8.

In the following, we will always assume that the requirements of Proposition 3.21 are satisfied and the frictionless optimization problem thus admits a solution φ^* which can be described by a continuous Itô process.
The following construction and the derivation of some of its most important consequences is a generalization of the results obtained in [KMK15] to the multidimensional case.

Proposition 4.10. *Let $\varphi = (\varphi^1, \dots, \varphi^n)$ be a trading strategy and $\widetilde{S}$ a price process such that $\widetilde{S}^j \in [\underline{S}^j, \overline{S}^j]$, $j \in \{1, \dots, n\}$, and satisfying $\widetilde{S}^j = \underline{S}^j$ and $\widetilde{S}^j = \overline{S}^j$ if and only if φ^j sells respectively buys j-th stock. Assume there exists a real-valued Itô process $\widetilde{Z}$ such that*

(i) $b^{\widetilde{Z}} = \mathcal{O}\left(\varepsilon^{2/3}\right)$,

(ii) $b^{\widetilde{Z}\widetilde{S}^j} = \mathcal{O}\left(\varepsilon^{2/3}\right)$, $j = 1, \dots, n$,

(iii) $\widetilde{Z}_T = u'(X(\varphi)_T) + \mathcal{O}\left(\varepsilon^{2/3}\right)$,

where $b^{\widetilde{Z}}$ and $b^{\widetilde{Z}\widetilde{S}^j}$ denote drift processes of $\widetilde{Z}$ and $\widetilde{Z}\widetilde{S}^j$, respectively. Then, the strategy φ is asymptotically optimal.

Proof. Let ψ be another trading strategy converging to the frictionless optimizer as ε tends to zero, i. e. $\psi \xrightarrow[\varepsilon \to 0]{} \varphi^*$. Since the utility function u is concave, we have

$$E\left[u(X(\varphi)_T)\right] \geq E\left[u(X(\psi)_T)\right] + E\left[u'(X(\varphi)_T)\left(X(\varphi)_T - X(\psi)_T\right)\right] .$$

According to Remark 4.7, the wealth processes of φ and ψ satisfy

$$X(\varphi)_T = x + \varphi \bullet \widetilde{S}_T, \quad X(\psi)_T \leq x + \psi \bullet \widetilde{S}_T .$$

Together with the condition (iii) this implies

$$E\left[u(X(\varphi)_T)\right] \geq E\left[u(X(\psi)_T)\right] + E\left[\widetilde{Z}_T\left((\varphi-\psi)\bullet\widetilde{S}_T\right)\right] + o\left(\varepsilon^{2/3}\right) \ .$$

For each $j \in \{1,\dots,n\}$, partial integration yields

$$\widetilde{Z}_T\left((\varphi-\psi)^j\bullet\widetilde{S}^j_T\right) = \left(\widetilde{Z}(\varphi-\psi)^j\right)\bullet\widetilde{S}^j_T + \left((\varphi-\psi)^j\bullet\widetilde{S}^j\right)\bullet\widetilde{Z}_T + (\varphi-\psi)^j\bullet[\widetilde{Z},\widetilde{S}^j]_T$$

and

$$[\widetilde{Z},\widetilde{S}^j] = \widetilde{Z}\widetilde{S}^j - \widetilde{Z}\bullet\widetilde{S}^j - \widetilde{S}^j\bullet\widetilde{Z} \ .$$

Altogether,

$$\widetilde{Z}_T\left((\varphi-\psi)^j\bullet\widetilde{S}^j_T\right) = \left((\varphi-\psi)^j\bullet\widetilde{S}^j - (\varphi-\psi)^j\widetilde{S}^j\right)\bullet\widetilde{Z}_T + (\varphi-\psi)^j\bullet(\widetilde{Z}\widetilde{S}^j)_T \ ,$$

and

$$\begin{aligned} E\left[\widetilde{Z}_T\left((\varphi-\psi)\bullet\widetilde{S}_T\right)\right] &= \sum_{j=1}^n E\left[\widetilde{Z}_T\left((\varphi-\psi)^j\bullet\widetilde{S}^j_T\right)\right] \\ &= \sum_{j=1}^n E\left[\left(\left((\varphi-\psi)^j\bullet\widetilde{S}^j - (\varphi-\psi)^j\widetilde{S}^j\right)b^{\widetilde{Z}}\right)\bullet I_T\right] \\ &+ \sum_{j=1}^n E\left[\left((\varphi-\psi)^j b^{\widetilde{Z}\widetilde{S}^j}\right)\bullet I_T\right] \ . \end{aligned}$$

By the conditions (i) and (ii) and the fact that, for all j, $\varphi^j - \psi^j \xrightarrow[\varepsilon\to 0]{} 0$, each expectation in the last two sums is of order $o\left(\varepsilon^{2/3}\right)$ which, in turn, implies the asymptotic optimality of φ. □

Remark 4.11. Note that Proposition 4.10 guarantees asymptotic optimality of φ also in the frictionless market with the price process $\widetilde{S}$ since $X(\varphi)_T = x + \varphi\bullet\widetilde{S}_T$; thus, the process $\widetilde{S}$ can be interpreted as a shadow price in the asymptotic sense. Moreover, conditions $(i), (ii)$ essentially mean that, up to some terms that are sufficiently small in ε (i. e., asymptotically), $\widetilde{Z}$ is the density of an equivalent martingale measure for the process $\widetilde{S}$, and condition (iii) implies that $\widetilde{Z}$ is an asymptotically optimal dual variable (cf. Proposition 3.21, Eq. (3.20)). Thus, Proposition 4.10 provides us with the desired dual characterization of asymptotic optimality in the sense of Definition 4.8.

We now assume that, for small transaction costs, our candidate policy φ (i. e., trading strategy whose optimality we wish to varify) and the shadow price $\widetilde{S}$ are of the form

$$\begin{aligned} \varphi &= \varphi^* + \Delta\varphi \ , \\ \widetilde{S} &= S + \Delta S \ , \end{aligned} \tag{4.11}$$

with S and φ^* being the mid-price process and the frictionless optimizer, respectively, and $\Delta\varphi$, ΔS two $\mathbb{R}^n$-valued semimartingales depending on ε. Since the shadow price lies inside the bid-ask spread, $\widetilde{S}^j \in [\underline{S}^j, \overline{S}^j]$, the process ΔS must obviously satisfy

$$-\varepsilon_j S^j \leq \Delta S^j \leq \varepsilon_j S^j, \quad j = 1, \ldots, n \ . \tag{4.12}$$

As already mentioned at the end of the previous section, an optimizing strategy φ^j changes iff the shadow price happens to coincide with either the bid or the ask price, which is characterized by the sets $\mathcal{C}^j$ defined in (4.8). Thus, on the compliment of $\mathcal{C}^j$ we have $d\varphi^j = 0$, and the semimartingale $\Delta\varphi^j$ has the dynamics

$$d\Delta\varphi_t^j = -d\varphi_t^{*,j} \tag{4.13}$$

We will consider processes $\Delta\varphi$ and ΔS with properties summarized in the following

Assumption A3.

(a) $\Delta\varphi = \mathcal{O}\left(\varepsilon^{1/3}\right)$;

(b) ΔS is an Itô process, i. e.

$$d\Delta S_t^j = b_t^{\Delta S,j} dt + \sum_{k=1}^n \sigma_t^{\Delta S,jk} dW_t^k,$$

such that $\Delta S = \mathcal{O}(\varepsilon)$ and its drift and diffusion coefficients satisfy

$$b^{\Delta S,j} = \mathcal{O}\left(\varepsilon^{1/3}\right), \quad \sigma^{\Delta S,jk} = \mathcal{O}\left(\varepsilon^{2/3}\right), \quad j,k = 1, \ldots, n \ .$$

As for the asymptotic behaviour of ΔS, the assumption $\Delta S = \mathcal{O}(\varepsilon)$ is rather obvious from (4.12). The statement in A3 (a) is based on existing results concerning the asymptotics of the boundaries of the no-trade region in the one-dimensional case (i. e. for markets with only one risky asset) [SS94, Rog04, WW97, HMS13, DP13, Bic14, Rog13, KMK15]. The assumptions about the drift and diffusion coefficients of ΔS are closely related to an equivalent formulation of the asymptotic optimality conditions that will be established in the following section.

4.3.1 Reformulating the optimization problem

Our next goal is to apply the asymptotic optimality condition to construct an optimizing strategy $\varphi = \varphi^* + \Delta\varphi$. Before moving on to actually determining the processes $\widetilde{Z}$, ΔS and $\Delta\varphi$ (cf. Proposition 4.10 and Assumption A3), we first examine the asymptotic behaviour of the payoff process of a candidate policy, $X(\varphi) = x + \varphi \bullet \widetilde{S}$, with $\widetilde{S}$ and x being a frictionless shadow-price process and the initial capital, respectively. Using the ansatz (4.11), the wealth process can be written as

$$X(\varphi) = x + \varphi \bullet \widetilde{S} = x + \varphi^* \bullet S + \varphi \bullet \Delta S + \Delta\varphi \bullet S \ .$$

The term $x + \varphi^* \bullet S$ does not depend on ε and describes the wealth generated by the frictionless optimizer. The behaviour of the last term is obvious from Assumption A3, $\Delta\varphi^j \bullet S^j = \mathcal{O}\left(\varepsilon^{1/3}\right)$ for all j. The remaining term satisfies

$$\varphi \bullet \Delta S = \varphi^* \bullet \Delta S + \Delta\varphi \bullet \Delta S = \mathcal{O}\left(\varepsilon^{2/3}\right) \ . \tag{4.14}$$

To see this, we first apply integration by parts to obtain

$$\varphi^{*,j} \bullet \Delta S^j = \underbrace{\varphi^{*,j}\Delta S^j}_{=\mathcal{O}(\varepsilon)} - \underbrace{\Delta S^j \bullet \varphi^{*,j}}_{=\mathcal{O}(\varepsilon)} - \underbrace{[\varphi^{*,j}, \Delta S^j]}_{=\mathcal{O}\left(\varepsilon^{2/3}\right)} = \mathcal{O}\left(\varepsilon^{2/3}\right) \ ,$$

where we used that

$$[\varphi^{*,j}, \Delta S^j] = \sum_k \sigma^{\varphi^*,jk}\sigma^{\Delta S,jk} \bullet I \ ,$$

and $\sigma^{\Delta S,jk} = \mathcal{O}\left(\varepsilon^{2/3}\right)$ by Assumption A3. Moreover, we have

$$\Delta\varphi^j \bullet \Delta S^j = \Big(\underbrace{\Delta\varphi^j b^{\Delta S,j}}_{=\mathcal{O}\left(\varepsilon^{2/3}\right)}\Big) \bullet I + \sum_k \Big(\underbrace{\Delta\varphi^j \sigma^{\Delta S,jk}}_{=\mathcal{O}(\varepsilon)}\Big) \bullet W^k = \mathcal{O}\left(\varepsilon^{2/3}\right)$$

Having understood the behaviour of the individual terms contributing to the wealth process, we first look at the process $\widetilde{Z}$.

$$\begin{aligned} u'\left(X(\varphi)_T\right) &= u'\left(x + \varphi^* \bullet S_T + \varphi \bullet \Delta S_T + \Delta\varphi \bullet S_T\right) \\ &= u'\left(x + \varphi^* \bullet S_T\right) \exp\left(-p\varphi \bullet \Delta S_T - p\Delta\varphi \bullet S_T\right) \\ &= y_* Z_T \left(1 - p\Delta\varphi \bullet S_T\right) + \mathcal{O}\left(\varepsilon^{2/3}\right) \ . \end{aligned}$$

In the last line, the frictionless optimality condition

$$u'\left(x + \varphi^* \bullet S_T\right) = y_* Z_T, \quad y_* = E\left[u'\left(x + \varphi^* \bullet S_T\right)\right] \ ,$$

was used with Z_T being the density of an equivalent martingale measure for S (cf. Equation (3.20)). The process $\widetilde{Z}$ can therefore be chosen as

$$\widetilde{Z} = Z\left(1 - p\Delta\varphi \bullet S\right) + \mathcal{O}\left(\varepsilon^{2/3}\right) \ . \tag{4.15}$$

We have constructed $\widetilde{Z}$ satisfying $(i), (iii)$ of Proposition 4.10. We now turn to the problem of finding $\Delta\varphi$ and ΔS. Applying integration by parts to the process $\widetilde{Z}\widetilde{S}^j$, $j \in \{1, \dots, n\}$, yields

$$\widetilde{Z}\widetilde{S}^j - \widetilde{Z}_0\widetilde{S}^j_0 = \widetilde{Z} \bullet \widetilde{S}^j + \widetilde{S}^j \bullet \widetilde{Z} + [\widetilde{Z}, \widetilde{S}^j]$$

Since $b^{\widetilde{Z}} = \mathcal{O}\left(\varepsilon^{2/3}\right)$, the drift coefficient of $\widetilde{Z}\widetilde{S}^j$ is given by

$$b^{\widetilde{Z}\widetilde{S}^j} = \widetilde{Z}\left(b^{S^j} + b^{\Delta S^j}\right) + c^{\widetilde{Z},\widetilde{S}^j} + \mathcal{O}\left(\varepsilon^{2/3}\right) \ . \tag{4.16}$$

Using the representation (4.15) and Assumption A3, we obtain

$$\begin{aligned} \widetilde{Z}\left(b^{S^j} + b^{\Delta S^j}\right) &= Z\left(1 - p\Delta\varphi \bullet S\right)\left(b^{S^j} + b^{\Delta S^j}\right) + \mathcal{O}\left(\varepsilon^{2/3}\right) \\ &= Z\Big(b^{S^j} + b^{\Delta S^j} - p(\Delta\varphi \bullet S)b^{S^j} - p\underbrace{(\Delta\varphi \bullet S)b^{\Delta S^j}}_{=\mathcal{O}(\varepsilon^{2/3})}\Big) + \mathcal{O}\left(\varepsilon^{2/3}\right) \\ &= Z\left(b^{\Delta S^j} + \left(1 - p\Delta\varphi \bullet S\right)b^{S^j}\right) + \mathcal{O}\left(\varepsilon^{2/3}\right) \ . \end{aligned} \tag{4.17}$$

To evaluate $c^{\widetilde{Z},\widetilde{S}^j}$, we look at the covariation process

$$[\widetilde{Z}, \widetilde{S}^j] = [\widetilde{Z}, S^j] + [\widetilde{Z}, \Delta S^j] \ .$$

Representing the frictionless density process as a stochastic exponential,

$$Z = 1 + Z \bullet N \ ,$$

and noting that, by Assumption A3, the diffusion coefficients of ΔS satisfy $\sigma^{\Delta S, ij} = \mathcal{O}\left(\varepsilon^{2/3}\right)$, yields

$$[\widetilde{Z}, \widetilde{S}^j] = [\widetilde{Z}, S^j] + \mathcal{O}\left(\varepsilon^{2/3}\right) = [Z(1 - p\Delta\varphi \bullet S), S^j] + \mathcal{O}\left(\varepsilon^{2/3}\right) \ .$$

Moreover,

$$\begin{aligned}
[Z(1-p\Delta\varphi \bullet S), S^j] &= Z \bullet [N, S^j] - p[Z(\Delta\varphi \bullet S), S^j] \\
&= Z \bullet \left([N, S^j] - p[\Delta\varphi \bullet S, S^j]\right) - p[(\Delta\varphi \bullet S) \bullet (Z \bullet N), S^j] \\
&= Z \bullet \left([N, S^j] - p(c^S\Delta\varphi)^j \bullet I\right) - p[(Z(\Delta\varphi \bullet S)) \bullet N, S^j] \\
&= Z \bullet \left([N, S^j] - p(c^S\Delta\varphi)^j \bullet I\right) - p[Z \bullet ((\Delta\varphi \bullet S) \bullet N), S^j] \\
&= Z \bullet \left([N, S^j] - p(c^S\Delta\varphi)^j \bullet I - p(\Delta\varphi \bullet S) \bullet [N, S^j]\right) \\
&= \left(Z\left(c^{N,S^j}(1-p\Delta\varphi \bullet S) - p(c^S\Delta\varphi)^j\right)\right) \bullet I \ .
\end{aligned}$$

The local covariation $c^{\widetilde{Z},\widetilde{S}^j}$ therefore reads as

$$c^{\widetilde{Z},\widetilde{S}^j} = Z\left(c^{N,S^j}(1-p\Delta\varphi \bullet S) - p(c^S\Delta\varphi)^j\right) \ .$$

Combining this expression with (4.17) and inserting their sum into (4.16) leaves us with

$$\begin{aligned}
b^{\widetilde{Z}\widetilde{S}^j} &= Z\left(b^{\Delta S^j} + (1-p\Delta\varphi \bullet S)\, b^{S^j}\right) \\
&\quad + Z\left(c^{N,S^j}(1-p\Delta\varphi \bullet S) - p(c^S\Delta\varphi)^j\right) + \mathcal{O}\left(\varepsilon^{2/3}\right) \\
&= Z\left(b^{\Delta S^j} - p(c^S\Delta\varphi)^j + (1-p\Delta\varphi \bullet S)\left(b^{S^j} + c^{N,S^j}\right)\right) + \mathcal{O}\left(\varepsilon^{2/3}\right) \\
&= Z\left(b^{\Delta S^j} - p(c^S\Delta\varphi)^j\right) + \mathcal{O}\left(\varepsilon^{2/3}\right) \ .
\end{aligned} \tag{4.18}$$

We conclude that in order for the condition (*ii*) in Proposition 4.10 to be satisfied, the optimizing strategy $\varphi = \varphi^* + \Delta\varphi$ and the shadow price $\widetilde{S} = S + \Delta S$ must be related via

$$b^{\Delta S^j} = p(c^S\Delta\varphi)^j, \quad j = 1, \ldots, n \ . \tag{4.19}$$

We now suppose that a connection between ΔS and $\Delta\varphi$ can be established via a sufficiently well-behaved function and write

$$\Delta S = f(\Delta\varphi) \ .$$

This ansatz will now be used to derive a partial differential equation for the function $f = (f_1, \ldots, f_n)$ from the above relation (4.19). We do this by applying Itô's formula, and we therefore require that the function f be C^2. We first make some remarks about the notation to be used in the following. As already mentioned at the end of the previous subsection, the no-trade region is a stochastic domain inside which, starting at the frictionless optimizer, an optimal trading strategy $\varphi = \varphi^+ - \varphi^-$ remains constant and whose boundary determines the behaviour of $\varphi^{\pm,j}$, $j = 1, \ldots, n$. This domain is implicitly determined by the sets

$$\mathcal{C}^{\pm,j} = \left\{ S^j = (1 \pm \varepsilon_j) S^j \right\}, \quad j = 1, \dots, n \ .$$

This means that, for each (t, ω), $\Delta S_t(\omega)$ lies inside the rectangular parallelotope

$$K_t(\omega) := [-\varepsilon_1 S^1_t(\omega), \varepsilon_1 S^1_t(\omega)] \times \ldots \times [-\varepsilon_n S^n_t(\omega), \varepsilon_n S^n_t(\omega)] \ .$$

For ease of notation we will often drop the variables (t, ω) and write

$$K = \prod_{j=1}^{n} [-\varepsilon_j S^j, \varepsilon_j S^j] \ .$$

The no-trade region constraining the movement of the optimizer φ will be denoted by $\widetilde{R} = \varphi^* + R$ with R being a stochastic domain inside which the process $\Delta\varphi$ is allowed to vary. More precisely, for each (t, ω), $R_t(\omega) \subset \mathbb{R}^n$ is a compact set containing $\Delta\varphi_t(\omega)$. Finally, for a function $f \in C^2(\mathbb{R}^n)$, $\partial_j f_k$ and $\partial_j^2 f_k$, $\partial_i \partial_j f_k$ will be used to denote the elements of the Jacobian and the Hessian of f, respectively. Having fixed the basic notation, we now apply Itô's formula to $f \colon \Delta\varphi \mapsto \Delta S$ to obtain an expression for the drift coefficients $b^{\Delta S, j}$.

$$d\Delta S^j_t = df_j(\Delta\varphi_t) = \sum_k \partial_k f_j(\Delta\varphi_t) d\Delta\varphi^k_t + \frac{1}{2} \sum_{k,l} \partial_k \partial_l f_j(\Delta\varphi_t) d[\Delta\varphi^k, \Delta\varphi^l]_t \ . \tag{4.20}$$

As stated in Assumption A3, the drift coefficients are of order $\mathcal{O}\left(\varepsilon^{1/3}\right)$. To understand the asymptotic behaviour of the first and the second partial derivatives of f, we argue heuristically and claim that

$$\partial_k f_j(\Delta\varphi) \overset{\varepsilon}{\sim} \frac{\Delta S^j}{\Delta\varphi^k}, \quad \partial_k \partial_l f_j(\Delta\varphi) \overset{\varepsilon}{\sim} \frac{\Delta S^j}{\Delta\varphi^k \Delta\varphi^l} \ .$$

The symbol $\overset{\varepsilon}{\sim}$ is meant to indicate that the quantities exhibit the same asymptotic behaviour as $\varepsilon \to 0$. Since $\Delta S = \mathcal{O}(\varepsilon)$ and $\Delta\varphi = \mathcal{O}\left(\varepsilon^{1/3}\right)$,

$$\partial_k f_j(\Delta\varphi) = \mathcal{O}\left(\varepsilon^{2/3}\right), \quad \partial_k \partial_l f_j(\Delta\varphi) = \mathcal{O}\left(\varepsilon^{1/3}\right), \quad j, k, l = 1, \dots, n \ .$$

Thus, since we are interested in computing the drift of ΔS, the first partial derivatives can be neglected at the leading order of $\mathcal{O}\left(\varepsilon^{1/3}\right)$ by Assumption A3. Moreover, due to φ being a finite-variation process, the quadratic covariations appearing in the above Itô formula are just those of the components of the frictionless optimizer. The process ΔS

therefore satisfies

$$\begin{aligned} d\Delta S_t^j &= \frac{1}{2}\sum_{k,l} \partial_k\partial_l f_j(\Delta\varphi_t) d[\varphi^{*,k},\varphi^{*,l}]_t + \mathcal{O}\left(\varepsilon^{2/3}\right) \\ &= \frac{1}{2}\sum_{k,l} \partial_k\partial_l f_j(\Delta\varphi_t) c_t^{\varphi^*,kl} dt + \mathcal{O}\left(\varepsilon^{2/3}\right) \ . \end{aligned}$$

We conclude that the drift coefficients of ΔS satisfy

$$b^{\Delta S,j} = \frac{1}{2}\sum_{k,l} \partial_k\partial_l f_j(\Delta\varphi) c^{\varphi^*,kl} + \mathcal{O}\left(\varepsilon^{2/3}\right) \ . \tag{4.21}$$

Combining this result with the expression (4.19) yields, at the leading order in ε, a system of partial differential equations for the function f,

$$\frac{1}{2}\sum_{k,l} \partial_k\partial_l f_j(x) c^{\varphi^*,kl} = p\sum_k c^{S,jk} x_k, \quad j = 1,\ldots,n \ . \tag{4.22}$$

If we denote the Hessian of f_j by H_{f_j}, the above system can be written in a more compact form as

$$\mathrm{Tr}\left(H_{f_j}(x) c^{\varphi^*}\right) = 2p\left(c^S x\right)^j, \quad j = 1,\ldots,n \ . \tag{4.23}$$

It remains to impose boundary conditions. Since the no-trade region, $\widetilde{R} = \varphi^* + R$, is not known (it is actually the object of our interest), imposing conditions at the boundary ∂R will turn the task of finding the function f into a *free-boundary problem.* To start with, we decompose the boundary ∂R and write

$$\begin{aligned} \partial R^{\pm,j} &= f_j^{-1}\left(\{\pm\varepsilon_j S^j\}\right), \\ \partial R^j &= \partial R^{+,j} \cup \partial R^{-,j}, \\ \partial R &= \bigcup_{j=1}^n \partial R^j \ . \end{aligned} \tag{4.24}$$

When $\varphi^{*,j} + \partial R^{\pm,j}$ is hit by the process $\varphi^j = \varphi^{+,j} - \varphi^{-,j}$, j-th asset is either bought by $\varphi^{+,j}$ or sold by $\varphi^{-,j}$; this, in turn, happens iff $\Delta S^j = \varepsilon_j S^j$ or $\Delta S^j = -\varepsilon_j S^j$. The above decomposition thus already contains the first collection of boundary conditions, namely

$$f_j\big|_{\partial R^{\pm,j}} = \pm\varepsilon_j S^j, \quad j = 1,\ldots,n \ .$$

Further boundary conditions can be referred to as arbitrage conditions. We need to make sure that there exists an equivalent martingale measure for the shadow-price process $\widetilde{S}$ so that the market is free of arbitrage. We again turn to the Itô formula for the process

$\widetilde{S} = S + f(\Delta\varphi)$,

$$\begin{aligned} d\widetilde{S}^j_t &= dS^j_t + d\Delta S^j_t = dS^j_t + \sum_k \partial_k f_j(\Delta\varphi_t) d\Delta\varphi^k_t + \frac{1}{2}\sum_{k,l} \partial_k\partial_l f_j(\Delta\varphi_t) d[\Delta\varphi^k, \Delta\varphi^l]_t \\ &= dS^j_t - \sum_k \partial_k f_j(\Delta\varphi_t) d\varphi^{*,k}_t + \frac{1}{2}\sum_{k,l} \partial_k\partial_l f_j(\Delta\varphi_t) d[\varphi^{*,k}, \varphi^{*,l}]_t + \sum_k \partial_k f_j(\Delta\varphi_t) d\varphi^k_t \ . \end{aligned}$$

The expression in the last line has the structure of an Itô process plus the sum containing the terms $d\varphi^j$ which only contribute at the boundary of the no-trade region. There is no way to determine an equivalent measure making such a process a martingale unless the contribution of each $d\varphi^j$, $j = 1, \dots, n$, is eliminated. We therefore demand that, for all $j, k = 1, \dots, n$, the partial derivatives $\partial_k f_j$ vanish at $\partial R^{\pm,j}$. We now give a rigorous statement of the free-boundary problem we finally obtain.

The free-boundary problem:
Find a C^2 function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ and a collection of hypersurfaces ∂R^j such that

$$\begin{aligned} \sum_{k,l} \partial_k\partial_l f_j(x) c^{\varphi^*,kl} &= 2p\sum_k c^{S,jk} x_k \ , \\ f_j\big|_{\partial R^{\pm,j}} &= \pm\varepsilon_j S^j \ , \\ \partial_k f_j\big|_{\partial R^{\pm,k}} &= 0 \ , \\ \text{for all } j,k &= 1,\dots,n \ . \end{aligned} \tag{4.25}$$

Having obtained a precise characterization of the boundary of the no-trade region, we can now describe an asymptotically optimal trading strategy in terms of a solution to a reflecting stochastic differential equation which is often also referred to as *stochastic Skorohod problem*. A precise formulation of this problem is provided in the following.

The Skorohod problem:
Let $\varphi^* = (\varphi^{*,1}, \dots, \varphi^{*,n})^\top$,

$$\varphi^{*,j} = \varphi^{*,j}_0 + b^{\varphi^*,j} \bullet I + \sum_{k=1}^m \sigma^{\varphi^*,jk} \bullet W^k, \quad j = 1,\dots,n \ ,$$

be the frictionless optimizer and let $(f, \partial R)$, $\partial R = \bigcup_{j=1}^n \partial R^{+,j} \cup \partial R^{-,j}$, be a solution to the free-boundary problem (4.25). Furthermore, define the function

$$\mu = (\mu_1, \dots, \mu_n), \qquad \mu_j(x) = \operatorname{sgn}(f_j(x)) \ .$$

A finite-variation process $\varphi = (\varphi^1, \dots, \varphi^n)^\top$ is an asymptotically optimal trading strategy if there exists a semimartingale $\Delta\varphi = (\Delta\varphi^1, \dots, \Delta\varphi^n)^\top = \mathcal{O}\left(\varepsilon^{1/3}\right)$ which, for all

$j = 1, \dots, n$ and $t \geq 0$, satisfies

$$\begin{aligned} \Delta\varphi_t &= -\varphi_t^* + \varphi_t \in R_t, \quad \Delta\varphi_0 = 0 \ , \\ \varphi_t^j &= \varphi_0^{*,j} + \int_0^t \mu_j(\Delta\varphi_u) d|D\varphi^j|_u \ , \\ d|D\varphi^j| \left(\left\{ t \in [0,T] \colon \Delta\varphi_t \notin \partial R_t^{\pm,j} \right\} \right) &= 0 \ . \end{aligned} \tag{4.26}$$

Remark 4.12. At this point it is important to remark that some caution must be taken when interpreting the above asymptotic results. All the way from introducing the function $f = (f_1, \dots, f_n)$ to formulating the free-boundary problem (4.25), the dependency of f on the proportionality factor ε was tacitly assumed to be only through the arguments $\Delta\varphi^1, \dots, \Delta\varphi^n$. However, if the starting point is the free-boundary problem, we should expect the boundary conditions to add terms to the solution, which can, in general, be stochastic processes that do not depend on $\Delta\varphi$ but depend on ε explicitly. These terms will contribute to the dynamics of $\Delta S = f(\Delta\varphi)$ and to that of $b^{\Delta S}$ in Equation (4.21). Taking this fact into account, the asymptotic results remain valid only if the effect of the additional terms is of order $\mathcal{O}(\varepsilon^{2/3})$; that is to say, if the terms do not contribute at the leading order in ε. As shown in [KMK15, KL13] and indicated in Subsection 4.3.3 of this thesis, this is the case for the one-dimensional problem. Moreover, this will also be the case for our approximate construction of the shadow price, which is to be presented in Section 5.2 of the next chapter. However, as long as we do not have the function f in an explicit form, the fact that all additional terms which depend on ε explicitly and may appear in the expression for f do not contribute at the leading order becomes a crucial assumption. We will express this assumption by claiming that f is *asymptotically independent of* ε.

To summarize the results and emphasize the connection between the free-boundary and the Skorohod problem on the one hand and the asymptotic optimization problem on the other hand, we formulate the following

Proposition 4.13. *Let f and $\partial R = \bigcup_{j\geq 1} \partial R^{-,j} \cup \partial R^{+,j}$ be a solution to the free-boundary problem* (4.25)*, and let f be asymptotically independent of ε in sense of Remark 4.12. Let φ^* be the frictionless optimizer and Z_T its dual counterpart, and let $\Delta\varphi = (\Delta\varphi^1, \dots, \Delta\varphi^n)$ be a solution to the stochastic Skorohod problem* (4.26) *with the initial condition $\Delta\varphi_0^j = 0$, $j = 1, \dots, n$. Then, $\widetilde{S} = S + f(\Delta\varphi)$ is a shadow price which satisfies the asymptotic optimality conditions of Proposition 4.10 with respect to the process $\widetilde{Z} = y_* Z(1 - p\Delta\varphi \bullet S)$, and the strategy $\varphi = \varphi^* + \Delta\varphi$ is therefore asymptotically optimal in the sense of Definition 4.8.*

Proof. Assume that $(f, \partial R)$, with f being asymptotically independent of ε, solves the free-boundary problem (4.25), and the process $\Delta\varphi = (\Delta\varphi^1, \ldots, \Delta\varphi^n)$ solves the Skorohod problem (4.26) with $\Delta\varphi_0^j = 0$, $j = 1, \ldots, n$. Then, the strategy $\varphi = \varphi^+ - \varphi^- = \varphi^* + \Delta\varphi$ starts at the frictionless optimizer φ^* and changes only in the j-th component only if $\Delta\varphi$ hits $\partial R^{\pm,j}$; φ^j increases if $\partial R^{+,j}$ is hit, meaning that $\varphi^{+,j}$ increases, and φ^j decreases if $\partial R^{-,j}$ is hit, meaning that $\varphi^{-,j}$ increases. Since $\partial R^{\pm,j} = f_j^{-1}(\{\pm\varepsilon_j S^j\})$, transactions are made if and only if $\widetilde{S}^j = S^j + f_j(\Delta\varphi) = (1 \pm \varepsilon_j)S^j$. Setting $\widetilde{Z} = Z(1 - p\Delta\varphi \bullet S) + \mathcal{O}\left(\varepsilon^{2/3}\right)$ automatically satisfies the items (i), (iii) of Proposition 4.10 (recall the derivation of Equation (4.15)). Finally, the fact that f solves the free-boundary problem (4.25) implies that $b^{\Delta S} = pc^S\Delta\varphi + \mathcal{O}\left(\varepsilon^{2/3}\right)$ for $\Delta S = f(\Delta\varphi)$ which, in turn, guarantees that $b^{\widetilde{Z}\widetilde{S}^j} = \mathcal{O}\left(\varepsilon^{2/3}\right)$, $j = 1, \ldots, n$, and thus satisfies item (ii). $\square$

Remark 4.14. As already mentioned in the introduction to this chapter, some of the existing results suggest ([AI12, ARS17]) that the boundary of the no-trade region is non-convex and non-smooth. Whereas non-convexity is less obvious, the fact that the boundary has "corners" is not surprising. To each asset there must correspond a part of the boundary controlling the behaviour of the associated component of the trading strategy, and this boundary part must be well-distinguished from the others, since the investor must always be able to tell what asset to trade when the boundary is crossed. This immediately suggests a non-smooth structure of the boundary. Thus, the existence of the process $\Delta\varphi$ determining the optimal investment policy $\varphi = \varphi^* + \Delta\varphi$ is linked to two difficult problems: a multidimensional free-boundary problem and a stochastic Skorohod problem inside a non-convex domain with a non-smooth and time-dependent reflecting boundary.

Before discussing Proposition 4.13 any further, we first apply the heuristic results to analyse the asymptotic behaviour of the utility of a candidate policy and to express the leading-order utility loss in terms of the processes $\Delta\varphi$ and ΔS.

4.3.2 The leading-order utility loss

Let $\varphi = \varphi^* + \Delta\varphi$ and $\widetilde{S} = S + \Delta S$ denote a candidate strategy and a shadow-price process satisfying the asymptotic optimality conditions of Proposition 4.10. To compute the leading-order utility loss, we consider the Taylor expansion of the expected utility of terminal wealth with respect to the frictionless price process $\widetilde{S}$. Recall that the leading-order loss is $\mathcal{O}\left(\varepsilon^{2/3}\right)$. In the following Taylor approximation, we will use the asymptotic estimates obtained at the beginning of Subsection 4.3.1.

$$
\begin{aligned}
E\left[u(x+\varphi \bullet \widetilde{S}_T)\right] &= E[u(x+\varphi^* \bullet S_T + \underbrace{\Delta\varphi \bullet S_T}_{=\mathcal{O}(\varepsilon^{1/3})} + \underbrace{\varphi \bullet \Delta S_T}_{=\mathcal{O}(\varepsilon^{2/3})})] \\
&= E\left[u(x+\varphi^* \bullet S_T)\right] + E\left[u'(x+\varphi^* \bullet S_T)(\Delta\varphi \bullet S_T + \varphi \bullet \Delta S_T)\right] \\
&+ \frac{1}{2}E\left[u''(x+\varphi^* \bullet S_T)(\Delta\varphi \bullet S_T + \varphi \bullet \Delta S_T)^2\right] + \dots \ .
\end{aligned}
$$

Now recall the frictionless optimality condition $u'(x+\varphi^* \bullet S_T) = y_* Z_T$. Here, Z_T is the density of an equivalent martingale measure for S, the minimum entropy martingale measure (cf. Proposition 3.18), which will henceforth be denoted by Q. Note also that $u''(x) = -pu'(x) = p^2 u(x)$. We thus obtain

$$
\begin{aligned}
E\left[u(x+\varphi \bullet \widetilde{S}_T)\right] &= E\left[u(x+\varphi^* \bullet S_T)\right] + y_* E_Q\left[\Delta\varphi \bullet S_T + \varphi \bullet \Delta S_T\right] \\
&- \frac{p y_*}{2} E_Q\left[(\Delta\varphi \bullet S_T + \varphi \bullet \Delta S_T)^2\right] + \dots \\
&= E\left[u(x+\varphi^* \bullet S_T)\right]\left(1 - pE_Q\left[\varphi \bullet \Delta S_T\right] + \frac{p^2}{2}E_Q\left[(\Delta\varphi \bullet S_T)^2\right]\right) \\
&+ \mathcal{O}(\varepsilon) \ .
\end{aligned}
\tag{4.27}
$$

Notice that since an optimizing strategy necessarily satisfies $X(\varphi)_T = x + \varphi \bullet \widetilde{S}_T$ with x denoting the investor's initial capital, and the process X describing the liquidation wealth introduced in Definition 4.3, the transaction costs paid up to time T can be expressed as

$$
\varphi \bullet \widetilde{S}_T - \varphi \bullet S_T = \varphi \bullet \Delta S_T \ .
$$

Recalling the definition of the liquidation wealth (cf. Equation (4.4), Definition 4.3 and Remark 4.4), we obtain, at the leading order in ε,

$$
\varphi \bullet \Delta S = -S \bullet \bar{\varphi} = -\sum_j \varepsilon_j S^j \bullet (\varphi^{-,j} + \varphi^{+,j}) \ . \tag{4.28}
$$

The first term contributing to the leading-order utility loss can therefore be interpreted as loss due to transactions, and we write

$$
L^U_{tc} = -pE_Q\left[\varphi \bullet \Delta S_T\right] \ . \tag{4.29}
$$

The deviation from the frictionless optimizer, $\Delta\varphi$, enters the expression for the utility loss in the form of a Q-martingale and thus appears only in the second-order Taylor expansion. Its contribution to the leading-order utility loss is the second term,

$$
L^U_{disp} = \frac{p^2}{2}E_Q\left[(\Delta\varphi \bullet S_T)^2\right] = \frac{p^2}{2}E_Q\left[\Delta\varphi^\top c^S \Delta\varphi \bullet I_T\right] \ , \tag{4.30}
$$

which can be interpreted as loss due to displacement. We also define

$$L^U_{tot} = L^U_{tc} + L^U_{disp} \tag{4.31}$$

and stress that L^U_{tot} provides the *total utility percentage lost* due to the presence of transaction costs.

We now introduce the notion of the *certainty equivalent* which will be used in the numerical analysis later in this work. When dealing with the utility of terminal wealth or, in particular, with the utility loss due to market frictions, it is rather difficult to comprehend how much is actually lost. We have no natural utility-based scale of measure. Instead, everyone is used to quantifying gains and losses in currency units, and this is the rationale for using certainty equivalent. Assume we have a money amount X at our disposal. The exponential utility associated with this amount is given by

$$U = -e^{-pX} \ .$$

To reverse things, if one starts with the utility, U, then the amount of money to which this utility corresponds is just

$$X = -\frac{1}{p}\ln(-U) \ .$$

This is the motivation behind the following

Definition 4.15. Let $X(\varphi)_T$ be the payoff generated by a trading strategy φ. The *certainty equivalent* associated with the expected utility of this payoff is defined as

$$CE(X(\varphi)_T) = -\frac{1}{p}\ln\left(-E\left[u(X(\varphi)_T)\right]\right) \ . \tag{4.32}$$

With the help of the certainty equivalent, the expected utility of the payoff $X(\varphi)_T$ can be given the following interpretation: *trading according to φ up to the time T is expected to be as good as getting $CE(X(\varphi)_T)$ currency units.*

To express Equation (4.27) in terms of the certainty equivalent, we first denote the certainty equivalent corresponding to the wealth generated by the frictionless optimizer by CE^*. Then, the certainty equivalent of $X(\varphi)_T$, the liquidation wealth at time T generated by a strategy φ trading with transaction costs, at the leading order is obtained by a Taylor expansion of (4.32) combined with (4.27):

$$\begin{aligned} CE &= CE^* + E_Q\left[\varphi \bullet \Delta S_T\right] - \frac{p}{2}E_Q\left[\Delta\varphi^\top c^S \Delta\varphi \bullet I_T\right] + o\left(\varepsilon^{2/3}\right) \\ &= CE^* - \frac{1}{p}L^U_{tot} + o\left(\varepsilon^{2/3}\right) \ . \end{aligned} \tag{4.33}$$

One can summarize the results of our asymptotic considerations as follows. The heuristic method presented in Subsection 4.3.1 allowed us to transform the optimization problem for small transaction costs into the multidimensional free-boundary problem (4.25), containing the information about the no-trade region, together with the system of stochastic differential equations with reflections at the boundary of the no-trade region (4.26). However, for the portfolio dimension of at least two, even the approximation for small transaction costs does not simplify the problem sufficiently so that it can be solved explicitly using any tractable method, at least to the knowledge of the author. As one can see from Proposition 4.13, the difficulties start with the free-boundary problem which is quite challenging in the multidimensional case. But even if one had the exact boundary, one would still need to show the existence of a process $\Delta\varphi$ with a prescribed dynamics in the interior and a proper componentwise reflection at this boundary. Put differently, to obtain an asymptotic optimizer $\varphi = \varphi^* + \Delta\varphi$, one requires a strong solution, the process $\Delta\varphi$, to the stochastic Skorohod problem (4.26) in a non-convex domain with a non-smooth and time-dependent reflecting boundary. Moreover, in order to make any quantitative statements about utility or certainty-equivalent loss (cf. (4.33), (4.31)), which is given in terms of Q-expectations of expressions depending on $\Delta\varphi$, some knowledge or at least additional assumptions about the distribution of $\Delta\varphi$ in the no-trade region are required. In the one-dimensional case (the case with only one risky asset), however, the asymptotic optimization problem can be solved explicitly. The complete heuristic treatment can be found in [KMK15], and in [KL13] a rigorous proof is given. The one-dimensional result can be easily extended to the multidimensional cases of zero correlation and of complete correlation between the assets. These three special cases will be discussed in the remainder of this section. As for the actually interesting case of arbitrarily correlated assets in high dimensions, our treatment of it will rely on an approximation of the no-trade region which will be based on the results obtained in this chapter; in particular, the exact one-dimensional solution, to be discussed in the next subsection, will play a crucial role. With the help of this approximation, several candidate strategies as well as an appropriately constructed dual variable will be derived. These, in turn, will be used to estimate the expected utility of the actual asymptotic optimizer numerically.

4.3.3 The one-dimensional case

In the case of a single risky asset, the system of partial differential equations of the free-boundary problem (4.25) is reduced to the single ordinary differential equation

$$f''(x) = \frac{2pc^S}{c^{\varphi^*}} x \ . \tag{4.34}$$

The no-trade region is then given by $\widetilde{R} = \varphi^* + R$ with $R = [R^-, R^+] \subset \mathbb{R}$ being an interval, and the boundary conditions read

$$f'(R^\pm) = 0, \quad f(R^\pm) = \mp\varepsilon S \ . \tag{4.35}$$

Integrating and using the conditions for the first derivative yields

$$R^+ = -R^- =: \Delta \ ,$$

and

$$f(x) = \frac{pc^S}{3c^{\varphi^*}}x^3 - \frac{pc^S}{c^{\varphi^*}}\Delta^2 x + const \ .$$

The remaining conditions imply that $const = 0$, and we obtain

$$\begin{aligned} f(x) &= \frac{\varepsilon S}{2}\left(\left(\frac{x}{\Delta}\right)^3 - 3\frac{x}{\Delta}\right), \\ \Delta &= \left(\frac{3}{2p}\frac{c^{\varphi^*}}{c^S}\varepsilon S\right)^{1/3} \ . \end{aligned} \tag{4.36}$$

Having fixed the boundary, we now turn to finding the asymptotic optimizer $\varphi = \varphi^* + \Delta\varphi$. The results from Section 4.3.1 show that the process $\Delta\varphi \in [-\Delta, \Delta]$ can be obtained as a solution to the stochastic Skorohod problem (4.26) for $n = 1$. The existence of such a solution is shown in [SW13] (see also [KL13], Lemma 5.5). With $\Delta S = f(\Delta\varphi)$, the process $\widetilde{S} = S + \Delta S$ is then the shadow price with respect to which $\varphi = \varphi^* + \Delta\varphi$ is proved to be asymptotically optimal in [KL13].

To see that the solution is asymptotically optimal in terms of the heuristic arguments made in Subsecion 4.3.1, we should recall Remark 4.12. The essence is that the free-boundary problem (4.25) and the Skorohod problem (4.26) guarantee asymptotic optimality only under the condition that f is asymptotically independent of ε. That is to say that, at the leading order of $\mathcal{O}(\varepsilon^{1/3})$, only the dynamics of $[\Delta\varphi^j, \Delta\varphi^k]$, $j,k = 1,\dots,n$, contributes to the drift of $\Delta S = f(\Delta\varphi)$; more formally,

$$b^{\Delta S,j} = \frac{1}{2}\sum_{k,l}\partial_k\partial_l f_j(\Delta\varphi)c^{\varphi^*,kl} + \mathcal{O}(\varepsilon^{2/3}), \quad j = 1,\dots,n \ .$$

This can be easily checked directly by computing $df(\Delta\varphi)_t$ using (4.36). Hence, for only one risky asset, the leading-order utility loss defined in (4.31) reads

$$L^U_{tot} = -pE_Q\left[\varphi \bullet \Delta S_T\right] + \frac{p^2}{2}E_Q\left[((\Delta\varphi)^2 c^S) \bullet I_T\right]$$

It can be shown that (cf. [KMK15], A.5)

$$L^U_{tot} = \frac{p^2}{2} E_Q \left[(\Delta^2 c^S) \bullet I_T \right] \ . \tag{4.37}$$

In terms of the certainty-equivalent loss one obtains (compare also [KL13], Corollary 5.18)

$$CE = CE^* - \frac{p}{2} E_Q \left[(\Delta^2 c^S) \bullet I_T \right] + o\left(\varepsilon^{2/3} \right) \ . \tag{4.38}$$

Note that in (4.37), (4.38) Δ denotes the one-dimensional stochastic boundary introduced in (4.36).

4.3.4 The uncorrelated multidimensional case

We consider a market with $n \geq 2$ risky assets with no correlation. The price processes as well as the frictionless optimizers are modelled as

$$dX^j_t = b^{X^j}_t dt + \sigma^{X^j}_t dW^j_t, \quad X \in \{S, \varphi^*\}$$

with the usual assumption that $W^1, \ldots, W^n$ are independent Brownian motions. The local quadratic covariation of $X = (X^1, \ldots, X^n)$ is given by the diagonal matrix

$$c^X = \operatorname{diag}\left((\sigma^{X^1})^2, \ldots, (\sigma^{X^n})^2 \right) \ .$$

The system of partial differential equations of the free-boundary problem (4.25) reduces to

$$\sum_{k=1}^{n} c^{\varphi^*,k} \partial^2_k f_j(x) = 2pc^{S^j} x^j, \quad j = 1, \ldots, n \ . \tag{4.39}$$

It appears reasonable to assume that in the case of zero correlation, when each price process is independent of the others, the problem can be tackled by an ansatz replicating the one-dimensional solution in each dimension. It can indeed be verified that the free-boundary problem (4.25) with the system of partial differential equations reduced to (4.39) is solved by

$$\begin{aligned} f(x) &= (f_1(x_1), \ldots, f_n(x_n)), \\ f_j(x_j) &= \frac{\varepsilon_j S^j}{2} \left(\left(\frac{x_j}{\Delta^j} \right)^3 - 3 \frac{x_j}{\Delta^j} \right), \\ \Delta^j &= \left(\frac{3}{2p} \frac{c^{\varphi^*,j}}{c^{S^j}} \varepsilon_j S^j \right)^{1/3} , \end{aligned} \tag{4.40}$$

and the boundary is given by

$$\partial R^{\pm,j} = \prod_{k=1}^{j-1} [-\Delta^k, \Delta^k] \times \{\mp\Delta^j\} \times \prod_{k=j+1}^{n} [-\Delta^k, \Delta^k] \; . \tag{4.41}$$

Put differently, the no-trade region is a rectangular parallelotope of the form

$$\varphi^* + R = \prod_{j=1}^{n} [\varphi^{*,j} - \Delta^j, \varphi^{*,j} + \Delta^j] \; . \tag{4.42}$$

The function f from (4.40) can be shown to be asymptotically independent of ε in the sense of Remark 4.12 by computing $df(\Delta\varphi)_t$ explicitly. Hence, the leading-order utility loss is given in terms of

$$L^U_{tc} = -p \sum_{j=1}^{n} E_Q \left[\varphi^j \bullet f_j(\Delta\varphi^j)_T\right]$$

and

$$L^U_{disp} = \frac{p^2}{2} \sum_{j=1}^{n} E_Q \left[\left((\Delta\varphi^j)^2 c^{S,jj}\right) \bullet I_T\right] \; ,$$

which is just the sum of the one-dimensional losses due to transactions and displacement, respectively. Thus, using (4.37), we expect the total utility loss in the uncorrelated multidimensional case to be of the form

$$L^U_{tot} = \frac{p^2}{2} \sum_{j=1}^{n} E_Q \left[\left((\Delta^j \sigma^{S^j})^2\right) \bullet I_T\right] \; . \tag{4.43}$$

4.3.5 The case of complete correlation

In the present study, the situation of our interest is that of long-term investments in large portfolios. This puts certain restrictions on portfolios with extreme correlations between the assets. Allowing several *different* assets to be highly correlated creates a sort of an arbitrage opportunity. Take two assets for simplicity, and let ρ denote their correlation coefficient. Now assume that the assets are almost completely correlated, $\rho \sim 1$, but, at the same time, let them have different expected rates of return and different volatilities. Then, depending on the ratio of the rate of return to the volatility of the individual assets, the optimal strategy will select one of the assets to go short in. Put differently, the investor can short one of the stocks and invest the money in the "better" one. After a certain time, the "better" stock can be sold so as to close the short position and make profit. If such investment opportunities exist, they do so for a very short time. They are purely speculative and have very little to do with long-term investment strategies we wish to consider. The situation can easiest be illustrated with the help of the Black-Scholes model. In Section 3.2.4 of Chapter 3, it was shown that for an investor whose preference is described by the exponential utility function, $u(x) = -e^{-px}$, $p > 0$, it is optimal to keep a constant amount of money in each stock,

$$\varphi^{*,j} S^j = \frac{1}{p}(c^{-1}b)_j \; .$$

The covariance matrix satisfies

$$c = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}, \qquad c^{-1} = \frac{1}{1-\rho^2}\begin{pmatrix} \frac{1}{\sigma_1^2} & -\frac{\rho}{\sigma_1\sigma_2} \\ -\frac{\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2} \end{pmatrix} \; ,$$

with σ_1, σ_2 being the volatilities of the assets. Simple matrix multiplication shows that if we wish to have a strictly positive amount of money invested in both assets, we must satisfy

$$\left(\frac{b_1}{\sigma_1} > \rho\frac{b_2}{\sigma_2}\right) \quad \wedge \quad \left(\frac{b_2}{\sigma_2} > \rho\frac{b_1}{\sigma_1}\right) \; .$$

Both inequalities can only be fulfilled for an arbitrary ρ if both assets have equal rates of return and equal volatilities. But splitting the amount of money one wishes to invest between multiple completely correlated and thus essentially identical assets is exactly as good as to invest the entire money in one of them. Thus, the case complete correlation (short-selling strategies excluded) is already covered by the one-dimensional solution discussed in Subsection 4.3.3. This fact will also be confirmed by the simulation results to be presented in Chapter 6.

We can define the no-trade region for a completely correlated portfolio by means of a rather heuristic argument. Assume that we invest in $n > 1$ completely correlated assets. As discussed above, an optimal *frictionless* investment strategy in this case is splitting the optimal one-stock investment equally between the completely correlated assets. In other words, in each portfolio component, we hold the n-th fraction of the optimal number prescribed by the one-dimensional frictionless solution. In the presence of proportional transaction costs, an optimal strategy will start at the frictionless optimizer and vary along the straight line with the directional vector $\mathbf{1}$, $\mathbf{1}_j = 1$ for all $j = 1, \ldots, n$, since each component is treated equally due to their indistinguishability. To obtain the asymptotically optimal maximum distance from the frictionless optimizer, we exploit the one-dimensional analogy. Recall that the boundary process Δ describing the half-width of the one-dimensional no-trade region measures the *number of shares* by which the portfolio can maximally deviate from the optimum (cf. Equation (4.36)). We now demand that the n-asset portfolio must not deviate farther from the frictionless optimum than $\pm\Delta$ shares *in total*. This implies that each portfolio component takes on its values in the interval $\left[-\frac{1}{n}\Delta, \frac{1}{n}\Delta\right]$. Therefore, in the limiting case of complete correlation, the no-trade region can be described as a line segment in $\mathbb{R}^n$ given by

$$\widetilde{R} = \left\{ x \in \mathbb{R}^n \colon x = \varphi^* + \lambda \Delta \mathbf{1},\ \lambda \in [-1, 1] \right\} . \tag{4.44}$$

Note also that the line segment has the length 2Δ with respect to the l_1-*distance*.

Chapter 5

Approximations

As discussed in the previous chapter, given the frictionless optimizing strategy φ^*, solving the portfolio-optimization problem with small proportional transaction costs, i. e. determining an asymptotically optimal trading strategy of the form $\varphi = \varphi^* + \Delta\varphi$, is equivalent to first finding the boundary of the no-trade region, given as a solution to the free-boundary problem (4.25), and then solving the stochastic Skorohod problem (4.26) to obtain the process $\Delta\varphi$. However, already the free-boundary problem presents a great challenge even in the case of low dimensionality, such as $n = 3$, not to mention the cases of moderate to high dimensionality, such as $n = 10$, $n = 30$ or higher. The next obstacle is the existence of the process $\Delta\varphi$ as a strong solution to a reflecting stochastic differential equation, since, as we already stressed in Remark 4.14, the no-trade region is expected to be non-convex with a non-smooth and time-dependent boundary. Finally, in order to calculate the utility loss of a candidate strategy $\varphi = \varphi^* + \Delta\varphi$, the knowledge of the distribution of the process $\Delta\varphi$ inside the no-trade region is required.

In this chapter, a construction scheme of an approximation to the no-trade region is proposed. Our primary goal is to obtain a tractable approximation which can easily be implemented in computer simulations. Important properties of no-trade regions were already discussed in Chapter 4; in particular, in Remark 4.14, we argued that the boundary of a no-trade region must be non-smooth. Our approximations will have the form of convex polyhedra constructed symmetrically around the frictionless optimizer. To be more precise, we will consider convex polyhedral domains which can be represented as linear transformations of rectangular parallelotops, which poses a great advantage in numerical applications. Candidate strategies with respect to approximate no-trade regions will be defined as solutions to Skorohod problems with appropriate boundary conditions (cf. Definition 5.1). Moreover, in Section 5.2, we will introduce a simple method for associating a dual upper bound with a given approximation of the no-trade region. As in the general asymptotic model, which was treated heuristically in the previous chapter, a dual variable will be a pair $(\widetilde{Q}, \widetilde{S})$ consisting of a shadow-price process $\widetilde{S} = S + \Delta S$ and a measure $\widetilde{Q} \sim P$ making $\widetilde{S}$ a martingale. The main difference to the previously discussed asymptotic theory is that there will be no connection between shadow prices and trading strategies via a free-boundary problem. It will be shown that ΔS can be constructed from an auxiliary process $\psi = \varphi^* + \Delta\psi$ via $\Delta S = f(\Delta\psi)$. The process ψ behaves almost like a trading strategy in the sense that it satisfies $d\Delta\psi = -d\varphi^*$ in the interior of the approximate no-trade region but its reflecting properties at the boundary are different from those of a usual trading strategy[1]. The unusual behaviour at the boundary will turn out to be necessary in order to ensure the existence of an equivalent martingale measure such that the frictionless market $\widetilde{S} = S + \Delta S$ is free of arbitrage. Finally, the dual variable $(\widetilde{Q}, \widetilde{S})$ will be used to derive an upper bound on the expected payoff generated by an arbitrary candidate strategy. The quality of such an upper bound is rather difficult to asses. The value of the upper bound alone does not tell us how much better than the trivial upper bound, the expected utility of the frictionless optimizer, it actually is. To deal with this problem, in the subsections 5.1.1 – 5.1.3, we introduce three different candidate strategies which will be analysed numerically in the Black-Scholes model in Chapter 6 to compare the expected utility of these policies with the value of the upper bound.

5.1 No-trade region and trading strategy

In this section, we introduce three domains to serve as candidates for the no-trade region, which will be basic to the numerical analysis of the next chapter. As already mentioned in the introduction to this chapter, the no-trade regions will have the form of linearly transformed rectangular parallelotops. In Subsection 4.3.1, we introduced trading strategies as solutions to Skorohod problems (cf. Equations (4.26)). The boundary conditions of the reflecting stochastic differential equations in (4.26) as well as the boundary of the no-trade region as such are determined by the shadow-price process linked to the trading strategy via the free-boundary problem (4.25) (see also Equations (4.24)). In our approach, this connection does not exist: we define trading strategies with respect to a given candidate domain, and the shadow price must then be found using different techniques. Whereas the latter will be the subject of the next section, we begin this section with a proper definition of a trading strategy with respect to a general parallelotope as the no-trade region. Our candidates will be introduced subsequently as special cases.

As already mentioned, when dealing with an asymptotically optimal trading strategy,

[1] A *usual* trading strategy is a process $\varphi = \varphi^* + \Delta\varphi$ satisfying Definition 5.1

the boundary of the no-trade region and the behaviour of the strategy at this boundary are both determined by the associated shadow price. In particular, as stated in (4.26), reflections of the trading strategy occur in a single component with probability one. Given a candidate domain approximating the no-trade region, the direction of reflection of the associated trading strategy may be arbitrary as long as the strategy stays inside the no-trade region after the reflection. One could, of course, argue that there must be at least one distinguished direction of reflection, namely the one leading to an optimal trading strategy with respect to a given domain. However, this argument does not help construct tractable approximations since it results in yet another optimization problem. We will construct our approximations by continuing to work with strategies reflecting in a single component at the boundary of the parallelotope we choose to be the no-trade region. That is to say that each side of the parallelotope corresponds to only one asset and vice versa. The notion of a trading strategy corresponding to a given candidate domain is made precise in the following

Definition 5.1. Let $\varphi^* = (\varphi^{*,1}, \dots, \varphi^{*,n})^\top$,

$$\varphi_t^{*,j} = \varphi_0^{*,j} + \int_0^t b_u^{\varphi^*,j} du + \sum_{k=1}^m \int_0^t \sigma_u^{\varphi^*,jk} dW_u^k$$

be the frictionless optimizing strategy and $\Delta = (\Delta^1, \dots, \Delta^n)^\top$ a stochastic process such that $\Delta^j \in \mathcal{O}\left(\varepsilon^{1/3}\right)$, $j = 1, \dots, n$. Set

$$\overline{R} = \prod_{j=1}^n \left[-\Delta^j, \Delta^j\right] .$$

For an invertible matrix $L \in \mathbb{R}^{n\times n}$, we define the no-trade region associated with L and Δ through

$$\widetilde{R} = \varphi^* + R, \qquad R = L\overline{R} . \tag{5.1}$$

The boundary of R then satisfies

$$\begin{aligned} \partial R &= \bigcup_{j=1}^n \partial R^{+,j} \cup \partial R^{-,j}, \qquad \partial R^{\pm,j} = L\, \partial \overline{R}^{\pm,j} \ , \\ \partial \overline{R}^{\pm,j} &= \prod_{k=1}^{j-1} \left[-\Delta^k, \Delta^k\right] \times \{\mp\Delta^j\} \times \prod_{k=j+1}^{n} \left[-\Delta^k, \Delta^k\right] . \end{aligned} \tag{5.2}$$

Let $\mu = (\mu_1, \dots, \mu_n)$ be a function defined by

$$\mu_j(x) = \operatorname{sgn}\left((L^{-1}x)_j\right) \ .$$

A finite-variation process $\varphi = (\varphi^1, \ldots, \varphi^n)^\top$ will be called a trading strategy to the no-trade region $\widetilde{R}$ if there exists a stochastic process $\Delta\varphi = (\Delta\varphi^1, \ldots, \Delta\varphi^n)^\top = \mathcal{O}\left(\varepsilon^{1/3}\right)$ which, for all $j = 1, \ldots, n$ and $t \geq 0$, satisfies

$$\begin{aligned} \Delta\varphi_t &= -\varphi_t^* + \varphi_t \in R_t, \quad \Delta\varphi_0 = 0 \ , \\ \varphi_t^j &= \varphi_0^{*,j} + \int_0^t \mu_j(\Delta\varphi_u) d|D\varphi^j|_u \ , \\ |D\varphi^j|_t &= \int_0^t I_{\partial R_u^{\pm,j}}(\Delta\varphi_u) d|D\varphi^j|_u \ , \end{aligned} \tag{5.3}$$

where $|D\varphi^j|$ denotes the total-variation process of φ^j, $j = 1, \ldots, n$.

Remark 5.2. Before introducing the candidate domains and the associated trading strategies, we recapitulate the results obtained in the previous chapter, which concern the two explicitly solvable multidimensional problems: the case of zero correlation and the case of complete correlation (cf. Subsections 4.3.4, 4.3.5). In the uncorrelated case, the no-trade region was shown to be a rectangular parallelotope $\varphi^* + \prod[-\Delta^j, \Delta^j]$ with Δ^j, $j = 1, \ldots, n$, being the boundary process associated with the one-dimensional solution corresponding to the price process S^j (cf. Subsection 4.3.3). As discussed in Subsection 4.3.5, for completely correlated assets, the no-trade region can be represented as the line segment connecting the points $\varphi^* \pm \Delta\mathbf{1}$, Δ being the half-width of the one-dimensional no-trade region (cf. Equation (4.44)). Thus, a candidate domain $R = L\overline{R}$, as introduced in Definition 5.1, should have the following properties:

(i) For n *uncorrelated* assets $S^1, \ldots, S^n$, R must coincide with the exact asymptotic solution, i. e.,

$$R = \prod_{j=1}^{n} [-\Delta^j, \Delta^j] \ , \tag{5.4}$$

Δ^j, $j = 1, \ldots, n$, being the half-width of the one-dimensional no-trade region associated with S^j (cf. Equations (4.40), (4.42)).

(ii) As already argued, in the limiting case of *complete correlation* we essentially deal with n identical assets, and the no-trade region can be described by a line segment (cf. Eqution (4.42)). Our candidate domain need not become a line segment in the limiting case. It must rather be such that its boundary limits the movement of a candidate strategy to a segment of the length 2Δ (with respect to the l_1-norm), Δ being the half-width of the asymptotically optimal one-dimensional no-trade region. More formally, the line along which the portfolio varies is given by $\varphi^* + \mathbb{R}\mathbf{1}$, and our candidate domain is of the form $\widetilde{R} = \varphi^* + R$, $R = L\overline{R}$ (cf. Definition 5.1). Let δ^1 and δ^2 be the two intersection points of the line $\mathbb{R}\mathbf{1}$ with the boundary of

R,

$$\{\delta^1, \delta^2\} = \mathbb{R}\mathbf{1} \cap \partial R \ .$$

Then, the candidate domain will have the property of the no-trade region in the limiting case of complete correlation if

$$\left\|\delta^1 - \delta^2\right\|_1 = 2\Delta \ , \tag{5.5}$$

with $\|x\|_1 = \sum_{j=1}^n |x_n|$ denoting the l_1-norm of $x \in \mathbb{R}^n$.

5.1.1 Naive candidate

Our first candidate for the no-trade region is probably the simplest construction incorporating the one-dimensional solution discussed in Subsection 4.3.3. Let $S = (S^1, \ldots, S^n)^\top$ be the price process of n arbitrarily correlated assets and $\varphi^* = (\varphi^{*,1}, \ldots, \varphi^{*,n})^\top$ be the frictionless optimizer for this market. Define the boundary process $\Delta = (\Delta^1, \ldots, \Delta^n)^\top$ by

$$\Delta^j = \left(\frac{3}{2p}\frac{c^{\varphi^*,jj}}{c^{S,jj}}\varepsilon_j S^j\right)^{1/3} \tag{5.6}$$

and let the transformation matrix be the identity, $L_{ij} = \delta_{ij}$ (compare the definition of the boundary process with the one-dimensional solution (4.36) in Subsection 4.3.3). The no-trade region thus is the rectangular parallelotope $\varphi^* + R$,

$$R = \prod_{j=1}^n [-\Delta^j, \Delta^j] \ . \tag{5.7}$$

Note that possible correlations between the assets are *not* neglected in this construction. The correlation coefficients enter into the above definition of the no-trade region via the local quadratic covariation of the frictionless optimizer, c^{φ^*}. For example, in the Black-Scholes model we have (cf. Equation (6.5))

$$c^{\varphi^*,jj} = \left(\frac{\sigma_j(c^{-1}b)_j}{pS^j}\right)^2 , \qquad c_{jk} = \sigma_j\sigma_k\rho_{jk} \ ,$$

with ρ being the correlation matrix. However, since the linear transformation L is trivial, asset correlations may affect only the size and proportions of the no-trade region which then still remains rectangular.

5.1.2 A more sophisticated construction

A more sophisticated construction of the approximate no-trade region proposed here is based on solving a series of one-dimensional problems. Consider the price process $S = (S^1, \dots, S^n)^\top$ of n arbitrarily correlated assets and let $\varphi^* = (\varphi^{*,1}, \dots, \varphi^{*,n})^\top$ be the frictionless optimizer for this market. For a fixed $i \in \{1, \dots, n\}$, let the processes $(\varphi^j)_{j \neq i}$ be of the form $\varphi^j = \varphi^{*,j} + \Delta\varphi^j$ where the deviations from the frictionless optimizers, $\Delta\varphi^j$, are taken to be small. Assume that the investor can only control the position in the i-th asset, the trades can be carried out without transaction costs, and the remaining portfolio positions are taken care of *optimally* by some individuals. Under these assumptions, we are first interested in the value of $\varphi^i = \varphi^{*,i} + \Delta\varphi^i$ which maximizes the expected utility of the terminal wealth of the entire portfolio at the leading order in $\Delta\varphi$. Put differently, we want to determine the optimal value of $\Delta\varphi^i$ given that $(\Delta\varphi^j)_{j \neq i}$ are already at the optimal position. We then denote the optimal position in the i-th asset by $\tilde{\varphi}^{*,i}$ and take it be the starting point for a constrained one-dimensional problem. Next, we impose boundaries around $\tilde{\varphi}^{*,i}$ that are known from the one-dimensional case (cf. Section 4.3.3, Equation (4.36)). For all possible values of $x_j = \varphi^j_t(\omega) = \varphi^{*,j}_t(\omega) + \Delta\varphi^j_t(\omega)$, $j \neq i$, and the associated optimal responses $x_i^* = \tilde{\varphi}^{*,i}_t(\omega)$, this construction defines a set $R^{(i)} \subset \mathbb{R}^n$ whose elements $x = (x_1, \dots, x_n)$ are such that x_i is confined to the boundaries of the one-dimensional solution around x_i^*. By repeating this procedure for each $i = 1, \dots, n$, we then define our approximation of the no-trade region as $\widetilde{R} = \bigcap R^{(i)}$. As we shall see in the following, at the leading order in ε, $\widetilde{R}$ will turn out to be a parallelotope.

In the above setting, let $i \in \{1, \dots, n\}$ be fixed, and let all $\Delta\varphi^j = \varphi^j - \varphi^{*,j}$, $j \neq i$, be at their optimal positions. We now determine the optimal value of $\Delta\varphi^i$ such as to maximize the expected utility of terminal wealth of the entire portfolio $(\varphi^{*,k} + \Delta\varphi^k)_{k=1,\dots,n}$. The terminal wealth associated with the strategy $\varphi = (\varphi^1, \dots, \varphi^n)^\top$ can be written as

$$V_T = V_T^0 + \Delta V_T = \sum_{k=1}^{n} \left(\varphi^{*,k} \bullet S_T^k + \Delta\varphi^k \bullet S_T^k\right) .$$

The Taylor expansion of the utility function in powers of ΔV_T at V_T^0 then reads as

$$u(V_T) = u(V_T^0) + u'(V_T^0)\Delta V_T + \frac{1}{2}u''(V_T^0)(\Delta V_T)^2 + \mathcal{O}\left((\Delta V_T)^3\right) .$$

Using $u(x) = -e^{-px}$ and recalling the frictionless optimality condition $u'(V_T^0) = Z_T E(u'(V_T^0))$, Z_T being the density of an equivalent martingale measure for S, one obtains

$$E(u(V_T)) = E(V_T^0)\left(1 - pE_Q(\Delta V_T) - \frac{p^2}{2}E_Q\left((\Delta V_T)^2\right)\right) + \mathcal{O}\left((\Delta V_T)^3\right) \ .$$

Since $\Delta\varphi^j \bullet S^j$ is a Q-martingale for all j, the term $E_Q(\Delta V_T)$ vanishes, and the problem is reduced to minimizing $E_Q\left((\Delta V_T)^2\right)$ with respect to $\Delta\varphi^i$.

$$\begin{aligned}
E_Q\left((\Delta V_T)^2\right) =& E_Q\left(\sum_{k,l}(\Delta\varphi^k \bullet S_T^k)(\Delta\varphi^l \bullet S_T^l)\right) \\
=& \sum_{k,l} E_Q\left(\left(\Delta\varphi^l(\Delta\varphi^k \bullet S^k)\right) \bullet S_T^l + \left(\Delta\varphi^k(\Delta\varphi^l \bullet S^l)\right) \bullet S_T^k\right) \\
&+ \sum_{k,l} E_Q\left((\Delta\varphi^k\Delta\varphi^l) \bullet [S^k, S^l]_T\right) \\
=& \sum_{k,l} E_Q\left(\Delta\varphi^k\Delta\varphi^l c^{S,kl} \bullet I_T\right) = E_Q\left(\int_0^T \Delta\varphi_t^\top c_t^S \Delta\varphi_t \, dt\right) \ .
\end{aligned}$$

Since the local quadratic covariation c^S is a positive semi-definite matrix for each (t,ω), the optimal value of $\Delta\varphi^i$ is obtained by minimizing the integrand for each (t,ω). Taking into account the symmetry of c^S, the value of $\Delta\varphi^i$ minimizing the quadratic form $\Delta\varphi^\top c^S \Delta\varphi$ is given through

$$\Delta\varphi^i = -\sum_{j\neq i} \frac{c^{S,ij}}{c^{S,ii}} \Delta\varphi^j \ . \tag{5.8}$$

Now let $\tilde{\varphi}^{*,i}$ denote the value of the strategy $\varphi^{*,i} + \Delta\varphi^i$ with $\Delta\varphi^i$ satisfying the optimality condition (5.8). This value then reads as

$$\tilde{\varphi}^{*,i} = \varphi^{*,i} - \sum_{j\neq i} \frac{c^{S,ij}}{c^{S,ii}} \Delta\varphi^j = \frac{1}{c^{S,ii}}\left(c^S\varphi^*\right)_i - \sum_{j\neq i} \frac{c^{S,ij}}{c^{S,ii}} \varphi^j. \tag{5.9}$$

Note that, for all possible choices of $(\varphi^j)_{j\neq i}$, the corresponding optimizers can be viewed as lying in the hyperplane $x_i^*\mathbf{e}^{(i)} + H^{(i)}$[2],

$$H^{(i)} = \left\{x \in \mathbb{R}^n : x_i = -\sum_{j\neq i} \frac{c^{S,ij}}{c^{S,ii}} x_j\right\}, \quad x_i^* = \frac{1}{c^{S,ii}}\left(c^S\varphi^*\right)_i \ . \tag{5.10}$$

Next, we take the point

$$\tilde{\varphi}^{(i)} = (\varphi^1, \dots, \varphi^{i-1}, \tilde{\varphi}^{*,i}(\varphi^1, \dots, \varphi^{i-1}, \varphi^{i+1}, \dots, \varphi^n), \varphi^{i+1}, \dots, \varphi^n) \tag{5.11}$$

[2] $\mathbf{e}^{(i)}$ denotes the i-th standard-basis vector of $\mathbb{R}^n$.

to be the optimal one for the one-dimensional **constrained** problem along the line $\tilde{\varphi}^{(i)} + \mathbb{R}\mathbf{e}^{(i)}$ and impose a boundary around $\tilde{\varphi}^{(i)}$ known from the solution of the one-dimensional free-boundary problem (4.36). In other words, the portfolio, now having only one degree of freedom, varies on the line segment $\tilde{\varphi}^{(i)} + r\mathbf{e}^{(i)}$, $r \in [-\Delta^i, \Delta^i]$, Δ^i being the half-width of the one-dimensional no-trade region (4.36). Thus, for all possible values of $(\varphi^j)_{j\neq i}$, the location of the portfolios under the one-dimensional constraint is confined to the space between the two hypersurfaces $H_\pm^{(i)}$ satisfying

$$H_\pm^{(i)} = \left\{ x \in \mathbb{R}^n \colon x_i = x_i^* - \sum_{j\neq i} \frac{c^{S,ij}}{c^{S,ii}}\, x_j \pm \Delta^i \right\} \ , \tag{5.12}$$

with x_i^* defined in (5.10).
By Equation (4.36), the results obtained for the one-dimensional constrained optimization problem imply that the boundary process Δ^i is given through

$$\Delta^i = \left(\frac{3}{2p} \frac{c^{\tilde{\varphi}^{*,i}}}{c^{S,ii}} \varepsilon_i S^i \right)^{1/3} \ , \tag{5.13}$$

where $c^{\tilde{\varphi}^{*,i}}$ refers to the local quadratic covariation of the optimizer (5.9), which is to distinguish from c^{φ^*} originally used in the one-dimensional solution (4.36). To compute $c^{\tilde{\varphi}^{*,i}}$, consider the process $\left[\tilde{\varphi}^{*,i}\right]$. From (5.9) one obtains

$$d\left[\tilde{\varphi}^{*,i}\right] = d\left[\varphi^{*,i}\right] - 2\sum_{j\neq i} d\left[\varphi^{*,i}, \frac{c^{S,ij}}{c^{S,ii}}\Delta\varphi^j\right] + \sum_{j,k\neq i} d\left[\frac{c^{S,ij}}{c^{S,ii}}\Delta\varphi^j, \frac{c^{S,ik}}{c^{S,ii}}\Delta\varphi^k\right] .$$

Applying integration by parts to the terms $\frac{c^{S,ij}}{c^{S,ii}}\Delta\varphi^j$ leads to

$$\begin{aligned} d\left[\tilde{\varphi}^{*,i}\right] &= d\left[\varphi^{*,i}\right] - 2\sum_{j\neq i} \Delta\varphi^j d\left[\varphi^{*,i}, \frac{c^{S,ij}}{c^{S,ii}}\right] - 2\sum_{j\neq i} \frac{c^{S,ij}}{c^{S,ii}} d\left[\varphi^{*,i}, \Delta\varphi^j\right] \\ &\quad + \sum_{j,k\neq i} \Delta\varphi^j \Delta\varphi^k d\left[\frac{c^{S,ij}}{c^{S,ii}}, \frac{c^{S,ik}}{c^{S,ii}}\right] + \sum_{j,k\neq i} \frac{c^{S,ij}c^{S,ik}}{(c^{S,ii})^2} d\left[\Delta\varphi^j, \Delta\varphi^k\right] . \end{aligned}$$

Since the boundary processes Δ^j as well as the candidate strategies $\Delta\varphi^j$ are of order $\mathcal{O}\left(\varepsilon^{1/3}\right)$, we must have $\left[\tilde{\varphi}^{*,i}\right] = \mathcal{O}(1)$ (cf. Eq. (5.13)). Since, in addition, $d\Delta\varphi^j = -d\varphi^{*,j}$ in the interior of the no-trade region, the local quadratic covariation $c_t^{\tilde{\varphi}^{*,i}} = \frac{d\left[\tilde{\varphi}^{*,i}\right]_t}{dt}$ of the optimizer $\tilde{\varphi}^{*,i}$ can be expressed as

$$c^{\tilde{\varphi}^{*,i}} = c^{\varphi^*,ii} + 2\sum_{j\neq i} \frac{c^{S,ij}}{c^{S,ii}} c^{\varphi^*,ij} + \sum_{j,k\neq i} \frac{c^{S,ij}c^{S,ik}}{(c^{S,ii})^2} c^{\varphi^*,jk} + \mathcal{O}\left(\varepsilon^{1/3}\right) \ . \tag{5.14}$$

Substituting this result into (5.13) gives (after an appropriate term rearrangement)

$$\Delta^i = \frac{1}{c^{S,ii}} \left[\frac{3\varepsilon_i S^i}{2p} \left(c^S c^{\varphi^*} c^S \right)^{ii} \right]^{1/3} + \mathcal{O}\left(\varepsilon^{2/3}\right) \ . \tag{5.15}$$

Hence, at the leading order in ε, $H^{(i)}_\pm$ in (5.12) define two hyperplanes and can therefore be written as

$$H^{(i)}_\pm = (x^*_i \pm \Delta^i)\mathbf{e}^{(i)} + H^{(i)}. \tag{5.16}$$

Define $R^{(i)}$ to be the set of all points lying between $H^{(i)}_+$ and $H^{(i)}_-$; more formally,

$$R^{(i)} = \left\{ x \in \mathbb{R}^n \colon \exists\, \delta \in [-\Delta^i, \Delta^i] \colon x \in (x^*_i + \delta)\mathbf{e}^{(i)} + H^{(i)} \right\} \ . \tag{5.17}$$

The set $R^{(i)}$ can be interpreted as the no-trade region of an investor which can only control the i-th component of the portfolio with the remaining components being adjusted exogenously. Finally, intersecting the sets $R^{(i)}$ for all $i \in \{1, \dots, n\}$ we obtain the no-trade region:

$$\widetilde{R} = \bigcap_{i=1}^{n} R^{(i)} \ . \tag{5.18}$$

Remark 5.3. At first glance, one might think that the set $\widetilde{R}$ defined in the above Equation 5.18 is already the asymptotically optimal no-trade region. Indeed, one could argue that $\widetilde{R}$ is obtained by successively optimizing each portfolio component, which must result in an optimal final solution. To see that this is not the case, we must keep in mind that in the above derivation we take $\varepsilon \sim 0$ only in one portfolio component, tacitly assuming that the asymptotic limit does not affect the rest of the portfolio. This assumption, of course, can be viewed only as an approximation allowing us to apply the one-dimensional asymptotic results established in Subsection 4.3.3.

We now represent the no-trade region $\widetilde{R}$ as a linear transformation of a rectangular parallelotope, in the sense of Definition 5.1. Define the transformation M,

$$M^{ij} = \frac{c^{S,ij}}{c^{S,ii}} \ , \tag{5.19}$$

and let

$$\overline{R} = [-\Delta^1, \Delta^1] \times \ldots \times [-\Delta^n, \Delta^n] \ , \tag{5.20}$$

with Δ^j, $j = 1, \dots, n$, being the boundary processes introduced in (5.15). The no-trade region $\widetilde{R}$ defined in (5.18) then satisfies

$$\begin{aligned}\widetilde{R} &= \bigcap_{i=1}^{n} \left\{ x \colon \exists \xi_i \in [-\Delta^i, \Delta^i] \colon x_i = x_i^* + \xi_i - \sum_{j \neq i} \frac{c^{S,ij}}{c^{S,ii}} x_j \right\} \\ &= \bigcap_{i=1}^{n} \left\{ x \colon \exists \xi_i \in [-\Delta^i, \Delta^i] \colon (Mx)_i = (M\varphi^*)_i + \xi_i \right\} \\ &= \left\{ x \colon \exists \xi \in \overline{R} \colon x = \varphi^* + M^{-1}\xi \right\} = \varphi^* + R \ ,\end{aligned} \tag{5.21}$$

with

$$R = M^{-1}\overline{R} \ .$$

Definition 5.1 now applies with $L = M^{-1}$.

5.1.3 Alternative candidate

To construct an alternative candidate, we begin by simplifying the construction of the *naive* candidate presented in Subsection 5.1.1. In the definition of the boundary process Δ, Equation (5.6), replace the local quadratic covariation of the frictionless optimizer, c^{φ^*}, by $c^{\widehat{\varphi}}$ with $\widehat{\varphi} = (\widehat{\varphi}^1, \dots, \widehat{\varphi}^n)^\top$ and $\widehat{\varphi}^j$, $j = 1, \dots, n$, being the *one-dimensional* frictionless optimizer to the price process S^j. In particular, this means that we assume the components of $\widehat{\varphi}$ to satisfy

$$d\widehat{\varphi}_t^j = b_t^{\widehat{\varphi}^j} dt + \sigma_t^{\widehat{\varphi}^j} dW_t^j, \quad j = 1, \dots, n \ ,$$

with $W^1, \dots, W^n$ being independent standard Brownian motions, which, in turn, implies that the local quadratic covariation of $\widehat{\varphi}$ is of the form

$$c^{\widehat{\varphi}} = \operatorname{diag}\left((\sigma^{\widehat{\varphi}^1})^2, \dots, (\sigma^{\widehat{\varphi}^n})^2 \right) \ .$$

We obtain the boundary processes (cf. Equation (5.6))

$$\Delta^j = \left(\frac{3\varepsilon_j S^j}{2p} \frac{(\sigma^{\widehat{\varphi}^j})^2}{c^{S,jj}} \right)^{1/3} \ . \tag{5.22}$$

The set $\varphi^* + \prod [-\Delta^j, \Delta^j]$ as a candidate for the no-trade region has the major drawback of being independent of possible stock correlations. This issue can be dealt with by choosing the linear transformation L from Definition 5.1 such as to multiply each boundary process Δ^j, $j = 1, \dots, n$, by a correlation-dependent scale factor κ^j. We define $\kappa = (\kappa^1, \dots, \kappa^n)^\top$ such that the resulting set $L\overline{R}$, $\overline{R} = \prod [-\Delta^j, \Delta^j]$, fulfils the conditions outlined in Remark 5.2. From Equation (5.22) we see that in the case of uncorrelated assets, κ must satisfy $\kappa^j \equiv 1$ since $\varphi^* + \prod[-\Delta^j, \Delta^j]$ coincides with the exact solution from Subsection 4.3.4, Equation (4.42). Now let the portfolio consist of $n > 1$ completely correlated assets. Then all components of Δ are identical, and we set $\Delta^j \equiv \widehat{\Delta}$. In this case, κ must be such that the condition (5.5) is satisfied. The line $\mathbb{R}\mathbf{1}$ intersects the boundary of $[-\widehat{\Delta}, \widehat{\Delta}]^n$ in the two vertices

$$\delta^{1,2} = \pm\widehat{\Delta}\mathbf{1} \ .$$

The l_1-length of the resulting line segment then reads $\|\delta^1 - \delta^2\|_1 = 2n\widehat{\Delta}$. Hence, to obtain the correct no-trade region, κ must fulfil $\kappa^j \equiv 1/n$ in the case of $n > 1$ completely correlated assets.

To construct such a scaling vector, let $\widehat{\varphi}^j$ again denote the *one-dimensional* optimal investment strategy to the price process S^j, for each $j = 1, \dots, n$. Define

$$\kappa^j = \frac{\varphi^{*,j}}{\widehat{\varphi}^j}, \qquad j = 1, \dots, n \ . \tag{5.23}$$

Then, for each j, κ has the desired property of taking on the value 1 and $1/n$ in the cases of zero and complete correlation, respectively. Thus, our alternative candidate is given by the pair (L, Δ) consisting of the linear transformation

$$L = \operatorname{diag}\left(\kappa^1, \dots, \kappa^n\right) \ , \tag{5.24}$$

and the boundary process defined in Equation (5.22).

5.2 Dual upper bound

The upper bound on the expected utility of candidate strategies we propose will be given in terms of the convex conjugate of the utility function. As demonstrated in Chapter 3 (cf. Proposition 3.21), when taking the convex-duality approach to the utility-maximization problem, the set of all equivalent (local) martingale measures (which we often identify with their densities with respect to the real-world measure, P) for the price process S turns out to be the right choice for the set of dual variables. To explain how the dual theory can be used to construct a non-trivial upper bound for candidate strategies, we briefly recapitulate the main assertions of Proposition 4.13 suggesting how asymptotic optimality can be obtained. Let $\varphi = \varphi^* + \Delta\varphi = \varphi^+ - \varphi^-$ be a candidate strategy. If a function $f = (f_1, \dots, f_n)$ and a boundary ∂R,

$$\partial R = \bigcup_{j \geq 1} f_j^{-1}\left(\{\varepsilon_j S^j\}\right) \cup f_j^{-1}\left(\{-\varepsilon_j S^j\}\right) ,$$

solve the free-boundary problem (4.25), then φ is optimal if $\Delta\varphi$ solves the Skorohod problem (4.26) with respect to ∂R. In other words, we have asymptotic optimality if ∂R is chosen to be the boundary of the no-trade region for φ. In this case, $d\Delta\varphi = -d\varphi^*$ is satisfied in the interior, and $\varphi^{\pm,j}$ increases when $f_j^{-1}(\{\pm\varepsilon_j S^j\})$ is hit causing φ to properly reflect from the boundary. The dual theory comes into play when proving optimality via a frictionless extension of the constrained problem by defining the process $\widetilde{S} = S + \Delta S$, $\Delta S = f(\Delta\varphi)$, and constructing an equivalent measure with the density $\widetilde{Z}_T$ such that the pair $(\widetilde{S}, \widetilde{Z})$ meets the requirements of the dual characterization of asymptotic optimality in the sense of Proposition 4.10 (see also Remark 4.11).

When dealing with an approximation of the no-trade region, the situation is different since we do not have an exact solution. Instead, we already start with a candidate domain and must then find a way to construct an arbitrage-free shadow-price process. Such a shadow price, in general, will have nothing to do with the trading strategy associated with the approximate no-trade region by Definition 5.3. This fact poses a problem since the very notion of the shadow price is linked to (asymptotically) optimal trading strategies (cf. Definition 4.6 and also Proposition 4.10, Remark 4.11). Nonetheless, there is a way out, and a key observation upon which the construction of a dual upper bound will be based is as follows. Consider a process $\widetilde{S} = (\widetilde{S}^1, \dots, \widetilde{S}^n)$ such that $\widetilde{S}^j \in [\underline{S}^j, \overline{S}^j]$, $j \in \{1, \dots, n\}$, evolving within the bid-ask spread. Assume there exists an equivalent martingale measure $\widetilde{Q} \sim P$ for $\widetilde{S}$ with the density $\widetilde{Z}_T = \frac{d\widetilde{Q}}{dP}$. Recall that, by Remark 4.7, any strategy φ satisfies $X(\varphi)_T \leq x + \varphi \bullet \widetilde{S}_T$, where $X(\varphi)$ is the payoff process generated by φ in the market $(\underline{S}, \overline{S})$ with transaction costs, and x is the investor's initial endowment. From the definition of the convex conjugate (cf. Definition 2.2) we immediately see that

$$\begin{aligned} E\left[u(X(\varphi)_T)\right] &\leq E\left[y\widetilde{Z}_T X(\varphi)_T + \tilde{u}(y\widetilde{Z}_T)\right] \\ &\leq E\left[y\widetilde{Z}_T(x + \varphi \bullet \widetilde{S}_T) + \tilde{u}(y\widetilde{Z}_T)\right] = xy + E\left[\tilde{u}(y\widetilde{Z}_T)\right], \quad y > 0 . \end{aligned}$$

Since this inequality holds for all $y > 0$, we have

$$E\left[u(X(\varphi)_T)\right] \leq \min_{y>0}\left\{xy + E\left[\tilde{u}(y\widetilde{Z}_T)\right]\right\} = -e^{-px - H(\widetilde{Q},P)} . \tag{5.25}$$

The right-hand side of the equality is the representation of the minimum in the special case of the exponential utility derived in Proposition 3.17. Thus, we see that *any* pair $(\widetilde{S}, \widetilde{Z})$ consisting of a price process $\widetilde{S}$, $\widetilde{S}^j \in [\underline{S}^j, \overline{S}^j]$, for which there exists an equivalent martingale measure with the density process $\widetilde{Z}$ can be used to upper-bound the primal functional. We will start by constructing a process $\widetilde{S} \in \prod [\underline{S}^j, \overline{S}^j]$ from the usual ansatz $\widetilde{S} = S + \Delta S$ and demand that ΔS satisfy the requirements of Assumption A3. That means that ΔS must take on its values inside $\prod [-\varepsilon_j S^j, \varepsilon_j S^j]$ and follow an Itô process,

$$\Delta S^j = b^{\Delta S, j} \bullet I + \sum_{k=1}^{m} \sigma^{\Delta S, jk} \bullet W^k \quad j = 1, \ldots, n \ ,$$

with $W = (W^1, \ldots, W^m)^\top$, $m \geq n$, being an $\mathbb{R}^m$-valued standard Brownian motion, and the drift and diffusion processes satisfying

$$b^{\Delta S} = \mathcal{O}\left(\varepsilon^{1/3}\right), \qquad \sigma^{\Delta S} = \mathcal{O}\left(\varepsilon^{2/3}\right) \ .$$

In the following, we will still refer to such a process $\widetilde{S} = S + \Delta S$ as a shadow price for simplicity. However, one must keep in mind that, strictly speaking, $\widetilde{S}$ is not a shadow price in the sense of Definition 4.6 as there is no trading strategy with the properties required to satisfy the item *(ii)* of that definition.

We begin by introducing a stochastic process $\Delta\psi = (\Delta\psi^1, \ldots, \Delta\psi^n)^\top$ taking on its values inside the parallelotope R constructed according to the scheme introduced in Definition 5.1. Let $\Delta\psi$ satisfy $d\Delta\psi = -d\varphi^*$ in the interior of R and reflect in the direction of $\mp L\mathbf{e}^{(j)}$ each time the boundary $\partial R^{\pm,j}$ is hit. Put differently, the direction of the reflection is given by the j-the column vector of the matrix associated with the linear mapping

$$L \colon \overline{R} = \prod_{k=1}^{n} [-\Delta^k, \Delta^k] \longrightarrow R \ ,$$

as introduced in Definition 5.1. Let $\tilde{f}$ be a function given through

$$\begin{aligned} \tilde{f} &= (\tilde{f}_1, \ldots, \tilde{f}_n) \colon \mathbb{R}^{3n} \longrightarrow \mathbb{R}^n \ , \\ \tilde{f}_j(x, y, z) &= \frac{\varepsilon_j y_j}{2} \left(\left(\frac{x_j}{z_j} \right)^3 - 3\, \frac{x_j}{z_j} \right) \ . \end{aligned} \tag{5.26}$$

Now define $\Delta\widetilde{\varphi} = L^{-1}\Delta\psi$ and set $\Delta S = \tilde{f}(\Delta\widetilde{\varphi}, S, \Delta)$. Observe that, since $\Delta^j, \Delta\psi^j = \mathcal{O}\left(\varepsilon^{1/3}\right)$, $j = 1, \ldots, n$, we have $\Delta S = \mathcal{O}(\varepsilon)$. The process $\Delta\widetilde{\varphi}$ varies inside the rectangular parallelotope $\overline{R}$, and $\Delta\widetilde{\varphi}^j$ reflects along $\mp\mathbf{e}^{(j)}$ when it hits $\pm\Delta^j$. Note that, for all $j, k = 1, \ldots, n$, the function $\tilde{f}$ satisfies

$$\tilde{f}_j(x, S, \Delta)\big|_{x_j = \pm\Delta^j} = \mp\varepsilon_j S^j \ , \tag{5.27}$$

$$\partial_{x_k} \tilde{f}_j(x, S, \Delta)\big|_{x_k = \pm\Delta^k} = 0 \ . \tag{5.28}$$

Define $\widetilde{S} = S + \Delta S$. The property (5.27) implies that the process $\widetilde{S}$ satisfies $\widetilde{S}^j \in [\underline{S}^j, \overline{S}^j]$, $j = 1, \ldots, n$.

The construction of ΔS can, of course, be viewed in terms of the process $\Delta\psi$ reflecting at the boundary of the parallelotope, ∂R, along the vectors $\mp L\mathbf{e}^{(j)}$ rather than in terms of $\Delta\widetilde{\varphi} = L^{-1}\Delta\psi$. Then, defining $f(\cdot, y, z) = \tilde{f}(\cdot, y, z) \circ L^{-1}$, for all $j, k = 1, \ldots, n$, we obtain the conditions

$$f_j(x, S, \Delta)\big|_{\partial R^{\pm,j}} = \mp\varepsilon_j S^j \ , \tag{5.29}$$

$$\sum_{l=1}^{n} L_{lk}\partial_{x_l} f_j(x, S, \Delta)\big|_{\partial R^{\pm,k}} = 0 \ . \tag{5.30}$$

The property (5.30) means that the directional derivatives of $f(\cdot, S, \Delta)$ along $\mp L\mathbf{e}^{(k)}$ vanish each time a reflection at $\partial R^{\pm,k}$ takes place.

The above construction scheme generates a process $\widetilde{S} = S + \Delta S$, with ΔS having the necessary property of evolving inside $\prod_{j=1}^{n}[-\varepsilon S^j, \varepsilon S^j]$, associated with an arbitrary linear approximation of the no-trade region determined by the pair (Δ, L) (cf. Definition 5.1). We next show that if we assume that c^S and c^{φ^*} are Itô processes, ΔS becomes an Itô process as well. Moreover, the drift and diffusion coefficients of ΔS then exhibit the desired asymptotic behaviour, as demanded in Assumption A3.

We first look at the dynamics of $\Delta S = f(\Delta\psi, S, \Delta)$ with $f(\cdot, S, \Delta) = \tilde{f}(\cdot, S, \Delta) \circ L^{-1}$ and $\tilde{f}$ being the function introduced in (5.26). Ito's formula yields

$$\begin{aligned}
d\Delta S_t^j =& \sum_{k=1}^{n}\Big(\partial_{x_k} f_j d\Delta\psi_t^k + \partial_{y_k} f_j dS_t^k + \partial_{z_k} f_j d\Delta_t^k\Big) \\
&+ \frac{1}{2}\sum_{k,l=1}^{n}\Big(\partial_{x_k}\partial_{x_l} f_j d[\Delta\psi^k, \Delta\psi^l]_t + \partial_{y_k}\partial_{y_l} f_j d[S^k, S^l]_t \\
&\qquad + \partial_{z_k}\partial_{z_l} f_j d[\Delta^k, \Delta^l]_t + \partial_{x_k}\partial_{y_l} f_j d[\Delta\psi^k, S^l]_t \\
&\qquad + \partial_{x_k}\partial_{z_l} f_j d[\Delta\psi^k, \Delta^l]_t + \partial_{y_k}\partial_{z_l} f_j d[S^k, \Delta^l]_t\Big) \ ,
\end{aligned} \tag{5.31}$$

for all $j = 1, \ldots, n$, where we used $f_j \equiv f_j(\Delta\psi, S, \Delta)_t$ for brevity. Since $f(\Delta\psi, S, \Delta) = \mathcal{O}(\varepsilon)$, the partial derivatives appearing in the above representation satisfy for all j, k

$$\begin{aligned}
\partial_{x_k} f_j(\Delta\psi, S, \Delta),\ \partial_{z_k} f_j(\Delta\psi, S, \Delta) &= \mathcal{O}\left(\varepsilon^{2/3}\right) \ , \\
\partial_{y_k} f_j(\Delta\psi, S, \Delta) &= \mathcal{O}\left(\varepsilon\right) \ ,
\end{aligned} \tag{5.32}$$

and

$$\begin{aligned}\partial_{x_k}\partial_{x_l} f_j(\Delta\psi, S, \Delta),\ \partial_{z_k}\partial_{z_l} f_j(\Delta\psi, S, \Delta),\ \partial_{x_k}\partial_{z_l} f_j(\Delta\psi, S, \Delta) = \mathcal{O}\left(\varepsilon^{1/3}\right)\ ,\\ \partial_{x_k}\partial_{y_l} f_j(\Delta\psi, S, \Delta),\ \partial_{y_k}\partial_{z_l} f_j(\Delta\psi, S, \Delta) = \mathcal{O}\left(\varepsilon^{2/3}\right)\ ,\end{aligned} \tag{5.33}$$

which can be verified directly by computing the derivatives. Noting that, for all j, the function f_j depends on y and z only through their j-th component and recalling that $\Delta\psi = \varphi - \varphi^*$ with φ being a finite-variation process, the expression (5.31) can be written as

$$\begin{aligned} d\Delta S_t^j = & \sum_{k=1}^n \partial_{x_k} f_j d\Delta\psi_t^k + \partial_{y_j} f_j dS_t^j + \partial_{z_j} f_j d\Delta_t^j \\ & + \frac{1}{2}\sum_{k,l=1}^n \partial_{x_k}\partial_{x_l} f_j d[\varphi^{*,k}, \varphi^{*,l}]_t + \frac{1}{2}\partial_{z_j}^2 f_j d[\Delta^j]_t + \frac{1}{2}\partial_{y_j}\partial_{z_j} f_j d[S^j, \Delta^j]_t \\ & - \sum_{k=1}^n \partial_{x_k}\partial_{y_j} f_j d[\varphi^{*,k}, S^j]_t - \sum_{k=1}^n \partial_{x_k}\partial_{z_j} f_j d[\varphi^{*,k}, \Delta^j]_t\ , \end{aligned} \tag{5.34}$$

for all $j = 1, \ldots, n$. Note that $\partial_{y_k}\partial_{y_l} f_j \equiv 0$. We now assume that the local quadratic covariations c^{φ^*} and c^S follow Itô processes. Then, by Equation (5.15), the boundary process $\Delta = (\Delta^1, \ldots, \Delta^n)^\top$, at the leading order in ε, can be written as

$$\Delta^j = \varepsilon_j^{1/3} Y^j, \quad j = 1, \ldots, n\ , \tag{5.35}$$

with $Y = (Y^1, \ldots, Y^n)^\top$ being an Itô process independent of ε. Thus, except for $\sum_{k=1}^n \partial_{x_k} f_j d\Delta\psi_t^k$, all terms on the right-hand side of (5.34) are Itô processes. In the interior of the no-trade region, the process $\Delta\psi$ satisfies $d\Delta\psi^j = -d\varphi^{*,j}$, $j = 1, \ldots, n$, and φ^* is again an Itô process by assumption. Hence, the only "critical" region is the boundary, ∂R, where the finite-variation part, $\varphi = \varphi^+ - \varphi^-$, contributes. Now observe that

$$\begin{aligned} \sum_{k=1}^n \partial_{x_k} f_j(\Delta\psi, S, \Delta)_t\, d\Delta\psi_t^k &= \sum_{l=1}^n \sum_{k=1}^n L_{kl}\partial_{x_k} f_j(\Delta\psi, S, \Delta)_t\, d\Delta\widetilde{\varphi}_t^l \\ &= \sum_{l=1}^n \partial_{x_l}\tilde{f}_j(\Delta\widetilde{\varphi}, S, \Delta)_t\, d\Delta\widetilde{\varphi}_t^l, \quad j = 1, \ldots, n\ . \end{aligned}$$

Hence, the boundary conditions for f respectively $\tilde{f}$, given by Equations (5.28), (5.30), force the contributions of φ to vanish, which, in turn, means that ΔS is an Itô process.

Using (5.35), Equation (5.34) can be rewritten as

$$
\begin{aligned}
d\Delta S_t^j = &-\sum_{k=1}^{n} \partial_{x_k} f_j d\varphi_t^{*,k} + \partial_{y_j} f_j dS_t^j + \varepsilon_j^{1/3} \partial_{z_j} f_j dY_t^j \\
&+ \frac{1}{2}\sum_{k,l=1}^{n} \partial_{x_k}\partial_{x_l} f_j d[\varphi^{*,k}, \varphi^{*,l}]_t + \frac{\varepsilon_j^{1/3}}{2}\partial_{z_j}^2 f_j d[Y^j]_t + \frac{\varepsilon_j^{1/3}}{2}\partial_{y_j}\partial_{z_j} f_j d[S^j, Y^j]_t \qquad (5.36) \\
&-\sum_{k=1}^{n} \partial_{x_k}\partial_{y_j} f_j d[\varphi^{*,k}, S^j]_t - \sum_{k=1}^{n} \varepsilon_j^{1/3} \partial_{x_k}\partial_{z_j} f_j d[\varphi^{*,k}, Y^j]_t, \quad j = 1, \ldots, n \; .
\end{aligned}
$$

Finally, for all $j = 1, \ldots, n$, using (5.32), (5.33) and

$$
d\varphi_t^{*,j} = b_t^{\varphi^*,j} dt + \sum_{k=1}^{m} \sigma_t^{\varphi^*,jk} dW_t^k \; ,
$$

the dynamics of ΔS can be represented as

$$
d\Delta S_t^j = b_t^{\Delta S,j} dt + \sum_{k=1}^{m} \sigma_t^{\Delta S,jk} dW_t^k
$$

with

$$
\begin{aligned}
b^{\Delta S,j} &= \frac{1}{2}\sum_{k,l=1}^{n} \partial_{x_k}\partial_{x_l} f_j(\Delta\psi, S, \Delta) c^{\varphi^*,kl} + \mathcal{O}\left(\varepsilon^{2/3}\right) \; , \\
\sigma^{\Delta S,jk} &= -\sum_{l=1}^{n} \partial_{x_l} f_j(\Delta\psi, S, \Delta) \sigma^{\varphi^*,lk} + \mathcal{O}\left(\varepsilon\right) \; .
\end{aligned}
\qquad (5.37)
$$

These expressions can also be written in a more compact form as

$$
\begin{aligned}
b^{\Delta S,j} &= \frac{1}{2}\mathrm{Tr}\left[H_{f_j}(\Delta\psi, S, \Delta) c^{\varphi^*}\right] + \mathcal{O}\left(\varepsilon^{2/3}\right) \; , \\
\sigma^{\Delta S} &= -J_f(\Delta\psi, S, \Delta)\sigma^{\varphi^*} + \mathcal{O}\left(\varepsilon\right) \; ,
\end{aligned}
\qquad (5.38)
$$

with H_{f_j}, $j = 1, \ldots, n$, and J_f being the Hessian of f_j and the Jacobian of f, respectively.

As already mentioned at the beginning of this section, although the process $\widetilde{S} = S + \Delta S$ is referred to as shadow price, it should not be understood as a shadow price in the sense of Definition 4.6. This is due to the fact that we do not have an optimizing strategy to satisfy item (*ii*) of that definition. However, the auxiliary process $\widetilde{S}$ of this section still exhibits the properties of evolving inside the bid-ask spread, $\widetilde{S} \in \prod [(1 - \varepsilon_j)S^j, (1 + \varepsilon_j)S^j]$, and giving rise to a dual variable which provides an upper bound on the expected utility of a candidate strategy (cf. Equation (5.25)). To eliminate the ambiguity in the notion of a shadow price and to summarize the main results of the above construction, we formulate the following

Definition 5.4. Let the pair (Δ, L) define a candidate domain in the sense of Definition 5.1. Assume that the frictionless optimizer, φ^*, follows an Itô process. Define the process

$$\widetilde{\varphi}^* = L^{-1}\varphi^*$$

and the function

$$\mu = (\mu_1, \dots, \mu_n)\colon \mathbb{R}^n \to \mathbb{R}^n \ ,$$

$$\mu_j(x) = \operatorname{sgn}(x_j) \ .$$

As in Definition 5.1, let

$$\overline{R} = \prod_{j=1}^{n} [-\Delta^j, \Delta^j] \ .$$

The boundary $\partial\overline{R}$ is given by

$$\partial\overline{R} = \bigcup_{j=1}^{n} \partial\overline{R}^{+,j} \cup \partial\overline{R}^{-,j} \ ,$$

$$\partial\overline{R}^{\pm,j} = \left\{x \in \mathbb{R}^n \colon x_j = \mp\Delta^j\right\} \ .$$

Let $\Delta\widetilde{\varphi} = (\Delta\widetilde{\varphi}^1, \dots, \Delta\widetilde{\varphi}^n)^\top$ and the finite-variation process $\widetilde{\varphi} = (\widetilde{\varphi}^1, \dots, \widetilde{\varphi}^n)^\top$ be a solution to the following stochastic Skorohod problem:

$$\begin{aligned} \Delta\widetilde{\varphi}_t &= -\widetilde{\varphi}^*_t + \widetilde{\varphi}_t \in \overline{R}_t, \quad \Delta\widetilde{\varphi}_0 = 0 \ , \\ \widetilde{\varphi}^j_t &= \widetilde{\varphi}^*_0 + \int_0^t \mu_j(\Delta\widetilde{\varphi}_u) d|D\widetilde{\varphi}^j|_u \ , \\ |D\widetilde{\varphi}^j|_t &= \int_0^t I_{\partial\overline{R}^{\pm,j}_u}(\Delta\widetilde{\varphi}_u) d|D\widetilde{\varphi}^j|_u \ , \end{aligned} \tag{5.39}$$

for all $j = 1, \dots, n$ and $t \geq 0$. Define

$$\Delta S = \tilde{f}(\Delta\widetilde{\varphi}, S, \Delta) \ ,$$

with $\tilde{f}$ being the function introduced in Equation (5.26). In the following, the process $\widetilde{S} = S + \Delta S$ will be referred to as the *shadow price* associated with the candidate domain (Δ, L).

Remark 5.5. In the remainder of this section, we will consider only the pair (Δ, L) corresponding to the candidate introduced in Subsection 5.1.2. This means that the process $\Delta = (\Delta^1, \dots, \Delta^n)^\top$ is given by (5.15), and the linear transformation is chosen as $L = M^{-1}$ with M defined in (5.19). The reason for this restriction is that, according to the simulation results, only this choice of (Δ, L) generates a non-trivial upper bound.

But it does so only among the candidates we have analysed in this study; this does not mean that this choice is the best possible.

Before moving on, we calculate the second partial derivatives in (5.37) to obtain an explicit expression for the drift process, $b^{\Delta S}$, in terms of the process $\Delta\psi = M^{-1}\Delta\widetilde{\varphi}$, which will be required in the following. For $f(\cdot, x, y) = \tilde{f}(\cdot, y, z) \circ M$, $\tilde{f}$ as in (5.26), $j, k, l = 1, \ldots, n$, we obtain

$$\partial_{x_k}\partial_{x_l} f_j(\Delta\psi, S, \Delta) = \frac{3\varepsilon_j S^j M^{jk} M^{jl}}{(\Delta^j)^3} \left(M\Delta\psi\right)^j .$$

Substituting this result into the expression (5.37), leads to

$$b^{\Delta S,j} = \frac{3\varepsilon_j S^j}{2(\Delta^j)^3} \left(M\Delta\psi\right)^j \sum_{k,l} M^{jk} M^{jl} c^{\varphi^*,kl} = \frac{3\varepsilon_j S^j}{2(\Delta^j)^3} \left(M c^{\varphi^*} M^\top\right)^{jj} \left(M\Delta\psi\right)^j .$$

Now recall that the linear transformation M is given by $M_{ij} = c^{S,ij}/c^{S,ii}$ (cf. Eq. (5.19)); thus,

$$\left(M c^{\varphi^*} M^\top\right)^{jj} \left(M\Delta\psi\right)^j = \frac{1}{(c^{S,jj})^3} \left(x^S c^{\varphi^*} c^S\right)^{jj} \left(c^S \Delta\psi\right)^j .$$

By Equation (5.15),

$$\left(\Delta^j\right)^3 = \frac{1}{(c^{S,jj})^3} \frac{3\varepsilon_j S^j}{2p} \left(x^S c^{\varphi^*} c^S\right)^{jj} ,$$

and we finally obtain

$$b^{\Delta S} = p c^S \Delta\psi . \tag{5.40}$$

As shown in Equation (5.25), to obtain an upper bound on the expected utility of the final payoff of a candidate strategy, we still require an equivalent probability measure, $\widetilde{Q} \sim P$, making $\widetilde{S}$ a martingale. Since $\widetilde{S}$ is shown to be an Itô process, an equivalent martingale measure can be obtained via a Girsanov transformation.

Let Q be a martingale measure for the mid-price process S. We start by representing the density $\widetilde{Z}_T$ of $\widetilde{Q}$ with respect to P as

$$\widetilde{Z}_T = \frac{dQ}{dP}\frac{d\widetilde{Q}}{dQ} = Z_T \widetilde{Z}^Q_T . \tag{5.41}$$

Now consider the **Q-dynamics** of the process $\widetilde{S}$,

$$\widetilde{S}^j = S^j + b^{\Delta S,j} \bullet I + \sum_{k=1}^{m} \sigma^{\Delta S,jk} \bullet W^k, \quad j = 1, \ldots, n , \tag{5.42}$$

where $W^1, \dots, W^m$, $m \geq n$, are independent Brownian motions **under Q**. Note that (5.42) is a decomposition of the semimartingale $\widetilde{S}$ into a Q-martingale

$$M^j = S^j + \sum_{k=1}^{m} \sigma^{\Delta S, jk} \bullet W^k$$

and a Q-finite-variation process

$$A^j = b^{\Delta S, j} \bullet I \ .$$

The goal is to determine an equivalent measure that "removes" $A = (A^1, \dots, A^n)$. We make the ansatz

$$\widetilde{Z}^Q = \mathcal{E}(N) = \exp\left(h \bullet S - \frac{1}{2}(h^\top c^S h) \bullet I \right) \ , \tag{5.43}$$

with $N = h \bullet S$ and $h = (h^1, \dots, h^n)$ being a stochastic process yet to be determined. By the Girsanov-Meyer Theorem[3], there exists a $\widetilde{Q}$-martingale L^j as well as a $\widetilde{Q}$-finite-variation process C^j such that, for each $j = 1, \dots, n$,

$$L^j = M^j - \frac{1}{\widetilde{Z}^Q} \bullet [\widetilde{Z}^Q, M^j], \quad C^j = \widetilde{S} - L^j \ , \tag{5.44}$$

Next, note that $\widetilde{Z}^Q$, as a stochastic exponential, can be written as

$$\widetilde{Z}^Q = 1 + \widetilde{Z}^Q \bullet N \ .$$

To determine the new semimartingale decomposition of $\widetilde{S}$, we calculate the processes introduced in (5.44). The $\widetilde{Q}$-martingale reads

$$\begin{aligned}
L^j &= M^j - \frac{1}{\widetilde{Z}^Q} \bullet [\widetilde{Z}^Q, M^j] = M^j - \frac{1}{\widetilde{Z}^Q} \bullet [1 + \widetilde{Z}^Q \bullet N, M^j] = M^j - [N, M^j] \\
&= M^j - \sum_k h^k \bullet [S^k, M^j] = M^j - \sum_k h^k \bullet [S^k, S^j] - \sum_k h^k \bullet [S^k, (\sigma^{\Delta S} \bullet W)^j] \\
&= M^j - \sum_k h^k \bullet [S^k, S^j] - \sum_k h^k \bullet [S^k, \Delta S^j - A^j] \\
&= M^j - \sum_k h^k \bullet [S^k, S^j] - \sum_k h^k \bullet [S^k, \Delta S^j] \\
&= M^j - \left(\sum_k (c^S + c^{S, \Delta S})^{jk} h^k \right) \bullet I, \quad j = 1, \dots, n \ .
\end{aligned}$$

Note that in the fourth line we used that A is a continuous finite-variation process, and its quadratic covariation with S thus is zero. The $\widetilde{Q}$-finite-variation process C^j is given

[3] cf. [Pro04], Theorem III.35.

through

$$C^j = \widetilde{S}^j - L^j = A^j + \left((c^S + c^{S,\Delta S})h\right)^j \bullet I = \left(b^{\Delta S,j} + (\widetilde{c}h)^j\right) \bullet I \ ,$$

for all $j = 1, \dots, n$, where $\widetilde{c} = c^S + c^{S,\Delta S}$. Hence, in order for $\widetilde{S}$ to be a $\widetilde{Q}$-martingale, we must have

$$b^{\Delta S,j} + (\widetilde{c}h)^j = 0, \quad j = 1, \dots, n \ ,$$

or, equivalently,

$$h = -\widetilde{c}^{\,-1} b^{\Delta S} \ . \tag{5.45}$$

We now take a closer look at the matrix $\widetilde{c}^{\,-1}$. As ΔS satisfies Assumption A3, we have $c^{S,\Delta S} = \mathcal{O}\left(\varepsilon^{2/3}\right)$. By setting $B = -(c^S)^{-1} c^{S,\Delta S}$, the inverse of $\widetilde{c}$ can be written as

$$\widetilde{c}^{\,-1} = (c^S + c^{S,\Delta S})^{-1} = (\mathrm{Id} - B)^{-1} (c^S)^{-1} \ .$$

Since

$$(\mathrm{Id} - B)^{-1} = \mathrm{Id} + \mathcal{O}\left(\varepsilon^{2/3}\right) \ ,$$

the asymptotic behaviour of the process h defined in (5.45) can be characterized as

$$h = -(c^S)^{-1} b^{\Delta S} + \mathcal{O}\left(\varepsilon\right) = -p\Delta\psi + \mathcal{O}\left(\varepsilon\right) \ . \tag{5.46}$$

The second equality follows from the expression (5.40) for the drift process of ΔS.
Our next goal is to express the dual upper bound provided in (5.25) in terms of the auxiliary process $\Delta\psi$. To accomplish this, we need to evaluate the entropy

$$H(\widetilde{Q}, P) = E\left[\widetilde{Z}_T \ln \widetilde{Z}_T\right] \ .$$

Note that, by Proposition 3.20, the density of the MEMM $Q \sim P$, Z_T, is connected with the frictionless optimizer φ^* via

$$Z_T = \exp\left\{H(Q, P) - p\varphi^* \bullet S_T\right\} \ .$$

We thus obtain

$$H(\widetilde{Q}, P) = E\left[Z_T \widetilde{Z}^Q_T \left(\ln Z_T + \ln \widetilde{Z}^Q_T\right)\right] = H(Q, P) + H(\widetilde{Q}, Q) - pE\left[\widetilde{Z}_T(\varphi^* \bullet S_T)\right]$$

The frictionless wealth process $\varphi^* \bullet S$ can be expressed as

$$\varphi^* \bullet S = (\psi - \Delta\psi) \bullet (\widetilde{S} - \Delta S) = \psi \bullet \widetilde{S} - \psi \bullet \Delta S - \Delta\psi \bullet S \ .$$

Since $\psi \bullet \widetilde{S}$ is a $\widetilde{Q}$-martingale, we have

$$E\left[\widetilde{Z}_T(\varphi^* \bullet S_T)\right] = -E_Q\left[\widetilde{Z}_T^Q\big(\underbrace{\psi \bullet \Delta S_T}_{=\mathcal{O}(\varepsilon^{2/3})} + \underbrace{\Delta\psi \bullet S_T}_{=\mathcal{O}(\varepsilon^{1/3})}\big)\right] \ .$$

Furthermore, by Equation (5.46), $h = \mathcal{O}\left(\varepsilon^{1/3}\right)$, and it follows

$$\begin{aligned}\widetilde{Z}_T^Q &= \exp\left\{h \bullet S_T - \frac{1}{2}h^\top c^S h \bullet I_T\right\} \\ &= 1 + h \bullet S_T - \frac{1}{2}h^\top c^S h \bullet I_T + \frac{1}{2}(h \bullet S_T)^2 + \mathcal{O}(\varepsilon) \ .\end{aligned} \tag{5.47}$$

The above expectation now reads as

$$E\left[\widetilde{Z}_T(\varphi^* \bullet S_T)\right] = -E_Q\big[\psi \bullet \Delta S_T + \Delta\psi \bullet S_T\big] - E_Q\left[\big(h \bullet S_T\big)\big(\Delta\psi \bullet S_T\big)\right] + \mathcal{O}(\varepsilon) \ .$$

Applying integration by parts to the last term on the right-hand side and using that S is a martingale with respect to Q yields

$$E\left[\widetilde{Z}_T(\varphi^* \bullet S_T)\right] = -E_Q\big[\psi \bullet \Delta S_T\big] - E_Q\big[h^\top c^S \Delta\psi \bullet I_T\big] \ .$$

Using the expansion of $\widetilde{Z}_T^Q$ given in (5.47), we now evaluate the entropy $H(\widetilde{Q}, Q)$.

$$\begin{aligned}\widetilde{Z}_T^Q \ln \widetilde{Z}_T^Q &= \Big(h \bullet S_T - \frac{1}{2}h^\top c^S h \bullet I_T\Big)\Big(1 + h \bullet S_T\Big) + \mathcal{O}(\varepsilon) \\ &= h \bullet S_T - \frac{1}{2}h^\top c^S h \bullet I_T + \big(h \bullet S_T\big)^2 + \mathcal{O}(\varepsilon) \ .\end{aligned}$$

Integration by parts and the martingale property of the stochastic integral yield

$$H(\widetilde{Q}, Q) = \frac{1}{2}E_Q\left[h^\top c^S h \bullet I_T\right] + \mathcal{O}(\varepsilon) \ .$$

Altogether we obtain

$$H(\widetilde{Q}, P) = H(Q, P) + pE_Q\big[\psi \bullet \Delta S_T\big] + pE_Q\big[h^\top c^S \Delta\psi \bullet I_T\big] + \frac{1}{2}E_Q\left[h^\top c^S h \bullet I_T\right] + \mathcal{O}(\varepsilon) \ .$$

Finally, we use the asymptotic expression for the process h given in (5.46), combine it with the leading-order representation (5.40) of the drift process $b^{\Delta S}$ to obtain

$$H(\widetilde{Q}, P) = H(Q, P) + pE_Q\big[\psi \bullet \Delta S_T\big] - \frac{p^2}{2}E_Q\left[\Delta\psi^\top c^S \Delta\psi \bullet I_T\right] + \mathcal{O}(\varepsilon) \ .$$

Substituting this entropy approximation into the expression for the dual upper bound given in (5.25) yields (up to $\mathcal{O}(\varepsilon)$ in the argument of the exponential)

$$-e^{-px-H(\widetilde{Q},P)} = -e^{-px-H(Q,P)} \exp\left\{-pE_Q\left[\psi \bullet \Delta S_T\right] + \frac{p^2}{2}E_Q\left[\Delta\psi^\top c^S \Delta\psi \bullet I_T\right]\right\} \tag{5.48}$$

The last step is to first note that, by the frictionless optimality condition,

$$-e^{-px-H(Q,P)} = E\left[u(x + \varphi^* \bullet S_T)\right]$$

and then expand the exponential on the right-hand side of (5.48) at the leading order in ε, which gives us

$$-e^{-px-H(\widetilde{Q},P)} = E\left[u(x + \varphi^* \bullet S_T)\right]\left(1 - pE_Q\left[\psi \bullet \Delta S_T\right] + \frac{p^2}{2}E_Q\left[\Delta\psi^\top c^S \Delta\psi \bullet I_T\right]\right) , \tag{5.49}$$

which, yet again, is to be understood as an approximation up to $\mathcal{O}(\varepsilon)$.

Now suppose that we are given an arbitrary candidate strategy, $\varphi = \varphi^* + \Delta\varphi$, $\Delta\varphi = \mathcal{O}\left(\varepsilon^{1/3}\right)$. The payoff generated by this strategy is given by (cf. Definition 4.3, Remark 4.4 and Equation (4.4))

$$X(\varphi)_T = V(\varphi)_T - \sum_{j=1}^n \varepsilon_j S_T^j |\varphi_T^j| = x + \varphi^* \bullet S_T + \Delta\varphi \bullet S_T - S \bullet \bar{\varphi}_T - \sum_{j=1}^n \varepsilon_j S_T^j |\varphi_T^j|$$

Note that, by Equations (4.14), (4.28), $S \bullet \bar{\varphi} = \mathcal{O}\left(\varepsilon^{2/3}\right)$. Hence, the utility of the final payoff reads as

$$\begin{aligned} u(X(\varphi)_T) &= u(x + \varphi^* \bullet S_T) + u'(x + \varphi^* \bullet S_T)(\Delta\varphi \bullet S_T - S \bullet \bar{\varphi}_T) \\ &\quad + \frac{1}{2}u''(x + \varphi^* \bullet S_T)(\Delta\varphi \bullet S_T - S \bullet \bar{\varphi}_T)^2 + \mathcal{O}(\varepsilon) \ . \end{aligned}$$

Taking the P-expectation on both sides and using $u''(x) = -pu'(x) = p^2 u(x)$ together with the frictionless optimality condition gives us the following expression for the expected utility at the leading order in ε:

$$\begin{aligned} E[u(X(\varphi)_T)] &= E[u(x + \varphi^* \bullet S_T)]\left(1 + pE_Q[S \bullet \bar{\varphi}_T] + \frac{p^2}{2}E_Q\left[(\Delta\varphi \bullet S_T)^2\right]\right) \\ &= E[u(x + \varphi^* \bullet S_T)]\left(1 + pE_Q[S \bullet \bar{\varphi}_T] + \frac{p^2}{2}E_Q\left[\Delta\varphi^\top c^S \Delta\varphi \bullet I_T\right]\right) . \end{aligned} \tag{5.50}$$

By the relation (5.25), the expression on the right-hand side of (5.49) is an upper bound for (5.50), the expected utility of the final payoff generated by an arbitrary candidate strategy. This fact is of crucial importance for the numerical analysis to be presented in the following chapter.

Remark 5.6. It is important to stress that, for each pair consisting of a trading strategy and a dual variable, the values of the corresponding primal and a dual functionals (given by (5.49) and (5.50), respectively) define an interval containing the value of the expected utility of terminal payoff generated by the exact asymptotic optimizer. Thus, the smaller is the primal-dual interval, the more precisely the exact value of the asymptotic optimization problem can be estimated.

Chapter 6

Numerical Analysis in the Black-Scholes Setting

In this chapter, the three primal candidates constructed in the subsections 5.1.1 – 5.1.3 and the dual candidate derived in Section 5.2 are analysed numerically in the Black-Scholes model. We begin by adapting all relevant expressions to the model of interest. Our next goal will be to select a realistic portfolio in $n = 30$ dimensions according to the formula for the frictionless optimizer in the Black-Scholes model. Such a portfolio is determined by a vector of drift coefficients and a covariance matrix. To obtain these parameters, we introduce an implicit parameter-estimation scheme in Section 6.2. In the following section, the estimation scheme will be applied to select a portfolio from the German stock market index. A discrete-time approximation of the problem, which is required for computer simulations, will be presented in the same section. Moreover, before running the multi-dimensional simulations, we will determine a suitable transaction-cost percentage, ε. To achieve this, the expected utility of a one-dimensional portfolio will be simulated and compared with the exact asymptotic solution presented in Subsection 4.3.3. In the final section, the results of multi-dimensional simulations are presented.

6.1 Approximations in the Black-Scholes model

A detailed discussion of the Black-Scholes model was already presented in Subsection 3.2.4. Here, we add a few explicit calculations concerning the frictionless optimizing strategy and use them to adapt the approximation results from the previous chapter to this specific model. We let $S^1, \ldots, S^n$ denote the *discounted* price processes following the dynamics

$$dS_t^j = S_t^j\Big(b_j dt + \sum_{k=1}^n \sigma_{jk} dW_t^k\Big), \quad t \in [0,T], \quad j = 1,\dots,n \ , \tag{6.1}$$

with $b = (b_1,\dots,b_n) \in \mathbb{R}^n$ and an invertible matrix $(\sigma_{jk})_{j,k=1,\dots,n}$, $\sigma_{jk} > 0$ for all j,k. We will refer to the constant b_j, $j = 1,\dots,n$, as the *mean rate of return* or the *drift coefficient* of the j-th asset. The matrix

$$c = \sigma\sigma^\top$$

will be called the *covariance matrix* of the assets. The covariance matrix is a symmetric, positive definite matrix which can be represented as

$$c_{jk} = \sigma_j \sigma_k \rho_{jk},$$

with $(\rho_{jk})_{j,k=1,\dots,n}$ being a symmetric, positive definite matrix with $\rho_{jk} \in [-1,1]$ and $\rho_{jj} = 1$ for all $(j,k) \in \{1,\dots,n\}^2$. The constant σ_j, $j = 1,\dots,n$, will be referred to as the *volatility* of the j-th asset, and the matrix ρ will be called the *correlation matrix* of the assets. Recall that, for each $j = 1,\dots,n$, the solution to the stochastic differential equation (6.1) reads as (cf. Equation (3.28))

$$\begin{aligned} S_t^j &= S_0^j \exp\Big(\nu_j t + \sum_{k=1}^n \sigma_{jk} W_t^k\Big), \quad t \in [0,T] \ , \\ \nu_j &= b_j - \frac{\sigma_j^2}{2} \ . \end{aligned} \tag{6.2}$$

As shown in Subsection 3.2.4, the frictionless optimizer in the Black-Scholes model, $\varphi^* = (\varphi^{*,1},\dots,\varphi^{*,n})$, with respect to the exponential utility function $u(x) = -e^{-px}$, $p > 0$, is given by

$$\varphi^{*,j} = \frac{(c^{-1}b)_j}{pS^j} \ . \tag{6.3}$$

To compute the boundary processes of the candidate domains introduced in the subsections 5.1.1 – 5.1.3, we also need the local quadratic covariations of the price process S and the frictionless optimizer φ^*, c^S and c^{φ^*}. The process c^S can be easily obtained from Equation (6.1). The quadratic covariations of the price processes S^j, S^k, $j,k \in \{1,\dots,n\}$, satisfy

$$d[S^j,S^k]_t = \sum_{l,m=1}^n S_t^j S_t^k \sigma_{jl}\sigma_{km} d[W^l,W^m]_t = \sum_{l=1}^n S_t^j S_t^k \sigma_{jl}\sigma_{kl} dt = S_t^j S_t^k c_{jk} dt \ ,$$

which implies

$$c^{S,jk} = S^j S^k c_{jk} \ . \tag{6.4}$$

To compute c^{φ^*}, we apply Itô's formula to the right-hand side of (6.3) to first obtain

$$d\varphi_t^{*,j} = \frac{(c^{-1}b)_j}{p} d\left(\frac{1}{S^j}\right)_t = \frac{(c^{-1}b)_j}{p}\left(-\frac{dS_t^j}{\left(S_t^j\right)^2} + \frac{d[S^j]_t}{\left(S_t^j\right)^3}\right)$$
$$= \varphi_t^{*,j}\left((\sigma_j^2 - b_j)dt - \sum_{k=1}^n \sigma_{jk} dW_t^k\right) ,$$

which then leads to

$$d[\varphi^{*,j}, \varphi^{*,k}]_t = \varphi_t^{*,j}\varphi_t^{*,k} \sum_{l,m=1}^n \sigma_{jl}\sigma_{km} d[W^l, W^m]_t = \varphi_t^{*,j}\varphi_t^{*,k} c_{jk} dt ,$$

and the local quadratic covariation therefore reads as

$$c^{\varphi^*,jk} = \varphi^{*,j}\varphi^{*,k} c_{jk} . \tag{6.5}$$

The main objects of our study, on the one hand, are the approximations of the no-trade region, together with the associated candidate strategies. As described in the previous chapter in Definition 5.1, the no-trade region is determined by a pair (Δ, L) consisting of a boundary process $\Delta = (\Delta^1, \ldots, \Delta^n)^\top$ and a linear transformation L, and the corresponding trading strategies are defined as solutions to the Skorohod problem (5.3)[1]. On the other hand, we have dual candidates $(\widetilde{S}, \widetilde{Z})$, $\widetilde{S}$ being an auxiliary price process and $\widetilde{Z}$ the density process of an equivalent martingale measure for $\widetilde{S}$, that can be associated with any linear no-trade region (Δ, L) via the algorithm introduced in Section 5.2. The role of the dual candidate is to generate an upper bound on the expected utility of the terminal payoff of a candidate strategy, as indicated in (5.25). Our goal now is to adapt the expressions describing our candidate domains and trading strategies, introduced in Subsections 5.1.1 – 5.1.3, to the Black-Scholes setting.

Primal candidates

1. *Naive candidate* $(\Delta^{(1)}, L^{(1)})$*:* The candidate domain introduced in Subsection 5.1.1 is characterized by a boundary process $\Delta^{(1)} = (\Delta^{(1),1}, \ldots, \Delta^{(1),n})^\top$ as defined in Equation (5.6) and the trivial linear transformation $L^{(1)} = \mathrm{Id}$. Using (6.4), (6.5), we obtain

$$\Delta^{(1),j} = \left(\frac{3}{2p}\frac{c^{\varphi^*,jj}}{c^{S,jj}}\varepsilon_j S^j\right)^{1/3} = \left(\frac{3}{2p}\frac{(\varphi^{*,j})^2}{(S^j)^2}\varepsilon_j S^j\right)^{1/3} = \frac{d_{1,j}}{S^j} ,$$
$$d_{1,j} = \frac{1}{p}\left(\frac{3}{2}\varepsilon_j (c^{-1}b)_j^2\right)^{1/3} . \tag{6.6}$$

[1] In the following, we will often refer to the trading strategies as the *primal candidates*.

2. *More sophisticated candidate* $(\Delta^{(2)}, L^{(2)})$*:* We now look at the candidate from Subsection 5.1.2. The boundary process $\Delta^{(2)}$ for this candidate is given by Equation (5.15), and the linear transformation, $L^{(2)}$, is defined by the inverse of the matrix $\widetilde{M}_{jk} = c^{S,jk}/c^{S,jj}$ (cf. Equation (5.19)). Equations (6.4), (6.5) yield

$$\widetilde{M}^{jk} = \frac{S^k}{S^j} M_{jk}, \qquad M_{jk} = \frac{c_{jk}}{c_{jj}} \ .$$

The inverse is given by

$$(M^{-1})^{jk} = \frac{S^j}{S^k} M^{-1}_{jk} \ .$$

Hence, the linear transformation associated with this candidate domain reads as

$$L^{(2),jk} = \frac{S^j}{S^k} \sigma_j^2 c^{-1}_{jk} \ . \tag{6.7}$$

To compute the boundary processes $\Delta^{(2),j}$, $j = 1, \dots, n$, which, at the leading order, read as (cf. Equation (5.15))

$$\Delta^{(2),j} = \frac{1}{c^{S,jj}} \left[\frac{3\varepsilon_j S^j}{2p} \left(c^S c^{\varphi^*} c^S \right)^{jj} \right]^{1/3} \ ,$$

we first evaluate the diagonal elements of the matrix product using Equations (6.4), (6.5):

$$\left(c^S c^{\varphi^*} c^S \right)^{jj} = \sum_{k,l=1}^{n} c^{S,jk} c^{\varphi^*,kl} c^{S,lj} = (S^j)^2 \sum_{k,l=1}^{n} S^k S^l \varphi^{*,k} \varphi^{*,l} c_{jk} c_{kl} c_{lj} \ ,$$

Setting $w_j = (c^{-1}b)_j$, $j = 1, \dots, n$, we obtain by Equation (6.3)

$$\left(c^S c^{\varphi^*} c^S \right)^{jj} = \frac{(S^j)^2}{p^2} v_j, \qquad v_j = \sum_{k,l=1}^{n} w_k w_l c_{jk} c_{kl} c_{lj} \ ,$$

which yields

$$\Delta^{(2),j} = \frac{d_{2,j}}{S^j}, \qquad d_{2,j} = \frac{1}{p\sigma_j^2} \left(\frac{3}{2} \varepsilon_j v_j \right)^{1/3} \ . \tag{6.8}$$

3. *Alternative candidate* $(\Delta^{(3)}, L^{(3)})$*:* The first ingredient of the construction presented in Subsection 5.1.3 is the process $\widehat{\varphi} = (\widehat{\varphi}^1, \dots, \widehat{\varphi}^n)^\top$ whose components are the one-dimensional optimal strategies. For each $j = 1, \dots, n$, we now have

$$\widehat{\varphi}^j = \frac{b_j}{p\sigma_j^2 S^j}$$

and

$$(\sigma^{\widehat{\varphi}^j})^2 = (\widehat{\varphi}^j)^2\sigma^2 = \left(\frac{b_j}{p\sigma_j S^j}\right)^2 .$$

Moreover, the scale factors $\kappa^1, \ldots, \kappa^n$, $\kappa^j = \varphi^{*,j}/\widehat{\varphi}^j$, $j = 1, \ldots, n$ (cf. Equation (5.23)), read as

$$\kappa^j = \frac{\sigma_j^2}{b_j}(c^{-1}b)_j .$$

Substituting these quantities into the equations (5.22) and (5.24) yields the following expressions for the linear transformation and the boundary process:

$$L^{(3)} = \operatorname{diag}\left(\frac{\sigma_1^2}{b_1}(c^{-1}b)_1, \ldots, \frac{\sigma_n^2}{b_n}(c^{-1}b)_n\right) , \tag{6.9}$$

$$\Delta^{(3),j} = \frac{d_{3,j}}{S^j}, \qquad d_{3,j} = \frac{1}{p}\left(\frac{3\varepsilon_j b_j^2}{2\sigma_j^4}\right)^{1/3} , \tag{6.10}$$

$j = 1, \ldots, n$.

Dual candidate

Dual candidate refers to the pair $(\widetilde{S}, \widetilde{Z})$ consisting of a shadow-price process $\widetilde{S}$ in the sense of Definition 5.4 and the density process $\widetilde{Z}$ of the equivalent martingale measure for $\widetilde{S}$, which was calculated in Section 5.2 (cf. Equation (5.41) and the following calculations). Given a candidate domain (Δ, L), both $\widetilde{S}$ and $\widetilde{Z}$ are determined by the process $\Delta\psi = L\Delta\widetilde{\varphi}$, with $\Delta\widetilde{\varphi}$ being a solution to the Skorohod problem (5.39) formulated in Definition 5.4. As already mentioned in Remark 5.5, only the candidate (Δ, M^{-1}) introduced in Subsection 5.1.2 and defined by the equations (5.15), (5.19) yields a non-trivial upper bound. Thus, the representation of the dual candidate in the Black-Scholes model is already determined by the candidate domain $(\Delta^{(2)}, L^{(2)})$ given by the equations (6.7), (6.8).

6.2 Replicating the DAX

The main goal of this section is to select a large portfolio using existing market data. This will provide us with a realistic framework within which the proposed upper bound (5.49) together with the three candidate strategies introduced in Subsections 5.1.1 – 5.1.3, whose utility loss at the leading order will be computed using (5.50), will be analysed numerically. Such a portfolio selection amounts to estimating the mean rate of return as well as the covariance matrix (cf. Section 6.1) of the assets to be included in the portfolio from the time series of these assets. The difficulties arising in this context are very well described in Chapter 4 of the monograph [Rog13] by L. C. G. Rogers, and the reader is referred to this work for a detailed overview. Our main concern will be the estimation of the annualized rate of return, the vector $b = (b_1, \ldots, b_n)$ introduced at the beginning of Section 6.1. The only quantity that actually carries information about the rate of return of an asset is the change in price of this asset over the whole observation period. This typically results in estimates having very wide confidence intervals. A discouraging example provided in Chapter 4 of [Rog13] shows that if one wishes to estimate the annualized volatility together with the annualized rate of return of a stock both with an accuracy of ± 0.01 and confidence level of 95% based on daily observations, one will require an observation period of over 1500 years to achieve this accuracy when estimating the rate of return, whereas 13 years are sufficient for an accurate volatility estimation.

To overcome the issue concerning the drift estimation, instead of choosing some particular stocks, we will select a portfolio based on the rank of the stocks with respect to their market capitalization. This means that the stocks will be identified not by their name but rather by their rank. This approach is inspired by the analysis of rank-based portfolios in *Stochastic Portfolio Theory* presented in [KF09] and in [Fer02]. As demonstrated in the latter monograph, the *capital distribution*, i. e. the relative capitalization of the highest-rank stocks, remains remarkably stable over long time periods — decades, in fact (cf. [Fer02], Figure 5.1). This stability property will be of crucial importance for the construction of our estimation scheme which will be based on the analysis of a stock market index consisting of largest-capitalization stocks. As we shall see in the following, the additional information coming from the dynamics of the entire index, rather than that of the individual assets, can then be used to estimate the drifts of the individual stocks in our rank-based portfolio.

6.2.1 Theoretical preliminaries

We begin with a few introductory remarks concerning the so-called *market portfolios*. For a detailed treatment of this subject, the reader is referred to [KF09], Section 2 of Chapter 1. Let the assets with the price processes $S^1, \ldots, S^n$ comprise the entire market. Moreover, assume that each of the stocks has only one share. Then, S^j_t, $j = 1, \ldots, n$, describes the capitalization of the j-th company at time t, and the capitalization of the whole market is given by $S = \sum_j S^j$. The *relative* capitalization of the j-th company at time t is defined as

$$\gamma^j_t = \frac{S^j_t}{S_t}, \quad j = 1, \ldots, n \; . \tag{6.11}$$

In the terminology of [KF09], the collection $\gamma = (\gamma^1, \ldots, \gamma^n)^\top$ is referred to as the *market portfolio*. γ can be thought of as a portfolio which, at each time t, invests the fixed proportion γ_t^j of the investor's capital in the j-th stock. We have the obvious relation

$$\frac{dS_t}{S_t} = \sum_{j=1}^{n} \gamma_t^j \frac{dS_t^j}{S_t^j} \ . \tag{6.12}$$

We now introduce the ordered market-weight distribution

$$\gamma^{(1)} \geq \gamma^{(2)} \geq \ldots \geq \gamma^{(n)}$$

starting with the largest relative capitalization,

$$\begin{aligned}
\gamma^{(1)} &= \max\{\gamma^1, \ldots, \gamma^n\} \ , \\
\gamma^{(2)} &= \max(\{\gamma^1, \ldots, \gamma^n\} \setminus \{\gamma^{(1)}\}) \ , \\
&\vdots \\
\gamma^{(n)} &= \min\{\gamma^1, \ldots, \gamma^n\} \ .
\end{aligned} \tag{6.13}$$

The collection $\{\gamma^{(1)}, \ldots, \gamma^{(n)}\}$ of the largest weights arranged in decreasing order is called the *capital distribution* of the market. The price processes corresponding to the ranks will be denoted by

$$S^{(1)}, \ldots, S^{(n)}$$

and referred to as the *rank processes*. In Chapter 5 of [Fer02], the author analyses the capital-distribution curves for the U.S. stock market in the time period from 1929 to 1999. In this time, the number of stocks increased from 700 to around 7500, and the overall stock-exchange composition had been changing permanently. Despite the dynamic structure of the stock market, the shape of the capital-distribution curves remained remarkably stable ([Fer02], pp. 93; Figure 5.1 on page 95).

To understand how the above observations actually pertain to our goal of estimating the parameters of a large number of stocks, we look at the German stock market index DAX (*Deutscher AktienindeX*). The DAX is comprised of 30 largest companies by market capitalization and trading volume. The stocks in the DAX are weighted according to their free-float capitalization. Both the index composition and the weighting of a given composition are subject to change. We adopt the perspective that the dynamics of the DAX represents the overall dynamics of a market containing only the 30 largest companies. The weights of the assets in the DAX are then viewed as building a market portfolio $\gamma = (\gamma^1, \ldots, \gamma^{30})^\top$ in the sense of (6.11). Let $\Gamma = (\gamma^{(1)}, \ldots, \gamma^{(30)})^\top$, $\gamma^{(1)} < \ldots < \gamma^{(30)}$, be the tuple of decreasingly ordered weights in γ, as introduced in (6.13). Then, at a given time t, Γ_t can be interpreted as a sample from the capital distribution of the 30 largest companies represented in the DAX. The rank processes corresponding to the ordered weights in Γ will be denoted by $S^{(1)}, \ldots, S^{(30)}$. The advantage of adopting a

rank-based perspective is twofold. As will be shown in the following, the drift coefficients of the individual assets can be estimated based on long-term observations of both the overall index dynamics and the dynamics of the assets. However, over the long term, the index composition changes, which complicates the approach significantly. This fact is the first reason we consider a rank-based model: it is immune to changes in the index composition. Another reason is the aforementioned stability of the capital distribution, presented and discussed in [Fer02]. Our rank-based framework will be used to analyse *long-term* investment strategies: the time horizon in the simulations will be set to $T = 30\,years$. Since $S^{(1)}, \dots, S^{(30)}$ represent the dynamics of the 30 largest companies, we expect the ranked weights Γ of these assets to exhibit a similar stability property over the long term. To make use of the advantages of rank-based models and keep the estimation algorithm tractable at the same time, we will introduce a simplified rank-based model. As a result, we will obtain a model which can only be viewed as a rather rough approximation of both, the real stock market index and the ranked assets included. Nonetheless, this will be sufficient since, for our purpose, we need a set of parameters that are only close to reality, and the index will rather play the role of a proxy providing us with the information required to obtain such parameters. The assumptions needed to keep the algorithm sufficiently simple will be made explicit in the following description of the parameter-estimation scheme. We will return to discussing our assumptions and their consequences after introducing the estimation algorithm.
We introduce our approach to estimating the mean rates of return in a general form and let the number of assets be $n > 1$. To emphasize different stages of the estimation procedure, the description is split into four steps.

1. Let the stock market consist of n stocks, $S^1, \dots, S^n$, and assume the rank processes of the assets follow geometric Brownian motions,

$$dS_t^{(j)} = S_t^{(j)} \left(b_{(j)} dt + \sum_{k=1}^{n} \sigma_{(jk)} dW_t^k \right) . \tag{6.14}$$

   Consider a portfolio composed of $S^{(1)}, \dots, S^{(n)}$ in the sense of the *Mutual-fund theorem (MFT)*. MFT, as initially formulated by Tobin in [Tob58], states that every economic agent seeking to maximize their expected utility of terminal wealth will achieve their goal by investing in the risk-free asset and a certain linear combination of risky assets available on the market. The combination of the risky assets is the same for all investors irrespective of their utility function and their initial endowment. Put differently, only the ratio of the risk-free to the overall risky investment may change; the basket of risky investments in the portfolio is the same for all agents [SST09]. Now let $\gamma_{(j)}$ denote the capital proportion invested in the j-th rank. By the MFT, this proportion is the same for all agents (i. e., for all utility functions). Noting that, e. g., in the case of the exponential utility the capital invested in the j-th asset is proportional to $(c^{-1}b)_j$, we can write

$$\gamma_{(j)} = \frac{(c^{-1}b)_{(j)}}{\sum_l (c^{-1}b)_l} \ ,$$

which can be rewritten as

$$b = Kc\gamma, \qquad K = \sum_{l=1}^{n} (c^{-1}b)_l \ . \tag{6.15}$$

Note that $\gamma = (\gamma_{(1)}, \ldots, \gamma_{(n)})^\top$ is constant. The fund describing the basket of risky investments in the sense of the MFT can now be represented by a process with the dynamics

$$dS_t = S_t \sum_{j=1}^{n} \gamma_{(j)} \frac{dS_t^{(j)}}{S_t^{(j)}} = S_t \sum_{j=1}^{n} \gamma_{(j)} \left(b_{(j)} dt + \sum_{k=1}^{n} \sigma_{(jk)} dW_t^k \right) \tag{6.16}$$

and some (constant) initial value S_0.

2. Equation (6.16) shows that the process S itself follows a geometric Brownian motion. We thus write the dynamics of S as

$$dS_t = S_t (b_S dt + \sigma_S d\overline{W}_t) \ , \tag{6.17}$$

and express the drift coefficient, b_S, the diffusion coefficient, σ_S, as well as the one-dimensional Brownian motion $\overline{W}$ in terms of $b = (b_{(1)}, \ldots, b_{(n)})^\top$, $\sigma = (\sigma_{(jk)})_{j,k=1,\ldots,n}$, $\gamma = (\gamma_{(1)}, \ldots, \gamma_{(n)})^\top$ and the independent Brownian motions $(W^j)_{j=1,\ldots,n}$. Comparing the drift terms in (6.16) and (6.17) and using the drift condition (6.15), we immediately get

$$b_S = \gamma^\top b = K\gamma^\top c\gamma \ . \tag{6.18}$$

Representation (6.17) then follows by defining

$$\overline{W} = \frac{1}{\sqrt{\gamma^\top c\gamma}} \sum_{j,k} \gamma_{(j)} \sigma_{(jk)} W^k \ ,$$

which is easily seen to be a Brownian motion by the Lévy characterization[2], and

$$\sigma_S = \sqrt{\gamma^\top c\gamma} \ . \tag{6.19}$$

Finally, note that Equation (6.18) together with the drift condition (6.15) form a system of $n + 1$ linear equations, which allows us to express the vector of the drift

[2] cf. Theorem II.39 in [Pro04]

coefficients of the assets, $b = (b_{(1)}, \ldots, b_{(n)})^\top$, as

$$b = \frac{b_S}{\gamma^\top c\gamma} c\gamma \ . \tag{6.20}$$

In Equation (6.20), the covariance matrix, c, is the only quantity which can be obtained within the framework of our model: it can be estimated based on long-term observations of the rank processes. There remain $n+1$ undetermined parameters: n relative capitalizations $\gamma_{(1)}, \ldots, \gamma_{(n)}$,

$$\gamma_{(1)} + \ldots + \gamma_{(n)} = 1 \ ,$$

determining the basket of risky investments in the sense of the MFT, and the drift coefficient b_S of the fund process S.

3. Let D denote the price process of a stock market index composed of n highest-rank stocks with respect to their capitalization. Let $\gamma^{(1)}, \ldots, \gamma^{(n)}$ denote the ranked weights of the stocks in the index. The $n+1$ free parameters from item 2 will now be chosen as follows.

   - Estimate b_S to be the mean rate of return of the stock index D .
   - Choose $\gamma_{(1)}, \ldots, \gamma_{(n)}$ as

   $$\gamma_{(j)} = \gamma_t^{(j)}, \quad j = 1, \ldots, n$$

   for some $t \geq 0$.

4. We still need to determine one important parameter: the risk aversion, p. There is no such thing as a typical risk aversion in the sense that this quantity depends decisively on the individual preference of the investor, on their perception of risk. Thus, we choose p as corresponding to an investor having a capital of $10000\,CU$[3] and investing half of it, $5000\,CU$, optimally in the fund S. We assume that the investor's preference is described by the exponential utility function. The process S is described by a one-dimensional geometric Brownian motion (cf. Equation (6.17)); thus, Merton's optimality condition implies

   $$5000 = \frac{b_S}{p\sigma_S^2} \ , \tag{6.21}$$

   which determines the risk aversion, p.

[3] CU =*Currency Unit*

The assumptions made in the above parameter-estimation scheme can be summarized as follows. We assume that the rank processes of all stocks available on the market follow geometric Brownian motions and expect the ranked weights of the corresponding market portfolio to be stable over the long term. By applying the Mutual-fund theorem to the rank processes, we construct a fund with constant relative capitalizations of the assets included and assume that, over the long term, this fund approximates the behaviour of the market portfolio. In the final step, we consider a stock market index weighted according to the capitalization of the member assets. The capital distribution of the index (being a subset of the market portfolio) is assumed to have a similar stability property, and the associated part of the MFT-fund is taken to be an approximation of the index over the long term. We conclude by elaborating on all these assumptions.

The first remark concerns the rank-based portfolio introduced in item 1. The fact that the rank processes follow geometric Brownian motions is necessary in order to apply the Mutual-fund Theorem and obtain constant proportions of capital, $\gamma_{(1)}, \dots, \gamma_{(n)}$, invested in risky assets. This, in turn, is required in order for the price process of the fund, S, to follow a geometric Brownian motion as well (cf. (6.16), (6.17)). But the question of whether the rank processes can be represented such as to satisfy (6.14) in continuous time is quite challenging. This is due to the fact that the stocks may swap their ranks. Each time the rank of a stock changes, the drift and diffusion coefficients of the associated price process must be adjusted accordingly. In a discrete-time setting, this does not seem to be an issue. The possible rank swaps, which can essentially be viewed as a "relabeling" of the stocks, do not change the fact that, at each instant, the logarithmic stock prices are described by normally distributed random variables, and the event of two stocks having equal ranks occurs with zero probability. The situation does not change if we make the interval between the time instants smaller by including more time steps in the discretization. The intuition suggests that there is a proper continuous-time limit. However, rigorous treatment requires a proof of existence and uniqueness of the limit of such discrete-time sequences. For details on this subject, the reader is referred to [BFK05, IPB$^+$11, Fer02]. As for the representation we obtain in Equations (6.16), (6.17), it can only be viewed as an approximation of what we assume to exist in the continuous time.

Next, we address the question of how we can justify identifying the constant weights of the MFT-fund with a sample from the capital distribution of the market portfolio. An essential ingredient of our estimation scheme is the empirical observation that the capital distribution of the highest-rank stocks remains stable over the long term (R. Fernholz, [Fer02]). This stability property can be interpreted as expressing the fact that the $\mathbb{R}^n$-valued process $\Gamma = (\gamma^{(1)}, \dots, \gamma^{(n)})^\top$ has a stationary distribution with small variance. By no means, however, does this stability property imply that the ranked weights are

constant. What we rather expect when equating the constant weights of the MFT-fund with the capital distribution of the entire market is that the distribution stabilizes around a value which is close to $(\gamma_{(1)}, \ldots, \gamma_{(n)})$. If it were not case, then, over the long term, the MFT-fund would beat the market creating a sort of an arbitrage opportunity. In our model, the fact that the ranked weights of the market portfolio are not constant can be interpreted as being due to the presence of agents following different investment strategies. However, we assume that the market is dominated by the agents investing optimally according to the Merton strategy in the sense of the MFT.
In the final step, we identify the stock market index, D, with a part of the market portfolio and assume that this part can be approximated by the corresponding part of the MFT-fund by means of the above arguments. By making this approximation, we expect that the part of the market portfolio associated with the member assets of the index is not influenced much by the rest of the market.
The estimation scheme presented is based on rather strong assumptions. Therefore, we stress once again that we do not claim that our scheme is suitable for obtaining precise estimates. However, we do expect that the estimation scheme yields drift coefficients that are close to reality, which is sufficient for our purposes.

6.2.2 Applying the estimation scheme

To select a portfolio replicating the DAX (German stock market index consisting of 30 largest stocks) using the approach presented in the previous subsection, we need to estimate 466 parameter: 465 elements of the (symmetric) covariance matrix plus the mean rate of return, b_D, of the index. To start with, we find the longest time period in the recent history, in which the DAX composition remained constant[4]. The data of the historical DAX compositions can be found on the official web page *www.dax-indices.com* provided by the *Deutsche Börse Group*. According to the data, the longest time period is

$$\text{21 June 2010} \quad — \quad \text{23 September 2012}\ . \tag{6.22}$$

Remark 6.1. In the following, we will always refer to the ranked statistics. That means, j-th component corresponds to the j-th largest weight, which, in the previous subsection, was indicated by putting parentheses around indices. In the following, we will drop the parentheses for ease of notation and understand all vectors and matrices of stock parameters as being rank ordered.

We begin with the estimation of the covariance matrix. This is a standard procedure, but, for the reader's convenience, we briefly review it here. The price process of each

[4]there have been quite a few; we will pick the longest.

asset is given by the solution (6.2). Consider an equidistant partition of the time interval $[0,T]$, $(t_k)_{k=0,\dots,N}$,

$$0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T \ ,$$
$$t_k - t_{k-1} \equiv \Delta t \ .$$

Equation (6.2) then implies

$$\begin{aligned} S^j_{t_k} &= S^j_0 \exp\left(\nu_j t_{k-1} + \nu_j \Delta t + \sum_l \sigma_{jl}(W^l_{t_k} - W^l_{t_{k-1}} + W^l_{t_{k-1}})\right) \\ &= S^j_0 \exp\left(\nu_j t_{k-1} + \sum_l \sigma_{jl} W^l_{t_{k-1}}\right) \exp\left(\nu_j \Delta t + \sum_l \sigma_{jl}(W^l_{t_k} - W^l_{t_{k-1}})\right) \\ &= S^j_{t_{k-1}} \exp\left(\nu_j \Delta t + \sum_l \sigma_{jl} Z_l\right) \ , \end{aligned}$$

for all $j,l = 1,\dots,n$ and $k = 1,\dots,N$, and $Z_1,\dots,Z_n$ are independent and identically distributed random variables such that $Z_1 \sim \mathcal{N}(0,\Delta t)$. For each $j = 1,\dots,n$, define the random variables $(X^j_k)_{k=1,\dots,N}$,

$$X^j_k = \log \frac{S^j_{t_k}}{S^j_{t_{k-1}}} \equiv \nu_j \Delta t + \sum_l \sigma_{jl} Z_l \ .$$

The expectations and the covariancies of these random variables then read as

$$\begin{aligned} E[X^j_k] &\equiv \nu_j \Delta t \ , \\ Cov(X^i_k, X^j_k) &\equiv \sum_{l,m=1}^{n} \sigma_{il}\sigma_{jm} Cov(Z_l, Z_m) = \sum_{l=1}^{n} \sigma_{il}\sigma_{jl}\Delta t = c_{ij}\Delta t \ . \end{aligned} \tag{6.23}$$

In this setting, for each $i,j = 1,\dots,n$, the sample mean and the sample covariancies of the sequence $(X^j_k)_{k=1,\dots,N}$,

$$\begin{aligned} \hat{\mu}_i &= \frac{1}{N}\sum_{k=1}^{N} X^i_k, \\ \hat{s}^2_{ij} &= \frac{1}{N-1}\sum_{k=1}^{N}(X^i_k - \hat{\mu}_i)(X^j_k - \hat{\mu}_j) \ , \end{aligned} \tag{6.24}$$

are used as estimators for the parameters ν_j and c_{jk}. By Equation (6.23), we obtain

$$\hat{\nu}_j = \frac{\hat{\mu}_j}{\Delta t}, \qquad \hat{c}_{jk} = \frac{\hat{s}^2_{jk}}{\Delta t} \ . \tag{6.25}$$

Our estimates will be based on data coming from daily observations of the stock prices. Thus, the time step Δt appearing in the above estimates must be understood as being equal to 1 *trading day*. Usually, however, the mean rates of return as well as the covariance matrix are considered with respect to the trading year rather than trading day. To *annualize* the estimates, we adopt the standard convention that one trading year equals 252 trading days. Hence, in annual terms, $\Delta t = \frac{1}{252}$ *years*. We indicate this symbolically by

$$\hat{\nu}_{year} = 252 \times \hat{\nu}_{day}, \qquad \hat{c}_{year} = 252 \times \hat{c}_{day} \ . \tag{6.26}$$

Remark 6.2. There are two different statistics of the DAX, the *Price Index* and the *Total Return Index*. Whereas the price index tracks only the capital gains of the stocks, the total return index is calculated under the assumption that the dividends on the stocks are reinvested back in the index. We will consider the total-return version of the DAX. This fact must be accounted for by considering time series of the stock prices that also include dividend payments.

In the following, we will make another simplifying assumption concerning in particular our estimate of the covariance matrix. As discussed at the end of the previous subsection, the process representing the stock index in our formal description of the estimation scheme, whose drift and diffusion coefficients we require to obtain estimates for the drift coefficients of the rank processes, is an approximation of the actual index, DAX. This is due to the assumption that the weights of the rank processes are constant. But even within this approximate framework, to estimate the *ranked* covariance matrix properly, one has to interchange the statistics of two stocks each time the two swap their ranks. To analyse the data obtained from time series in this regard would require a considerable amount of work. To get a sense of how often rank swaps may have occurred in the observation time, one can look directly at relative stock capitalization in the index. More precisely, let $S^1, \dots, S^{30}$ be the price processes of the stocks (not the ranks) in the index, and let D denote the index price[5]. The weights $\gamma^1, \dots, \gamma^{30}$ of the assets are proportional to the relative capitalization of these assets, and we therefore assume that there exist constants $C_1, \dots, C_{30} \in \mathbb{R}$ such that

$$\gamma^j = C_j \frac{S^j}{D}, \qquad j = 1, \dots, 30 \ .$$

We know the weights at the last day of the observation period (6.22), and we denote them here by $\widehat{\gamma}_1, \dots, \widehat{\gamma}_{30}$. The observed weights can now be used to determine the constants $C_1, \dots, C_{30}$, which then gives us the dynamics of $\gamma^1, \dots, \gamma^{30}$ by the above equation. Implementing this with the real data shows that, in the observation time (6.22), rank swaps did occur. For the purpose of the present study, however, we need a set of parameters that are only close to those of the real stocks. Therefore, in our estimation, we will not account for the rank changes of the assets. This is equivalent to

[5] Here we mean the price that is actually observed, not an approximation in any sense.

the simplifying assumption that the stocks kept their ranks over the entire observation period.

The time series of the stocks that are contained in the DAX in the time period indicated in (6.22) are taken from *www.ariva.de*. The DAX composition in this time period as well as the associated asset weights are shown in Table A.1. The estimates of the *annualized* covariance and the associated correlation matrix are presented in Table A.2 and Table A.3, respectively.
As one can see from the tables A.2, A.3, the minimum/maximum volatilities and correlation coefficients are

$$\sigma_{min} = \sigma_{18} \approx 0.19, \qquad \sigma_{max} = \sigma_{22} \approx 0.5834 \ ,$$

$$\rho_{min} = \rho_{18,22} = \rho_{22,18} \approx 0.1533, \qquad \rho_{max} = \rho_{7,14} = \rho_{14,7} \approx 0.8588 \ .$$

The estimated correlation matrix allows us to determine the volatility of the stock index. Equation (6.19) yields the value

$$\sigma_D = \sqrt{\gamma^\top c \gamma} \approx 0.2222 \ . \tag{6.27}$$

To obtain an estimate of the drift coefficients of the stocks, by Equation (6.20), we require the annualized mean rate of return of the DAX, b_D. This quantity is estimated from a DAX time series in the time period

$$\text{7 October 1977} \quad — \quad \text{29 August 2016} \ . \tag{6.28}$$

We obtain the following estimated value:

$$b_D \approx 0.0992 \ . \tag{6.29}$$

Notice that the drift coefficients $b_1, \dots, b_n$ as well as b_D appearing in the equations of Section 6.2 refer to those of the *discounted* rank processes. As discussed in Subsection 3.2.4, in the Black-Scholes model, discounting amounts to reducing the drift coefficients by the (annualized) risk-free interest rate (cf. Equation (3.29)), which, in the following, will be denoted by r. An estimate of the interest rate can be obtained from time series of the German bond index *REX (REntenindeX)*. As in the case of the DAX, there are two versions of the REX, the price index and the total return index. We will opt for the latter version. Denoting the bond price by $B = (B_t)_{t\in[0,T]}$ and assuming

$$B_t = B_0 e^{rt} \ ,$$

Rank	**MROR**	**Rank**	**MROR**
(1)	0.0533	(16)	0.0396
(2)	0.0427	(17)	0.0192
(3)	0.0206	(18)	0.0169
(4)	0.0319	(19)	0.0292
(5)	0.0582	(20)	0.0372
(6)	0.0589	(21)	0.0536
(7)	0.0487	(22)	0.0731
(8)	0.0689	(23)	0.0443
(9)	0.0322	(24)	0.0556
(10)	0.0352	(25)	0.0280
(11)	0.0455	(26)	0.0660
(12)	0.0569	(27)	0.0624
(13)	0.0422	(28)	0.0541
(14)	0.0507	(29)	0.0489
(15)	0.0391	(30)	0.0416

TABLE 6.1: Anualized estimated mean rates of return (MROR) of the discounted rank processes.

the interest rate can be estimated by the slope of the line connecting the initial and the terminal value of the logarithmic bond price, i. e., the values $\ln B_0$ and $\ln B_0 + rT$. From a series of daily observations over the time period

$$\text{27 April 1992} \quad — \quad \text{26 August 2016} \ , \tag{6.30}$$

the annualized interest rate can be estimated to be

$$r \approx 0.055 \ . \tag{6.31}$$

We now have all the ingredients to calculate the drift coefficients of the (discounted) rank processes, $b_1, \ldots, b_n$, as well as the risk aversion, p. The latter follows from Equation (6.21) after replacing b_D by $b_D - r$ and using (6.27), which yields

$$p \approx 1.7906 \cdot 10^{-4} \ . \tag{6.32}$$

The mean rates of return are obtained from Equation (6.20) where, yet again, discounting must be accounted for by $b_D \mapsto b_D - r$. The estimated values are presented in Table 6.1.

6.3 Implementation algorithm

Our goal is to calculate the values of the primal and the dual functional derived in Section 5.2 of the previous chapter and given by the equations (5.49), (5.50). To achieve this, numerical methods will be used. Recall that the functional (5.50) describes the expected utility $E[u(X(\varphi)_T)]$ of terminal payoff generated by a candidate strategy φ. The functional $-e^{-px-H(\widetilde{Q},P)}$ evaluated in (5.49), for each appropriately constructed EMM $\widetilde{Q} \sim P$, yields an upper bound on the expected utility of any candidate strategy[6],

$$E[u(X(\varphi)_T)] \leq -e^{-px-H(\widetilde{Q},P)} \ . \tag{6.33}$$

As emphasized in Remark 5.6, the relation (6.33), for all strategies and appropriate martingale measures, defines an interval containing the exact value of the asymptotic optimization problem. Thus, the quality of a primal-dual pair can be assessed from two perspectives, a practical and a theoretical one. From the practical perspective, the information provided by (6.33) is a direct measure of the performance of an investor following one of our candidate strategies. From the theoretical point of view, (6.33) yields an estimate of the expected utility of the (unknown) asymptotic optimizer. In the following, we will provide the values of primal and dual functionals resulting from our numerical simulations, which should always be interpreted from the two aforementioned complementary perspectives.
For the purpose of numerical calculations, it will be more convenient to analyse our candidates in terms of *utility loss* rather than total utility generated. We therefore begin this section by defining and explaining the meaning of the quantities we will use to measure the investor's loss due to the presence of proportional transaction costs. Subsequently, we introduce the discrete-time version of the model and explain how these quantities can be obtained by means numerical simulations. This section will be concluded by presenting and discussing the results of our numerical analysis.

6.3.1 Measures of loss and their dual bounds

In Chapter 4, Section 4.3.2, we discussed the leading-order contributions to the utility loss of a strategy trading in a bid-ask spread. We briefly recapitulate the key observations here for convenience. At the leading order in ε, the total utility percentage lost by trading according to a strategy $\varphi = \varphi^* + \Delta\varphi = \varphi^+ - \varphi^-$ in a bid-ask spread is the sum of the

[6]See Equation (5.25) and its derivation for details.

utility loss due to fees levied on each transaction,

$$L^U_{tc} = pE_Q[S \bullet \bar{\varphi}_T] = p\sum_{j=1}^{n} \varepsilon_j E_Q[S^j \bullet (\varphi^{+,j} + \varphi^{-,j})_T] \ ,$$

and the utility loss due to the deviation of the candidate strategy from the frictionless optimizer,

$$L^U_{disp} = \frac{p^2}{2} E_Q[\Delta\varphi^\top c^S \Delta\varphi \bullet I_T] \ .$$

Our main measure of the investor's loss will be the functional

$$L^U_{tot}(\varphi) = L^U_{tc}(\varphi) + L^U_{disp}(\varphi) \ , \tag{6.34}$$

with φ standing for one of the candidates introduced in the subsections 5.1.1 – 5.1.3. The dual upper bound on the expected utility, which is described by the functional $-e^{-px-H(\widetilde{Q},P)}$, can be transformed into a dual *lower bound on the utility loss* in a rather natural way. By Equation (5.49), the dual functional

$$D^U(\psi) = -pE_Q[\psi \bullet \Delta S_T] + \frac{p^2}{2} E_Q[\Delta\psi^\top c^S \Delta\psi \bullet I_T] \tag{6.35}$$

satisfies

$$L^U_{tot}(\varphi) \geq D^U(\psi) \tag{6.36}$$

for any candidate strategy φ and the auxiliary process $\psi = \varphi^* + \Delta\psi$, $\Delta\psi = M^{-1}\Delta\widetilde{\varphi}$, used to generate the shadow price process in the sense of Definition 5.4 (see also Remark 5.5).

Another measure of loss is given in terms of the certainty equivalent introduced Chapter 4, Definition 4.15. For the exponential utility function, the certainty equivalent reads as

$$CE(\varphi)_T = -\frac{1}{p} \ln\left(-E\left[u(X(\varphi)_T)\right]\right) \ ,$$

with φ being a trading strategy and X its payoff process. Let CE^* denote the certainty equivalent of the frictionless optimizer φ^*. By Equation (3.34) from Subsection 3.2.4, CE^* in the Black-Scholes model can easily be shown to satisfy

$$CE^* = x + \frac{T}{2p} b^\top c^{-1} b \ . \tag{6.37}$$

As shown in Subsection 4.3.2, Equation (4.33), the certainty equivalent of a strategy $\varphi = \varphi^* + \Delta\varphi$, $\Delta\varphi = \mathcal{O}(\varepsilon^{1/3})$, at the leading order in ε, reads as

$$CE = CE^* - \frac{1}{p} L^U_{tot} \ .$$

The second functional we will use to quantify the investor's loss is the relative certainty-equivalent loss,

$$L_{rel}^{CE}(\varphi) = \frac{CE^* - CE(\varphi)}{CE^*} = \frac{L_{tot}^U(\varphi)}{px + \frac{T}{2} b^\top c^{-1} b} \ . \tag{6.38}$$

This quantity measures the percentage of the certainty equivalent lost due to the presence of the bid-ask spread. By the relation (6.36), the functional

$$D_{rel}^{CE}(\psi) = \frac{D^U(\psi)}{px + \frac{T}{2} b^\top c^{-1} b} \tag{6.39}$$

satisfies

$$L_{rel}^{CE}(\varphi) \geq D_{rel}^{CE}(\psi) \ . \tag{6.40}$$

The certainty-equivalent loss, $CE^* - CE$, can be related to the total amount invested in the optimal portfolio at the beginning. In the Black-Scholes model, the total initial investment is given by $\frac{1}{p} \sum (c^{-1} b)_j$, and we obtain our third measure of loss, the functional

$$L_{inv}^{CE}(\varphi) = \frac{CE^* - CE(\varphi)}{\frac{1}{p} \sum (c^{-1} b)_j} = \frac{L_{tot}^U(\varphi)}{\sum (c^{-1} b)_j} \ . \tag{6.41}$$

The corresponding dual lower bound for the loss functional L_{inv}^{CE} is given by

$$D_{inv}^{CE}(\psi) = \frac{D^U(\psi)}{\sum (c^{-1} b)_j} \ , \tag{6.42}$$

so that

$$L_{inv}^{CE}(\varphi) \geq D_{inv}^{CE}(\psi) \ . \tag{6.43}$$

As already mentioned in the introduction of Chapter 4, the presence of a no-trade region in markets with a bid-ask spread is a natural consequence of two opposite effects balancing out: trading so as to stay close to the frictionless optimizer, and keeping the position unchanged in order to avoid unnecessary transaction fees. Thus, an optimal no-trade region will necessarily result in an optimal trade-off between the displacement and transaction loss. It will therefore be instructive to compare the overall performance of a candidate strategy with the ratio of the displacement loss to the transaction loss it generates. To measure this ratio, we introduce the functional

$$F(\varphi) = \frac{L_{tc}^U(\varphi)}{L_{disp}^U(\varphi)} \ . \tag{6.44}$$

6.3.2 Discretization scheme

Let $[0,T]$ be the time interval of the optimization problem. Let $N \in \mathbb{N}$ and set

$$\Delta t = \frac{T}{N} \ .$$

We will use an equidistant partition $(t_k)_{k=0,\ldots,N}$ of the time interval given by

$$0 = t_0 < t_1 < \ldots < t_N = T \ ,$$

$$t_k - t_{k-1} \equiv \Delta t \ .$$

The processes we need to simulate in order to implement our approximation scheme fall into two categories. First, there are the processes associated with the frictionless market: the rank processes $S^1, \ldots, S^n$ following geometric Brownian motions (cf. Equation (6.14)), and the optimizer $\varphi^* = (\varphi^{*,1}, \ldots, \varphi^{*,n})^\top$ defined in (6.3). The second category comprises primal and dual candidates which are essentially determined by the candidate domains $(\Delta^{(i)}, L^{(i)})$, $i = 1,2,3$, (cf. equations (6.6) – (6.10)) and the associated Skorohod problems (5.3), (5.39). Since $S^1, \ldots, S^n$ are known explicitly (cf. Equation (6.2)), simulating these processes essentially amounts to generating n independent and normally distributed random variables and discretizing the explicit solution. Once the values of the rank processes are obtained, the frictionless optimizing strategy follows by Equation (6.3). By (6.6) – (6.10), the processes $S^1, \ldots, S^n$ also determine the candidate domains. To obtain the trading strategies $\varphi = \varphi^* + \Delta\varphi$ to each candidate domain, and the dual candidate $(\widetilde{S}, \widetilde{Z})$ to the domain $(\Delta^{(2)}, L^{(2)})$, the Skorohod problems (5.3), (5.39) must be solved, which turns out to be the most challenging part of the discretization procedure. To be more precise, we need a discrete-time approximation of a strong solution (if it exists) to a Skorohod problem in a *convex* domain with a non-smooth and time-dependent boundary. As already mentioned, the one-dimensional case is covered in [SW13]. As for the multi-dimensional case, there exist many results on the existence of strong solutions and different approximation techniques for Skorohod problems in general, *time-independent* domains [DI93, DI08, Sł01, Sł13, Sł14]. The existence of a *weak* solutions to stochastic differential equations with oblique reflection in time-dependent domains is proved in [Nn10a], and in the subsequent article [Nn10b], the authors develop an algorithm for weak approximations of such solutions. This algorithm as well as the aforementioned approximations in time-independent domains, [Sł01], suggest that an Euler scheme with projections onto the boundary of the no-trade region along the direction of reflection may provide a good algorithm to implement a time-discrete approximation in our case. However, when one attempts to implement such a projection scheme in numerical simulations with finite computational resources, one immediately faces a problem connected with the non-smooth structure of the boundary of the domain. To understand the source of the problem, recall the formulation of the Skorohod problem provided in Definition 5.1. A candidate domain, R, is determined by a pair (Δ, L) via

$$R = L\overline{R}, \quad \overline{R} = \prod_{j=1}^{n} [-\Delta^j, \Delta^j] \ ,$$

L being a linear transformation. The boundary of R, ∂R, satisfies

$$\partial R = \bigcup_{j=1}^{n} \partial R^{+,j} \cup \partial R^{-,j} \ ,$$

$$\partial R^{\pm,j} = \left\{ x \in R \colon (L^{-1}x)_j = \mp\Delta^j \right\} \ .$$

For $j = 1, \dots, n$, define the sets

$$Q^{+,j} = \left\{ x \in \mathbb{R}^n \colon (L^{-1}x)_j < -\Delta^j \right\}, \quad Q^{-,j} = \left\{ x \in \mathbb{R}^n \colon (L^{-1}x)_j > \Delta^j \right\} \ , \qquad (6.45)$$

$$P^{\pm,j} = \left\{ x \in Q^{\pm,j} \colon \exists\, \lambda > 0 \colon x \pm \lambda \mathbf{e}^{(j)} \in \partial R^{\pm,j} \right\} \ ,$$

$$P = \bigcup_{j=1}^{n} P^{+,j} \cup P^{-,j} \ . \qquad (6.46)$$

The elements of P will be referred to as *simply projectable*. Note that the sets $Q^{\pm,j}$ describe half-spaces; they are not restricted to R in any other than the defining direction. This is not the case for the sets $P^{\pm,j}$. A discrete-time approximation of $\Delta\varphi$, starting at the origin and evolving according to $d\Delta\varphi = -d\varphi^*$ in the interior of R, will eventually cross the boundary by jumping from an interior point to an element of R^c. With positive probability, this element can also be an element of $R^c \setminus P$. It is clear that for a non-degenerate portfolio, i. e. a portfolio of assets whose correlation matrix ρ satisfies $\rho_{ij} < 1$ for all $i, j = 1, \dots, n$, the probability of jumping to an element of $R^c \setminus P$ will decrease to zero as $\Delta t \to 0$. However, in computer simulations, the step size is finite, and we have to construct our algorithm such as to account for those events. Assume that the process $\Delta\varphi$ jumps to a point $x \in R^c \setminus P$. What point on the boundary of R do we associate with x? Obviously, there are many possibilities to map such an x onto the boundary of R. It is important to notice that from the region $R^c \setminus P$, the boundary of the domain can be reached only by adjusting at least two components of x. This is not an immediate consequence of the definition of P since, in a non-rectangular domain, there exist $x \in (Q^{-,j} \setminus P^{-,j}) \cup (Q^{+,j} \setminus P^{+,j})$ satisfying $x + \lambda \mathbf{e}^{(j)} \in \partial R \setminus (\partial R^{+,j} \cup \partial R^{-,j})$, for some $\lambda \in \mathbb{R}$. However, such projections should be forbidden since it appears reasonable to demand that an exterior point satisfying $x \in Q^{+,j} \cup Q^{-,j}$ be projected onto a boundary element contained in $\partial R^{+,j} \cup \partial R^{-,j}$. We propose here a projection scheme involving at most n iteration steps. Simply projectable points are detected automatically in the sense that for all $x \in P$, the algorithm terminates after the first iteration. The details of the algorithm are presented in the following

Definition 6.3. Let (Δ, L) define a candidate domain. We define the projection

$$\pi^R \colon \mathbb{R}^n \longrightarrow R$$

by the following algorithm. Let $y \in \mathbb{R}^n$. Set $y^{(0)} = y$ and $x^{(0)} = L^{-1}y^{(0)}$. Define

$$I^{(0)} = \left\{ i \in \{1, \ldots, n\} \colon |x_i^{(0)}| > \Delta^i \right\} .$$

If $I^{(0)} = \emptyset$,

$$\pi^R(y) = y^{(0)} = y \ .$$

Otherwise, let

$$x_{m_1}^{(0)} = \max\left\{ |x_i^{(0)}| - \Delta^i \colon i \in I^{(0)} \right\} .$$

Set

$$y^{(1)} = y^{(0)} + \lambda^{(1)} \mathbf{e}^{m_1} \ ,$$

$$x^{(1)} = L^{-1}y^{(1)} = x^{(0)} + \lambda^{(1)}(L^{-1})^{(m_1)} \ ,$$

with $(L^{-1})^{(j)}$, $j = 1, \ldots, n$, denoting the j-th column vector of L^{-1}. Demand

$$x_{m_1}^{(1)} = x_{m_1}^{(0)} + \lambda^{(1)} L^{-1}_{m_1,m_1} \stackrel{!}{=} \operatorname{sgn}(x_{m_1}^{(0)}) \Delta^{m_1} \ ,$$

wich implies

$$x^{(1)} = x^{(0)} + \frac{\operatorname{sgn}(x_{m_1}^{(0)})\Delta^{m_1} - x_{m_1}^{(0)}}{L^{-1}_{m_1,m_1}} (L^{-1})^{(m_1)} \ ,$$

$$y^{(1)} = y^{(0)} + \frac{\operatorname{sgn}(x_{m_1}^{(0)})\Delta^{m_1} - x_{m_1}^{(0)}}{L^{-1}_{m_1,m_1}} \mathbf{e}^{(m_1)} \ .$$

Define

$$I^{(1)} = \left\{ i \in \{1, \ldots, n\} \colon |x_i^{(1)}| > \Delta^i \right\} .$$

If $I^{(1)} = \emptyset$, set

$$\pi^R(y) = y^{(1)} \ .$$

Otherwise, let $k > 1$ and set

$$\begin{aligned} y^{(k)} &= y^{(k-1)} + \sum_{j=1}^{k} \lambda_j^{(k)} \mathbf{e}^{(m_j)} \ , \\ x^{(k)} &= L^{-1} y^{(k)} = x^{(k-1)} + \sum_{j=1}^{k} \lambda_j^{(k)} (L^{-1})^{(m_j)} \ , \end{aligned} \tag{6.47}$$

with $m_1, \ldots, m_{k-1}$ being the old maximum indeces and m_k defined through

$$x_{m_k}^{(k-1)} = \max_{i \in I^{(k-1)}} \left\{ |x_i^{(k-1)}| > \Delta^i \right\} \ .$$

Demand

$$x_{m_l}^{(k)} = x_{m_l}^{(k-1)} + \sum_{j=1}^{k} \lambda_j^{(k)} L_{m_l,m_j}^{-1} \stackrel{!}{=} x_{m_l}^{(k-1)}, \quad l = 1, \ldots, k-1 \ ,$$

$$x_{m_k}^{(k)} = x_{m_k}^{(k-1)} + \sum_{j=1}^{k} \lambda_j^{(k)} L_{m_k,m_j}^{-1} \stackrel{!}{=} \operatorname{sgn}(x_{m_k}^{(k-1)}) \Delta^{m_k} \ .$$

This yields the following system of k linear equations:

$$\begin{aligned} \sum_{j=1}^{k} \lambda_j^{(k)} L_{m_l,m_j}^{-1} &= 0, \quad l = 1, \ldots, k-1 \ , \\ \sum_{j=1}^{k} \lambda_j^{(k)} L_{m_k,m_j}^{-1} &= \operatorname{sgn}(x_{m_k}^{(k-1)}) \Delta^{m_k} - x_{m_k}^{(k-1)} \ . \end{aligned} \tag{6.48}$$

Solve (6.48) for $\lambda_1^{(k)}, \ldots \lambda_k^{(k)}$ and obtain $y^{(k)}$ from (6.47). Define

$$I^{(k)} = \left\{ i \in \{1, \ldots, n\} \colon |x_i^{(k)}| > \Delta^i \right\} \ .$$

If $I^{(k)} = \emptyset$, set

$$\pi^R(y) = y^{(k)} \ .$$

Otherwise, $k \mapsto k+1$ and proceed to compute the next iteration starting from (6.47).

Remark 6.4. In the case of a rectangular domain, the algorithm can be simplified significantly. This is due to the fact that each point $x \in R^c$ can be uniquely identified with an element of the boundary of R in a natural way. The domain (Δ, L) is recatangular if the matrix L is diagonal, $L_{ij} = L_i \delta_{ij}$. The no-trade region is given by

$$R = \prod_{j=1}^{n} [-\widetilde{\Delta}^j, \widetilde{\Delta}^j], \quad \widetilde{\Delta}^j = L_j \Delta^j \ .$$

For each $x \in R^c$, there exists a tuple $(i_1, \dots, i_k) \in \{1, \dots, n\}^k$, $k \le n$, such that

$$x \in \bigcap_{l=1}^{k} Q^{\mu_l, i_l} \ ,$$

with $\mu_1, \dots, \mu_k \in \{-1, 1\}$ symbolizing $+$, $-$, as introduced in Equation (6.45). This can also be expressed as

$$|x_{i_l}| > \widetilde{\Delta}^{i_l} \ \wedge \ \text{sgn}(x_{i_l}) = \mu_l, \quad l = 1, \dots, k \ .$$

The point x is then simply projectable (cf. Equation (6.46)) if and only if $k = 1$, meaning that x is contained in precisely one of the $2n$ half-spaces $Q^{\pm, j}$, $j = 1, \dots, n$. The projection of x onto the boundary of R, $\pi^R(x)$, can be defined as

$$\left(\pi^R(x)\right)_j = \begin{cases} \mu_j \widetilde{\Delta}^j, \ j \in \{i_1, \dots, i_k\} \ , \\ x_j, \ \text{otherwise} \ . \end{cases}$$

This can also be viewed as being defined on the whole of $\mathbb{R}^n$ since $\pi^R\big|_R = \text{Id}$. In this case, a more appropriate definition would be

$$\left(\pi^R(x)\right)_j = \max\{\min\{x_j, \widetilde{\Delta}^j\}, -\widetilde{\Delta}^j\}, \quad j \in \{1, \dots, n\} \ . \tag{6.49}$$

This equation gives us a direct expression for the projection map π^R, introduced in Definition 6.3, without any iterations.

The projection scheme presented in Definition 6.3 and Remark 6.4 allows us to construct discrete-time approximations of the primal and dual candidates with respect to the domains defined by (6.6) – (6.10). The algorithm presented in Definition 6.3 is required only for the primal candidate corresponding to the domain $(\Delta^{(2)}, L^{(2)})$ given by a non-diagonal linear transformation (cf. Subsection 5.1.2 and Equation (6.7), (6.8)). The domains $(\Delta^{(i)}, L^{(i)})$, $i = 1, 3$, are rectangular, and the corresponding candidates can be obtained using the simplified projection scheme from Remark 6.4. This is also the case for the dual candidate. It is generated by the process $\Delta\psi = M^{-1}\Delta\widetilde{\varphi}$, with $M^{-1} = L^{(2)}$ given by (6.7) and $\Delta\widetilde{\varphi}$ solving the Skorohod problem (5.39) which is formulated with respect to a rectangular domain. In Appendix B, we discuss an alternative projection scheme based on l_1-minimization. The reader can see the surprisingly big difference in the performance of both schemes by comparing the simulation results presented in Table B.1.

We can now present the algorithm for simulating the trading strategies and computing the values of the primal and dual functionals introduced in Subsection 6.3.1, Equation (6.34) – (6.44). Being different measures of the investor's loss due to the presence of proportional transaction costs, all these functionals are given in terms of expected values of a few (stochastic) integrals with respect to the EMM $Q \sim P$. To avoid changing the probability measure when evaluating the expectations, the values of the price processes will be simulated with respect to the measure Q. With all relevant processes computed, the integrals can be easily calculated by a standard discretization procedure. To obtain the respective expectations, Monte-Carlo method will be employed.

In the following, let $\varphi^{(i)} = \varphi^* + \Delta\varphi^{(i)}$ be the trading strategy corresponding to the domain $(\Delta^{(i)}, L^{(i)})$, $i = 1, 2, 3$, as introduced in Definition 5.1, and let $\psi = \varphi^* + \Delta\psi$, $\Delta\psi = L^{(2)}\Delta\widetilde{\varphi}$, be the auxiliary process generating the shadow price (in the sense of Definition 5.4) with respect to the domain $(\Delta^{(2)}, L^{(2)})$. For convenience, we recapitulate briefly that the shadow price $\widetilde{S} = S + \Delta S$ is determined by $\Delta S = \tilde{f}(\Delta\widetilde{\varphi}, S, \Delta^{(2)})$, with

$$\tilde{f}_j(x, y, z) = \frac{\varepsilon_j y_j}{2} \left(\left(\frac{x_j}{z_j} \right)^3 - 3 \, \frac{x_j}{z_j} \right) . \tag{6.50}$$

The implementation algorithm consists of the following steps:

1. <u>Time step $t_0 = 0$:</u> Initialize the stock prices:

   $$S_0 = (S_0^1, \ldots, S_0^n)^\top .$$

   By (6.3) and (6.6) – (6.10) obtain the frictionless optimizer and the candidate domains,

   $$\varphi_0^* = (\varphi_0^{*,1}, \ldots, \varphi_0^{*,n})^\top, \quad (\Delta_0^{(i)}, L_0^{(i)}), \quad i = 1, 2, 3 .$$

   For $j = 1, \ldots, n$, $i = 1, 2, 3$, set $\Delta\varphi_0^{(i),j} = \Delta\widetilde{\varphi}_0^j = 0$, which yields $\Delta\psi_0^j = L_0^{(2)}\Delta\widetilde{\varphi}_0^j = 0$ and

   $$\varphi_0^{(i),j} = \psi_0^j = \varphi_0^{*,j}$$

   For the shadow price, this implies $\Delta S_0^j = \tilde{f}_j\left(\Delta\widetilde{\varphi}_0^j, S_0, \Delta_0^{(2)}\right) = 0$, $j = 1, \ldots, n$, and $\widetilde{S}_0 = S_0$.

2. <u>Time steps $t_k > t_0$:</u> Generate n independent and identically distributed random variables $Z_1, \ldots, Z_n$ such that $Z_1 \sim N(0, 1)$. For $k = 1, \ldots, N - 1$, obtain the price value at the time t_k, $S_k = (S_k^1, \ldots, S_k^n)^\top$, via (cf. Section 6.1, Equation (6.2))

   $$S_k^j = S_{k-1}^j \exp\left\{ -\frac{\sigma_j^2}{2}\Delta t + \sum_{l=1}^n \sigma_{jl} Z_l \right\} , \tag{6.51}$$

   which, by the equations (6.3), (6.6) – (6.10), then yields the associated frictionless optimizer and the candidate domains,

   $$\varphi_k^* = (\varphi_k^{*,1}, \ldots, \varphi_k^{*,n})^\top, \quad (\Delta_k^{(i)}, L_k^{(i)}), \quad i = 1, 2, 3 .$$

Note that the right-hand side of (6.51) is the representation of the price processes with respect to a Q-Brownian motion, which can be obtained from Equation (6.2) by

$$W^j \longmapsto W^j - (\sigma^{-1}b)_j I \ ,$$

for all $j = 1, \dots, n$, with σ being the Cholesky factor of the covariance matrix (cf. Subsection 3.2.4, Equation (3.33)). Set

$$\widetilde{\varphi}_k^* = (L^{(2)})^{-1} \varphi_k^* \ .$$

For each $i = 1, 2, 3$, let

$$\overline{R}_k^{(i)} = \prod_{j=1}^{n} [-\Delta_k^{(i)}, \Delta_k^{(i)}], \quad R_k = L_k^{(i)} \overline{R}_k^{(i)}$$

Define

$$\xi_k = \Delta\varphi_{k-1} - \varphi_k^* + \varphi_{k-1}^* \ ,$$

$$\eta_k = \Delta\widetilde{\varphi}_{k-1} - \widetilde{\varphi}_k^* + \widetilde{\varphi}_{k-1}^* \ ,$$

and set, for $i = 1, 2, 3$,

$$\Delta\varphi_k^{(i)} = \pi^{R_k}(\xi_k), \quad \Delta\widetilde{\varphi}_k = \pi^{R_k}(\eta_k), \quad \Delta\psi_k = L_k^{(2)} \Delta\widetilde{\varphi}_k$$

$$\varphi_k^{(i)} = \varphi_k^* + \Delta\varphi_k^{(i)}, \quad \psi_k = \varphi_k^* + \Delta\psi_k \ .$$

Recall that $\Delta\widetilde{\varphi}$ and $\Delta\varphi^{(i)}$, $i = 1, 3$, evolve in rectangular domains. Hence, for these processes, the simplified version of the projection map, introduced in Remark 6.4, should be used. The value of the shadow price reads as

$$\widetilde{S}_k = S_k + \Delta S_k, \quad \Delta S_k = \tilde{f}(\Delta\widetilde{\varphi}_k, S_k, \Delta_k^{(2)}) \ ,$$

with $\tilde{f}$ given by (6.50).
Liquidate the portfolio at the time $t_N = T$.

3. Integral computation: Introduce the following discretizations of the integrals appearing in the functionals (6.34), (6.35):

$$I_{tc}(\varphi)_t \approx \int_0^t S_u \, d\bar{\varphi}_u, \quad I_{disp}(\beta)_t \approx \int_0^t \beta_u^\top c_u^S \beta_u \, du, \quad I(\psi) \approx \int_0^t \psi_u \, d\Delta S_u \ ,$$

$\beta \in \{\Delta\psi, \Delta\varphi^{(1)}, \Delta\varphi^{(2)}, \Delta\varphi^{(3)}\}$. Recall that

$$d\bar{\varphi}^j = \varepsilon_j d|D\varphi^j| = \varepsilon_j (d\varphi^{+,j} + d\varphi^{-,j}) \ .$$

Noting that, for $j = 1, \ldots, n$, $k = 1, \ldots, N$,

$$\bar{\varphi}^j_k - \bar{\varphi}^j_{k-1} = \varepsilon_j(\varphi^{+,j}_k - \varphi^{+,j}_{k-1}) + \varepsilon_j(\varphi^{-,j}_k - \varphi^{-,j}_{k-1}) = \varepsilon_j\big|\varphi^j_k - \varphi^j_{k-1}\big| ,$$

the first term reads as

$$\begin{aligned} I_{tc}(\varphi)_T &= \sum_{j=1}^{n}\sum_{k=1}^{N-1} \varepsilon_j S^j_k \big|\varphi^j_k - \varphi^j_{k-1}\big| \\ &= \sum_{j=1}^{n}\sum_{k=1}^{N-1} \varepsilon_j S^j_k \big|\pi^{R^k}_1(\xi_k)^j - \xi^j_k\big| \end{aligned} \tag{6.52}$$

Notice that the liquidation costs, $\sum_{j=1}^n \varepsilon_j S^j_N |\varphi_{N-1}|$, are dropped in (6.52). The remaining integrals are discretized as

$$I_{disp}(\beta)_T = \Delta t \sum_{k=1}^{N-1} \beta^\top_k c^S_k \beta_k = \Delta t \sum_{i,j=1}^{n} c_{ij} \sum_{k=1}^{N-1} \beta^i_k \beta^j_k S^i_k S^j_k \, . \tag{6.53}$$

$$I(\psi)_T = \sum_{j=1}^{n}\sum_{k=1}^{N} \psi^j_{k-1}(\Delta S^j_k - \Delta S^j_{k-1}) \, . \tag{6.54}$$

4. <u>Monte Carlo:</u> Let L be a large integer (typically, $L \gg 10^3$). Repeat steps 1 – 3 L times and obtain L realizations of the random variables $I_{tc}(\varphi)_T$, $I(\psi)_T$, $I_{disp}(\beta)_T$, $\beta \in \{\Delta\psi, \Delta\varphi^{(1)}, \Delta\varphi^{(2)}, \Delta\varphi^{(3)}\}$. The Q-expectations of the integrals can then be estimated by the sample means of the Monte-Carlo realizations. More precisely, let $Y^{(1)}, \ldots, Y^{(L)}$ be the outcomes of L independent Monte-Carlo simulations of a random variable Y. The expectation of Y can be estimated by

$$\widehat{Y} = \frac{1}{L}\sum_{l=1}^{L} Y^{(l)} \, . \tag{6.55}$$

The standard deviation of the Monte-Carlo estimates will be computed by

$$\hat{s}_Y = \sqrt{\frac{1}{L-1}\sum_{l=1}^{L}(Y^{(l)} - \widehat{Y})^2} \, . \tag{6.56}$$

6.3.3 Asymptotics

In the present study we investigate the *asymptotic* behaviour of the expected utility loss due to the presence of a bid-ask spread. In theory, investigating the asymptotics amounts to analysing the leading-order contributions of all relevant terms to the utility

functional as ε tends to zero. In practice, in order to obtain the utility loss via numerical simulations, the proportionality factor determining the transaction costs must be assigned a certain value. In order for the simulation results to be consistent with the theoretical asymptotic approximations, the value of ε we choose must have an appropriate order of magnitude. Thus, when implementing the approximation scheme practically, the following question must be addressed: *How small do we have to choose ε to ensure the applicability of the theoretical asymptotic approximations?* To answer this question, we first simulate a one-dimensional portfolio (portfolio with only one risky asset) for different values of ε. Then, we compute numerically the expected utility of terminal wealth of the portfolio without any asymptotic approximations and compare the results with the exact asymptotic solution presented in Subsection 4.3.3.
According to Equation (4.37), at the leading order in ε, the utility loss of a single-stock portfolio is given by

$$L^U_{tot} = \frac{p^2}{2} E_Q\left[(\Delta^2 c^S) \bullet I_T\right] \,, \tag{6.57}$$

with

$$\Delta = \left(\frac{3}{2p}\frac{c^{\varphi^*}}{c^S}\varepsilon S\right)^{1/3}$$

being the half-width of the asymptotically optimal no-trade region. For a Black-Scholes asset with the drift coefficient b and the volatility σ, we have (cf. Subsection 6.1)

$$\varphi^* = \frac{b}{p\sigma^2 S^2}, \quad c^S = \sigma^2 S^2, \quad c^{\varphi^*} = \sigma^2(\varphi^*)^2 \,,$$

which yields

$$\Delta = \frac{1}{pS}\left(\frac{3\varepsilon b^2}{2\sigma^4}\right)^{1/3} \tag{6.58}$$

and

$$\Delta^2 c^S = \frac{\sigma^2}{p^2}\left(\frac{3\varepsilon b^2}{2\sigma^4}\right)^{2/3} \,.$$

Inserting this result into (6.57) leads to the following expression for the leading-order utility loss in the Black-Scholes model:

$$L^U_{tot} = \left(\frac{9\varepsilon^2 b^4}{32\sigma^2}\right)^{1/3} T \,. \tag{6.59}$$

Equation (6.59) is used to produce the analytic reference plot presented in Figure 6.1. To simulate the expected utility loss without any asymptotic approximations, we essentially follow the algorithm presented in Subsection 6.3.2. A few adjustments must, however, be made, and we therefore present the main steps here once again for clarity.
The exact asymptotic solution to the one-dimensional optimization problem is the pair $(\Delta\varphi, \varphi)$, $\Delta\varphi$ being a semimartingale and φ a finite-variation process, which solves the

Skorohod problem

$$\begin{aligned}\Delta\varphi_t &= -\varphi_t^* + \varphi_t \in [-\Delta_t, \Delta_t], \quad \Delta\varphi_0 = 0 \ , \\ \varphi_t &= \int_0^t -\mathrm{sgn}(\Delta\varphi_u) d|D\varphi|_u \ , \\ |D\varphi|_t &= \int_0^t I_{\{\pm\Delta_u\}}(\Delta\varphi_u) d|D\varphi|_u \ ,\end{aligned} \tag{6.60}$$

with Δ being the boundary process (6.58). This problem is shown to have a unique strong solution in [SW13]. Our goal is to estimate the value of the functional $E[u(X(\varphi)_T)]$ by means of numerical simulations, with $X(\varphi)$ denoting the payoff process generated by φ. Consider the time-interval partition $0 = t_0 < \ldots < t_N = T$, and let S_0 denote the initial value of the stock price which we assume to follow a one-dimensional geometric Brownian motion (cf. Equation (6.2)). The discrete-time approximation of the price process and that of the frictionless optimizer read as

$$\begin{aligned}S_k &= S_{k-1} \exp\left(\nu\Delta t + \sigma Z\right), \quad \nu = b - \frac{\sigma^2}{2}, \quad Z \sim N(0,1) \ , \\ \varphi_k^* &= \frac{b}{p\sigma^2 S_k} \ ,\end{aligned} \tag{6.61}$$

for all $k = 1, \ldots, N$. Note that, unlike in (6.51), ν is used in the deterministic part of the exponent. This is due to the fact that the simulations are performed with respect to a P-Brownian motion, which makes sense since we are interested in calculating a P-expectation at the end. To discretize the startegy $\varphi = \varphi^* + \Delta\varphi$, we use the Euler projection scheme proposed in [SW13]. Let $[-a, a] \subset \mathbb{R}$ be an interval. Define

$$\pi^a(x) = \max\{\min\{a, x\}, -a\} \ . \tag{6.62}$$

Set $\Delta\varphi_0 = 0$ and

$$\begin{aligned}\xi_k &= \Delta\varphi_{k-1} - \varphi_k^* + \varphi_{k-1}^*, \quad \Delta\varphi_k = \pi^{\Delta_k}(\xi_k) \ , \\ \varphi_k &= \varphi_k^* + \Delta\varphi_k \ .\end{aligned} \tag{6.63}$$

Assuming the investor's initial endowment to be zero, the discretized terminal payoff is given by

$$X(\varphi)_T = \sum_{k=1}^{N} \varphi_{k-1}(S_k - S_{k-1}) - \varepsilon \sum_{k=1}^{N-1} S_k |\varphi_k - \varphi_{k-1}| - \varepsilon S_N |\varphi_{N-1}| \ .$$

p	b	σ	N	L	$T\,(yrs.)$
10^{-4}	0.05	0.3	10^4	10^7	30

Table 6.2: Simulation parameters.

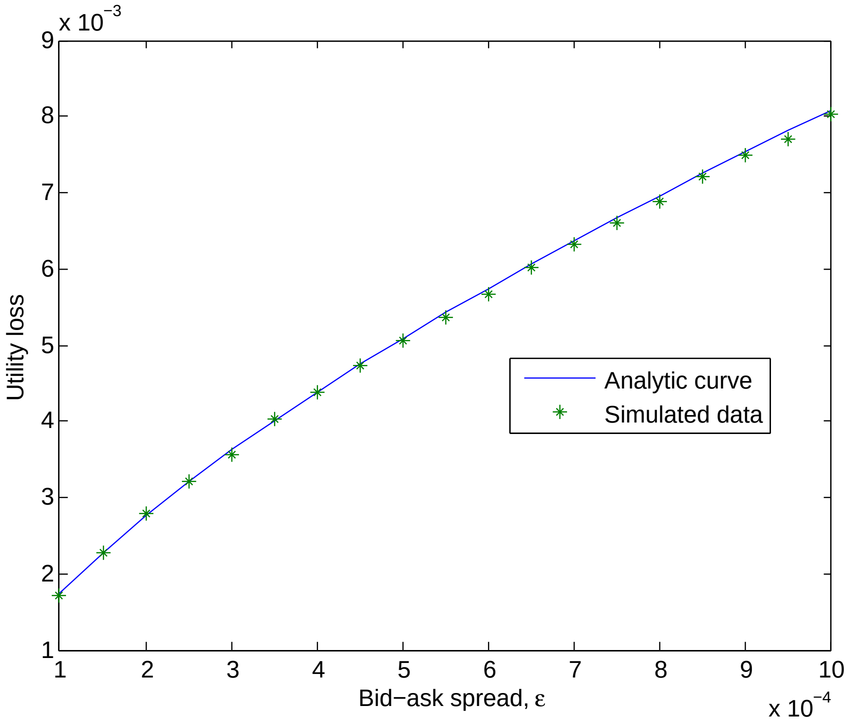

Figure 6.1: Utility loss for different bid-ask spreads: Exact asymptotics vs. simulated loss without asymptotic approximations.

The P-expectation $E[u(X(\varphi)_T)]$ will be estimated by the Monte-Carlo method. Recalling that, by Equation (3.34) from Subsection 3.2.4,

$$E[u(x + \varphi^* \bullet S_T)] = -e^{-px - \frac{T}{2}\frac{b^2}{\sigma^2}} \ ,$$

we calculate

$$L^U_{tot} = -e^{px + \frac{T}{2}\frac{b^2}{\sigma^2}} E[u(X(\varphi)_T)] - 1 \ .$$

to obtain the value of utility loss. The parameters we choose for the numerical simulation are presented in Table 6.2. The (anualized) drift coefficient, b, and the volatility, σ, are chosen close to the average of the estimated values of the portfolio selected in Subsection 6.2.2 (cf. Table 6.1 and Table A.2). The parameters N and L describe the number of time intervals in the partition and the number of Monte-Carlo samples, respectively. Simulation results are shown in Table 6.3, Figure 6.1. We can see from the data that the asymptotic results can be considered to be applicable even at the largest value of the transaction-cost percentage, $\varepsilon = 10^{-3}$. We therefore expect that all the asymptotic approximations used in the theoretical preliminaries of Chapter 5 to obtain leading-order

$\varepsilon \times 10^{-4}$	REF	L^U_{tot}	$SD \times 10^{-5}$
1.0	0.001741	0.001720	1.0744
1.5	0.002280	0.002268	1.2326
2.0	0.002763	0.002793	1.3514
2.5	0.003206	0.003214	1.4541
3.0	0.003621	0.003566	1.5490
3.5	0.004012	0.004034	1.6257
4.0	0.004386	0.004389	1.6999
4.5	0.004744	0.004721	1.7676
5.0	0.005090	0.005049	1.8298
5.5	0.005423	0.005370	1.8920
6.0	0.005747	0.005666	1.9463
6.5	0.006062	0.006015	1.9997
7.0	0.006369	0.006322	2.0450
7.5	0.006669	0.006604	2.0926
8.0	0.006962	0.006874	2.1439
8.5	0.007250	0.007205	2.1863
9.0	0.007531	0.007479	2.2267
9.5	0.007808	0.007705	2.2600
10.0	0.008079	0.008028	2.3098

Table 6.3: Simulated utility loss for different bid-ask spreads.
SD = Standard Deviation; REF = exact value obtained from (6.59).

expressions for the primal and dual functional are valid. Since with a shrinking bid-ask spread its effect on the utility loss decreases, we will use the largest value of ε so as to obtain numerical values that are large enough to be well distinguished from the noise introduced by the standard deviation of the Monte-Carlo simulations. Hence, we will set $\varepsilon = 10^{-3}$ in the following simulations.

Remark 6.5. To obtain the results presented in Table 6.3, the values of the expected utility loss were calculated from the values of the total expected utility, $E[u(X(\varphi)_T)]$, which exceed the loss values by approximately two orders of magnitude. This resulted in very large standard deviations of the Monte-Carlo estimates. To cope with this issue, we had to produce $L = 10^7$ Monte-Carlo samples and use variance-reduction techniques in addition[7]. We opted for the Control-Variate Method. As expected, choosing the

[7]The reader is referred to the detailed discussion of the basic variance-reduction techniques presented in Chapter 4 of [Gla04].

frictionless utility, $u(x+\varphi^* \bullet S_T)$, as a control variate for the random variable of interest, $u(X(\varphi)_T)$, yielded satisfactory results.

6.4 Portfolio performance

In this section we present the results of numerical simulations for two types of portfolios. First, we discuss the performance of the DAX portfolio, a 30-asset portfolio selected according to the scheme introduced in Section 6.2. In addition, we analyse the *symmetric portfolio* in different dimensions. By symmetric portfolio we mean a multi-asset portfolio characterized by the drift vector

$$b_{sym} = b\mathbf{1}, \quad b \in \mathbb{R} \ ,$$

the volatility vector

$$\sigma_{sym} = \sigma\mathbf{1}, \quad \sigma > 0 \ ,$$

the correlation matrix

$$(\rho_{sym})_{ij} = \begin{cases} 1 \ , & i = j \\ \rho \ , & i \neq j \end{cases} , \quad \rho \in [0, 1) \ ,$$

and the covariance matrix

$$c_{sym} = \sigma^2 \rho_{sym} \ .$$

In all simulations, we assume the transaction-cost percentage to be the same for all assets and denote it by

$$\varepsilon_j = \varepsilon, \quad j = 1, \ldots, n \ .$$

Table 6.4 shows the values of the parameters which will remain unchanged for all numerical simulations of the multi-dimensional portfolios discussed in this section.

p	N	L	$T(yrs.)$	ε
1.79×10^{-4}	10^4	10^4	30	10^{-3}

Table 6.4: Simulation parameters.

The value of the risk aversion p was determined in Subsection 6.2.2. The choice of the transaction-cost percentage, ε, was discussed in Subsection 6.3.3. N and L denote the number of time intervals in the partition and the number of Monte-Carlo samples, respectively.

6.4.1 One-dimensional portfolio

In this subsection, we test the algorithm by simulating a portfolio with only one risky asset and comparing the results with the exact asymptotic solution. Basically, we are in the setting of Subsection 6.3.3, but instead of simulating the complete functional $E[X(\varphi)_T]$ for different values of the bid-ask spread, we simulate the asymptotic expansions of the primal and dual functionals, Equation (5.50), (5.49), for a fixed bid-ask spread, $\varepsilon = 10^{-3}$. An important part of this benchmark is to compare two discretization schemes for solutions of one-dimensional Skorohod problems — the *projection scheme* and the *reflection scheme*. Recall that, in Subsection 6.3.3, the projection scheme was used to analyse the asymptotic behaviour of the loss functional. That means, the discretization was implemented using the projection mapping

$$\pi^a(x) = \max\{\min\{a, x\}, -a\}, \quad a > 0 \; . \tag{6.64}$$

To implement the reflection scheme, we simply replace π^a with the mapping

$$r^a(x) = 2\pi^a(x) - x, \quad a > 0 \; . \tag{6.65}$$

Notice that for $n = 1$ all primal candidates coincide. Moreover, the auxiliary process ψ generating the shadow-price process and appearing in the expression (5.49) for the dual functional is equal to the trading strategy φ.
For both discretization schemes, we compute the total utility-loss percentage, L^U_{tot}, its dual bound, D^U, as well as the ratio of the transaction loss to the displacement loss, F (cf. Equations (6.34), (6.35) and (6.44), respectively). To get a sense of the quality of the discretization, we will perform three simulations each time increasing the number of intervals in the partition of the time interval, N, by one order of magnitude. The drift and the diffusion coefficient are $b = 0.05$ and $\sigma = 0.3$, respectively. Apart from N, all other parameters are chosen as presented in Table 6.4. The exact asymptotic values of the functionals are

$$L^U_{tot} = D^U \approx 0.008079, \quad F = 2 \; . \tag{6.66}$$

The fact that the exact value of the (asymptotic) transaction-to-displacement ratio is $F = 2$ does not follow from the rather qualitative discussion of the one-dimensional solution presented in Subsection 4.3.3. However, this can be shown to hold true by analysing the stationary distribution of the process $\Delta\varphi = \varphi - \varphi^*$. As shown in [KMK15] (see also [KL13], Lemma 5.17), asymptotically, the transaction loss and the displacement loss satisfy

$$L^U_{tc} = \frac{2}{3} L^U_{tot}, \quad L^U_{disp} = \frac{1}{3} L^U_{tot} \; .$$

N	F	L^U_{tot}	$SD \times 10^{-6}$	D^U	$SD \times 10^{-6}$
10^4	1.5984	0.008006	6.58	0.007984	6.55
10^5	1.8118	0.007998	6.36	0.007995	6.35
10^6	1.8901	0.007989	6.32	0.007989	6.31

TABLE 6.5: Performance of the projection scheme.

N	F	L^U_{tot}	$SD \times 10^{-6}$	D^U	$SD \times 10^{-6}$
10^4	1.928	0.0079921	6.30	0.007941	6.24
10^5	1.9307	0.008003	6.29	0.007998	6.28
10^6	1.9283	0.007997	6.27	0.007997	6.27

TABLE 6.6: Performance of the reflection scheme.

As one can see from Table 6.5, in terms of the values of F, the projection scheme is not stable with respect to the number of partition intervals, N, but it does (rather slowly) approach the exact value as N increases. However, increasing N has no effect on the utility loss, and the primal and dual values are indistinguishable. All this, of course, is true only at the precision level provided by the standard diviation, SD. As for the reflection scheme, Table 6.6 shows stable values of the ratio F, which are very close to the exact value of 2. In the case of this discretization scheme, increasing N appears to affect neither the ratio F nor the total utility-loss percentage. However, for $N = 10^4$, there is a small difference between the primal and the dual value, which is larger than two standard deviations. This discrepancy does not occur for larger values of N. Both discretization schemes slightly underestimate the exact value shown in (6.66); the average difference is approximately 9×10^{-5}. Unfortunately, we were unable to explain this difference. There are three sources of possible approximation errors in our simulation. The first one is the asymptotic approximation for $\varepsilon \sim 0$. This cannot be the reason since we simulate the value of an asymptotic expansion. The second source is the Monte-Carlo approximation. This can also be excluded since the difference cannot be explained by the standard deviation. The third one is the Euler approximation of the stochastic differential equation. This cannot be excluded with certainty. However, taking into account that the values remain stable as the number of time steps increases, this type of error does not seem to explain the difference either.

In the following, when simulating multi-dimensional portfolios, we will opt for the projection scheme. If we were to implement the reflection scheme in multi-dimensional domains, it would be much more difficult to decide what points inside a given domain

R to assign to the elements of R^c. To see this, let (Δ, L) define a candidate domain R, and let π^R be the projection algorithm introduced in Definition 6.3. At first it appears reasonable to implement the reflection scheme simply by setting

$$r^R(x) = 2\pi^R(x) - x \ .$$

But if the matrix L defining the domain is non-diagonal, then, in general, there exist elements $x \in R^c$ such that $r^R(x) \in R^c$, meaning that the algorithm fails to find an element of R. This can happen even if x is very close to the boundary. A further argument for choosing the simpler discretization algorithm is that, although it fails to reproduce the exact value of the transaction-to-displacement ratio F, the values it calculates for the utility loss are indistinguishable at the given precision level from those generated by the reflection scheme.

6.4.2 DAX portfolio

The DAX portfolio approximates the 30 highest ranks (with respect to the market capitalization) of the German stock market index (cf. Section 6.2 for details). The estimated drift coefficients and covariance matrix are presented in Table 6.1 and Table A.2, respectively. The remaining simulation parameters are presented in Table 6.4. Table 6.7 shows the values of the loss functionals introduced in Subsection 6.3.1 for the three primal candidates, $P^{(i)}$, $i = 1, 2, 3$, and the dual bound, D. The second primal candidate corresponding to the domain constructed in Subsection 5.1.2 of Chapter 5 shows the best performance. The value of the total utility loss of $P^{(2)}$ is less than one percent. However, this value is still roughly 32% higher than that of the dual lower bound. Alternatively, we can also interpret the results by concluding that the value of the total utility loss of the exact asymptotic optimizer lies inside the interval $(0.007344, 0.009703)$. The functional L^{CE}_{rel} measures the loss generated by the candidate in terms of the certainty equivalent, in relation to the certainty equivalent of the frictionless optimizer. In the case of L^{CE}_{inv}, the certainty-equivalent loss of the candidate is related to the total amount invested in the portfolio. The primal candidate $P^{(1)}$ corresponding to the domain constructed in Subsection 5.1.1 generates loss values roughly twice as high as those of $P^{(2)}$. The candidate $P^{(3)}$ trading with respect to the domain from Subsection 5.1.3 performs surprisingly well: its loss values are only $\sim$ 4.3% worse than those of $P^{(2)}$. Taking into account the simple construction of $P^{(3)}$ and the resulting straightforward numerical implementation due to the rectangular no-trade region, the high performance of this candidate appears even more remarkable. The value of the functional F shows how the total utility loss is distributed between the transaction and the displacement loss. In the case of the candidate $P^{(1)}$, the displacement loss is almost five times the

	$P^{(1)}$	$P^{(2)}$	$P^{(3)}$	D
L^U_{tot}	0.01889	0.009703	0.010124	0.007344
SD	3.52×10^{-5}	7.15×10^{-6}	6.35×10^{-6}	8.37×10^{-6}
L^{CE}_{rel}	0.03183	0.016347	0.017055	0.012373
L^{CE}_{inv}	0.02110	0.010838	0.011307	0.008203
F	0.18754	1.227002	2.805036	—

TABLE 6.7: Performance of the DAX portfolio.

transaction loss. If we look at $P^{(3)}$, we see that for this candidate, the utility loss due to transactions is greater than the loss due to displacement by the factor of 2.81, which yields a significantly better result.

6.4.3 Symmetric portfolio

The purpose of the symmetric portfolio is to demonstrate the effect of asset correlations in different dimensions. As mentioned at the beginning of this subsection, a symmetric portfolio consists of n assets having equal drift and diffusion coefficients. In addition, each asset pair is assumed to have the same correlation coefficient. Simulations were run in four different dimensions, $n \in \{2, 5, 10, 30\}$. The drift and the diffusion coefficient were chosen as in the one-dimensional simulation in Subsection 6.3.3, Table 6.2, namely $b = 0.05$, $\sigma = 0.3$. The remaining simulation parameters are presented in Table 6.4. The only analytic value with which we can compare the simulation results to assess their quality is the one-dimensional asymptotic solution presented in Subsection 4.3.3. In Subsection 6.3.3, the solution was calculated for the Black-Scholes model, and the resulting total utility-loss percentage is given by Equation (6.59). For the simulation parameters from Table 6.4 and $b = 0.05$, $\sigma = 0.3$, the utility loss reads as

$$L^{U,1}_{tot} \approx 0.008079 \,, \tag{6.67}$$

where the superscript 1 is meant to indicate the exact *one-dimensional* asymptotic solution. As discussed in Subsection 4.3.4 and 4.3.5, in the uncorrelated case, $\rho = 0$, we expect the symmetric portfolio to generate the total utility-loss percentage

$$L^U_{tot} \approx nL^{U,1}_{tot} \,.$$

In the case of (almost) complete correlation, we expect

$$L^U_{tot} \approx L^{U,1}_{tot} \,.$$

n	2	5	10	30
Exact value	0.016158	0.040395	0.08079	0.24237
Numerical value	0.016019	0.040025	0.080115	0.240298

TABLE 6.8: Expected utility loss of the uncorrelated portfolio in different dimension.

The data presented in the tables 6.9, 6.13, 6.17, 6.21 shows that the simulation results provide a good approximation of the exact values. However, we can see that the numerical values underestimate the exact values. To emphasize this observation, the simulated expected utility loss of the uncorrelated, symmetric portfolio is compared with the exact values in Table 6.8. The same effect is also present in the case of complete correlation. We remark that the effect was already observed in the benchmark of the one-dimensional portfolio. We refer to Subsection 6.4.1 for discussion.

In the discussion of the simulation results obtained for the DAX portfolio, we already mentioned that the three primal candidates are quite different in terms of the transaction-to-displacement ratio, F. A similar behaviour of the candidates can be observed in the case of the symmetric portfolio. The values of F for the candidate $P^{(1)}$ decrease as the correlation increases. This indicates that the rectangular domain corresponding to this candidate is too big, and, as a consequence, the trading strategy deviates too far from the frictionless optimizer. The result is the extremely low performance of this candidate in terms of utility and certainty-equivalent loss in the range from moderate to high correlations. The candidate $P^{(2)}$ shows the best performance in all dimensions of the symmetric portfolio. The values of the functional F generated by this candidate remain rather stable. The candidate $P^{(3)}$ is placed second in terms of its overall performance. The values of the ratio F generated by $P^{(3)}$ are very large compared to the other two candidates, which indicates that trades are carried out very often.

Dimension $n = 2$.

ρ	$P^{(1)}$	$SD^{(1)}$	$P^{(2)}$	$SD^{(2)}$	$P^{(3)}$	$SD^{(3)}$	D	SD
0.0	0.016019	9.16×10^{-6}	0.016019	9.16×10^{-6}	0.016019	9.16×10^{-6}	0.015992	9.06×10^{-6}
0.1	0.014141	8.45×10^{-6}	0.014131	8.41×10^{-6}	0.014093	8.30×10^{-6}	0.014085	8.27×10^{-6}
0.2	0.012741	8.02×10^{-6}	0.012700	7.87×10^{-6}	0.012666	7.75×10^{-6}	0.012590	7.94×10^{-6}
0.3	0.011632	7.67×10^{-6}	0.011541	7.38×10^{-6}	0.011533	7.26×10^{-6}	0.011378	7.60×10^{-6}
0.4	0.010766	7.67×10^{-6}	0.010595	7.18×10^{-6}	0.010634	7.11×10^{-6}	0.010417	7.39×10^{-6}
0.5	0.010104	7.59×10^{-6}	0.009840	6.83×10^{-6}	0.009925	6.85×10^{-6}	0.009676	7.08×10^{-6}
0.6	0.009588	7.52×10^{-6}	0.009221	6.70×10^{-6}	0.009345	6.69×10^{-6}	0.009120	6.96×10^{-6}
0.7	0.009187	7.56×10^{-6}	0.008740	6.66×10^{-6}	0.008868	6.58×10^{-6}	0.008701	6.89×10^{-6}
0.8	0.008931	7.70×10^{-6}	0.008414	6.70×10^{-6}	0.008507	6.59×10^{-6}	0.008371	6.74×10^{-6}
0.9	0.008753	7.75×10^{-6}	0.008163	6.64×10^{-6}	0.008202	6.55×10^{-6}	0.008150	6.60×10^{-6}
0.999999	0.008730	7.87×10^{-6}	0.008013	6.59×10^{-6}	0.008013	6.58×10^{-6}	0.007984	6.58×10^{-6}

TABLE 6.9: Total utility loss, L^U_{tot}, for a symmetric portfolio in $n = 2$ dimensions.

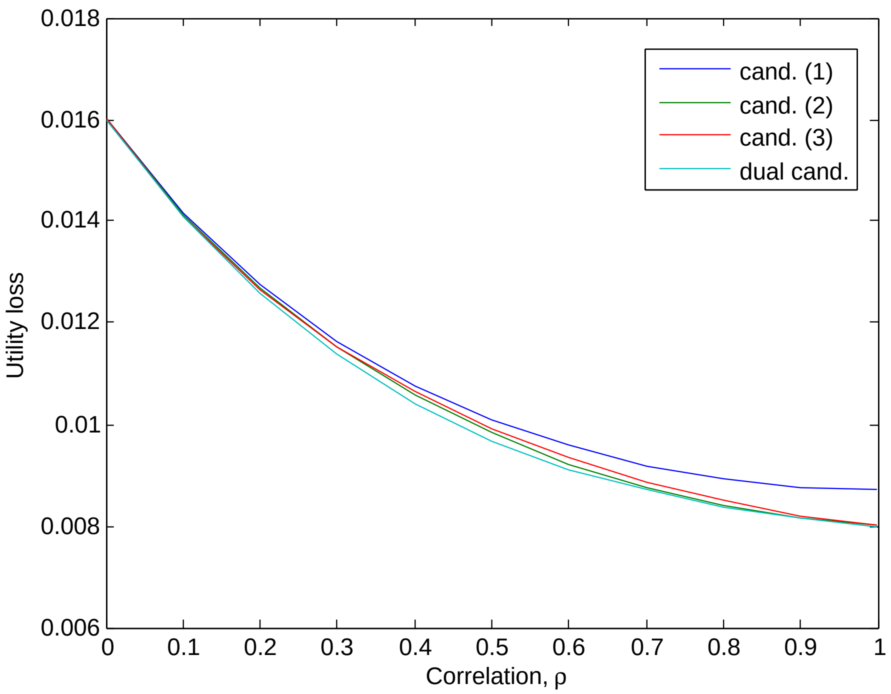

FIGURE 6.2: Total utility loss, L^U_{tot}, corresponding to Table 6.9.

ρ	$P^{(1)}$	$P^{(2)}$	$P^{(3)}$	D
0.0	0.019223	0.019223	0.019223	0.019191
0.1	0.018667	0.018653	0.018603	0.018593
0.2	0.018347	0.018288	0.018239	0.018130
0.3	0.018145	0.018003	0.017992	0.017750
0.4	0.018087	0.017800	0.017866	0.017501
0.5	0.018187	0.017712	0.017865	0.017417
0.6	0.018410	0.017705	0.017944	0.017511
0.7	0.018742	0.017830	0.018091	0.017751
0.8	0.019291	0.018174	0.018376	0.018083
0.9	0.019958	0.018612	0.018702	0.018582
0.999999	0.020954	0.019233	0.019233	0.019163

TABLE 6.10: Relative certainty-equivalent loss, L_{rel}^{CE}, for a symmetric portfolio in $n = 2$ dimensions.

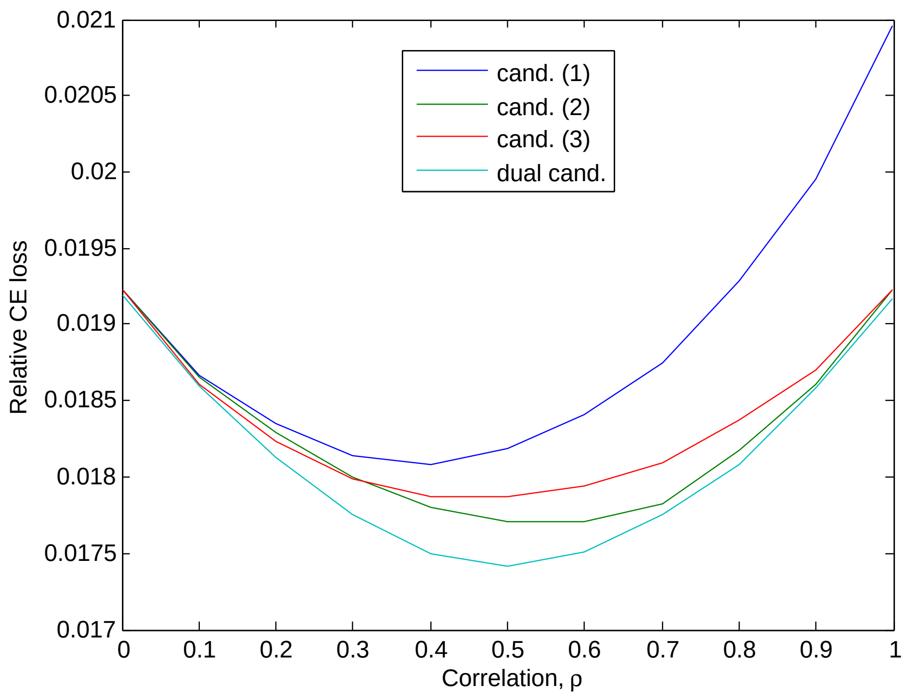

FIGURE 6.3: Relative certainty-equivalent loss, L_{rel}^{CE}, corresponding to Table 6.10.

ρ	$P^{(1)}$	$P^{(2)}$	$P^{(3)}$	D
0.0	0.014417	0.014417	0.014417	0.014393
0.1	0.014000	0.013990	0.013952	0.013944
0.2	0.013760	0.013716	0.013679	0.013597
0.3	0.013609	0.013502	0.013494	0.013313
0.4	0.013565	0.013350	0.013399	0.013125
0.5	0.013640	0.013284	0.013399	0.013063
0.6	0.013807	0.013278	0.013458	0.013133
0.7	0.014057	0.013372	0.013568	0.013313
0.8	0.014468	0.013630	0.013782	0.013562
0.9	0.014968	0.013959	0.014027	0.013936
0.999999	0.015715	0.014425	0.014425	0.014372

TABLE 6.11: Certainty-equivalent loss on investment, L_{inv}^{CE}, for a symmetric portfolio in $n = 2$ dimensions.

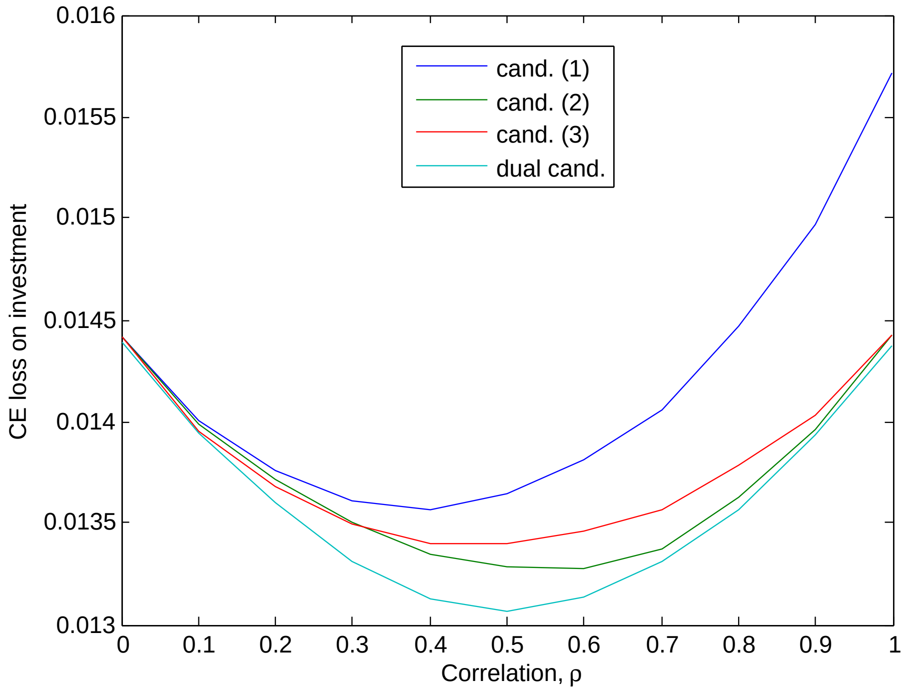

FIGURE 6.4: Certainty-equivalent loss on investment, L_{inv}^{CE}, corresponding to Table 6.11.

ρ	$P^{(1)}$	$P^{(2)}$	$P^{(3)}$
0.0	1.599294	1.599294	1.599294
0.1	1.585011	1.597893	1.738152
0.2	1.550903	1.583172	1.850681
0.3	1.490050	1.561924	1.922599
0.4	1.410580	1.550290	1.957243
0.5	1.324113	1.556935	1.965099
0.6	1.222330	1.572553	1.933942
0.7	1.119939	1.589434	1.880565
0.8	1.019584	1.603606	1.809791
0.9	0.915674	1.601312	1.713697
0.999999	0.814292	1.600387	1.600388

TABLE 6.12: Transaction loss to displacement loss, F, for a symmetric portfolio in $n = 2$ dimensions.

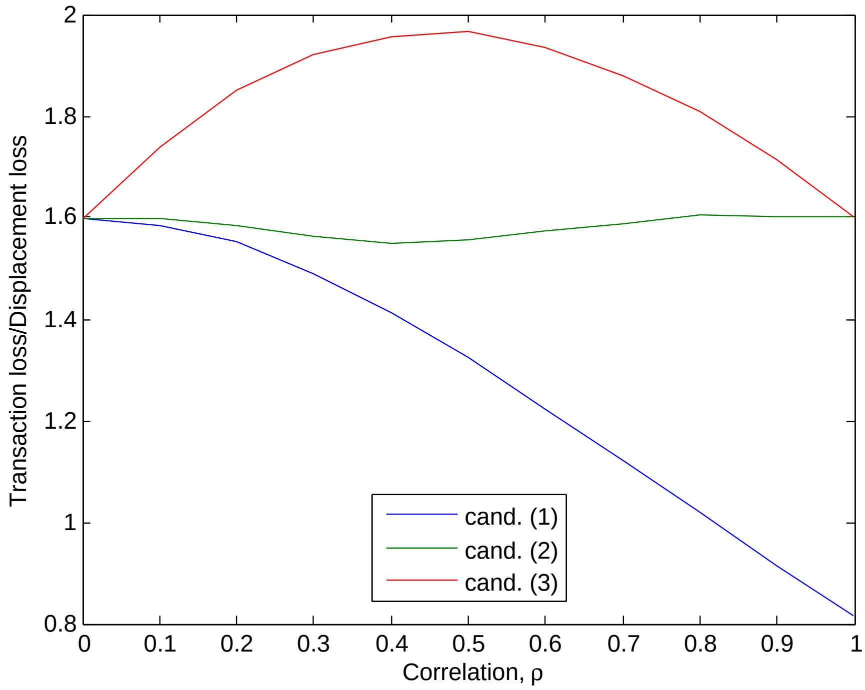

FIGURE 6.5: Transaction loss to displacement loss, F, corresponding to Table 6.12.

Dimension $n = 5$.

ρ	$P^{(1)}$	$SD^{(1)}$	$P^{(2)}$	$SD^{(2)}$	$P^{(3)}$	$SD^{(3)}$	D	SD
0.0	0.040025	1.46×10^{-5}	0.040025	1.46×10^{-5}	0.040025	1.46×10^{-5}	0.039928	1.45×10^{-5}
0.1	0.025931	1.08×10^{-5}	0.025826	1.06×10^{-5}	0.025788	1.01×10^{-5}	0.025540	1.06×10^{-5}
0.2	0.019295	9.71×10^{-6}	0.018982	8.75×10^{-6}	0.019160	8.43×10^{-6}	0.018305	8.89×10^{-6}
0.3	0.015694	9.67×10^{-6}	0.015044	7.80×10^{-6}	0.015422	7.62×10^{-6}	0.014263	8.24×10^{-6}
0.4	0.013612	1.00×10^{-5}	0.012529	7.25×10^{-6}	0.013061	7.21×10^{-6}	0.011917	7.97×10^{-6}
0.5	0.012358	1.06×10^{-5}	0.010865	7.23×10^{-6}	0.011449	7.08×10^{-6}	0.010486	7.69×10^{-6}
0.6	0.011610	1.11×10^{-5}	0.009757	7.13×10^{-6}	0.010287	6.89×10^{-6}	0.009586	7.66×10^{-6}
0.7	0.011258	1.14×10^{-5}	0.009054	7.09×10^{-6}	0.009449	6.73×10^{-6}	0.008970	7.24×10^{-6}
0.8	0.011123	1.17×10^{-5}	0.008575	6.97×10^{-6}	0.008801	6.64×10^{-6}	0.008556	7.00×10^{-6}
0.9	0.011246	1.18×10^{-5}	0.008253	6.70×10^{-6}	0.008327	6.53×10^{-6}	0.008232	6.76×10^{-6}
0.999999	0.011647	1.19×10^{-5}	0.008019	6.52×10^{-6}	0.008019	6.52×10^{-6}	0.007982	6.48×10^{-6}

TABLE 6.13: Total utility loss, L^U_{tot}, for a symmetric portfolio in $n = 5$ dimensions.

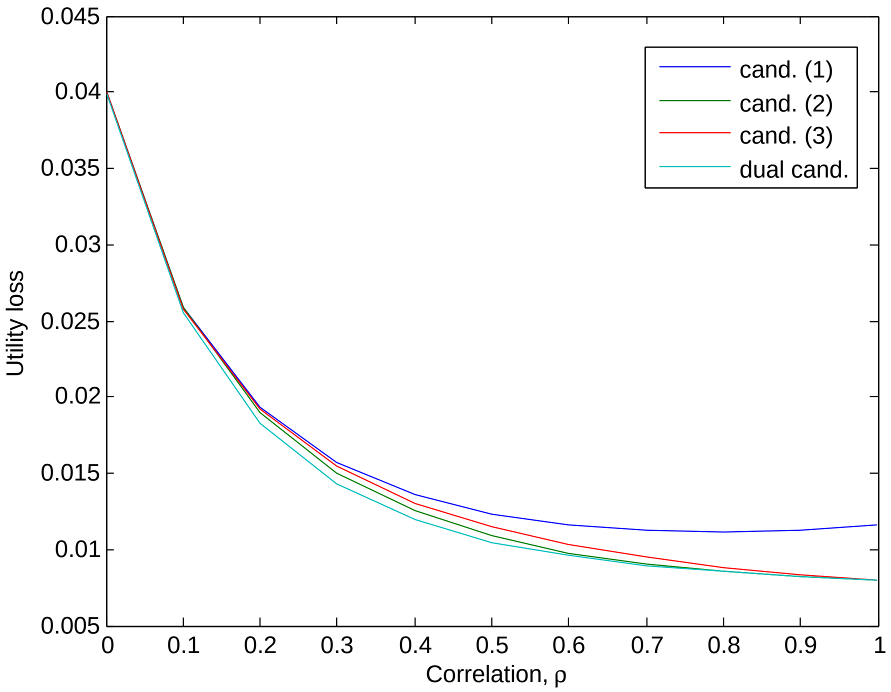

FIGURE 6.6: Total utility loss, L^U_{tot}, corresponding to Table 6.13.

ρ	$P^{(1)}$	$P^{(2)}$	$P^{(3)}$	D
0.0	0.019212	0.019212	0.019212	0.019218
0.1	0.017425	0.017355	0.017330	0.017325
0.2	0.016671	0.016401	0.016554	0.016550
0.3	0.016573	0.015886	0.016286	0.016298
0.4	0.016988	0.015637	0.016301	0.016307
0.5	0.017795	0.015646	0.016487	0.016499
0.6	0.018947	0.015924	0.016788	0.016807
0.7	0.020535	0.016515	0.017235	0.017226
0.8	0.022425	0.017287	0.017744	0.017745
0.9	0.024831	0.018224	0.018387	0.018375
0.999999	0.027954	0.019247	0.019247	0.019234

TABLE 6.14: Relative ceretainty-equivalent loss, L_{rel}^{CE}, for a symmetric portfolio in $n = 5$ dimensions.

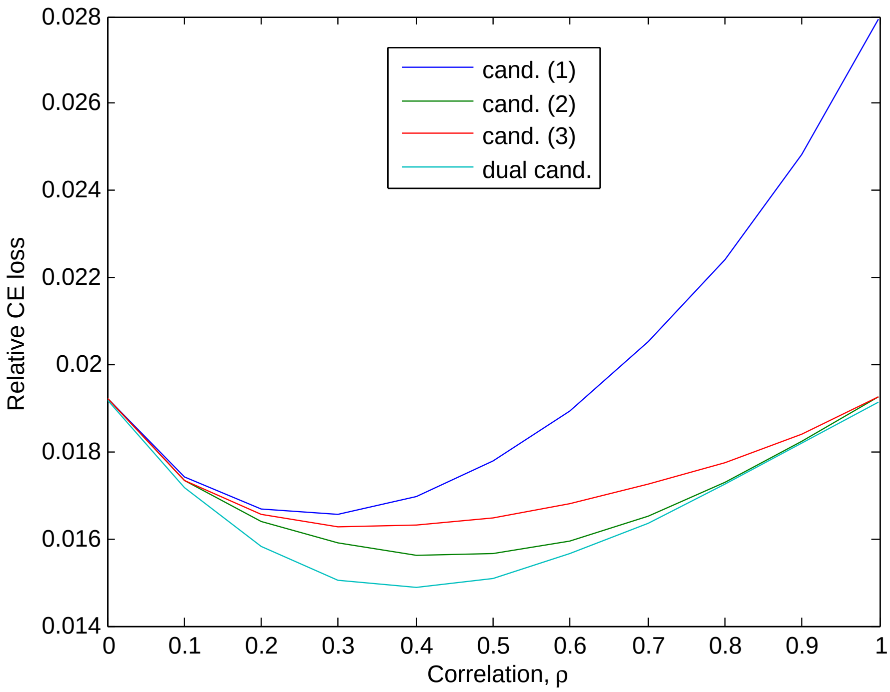

FIGURE 6.7: Relative ceretainty-equivalent loss, L_{rel}^{CE}, corresponding to Table 6.14.

ρ	$P^{(1)}$	$P^{(2)}$	$P^{(3)}$	D
0.0	0.014409	0.014409	0.014409	0.014374
0.1	0.013069	0.013016	0.012997	0.012872
0.2	0.012503	0.012300	0.012415	0.011861
0.3	0.012430	0.011914	0.012214	0.011926
0.4	0.012741	0.011727	0.012225	0.011155
0.5	0.013346	0.011734	0.012365	0.011325
0.6	0.014210	0.011943	0.012591	0.011733
0.7	0.015401	0.012386	0.012926	0.012271
0.8	0.016819	0.012965	0.013308	0.012938
0.9	0.018623	0.013668	0.013790	0.013633
0.999999	0.020965	0.014435	0.014435	0.014367

TABLE 6.15: Certainty-equivalent loss on investment, L_{inv}^{CE}, for a symmetric portfolio in $n = 5$ dimensions.

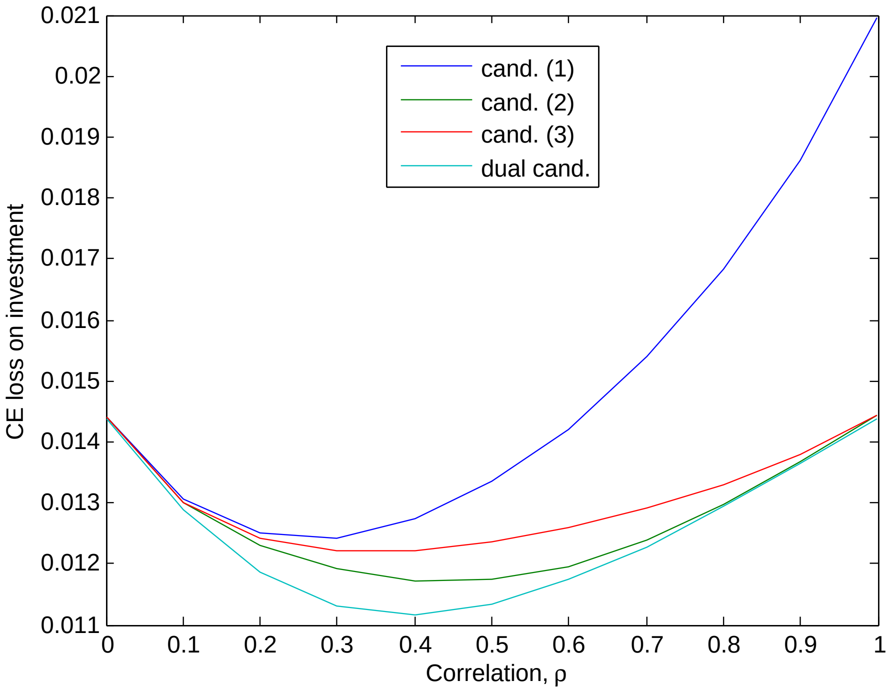

FIGURE 6.8: Certainty-equivalent loss on investment, L_{inv}^{CE}, corresponding to Table 6.15.

ρ	$P^{(1)}$	$P^{(2)}$	$P^{(3)}$
0.0	1.598927	1.598927	1.598927
0.1	1.545858	1.585153	2.146752
0.2	1.401330	1.489796	2.500660
0.3	1.214402	1.432152	2.652685
0.4	1.020659	1.443862	2.642768
0.5	0.847118	1.498835	2.534195
0.6	0.699780	1.554514	2.373108
0.7	0.577451	1.593969	2.186458
0.8	0.476743	1.606409	1.994438
0.9	0.394267	1.606062	1.802027
0.999999	0.323233	1.600150	1.600153

TABLE 6.16: Transaction loss to displacement loss, F, for a symmetric portfolio in $n = 5$ dimensions.

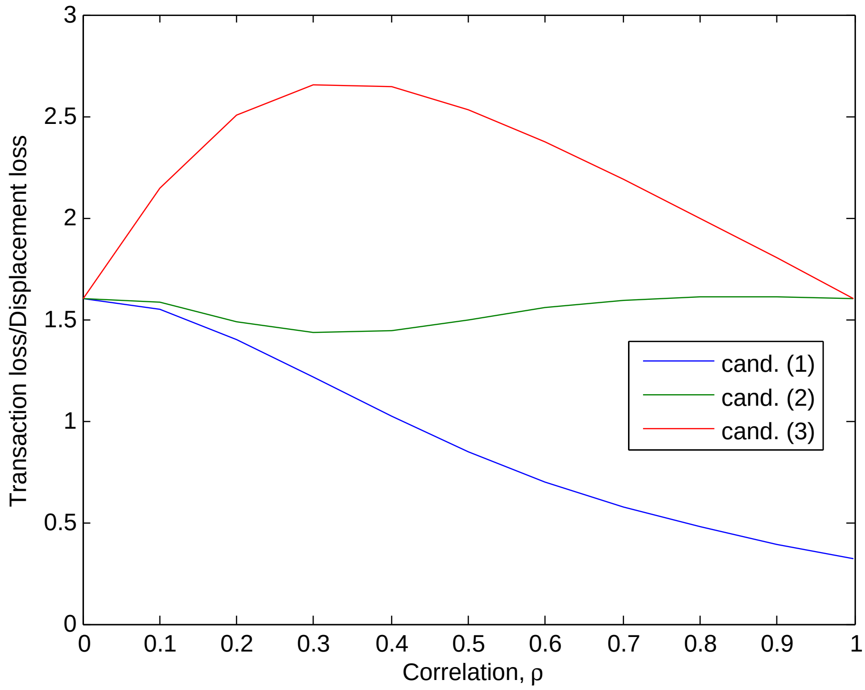

FIGURE 6.9: Transaction loss to displacement loss, F, corresponding to Table 6.16.

Dimension $n = 10$.

ρ	$P^{(1)}$	$SD^{(1)}$	$P^{(2)}$	$SD^{(2)}$	$P^{(3)}$	$SD^{(3)}$	D	SD
0.0	0.080115	2.08×10^{-5}	0.080115	2.08×10^{-5}	0.080115	2.08×10^{-5}	0.079886	2.06×10^{-5}
0.1	0.035194	1.23×10^{-5}	0.034806	1.15×10^{-5}	0.035275	1.05×10^{-5}	0.033754	1.13×10^{-5}
0.2	0.022942	1.20×10^{-5}	0.021885	8.80×10^{-6}	0.022862	8.37×10^{-6}	0.020037	9.26×10^{-6}
0.3	0.018010	1.34×10^{-5}	0.015977	7.75×10^{-6}	0.017224	7.58×10^{-6}	0.014498	8.82×10^{-6}
0.4	0.015733	1.50×10^{-5}	0.012735	7.57×10^{-6}	0.014038	7.31×10^{-6}	0.011882	8.52×10^{-6}
0.5	0.014695	1.65×10^{-5}	0.010880	7.49×10^{-6}	0.012029	7.00×10^{-6}	0.010480	8.21×10^{-6}
0.6	0.014256	1.75×10^{-5}	0.009769	7.41×10^{-6}	0.010634	6.79×10^{-6}	0.009604	7.93×10^{-6}
0.7	0.014332	1.89×10^{-5}	0.009073	7.42×10^{-6}	0.009638	6.83×10^{-6}	0.009020	7.47×10^{-6}
0.8	0.014661	1.95×10^{-5}	0.008620	7.15×10^{-6}	0.008913	6.69×10^{-6}	0.008587	7.15×10^{-6}
0.9	0.015271	2.03×10^{-5}	0.008269	6.84×10^{-6}	0.008362	6.62×10^{-6}	0.008251	6.78×10^{-6}
0.999999	0.016201	1.98×10^{-5}	0.008012	6.50×10^{-6}	0.008012	6.50×10^{-6}	0.007991	6.42×10^{-6}

TABLE 6.17: Total utility loss, L^U_{tot}, for a symmetric portfolio in $n = 10$ dimensions.

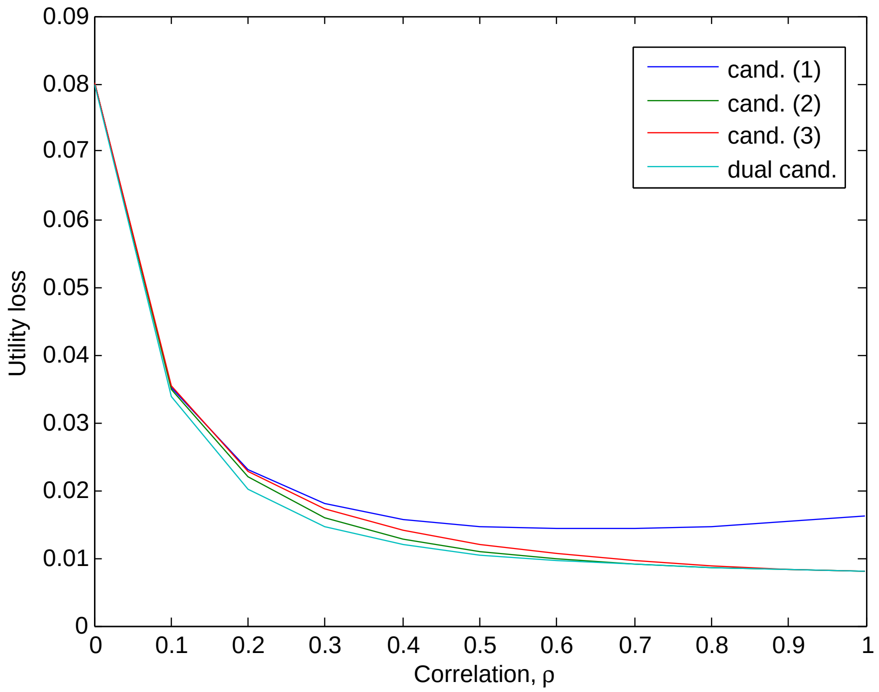

FIGURE 6.10: Total utility loss, L^U_{tot}, corresponding Table 6.17.

ρ	$P^{(1)}$	$P^{(2)}$	$P^{(3)}$	D
0.0	0.019227	0.019227	0.019227	0.019172
0.1	0.016048	0.015871	0.016085	0.015391
0.2	0.015417	0.014706	0.015363	0.013465
0.3	0.015993	0.014187	0.015295	0.012874
0.4	0.017369	0.014059	0.015498	0.013117
0.5	0.019397	0.014362	0.015878	0.013833
0.6	0.021897	0.015005	0.016335	0.014752
0.7	0.025110	0.015897	0.016886	0.015803
0.8	0.028854	0.016964	0.017541	0.016899
0.9	0.033353	0.018061	0.018262	0.018022
0.999999	0.038884	0.019230	0.019230	0.019178

TABLE 6.18: Relative certainty-equivalent loss, L_{rel}^{CE}, for a symmetric portfolio in $n = 10$ dimensions.

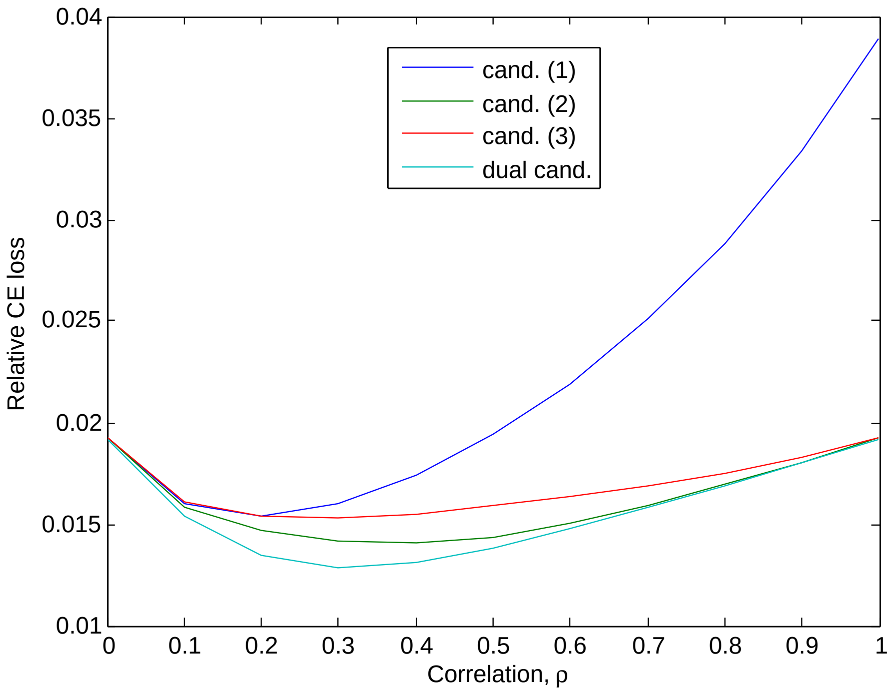

FIGURE 6.11: Relative certainty-equivalent loss, L_{rel}^{CE}, corresponding to Table 6.18.

ρ	$P^{(1)}$	$P^{(2)}$	$P^{(3)}$	D
0.0	0.014420	0.014420	0.014420	0.014379
0.1	0.012036	0.011903	0.012064	0.011543
0.2	0.011563	0.011030	0.011522	0.010099
0.3	0.011995	0.010640	0.011471	0.009656
0.4	0.013027	0.010544	0.011623	0.009838
0.5	0.014548	0.010771	0.011909	0.010375
0.6	0.016423	0.011254	0.012251	0.011064
0.7	0.018833	0.011923	0.012665	0.011852
0.8	0.021640	0.012723	0.013155	0.012674
0.9	0.025014	0.013546	0.013696	0.013516
0.999999	0.029163	0.014423	0.014423	0.014384

TABLE 6.19: Certainty-equivalent loss on investment, L^{CE}_{inv}, for a symmetric portfolio in $n = 10$ dimensions.

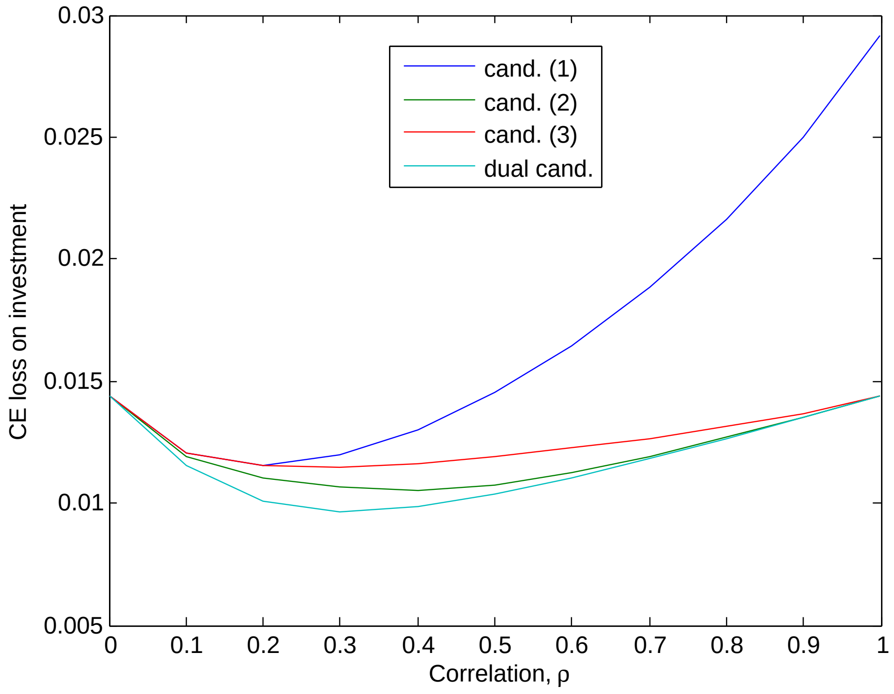

FIGURE 6.12: Certainty-equivalent loss on investment, L^{CE}_{inv}, corresponding to Table 6.19.

ρ	$P^{(1)}$	$P^{(2)}$	$P^{(3)}$
0.0	1.599722	1.599722	1.599722
0.1	1.470716	1.524523	2.775319
0.2	1.185927	1.330311	3.344850
0.3	0.902468	1.307875	3.408993
0.4	0.677362	1.396126	3.211834
0.5	0.513121	1.500972	2.923836
0.6	0.396000	1.570124	2.622711
0.7	0.307890	1.602652	2.334245
0.8	0.244383	1.612467	2.073626
0.9	0.195012	1.605554	1.833249
0.999999	0.155481	1.598968	1.598971

TABLE 6.20: Transaction loss to displacement loss, F, for a symmetric portfolio in $n = 10$ dimensions.

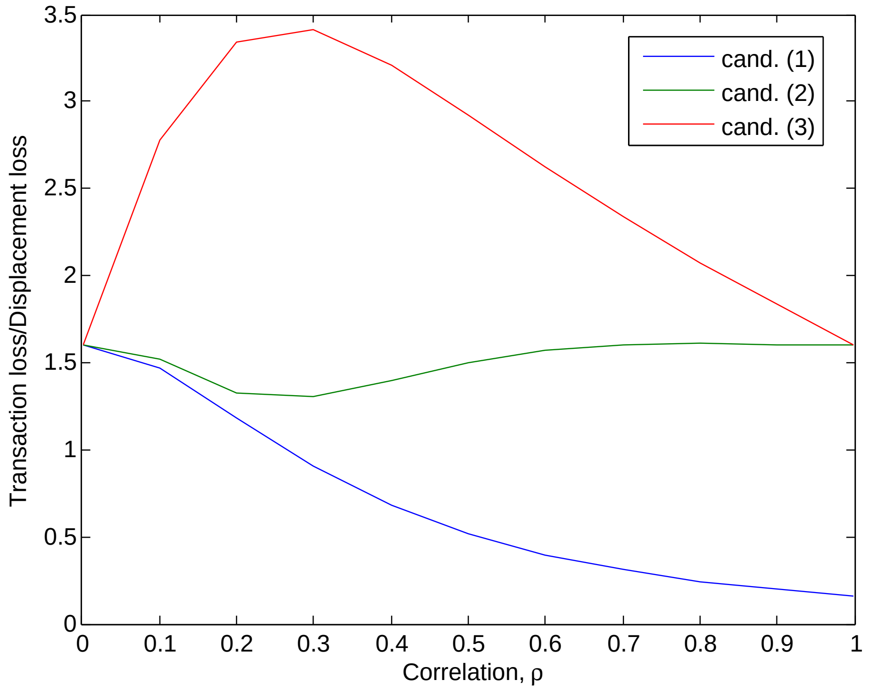

FIGURE 6.13: Transaction loss to displacement loss, F, corresponding to Table 6.20.

Dimension $n = 30$.

ρ	$P^{(1)}$	$SD^{(1)}$	$P^{(2)}$	$SD^{(2)}$	$P^{(3)}$	$SD^{(3)}$	D	SD
0.0	0.240297	3.60×10^{-5}	0.240298	3.60×10^{-5}	0.240297	3.60×10^{-5}	0.239572	3.53×10^{-5}
0.1	0.044463	1.86×10^{-5}	0.042040	1.11×10^{-5}	0.045868	9.86×10^{-6}	0.036514	1.05×10^{-5}
0.2	0.027924	2.66×10^{-5}	0.022135	8.27×10^{-6}	0.025937	7.93×10^{-6}	0.018114	1.06×10^{-5}
0.3	0.023864	3.39×10^{-5}	0.015097	7.98×10^{-6}	0.018555	7.44×10^{-6}	0.013393	1.02×10^{-5}
0.4	0.023101	3.99×10^{-5}	0.012042	8.23×10^{-6}	0.014730	7.11×10^{-6}	0.011390	9.38×10^{-6}
0.5	0.023465	4.56×10^{-5}	0.010519	8.26×10^{-6}	0.012411	6.93×10^{-6}	0.010280	8.82×10^{-6}
0.6	0.024486	4.98×10^{-5}	0.009646	8.02×10^{-6}	0.010871	6.86×10^{-6}	0.009554	8.14×10^{-6}
0.7	0.025789	5.35×10^{-5}	0.009050	7.66×10^{-6}	0.009765	6.78×10^{-6}	0.009003	7.71×10^{-6}
0.8	0.027276	5.60×10^{-5}	0.008636	7.25×10^{-6}	0.008981	6.72×10^{-6}	0.008610	7.23×10^{-6}
0.9	0.029169	5.74×10^{-5}	0.008297	6.86×10^{-6}	0.008398	6.60×10^{-6}	0.008260	6.84×10^{-6}
0.999999	0.31630	5.65×10^{-5}	0.008005	6.54×10^{-6}	0.008005	6.54×10^{-6}	0.007995	6.50×10^{-6}

TABLE 6.21: Total utility loss, L^U_{tot}, for a symmetric portfolio in $n = 30$ dimensions.

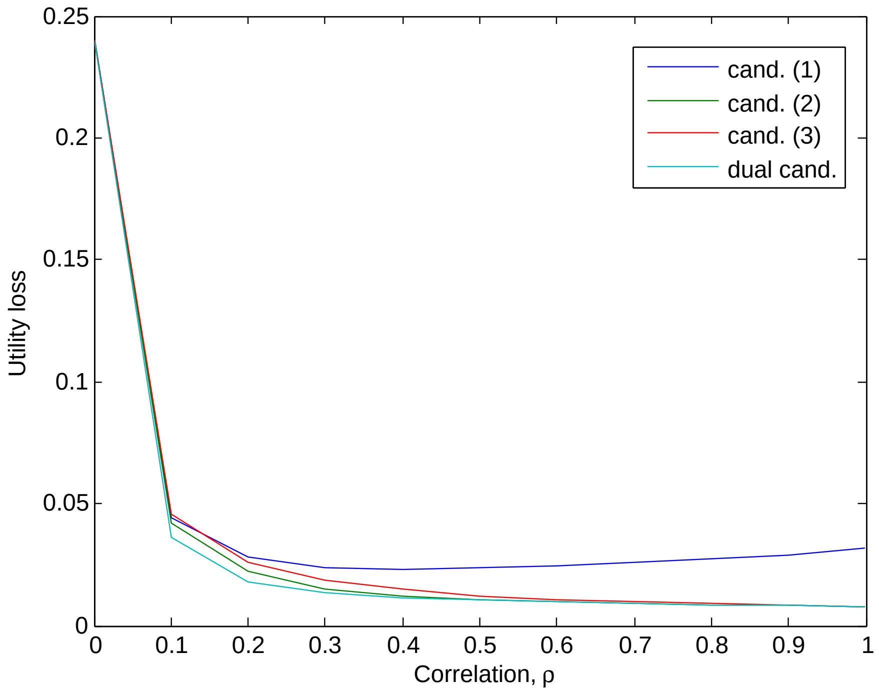

FIGURE 6.14: Total utility loss, L^U_{tot}, corresponding to Table 6.21.

ρ	$P^{(1)}$	$P^{(2)}$	$P^{(3)}$	D
0.0	0.019223	0.019223	0.019223	0.019165
0.1	0.013872	0.013116	0.014311	0.011392
0.2	0.015191	0.012041	0.014110	0.009854
0.3	0.018518	0.011715	0.014399	0.010393
0.4	0.023268	0.012138	0.014848	0.011481
0.5	0.029097	0.013043	0.015390	0.012748
0.6	0.036044	0.014200	0.016002	0.014063
0.7	0.043945	0.015422	0.016641	0.015341
0.8	0.052808	0.016720	0.017387	0.016669
0.9	0.063239	0.017988	0.018207	0.017909
0.999999	0.075913	0.019212	0.019212	0.019189

TABLE 6.22: Relative certainty-equivalent loss, L_{rel}^{CE}, for a symmetric portfolio in $n = 30$ dimensions.

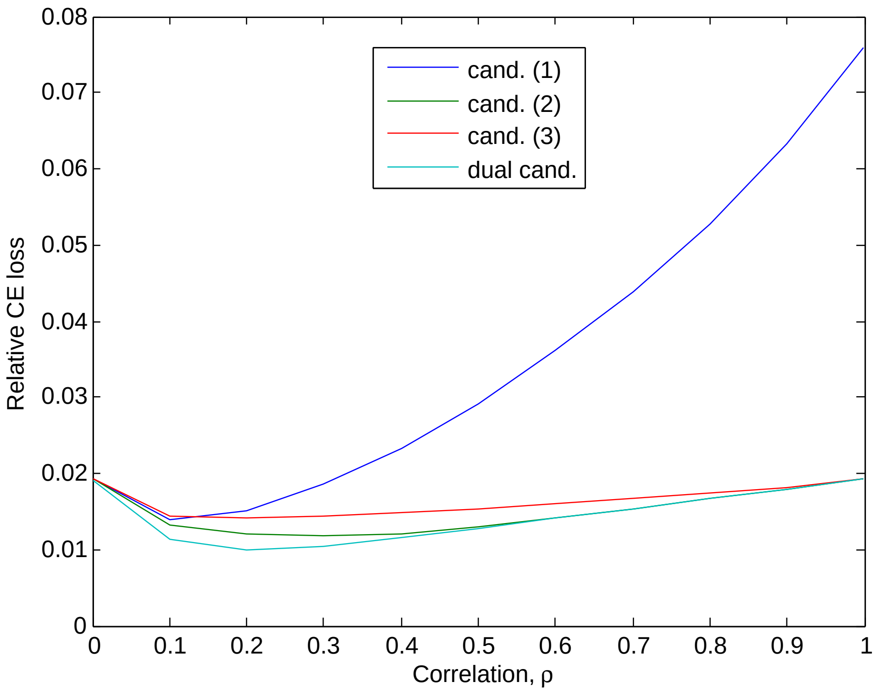

FIGURE 6.15: Relative certainty-equivalent loss, L_{rel}^{CE}, corresponding to Table 6.22.

ρ	$P^{(1)}$	$P^{(2)}$	$P^{(3)}$	D
0.0	0.014417	0.014417	0.014417	0.014374
0.1	0.010404	0.009837	0.010733	0.008544
0.2	0.011393	0.009031	0.010582	0.007390
0.3	0.013889	0.008786	0.010799	0.007795
0.4	0.017464	0.009104	0.011136	0.008611
0.5	0.021822	0.009782	0.011543	0.009561
0.6	0.027033	0.010650	0.012002	0.010547
0.7	0.032958	0.011566	0.012480	0.011506
0.8	0.039606	0.012540	0.013040	0.012502
0.9	0.047429	0.013491	0.013655	0.013432
0.999999	0.056934	0.014409	0.014409	0.014391

TABLE 6.23: Certainty-equivalent loss on investment, L_{inv}^{CE}, for a symmetric portfolio in $n = 30$ dimensions.

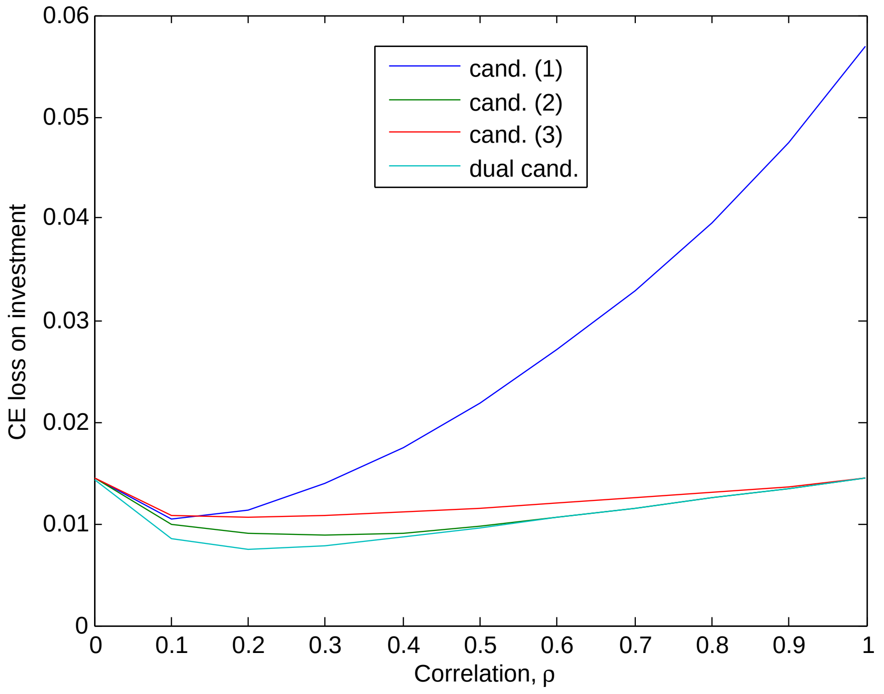

FIGURE 6.16: Certainty-equivalent loss on investment, L_{inv}^{CE}, corresponding to Table 6.23.

ρ	$P^{(1)}$	$P^{(2)}$	$P^{(3)}$
0.0	1.598988	1.598988	1.598988
0.1	1.159087	1.210734	4.767740
0.2	0.667479	1.075889	5.186914
0.3	0.401388	1.260496	4.602253
0.4	0.258220	1.449775	3.908813
0.5	0.177786	1.559024	3.318750
0.6	0.127789	1.605658	2.838823
0.7	0.095028	1.614207	2.449766
0.8	0.073580	1.616192	2.133558
0.9	0.057437	1.609879	1.859427
0.999999	0.045046	1.599665	1.599668

TABLE 6.24: Transaction loss to displacement loss, F, for a symmetric portfolio in $n = 30$ dimensions.

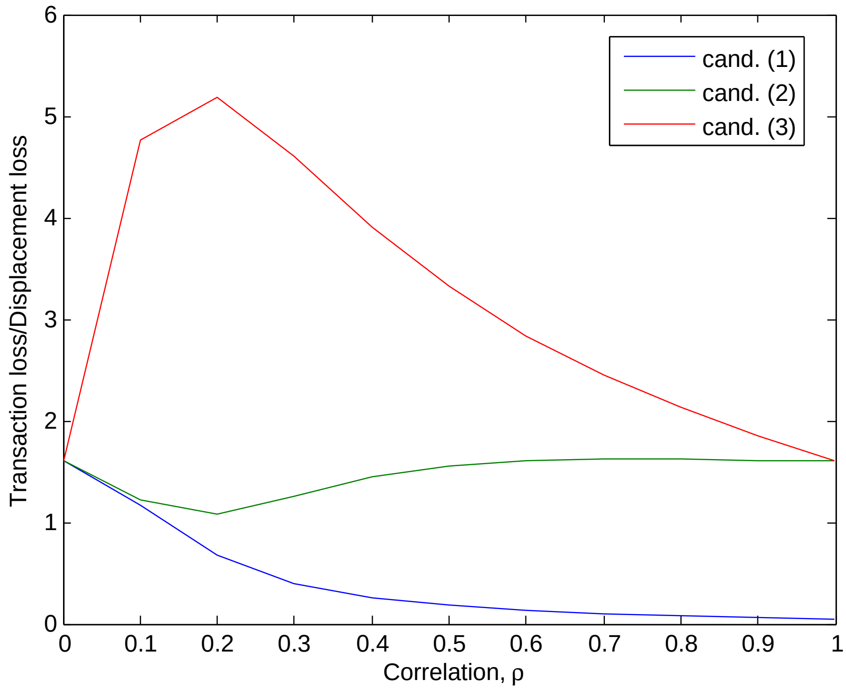

FIGURE 6.17: Transaction loss to displacement loss, F, from Table 6.24.

Conclusion

This thesis addressed the problem of maximizing the expected utility of terminal wealth in arbitrary dimensions in the presence of small proportional transaction costs. We used the shadow-price method to formulate a sufficient condition for asymptotic optimality of a trading strategy. We showed that an asymptotically optimal trading strategy can be described as solution to a reflecting stochastic differential equation. The reflecting boundary defining the no-trade region of the strategy was characterized as solution to a free-boundary problem. Instead of attempting to solve the problem exactly, we proposed three candidate domains as possible no-trade regions. With each candidate domain, we associated a trading strategy by defining it as solution to a stochastic Skorohod problem with reflections at the boundary of the domain. By modifying the notion of the shadow price, we established a duality relation between trading strategies and martingale measures for shadow-price processes. As a result, we obtained an upper bound on the expected utility of an arbitrary trading strategy. The values of the expected utility of each of our candidate strategies as well as the value of the dual upper bound were obtained by means of numerical simulations. The simulations were run on the Black-Scholes model for two different portfolios. The first portfolio consisted of 30 stocks replicating the German stock market index DAX. The second portfolio contained identical assets; it was analysed to demonstrate the effect of asset correlations on the expected utility of trading strategies in different dimensions.

The numerical analysis of the DAX portfolio showed that the long-term impact of small proportional transaction costs is rather small. Our best candidate strategy generated a total utility loss percentage of $\sim 0.97\%$. In terms of the certainty equivalent, this corresponds to a loss of $\sim 1.63\%$ compared to the frictionless market. The value of the dual bound on the utility loss is $\sim 0.73\%$, or $\sim 1.24\%$ in terms of the certainty equivalent. The meaning of the dual values is that no strategy can generate less utility respectively certainty-equivalent loss than the values provided by the dual bounds.

A further interesting result is the observation that one can already do remarkably well when trading according to a strategy defined in a very simple rectangular domain. The values we obtained were $\sim 1.01\%$ utility loss and $\sim 1.7\%$ certainty-equivalent loss.

Summarizing the results of this study, we can say that solving the problem exactly is a very challenging task. This is primarily due to the fact that the no-trade region, which essentially defines an asymptotically optimal trading strategy, is characterized by a multi-dimensional free-boundary problem. But even if one had the exact boundary, one would still have to solve a Skorohod problem in a non-convex domain with a non-smooth stochastic boundary to obtain an asymptotically optimal trading strategy. If one also wants to compute the expected utility loss, one immediately faces another problem: one requires the knowledge of the distribution of the trading strategy inside the no-trade region. Considering all the challenges described, it seems to be reasonable to use numerical methods to analyse the problem in high dimensions.

The numerical approach we described can be extended to other utility function. The properties of the exponential utility were used to obtain a simple asymptotic expansion of both the primal and the dual functional. This, in turn, allowed us to simulate the small first-order contributions to the utility loss directly as Q-expectations of certain (stocahstic) integrals. Nonetheless, all approximations of Chapter 5 and the implementation scheme presented in Chapter 6 do not depend on the particular choice of the utility function.

As discussed in Section 6.4, our algorithm for simulating the asymptotic expansions of the loss functionals slightly underestimated the existing exact values. Unfortunately, we were not able to resolve this issue. Hence, one should attempt to improve the numerical implementation, which might lead to better results.

Appendix A

DAX Data

In this appendix, we collect the data that was used in Subsection 6.2.2 to replicate the DAX.

Company	Weight(%)	Company	Weight(%)
BASF	9.772	Deutsche Post	2.1487
Siemens	9.5991	Fresenius	2.0009
Beiersdorf	8.3453	Fresenius med. care	1.7545
SAP	7.8424	Henkel Vz.	1.6636
Allianz	6.5383	Deutsche Boerse	1.4379
Daimler	5.8741	Infineon	1.1726
E.ON	5.6357	Commerzbank	1.0903
Deutsche Bank	4.8395	K+S	1.0158
Deutsche Telekom	4.7424	BMW St.	0.9522
Linde	3.8096	Merck	0.9167
Beyer	3.41	Heidelberg Cement	0.8795
Volkswagen Vz.	3.3324	Thyssen Krupp	0.8422
Munich Re	3.198	MAN	0.8378
RWE St.	2.8185	Lufthansa	0.7041
Adidas	2.25356	Metro	0.5723

Table A.1: DAX composition in the time 21 June 2010 — 23 September 2012. The weights of the stocks in the table are those of the last trading day of the time period. *Source: www.dax-indeces.de*

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)	(26)	(27)	(28)	(29)	(30)
(1)	0.0897	0.0598	0.0273	0.0403	0.075	0.0765	0.0581	0.0851	0.0383	0.0508	0.0633	0.0754	0.0537	0.0607	0.0527	0.0508	0.0257	0.022	0.0395	0.0492	0.0724	0.0897	0.0601	0.075	0.0337	0.0895	0.0867	0.0725	0.0654	0.0545
(2)	0.0598	0.0627	0.0175	0.0342	0.0604	0.0619	0.0486	0.0693	0.0323	0.0388	0.0464	0.0619	0.0441	0.0478	0.0409	0.0415	0.0179	0.0158	0.0294	0.0362	0.0583	0.0731	0.0473	0.0616	0.0273	0.0729	0.0686	0.0611	0.0529	0.0449
(3)	0.0273	0.0175	0.0373	0.0183	0.0248	0.0274	0.0206	0.0259	0.0167	0.0179	0.021	0.0235	0.02	0.0232	0.0229	0.022	0.0138	0.014	0.0235	0.0146	0.0258	0.0231	0.0224	0.0232	0.0162	0.0288	0.0287	0.0259	0.0229	0.0201
(4)	0.0403	0.0342	0.0183	0.0487	0.0415	0.0456	0.0362	0.047	0.0242	0.0289	0.0349	0.0427	0.0308	0.0359	0.0322	0.03	0.0187	0.015	0.0245	0.0269	0.0463	0.0504	0.0386	0.0432	0.0235	0.0505	0.0466	0.0423	0.034	0.0348
(5)	0.075	0.0604	0.0248	0.0415	0.1103	0.0787	0.0784	0.1156	0.0482	0.0462	0.065	0.0729	0.0711	0.0822	0.049	0.059	0.0237	0.0202	0.0372	0.0534	0.0682	0.1235	0.0598	0.0713	0.0396	0.0985	0.0879	0.0706	0.072	0.0587
(6)	0.0765	0.0619	0.0274	0.0456	0.0787	0.1244	0.0646	0.0951	0.0409	0.0502	0.0663	0.1018	0.0576	0.0683	0.0603	0.0591	0.0257	0.0223	0.0426	0.0517	0.088	0.1008	0.067	0.1062	0.0403	0.0995	0.0952	0.0897	0.0732	0.0628
(7)	0.0581	0.0486	0.0206	0.0362	0.0784	0.0646	0.095	0.0903	0.0451	0.0373	0.0544	0.0587	0.0551	0.0864	0.041	0.048	0.0198	0.019	0.0296	0.0464	0.0516	0.0916	0.0472	0.0564	0.0361	0.0775	0.071	0.0605	0.0581	0.0513
(8)	0.0851	0.0693	0.0259	0.047	0.1156	0.0951	0.0903	0.1784	0.0569	0.0516	0.077	0.0867	0.0788	0.0945	0.0591	0.0674	0.0219	0.0197	0.04	0.0721	0.0861	0.1722	0.0678	0.088	0.0486	0.1226	0.108	0.0912	0.0893	0.0716
(9)	0.0383	0.0323	0.0167	0.0242	0.0482	0.0409	0.0451	0.0569	0.0609	0.0248	0.0381	0.0374	0.0377	0.0476	0.0291	0.0303	0.0166	0.0138	0.0206	0.0328	0.0368	0.0551	0.0306	0.0361	0.0247	0.0437	0.0444	0.0358	0.0342	0.0296
(10)	0.0508	0.0388	0.0179	0.0289	0.0462	0.0502	0.0373	0.0516	0.0248	0.0533	0.0424	0.0525	0.0343	0.0382	0.0368	0.0354	0.0214	0.0189	0.0315	0.0331	0.0493	0.0548	0.0444	0.0515	0.027	0.0605	0.0577	0.0501	0.0439	0.0382
(11)	0.0633	0.0464	0.021	0.0349	0.065	0.0663	0.0544	0.077	0.0381	0.0424	0.0824	0.0624	0.0468	0.0585	0.0456	0.0445	0.0248	0.0217	0.0348	0.0401	0.0556	0.0766	0.0517	0.0609	0.0332	0.0728	0.0682	0.0619	0.0529	0.0429
(12)	0.0754	0.0619	0.0235	0.0427	0.0729	0.1018	0.0587	0.0867	0.0374	0.0525	0.0624	0.144	0.0532	0.0623	0.0641	0.0558	0.0254	0.0205	0.0434	0.0523	0.0904	0.0967	0.0693	0.1081	0.0368	0.0969	0.0988	0.0901	0.0748	0.0613
(13)	0.0537	0.0441	0.02	0.0308	0.0711	0.0576	0.0551	0.0788	0.0377	0.0343	0.0468	0.0532	0.0658	0.0592	0.0369	0.0421	0.0207	0.0174	0.0288	0.0387	0.0502	0.0832	0.0436	0.0524	0.0296	0.0685	0.0617	0.0503	0.0519	0.0427
(14)	0.0607	0.0478	0.0232	0.0359	0.0822	0.0683	0.0864	0.0945	0.0476	0.0382	0.0585	0.0623	0.0592	0.1066	0.0429	0.0524	0.0217	0.0194	0.0328	0.0488	0.0548	0.0947	0.051	0.0588	0.036	0.0809	0.0759	0.0637	0.062	0.0546
(15)	0.0527	0.0409	0.0229	0.0322	0.049	0.0603	0.041	0.0591	0.0291	0.0368	0.0456	0.0641	0.0369	0.0429	0.0758	0.0399	0.0216	0.0205	0.0356	0.0388	0.0618	0.0631	0.0478	0.0618	0.0266	0.0614	0.0631	0.0564	0.0468	0.041
(16)	0.0508	0.0415	0.022	0.03	0.059	0.0591	0.048	0.0674	0.0303	0.0354	0.0445	0.0558	0.0421	0.0524	0.0399	0.0642	0.019	0.0164	0.0291	0.0382	0.0553	0.0701	0.0452	0.0544	0.029	0.0688	0.0624	0.0544	0.0539	0.0389
(17)	0.0257	0.0179	0.0138	0.0187	0.0237	0.0257	0.0198	0.0219	0.0166	0.0214	0.0248	0.0254	0.0207	0.0217	0.0216	0.019	0.0422	0.024	0.0197	0.0173	0.0281	0.0209	0.0258	0.0247	0.0182	0.0268	0.0271	0.0209	0.0214	0.0182
(18)	0.022	0.0158	0.014	0.015	0.0202	0.0223	0.019	0.0197	0.0138	0.0189	0.0217	0.0205	0.0174	0.0194	0.0205	0.0164	0.024	0.0361	0.0196	0.0144	0.0239	0.017	0.023	0.0218	0.0182	0.0214	0.0245	0.0209	0.0189	0.0161
(19)	0.0395	0.0294	0.0235	0.0245	0.0372	0.0426	0.0296	0.04	0.0206	0.0315	0.0348	0.0434	0.0288	0.0328	0.0356	0.0291	0.0197	0.0196	0.0553	0.0229	0.0421	0.0395	0.0358	0.0414	0.0226	0.0473	0.0425	0.0428	0.0357	0.0322
(20)	0.0492	0.0362	0.0146	0.0269	0.0534	0.0517	0.0464	0.0721	0.0328	0.0331	0.0401	0.0523	0.0387	0.0488	0.0388	0.0382	0.0173	0.0144	0.0229	0.1026	0.0555	0.0739	0.0416	0.052	0.0244	0.0662	0.06	0.0474	0.0507	0.0382
(21)	0.0724	0.0583	0.0258	0.0463	0.0682	0.088	0.0516	0.0861	0.0368	0.0493	0.0556	0.0904	0.0502	0.0548	0.0618	0.0553	0.0281	0.0239	0.0421	0.0555	0.1565	0.1012	0.0671	0.0889	0.0373	0.0937	0.0955	0.0868	0.0707	0.0585
(22)	0.0897	0.0731	0.0231	0.0504	0.1235	0.1008	0.0916	0.1722	0.0551	0.0548	0.0766	0.0967	0.0832	0.0947	0.0631	0.0701	0.0209	0.017	0.0395	0.0739	0.1012	0.3404	0.0822	0.0983	0.048	0.1345	0.1201	0.0963	0.1005	0.0804
(23)	0.0601	0.0473	0.0224	0.0386	0.0598	0.067	0.0472	0.0678	0.0306	0.0444	0.0517	0.0693	0.0436	0.051	0.0478	0.0452	0.0258	0.023	0.0358	0.0416	0.0671	0.0822	0.0972	0.0689	0.0316	0.0789	0.0809	0.0638	0.0588	0.0498
(24)	0.075	0.0616	0.0232	0.0432	0.0713	0.1062	0.0564	0.088	0.0361	0.0515	0.0609	0.1081	0.0524	0.0588	0.0618	0.0544	0.0247	0.0218	0.0414	0.052	0.0889	0.0983	0.0689	0.1293	0.0384	0.097	0.095	0.0876	0.0706	0.0609
(25)	0.0337	0.0273	0.0162	0.0235	0.0396	0.0403	0.0361	0.0486	0.0247	0.027	0.0332	0.0368	0.0296	0.036	0.0266	0.029	0.0182	0.0182	0.0226	0.0244	0.0373	0.048	0.0316	0.0384	0.0619	0.0475	0.0414	0.0348	0.0327	0.0311
(26)	0.0895	0.0729	0.0288	0.0505	0.0985	0.0995	0.0775	0.1226	0.0437	0.0605	0.0728	0.0969	0.0685	0.0809	0.0614	0.0688	0.0268	0.0214	0.0473	0.0662	0.0937	0.1345	0.0789	0.097	0.0475	0.1648	0.1158	0.0989	0.088	0.0738
(27)	0.0867	0.0686	0.0287	0.0466	0.0879	0.0952	0.071	0.108	0.0444	0.0577	0.0682	0.0988	0.0617	0.0759	0.0631	0.0624	0.0271	0.0245	0.0425	0.06	0.0955	0.1201	0.0809	0.095	0.0414	0.1158	0.1541	0.0966	0.0851	0.0715
(28)	0.0725	0.0611	0.0259	0.0423	0.0706	0.0897	0.0605	0.0912	0.0358	0.0501	0.0619	0.0901	0.0503	0.0637	0.0564	0.0544	0.0209	0.0209	0.0428	0.0474	0.0868	0.0963	0.0638	0.0876	0.0348	0.0989	0.0966	0.1313	0.0746	0.0636
(29)	0.0654	0.0529	0.0229	0.034	0.072	0.0732	0.0581	0.0893	0.0342	0.0439	0.0529	0.0748	0.0519	0.062	0.0468	0.0539	0.0214	0.0189	0.0357	0.0507	0.0707	0.1005	0.0588	0.0706	0.0327	0.088	0.0851	0.0746	0.1059	0.0574
(30)	0.0545	0.0449	0.0201	0.0348	0.0587	0.0628	0.0513	0.0716	0.0296	0.0382	0.0429	0.0613	0.0427	0.0546	0.041	0.0389	0.0182	0.0161	0.0322	0.0382	0.0585	0.0804	0.0498	0.0609	0.0311	0.0738	0.0715	0.0636	0.0574	0.1082

Table A.2: Structure of the (anualized) ranked DAX covariance matrix.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)	(26)	(27)	(28)	(29)	(30)
(1)	1	0.7977	0.4721	0.6094	0.7542	0.7243	0.6292	0.6724	0.518	0.7348	0.7366	0.6634	0.6991	0.6206	0.6391	0.6699	0.4168	0.386	0.5615	0.5126	0.6109	0.5133	0.6431	0.6968	0.452	0.7359	0.7373	0.6683	0.6714	0.5531
(2)	0.7977	1	0.3611	0.6182	0.7259	0.7011	0.6301	0.6549	0.5228	0.6709	0.6449	0.6519	0.6875	0.5847	0.5934	0.6542	0.348	0.3327	0.499	0.4515	0.5884	0.5004	0.606	0.6838	0.4374	0.7177	0.6978	0.6737	0.6487	0.5448
(3)	0.4721	0.3611	1	0.4287	0.386	0.4026	0.3461	0.3175	0.3494	0.4009	0.3778	0.3209	0.4042	0.3678	0.4307	0.449	0.3483	0.3827	0.5168	0.2351	0.3376	0.2046	0.3721	0.3344	0.3361	0.3671	0.3783	0.3702	0.3646	0.3159
(4)	0.6094	0.6182	0.4287	1	0.5657	0.5862	0.5326	0.5042	0.4438	0.5671	0.5501	0.5101	0.5444	0.4979	0.5299	0.5375	0.4119	0.3581	0.4719	0.3799	0.5303	0.3917	0.5603	0.5441	0.4271	0.5639	0.5383	0.5288	0.4732	0.4787
(5)	0.7542	0.7259	0.386	0.5657	1	0.6719	0.7657	0.824	0.5876	0.6025	0.6814	0.5787	0.8354	0.7584	0.5358	0.7008	0.3472	0.3205	0.4769	0.5025	0.5193	0.6375	0.5772	0.5974	0.4791	0.7306	0.6748	0.5868	0.6666	0.5378
(6)	0.7243	0.7011	0.4026	0.5862	0.6719	1	0.5946	0.6383	0.4699	0.6169	0.6545	0.7607	0.6369	0.5931	0.6203	0.661	0.3542	0.333	0.5136	0.4574	0.6307	0.4899	0.6091	0.8377	0.4591	0.6949	0.6876	0.7018	0.6376	0.5414
(7)	0.6292	0.6301	0.3461	0.5326	0.7657	0.5946	1	0.6932	0.5931	0.5242	0.6151	0.5018	0.6972	0.8588	0.4835	0.6142	0.3132	0.3251	0.4085	0.4699	0.4234	0.5092	0.4915	0.5085	0.4701	0.619	0.5869	0.5417	0.5796	0.5058
(8)	0.6724	0.6549	0.3175	0.5042	0.824	0.6383	0.6932	1	0.5453	0.5295	0.6349	0.5411	0.7278	0.6853	0.508	0.6296	0.2517	0.2456	0.4029	0.533	0.5153	0.6989	0.5149	0.5796	0.4625	0.7148	0.6513	0.5961	0.6496	0.5153
(9)	0.518	0.5228	0.3494	0.4438	0.5876	0.4699	0.5931	0.5453	1	0.435	0.5375	0.3988	0.5955	0.5913	0.4288	0.484	0.328	0.2941	0.3556	0.4147	0.3769	0.3826	0.3982	0.4063	0.4024	0.4359	0.4585	0.4005	0.4263	0.3641
(10)	0.7348	0.6709	0.4009	0.5671	0.6025	0.6169	0.5242	0.5295	0.435	1	0.6397	0.5993	0.5789	0.5069	0.5787	0.6052	0.4505	0.4303	0.58	0.4474	0.54	0.4067	0.6168	0.6198	0.4706	0.6452	0.6371	0.5993	0.5847	0.5031
(11)	0.7366	0.6449	0.3778	0.5501	0.6814	0.6545	0.6151	0.6349	0.5375	0.6397	1	0.5728	0.6352	0.6242	0.5771	0.6115	0.4202	0.3972	0.5156	0.4356	0.49	0.4573	0.5774	0.5896	0.4647	0.6244	0.6053	0.5947	0.5658	0.4541
(12)	0.6634	0.6519	0.3209	0.5101	0.5787	0.7607	0.5018	0.5411	0.3988	0.5993	0.5728	1	0.5463	0.5033	0.6131	0.5804	0.3252	0.2846	0.4869	0.4304	0.6021	0.4366	0.586	0.792	0.3891	0.6288	0.6633	0.6555	0.6055	0.4908
(13)	0.6991	0.6875	0.4042	0.5444	0.8354	0.6369	0.6972	0.7278	0.5955	0.5789	0.6352	0.5463	1	0.7073	0.523	0.6488	0.3936	0.3579	0.4774	0.4715	0.4948	0.5558	0.5458	0.5682	0.4639	0.6585	0.6127	0.5414	0.6224	0.5059
(14)	0.6206	0.5847	0.3678	0.4979	0.7584	0.5931	0.8588	0.6853	0.5913	0.5069	0.6242	0.5033	0.7073	1	0.4769	0.6331	0.3238	0.3122	0.4275	0.4667	0.4241	0.4975	0.5013	0.5013	0.4434	0.6104	0.5926	0.5382	0.5838	0.5083
(15)	0.6391	0.5934	0.4307	0.5299	0.5358	0.6203	0.4835	0.508	0.4288	0.5787	0.5771	0.6131	0.523	0.4769	1	0.572	0.3819	0.3921	0.5494	0.4399	0.5676	0.3928	0.5568	0.6243	0.3887	0.5496	0.5839	0.5651	0.5227	0.4522
(16)	0.6699	0.6542	0.449	0.5375	0.7008	0.661	0.6142	0.6296	0.484	0.6052	0.6115	0.5804	0.6488	0.6331	0.572	1	0.3643	0.3405	0.4881	0.4704	0.5514	0.4744	0.5723	0.5972	0.4606	0.6688	0.628	0.5931	0.6533	0.4668
(17)	0.4168	0.348	0.3483	0.4119	0.3472	0.3542	0.3132	0.2517	0.328	0.4505	0.4202	0.3252	0.3936	0.3238	0.3819	0.3643	1	0.6149	0.408	0.2622	0.3457	0.1745	0.4025	0.3338	0.3563	0.3207	0.3364	0.2813	0.32	0.2695
(18)	0.386	0.3327	0.3827	0.3581	0.3205	0.333	0.3251	0.2456	0.2941	0.4303	0.3972	0.2846	0.3579	0.3122	0.3921	0.3405	0.6149	1	0.4395	0.2373	0.3176	0.1533	0.3888	0.3192	0.3843	0.2769	0.3285	0.304	0.3054	0.2573
(19)	0.5615	0.499	0.5168	0.4719	0.4769	0.5136	0.4085	0.4029	0.3556	0.58	0.5156	0.4869	0.4774	0.4275	0.5494	0.4881	0.408	0.4395	1	0.3036	0.4526	0.288	0.4879	0.4896	0.3855	0.4952	0.461	0.5025	0.467	0.4163
(20)	0.5126	0.4515	0.2351	0.3799	0.5025	0.4574	0.4699	0.533	0.4147	0.4474	0.4356	0.4304	0.4715	0.4667	0.4399	0.4704	0.2622	0.2373	0.3036	1	0.4376	0.3955	0.4168	0.4512	0.3064	0.5091	0.477	0.4081	0.4867	0.3622
(21)	0.6109	0.5884	0.3376	0.5303	0.5193	0.6307	0.4234	0.5153	0.3769	0.54	0.49	0.6021	0.4948	0.4241	0.5676	0.5514	0.3457	0.3176	0.4526	0.4376	1	0.4384	0.5436	0.6248	0.3787	0.5833	0.6153	0.6053	0.5493	0.4497
(22)	0.5133	0.5004	0.2046	0.3917	0.6375	0.4899	0.5092	0.6989	0.3826	0.4067	0.4573	0.4366	0.5558	0.4975	0.3928	0.4744	0.1745	0.1533	0.288	0.3955	0.4384	1	0.452	0.4684	0.3309	0.5679	0.5246	0.4556	0.5292	0.4189
(23)	0.6431	0.606	0.3721	0.5603	0.5772	0.6091	0.4915	0.5149	0.3982	0.6168	0.5774	0.586	0.5458	0.5013	0.5568	0.5723	0.4025	0.3888	0.4879	0.4168	0.5436	0.452	1	0.6143	0.4069	0.6236	0.6607	0.5645	0.5796	0.4859
(24)	0.6968	0.6838	0.3344	0.5441	0.5974	0.8377	0.5085	0.5796	0.4063	0.6198	0.5896	0.792	0.5682	0.5013	0.6243	0.5972	0.3338	0.3192	0.4896	0.4512	0.6248	0.4684	0.6143	1	0.4294	0.6646	0.673	0.6726	0.603	0.5147
(25)	0.452	0.4374	0.3361	0.4271	0.4791	0.4591	0.4701	0.4625	0.4024	0.4706	0.4647	0.3891	0.4639	0.4434	0.3887	0.4606	0.3563	0.3843	0.3855	0.3064	0.3787	0.3309	0.4069	0.4294	1	0.4698	0.4234	0.386	0.4037	0.3796
(26)	0.7359	0.7177	0.3671	0.5639	0.7306	0.6949	0.619	0.7148	0.4359	0.6452	0.6244	0.6288	0.6585	0.6104	0.5496	0.6688	0.3207	0.2769	0.4952	0.5091	0.5833	0.5679	0.6236	0.6646	0.4698	1	0.7266	0.6727	0.6663	0.5529
(27)	0.7373	0.6978	0.3783	0.5383	0.6748	0.6876	0.5869	0.6513	0.4585	0.6371	0.6053	0.6633	0.6127	0.5926	0.5839	0.628	0.3364	0.3285	0.461	0.477	0.6153	0.5246	0.6607	0.673	0.4234	0.7266	1	0.679	0.6665	0.5536
(28)	0.6683	0.6737	0.3702	0.5288	0.5868	0.7018	0.5417	0.5961	0.4005	0.5993	0.5947	0.6555	0.5414	0.5382	0.5651	0.5931	0.2813	0.304	0.5025	0.4081	0.6053	0.4556	0.5645	0.6726	0.386	0.6727	0.679	1	0.6328	0.5339
(29)	0.6714	0.6487	0.3646	0.4732	0.6666	0.6376	0.5796	0.6496	0.4263	0.5847	0.5658	0.6055	0.6224	0.5838	0.5227	0.6533	0.32	0.3054	0.467	0.4867	0.5493	0.5292	0.5796	0.603	0.4037	0.6663	0.6665	0.6328	1	0.5362
(30)	0.5531	0.5448	0.3159	0.4787	0.5378	0.5414	0.5058	0.5153	0.3641	0.5031	0.4541	0.4908	0.5059	0.5083	0.4522	0.4668	0.2695	0.2573	0.4163	0.3622	0.4497	0.4189	0.4859	0.5147	0.3796	0.5529	0.5536	0.5339	0.5362	1

TABLE A.3: Structure of the ranked DAX correlation matrix.

Appendix B

Alternative Projection Scheme

In Subsection 6.3.2, we introduced the discretization used for computer simulations. We defined the projection scheme, π^R, for simulating reflected processes in Definition 6.3. However, there is no rule which determines what elements of the boundary of the no-trade region R should be assigned to the points $x \in R^c$. For this reason, we introduce here an alternative projection scheme which we compare with the scheme from Definition 6.3 by running a simulation of the DAX portfolio. The alternative scheme is based on the l_1-projection onto the boundary. More formally, define

$$\pi_1^R \colon \mathbb{R}^n \to R, \quad x \mapsto \arg\min_{y \in R} \|x - y\|_1 \ .$$

At first glance, projecting onto the boundary by minimizing the l_1-distance appears very reasonable since it minimizes the investor's transaction costs. However, the simulation results show that, in general, π_1^R leads to trading strategies generating significantly higher values of the expected utility loss. We compare π_1^R with π^R by simulating the primal candidate $P^{(2)}$ (cf. Table 6.7). The results are shown in Table B.1. Notice the difference in the values of the functional F. The proportion of the displacement loss in the case of π_1^R is much larger than in the case of π^R. It is worth mentioning that both schemes perform equally well for rectangular domains. Moreover, in the case of the symmetric portfolio, the results are also equal for the candidate $P^{(2)}$.

	π^R	π_1^R
L^U_{tot}	0.009703	0.035429
SD	7.15×10^{-6}	1.43×10^{-4}
F	1.227002	0.492674

TABLE B.1: Performance of the l_1-projection.

Bibliography

[ABG12] A. Arapostathis, V.S. Borkar, and M.K. Ghosh. *Ergodic Control of Diffusion Processes*. Encyclopedia of Mathematics an. Cambridge University Press, 2012.

[AI12] C. Atkinson and P. Ingpochai. The effect of correlation and transaction costs on the pricing of basket options. *Applied Mathematical Finance*, 19(2):131–179, 2012.

[AMS96] Marianne Akian, José Luis Menaldi, and Agnès Sulem. On an investment-consumption model with transaction costs. *SIAM Journal on Control and Optimization*, 34(1):329–364, 1 1996.

[Ans94] Christophe Ansel, Jean-Pascal, Stricker. Couverture des actifs contingents et prix maximum. *Annales de l'I.H.P. Probabilités et statistiques*, 30(2):303–315, 1994.

[ARS17] Albert Altarovici, Max Reppen, and H. Mete Soner. Optimal consumption and investment with fixed and proportional transaction costs. *SIAM Journal on Control and Optimization*, 55(3):1673–1710, 2017.

[Be11] Sara Biagini and Aleš Černý. Admissible strategies in semimartingale portfolio selection. *SIAM Journal on Control and Optimization*, 49(1):42–72, 2011.

[BF02] Fabio Bellini and Marco Frittelli. On the existence of minimax martingale measures. *Mathematical Finance*, 12(1):1–21, 2002.

[BF07] Sara Biagini and Marco Frittelli. The supermartingale property of the optimal wealth process for general semimartingales. *Finance and Stochastics*, 11(2):253–266, 2007.

[BF08] Sara Biagini and Marco Frittelli. A unified framework for utility maximization problems: An orlicz space approach. *Ann. Appl. Probab.*, 18(3):929–966, 06 2008.

[BFK05] Adrian D. Banner, Robert Fernholz, and Ioannis Karatzas. Atlas models of equity markets. *Ann. Appl. Probab.*, 15(4):2296–2330, 11 2005.

[Bic14] Maxim Bichuch. Pricing a contingent claim liability with transaction costs using asymptotic analysis for optimal investment. *Finance and Stochastics*, 18(3):651–694, Jul 2014.

[BP10] Viorel Barbu and Teodor Precupanu. *Convexity and Optimization in Banach Spaces, 4th Edition.* Springer Monographs in Mathematics. Springer, 2010.

[DGR+02] Freddy Delbaen, Peter Grandits, Thorsten Rheinländer, Dominick Samperi, Martin Schweizer, and Christophe Stricker. Exponential hedging and entropic penalties. *Mathematical Finance*, 12(2):99–123, 2002.

[DI93] Paul Dupuis and Hitoshi Ishii. Sdes with oblique reflection on nonsmooth domains. *The Annals of Probability*, 21(1):554–580, 1993.

[DI08] Paul Dupuis and Hitoshi Ishii. Sdes with oblique reflections on nonsmooth domains. *Ann. Probab.*, 36(5):1992–1997, 09 2008.

[DN90] M. H. A. Davis and A. R. Norman. Portfolio selection with transaction costs. *Mathematics of Operations Research*, 15(4):676–713, 1990.

[DP13] Nizar Touzi Dylan Possamaï, H. Mete Soner. Homogenization and asymptotics for small transaction costs: the multidimensional case. *ArXiv e-prints*, January 2013.

[DR03] Philip Dybvig and Stephen A. Ross. *Handbook of the Economics of Finance. Financial Markets and Asset Pricing. Arbitrage, State Prices and Portfolio Theory.* Elsevier, 2003.

[DS94] Freddy Delbaen and Walter Schachermayer. A general version of the fundamental theorem of asset pricing. *Mathematische Annalen*, 300(3):463–520, 1994.

[DS96] F. Delbaen and W. Schachermayer. The fundamental theorem of asset pricing for unbounded stochastic processes. *MATHEMATISCHE ANNALEN*, 312:215–250, 1996.

[DS06] Freddy Delbaen and Walter Schachermayer. *The mathematics of arbitrage.* Springer finance. Springer, Berlin, 2006.

[Fer02] E. Robert Fernholz. *Stochastic Portfolio Theory.* Stochastic Modelling and Applied Probability. Springer-Verlag New York, 2002.

[Fri00] Marco Frittelli. The minimal entropy martingale measure and the valuation problem in incomplete markets. *Mathematical Finance*, 10(1):39–52, 2000.

[GK12] Julien Grépat and Yuri Kabanov. Small transaction costs, absence of arbitrage and consistent price systems. *Finance and Stochastics*, 16(3):357 – 368, 2012.

[Gla04] P. Glasserman. *Monte Carlo Methods in Financial Engineering*. Applications of mathematics : stochastic modelling and applied probability. Springer, 2004.

[GO10] Jonathan Goodman and Daniel N. Ostrov. Balancing small transaction costs with loss of optimal allocation in dynamic stock trading strategies. *SIAM Journal on Applied Mathematics*, 70(6):1977–1998, 2010.

[GR01] Thomas Goll and Ludger Rüschendorf. Minimax and minimal distance martingale measures and their relationship to portfolio optimization. *Finance and Stochastics*, 5(4):557–581, 2001.

[GR02] Peter Grandits and Thorsten Rheinländer. On the minimal entropy martingale measure. *Ann. Probab.*, 30(3):1003–1038, 07 2002.

[GRS08] Paolo Guasoni, Miklós Rásonyi, and Walter Schachermayer. Consistent price systems and face-lifting pricing under transaction costs. *Ann. Appl. Probab.*, 18(2):491–520, 04 2008.

[Gua06] Paolo Guasoni. No arbitrage under transaction costs, with fractional brownian motion and beyond. *Mathematical Finance*, 16(3):569–582, 2006.

[HK79] J. Michael Harrison and David Kreps. Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory*, 20(3):381–408, 1979.

[HMS13] Nizar Touzi H. Mete Soner. Homogenization and asymptotics for small transaction costs. *ArXiv e-prints*, June 2013.

[HP81] J.Michael Harrison and Stanley R. Pliska. Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and their Applications*, 11(3):215 – 260, 1981.

[HP91] Hua He and Neil D. Pearson. Consumption and portfolio policies with incomplete markets and short-sale constraints: The infinite dimensional case. *Journal of Economic Theory*, 54(2):259–304, 1991.

[IP08] Albrecht Irle and Claas Prelle. A note on arbitrage under transaction costs. *Kiel Institute for the World Economy. Kiel Working Papers 1450*, 2008.

[IPB+11] Tomoyuki Ichiba, Vassilios Papathanakos, Adrian Banner, Ioannis Karatzas, and Robert Fernholz. Hybrid atlas models. *Ann. Appl. Probab.*, 21(2):609–644, 04 2011.

[Irl12] A. Irle. *Finanzmathematik. Die Bewertung von Derivaten.* Springer Spektrum, Wiesbaden, 2012.

[Jac92] S. D. Jacka. A martingale representation result and an application to incomplete financial markets. *Mathematical Finance*, 2(4):239–250, 1992.

[JK95] Elyès Jouini and Hédi Kallal. Martingales and arbitrage in securities markets with transaction costs. *Journal of Economic Theory*, 66(1):178 – 197, 1995.

[JS13] J. Jacod and A.N. Shiryaev. *Limit Theorems for Stochastic Processes.* Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2013.

[Øk92] Bernt Øksendal. *Stochastic Differential Equations (3rd Ed.): An Introduction with Applications.* Springer-Verlag New York, Inc., New York, NY, USA, 1992.

[KF09] Ioannis Karatzas and Robert Fernholz. Stochastic portfolio theory: an overview. In Alain Bensoussan and Qiang Zhang, editors, *Special Volume: Mathematical Modeling and Numerical Methods in Finance*, volume 15 of *Handbook of Numerical Analysis*, pages 89 – 167. Elsevier, 2009.

[KL13] J. Kallsen and S. Li. Portfolio Optimization under Small Transaction Costs: a Convex Duality Approach. *ArXiv e-prints*, September 2013.

[KLSX91] Ioannis Karatzas, John P. Lehoczky, Steven E. Shreve, and Gan-Lin Xu. Martingale and duality methods for utility maximization in an incomplete market. *SIAM Journal on Control and Optimization*, 29(3):702–730, 1991.

[KMK10] J. Kallsen and J. Muhle-Karbe. On using shadow prices in portfolio optimization with transaction costs. *Ann. Appl. Probab.*, 20(4):1341–1358, 08 2010.

[KMK15] Jan Kallsen and Johannes Muhle-Karbe. Option pricing and hedging with small transaction costs. *Mathematical Finance*, 25(4):702–723, 2015.

[KP11] P.E. Kloeden and E. Platen. *Numerical Solution of Stochastic Differential Equations.* Stochastic Modelling and Applied Probability. Springer Berlin Heidelberg, 2011.

[KS99] D. Kramkov and W. Schachermayer. The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Ann. Appl. Probab.*, 9(3):904–950, 08 1999.

[KS02] Yuri M. Kabanov and Christophe Stricker. On the optimal portfolio for the exponential utility maximization: remarks to the six-author paper. *Mathematical Finance*, 12(2):125–134, 2002.

[KS10] Y. Kabanov and M. Safarian. *Markets with Transaction Costs: Mathematical Theory.* Springer Finance. Springer Berlin Heidelberg, 2010.

[Liu04] Hong Liu. Optimal consumption and investment with transaction costs and multiple risky assets. *The Journal of Finance*, 59(1):289–338, 2004.

[LT10] Anthony W. Lynch and Sinan Tan. Multiple risky assets, transaction costs, and return predictability: Allocation rules and implications for u.s. investors. *Journal of Financial and Quantitative Analysis*, 45(4):1015–1053, 2010.

[Ém80] Michel Émery. Compensation de processus à variation finie non localement intégrables. *Séminaire de probabilités de Strasbourg*, 14:152–160, 1980.

[MC76] Michael Magill and George Constantinides. Portfolio selection with transactions costs. *Journal of Economic Theory*, 13(2):245–263, 1976.

[Mer69] Robert Merton. Lifetime portfolio selection under uncertainty: The continuous-time case. *The Review of Economics and Statistics*, 51(3):247–57, 1969.

[Mer71] Robert Merton. Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory*, 3(4):373–413, 1971.

[MK06] Kumar Muthuraman and Sunil Kumar. Multidimensional portfolio optimization with proportional transaction costs. *Mathematical Finance*, 16(2):301–335, 2006.

[Nn10a] Kaj Nyström and Thomas Önskog. The skorohod oblique reflection problem in time-dependent domains. *Ann. Probab.*, 38(6):2170–2223, 11 2010.

[Nn10b] Kaj Nyström and Thomas Önskog. Weak approximation of obliquely reflected diffusions in time-dependent domains. *Journal of Computational Mathematics*, 28(5):579–605, 2010.

[Pro04] Philip E. Protter. *Stochastic integration and differential equations.* Applications of mathematics. Springer, Berlin, Heidelberg, New York, 2004.

[Roc70] R. Tyrrell Rockafellar. *Convex Analysis.* Princeton University Press, 1970.

[Rog04] L. C. G. Rogers. Why is the effect of proportional transaction costs $\mathcal{O}(\delta^{2/3})$? *Mathematics of Finance*, pages 303–308, 2004.

[Rog13] L.C.G. Rogers. *Optimal Investment.* SpringerBriefs in Quantitative Finance. Springer Berlin Heidelberg, 2013.

[Sł01] Leszek Słomiński. Euler's approximations of solutions of sdes with reflecting boundary. *Stochastic Processes and their Applications*, 94(2):317 – 337, 2001.

[Sł13] Leszek Słomiński. Weak and strong approximations of reflected diffusions via penalization methods. *Stochastic Processes and their Applications*, 123(3):752 – 763, 2013.

[Sł14] Leszek Słomiński. On wong-zakai type approximations of reflected diffusions. *Electron. J. Probab.*, 19:15 pp., 2014.

[Sch01] Walter Schachermayer. Optimal investment in incomplete markets when wealth may become negative. *Ann. Appl. Probab.*, 11(3):694–734, 08 2001.

[Sch03] Walter Schachermayer. A super-martingale property of the optimal portfolio process. *Finance and Stochastics*, pages 433–456, 2003.

[Sch04] Walter Schachermayer. The fundamental theorem of asset pricing under proportional transaction costs in finite discrete time. *Mathematical Finance*, 14(1):19–48, 2004.

[SS94] S. E. Shreve and H. M. Soner. Optimal investment and consumption with transaction costs. *Ann. Appl. Probab.*, 4(3):609–692, 08 1994.

[SST09] Walter Schachermayer, Mihai Sîrbu, and Erik Taflin. In which financial markets do mutual fund theorems hold true? *Finance and Stochastics*, 13(1):49–77, Jan 2009.

[Str02] Christophe Stricker. Simple strategies in exponential utility maximization. *Séminaire de probabilités de Strasbourg*, 36:415–418, 2002.

[SW13] Leszek Słomiński and Tomasz Wojciechowski. Stochastic differential equations with time-dependent reflecting barriers. *Stochastics*, 85(1):27–47, 2013.

[Tob58] James Tobin. Liquidity preference as behavior towards risk. *Review of Economic Studies*, 25(2):65–86, 1958.

[WW97] A. E. Whalley and P. Wilmott. An asymptotic analysis of an optimal hedging model for option pricing with transaction costs. *Mathematical Finance*, 7(3):307–324, 1997.