Long Time Behavior of the Spinor Flow

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Abstract

The subject of this thesis is the long time behavior of the spinor flow in two situations. The spinor flow is a geometric flow which arises as the negative gradient flow of the functional which associates to a metric g and a unit spinor field $\varphi \in \Gamma(\Sigma_g M)$ on a manifold M the spinorial energy

$$\mathcal{E}(g,\varphi) = \frac{1}{2}\int_M |\nabla^g \varphi|^2 \operatorname{vol}_g .$$

The geometric interpretation of this energy depends on the dimension of the manifold. If the manifold has dimension at least three, critical points of this functional are Ricci-flat special holonomy metrics. In dimension two, a pair (g,φ) can be interpreted as a generalized isometric immersion and the spinorial energy as a generalized Willmore energy.

The first theme is the stability of the spinor flow in dimension three and up. Given a critical point (g,φ) of $\mathcal{E}$ it is found that the spinor flow with initial condition close enough to (g,φ) exists for all times and converges to a critical point of $\mathcal{E}$ at an exponential rate. The critical points of the spinorial energy restricted to metrics of constant volume are also geometrically very interesting. A volume constrained critical point (g,φ) is shown to be stable in the above sense if it is a minimizer, the critical set at that point satisfies some regularity constraint and g has a discrete isometry group. The rate of convergence depends on the regularity of the critical set.

The second theme is the behavior of the spinor flow on closed surfaces of positive genus. If (g_t,φ_t) is a solution of the spinor flow on an interval $[0,T)$, then one may ask under what conditions the flow can be continued beyond time T. The criterium

$$\inf_{0\le t<T} \operatorname{inj}(M,g_t) > 0 \qquad \text{and} \qquad \sup_{0\le t<T} \int_M |\nabla^2 \varphi_t|^p \operatorname{vol}_{g_t} < \infty$$

suffices to continue the flow. An alternative sufficient condition is

$$\sup_{\substack{x \in M,\\ 0 \le t < T}} |\nabla^2 \varphi_t(x)| < \infty.$$

The proofs are based on a new compactness theorem for families of metrics on closed surfaces of positive genus.

Zusammenfassung

Gegenstand dieser Arbeit ist das Langzeitverhalten des Spinorflusses in zwei verschiedenen Situationen. Der Spinorfluss ist ein geometrischer Fluss, der als der negative Gradientenfluss der spinoriellen Energie

$$\mathcal{E}(g,\varphi) = \frac{1}{2}\int_M |\nabla^g\varphi|^2 \operatorname{vol}_g$$

definiert ist, wobei g eine Riemannsche Metrik und $\varphi \in \Gamma(\Sigma_g M)$ ein Spinorfeld von Einheitslänge auf einer Mannigfaltigkeit M sind. Die geometrische Interpretation der spinoriellen Energie hängt von der Dimension der Mannigfaltigkeit ab. Ab Dimension drei sind kritische Punkte von $\mathcal{E}$ Ricci-flache Metriken mit spezieller Holonomie. In Dimension zwei kann man ein Paar (g,φ) als eine verallgemeinerte isometrische Immersion und die Spinorenergie als eine verallgemeinerte Willmoreenergie interpretieren.

Das erste Themengebiet der Dissertation ist die Stabilität des Spinorflusses ab Dimension drei. Es wird bewiesen, dass kritische Punkte (g,φ) der spinoriellen Energie $\mathcal{E}$ stabil sind. Das heißt, dass der Spinorfluss mit Anfangswerten in der Nähe von (g,φ) unendlich lange existiert und gegen einen kritischen Punkt konvergiert. Die kritischen Punkte der spinoriellen Energie eingeschränkt auf Metriken konstanten Volumens sind geometrisch ebenfalls von großem Interesse. Es wird gezeigt, dass ein solcher kritischer Punkt (g,φ) stabil im obigen Sinne ist, falls er ein lokales Minimum ist, die kritische Menge nahe diesem Punkt hinreichend glatt ist und die Isometriegruppe von g diskret ist. Die Konvergenzgeschwindigkeit hängt von der Regularität der kritischen Menge ab.

Das zweite Themengebiet ist der Spinorfluss auf geschlossenen Flächen positiven Geschlechts. Falls (g_t,φ_t) eine Lösung des Spinorflusses auf einem Intervall $[0,T)$ ist, dann ist die Frage naheliegend, unter welchen Bedingungen sich der Fluss über die Intervallgrenze T hinaus fortsetzen lässt. Die folgende Bedingung

$$\inf_{0\le t<T} \operatorname{inj}(M,g_t) > 0 \qquad \text{und} \qquad \sup_{0\le t<T}\int_M |\nabla^2\varphi_t|^p \operatorname{vol}_{g_t} < \infty, p > 8$$

ist ausreichend um dies sicherzustellen. Ein weiteres mögliches Kriterium ist

$$\sup_{\substack{x\in M,\\ 0\le t<T}} |\nabla^2\varphi_t(x)| < \infty.$$

Die Beweise dieser Kriterien basieren auf einem neuen Kompaktheitssatz für Familien von Metriken auf geschlossenen Flächen positiven Geschlechts.

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Contents

Introduction

A fundamental result of differential geometry is the classification of complete Riemannian manifolds of constant sectional curvature. After rescaling, a Riemannian manifold of constant sectional curvature is isometric to a quotient of hyperbolic space $\mathbb{H}^n$, Euclidean space $\mathbb{R}^n$ or the round sphere S^n by a discrete group of isometries. Given one of those Riemannian manifolds, the metric of constant sectional curvature can be considered to be a canonical metric of the underlying smooth manifold. However, most smooth manifolds do not admit any metric of constant sectional curvature. It is natural to ask for a weaker condition on the metric, which still yields metrics which can be considered to be canonical in some sense. The *Ricci curvature* is obtained from the sectional curvatures by an averaging process. For a Riemannian metric g and a constant $\lambda \in \mathbb{R}$ the condition

$$\mathrm{Ric}_g = \lambda g$$

can be interpreted as saying that the Ricci curvature is constant. A manifold satisfying this condition is called an *Einstein manifold* and λ is called the *Einstein constant* of g. The Einstein condition can be interpreted as saying that the Ricci curvature is constant. While much is known about Einstein manifolds, much more remains unknown. In dimensions 2 and 3 the Einstein condition coincides with the constant curvature condition. In dimension 4 the curvature tensor of Einstein manifolds is still fairly restricted. This leads, for example, to the Hitchin–Thorpe inequality which says that if M admits an Einstein metric, then two topological invariants of the manifold, the Euler characteristic $\chi(M)$ and the signature $\tau(M)$, satisfy the inequality $2\chi(M) \geq 3|\tau(M)|$. This inequality constrains the topological type of Einstein manifolds in four dimensions. From dimension five, it is not known whether there are any manifolds which do not admit a Einstein metric. Finding Einstein metrics is a major research field of differential geometry. Within the class of Einstein metrics, the *Ricci-flat metrics* — i.e. those with Einstein constant $\lambda = 0$ — are especially interesting, because they have a deep relationship with the holonomy group.

The *holonomy group* $\mathrm{Hol}_x(M, g)$ of a Riemannian manifold (M, g) at a point $x \in M$ is the group of automorphisms of T_xM which arise as parallel transport maps of loops which are based at $x \in M$. Because any parallel transport map is an orthogonal transformation of T_xM, $\mathrm{Hol}_x(M, g)$ is a subgroup of the orthogonal group $O(T_xM, g_x)$. A famous theorem due to Berger from 1955 lists all possible groups which can appear as holonomy groups. He found that if (M, g) is an irreducible non-symmetric simply-connected Riemannian manifold, then the holonomy group must be isomorphic to $\mathrm{SO}(n), U(n), \mathrm{SU}(n), \mathrm{Sp}(n)$, $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$, G_2 or

Spin(7).[5] The holonomy group of a generic metric on an orientable manifold is isomorphic to $\mathrm{SO}(n)$.

The *holonomy principle* states that a reduction of the holonomy group to a proper subgroup of $\mathrm{SO}(n)$ is equivalent to the existence of certain parallel tensor fields. These tensor fields describe geometric structures on the manifold beyond the Riemannian metric. For example, if (M, g) has holonomy $U(n)$, then there exists a parallel complex structure J, i.e. a section of the endomorphism bundle of TM, such that $J^2 = -\operatorname{id}_{TM}$. Such a manifold is called a *Kähler manifold* and they arise in abundance in algebraic geometry as (smooth) projective varieties in $\mathbb{C}P^n$. Manifolds with holonomy $\mathrm{SU}(n) \subset U(n)$ are particularly well understood. Yau's solution of the Calabi conjecture yields a precise criterium, when a manifold with holonomy $U(n)$ can be deformed to a manifold with holonomy $\mathrm{SU}(n)$.[48] Honoring this result, such manifolds are called *Calabi–Yau manifolds*.

The other subgroups of $\mathrm{SO}(n)$ also correspond to certain geometric structures on the manifold. Manifolds with holonomy $\mathrm{Sp}(n)$ are called *hyperkähler*, whereas manifolds with holonomy $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$ are called *quaternion Kähler*. Finally, manifolds with holonomy G_2 or Spin(7) are simply called G_2-manifolds or Spin(7)-manifolds. As the names suggest, hyperkähler and quaternion Kähler manifolds are related to Kähler geometry and as such techniques from algebraic geometry can be applied to study them. For G_2- and Spin(7)-manifolds the situation is more complicated. Giving examples of such manifolds is difficult. The first manifolds with holonomy G_2 and Spin(7) were constructed by Bonan([6]) in 1966, the first complete examples by Bryant and Salamon([10]) in 1989 and the first compact examples by Joyce([24], [25]) in 1996.

Calabi–Yau, hyperkähler, G_2 and Spin(7) manifolds all are Ricci-flat. Indeed, all known compact Ricci-flat manifolds are of this type, but it is not known if the holonomy group of a compact Ricci-flat manifold must be a proper subgroup of $\mathrm{SO}(n)$. Remarkably, all these cases admit a uniform description in terms of spin geometry. Spin geometry exploits the fact that the universal cover of $\mathrm{SO}(n)$, called the spin group $\mathrm{Spin}(n)$, admits a representation which does not factor through $\mathrm{SO}(n)$. Under certain topological conditions on a manifold (M, g) one can then form a vector bundle $\Sigma_g M$ associated to this representation. The vector bundle $\Sigma_g M$ is called the spinor bundle. The Levi–Civita connection of (M, g) induces a connection ∇^g on $\Sigma_g M$. It turns out that an irreducible, simply connected manifold admits a parallel spinor field, i.e. a section $\varphi \in \Gamma(\Sigma_g M)$ such that $\nabla^g \varphi = 0$, if and only if (M, g) has holonomy $\mathrm{SU}(n), \mathrm{Sp}(n), G_2$ or Spin(7).

It is not known under which conditions a manifold admits such a metric. One approach is to deform a given metric g by some procedure towards a metric of this type. Typically the procedure is some kind of parabolic partial differential equation on the metric g or a geometric structure. The *Kähler–Ricci flow* for example takes an initial Kähler metric g_0 and deforms this metric by means of the equation

$$\partial_t g_t = -2 \operatorname{Ric}_{g_t} .$$

In this case, an analogue of Yau's theorem exists, giving a precise condition for convergence of this flow towards a Calabi–Yau metric.[13] In the G_2 setting, Bryant defined the *Laplacian* *flow*.[9] Here, it is not the metric, which is deformed, but rather a G_2-form. A G_2-form is a 3-form Ω whose stabilizer group $\mathrm{Stab}(\Omega) \subset GL(T_xM)$ is isomorphic to G_2 at any point $x \in M$. Because G_2 is a subgroup of $\mathrm{SO}(n)$, a G_2 form Ω induces a metric g_Ω. If

$$\Delta_{g_\Omega}\Omega = 0$$

or equivalently

$$d\Omega = 0 \text{ and } d^{*g_\Omega}\Omega = 0,$$

then (M, g_Ω) is a G_2-manifold. The Laplacian flow is defined by the equation

$$\partial_t\Omega_t = \Delta_{g_{\Omega_t}}\Omega_t$$

under the assumption that Ω_0 is closed, i.e. $d\Omega_0 = 0$. The Laplacian flow is the gradient flow of the Hitchin functional

$$\mathcal{H} : \{\Omega \in [\bar{\Omega}] : \Omega \text{ is a } G_2 \text{ form}\} \to \mathbb{R}$$

$$\Omega \mapsto \int_M \mathrm{vol}_{g_\Omega},$$

where $\bar{\Omega}$ is a given closed G_2-form.

Another flow on G_2 forms is the G_2 heat flow introduced by Weiß and Witt.[46] This is the negative gradient flow of the functional

$$\mathcal{D} : \{\Omega \in \Omega^3(M) : \Omega \text{ is a } G_2 \text{ form}\} \to \mathbb{R}$$

$$\mathcal{D}(\Omega) = \frac{1}{2}\int_M |d\Omega|^2_{g_\Omega} + |d^{*g_\Omega}\Omega|^2_{g_\Omega}\,\mathrm{vol}_{g_\Omega},$$

i.e. given any G_2-form Ω_0 (not necessarily closed) the G_2 heat flow is defined by the equation

$$\partial_t\Omega_t = -\operatorname{grad}\mathcal{D}(\Omega_t).$$

It turns out that the functional $\mathcal{D}$ admits a generalization to any dimension using spin geometry. We have already seen that to every G_2-form Ω there is associated a metric g_Ω. One can also construct a spinor field $\varphi_\Omega \in \Gamma(\Sigma_g M)$ which satisfies $|\varphi_\Omega| = 1$. Conversely, on a 7-manifold given a metric g and a spinor field $\varphi \in \Gamma(\Sigma_g M)$ with $|\varphi| = 1$, one can construct a G_2 form $\Omega_{g,\varphi}$. It turns out that the functional $\mathcal{D}(\Omega)$ is given by

$$\int_M 8|\nabla^{g_\Omega}\varphi_\Omega|^2_{g_\Omega} + R_{g_\Omega}\,\mathrm{vol}_{g_\Omega}.$$

Motivated by this, Ammann, Weiß and Witt ([3]) define the *spinorial energy functional*

$$\mathcal{E} : \mathcal{N} \to \mathbb{R}$$

$$\mathcal{E}(g,\varphi) = \frac{1}{2}\int_M |\nabla^g\varphi|^2\,\mathrm{vol}_g$$

on the set

$$\mathcal{N} = \{(g, \varphi) : g \text{ is a metric on } M, \varphi \in \Gamma(\Sigma_g M) \text{ and } |\varphi| = 1\}.$$

If $n = \dim M \geq 3$, then every critical point of (g, φ) satisfies

$$\nabla^g \varphi = 0.$$

In particular, if g is irreducible and M simply connected, g is a Ricci-flat metric with holonomy $\mathrm{SU}(n), \mathrm{Sp}(n), G_2$ or $\mathrm{Spin}(7)$. The *spinor flow* is then the negative gradient flow of $\mathcal{E}$, i.e. for a given element $(g_0, \varphi_0) \in \mathcal{N}$ the spinor flow is defined by the evolution equation

$$\partial_t (g_t, \varphi_t) = - \operatorname{grad} \mathcal{E}(g_t, \varphi_t).$$

Short time existence of the spinor flow has been established in [46]: for any $(g_0, \varphi_0) \in \mathcal{N}$, there exists a solution of the spinor flow equation with initial condition (g_0, φ_0) on a maximal time interval $[0, T)$. The maximal time of existence T may be finite or infinite and depends on (g_0, φ_0).

In the best case the spinor flow succeeds in deforming the initial metric towards a critical point of $\mathcal{E}$. To be precise, this means that the flow exists on the interval $[0, \infty)$ and that (g_t, φ_t) converges towards a critical point as $t \to \infty$. The property that a solution exists on the interval $[0, \infty)$ is usually called *long time existence* or *global existence*.

The subject of this thesis is to give conditions for long time existence and convergence of the spinor flow.

Stability of the spinor flow

The first major result of this thesis is a stability result. We introduce some terminology before stating the result. Let $\mathcal{M}$ be a manifold and let $X : \mathcal{M} \to T\mathcal{M}$ be a vector field. Then X defines the differential equation

$$\frac{d}{dt} \Phi_t = X(\Phi_t), \text{ where } \Phi_t \in \mathcal{M}.$$

A critical point of this equation is a point $\Phi \in \mathcal{M}$, such that

$$X(\Phi) = 0.$$

Consequently, the constant map $t \mapsto \Phi$ solves the differential equation above. A critical point Φ is called *stable*, if there is a neighborhood U of Φ, such that for any $\tilde{\Phi} \in U$ the solution of the equation above with initial value $\tilde{\Phi}$ remains nearby Φ for all times and moreover converges to a critical point as $t \to \infty$.

Since our goal is to find critical points of the spinorial energy functional, it would be quite unsettling if critical points were not stable. This would mean that there are points $(\tilde{g}, \tilde{\varphi}) \in \mathcal{N}$

close to such a critical point, so that the spinor flow with initial condition $(\tilde{g}, \tilde{\varphi})$ fails to converge to a critical point. This would in particular exclude the possibility of giving any simple criterium for convergence of the spinor flow. Fortunately, in dimension three and above, all critical points of the spinor flow are indeed stable.

Let M be a compact spin manifold of dimension $n \geq 3$.

Theorem (Stability of the spinor flow)**.**
Let $(g, \varphi) \in \mathcal{N}$ be a critical point of $\mathcal{E}$, i.e. $\nabla^g \varphi = 0$. Suppose moreover that the isometry group of g is discrete.

Then there exists a C^∞ neighborhood $U \subset \mathcal{N}$ of (g, φ), such that for any $(\tilde{g}, \tilde{\varphi}) \in U$ the spinor flow with initial conditions $(\tilde{g}, \tilde{\varphi})$ exists for all time and smoothly converges to a critical point. In any C^k norm the speed of convergence is exponential.

This is theorem 4.9 in chapter 4. Two comments are in order. This formulation of the theorem makes no statement on the size of the neighborhood and uses the very fine $\mathcal{C}^\infty$ topology. Theorem 4.9 also makes no statement on the size of the neighborhood, but uses a coarser topology.

The second comment regards the assumption on the isometry group. Although we make this assumption for technical reasons, it is actually a very natural assumption. By assumption, the manifold (M, g) is compact and Ricci flat. It can be shown that the isometry group of a Ricci flat manifold is discrete, if the universal cover contains no Euclidean factors. This can for instance be excluded by assuming that the fundamental group of M is finite. Since we are interested in Ricci flat manifolds which are not Euclidean, this is an assumption we would typically make to ensure convergence towards such a metric.

A Riemannian manifold (M, g) can be rescaled to $(M, \mu g)$ for any $\mu \in \mathbb{R}^{>0}$. Let (g_t, φ_t) be a solution of the spinor flow. If $g_t = \mu(t) g_0$, the spinor flow evolves by rescaling the metric. This is a rather special situation and it is expected that such solutions play a role in the formation of singularities along the flow. A solution of this type is not a critical point of the spinorial energy functional, unless $\mu(t) \equiv 1$. It is, however, a critical point of the energy functional restricted to the set

$$\mathcal{N}^1 = \left\{ (g, \varphi) \in \mathcal{N} : \int_M \mathrm{vol}_g = 1 \right\}.$$

In general, not much is known about these critical points, but it contains the rich class of metrics with real Killing spinors. A *Killing spinor* is a spinor field φ on a manifold (M, g), such that

$$\nabla_X \varphi = \lambda X \cdot \varphi$$

for some $\lambda \in \mathbb{C}$. This condition implies that g is an Einstein manifold. In fact, just as the existence of a parallel spinor field implies reduced holonomy and thus existence of certain geometric structures on a Riemannian manifold, the existence of a Killing spinor also implies existence of certain geometric structures. This is explained by the cone construction: If (M, g) is a Riemannian manifold and $\varphi \in \Gamma(\Sigma_g M)$ a Killing field, then the cone $((0, \infty) \times M, dr^2 + r^2 g)$ carries a parallel spinor field induced by φ.[12]

In this context there is a second stability result. We introduce some terminology first. A *volume constrained critical point* is a critical point of $\mathcal{E}$ restricted to $\mathcal{N}^1$. Likewise, a *volume constrained minimizer* is a local minimum of $\mathcal{E}$ restricted to $\mathcal{N}^1$. A solution of the *volume normalized spinor flow* is given by a solution of the spinor flow, where the metrics are rescaled, such that their volume is 1. It can be shown that the volume normalized spinor flow coincides with the gradient flow of $\mathcal{E}$ restricted to $\mathcal{N}^1$. With these definitions at hand, we state the following stability theorem.

Theorem (Stability of volume contrained minimizers)**.**
Let $(g, \varphi) \in \mathcal{N}^1$ be a volume constrained minimizer. Suppose moreover that the isometry group of g is discrete and that the critical set of $\mathcal{E}$ near (g, φ) is smooth. Then there exists a $\mathcal{C}^\infty$ neighborhood $U \subset \mathcal{N}^1$ of (g, φ), such that for any initial condition $(\tilde{g}, \tilde{\varphi})$ the volume normalized spinor flow exists for all time and smoothly converges to a volume constrained minimizer. The convergence speed is exponential in all C^k norms.

This is theorem 4.13 in chapter 4.

In this case, the assumption on the isometry group is a strong condition, since many examples of Riemannian manifolds carrying Killing spinors do have symmetries. The proof requires the assumption that the isometry group is discrete, but there is no indication that this is required for the theorem to hold. The smoothness of the critical set can be dispensed with to some degree, but the method of proof does not allow for arbitrary critical sets. This is made more precise in the statement of theorem 4.13. There are indeed counterexamples where the critical set is not smooth. Van Coevering shows that the moduli space of Killing spinors on some toric Sasaki–Einstein manifolds does not have constant dimension.[42] The assumption that (g, φ) is a minimizer rather than a general volume constrained critical point can clearly not be removed.

The spinor flow on surfaces

The other major results of this thesis concern the spinor flow on surfaces. In dimensions 3 and higher the scaling properties of the spinorial energy imply that any critical point must be an absolute minimizer. In dimension 2 the spinorial energy is scale invariant, allowing for a richer set of critical points and energies. Even the structure of absolute minimizers is more complicated and depends on the topological type of the surface. This is evident in the following formula:

$$\mathcal{E}(g, \varphi) = \frac{1}{2} \int_M |D_g \varphi|^2 \operatorname{vol}_g - \frac{\pi}{2} \chi(M).$$

The operator $D_g : \Gamma(\Sigma_g M) \to \Gamma(\Sigma_g M)$ is the *Dirac operator* and $\chi(M)$ is the Euler characteristic of M. Since the Euler characteristic is a purely topological term and does not depend on the metric, minimizing the spinorial energy amounts to minimizing the L^2 norm of $D_g \varphi$. The Dirac operator arises from the spin connection by a kind of spinorial trace. In particular a parallel spinor field φ also satisfies $D_g\varphi = 0$, but the converse is true only in very special cases. The structure of minimizers depends on the topology of the surface. The topology of a closed surface can be read from its Euler characteristic: the genus of a closed surface M is $\gamma = 1 - \chi(M)/2$. In particular, M is diffeomorphic to the sphere S^2 if $\chi(M) = 2$, and M is diffeomorphic to the torus T^2 if $\chi(M) = 0$. We get the following trichotomy for absolute minimizers: a pair (g, φ) is an absolute minimizer if

$$\begin{aligned} P^g\varphi &= 0, \quad \text{if } \chi(M) = 2, \\ \nabla^g\varphi &= 0, \quad \text{if } \chi(M) = 0, \\ D_g\varphi &= 0, \quad \text{if } \chi(M) < 0. \end{aligned}$$

The twistor operator P^g has nontrivial kernel on S^2 if and only if g is isometric to the sphere. The spinorial energy functional admits a further geometric interpretation. The *spinorial Weierstraß representation* is a parametrisation of a surface by means of a unit spinor field. Suppose φ is a unit spinor field which satisfies $D_g\varphi = H\varphi$ for some $H \in \mathcal{C}^\infty(M)$. Then one can construct an immersion of the universal Riemannian cover of (M, g) into $\mathbb{R}^3$, such that the immersion has mean curvature function H. In that sense the spinorial energy functional is a generalization of the *Willmore energy*

$$\int_M |H|^2 \operatorname{vol}_g$$

of an immersion. It should be noted that the spinor flow does not preserve the condition $D_g\varphi = H\varphi$, in particular the spinor flow is not tangent to the Willmore flow. The results in this thesis regarding the spinor flow on surfaces concern the formation of singularities. More precisely, criteria that exclude the formation of singularities are found. The first such criterium is closely related to an analogous result for the Ricci flow. For the Ricci flow it is known that if the flow becomes singular, then the norm $|\operatorname{Rm}(g)|$ has to diverge. For the spinor flow an analogous criterium is that $|\nabla^2\varphi| = |\nabla^g\nabla^g\varphi|$ diverges as the flow becomes singular. It can be shown that

$$|\nabla^2\varphi|^2 = \frac{1}{16}R_g^2 + |(\nabla^2\varphi)^{\mathrm{sym}}|^2,$$

i.e. a bound on $\nabla^2\varphi$ implies a bound on the curvature. The criterium can then be formulated as follows.

Theorem.
Suppose M is a closed surface of genus $\gamma > 0$ and suppose that (g_t, φ_t) solves the spinor flow on an interval $[0, T)$. If

$$\sup_{\substack{x \in M, \\ 0 \leq t < T}} |\nabla^2\varphi_t(x)| < \infty,$$

then the spinor flow solution (g_t, φ_t) can be smoothly extended to a solution on an interval $[0, T + \delta)$ for some $\delta > 0$.

This is theorem 5.24 in chapter 5. It is also possible to give a criterium in terms of an integral bound on $\nabla^2\varphi$ and a lower injectivity radius bound on the metrics.

Theorem.
Suppose M is a closed surface of genus $\gamma > 0$ and suppose that (g_t, φ_t) solves the spinor flow on an interval $[0, T)$. If

$$\sup_{0<t<T} \int_M |\nabla^2 \varphi_t|^q \operatorname{vol}_{g_t} < \infty$$

for some $q > 8$ and

$$\inf_{0<t<T} \operatorname{inj}(M, g_t) > 0,$$

then the spinor flow solution (g_t, φ_t) can be smoothly extended to a solution on an interval $[0, T+\delta)$ for some $\delta > 0$.

This is theorem 5.23 in chapter 5. We will prove the theorems in the reverse order. It turns out that the pointwise bound on $\nabla^2\varphi$ can be used to prove a lower injectivity radius bound, and thus the first theorem follows from the second theorem. We will also consider the *conformal spinor flow*, the spinor flow restricted to a conformal class of metrics. There we obtain a significantly better blow up criterium.

Theorem.
Suppose M is a closed surface of genus $\gamma > 0$ and suppose that (g_t, φ_t) solves the conformal spinor flow on an interval $[0, T)$. If

$$\sup_{0<t<T} \int_M |R_{g_t}|^2 + |\nabla^{g_t} \varphi_t|^q \operatorname{vol}_{g_t} < \infty$$

for some $q > 4$, then the conformal spinor flow solution (g_t, φ_t) can be smoothly extended to a solution on an interval $[0, T+\delta)$ for some $\delta > 0$.

This is theorem 5.16 in chapter 5.

In the following we briefly explain the techniques used to show these criteria, which are of some independent interest. A fundamental idea, due to Buzano and Rupflin in the case of the Ricci harmonic flow, is to use the decomposition of the space of Riemannian metrics on surfaces into constant curvature metrics and conformal classes. Using this decomposition a family of metrics can be split into a family of conformal factors and constant curvature metrics, allowing us to analyse them independently. The following new compactness theorem is of basic importance for the proof of the blow up criteria.

Theorem.
Suppose M is a closed surface and suppose $\chi(M) \leq 0$. Let g_n be a sequence of Riemannian metrics with

$$\operatorname{Vol}(M, g_n) < V, \int_M |R_{g_n}|^2 \operatorname{vol}_{g_n} < K \text{ and } \operatorname{inj}(M, g_n) > \epsilon.$$

Then there exists a subsequence g_{n_k}, a family of $\mathcal{C}^\infty$ diffeomorphisms φ_k with the following significance. Let $\tilde{g}_k = \varphi_k^* g_{n_k}$ and suppose that $\bar{g}_k = e^{-2u_k}\tilde{g}_k$ is the uniformization of $\tilde{g}_k$. Then the sequence $\bar{g}_k$ converges in the $\mathcal{C}^\infty$ topology to a metric $\bar{g}$ and the sequence u_k converges weakly in the H^2 norm.

This is theorem 3.15 in chapter 3.

Outline of the thesis

We conclude the introduction with an overview over the chapters of the thesis:

Chapter 1 provides tools from analysis which we will need to study the spinor flow. This includes definition of isotropic and anisotropic function spaces, embedding and multiplication theorems, regularity theory for elliptic and parabolic equations and short time existence for quasilinear parabolic equations. The results are only proven when no adequate reference has been found.

Chapter 2 introduces the basic concepts of spin geometry, the spinorial energy functional and the spinor flow. The material on spin geometry is a refresher of standard material, but includes less well known constructions, such as the universal spinor bundle and the Bourguignon–Gauduchon connection. In a second part we introduce the spinorial energy functional and the spinor flow.

Chapter 3 provides some general tools to analyse geometric flows on surfaces. In particular the compactness theorem above is proven in this chapter.

Chapter 4 presents the proof of the stability theorems.

Chapter 5 is dedicated to the spinor flow on surfaces. First, the flow is restricted to a conformal class and its behavior is examined in this setting. In a second step, this restriction is removed and the blow up criteria are proven.

Chapter 1

Linear and quasilinear parabolic equations

This chapter discusses linear and quasilinear parabolic equations. Geometric flows are defined by some process where a geometric quantity such as the curvature gives rise to a "force" which deforms the geometry. The geometry may be encoded by a Riemannian metric, a differential form or other objects. The geometric quantity is usually a nonlinear combination of derivatives of the object in question. For example, the Ricci tensor Ric_g of a Riemannian metric g is a nonlinear combination of the metric, its first and second derivatives. The *Ricci flow* is given by solutions of

$$\partial_t g_t = -2\,\mathrm{Ric}_{g_t},$$

where g_t is a family of metrics and g_0 is the initial metric. This is a nonlinear parabolic system of partial differential equations. Most geometric flows, including the spinor flow, are defined by such parabolic systems. Linear parabolic equations have the useful property that their solutions become more regular over time. To a degree, this fact can be exploited also in the study of nonlinear parabolic equation. The majority of this chapter is used to make this notion precise and cite the relevant results.

1.1 Isotropic and anisotropic function spaces

In this chapter the function spaces relevant to the study of elliptic and parabolic partial differential equations are introduced. In the following (M, g) will always be a compact Riemannian manifold of dimension n, E a rank k (real or complex) vector bundle with fiber metric h and $\nabla : \Gamma(E) \to \Omega^1(E)$ a metric connection.

1.1.1 Metrics and connections on tensor bundles

Let E_1, E_2 be vector bundles with fiber metrics h_1 and h_2. On $E_1 \oplus E_2$ there is the natural fiber metric $h_1 \oplus h_2$, acting on sections $r = (r_1, r_2), s = (s_1, s_2) \in \Gamma(E_1 \oplus E_2)$ via

$$(r, s)_{h_1 \oplus h_2} = (r_1, s_1)_{h_1} + (r_2, s_2)_{h_2}.$$

Likeweise on $E_1 \otimes E_2$ there is a natural fiber metric acting on sections $r = r_1 \otimes r_2, s = s_1 \otimes s_2 \in \Gamma(E_1 \otimes E_2)$ via

$$(r, s)_{h_1 \otimes h_2} = (r_1, s_1)_{h_1}(r_2, s_2)_{h_2}.$$

These constructions clearly generalize to multiple sums or products. For the exterior bundle ΛE associated to E a fiber metric is defined by

$$(r, s)_{\Lambda h} = 0 \text{ if } p \neq q$$

$$(r, s)_{\Lambda h} = (r_1, s_1)_h \dots (r_p, s_p)_h$$

for $r = r_1 \wedge ... \wedge r_p \in \Gamma(\Lambda^p E), s = s_1 \wedge \cdots \wedge s_q \in \Gamma(\Lambda^q E)$. Analogously for the symmetric product $\odot^2 E$

$$(r, s)_{\odot^2 h} = (r_1, s_1)_h (r_2, s_2)_h$$

defines a fiber metric for sections $r = r_1 \odot r_2, s = s_1 \odot s_2 \in \Gamma(\odot^2 E)$.

Given a metric connection ∇, the higher derivatives

$$\nabla^k : \Gamma(E) \to \Gamma((T^*M)^{\otimes k} \otimes E)$$

are defined iteratively by

$$\nabla^0 s = s$$

$$\nabla^1 s = \nabla s$$

$$\nabla^k s = \nabla^{(T*M)^{\otimes k} \otimes E} \nabla^{k-1} s.$$

1.1.2 Hölder spaces

The space of continuous functions on M is denoted by $C^0(M)$. The norm

$$\|f\|_{C^0} = \sup_{x \in M} |f(x)|$$

turns C^0 into a Banach space. Given $\alpha \in (0, 1)$, a function $f : M \to \mathbb{R}$ is called α-Hölder continuous, if

$$[f]_\alpha = \sup_{\substack{x,y \in M, \\ x \neq y}} \frac{|f(x) - f(y)|}{d(x, y)^\alpha} < \infty.$$

The number $[f]_\alpha$ is called the Hölder coefficient of f. The space of Hölder continuous functions is then

$$C^\alpha(M) = \{f : M \to \mathbb{R} | [f]_\alpha < \infty\}.$$

Equipped with the norm

$$\|f\|_{C^\alpha} = \|f\|_{C^0} + [f]_\alpha,$$

the space C^α is a Banach space. Clearly, there exist analogous definitions for $\mathbb{R}^n$ or vector space valued functions. The space of continuous sections of E is defined as

$$C^0(E) = \{s : M \to E : s \text{ is a continuous section of } E\}$$

with norm

$$\|s\|_{C^0} = \sup_{x \in M} |s(x)|_h.$$

Slightly more care is needed when introducing the notion Hölder continuous sections of vector bundles. For this we need to choose a covering (U_i, φ_i) of Hermitian trivializations of E. Given $s \in \Gamma(E)$, $\varphi_i \circ s|_{U_i}$ is a $\mathbb{R}^n$ or $\mathbb{C}^n$ valued function and

$$[s]_\alpha = \max_{i \in I} [\varphi_i \circ s|_{U_i}]_\alpha$$

defines the Hölder coefficient of s. With this definition the Hölder spaces of sections $C^{0,\alpha}(E)$ is defined as above. This definition depends on the covering (U_i, φ_i). However since M is compact, it can be shown that different choices lead to equivalent norms.

The space of k times continuously differentiable functions $C^k(M)$ is equipped with the norm

$$\|f\|_{C^k} = \sum_{j=0}^{k} \|\nabla^j s\|_{C^0}.$$

This is a Banach space. Analogously the space of k times α-Hölder differentiable functions

$$C^{k,\alpha} = \{f \in C^k(M) : \nabla^k f \text{ is } \alpha\text{-Hölder continuous}\}$$

is a Banach space with the norm

$$\|f\|_{C^{k,\alpha}} = \|f\|_{C^k} + [\nabla^k f]_\alpha.$$

The spaces $C^k(E)$ and $C^{k,\alpha}(E)$ of sections and their norms are defined similarly.

For parabolic equations, defining a slightly different family of spaces is convenient. Given $\alpha \in (0,1)$ the parabolic Hölder space is the space

$$C^{\alpha,\alpha/2}(M \times [T_1, T_2]) = \{f \in C^0(M \times [T_1, T_2] : [f]_{\alpha,\alpha/2} < \infty\}$$

where

$$[f]_{\alpha,\alpha/2} = \sup_{\substack{(x_1,t_1),(x_2,t_2) \in M \times [T_1,T_2] \\ (x_1,t_1) \neq (x_2,t_2)}} \frac{|f(x_1,t_1) - f(x_2,t_2)|}{d((x_1,t_1),(x_2,t_2))^\alpha}.$$

Here d denotes the *parabolic distance*

$$d((x_1,t_1),(x_2,t_2)) = \left(d^g(x_1,x_2)^2 + |t_1 - t_2|)\right)^{1/2}.$$

With the norm

$$\|f\|_{C^{\alpha,\alpha/2}} = \|f\|_{C^0} + [f]_{\alpha,\alpha/2}$$

the space $C^{\alpha,\alpha/2}$ becomes a Banach space. The parabolic Hölder spaces of sections $C^{\alpha,\alpha/2}(E \times [T_1, T_2])$ are defined analogously. If the data of a parabolic equation is $C^{\alpha,\alpha/2}$, then the solutions will lie in the space

$$C^{2+\alpha,1+\alpha/2}(M \times [T_1, T_2]) = \{f \in C^0(M \times [T_1, T_2]) : f, \partial_t f, \nabla f, \nabla^2 f \in C^{\alpha,\alpha/2}\}.$$

The norm

$$\|f\|_{2+\alpha,1+\alpha/2} = \|f\|_{\alpha,\alpha/2} + \|\partial_t f\|_{\alpha,\alpha/2} + \|\nabla f\|_{\alpha,\alpha/2} + \|\nabla^2 f\|_{\alpha,\alpha/2}$$

turns $C^{2+\alpha,1+\alpha/2}$ into a Banach space. For higher regularity of solutions the following spaces are useful. If data and coefficients of parabolic equations lie in the space

$$C_k^{\alpha,\alpha/2}(M \times [T_1,T_2]) = \{f \in C^0(M \times [T_1,T_2]) : \nabla^i f \in C^{\alpha,\alpha/2} \text{ for } 0 \le i \le k\}$$

with the norm

$$\|f\|_{C_k^{\alpha,\alpha/2}} = \sum_{i=0}^{k} \|\nabla^i f\|_{C^{\alpha,\alpha/2}},$$

we expect solutions to lie in the space

$$C_k^{2+\alpha,1+\alpha/2}(M \times [T_1,T_2]) = \{f \in C^0(M \times [T_1,T_2]) : \nabla^i f \in C^{2+\alpha,1+\alpha/2} \text{ for } 0 \le i \le k\}$$

with the norm

$$\|f\|_{C_k^{2+\alpha,1+\alpha/2}} = \sum_{i=0}^{k} \|\nabla^i f\|_{C^{2+\alpha,1+\alpha/2}}.$$

1.1.3 Sobolev spaces

Like the $C^{k,\alpha}$ spaces the Sobolev spaces $W^{s,p}$ are a family of Banach spaces whose members have certain differentiability properties dependent on s and p. In contrast to the $C^{k,\alpha}$ spaces, the Sobolev spaces do not force differentiability at every point.

The space $L^p(E)$ is the completion of $\Gamma(E)$ with respect to the norm

$$\|s\|_{L^p} = \left(\int_M |s|^p \operatorname{vol}_g\right)^{1/p}.$$

It turns out that L^2 is a Hilbert space with the inner product

$$(s_1, s_2)_{L^2} = \int_M h(s_1, s_2) \operatorname{vol}_g .$$

The Sobolev spaces $W^{k,p}(M)$ with $k \in \mathbb{N}$ respectively $W^{k,p}(E)$ are the completions of $\mathcal{C}^\infty(M)$ respectively $\Gamma(E)$ with respect to the norms

$$\|f\|_{W^{k,p}} = \sum_{j=0}^{k} \|\nabla^k f\|_{L^p}.$$

In the special case $p = 2$ we instead choose the norm

$$\|f\|_{W^{k,2}}^2 = \sum_{j=0}^{k} \|\nabla^k f\|_{L^2}^2.$$

With this choice $H^k = W^{k,2}$ is a Hilbert space with the inner product

$$(f_1, f_2)_{H^k} = \sum_{j=0}^{k} (\nabla^k f_1, \nabla^k f_2)_{L^2}.$$

The definition can be extended to negative integers as follows. Let $k \in \mathbb{Z}$, $k < 0$. We restrict to the case $p = 2$. Then define the following norm for $f \in \mathcal{C}^\infty(M)$:

$$\|f\|_{W^{k,2}} = \sup_{g \in W^{-k,2} \setminus \{0\}} \frac{(f, g)_{L^2}}{\|g\|_{W^{-k,2}}}.$$

As in the case of Hölder spaces, working with parabolic equations requires slightly different function spaces. Given an interval $[T_1, T_2]$ denote by $E_{[T_1,T_2]}$ the pullback bundle $\pi^*_{[T_1,T_2]} E$, where $\pi_{[T_1,T_2]} : M \times [T_1, T_2] \to M$. For $p \in [1, \infty]$, the space $W^{2,1}_p(E_{[T_1,T_2]})$ is the completion of $\Gamma(E_{[T_1,T_2]})$ with respect to the norm

$$\|s\|_{W^{2,1}_p} = \|s\|_{L^p} + \|\partial_t s\|_{L^p} + \|\nabla s\|_{L^p} + \|\nabla^2 s\|_{L^p}.$$

Analogously to the $W^{k,2}$ spaces, a slight adaptation of the definition for $p = 2$ turns $W^{2,1}_2$ into a Hilbert space.

1.1.4 Sobolev embedding theorems

The Sobolev embedding theorems are two sets of theorems. The first type allows to pass from weak differentiability to integrability in a higher L^p norm. This is a highly useful property, especially in the study of nonlinear partial differential equations. The other type of theorem allows us to pass from weak differentiability to classical Hölder continuity or differentiability. This enables us to study partial differential equations in Sobolev spaces and then to draw conclusions about classical solutions of the equation. This is of fundamental importance to the subject.

In the following theorems all Sobolev spaces are understood to be defined on a compact n dimensional manifold (M, g).

Theorem 1.1.
Let $k, l \in \mathbb{N}$, $p, q \in [1, \infty)$. There is a continuous inclusion

$$W^{k,p}(M) \hookrightarrow W^{l,q}(M) \qquad \text{if} \qquad \frac{1}{q} \geq \frac{1}{p} - \frac{k-l}{n},$$

i.e. there exists a $C > 0$, such that

$$\|f\|_{W^{l,q}} \leq C \|f\|_{W^{k,p}}.$$

Theorem 1.2.
Let $k \in \mathbb{N}$, $p \in [1, \infty)$, $l \in \mathbb{N}_0, \alpha \in (0,1)$. There are continuous inclusions

$$W^{k,p}(M) \hookrightarrow C^l(M) \qquad \text{if} \qquad k - \frac{n}{p} > l$$

and

$$W^{k,p}(M) \hookrightarrow C^{l,\alpha}(M) \qquad \text{if} \qquad k - \frac{n}{p} \geq l + \alpha.$$

Furthermore, the following inequality is known as the Moser–Trudinger inequality and can be considered as a critical case of the Sobolev embedding.

Theorem 1.3.
There exists $\alpha, C > 0$ such that

$$\int_M \exp\left(\alpha \left(\frac{u}{\|Du\|_{L^n}}\right)^{\frac{n}{n-1}}\right) \mathrm{vol}_g \leq C$$

for all $u \in W^{1,n}(M)\backslash\{0\}$ with $\int_M u \,\mathrm{vol}_g = 0$.

The following inequality is a simple corollary of the Sobolev embedding theorem.

Corollary 1.4.
If $n = \dim M = 2$

$$\|u\|_{L^4}^4 \leq C\|u\|_{L^2}^2 \|u\|_{H^1}^2$$

for all $u \in H^1(M)$.

Proof. In two dimensions, the Sobolev embedding theorem yields the embedding

$$W^{1,1} \hookrightarrow L^2.$$

Let $u \in H^1$. Using this embedding and the Hölder inequality we calculate

$$\begin{aligned}
\|u\|_{L^4}^2 = \|u^2\|_{L^2} &\leq C\|u^2\|_{W^{1,1}} \\
&\leq C(\|u^2\|_{L^1} + \|\nabla(u^2)\|_{L^1}) \\
&\leq C(\|u\|_{L^2}^2 + \|2u\nabla u\|_{L^1}) \\
&\leq C(\|u\|_{L^2}^2 + 2\|u\|_{L^2}\|\nabla u\|_{L^2}) \\
&\leq C(\|u\|_{L^2}\|u\|_{H^1})
\end{aligned}$$

□

There are also Sobolev embedding theorems for the anisotropic Sobolev spaces.

Theorem 1.5.
Let n be the dimension of M. If $1 \leq p < \frac{n+2}{2}$, then

$$W_p^{2,1}(M \times [T_1, T_2]) \hookrightarrow L^q(M \times [T_1, T_2])$$

with

$$q = \frac{(n+2)p}{n+2-2p}.$$

If $p > \frac{n+2}{2}$, *then*

$$W_p^{2,1}(M \times [T_1, T_2]) \hookrightarrow C^{\alpha,\alpha/2}(M \times [T_1, T_2])$$

with

$$\alpha = 2 - \frac{n+2}{p}.$$

If $p > n+2$, *then*

$$W_p^{2,1}(M \times [T_1, T_2]) \hookrightarrow C_1^{\beta,\beta/2}(M \times [T_1, T_2])$$

with $\beta = 1 - \frac{n+2}{p}$.

See [29], Lemma II.4.3.

1.1.5 A multiplication theorem

In Hölder spaces the question when pointwise multiplication of functions is a continuous map between Banach spaces has a fairly obvious answer. The situation is much more complicated in Sobolev spaces. Indeed, it is often not even clear whether multiplication of two members of a Sobolev space is well defined. In this thesis we will only need the following result.

Theorem 1.6.
Suppose M *is a compact manifold,* $n = \dim M$*. Let* $s \in \mathbb{R}$*,* $k \in \mathbb{R}$*, such that*

$$s < 0$$

and

$$k > \frac{n}{2} \text{ and } k > |s|.$$

Then the multiplication map

$$\mu : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$$
$$(f, g) \mapsto fg$$

extends to a continuous map

$$\mu : H^s(M) \times H^k(M) \to H^s(M).$$

This is a special case of theorem 2 in section 4.4.3 in [34]. Indeed, there it is shown that multiplication extends continuously to a map

$$F^{s_1}_{p_1,q_1} \times F^{s_2}_{p_2,q_2} \to F^{s_1}_{p,q},$$

if $s_1 < 0 < s_2$, $s_1+s_2 > 0$, $\frac{1}{p} \leq \frac{1}{p_1}+\frac{1}{p_2}$, $s_1+s_2 > \frac{n}{p_1}+\frac{n}{p_2}-n$, $q \geq q_1$ and $\frac{1}{p} > \frac{1}{p_1} + \frac{1}{n}\max\{\frac{n}{p_2} - s_2, 0\}$. The space $F^s_{p,q}$ are called Triebel–Lizorkin spaces. For $p = q = 2$, these spaces are the Sobolev spaces H^s. Checking all the conditions in this case with $s = s_1$ and $k = s_2$ then yields the theorem.

1.2 Linear elliptic and parabolic systems

1.2.1 Regularity theory on domains

In this section we review the regularity theory for elliptic and parabolic systems. We state the theorems for bounded domains, because there the notion of regularity of the coefficients is unambiguous. The results can then be transferred to manifolds by covering arguments.

Definition 1.7. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and suppose its boundary is a smooth submanifold. Let $A_{ij} : \Omega \to \mathbb{R}^{k\times k}$, $B_i : \Omega \to \mathbb{R}^{k\times k}$, $C : \Omega \to \mathbb{R}^{k\times k}$ be functions. Suppose there exist *ellipticity constants* $\lambda, \Lambda > 0$ such that

$$\lambda|\xi|^2|v|^2 \leq A_{ij}(x)^{\alpha\beta}\xi^i\xi^j v_\alpha v_\beta \leq \Lambda|\xi|^2|v|^2$$

holds for all $x \in \Omega$, $\xi \in \mathbb{R}^n, v \in \mathbb{R}^k$. In this case we call the operator L on $\mathbb{R}^k$ valued functions defined by

$$Lf(x) = \sum_{ij} A_{ij}(x)\partial_i\partial_j f(x) + \sum_i B_i(x)\partial_i f(x) + C(x)f(x)$$

strongly elliptic (in the sense of Legendre–Hadamard).

The first theorem in this section concerns the regularity of solutions in L^2 Sobolev spaces.

Theorem 1.8 ([19], Thm. 2.3)**.**
Let $k \in \mathbb{N}$. Suppose L is an elliptic operator and the coefficients are C^k. Then there exists a constant $C > 0$ depending on the domain, the ellipticity constants and the C^k norms of the coefficients, such that

$$\|f\|_{W^{k+2,2}} \leq C\left(\|Lf\|_{W^{k,2}} + \|f\|_{W^{k,2}}\right).$$

If L is in divergence form, i.e.

$$Lf(x) = \sum_{ij} \partial_i(A_{ij}(x)\partial_j f(x)) + \sum_i B_i(x)\partial_i f(x) + C(x)f(x)$$

and

$$Lu = f + \sum_i \partial_i F_i,$$

then

$$\|u\|_{W^{k+1}} \leq C\left(\|u\|_{W^{k-1}} + \sum_i \|F_i\|_{W^k} + \|f\|_{W^{k-1}}\right).$$

There is also the following version for Sobolev spaces with negative Sobolev exponent.

Theorem 1.9 ([39], Thm. 1.2.A)**.**
Let $k \in \mathbb{N}$. Suppose L is an elliptic operator and the coefficients are H^k. Then there exists a constant $C > 0$ depending on the domain, the ellipticity constants and the C^k norms of the coefficients, such that

$$\|f\|_{H^{-k+2}} \leq C\left(\|Lf\|_{H^{-k}} + \|f\|_{H^{-k-1}}\right).$$

We will also need the following result for elliptic operators on Hölder spaces.

Theorem 1.10 ([19], Thm. 3.5,3.6).
Let $k \in \mathbb{N}$. Suppose L is an elliptic operator and the coefficients are $C^{k,\alpha}$. Then there exists a constant $C > 0$ depending on the domain, the ellipticity constants and the $C^{k,\alpha}$ norms of the coefficients, such that

$$\|f\|_{C^{k+2,\alpha}} \leq C\left(\|Lf\|_{C^{k,\alpha}} + \|f\|_{L^2}\right).$$

If L is in divergence form, i.e.

$$Lf(x) = \sum_{ij} \partial_i(A_{ij}(x)\partial_j f(x)) + \sum_i B_i(x)\partial_i f(x) + C(x)f(x)$$

and

$$Lu = \sum_i \partial_i F_i,$$

then

$$\|u\|_{C^{1,\alpha}} \leq C\left(\sum_i \|F_i\|_{C^\alpha} + \|u\|_{L^2}\right).$$

For L^p spaces the following result holds.

Theorem 1.11 ([32], Thm. 7.2, 7.3).
Suppose L is an elliptic operator and the coefficients are C^0. Then there exists a constant $C > 0$ depending on the domain, the ellipticity constants and the modulus of continuity of the coefficients, such that

$$\|D^2 f\|_{L^p} \leq C\left(\|Lf\|_{L^p} + \|D^2 f\|_{L^2}\right).$$

If $k = 0$ and L is in divergence form, i.e.

$$Lf(x) = \sum_{ij} \partial_i(A_{ij}(x)\partial_j f(x)) + \sum_i B_i(x)\partial_i f(x) + C(x)f(x)$$

and

$$Lu = f + \sum_i \partial_i F_i,$$

then

$$\|Du\|_{L^p} \leq C\left(\|f\|_{L^p} + \sum_i \|F_i\|_{L^{np/(n+p)}} + \|Du\|_{L^2}\right).$$

The next definition concerns parabolic operators.

Definition 1.12. Let $T_1, T_2 \in \mathbb{R}, T_2 > T_1$. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. Let $A_{ij} : \Omega \times [T_1, T_2] \to \mathbb{R}^{k\times k}$, $B_i : \Omega \times [T_1, T_2] \to \mathbb{R}^{k\times k}$, $C : \Omega \times [T_1, T_2] \to \mathbb{R}^{k\times k}$ be functions. Suppose there exist constants $\lambda, \Lambda > 0$ such that

$$\lambda|\xi|^2 \operatorname{id} \leq A_{ij}(x,t)\xi_i\xi_j \leq \Lambda|\xi|^2 \operatorname{id}$$

holds for all $(x,t) \in \Omega \times [T_1, T_2]$, $\xi \in \mathbb{R}^n$. In this case we call the operator P on $\mathbb{R}^k$ valued functions defined by

$$Pf(x,t) = \partial_t f(x,t) - \sum_{ij} A_{ij}(x,t)\partial_i\partial_j f(x,t) + \sum_i B_i(x,t)\partial_i f(x,t) + C(x,t)f(x,t)$$

strongly parabolic (in the sense of Legendre–Hadamard).

For parabolic operators we obtain the following basic regularity result.

Theorem 1.13.
Suppose $\Omega \subset \mathbb{R}^n$ is a domain and $\tilde{\Omega} \subset\subset \Omega$ is a compactly contained subdomain, $T \in (0, \infty)$. Suppose P is a parabolic operator defined on $\Omega \times [0,T]$. Suppose the coefficients are C^k uniformly in time. Then there exists a constant C depending on the domains Ω and $\tilde{\Omega}$, the constants λ, Λ and the C^k norm of the coefficients, such that for any $s \in 2\mathbb{Z}$ with $|s| \leq k$ the following equation holds

$$\begin{aligned}\|\partial_t f\|_{L^2([0,T],H^s(\tilde{\Omega}))} &+ \|f\|_{L^2([0,T],H^{s+2}(\tilde{\Omega}))} + \|f\|_{L^\infty([0,T],H^{s+1}(\tilde{\Omega}))} \\ &\leq C\left(\|Pf\|_{L^2([0,T],H^s(\Omega))} + \|f\|_{L^2([0,T],H^s(\Omega))} + \|f_0\|_{H^{s+1}(\Omega)}\right).\end{aligned}$$

We will need the following lemma in the proof of this theorem.

Lemma 1.14.
Suppose $k \in \mathbb{Z}$ and that L is a differential operator of order r on a domain $\Omega \subset \mathbb{R}^n$ with smooth boundary with coefficients in $C^{2|k|}(\bar{\Omega})$. Then for any $s \in \mathbb{R}$

$$[L, \Lambda_{2k}] : H^{2k+r-1+s}(\Omega) \to H^s(\Omega)$$

is continuous, where $\Lambda_{2k} = (\operatorname{id} + \Delta)^k$.

Proof. We can extend L to an operator on $\mathbb{R}^n$ with coefficients in $C^{2|k|}$. Let $\tilde{L}$ be such an extension to $\mathbb{R}^n$. Since Ω has smooth boundary, there exist continuous extension and restriction operators

$$E : H^s(\Omega) \to H^s(\mathbb{R}^n) \text{ and } R : H^s(\mathbb{R}^n) \to H^s(\Omega).$$

Thus it suffices to show that the commutator $[\tilde{L}, \Lambda_{2k}]$ is continuous, since $[L, \Lambda_{2k}] = R[\tilde{L}, \Lambda_{2k}]E$. Moreover, due to the following observation, the continuity of the commutator for negative k follows from the continuity of the commutator for positive k. Assume $k < 0$ and assume that we know the statement for positive k. In particular $[L,\Lambda_{-2k}] : H^{r-1} \to H^{2k}$ is continuous. The commutator $[L,\Lambda_{2k}]$ can be written as

$$\Lambda_{2k}[L,\Lambda_{-2k}]\Lambda_{2k}.$$

The operator

$$\Lambda_{2k} : H^{s+2k} \to H^s$$

is continuous for any k and any s. Hence

$$[L,\Lambda_{2k}] : H^{2k+r-1+s} \xrightarrow{\Lambda_{2k}} H^{r-1+s} \xrightarrow{[L,\Lambda_{-2k}]} H^{2k+s} \xrightarrow{\Lambda_{2k}} H^s$$

is continuous, as claimed.

The case of positive k remains to be shown. We may assume L consists only of highest order terms, i.e.

$$Lf = \sum_{|\alpha|=r} A_\alpha \partial^\alpha.$$

Since

$$\Lambda_{2k} = (\mathrm{id} + \Delta)^k = \sum_{i=0}^{k} \binom{k}{i} \Delta^i,$$

it suffices to show

$$[\Delta^k, L] : H^{2k+r-1+s} \to H^s$$

is continuous. Observe that

$$\Delta^k(A_\alpha \partial^\alpha) = A_\alpha \Delta^k \partial^\alpha + \sum_{j<2k} B_j,$$

where B_j is a differential operator of order $j+r$, whose coefficents contain $2k-j$ derivatives of A_α. Thus if $A_\alpha \in C^{2k}$, the commutator

$$[\Delta^k, A_\alpha \partial^\alpha] = \sum_{j<2k} B_j : H^{2k+r-1+s} \to H^s$$

is continuous. □

Proof. The standard estimate for parabolic operators is

$$\begin{aligned}\|\partial_t f\|_{L^2([0,T],L^2(\tilde\Omega))} + \|f\|_{L^2([0,T],H^2(\tilde\Omega))} + \|f\|_{L^\infty([0,T],H^1(\tilde\Omega))} \\ \leq C\left(\|Pf\|_{L^2([0,T],L^2(\Omega))} + \|f\|_{L^2([0,T],L^2(\Omega))} + \|f_0\|_{H^1(\Omega)}\right),\end{aligned}$$

see for example [18], p. 360. Denote by Λ_s the operator

$$\Lambda_s = (\mathrm{id} + \Delta)^{s/2} : H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^n).$$

This is an isomorphism. Indeed, we can define

$$(f,g)_{H^s} = (\Lambda_s f, \Lambda_s g)_{L^2}.$$

For the interior estimates it suffices to consider functions with compact support in Ω, and we have an embedding

$$H_0^s(\Omega) \hookrightarrow H^s(\mathbb{R}^n),$$

thus it does not matter whether we take the norm in $H^s(\Omega)$ or $H^s(\mathbb{R}^n)$. If we plug in $\Lambda_s f$ in the estimate above we obtain by $\|\Lambda_s f\|_{L^2} = \|f\|_{H^s}$

$$\begin{aligned}\|\partial_t f\|_{L^2([0,T],H^s)} &+ \|f\|_{L^2([0,T],H^{s+2})} + \|f\|_{L^\infty([0,T],H^{s+1})} \\ &\leq C\left(\|P\Lambda_s f\|_{L^2([0,T],L^2)} + \|f\|_{L^2([0,T],H^s)} + \|f_0\|_{H^{s+1}}\right).\end{aligned}$$

This is almost the desired inequality, except that we have the term $\|P\Lambda_s f\|_{L^2([0,T],L^2)}$ instead of $\|Pf\|_{L^2([0,T],H^s)}$. Denote by L_t the operator defined by

$$L_t f(x) = -\sum_{ij} A_{ij}(x,t)\partial_i\partial_j f(x) + \sum_i B_i(x,t)\partial_i f(x) + C(x,t)f(x).$$

Then we can write $Pf = \partial_t f + L_t f$. We compute at a fixed time t

$$\begin{aligned}\|P\Lambda_s f\|_{L^2} &= \|\partial_t \Lambda_s f + L_t \Lambda_s f\|_{L^2} \\ &= \|\Lambda_s\left(\partial_t f + L_t f\right) + [L_t, \Lambda_s] f\|_{L^2} \\ &\leq \|Pf\|_{H^s} + C\|f\|_{H^{s+1}},\end{aligned}$$

where the last inequality follows from previous lemma. For any $\epsilon > 0$ there exists a constant $C_\epsilon > 0$, such that

$$\|f\|_{H^{s+1}} \leq \epsilon\|f\|_{H^{s+2}} + C_\epsilon\|f\|_{H^s}.$$

The term $\epsilon\|f\|_{H^{s+2}}$ can be absorbed into the left hand side, if $\epsilon > 0$ is small enough, whereas the term $C_\epsilon\|f\|_{H^s}$ can be combined with the term of that form on the right hand side, and thus we get the desired inequality. □

Theorem 1.15 ([36]).
Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, $T_1 < T_2 \in \mathbb{R}$. Suppose P is a parabolic operator defined on $\Omega \times [T_1, T_2]$. Suppose the coefficients are $C^{\alpha,\alpha/2}$. Then there exists a constant C depending on the domain, the constants λ, Λ and the $C^{\alpha,\alpha/2}$ norm of the coefficients, such that

$$\|f\|_{C^{2+\alpha,1+\alpha/2}} \leq C\left(\|Pf\|_{C^{\alpha,\alpha/2}} + \|f\|_{C^{\alpha,\alpha/2}}\right)$$

Remark: Schlag proves these results under the stronger Legendre conditions on the coefficients. The proof rests on parabolic versions of Cacciopoli estimates for constant coefficient equations. These do not actually require Legendre conditions, as can be seen in the elliptic case in [32], Theorem 4.4. There is also the following higher regularity version of this theorem.

Theorem 1.16 ([27], Cor. 8.12.2).
Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, $T_1 < T_2 \in \mathbb{R}$. Suppose P is a parabolic operator defined on $\Omega \times [T_1, T_2]$. Suppose the coefficients lie in $C_k^{\alpha,\alpha/2}$. Then there exists a constant C depending on the domain, the constants λ, Λ and the $C_k^{\alpha,\alpha/2}$ norm of the coefficients, such that

$$\|f\|_{C_k^{2+\alpha,1+\alpha/2}} \leq C \left(\|Pf\|_{C_k^{\alpha,\alpha/2}} + \|f\|_{C_k^{\alpha,\alpha/2}} \right)$$

Finally, we turn to estimates for data in L^p spaces, $2 \leq p < \infty$.

Theorem 1.17 ([36]).
Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, $T_1 < T_2 \in \mathbb{R}$. Suppose P is a parabolic operator defined on $\Omega \times [T_1, T_2]$. Suppose the coefficients are $C^{\alpha,\alpha/2}$. Then there exists a constant C depending on the domain, the constants λ, Λ and the $C^{\alpha,\alpha/2}$ norm of the coefficients, such that

$$\|f\|_{W_p^{2,1}(\Omega \times [T_1,T_2])} \leq C \left(\|Pf\|_{L^p(\Omega \times [T_1,T_2])} + \|f\|_{L^p(\Omega \times [T_1,T_2])} \right)$$

We will also need the following estimate for bounded measurable coefficients, which is specific to scalar parabolic equations.

Theorem 1.18 ([28], Thm. 4.3).
Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, $\tilde{\Omega} \subset\subset \Omega$ a compactly contained subdomain and $T_1 < T_2 \in \mathbb{R}$. Suppose P is a scalar parabolic operator, i.e. $k = 1$, and suppose that the coefficients are measurable and bounded. Then there exist constants $C, \alpha > 0$ depending on the domains Ω and $\tilde{\Omega}$, the constants λ, Λ and the L^∞ norm of the coefficients, such that if $f \in W_{n+1}^{2,1}$, then

$$\|f\|_{C^{\alpha,\alpha/2}(\Omega \times [T_1,T_2])} \leq C \left(\|f\|_{L^\infty(\Omega \times [T_1,T_2])} + \|Pf\|_{L^{n+1}(\Omega \times [T_1,T_2])} \right)$$

1.2.2 Elliptic and parabolic operators on manifolds

In this section we review elliptic operators on manifolds. Many textbooks treat this material, for instance [30], Chapter III.1–4. Let $U \subset \mathbb{R}^n$ be open and let $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ be a multi index. Denote by $|\alpha|$ the sum $\alpha_1 + ... + \alpha_n$. Then we define the operator

$$D^\alpha = \frac{1}{i^{|\alpha|}} \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 ... \partial^{\alpha_n} x_n}.$$

A *linear differential operator* L of order l on the trivial vector bundles $E_i = U \times \mathbb{C}^{r_i}$, $i = 1, 2$ is a $\mathbb{C}$-linear operator

$$L : \Gamma(E_1) \to \Gamma(E_2)$$

of the form

$$Ls = \sum_{|\alpha| \leq l} A_\alpha D^\alpha s,$$

where $s \in \Gamma(E_1)$ and $A_\alpha : U \to \mathrm{Hom}(\mathbb{C}^{r_1}, \mathbb{C}^{r_2})$ is a smooth map for every α with $|\alpha| \leq l$.

This definition can be generalized to the setting of smooth manifolds and vector bundles by covering the manifold with charts and trivializations of the vector bundles. Instead, we give an equivalent but invariant definition.

Let M be a manifold and let E_1, E_2 be complex vector bundles of rank r_1, r_2. For any $\mathbb{C}$-linear operator

$$L : \Gamma(E_1) \to \Gamma(E_2)$$

and any $f \in \mathcal{C}^\infty(M, \mathbb{C})$ define $\mathrm{ad}(f)L : \Gamma(E_1) \to \Gamma(E_2)$ by

$$\mathrm{ad}(f)Ls = fLs - L(fs).$$

Definition 1.19. A (linear) *differential operator* of *order* 0 is a $\mathcal{C}^\infty(M, \mathbb{C})$-linear operator $L : \Gamma(E_1) \to \Gamma(E_2)$.

A differential operator of *at most order* l is a $\mathbb{C}$-linear operator $L : \Gamma(E_1) \to \Gamma(E_2)$, such that $\mathrm{ad}(f)L$ is a differential operator of order $l-1$ for every $f \in \mathcal{C}^\infty(M, \mathbb{C})$.

A differential operator of at most order l, which is not of order $l-1$, is a differential operator of *order* l.

Notice that all definitions work equally well for real vector bundles.

The *symbol* of the operator reflects the highest order terms of a differential operator. In order to define it, we first need a few observations. The first observation is that

$$\mathrm{ad}(f)\,\mathrm{ad}(g)L = \mathrm{ad}(g)\,\mathrm{ad}(f)L$$

and hence the expression

$$\mathrm{ad}(f_1, ..., f_l)L = \mathrm{ad}(f_1) ... \mathrm{ad}(f_l)L$$

is independent of the order. The second observation is that the cotangent space T_x^*M at a point $x \in M$ is canonically isomorphic to I_x/I_x^2, where

$$I_x = \{f \in \mathcal{C}^\infty(M) : f(x) = 0\}.$$

The isomorphism is given by

$$I_x/I_x^2 \to T_x^*M$$
$$[f] \mapsto df(x).$$

Finally, one can show that for a differential operator $L : \Gamma(E_1) \to \Gamma(E_2)$ of order k

$$\mathrm{ad}(f_1, \ldots, f_k)L(x) = 0$$

if $f_i \in I_x^2$ for any $x \in T_x^*M$. Thus the map

$$\mathrm{ad}(\cdot, ..., \cdot)L : (I_x \otimes \mathbb{C})^l \to \mathrm{Hom}(E_1, E_2)_x$$

induces the map

$$\sigma(L)(x) : \odot^l(T_x^*M \otimes \mathbb{C}) \to \mathrm{Hom}(E_1, E_2)_x$$
$$df_1 \odot \ldots \odot df_l \mapsto \frac{1}{i^l l!}\,\mathrm{ad}(f_1, \ldots, f_l)L(x).$$

Since the symmetric power of order k on a vector space V can be identified with the space of homogeneous polynomials of order k on V, we can also view $\sigma(L)(x)$ as a homogeneous polynomial of order k on $T_x^*M \otimes V$ with values in $\mathrm{Hom}(E_1, E_2)_x$.

The map $\sigma(L)(x)$ is the *symbol* of L at $x \in M$. It can be shown that $\sigma(L)$ depends smoothly on x and hence defines a section

$$\sigma(L) \in \Gamma\left(\mathrm{Hom}(\odot^l T^*M \otimes \mathbb{C}, \mathrm{Hom}(E_1, E_2)\right).$$

The symbol of $D^\alpha : \mathcal{C}^\infty(M, C) \to \mathcal{C}^\infty(M, C)$, $|\alpha| = l$, can be computed as

$$\sigma(D^\alpha)(x)\xi = \xi_1^{\alpha_1} \cdot \ldots \cdot \xi_n^{\alpha_n}$$

for $\xi \in T_x^*U = \mathbb{R}^n$. Here we consider $\sigma(D^\alpha)(x)$ to be a homogeneous polynomial. This result is the reason for the normalization factor in the definition of the symbol. More generally, given a differential operator of the form

$$Ls = \sum_{|\alpha| \leq l} A_\alpha D^\alpha s,$$

its symbol is given by

$$\sigma(L)(x)\xi = \sum_{|\alpha| = l} A_\alpha(x)\xi_1^{\alpha_1} \cdot \ldots \cdot \xi_n^{\alpha_n}.$$

Definition 1.20. A differential operator L of order l is called *elliptic*, if

$$\sigma(L)(x)\xi \in \mathrm{Hom}(E_1, E_2)_x$$

is an isomorphism for every $x \in M$, $\xi \in T_x^*M \backslash \{0\}$.

Any differential operator L of order l induces a continuous map of Banach spaces

$$L : H^{s+l}(E_1) \to H^s(E_2)$$

for every $s \in \mathbb{R}$. This means

$$\|Lu\|_{H^s} \leq C\|s\|_{H^{s+l}}$$

for some $C > 0$ and every $u \in H^{s+l}(E_1)$. The next theorem is a generalization of the elliptic regularity theory in the previous section for elliptic operators on manifolds. This theorem has model character for us: the other regularity theorems can be generalized to manifolds in a similar manner.

Theorem 1.21.
Suppose $L : \Gamma(E_1) \to \Gamma(E_2)$ is an elliptic operator of order l. Then for any $s \in \mathbb{R}$ there exists a constant C_s, such that

$$\|u\|_{H^{s+l}} \leq C_s \left(\|u\|_{H^s} + \|Lu\|_{H^s}\right)$$

for every $u \in H^{s+l}(E_1)$.
If $\ker L = 0$, the inequality simplifies to

$$\|u\|_{H^{s+l}} \leq C_s \|Lu\|_{H^s}.$$

In particular, for an elliptic operator L of order l, the norm $\|\cdot\|_{H^l}$ is equivalent to $\|L\cdot\|_{L^2} + \|\cdot\|_{L^2}$ and for suitable $\lambda \in \mathbb{R}$

$$\lambda\iota + L : H^l \to L^2$$

is an isomorphism. We can define an equivalent norm and Hilbert space structure on H^l by

$$(u, v)_{H^l} = ((\lambda + L)u, (\lambda + L)v)_{L^2}.$$

We now turn to parabolic systems on manifolds. Let (M, g) be a closed Riemannian manifold and (E, h) be a Hermitian vector bundle.

Definition 1.22. A *parabolic operator* (of second order) on $E_{[T_1,T_2]}$ is an operator acting on $\Gamma(E)$ of the form

$$\partial_t + L_t,$$

where

$$L_t : \Gamma(E) \to \Gamma(E)$$

is a family of elliptic operators (of second order) depending smoothly on t, such that the symbol

$$\sigma(L_t)(x)\xi : E_x \to E_x$$

satisfies

$$\sigma(L_t)(x)\xi \geq \lambda |\xi|^2 \,\mathrm{id}_{E_x}$$

for some $\lambda \in (0, \infty)$.

Locally, a parabolic operator thus has the form

$$Ls = \sum_{|\alpha| \leq l} A_\alpha(x, t) D^\alpha s,$$

with $A_\alpha : U \times [T_1, T_2] \to \mathrm{Hom}(\mathbb{C}^{r_1}, \mathbb{C}^{r_2})$ a smooth map and $A_\alpha(x, t)$ positive definite with respect to the metrics h_1, h_2. In particular a second order parabolic operator is locally a parabolic operator on a domain in the sense of the previous section.

Many of the results for elliptic equations have analogues in the parabolic case. We state the analogues to the regularity results in the previous section.

Theorem 1.23 (Schauder estimate).
Suppose P is a parabolic operator. Then for every $\delta > 0$ there exists $C > 0$, such that

$$\|s\|_{C^{2+\alpha,1+\alpha}(E_{[T_1+\delta,T_2]})} \leq C\left(\|s\|_{C^0(E_{[T_1,T_2]})} + \|Ps\|_{C^\alpha(E_{[T_1,T_2]})}\right)$$

for every $s \in C^{2,1}(E_{[T_1,T_2]})$.

Theorem 1.24 (H^r estimates).
Suppose P is a parabolic operator. Then for every $r \in \mathbb{Z}$ there exists $C > 0$, such that

$$\|\partial_t s\|_{L^2([0,T],H^r)}+\|s\|_{L^2([0,T],H^{r+2})}+\|s\|_{L^\infty([0,T],H^{r+1})} \leq C\left(\|Ps\|_{L^2([0,T],H^r)} + \|s\|_{L^2([0,T],H^r)} + \|s_0\|_{H^{r+1}}\right).$$

Theorem 1.25.
Suppose P is a parabolic operator. Then for any $\delta > 0$ there exists $C > 0$, such that

$$\|s\|_{W^{2,1}_p(E_{[T_1+\delta,T_2]})} \leq C\left(\|Ps\|_{L^p(E_{[T_1,T_2]})} + \|s\|_{L^p(E_{[T_1,T_2]})}\right)$$

for every $s \in C^{2,1}(E_{[T_1,T_2]})$

Notice that in all three cases we have assumed smooth coefficents and we have made no assertion about the dependence of the constants on the manifold or the coefficients. In practice it will be necessary to do this, but for that we refer back to the theorems about parabolic systems on domains.

1.2.3 Time dependent solutions of elliptic equations

Suppose $P : C^{2,\alpha}(M) \to C^\alpha(M)$ is an invertible elliptic operator. Suppose furthermore that $f \in C^{\alpha,\alpha/2}(I \times M)$. Then let u_t be the solution of

$$Pu_t = f_t$$

for every $t \in I$. If Q is the inverse of P, then $u_t = Qf_t$. In particular $u_t \in C^{2,\alpha}(M)$. We claim that moreover

$$u \in C_2^{\beta,\beta/2},$$

i.e. the temporal Hölder continuity of f is also preserved, albeit with a potentially worse Hölder exponent. For this we make use of the following alternative characterization of the space $C^{\alpha,\alpha/2}$:

$$C^{\alpha,\alpha/2}(I \times M) = C^{\alpha/2}(I, C^0(M)) \cap C^0(I, C^\alpha(M)).$$

Similarly,

$$C_2^{\alpha,\alpha/2}(I \times M) = C^{\alpha/2}(I, C^2(M)) \cap C^0(I, C^{2,\alpha}(M)).$$

Notice that Q is a bounded linear operator. This immediately yields

$$u \in C^0(I, C^{2,\alpha}(M)).$$

The Hölder continuity in time requires slightly more work. The following inequality is proven in [20], Lemma 6.32. (The explicit constant can be found by following the proof.) For every $h \in C^{2,\alpha}$ and every $\epsilon > 0$

$$\|h\|_{C^2(M)} \leq \epsilon \sum_{|\gamma|\leq 2} [D^\gamma h]_\alpha + \frac{1}{\epsilon^{1+1/\alpha}} \|h\|_{C^0}.$$

In general, an inequality

$$f(a,b) \leq \epsilon a + \frac{1}{\epsilon^k} b$$

implies that

$$f(a,b) \leq a b^{1/(2k)} + b^{1/2},$$

by substituting $\epsilon = b^{1/(2k)}$. Applying this to the inequality above, we obtain

$$\|h\|_{C^2(M)} \leq \left(\sum_{|\gamma|\leq 2} [D^\gamma h]_\alpha \right) \|h\|_{C^0}^{1/(2\delta)} + \|h\|_{C^0}^{1/2}$$

with $\delta = 1 + 1/\alpha$. Since we already know $u \in C^0(I, C^{2,\alpha}(M))$, we conclude that

$$\sum_{|\gamma|\leq 2} [D^\gamma h]_\alpha$$

is uniformly bounded in time. In particular we get a uniform inequality

$$\|u_{t_1} - u_{t_2}\|_{C^2(M)} \leq C \|u_{t_1} - u_{t_2}\|_{C^0(M)}^{1/(2\delta)}.$$

Using the C^0 estimate

$$\|g\|_{C^0(M)} \leq C \|Pg\|_{C^0(M)},$$

which we will justify below, we obtain in particular

$$\|u_{t_1} - u_{t_2}\|_{C^0(M)} \leq C \|Pu_{t_1} - Pu_{t_2}\|_{C^0} = C \|f_{t_1} - f_{t_2}\|_{C^0}.$$

That f is in $C^{\alpha/2}(I, C^0(M))$ means that there exists $C > 0$, such that

$$\|f_{t_1} - f_{t_2}\|_{C^0(M)} \leq C |t_1 - t_2|^{\alpha/2}.$$

Combining all these inequalities, we obtain

$$\|u_{t_1} - u_{t_2}\|_{C^2(M)} \leq C |t_1 - t_2|^{\alpha/(4\delta)},$$

i.e.

$$u \in C^{\alpha/(4\delta)}(I, C^2(M)).$$

On the other hand, we already know $u \in C^0(I, C^{2,\alpha}(M)) \subset C^0(I, C^{2,\alpha/(2\delta)}(M))$. This implies

$$u \in C^{\alpha/(2\delta),\alpha/(4\delta)}(I \times M).$$

The C^0 estimate for invertible elliptic operators can be deduced from Sobolev theory as follows: on the one hand

$$\|g\|_{W^{2,p}} \lesssim \|Pg\|_{L^p} \lesssim \|Pg\|_{C^0},$$

on the other hand by Sobolev embedding

$$\|g\|_{C^0} \lesssim \|g\|_{W^{2,p}}$$

for appropiate choices of p. Such an approach also works for divergence form operators: if

$$u \mapsto \operatorname{div}(Au)$$

induces an invertible elliptic differential operator and

$$\operatorname{div}(Au) = \operatorname{div} f,$$

then

$$\|u\|_{W^{1,p}} \lesssim \|f\|_{L^p}.$$

Since $W^{1,p} \hookrightarrow C^\alpha$ for sufficiently large p, the same approach works.

1.3 Quasilinear parabolic systems

In this section we discuss quasilinear parabolic systems and existence of solutions for short time. This result is foundational to the study of the spinor flow, whose flow equations form a quasilinear parabolic system.

Definition 1.26. A quasilinear differential operator (of second order) is an operator

$$Q : \Gamma(E) \to \Gamma(E),$$

such that over a coordinate patch U it can be written as

$$Qs(x) = \sum_{i,j} A_{ij}(x, s(x), ds(x)) \frac{\partial^2}{\partial x_i \partial x_j} s + b(x, s(x), ds(x)),$$

where $A_{ij}, b : U \times E|_U \times T^*M \otimes E|_U \to \operatorname{End}(E|_U)$ are smooth maps.

Definition 1.27. A quasilinear differential parabolic equation is an equation of the form

$$\partial_t s_t + Q_t(s_t) = 0,$$

where $Q_t : \Gamma(E) \to \Gamma(E)$ is a quasilinear operator, whose differential

$$DQ_t(s) : \Gamma(E) \to \Gamma(E)$$

is elliptic for every $s \in \Gamma(E)$.

Then we have the following existence and uniqueness theorem.

Theorem 1.28.
Suppose

$$\partial_t s_t + Q_t(s_t) = 0$$

is a quasilinear differential parabolic equation and $s \in \Gamma(E)$. *Then there exists a unique maximal solution* $s_t \in \Gamma(E)$ *with* $s_0 = s$ *on some interval* $[0, T)$, $T > 0$.

For a reference, see for instance [39], chapter 7.

Chapter 2

Spin geometry and the spinorial energy functional

2.1 Principal bundles and connections

This section outlines the basic definitions and constructions in the theory of principal bundles, which are required for spin geometry. This is standard material. The purpose of this chapter is to remind readers and to fix notation. Details may be found in many books, for example [4], [37].

For a fibration $\pi : F \to X$, $x \in X, f \in F$ we denote the fibers by $F_x = \pi^{-1}(x)$, $F_f = \pi^{-1}(\pi(f))$. If a group G acts on a space X from the right, we denote the action of $g \in G$ on $x \in X$ by $x \cdot g$.

Let M be a manifold, G a Lie group. A *principal bundle* over M with structure group G is a fibration $P \to M$ with a right action of G on P, such that

1. The group action preserves fibers, i.e. if $p \in P$ and $g \in G$, then $p \cdot g \in P_p$.
2. The group action is free, i.e. if $g \in G$ and $g \neq e$, then $x \cdot g \neq x$.
3. The group action is transitive on fibers, i.e. if $p \in P$, then $G \cdot p = P_p$.
4. The base M can be covered by *local trivializations* of P. A local trivialization of P is an open set $U \subset M$ and a diffeomorphism

   $$\varphi : U \times G \to P_U,$$

   such that

   $$\varphi(x, g \cdot h) = \varphi(x, g) \cdot h$$

   for all $x \in U, g, h \in G$.

The central example of a principal bundle is the *frame bundle* of a rank k vector bundle $E \to M$. The frame bundle is given by the set

$$F(E) = \{(e_1, ..., e_n) \in E_x : x \in M, e_1, ..., e_n \text{ forms a basis of } E_x\}$$

with the obvious projection to M and a right action of $\mathrm{GL}(k)$ defined by

$$(e_1, ..., e_n) \cdot A = \left(\sum_{j=1}^{n} a_{j1} e_j, ..., \sum_{j=1}^{n} a_{jn} e_n \right).$$

For us the most interesting case is $E = TM$ and we write $F(M)$ instead of $F(TM)$. Additional structures on M often correspond to principal subbundles of the frame bundle $F(M)$. If M is oriented, we define the *oriented frame bundle*

$$F^+(M) = \{(e_1, ..., e_n) \in F(M) : x \in M, e_1, ..., e_n \text{ forms an oriented basis of } T_x M\}$$

together with the right action of $\mathrm{GL}^+(n)$ coming from the restriction of the $\mathrm{GL}(n)$ action on $F(M)$ to $\mathrm{GL}_+(n)$. If g is a Riemannian metric on M, the *oriented orthonormal frame bundle* of M is given by

$$F(M, g) = \{(e_1, ..., e_n) \in F(M) : e_1, ..., e_n \text{ forms an oriented, orthonormal basis of } T_x M\}$$

together with the group action of $\mathrm{SO}(n)$.

The principal bundle $F(M)$ arises from the vector bundle TM. In the other direction, given a principal G-bundle $P \to M$ and a linear representation $\rho : G \to \mathrm{Aut}(V)$, one can form an *associated vector bundle*

$$P \times_G V = (P \times V)/G,$$

where G acts from the right on $P \times V$ via

$$(p, v) \cdot g = (p \cdot g, \rho(g)^{-1} v).$$

Using the standard representation $\rho = \mathrm{id}$, the tangent bundle TM is isomorphic to the associated bundle $F(M) \times_{\mathrm{GL}(n)} \mathbb{R}^n$. The sections of the associated bundle $P \times_G V$ correspond to G-equivariant maps $P \to V$, i.e. a map $s : P \to V$ defines a section of $P \times_G V$, if

$$s(p \cdot g) = \rho(g)^{-1} s(p)$$

for all $p \in P, g \in G$.

Recall that the adjoint representation of G on $\mathrm{Aut}(\mathfrak{g})$ is defined by

$$\mathrm{Ad}_g : G \to \mathrm{Aut}(\mathfrak{g})$$

$$g \mapsto Dc_g(e),$$

where $c_g : G \to G$ is the the conjugation map $c_g(h) = ghg^{-1}$.

A *connection* on a principal G-bundle $P \to M$ is given by a 1-form $\omega \in \Omega^1(M, \mathfrak{g})$, such that

1. $R_g^* \omega = \mathrm{Ad}_{g^{-1}} \omega$ for all $g \in G$
2. $\omega \left(\frac{d}{dt}\big|_{t=0} p \cdot \exp(tv) \right) = v$ for all $p \in P, v \in \mathfrak{g}$.

A connection ω defines the space $H_p = \ker \omega_p \subset T_pP$ at every point $p \in P$. The space H_p is called the horizontal space at p and it is a complement to $V_p = \ker d\pi_p$, i.e. we have the splitting

$$T_pP = H_p \oplus V_p.$$

The map

$$d\pi(p)|_{H_p} : H_p \to T_{\pi(p)}M$$

is an isomorphism and we call its inverse $h_p : T_{\pi(p)}M \to H_p$. Given $v \in T_{\pi(p)}M$, $h_p(v)$ is called the horizontal lift of v. The map h_p is equivariant in p in the sense that

$$h_{p\cdot g} = dR_g \circ h_p.$$

A *covariant derivative* on a vector bundle E is a linear operator

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E),$$

satisfying the product rule

$$\nabla(fs) = df \otimes s + f\nabla s$$

for all $f \in \mathcal{C}^\infty(M)$ and $s \in \Gamma(E)$.

Given a principal G-bundle $P \to M$, a connection ω on P and a representation (V, ρ) of G, there is a natural construction of a covariant derivative on $P \times_G V$. Let $s \in \Gamma(P \times_G V)$ be a section. As mentioned above, s can be considered as an equivariant map $P \to V$. Thus we may define for $v \in T_xM$

$$(\nabla^\omega_v s)(p) = ds(p)h_p(v) \tag{2.1}$$

This defines a covariant derivative on $P \times_G V$.

Suppose now that $\sigma : U \subset M \to P|_U$ is a local section. For a given a section $s \in \Gamma(P \times_G V)$ there exists a function $f : U \to V$, such that

$$s(x) = [\sigma(x), f(x)] \text{ for every } x \in U.$$

Denoting by ρ_* the induced representation

$$\rho_* : \mathfrak{g} \to \mathrm{End}(V),$$

one can show that

$$\nabla^\omega_X s = [\sigma(x), Xf(x) + (\rho_*(\sigma^*\omega(X(x))))f(x)] \tag{2.2}$$

for a local vector field $X \in \Gamma(TM|_U)$.

Conversely, given a metric covariant derivative ∇ on TM, we can construct a connection on $F(M, g)$ as follows. Suppose $\sigma_\alpha = (e^\alpha_1, ..., e^\alpha_n) : U_\alpha \to F(M, g)$ are local sections, where $U_\alpha \subset M$ is a covering of M. Now define $\omega_i \in \Omega^1(U_i, \mathfrak{o}(n))$ via

$$\omega_i(X) = \sum_{i<j} g(\nabla_X e^\alpha_i, e^\alpha_j)E_{ij} \tag{2.3}$$

where $E_{ij} \in \mathfrak{o}(n)$ is the matrix, which is given by

$$(E_{ij})_{ij} = 1$$

$$(E_{ij})_{ji} = -1$$

and

$$(E_{ij})_{kl} = 0$$

in all other cases. One can check that the 1-forms $(s^{-1})^*\omega_i \in \Omega^1(P|_{U_i}, \mathfrak{o}(n))$ coincide on all intersections $U_i \cap U_j$ and hence define a global connection 1-form $\omega \in \Omega^1(P, \mathfrak{o}(n))$. Moreover, from formula 2.2 it follows that the covariant derivative induced by ω is the original covariant derivative ∇. In other words, we have inverted the above construction in this special case.

2.2 The spin group and its representations

The material in this and the next section can be found in many sources, for example [30]. Let V be a real or a complex vector space and let q be a symmetric, bilinear form on V. Denote by $\mathcal{T}V$ the tensor algebra of V and define the two-sided ideal

$$\mathcal{I}(V,q) = \{\alpha \otimes (v \otimes v + q(v,v)1) \otimes \beta : \alpha, \beta \in \mathcal{T}V\}.$$

The quotient algebra

$$\mathrm{Cl}(V,q) = \mathcal{T}V/\mathcal{I}(V,q)$$

is called the *Clifford algebra* of (V,q). The embedding of $V \hookrightarrow \mathcal{T}V$ descends to an embedding $\iota : V \hookrightarrow \mathrm{Cl}(V,q)$. Denoting multiplication in $\mathrm{Cl}(V,q)$ by $\cdot$, we have the important relationships

$$v \cdot v = -q(v,v)1$$

and by polarization

$$v \cdot w + w \cdot v = -2q(v,w)1.$$

Given a basis $e_1, ..., e_n$ of V, the set

$$\{1 \cdot e_{i_1} \cdot ... \cdot e_{i_k} : 0 \le k \le n, 1 \le i_1 < ... < i_k \le n\}$$

forms a vector space basis of $\mathrm{Cl}(V,q)$. Moreover, there is a canonical vector space isomorphism between the exterior algebra and the Clifford algebra on V

$$\Lambda^* V \to \mathrm{Cl}(V,q)$$

$$e_{i_1} \wedge ... \wedge e_{i_k} \mapsto e_{i_1} \cdot ... \cdot e_{i_k}.$$

This is never an algebra isomorphism, unless $q = 0$. Another useful map is the extension of $-\mathrm{id} : V \to V$ to a map $\alpha : \mathrm{Cl}(V,q) \to \mathrm{Cl}(V,q)$. The map α is an involution, i. e. $\alpha^2 = \mathrm{id}$. This involution induces a decomposition of $\mathrm{Cl}(V,q)$ into eigenspaces as follows:

$$\mathrm{Cl}^0(V,q) = \{x \in \mathrm{Cl}(V,q) : \alpha(x) = x\},$$

$$\mathrm{Cl}^1(V,q) = \{x \in \mathrm{Cl}(V,q) : \alpha(x) = -x\}.$$

Considering $0, 1$ as elements in $\mathbb{Z}_2$ we have

$$\mathrm{Cl}^i(V,q) \cdot \mathrm{Cl}^j(V,q) \subset \mathrm{Cl}^{i+j}(V,q),$$

i.e. $\mathrm{Cl}(V,q)$ is a $\mathbb{Z}_2$ graded algebra. We define

$$\mathrm{Cl}_n = \mathrm{Cl}(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$$

and

$$\mathbb{C}\mathrm{l}_n = \mathrm{Cl}(\mathbb{C}^n, \langle\cdot,\cdot\rangle)$$

where $\langle\cdot,\cdot\rangle$ denote the standard symmetric, bilinear forms on $\mathbb{R}^n$ and $\mathbb{C}^n$ respectively. We can also consider Cl_n and $\mathbb{C}\mathrm{l}_n$ to be the algebras generated by the relations

$$e_i \cdot e_j + e_j \cdot e_i = -2\delta_{ij}1 \text{ for } 1 \le i,j \le n$$

over $\mathbb{R}$ or $\mathbb{C}$ respectively.

The following algebra isomorphisms

$$\mathbb{C}\mathrm{l}_n \cong \mathrm{Cl}_n \otimes \mathbb{C}$$

and

$$\mathbb{C}\mathrm{l}_{n+2} \cong \mathbb{C}\mathrm{l}_n \otimes \mathbb{C}\mathrm{l}_2$$

together with the calculations of the first and second real Clifford algebras

$$\mathrm{Cl}_1 \cong \mathbb{C} \text{ and } \mathrm{Cl}_2 \cong \mathbb{H}$$

yield a classification of the complex Clifford algebras. Indeed, we conclude

$$\mathbb{C}\mathrm{l}_1 \cong \mathbb{C} \oplus \mathbb{C} \text{ and } \mathbb{C}\mathrm{l}_2 \cong \mathrm{End}(\mathbb{C}^2).$$

and by induction

$$\begin{aligned} \mathbb{C}\mathrm{l}_{2n+1} &\cong \mathrm{End}(\mathbb{C}^{2^n}) \oplus \mathrm{End}(\mathbb{C}^{2^n}), \\ \mathbb{C}\mathrm{l}_{2n+2} &\cong \mathrm{End}(\mathbb{C}^{2^{n+1}}). \end{aligned}$$

We can also rewrite this as

$$\begin{array}{rccl} \mathbb{C}\mathrm{l}_n \cong & \mathrm{End}(\Sigma_n) \oplus \mathrm{End}(\Sigma_n), & n \text{ odd}, \\ \mathbb{C}\mathrm{l}_n \cong & \mathrm{End}(\Sigma_n), & n \text{ even}, \end{array}$$

where

$$\Sigma_n = \mathbb{C}^{2^{[n/2]}}.$$

The vector space Σ_n is called the *spinor module.*

The group of invertible elements of Cl_n is denoted by Cl_n^*. The spin group is the subgroup $\mathrm{Spin}(n) \subset \mathrm{Cl}_n^*$ generated by the elements of the form

$$v_1 \cdot \ldots \cdot v_{2k} \in \mathbb{R}^n, k \in \mathbb{N}, |v_i| = 1 \text{ for } 1 \leq i \leq 2k.$$

For $g \in \mathrm{Cl}_n^*$ consider the map

$$\tilde{c}_g : \mathrm{Cl}_n \to \mathrm{Cl}_n$$
$$x \mapsto \alpha(g) \cdot x \cdot g^{-1}.$$

If $g \in \mathrm{Spin}(n)$, $\tilde{c}_g$ leaves $\mathbb{R}^n$ invariant, i.e. $\tilde{c}_g(\mathbb{R}^n) \subset \mathbb{R}^n \subset \mathrm{Cl}_n$. Thus $\tilde{c}_g$ can be restricted to an automorphism of $\mathbb{R}^n$. In fact, it can be shown that $\tilde{c}_g \in \mathrm{SO}(n)$. Hence we get a map

$$\kappa_n : \mathrm{Spin}(n) \to \mathrm{SO}(n)$$

$$g \mapsto \tilde{c}_g.$$

It turns out that κ_n is a non-trivial double covering map. Since $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$, κ_n is the universal covering of $\mathrm{SO}(n)$.

The Lie algebra of Cl_n^* is Cl_n with the Lie bracket given by

$$[\varphi, \psi] = \varphi \cdot \psi - \psi \cdot \varphi.$$

The Lie algebra $\mathfrak{spin}(n)$ of $\mathrm{Spin}(n)$ is spanned by the elements $e_i \cdot e_j$, $1 \leq i < j \leq n$. The map κ_n induces an isomorphism of Lie algebras

$$\kappa_{n*} : \mathfrak{spin}(n) \to \mathfrak{so}(n).$$

The Lie algebra $\mathfrak{so}(n)$ is the subalgebra of antisymmetric matrices in the matrix Lie algebra $\mathbb{R}^{n \times n}$. It is spanned by the matrices E_{ij} introduced in the previous section. The action of κ_n is then given by

$$\kappa_{n*}(e_i \cdot e_j) = 2E_{ij}. \tag{2.4}$$

More invariantly, we may write

$$\kappa_{n*}([u, v])w = 4(\langle u, w \rangle v - \langle u, v \rangle w) \tag{2.5}$$

for $u, v, w \in \mathbb{R}^n$.

We will now study the complex representations of the spin group. It should be noticed that any representation of $\mathrm{SO}(n)$ gives rise to a representation of $\mathrm{Spin}(n)$. We will not consider these representations here, but only consider those representations which do not arise from representations of $\mathrm{SO}(n)$. These representations come from restrictions of representations of $\mathbb{C}\mathrm{l}_n$. Again, the situation depends on the parity of n. Indeed, we have the following facts: If n is even, $\mathbb{C}\mathrm{l}_n$ has a unique irreducible representation

$$\chi_n : \mathbb{C}\mathrm{l}_n \to \mathrm{End}(\Sigma_n),$$

which is given by the isomorphism which we constructed above. If n is odd, $\mathbb{C}\mathrm{l}_n$ has two inequivalent irreducible representations

$$\chi_n^{\pm} : \mathbb{C}\mathrm{l}_n \to \mathrm{End}(\Sigma_n).$$

These are given by taking the isomorphism $\chi_n : \mathbb{C}\mathrm{l}_n \to \mathrm{End}(\Sigma_n) \oplus \mathrm{End}(\Sigma_n)$ and projecting onto the first or second factor, i.e. $\chi_n^+ = \pi_1 \circ \chi_n$ and $\chi_n^- = \pi_2 \circ \chi_n$, where

$$\pi_i : \mathrm{End}(\Sigma_n) \oplus \mathrm{End}(\Sigma_n) \to \mathrm{End}(\Sigma_n)$$

$$(A_1, A_2) \mapsto A_i.$$

We now define the *standard spin representation* $\rho_n : \mathrm{Spin}(n) \to \mathrm{End}(\Sigma_n)$ via

$$\rho_n = \chi_n|_{\mathrm{Spin}(n)} \text{ if } n \text{ even}$$

and

$$\rho_n = \chi_n^+|_{\mathrm{Spin}(n)} \text{ if } n \text{ odd}.$$

We also define the following map

$$\mu : \mathbb{C}\mathrm{l}_n \otimes \Sigma_n \to \Sigma_n$$

$$x \otimes \varphi \mapsto \chi_n(x)\varphi =: x \cdot \varphi, \text{ if } n \text{ even}$$

$$x \otimes \varphi \mapsto \chi_n^+(x)\varphi =: x \cdot \varphi, \text{ if } n \text{ odd}$$

This map is called the *Clifford multiplication* of spinors.

Up to scaling, there is a unique $\mathrm{Spin}(n)$ invariant Hermitian product on Σ_n. Fix one such Hermitian product and call it $\langle \cdot, \cdot \rangle_{\mathbb{C}}$. The real part of this Hermitian product is a real inner product on Σ_n and we denote it by

$$\langle \cdot, \cdot \rangle = \mathrm{Re}\langle \cdot, \cdot \rangle_{\mathbb{C}}.$$

Notice that for $v \in \mathbb{R}^n$ it follows that

$$\langle v \cdot \varphi, \psi \rangle_{\mathbb{C}} = -\langle \varphi, v \cdot \psi \rangle_{\mathbb{C}}.$$

To study the Clifford multiplication on bundles, it will be useful to have a representation of $\mathrm{SO}(n)$ on the Clifford algebra

$$\mathrm{cl}_n : \mathrm{SO}(n) \to \mathrm{Aut}(\mathrm{Cl}_n).$$

This representation arises as follows. Let $A \in \mathrm{SO}(n)$. Then A induces an automorphism of the tensor algebra $\mathcal{T}A : \mathcal{T}\mathbb{R}^n \to \mathcal{T}\mathbb{R}^n$. It can be checked that this map descends to a map $\mathrm{cl}_n(A) : \mathrm{Cl}_n \to \mathrm{Cl}_n$. In the same manner, a representation of $\mathrm{SO}(n)$ on the complex Clifford algebra

$$\mathrm{cl}_n^{\mathbb{C}} : \mathrm{SO}(n) \to \mathrm{Aut}(\mathbb{C}\mathrm{l}_n)$$

can be defined.

2.3 Spin structures and the spinor bundle

Let (M, g) be an n dimensional, orientable Riemannian manifold. The oriented orthonormal frame bundle $F = F(M, g)$ is the $SO(n)$ principal bundle

$$\{(e_1, ..., e_n) : e_1, ..., e_n \text{ is an oriented, orthonormal basis of } T_xM, x \in M\}.$$

The bundle projection is denoted by $\pi_F : F \to M$ and the right action of $\mathrm{SO}(n)$ is denoted by $\lambda_F : F \times \mathrm{SO}(n) \to F$.

Definition 2.6 (Spin structure)**.** A *spin structure* on (M, g) is a principal $\mathrm{Spin}(n)$ bundle $P \xrightarrow{\pi_P} M$ and a 2:1 covering map $\pi : P \to F$, such that the following diagram commutes

$$\begin{tikzcd}
P \times \mathrm{Spin}(n) \arrow[r, "\lambda_P"] \arrow[d, "\pi\times\kappa"] & P \arrow[d, "\pi"] \arrow[dr, "\pi_P"] & \\
F \times \mathrm{SO}(n) \arrow[r, "\lambda_F"] & F \arrow[r, "\pi_F"] & M
\end{tikzcd}$$

where $\kappa_n : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is the standard 2:1 covering of the special orthogonal group.

In other words, a spin structure is a fiberwise lift of the structure group of the oriented orthonormal frame bundle from $\mathrm{SO}(n)$ to $\mathrm{Spin}(n)$, which is compatible with the standard covering $\mathrm{Spin}(n) \xrightarrow{\kappa_n} \mathrm{SO}(n)$.

Definition 2.7 (Isomorphism of spin structures)**.** An isomorphism of spin structures P and $\tilde{P}$ is an isomorphism of principal bundles $A : P \to \tilde{P}$, such that $\pi_{\tilde{P}} \circ A = \pi_P$.

Existence of spin structures and their classification is a topological problem and its solution is given by the following theorem.

Theorem 2.8.
An orientable manifold M admits a spin structure, if and only if its second Stiefel–Whitney class $w_2(M) \in H^2(M, \mathbb{Z}_2)$ vanishes. The set of isomorphism classes of spin structures is in bijective correspondence to $H^1(M, \mathbb{Z}_2)$.

A manifold which admits a spin structure is called a *spin manifold*. Note that any simply connected manifold with $w_2(M) = 0$ has a unique spin structure. In the context of special holonomy the following result is also useful.

Theorem 2.9.
If the structure group of $F^+(M, g)$ reduces to a simply connected subgroup of $\mathrm{SO}(n)$*, then M admits a spin structure P, such that the reduction lifts to P.*

Since the structure group of a Riemannian manifold can be reduced to a group isomorphic to the holonomy group, any manifold with a simply connected holonomy group admits a spin structure. With the exception of $U(n)$ and $\mathrm{SO}(n)$ all holonomy groups appearing in Berger's classification are simply connected. As we will see later, Ricci flat special holonomy metrics can be characterized completely in terms of spin geometry.

The tangent bundle and the exterior algebra bundles over M can be written as associated bundles to the frame bundle with respect to certain representations of $\mathrm{SO}(n)$. In analogy to these constructions we can now define the spinor bundle as an associated bundle of a spin structure with respect to the spin representation.

Definition 2.10 (Spinor bundle)**.** Let (M, g) be a spin manifold with spin structure P. The *spinor bundle* is the associated bundle

$$\Sigma_g M = P \times_{\rho_n} \Sigma_n,$$

where (Σ_n, ρ_n) is the spinor module. Explicitly,

$$\Sigma_g M = (P \times \Sigma_n)/\operatorname{Spin}(n)$$

where $\mathrm{Spin}(n)$ acts on $P \times \Sigma_n$ via

$$(p, \varphi) \cdot \sigma = (p \cdot \sigma, \rho_n(\sigma)^{-1}\varphi).$$

Suppose now that $U \subset M$ is an open set and $e_1, ..., e_n$ is an oriented, orthonormal frame of TM over U. Then $b = (e_1, ..., e_n)$ is a section of $F|_U$. Given a spin structure P and any point $x \in U$, the set $\pi^{-1}((e_1(x), ..., e_n(x)))$ consists of two elements. A choice of one of these elements determines a lift $\tilde{b}$ of b to $P|_U$. Thus b induces a local trivialization $\tilde{b}$ of the spin structure P over U.

Any local section of $\Sigma_g M$ over U can be described by $[\tilde{b}, \varphi]$, where φ is a smooth map $U \to \Sigma_n$. Another way to view sections of $\Sigma_g M$ is as smooth equivariant maps

$$\varphi : P \to \Sigma_n,$$

i.e. such that

$$\varphi(p \cdot \sigma) = \rho_n(\sigma)^{-1}\varphi(p) \text{ for all } p \in P, \sigma \in \operatorname{Spin}(n).$$

As we saw in the last section, the spinor module Σ_n carries an invariant Hermitian metric and the Clifford algebra acts via Clifford multiplication. Both structures can be transferred to the spinor bundle.

Definition 2.11 (Hermitian structure on the spinor bundle)**.** Let $\langle \cdot, \cdot \rangle$ be a $\mathrm{Spin}(n)$-invariant inner product on Σ_n. Then ρ_n is a unitary representation with respect to that structure and

$$\Sigma_g M = P \times_{\rho_n} (\Sigma_n, \langle \cdot, \cdot \rangle)$$

carries the structure of a Hermitian bundle. Explicitly, if $[p_1, \varphi_1], [p_2, \varphi_2] \in \Sigma_g M_x$, then

$$\langle [p_1, \varphi_1], [p_2, \varphi_2] \rangle = \langle \varphi_1, \rho_n(\sigma)^{-1}\varphi_2 \rangle,$$

where $\sigma \in \operatorname{Spin}(n)$ is such that $p_2 \cdot \sigma = p_1$.

To transfer the Clifford multiplication to $\Sigma_g M$, we first need to define the real Clifford bundle

$$\mathrm{Cl}(M) = P \times_{\mathrm{cl}_n \circ \kappa_n} \mathrm{Cl}_n \cong F(M,g) \times_{\mathrm{cl}_n} \mathrm{Cl}_n$$

and the complex Clifford bundle

$$\mathbb{C}\mathrm{l}(M) = P \times_{\mathrm{cl}_n^{\mathbb{C}} \circ \kappa_n} \mathbb{C}\mathrm{l}_n \cong F(M,g) \times_{\mathrm{cl}_n^{\mathbb{C}}} \mathbb{C}\mathrm{l}_n \, .$$

Clifford multiplication on $\Sigma_g M$ by elements of $\mathrm{Cl}(M)$ is then defined via

$$\mu : \mathrm{Cl}(M) \otimes \Sigma_g M \to \Sigma_g M$$
$$\mu([\tilde{b}, c], [\tilde{b}, \varphi]) = [\tilde{b}, c \cdot \varphi]$$

where $\tilde{b} \in P$, $c \in \mathrm{Cl}_n$, $\varphi \in \Sigma_n$. Clifford multiplication for $\mathbb{C}\mathrm{l}(M)$ is defined analogously. Elements of $\Lambda^* T^* M$ also act by Clifford multiplication on $\Sigma_g M$, by means of the canonical isomorphism

$$\Lambda^* T^* M \to \mathrm{Cl}(M).$$

Suppose the dimension of M is even, i.e. $\dim M = n = 2m$. Then we can define a global section $\omega_{\mathbb{C}}$ of $\mathbb{C}\mathrm{l}(M)$ via

$$\omega_{\mathbb{C}} = i^m e_1 \cdot \ldots \cdot e_n,$$

or alternatively as i^m times the image of the Riemannian volume form under the canonical isomorphism above. The section $\omega_{\mathbb{C}}$ is called the complex volume form. It can then be shown that

$$\omega_{\mathbb{C}}^2 = 1,$$

i.e. $\omega_{\mathbb{C}}$ acts as an involution on $\Sigma_g M$. Thus $\omega_{\mathbb{C}}$ induces a splitting of $\Sigma_g M$ into two subbundles:

$$\Sigma_g^+ M = \{\varphi \in \Sigma_g M : \omega_{\mathbb{C}} \cdot \varphi = \varphi\}$$
$$\Sigma_g^- M = \{\varphi \in \Sigma_g M : \omega_{\mathbb{C}} \cdot \varphi = -\varphi\}.$$

2.4 The spinor bundle on a hypersurface

Suppose N is an oriented codimension 1 submanifold of a spin manifold (M, g). Let $\iota : N \to M$ denote the inclusion and $g_N = \iota^* g$ the induced metric on N. Then the spin structure P on M induces a spin structure on N. Suppose $\nu : N \to \Gamma(\iota^* TM)$ is a normal vector field, such that for any oriented orthonornomal basis $e_1, \ldots, e_{n-1}$ of $T_x N$ the basis $e_1, \ldots, e_{n-1}, \nu(x)$ is also an oriented orthonormal basis of $T_x M$. We can define an inclusion

$$F^+(N, g_N) \to F^+(M, g)$$
$$(e_1, \ldots, e_{n-1}) \mapsto (e_1, \ldots, e_{n-1}, \nu).$$

Denote its image by F_N. Notice that this is a $\mathrm{SO}(n-1)$ subbundle of $F^+(M,g)|_N$. Then we define $P_N = \pi^{-1}(F_N) \subset P$, where π is the projection $\pi : P \to F^+(M, g)$. This is a $\mathrm{Spin}(n-1)$ bundle over N and by construction it covers F_N and hence $F^+(N, g_N)$. This is the induced spin structure. One can then show the following fact about the associated spinor bundle.

Proposition 2.12.
There is an identification

$$\Sigma_{g_N} N \cong \begin{cases} \iota^* \Sigma_g^+ M, & n \text{ even} \\ \iota^* \Sigma_g M, & n \text{ odd} \end{cases}$$

Furthermore, under this identification the Clifford multiplication can be expressed as

$$X \cdot \iota^* \varphi = \iota^*(X \cdot \nu \cdot \varphi)$$

for any $X \in TN$.

This result can be found in [8], section 2.4.1.

2.5 The spin connection and the Dirac operator

A fundamental fact of Riemannian geometry is the existence and uniqueness of a torsion-free metric connection on the tangent bundle, the *Levi–Civita connection*. This connection induces a connection on the spinor bundle, the *spin connection*. It turns out that there is another highly interesting natural first order differential operator on the spinor bundle, the *Dirac operator*. A third operator is the *twistor operator*, which encodes in some sense the difference between the spin connection and the Dirac operator. Beyond defining these operators, this section discusses the curvature of the spin connection and the very useful Lichnerowicz formula, which relates the square of the Dirac operator to the spin connection Laplacian and the scalar curvature of the underlying Riemannian manifold. Finally, we characterize Ricci flat special holonomy manifolds in terms of the existence of parallel spinor fields. This is standard material which can be found in many sources, such as [30] or [8].

2.5.1 The spin connection

The spin connection is the lift of the Levi–Civita connection to the spinor bundle. The Levi–Civita connection

$$\nabla^g : \Gamma(TM) \to \Gamma(T^*M \otimes TM)$$

is the unique metric, torsion free connection on (M, g). This induces a connection $\omega \in \Omega^1(F, \mathfrak{so}(n))$ on $F = F^+(M, g)$ via equation 2.3. Now let P be a spin structure and let $\pi : P \to F$ be the covering map. Then $\pi^*\omega$ is a 1 form on P with values in $\mathfrak{so}(n)$. As we have seen, the standard covering $\kappa_n : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ induces an isomorphism of Lie algebras

$$\kappa_{n*} : \mathfrak{spin}(n) \to \mathfrak{so}(n).$$

Denote by $\tilde{\omega} \in \Omega^1(P, \mathfrak{spin}(n))$ the connection form $\kappa_{n*}^{-1} \circ \pi^*\omega$.

Definition 2.13 (Spin connection)**.** On a manifold (M, g) with spin structure P the spin connection is the covariant derivative

$$\nabla^g : \Gamma(\Sigma_g M) \to \Gamma(T^*M \otimes \Sigma_g M)$$

induced by $\tilde{\omega}$ on the spinor bundle $\Sigma_g M = P \times_{\rho_n} \Sigma_n$.

(The Levi–Civita connection and the spin connection are both named ∇^g. It will always be clear from the context which connection is meant.)

The formulas 2.2 and 2.3 applied to this situation yield the following local description of the spin connection. Let $e_1, ..., e_n$ be an oriented orthonormal frame over an open set $U \subset M$. Let $\sigma : U \to P$ be a lift of the corresponding local section $(e_1, ..., e_n)$ of the frame bundle. Let $s \in \Gamma(\Sigma_g M|_U)$. Then there exists $\varphi : U \to \Sigma_n$, such that

$$s(x) = [\sigma(x), \varphi(x)] \text{ for every } x \in U.$$

To apply the formula 2.2, we first compute

$$\begin{aligned} \kappa_{n*}^{-1}(\sigma^*\omega(X)) &= \kappa_{n*}^{-1}\left(\sum_{i<j} g(\nabla^g_X e_i, e_j) E_{ij}\right) \\ &= \frac{1}{2}\sum_{i<j} g(\nabla^g_X e_i, e_j) e_i \cdot e_j. \end{aligned}$$

This yields the local formula

$$\nabla^g_X s = \left[\sigma, X\varphi + \frac{1}{2}\sum_{i<j} g(\nabla^g_X e_i, e_j) e_i \cdot e_j \cdot \varphi\right]. \tag{2.14}$$

It should be observed that the e_i appear in the formula in two distinct meanings: first as local sections of the tangent bundle and then as elements of $\mathbb{R}^n \subset \mathrm{Cl}_n$, acting on Σ_n.

Next we compute the curvature of the spin connection. Suppose $X, Y \in \Gamma(TM)$. Then the spinorial term in the local expression of $\nabla^g_X \nabla^g_Y s$ is given by

$$\begin{aligned} XY\varphi &+ \frac{1}{2}\sum_{i<j} Xg(\nabla^g_Y e_i, e_j) e_i \cdot e_j \cdot \varphi \\ &+ \frac{1}{2}\sum_{i<j} g(\nabla^g_Y e_i, e_j) e_i \cdot e_j \cdot X\varphi \\ &+ \frac{1}{2}\sum_{i<j} g(\nabla^g_X e_i, e_j) e_i \cdot e_j \cdot Y\varphi \\ &+ \frac{1}{4}\sum_{i<j}\sum_{k<l} g(\nabla^g_X e_i, e_j) g(\nabla^g_Y e_k, e_l) e_i \cdot e_j \cdot e_k \cdot e_l \varphi. \end{aligned}$$

Now consider $\nabla^g_X\nabla^g_Y s - \nabla^g_Y\nabla^g_X s$. The last three terms in the above calculation are symmetric in X and Y, hence they drop out of $\nabla^g_X\nabla^g_Y s - \nabla^g_Y\nabla^g_X s$. The same is true for the second term in

$$Xg(\nabla^g_Y e_i, e_j) = g(\nabla^g_X\nabla^g_Y e_i, e_j) + g(\nabla^g_Y e_i, \nabla^g_X e_j).$$

Assuming at a point that $[X,Y]=0$, the curvature of ∇^g is thus

$$R^g(X,Y)s = \left[\sigma, \frac{1}{2}\sum_{i<j} g(R^g(X,Y)e_i, e_j)e_i \cdot e_j \cdot \varphi\right] \tag{2.15}$$

For a more invariant description of the curvature, consider the canonical isomorphism

$$\Lambda^*\mathbb{R}^n \to \mathrm{Cl}_n\,.$$

This induces an isomorphism of vector bundles

$$\theta : \Lambda^* T^*M \to \mathrm{Cl}(M).$$

In particular considering the curvature of the Levi–Civita connection as the curvature operator

$$R : \Lambda^2 T^*M \to \Lambda^2 T^*M,$$

one can rewrite the above formula as

$$R^g(X,Y)s = \theta(R(X\wedge Y))\cdot s.$$

We will suppress the isomorphism θ from the notation and simply write

$$R^g(X,Y)s = R(X,Y)\cdot s \tag{2.16}$$

The following proposition gives the L^2 adjoint of the spin connection.

Proposition 2.17.
Suppose (M,g) is closed. The operator

$$\nabla^* : \Gamma(T^*M\otimes\Sigma_g M) \to \Gamma(\Sigma_g M)$$

$$\nabla^* = -\operatorname{tr}\circ(\mathrm{id}_{T^*M}\otimes \cdot^\flat \otimes \mathrm{id}_{\Sigma_g M})\circ\nabla^{T^*M\otimes\Sigma_g M}$$

is formally L^2 adjoint to ∇ on smooth sections, i.e.

$$\int_M \langle\nabla\varphi, \alpha\otimes\psi\rangle_{T^*M\otimes\Sigma_g M}\,\mathrm{vol}_g = \int_M \langle\varphi, \nabla^*(\alpha\otimes\psi)\rangle\,\mathrm{vol}_g\,.$$

Using this adjoint we introduce the *connection Laplacian*

$$\nabla^*\nabla : \Gamma(\Sigma_g M) \to \Gamma(\Sigma_g M).$$

Given a synchronous orthonormal frame at p we have the following formula for the connection Laplacian at p:

$$\nabla^*\nabla\varphi = -\sum_i \nabla_{e_i}\nabla_{e_i}\varphi.$$

If $N \subset M$ is a hypersurface of M, then the spinor bundle of N can be identified with the spinor bundle of M (or the subbundle $\Sigma_g^+ M$, depending on the dimension), as we saw previously. Denote by $\iota : N \hookrightarrow M$ the inclusion. Then we have a pull-back map

$$\iota^* : \Gamma(\Sigma_g M) \to \Gamma(\Sigma_{g_N} N).$$

There is a useful formula for the spin derivative in this setting, given in the next proposition.

Proposition 2.18.
Let $\varphi \in \Gamma(\Sigma_g M)$. *Then*

$$\iota^*(\nabla^g_X \varphi) = \nabla^{g_N}_X \iota^*\varphi + \frac{1}{2}\,\mathrm{II}(X, X)\cdot\iota^*\varphi \tag{2.19}$$

for every $X \in TN$. *Here* $\mathrm{II} : TN \times TN \to TN^\perp$ *denotes the second fundamental form of* N.

This result can be found in sect. 2.4.1 in [8].

2.5.2 The Dirac operator and the Lichnerowicz formula

Definition 2.20. The *Dirac operator* on (M, g) is the first order differential operator

$$D = \mu \circ (\cdot^\sharp \otimes \mathrm{id}) \circ \nabla^g : \Gamma(\Sigma_g M) \to \Gamma(\Sigma_g M),$$

where $\mu : \Gamma(TM \otimes \Sigma_g M) \to \Gamma(\Sigma_g M)$ denotes Clifford multiplication and $\cdot^\sharp : \Gamma(T^*M) \to \Gamma(TM)$ is the canonical identification with respect to the metric g.

Given an orthonormal frame $e_1, ..., e_n$, we calculate the action of the Dirac operator on the spinor field φ as

$$D\varphi = \mu\left(\sum_i e_i \otimes \nabla_{e_i}\varphi\right) = \sum_i e_i \cdot \nabla_{e_i}\varphi.$$

It is a simple matter to compute the following identities using the previous formula. Let $f \in \mathcal{C}^\infty(M)$, $X \in \Gamma(TM)$, $\varphi \in \Sigma_g M$. Then

$$D(f\varphi) = \operatorname{grad} f \cdot \varphi + fD\varphi \tag{2.21}$$

$$D(X\cdot\varphi) = -X\cdot D\varphi - \nabla_{2X}\varphi + \mathrm{CC}(X)\cdot\varphi \tag{2.22}$$

$$D^2(f\varphi) = fD^2\varphi - \nabla_{2\operatorname{grad} f}\varphi + (\Delta f)\varphi \tag{2.23}$$

where

$$\mathrm{CC} : \Gamma(TM) \to \Gamma(\mathbb{C}\mathrm{l}(M))$$

is given by

$$\mathrm{CC}(X) = \sum_i e_i \cdot \nabla_{e_i} X.$$

A slightly more involved calculation using the local formula for the Dirac operator and the symmetries of the Riemannian curvature yields the well known Lichnerowicz formula.

$$D^2\varphi = \nabla^*\nabla\varphi + \frac{\mathrm{scal}}{4}\varphi \tag{2.24}$$

2.5.3 Parallel spinors and holonomy reduction

The subject of this section is the relationship between spin geometry and special holonomy. Although we will not directly use these results, they motivate much of the following work as explained in the introduction.

Let (M, g) be a Riemannian manifold and let $\mathrm{Hol}_x(M, g)$ be its holonomy group based at $x \in M$. The holonomy principle states that a tensor in the tensor algebra $\mathcal{T}T_xM$, which is invariant under the action of the holonomy group $\mathrm{Hol}_x(M, g)$, corresponds to a parallel tensor field and vice versa. Thus Riemannian manifolds with special holonomy can be characterized by the kind of parallel tensor fields they carry.

Suppose $\varphi \in \Gamma(\Sigma_g M)$ is a non-trivial parallel spinor field, i.e. $\nabla^g\varphi = 0$ and $\varphi \neq 0$. How does this fact constrain the holonomy group of (M, g)? Before we turn to this question, we note that this condition implies that the manifold is Ricci flat, indicating the strength of this assumption.

Theorem 2.25.
Suppose (M, g) is a spin manifold and suppose $\varphi \in \Gamma(\Sigma_g M)$ is a non-trivial parallel spinor field, i.e. $\nabla^g\varphi = 0$. Then

$$\mathrm{Ric}_g = 0.$$

This is an immediate consequence of the following useful identity

$$\sum_i e_i \cdot R(e_i, X)\varphi = \frac{1}{2}\,\mathrm{ric}_g(X) \cdot \varphi, \tag{2.26}$$

which also shows that one can recover the Ricci curvature of the manifold from the curvature of the spin connection. A proof of this identity can be found in [8], Cor 2.8.

The existence of parallel spinor fields implies the existence of certain parallel tensor fields on M. Assume for simplicity that M is simply connected. Then the next theorem states that the Ricci flat metrics of special holonomy are characterized by existence of a parallel spinor field. Notice that these manifolds are automatically spin by theorem 2.9, since their holonomy groups are simply connected.

Theorem 2.27.
Suppose (M, g) is a compact, irreducible, simply-connected Riemannian spin manifold. Let n be the dimension of M and let k be the dimension of the vector space of parallel spinor fields. Then

- *If $n = 2m$, $k = 2$, then $\mathrm{Hol}(M, g) \cong \mathrm{SU}(m)$.*
- *If $n = 4m$, $k = m + 1$, then $\mathrm{Hol}(M, g) \cong \mathrm{Sp}(m)$.*
- *If $n = 7$, $k = 1$, then $\mathrm{Hol}(M, g) \cong G_2$.*
- *If $n = 8$, $k = 1$, then $\mathrm{Hol}(M, g) \cong \mathrm{Spin}(7)$.*

Conversely, if the holonomy group of (M, g) is isomorphic to any of the groups above, then the space of parallel spinor fields has the dimension as above.

This theorem is due to Wang, see [45].

2.6 The universal spinor bundle

The spinorial characterization of special holonomy metrics suggests a natural approach to finding such metrics in the space of Riemannian metrics: look for metrics with parallel spinor fields. The spinor flow implements this idea by trying to minimize the spinorial energy functional, whose minima are precisely pairs of metrics and parallel spinor fields. This requires comparing spinor fields in spinor bundles associated with different metrics. To this end, we construct the *universal spinor bundle*. Any pair of a metric and spinor field in the associated spinor bundle can be considered to be a section of the universal spinor bundle. The Bourguignon–Gauduchon connection, which we construct next, defines isomorphisms of spinor bundles of different metrics. Infinitesimally, it also splits the tangent bundle of the universal spinor bundle into a subspace of symmetric two forms and a subspace of spinors. The spin diffeomorphism group acts on the sections of the universal spinor bundle in a natural way. In particular the notion of the Lie derivative can be transferred to the universal spinor bundle. Finally, the derivative of the spin connection with respect to metric and spinorial variations is computed. Most of the material in this section can be found in [3].

Denote by $\widetilde{\mathrm{GL}_+}(n)$ the universal (double) cover of $\mathrm{GL}_+(n)$. Considering $\mathrm{Spin}(n)$ as a subgroup of $\widetilde{\mathrm{GL}_+}(n)$, this can be chosen to be an extension of the double cover $\kappa_n : \mathrm{Spin}(n) \to \mathrm{SO}(n)$, as the following diagram shows.

$$\begin{tikzcd}
\mathrm{Spin}(n) \arrow[r, phantom, "\subset"] \arrow[d, "\kappa"'] & \widetilde{\mathrm{GL}_+}(n) \arrow[d] \\
\mathrm{SO}(n) \arrow[r, phantom, "\subset"] & \mathrm{GL}_+(n)
\end{tikzcd}$$

The definition of spin structures can then be generalized as follows: Let M be an orientable manifold. The bundle $F^+(M)$ of oriented frames of M is a $\mathrm{GL}_+(n)$ principal bundle with the obvious action. A *topological spin structure* is a $\widetilde{GL_+}(n)$ principal bundle P together with a 2:1 covering $\pi : P \to F^+(M)$, such that the diagram

$$\begin{tikzcd}
P \times \widetilde{\mathrm{GL}_+}(n) \arrow[rr] \arrow[d] & & P \arrow[d, "\pi"] \arrow[dr] & \\
F_+(M) \times \mathrm{GL}_+(n) \arrow[rr] & & F_+(M) \arrow[r] & M
\end{tikzcd}$$

commutes. The manifold M admits a topological spin structure if and only if $w_2(M) = 0$, as was the case for spin structures. If P is a topological spin structure and g is a metric on M, then $P_g = \pi^{-1}(F(M,g))$ is a spin structure on (M,g), i.e. we find spin structures for every metric g as a subbundle of P.

Observe that

$$\widetilde{\mathrm{GL}_+}(n)/\operatorname{Spin}(n) = \mathrm{GL}_+(n)/\operatorname{SO}(n) \cong \odot^2_+(\mathbb{R}^n),$$

where $\odot^2_+(\mathbb{R}^n)$ is the space of positive definite bilinear forms, i.e. metrics on $\mathbb{R}^n$. The space

$$\widetilde{\mathrm{GL}_+}(n) \times_{\operatorname{Spin}(n)} \Sigma_n$$

is a vector bundle over $\widetilde{\mathrm{GL}_+}(n)/\operatorname{Spin}(n)$.

The fiber bundle

$$\Sigma M = P \times_{\operatorname{Spin}(n)} \Sigma_n.$$

is called the *universal spinor bundle* and its fibres are diffeomorphic to $\widetilde{\mathrm{GL}_+}(n) \times_{\operatorname{Spin}(n)} \Sigma_n$. This suggests the alternative definition

$$\begin{aligned}
\Sigma M &= (P \times \Sigma_n)/\operatorname{Spin}(n) \\
&= \Big((P \times \widetilde{\mathrm{GL}_+}(n))/\widetilde{\mathrm{GL}_+}(n) \times \Sigma_n\Big)/\operatorname{Spin}(n) \\
&= P \times_{\widetilde{\mathrm{GL}_+}(n)} \Big((\widetilde{\mathrm{GL}_+}(n) \times \Sigma_n)/\operatorname{Spin}(n)\Big)
\end{aligned}$$

The bundle ΣM is also a Hermitian vector bundle with fibers isomorphic to Σ_n over $P/\operatorname{Spin}(n)$. The bundle $P/\operatorname{Spin}(n)$ is isomorphic to $\odot^2_+ T^*M$. Thus, there is a projection

$$\pi_\Sigma : \Sigma M \to \odot^2_+ T^*M.$$

Given $g_x \in \odot^2_+ T^*_x M$, the spinor module over g_x can be defined as

$$\Sigma_{g_x} M = \pi_\Sigma^{-1}(g_x).$$

Now suppose Φ is a section of ΣM. There are different ways to view a section of the universal spinor bundle. The conventional view is to see a section $\Phi \in \Gamma(\Sigma M)$ as a map

$$\Phi : M \to \Sigma M$$

satisfying $\pi \circ \Phi = \mathrm{id}_M$. Alternatively, we can view it as a $\widetilde{\mathrm{GL}_+}(n)$ equivariant map

$$\Phi : P \to (\widetilde{\mathrm{GL}_+}(n) \times \Sigma_n)/\operatorname{Spin}(n).$$

In the same vein, we can consider it as a $\operatorname{Spin}(n)$-equivariant map $\Phi : P \to \Sigma_n$. We will freely pass between these different point of views.

If we consider Φ to be a map $\Phi : M \to \Sigma M$, then $\pi_\Sigma \circ \Phi$ is a section of $\odot^2_+ T^*M$ and corresponds to a Riemannian metric. We call this Riemannian metric g_Φ. Alternatively we can define the projection

$$\pi_S : (\widetilde{\mathrm{GL}_+}(n) \times \Sigma_n)/\operatorname{Spin}(n) \to \widetilde{\mathrm{GL}_+}(n)/\operatorname{Spin}(n).$$

Viewing Φ as a $\widetilde{\mathrm{GL}_+}(n)$ equivariant map $\Phi : P \to (\widetilde{\mathrm{GL}_+}(n) \times \Sigma_n)/\operatorname{Spin}(n)$, the composition $\pi_S \circ \Phi$ is a $\widetilde{\mathrm{GL}_+}(n)$ equivariant map

$$\pi_S \circ \Phi : P \to \widetilde{\mathrm{GL}_+}(n)/\operatorname{Spin}(n),$$

which can be considered to be a section of

$$P \times_{\widetilde{\mathrm{GL}_+}(n)} \widetilde{\mathrm{GL}_+}(n)/\operatorname{Spin}(n) \cong \odot^2_+(T^*M),$$

i.e. a Riemannian metric.

A useful remark concerning this picture is that there is a unique $GL_+(n)$ equivariant map $m : F^+(M) \to \mathrm{GL}_+(n)/\operatorname{SO}(n)$ completing the diagram

$$\begin{tikzcd}
P \arrow[r, "\pi_S \circ \Phi"] \arrow[d, "\pi"'] & \widetilde{\mathrm{GL}_+}(n)/\operatorname{Spin}(n) \arrow[d, "\cong"] \\
F^+(M) \arrow[r, "m"] & \mathrm{GL}_+(n)/\operatorname{SO}(n)
\end{tikzcd}.$$

The map m is yet another way to view a Riemannian metric on M.

Now consider Φ to be a $\operatorname{Spin}(n)$ equivariant map $\Phi : P \to \Sigma_n$. Restricting this map to P_{g_Φ} yields a $\operatorname{Spin}(n)$-equivariant map $P_{g_\Phi} \to \Sigma_n$. This is a section of $\Sigma_{g_\Phi} M$, and this section is denoted by φ_Φ.

To sum up, sections of the universal spinor bundle correspond to pairs of Riemannian metrics and a spinor field in the spinor bundle associated to that metric. By abuse of notation we express sections of the universal spinor bundle as pairs, i.e.

$$\Phi = (g_\Phi, \varphi_\Phi) = (\pi_\Sigma \circ \Phi, \Phi|_{P_{g_\Phi}}).$$

The space of sections of the universal spinor bundle is also denoted $\mathcal{F} = \Gamma(\Sigma M)$ and the space of unit sections

$$\mathcal{N} = \{(g, \varphi) \in \mathcal{F} : |\varphi| = 1\}.$$

2.6.1 The Bourguignon–Gauduchon connection

The Bourguignon–Gauduchon connection is a choice of horizontal subspaces with respect to the fibration $\pi_{\Sigma,x} : \Sigma M_x \to \odot^2_+ T^*M_x$, i.e. a family of vector spaces $H_{(g,\varphi)} \subset T_{(g,\varphi)}\Sigma M_x$ complementary to $\ker d\pi_{\Sigma,x}(g,\varphi)$, depending smoothly on $x \in M$ and $(g,\varphi) \in \Sigma M_x$. Notice that this is *not* a connection of the bundle $\pi_\Sigma : \Sigma M \to \odot^2_+ T^*M$, but only a connection along the fibers ΣM_x! This is no limitation for us, because this connection will be used to compare different sections of ΣM and for a family of sections $\phi_t \in \Gamma(\Sigma M)$ we have that $\phi_t(x) \in \Sigma M_x$.

The connection arises from a connection on the principal bundle

$$\mathrm{SO}(n) \hookrightarrow \mathrm{GL}_+(n) \to \mathrm{GL}_+(n)/\,\mathrm{SO}(n).$$

Given $A \in \mathrm{GL}_+(n)$, the tangent space to the fibers is given by

$$V_A = \{M \in T_A\,\mathrm{GL}_+(n) = \mathbb{R}^{n\times n} : (A^{-1}M)^T = -A^{-1}M\}.$$

We can define a complement

$$H_A = \{M \in T_A\,\mathrm{GL}_+(n) = \mathbb{R}^{n\times n} : (A^{-1}M)^T = A^{-1}M\}.$$

This corresponds to the decomposition of matrices into symmetric and antisymmetric matrices. It can be easily checked that H_A is right invariant and hence defines a connection.

Given $x \in M$, this horizontal distribution can be transfered to

$$F^+(M)_x \to F^+(M)_x/\,\mathrm{SO}(n) \cong \odot^2_+ T^*M_x.$$

Let $b \in F^+(M)_x$. The element b can be considered as an isomorphism $\mathbb{R}^n \to T_xM$ and tangent vectors $v \in T_bF^+(M)_x$ can be considered to be homomorphisms $\mathbb{R}^n \to T_xM$. Thus we can define a horizontal subspace in the same manner as above by

$$H_b = \{v \in T_bF^+(M)_x : (b^{-1}v)^T = b^{-1}v\}.$$

If P is a topological spin structure on M, then this distribution lifts to a distribution

$$H_{\tilde b} \subset T_{\tilde b}P_x$$

for any $x \in M$, $\tilde b \in P_x$. To pass to ΣM_x, consider the canonical isomorphism

$$T_{[\tilde b,\varphi]}\Sigma M_x \to T_{\tilde b}P_x/T_{\tilde b}(\tilde b \cdot \mathrm{Spin}(n)) \oplus \Sigma_{g_x}.$$

By construction the factor $T_{\tilde b}P_x/T_{\tilde b}(\tilde b \cdot \mathrm{Spin}(n))$ is isomorphic to $H_{\tilde b}$. The map $P_x \to P_x/\,\mathrm{Spin}(n) \cong \odot^2_+ T^*_xM$ induces an isomorphism

$$H_{\tilde b} \to \odot^2 T^*_xM.$$

Furthermore, $[\tilde{b}, \varphi] \in \Sigma M_x$ corresponds to a pair (g, φ) with $g \in \odot^2_+ T^*_x M$ and $\varphi \in \Sigma_{g_x} M$. We have finally arrived at an isomorphism

$$T_{(g,\varphi)} \Sigma M_x \to \odot^2 T^*_x M \oplus \Sigma_{g_x} M.$$

In particular we can define the horizontal complement of $\Sigma_{g_x} M = \ker d\pi_{\Sigma,x}$ as the preimage of $\odot^2 T^*_x M$ under this isomorphism.

This isomorphism induces a splitting of the tangent bundle of the space of sections. Given $\Phi = (g, \varphi) \in \mathcal{F}$, the tangent space at Φ splits as follows

$$T_\Phi \mathcal{F} \cong \Gamma(\odot^2 T^* M \oplus \Sigma_g M).$$

This follows, because for a curve of sections $\Phi_t \in \mathcal{F}$ with $\Phi_0 = \Phi$, the time derivative of Φ_t at a point $x \in M$ satisfies

$$\frac{d}{dt}\Big|_{t=0} \Phi_t(x) \in T_\Phi \Sigma M_x \cong \odot^2 T^*_x M \oplus \Sigma_{g_x} M$$

by the definition of sections. Likewise, the tangent space of $\mathcal{N}$ at $\Phi \in \mathcal{N}$ splits as

$$T_\Phi \mathcal{N} \cong \Gamma(\odot^2 T^* M \oplus \varphi^\perp),$$

where $\varphi^\perp$ is the subbundle

$$\{\psi \in \Sigma_g M : \langle \varphi, \psi \rangle = 0\} \subset \Sigma_g M.$$

Suppose (g_0, g_1) are two metrics on M. The path $g_t = g_0 + t(g_1 - g_0)$ linearly interpolates between the two metrics. For every $x \in M$ and $\varphi \in \Sigma_{g_0} M_x$ one can consider the parallel transport along the horizontal distribution to $\Sigma_{g_1} M_x$. This defines an isomorphism

$$\Sigma_{g_0} M_x \to \Sigma_{g_1} M_x.$$

These isomorphisms form a bundle isomorphism, which we call

$$\hat{B}^{g_0}_{g_1} : \Sigma_{g_0} M \to \Sigma_{g_1} M.$$

It turns out that there is another description of these isomorphisms, without reference to the horizontal distribution. Given two inner products $\langle \cdot, \cdot \rangle_i$, $i = 0, 1$, on a vector space V, there exists a unique endomorphism $A^0_1 : V \to V$, such that

$$\langle v, w \rangle_1 = \langle A^0_1 v, w \rangle_0.$$

Since A^0_1 is positive and symmetric, it has a well-defined square root B^0_1. Applying this observation to the metrics g_0 and g_1 yields an endomorphism $B^{g_0}_{g_1} \in \Gamma(\operatorname{End} TM)$, which relates the metrics g_0 and g_1 in this way. This endomorphism induces a map between the oriented, orthonormal frame bundles of g_0 and g_1

$$B^{g_0}_{g_1} : F^+(M, g_0) \subset F^+(M) \to F^+(M, g_1) \subset F^+(M).$$

This map can be lifted to a map

$$\tilde{B}^{g_0}_{g_1} : P_{g_0} \to P_{g_1},$$

which then induces the isomorphism

$$\hat{B}^{g_0}_{g_1} : \Sigma_{g_0} M \to \Sigma_{g_1} M$$

$$[\tilde{b}, \varphi] \mapsto [\tilde{B}^{g_0}_{g_1}(\tilde{b}), \varphi].$$

It can be shown that both definitions of $\hat{B}^{g_0}_{g_1}$ coincide.

2.6.2 The L^2 metric on the space of sections

Given the splitting of the tangent space of $\mathcal{N}$, it is now easy to define a metric. Let $\Phi = (g, \varphi) \in \mathcal{N}$ and $\Psi_1, \Psi_2 \in T_\Phi \mathcal{N}$. With respect to the splitting, we can write

$$\Psi_i = (h_i, \psi_i) \in \Gamma(\odot^2 T^* M \oplus \Sigma_g M)$$

for $i = 1, 2$. Then we define a metric on $T_\Phi \mathcal{N}$ by

$$(\Psi_1, \Psi_2) = \int_M g(h_1, h_2) + \langle \psi_1, \psi_2 \rangle \operatorname{vol}_g .$$

While this definition seems fairly natural, there is actually a choice of scaling involved in the spinorial part. Recall that the metric on the spinor bundle was defined by extending any $\mathrm{Spin}(n)$ invariant inner product on the spinor module. This inner product is only unique up to scaling. To allow for a different choice of inner product we introduce the scaling parameter $\mu \in (0, \infty)$ and define a family of L^2 metrics via

$$(\Psi_1, \Psi_2)_\mu = \int_M g(h_1, h_2) + \mu^{-1} \langle \psi_1, \psi_2 \rangle \operatorname{vol}_g .$$

2.6.3 Spin diffeomorphisms and the spinorial Lie derivative

As before let M be a connected spin manifold and let P be a topological spin structure. Any orientation preserving diffeomorphism $f \in \mathrm{Diff}_+(M)$ induces a map

$$f_* : F^+(M) \to F^+(M)$$

$$(e_1, ..., e_n) \mapsto (Df(e_1), ..., Df(e_n)).$$

Notice that this is not a principal bundle isomorphism in the normal sense, because it is not fibre preserving. Denote by $\mathrm{Diff}_s(M) \subset \mathrm{Diff}_+(M)$ the set of diffeomorphisms, which lift to P. Since P covers $F^+(M)$ two-to-one, there is a choice of lifts involved, determined up to an element in $\mathbb{Z}_2$. Denoting the set of lifts of diffeomorphisms in $\operatorname{Diff}_s(M)$ to P by $\widehat{\operatorname{Diff}}_s(M)$, there is then an exact sequence of groups

$$0 \to \mathbb{Z}_2 \to \widehat{\operatorname{Diff}}_s(M) \to \operatorname{Diff}_s(M) \to 0.$$

The group $\widehat{\operatorname{Diff}}_s(M)$ acts naturally on spinor fields $\Phi \in \mathcal{F}$. Since $\Phi = (g, \varphi) \in \mathcal{F}$ is a $\widetilde{\mathrm{GL}_+}(n)$ equivariant map

$$\Phi : P \to (\widetilde{\mathrm{GL}_+}(n) \times \Sigma_n)/\operatorname{Spin}(n),$$

we can define for $F \in \widehat{\operatorname{Diff}}_s(M)$ the pull back of a spinor field by

$$F^*\Phi = \Phi \circ F.$$

Clearly,

$$(F_1 \circ F_2)^*\Phi = F_2^* F_1^* \Phi.$$

Another simple observation is that

$$|F^*\Phi|(x) = |\Phi|(f(x)).$$

The metric component g of Φ is given by the map $\pi_S \circ \Phi : P \to \widetilde{\mathrm{GL}_+}(n)/\operatorname{Spin}(n)$, which induces a map $m : F^+(M) \to \mathrm{GL}_+(n)/\operatorname{SO}(n)$. The metric component of $F^*\Phi$ is given by

$$g_{F^*\Phi} = \pi_S \circ \Phi \circ F : P \to \widetilde{\mathrm{GL}_+}(n)/\operatorname{Spin}(n),$$

which induces a map $\tilde{m} : P \to \widetilde{\mathrm{GL}_+}(n)/\operatorname{Spin}(n)$. It turns out that

$$\tilde{m} = m \circ f_*,$$

where f is the diffeomorphism corresponding to F. Furthermore, $m \circ f_*$ corresponds to the metric f^*g. In conclusion

$$g_{F^*\Phi} = f^* g_\Phi.$$

This also implies that the spinorial component $\varphi_{F^*\Phi}$ is a section of the spinor bundle $\Sigma_{f^*g}M$. If $[\tilde{b}, \varphi]$ defines a local section of ΣM, $\tilde{b} : U \to P|_U$ and $\varphi : U \to \Sigma_n$, then $F^*\Phi$ is given over U by

$$[F^{-1} \circ \tilde{b} \circ f, \varphi \circ f].$$

Next, we study the infinitesimal action of the diffeomorphism group on $\mathcal{F}$. To this end, suppose that $X \in \Gamma(TM)$. Let f_t be the corresponding 1-parameter family of diffeomorphisms. Since $f_0 = \mathrm{id}_M$, all f_{t*} are homotopic to the identity and hence lift to P, i.e. $f_t \in \operatorname{Diff}_s(M)$ for every t. We choose the lift satisfying $F_0 = \mathrm{id}_P$. Denote by F_t this choice of lifts of f_t and let $\Phi \in \mathcal{F}$.

Definition 2.28. The *spinorial Lie derivative* of Φ in the direction X is defined by

$$\mathcal{L}_X \Phi = \frac{d}{dt}\Big|_{t=0} F_t^* \Phi \in T_\Phi \mathcal{F}.$$

A natural question is what the metric and the spinorial components of $\mathcal{L}_X\Phi$ are. For the metric part, the answer is simple. Since $g_{F_t^*\Phi} = f_t^* g_\Phi$, it follows that

$$g_{\mathcal{L}_X\Phi} = \mathcal{L}_X g_\Phi.$$

Proposition 2.29.
Let $X \in \Gamma(TM)$ and $\Phi = (g, \varphi) \in \mathcal{F}$. Then

$$\mathcal{L}_X\Phi = \left(\mathcal{L}_X g, \nabla^g_X\varphi - \frac{1}{4}dX^\flat \cdot \varphi\right).$$

This result can be found in [7], prop. 16. We now have a description of the tangent space to the diffeomorphism action on $\mathcal{F}$ at $\Phi \in \mathcal{F}$:

$$\{\mathcal{L}_X\Phi \in T_\Phi\mathcal{F} : X \in \Gamma(TM)\}.$$

A complement of this space in $T_\Phi\mathcal{F}$ could be considered an infinitesimal slice of the diffeomorphism action. There is a natural way to construct such a slice. First we review the situation for metric tensors. Let g be a metric. Then let

$$\delta_g : \Gamma(\odot^2 T^*M) \to \Gamma(T^*M)$$

be the divergence operator. With respect to an orthonormal frame e_i, it is given as

$$\delta_g h = \sum_i -(\nabla^g_{e_i} h)(e_i, \cdot).$$

Consider its formal adjoint

$$\delta_g^* : \Gamma(T^*M) \to \Gamma(\odot^2 T^*M).$$

It turns out that

$$\delta_g^*\alpha(X, Y) = \frac{1}{2}\left((\nabla^g\alpha)(X, Y) + (\nabla^g\alpha)(Y, X)\right).$$

A simple calculation then shows

$$\delta_g^* X^\beta = \frac{1}{2}\mathcal{L}_X.$$

Thus $\operatorname{im}\delta_g^*$ coincides with the space tangent to the diffeomorphism action. One can show that δ_g^* is overdetermined elliptic, and hence by standard theory there is an orthogonal decomposition

$$\Gamma(\odot^2 T^*M) = \operatorname{im}\delta_g^* \oplus \ker\delta_g.$$

The space $\ker\delta_g$ is sometimes called the *Ebin slice* at g.

We now repeat this argument for $T_\Phi\mathcal{F}$. In analogy to δ_g^*, we define

$$\lambda_\Phi^* : \Gamma(TM) \to T_\Phi\mathcal{F}$$

$$X \mapsto \mathcal{L}_X\Phi.$$

Its formal adjoint is called $\lambda_\Phi : T_\Phi\mathcal{F} \to \Gamma(TM)$. The precise form of λ_Φ is not important to us. It can be shown that λ_Φ^* is overdetermined elliptic and hence we get an orthogonal splitting

$$T_\Phi\mathcal{F} = \operatorname{im}\lambda_\Phi^* \oplus \ker\lambda_\Phi.$$

The space $\operatorname{im}\lambda_\Phi^*$ is tangent to the orbit of the diffeomorphism group and $\ker\lambda_\Phi$ is an infinitesimal slice of this action.

2.6.4 Conformal changes

A *conformal change* of a given metric g is a metric that arises as the product of g with a function e^{2u}, where $u \in \mathcal{C}^\infty(M)$. We denote this metric by g^u, i.e. $g^u = e^{2u}g$. Let ΣM be the universal spinor bundle associated with a topological spin structure P. Then $\Sigma_g M$ and $\Sigma_{g^u}M$ are both subsets of ΣM and there is an isometry

$$C_u = \hat{B}^g_{g^u} : \Sigma_g M \to \Sigma_{g^u}M$$

between these vector bundles. The aim of this section is to show how different objects, such as the spin connection and the Dirac operator behave under conformal change.

For this, it will be necessary to understand how orthonormal frames behave under conformal change. Let $b = (e_1, ..., e_n)$ be an orthonormal frame of g. Then $e_i^u = e^{-u}e_i$, $1 \le i \le n$, is an orthonormal frame of g^u. This frame will be denoted by b_u. Suppose $\tilde{b}$ is a spin frame covering b. Then $\tilde{b}_u = \tilde{B}^g_{g^u}\tilde{b}$ covers b_u. Locally, any spinor field can be written as $[\tilde{b}, \varphi]$. The map C_u acts on such a spinor field via

$$C_u([\tilde{b}, \varphi]) = [\tilde{b}_u, \varphi].$$

Notice that the tangent bundle TM is isomorphic to the associated product $P \times_{\widetilde{\mathrm{GL}_+}(n)} \mathbb{R}^n$. The element $[\tilde{b}(x), v]$, $v \in \mathbb{R}^n$ at $x \in M$ corresponds to $\sum_i v_i e_i(x)$. By abuse of notation, we will write

$$[\tilde{b}, v] = \sum_i v_i e_i.$$

Notice that $[\tilde{b}_u, v] = e^{-u}[\tilde{b}, v]$.

Theorem 2.30.
Let g be a metric and let $g^u = e^{2u}g$. Let $n = \dim M$ and $X \in TM$. Suppose $\Phi = (g, \varphi) \in \mathcal{N}$. Then the Clifford multiplication behaves under the conformal change as follows:

$$C_u(X \cdot \varphi) = e^{-u}X \cdot C_u(\varphi) \tag{2.31}$$

The spin connection and its norm behave as:

$$\begin{aligned} \nabla^{g^u}_X C_u(\varphi) &= C_u\left(\nabla^g_X\varphi - \tfrac{1}{2}X \cdot \operatorname{grad}_g u \cdot \varphi - \tfrac{1}{2}(Xu)\varphi\right) \\ |\nabla^{g^u}_X C_u(\varphi)|^2_{g^u} &= e^{-2u}\left(|\nabla^g\varphi|^2_g + \tfrac{n-1}{4}|du|^2_g + \langle D_g\varphi, \operatorname{grad}_g u \cdot \varphi\rangle\right) \end{aligned} \tag{2.32}$$

and the Dirac operator and its norm behave as

$$\begin{aligned} D_{g^u}C_u(\varphi) &= C_u\left(e^{-u}\left(D_g\varphi + \tfrac{n-1}{2}\operatorname{grad}_g u\cdot\varphi\right)\right) \\ |D_{g^u}C_u(\varphi)|^2 &= e^{-2u}\left(|D_g\varphi|^2 + \left(\tfrac{n-1}{2}\right)^2|du|_g^2 + (n-1)\langle D_g\varphi, \operatorname{grad}_g u\cdot\varphi\rangle\right) \end{aligned} \tag{2.33}$$

In particular, if $n=2$*,*

$$|D_{g^u}C_u(\varphi)|^2 - |\nabla^{g^u}C_u(\varphi)|^2_{g^u} = e^{-2u}(|D_g\varphi|^2 - |\nabla^g\varphi|^2) \tag{2.34}$$

Proof. Throughout the whole proof, let e_i, e_i^u and $b, \tilde{b}, b_u, \tilde{b}_u$ be as above. Furthermore, we let $\varphi = [\tilde{b}, \varphi]$ by abuse of notation. For the proof of equation 2.31, let $X = [\tilde{b}, v]$ with $v \in \mathbb{R}^n$. Then

$$\begin{aligned} C_u(X\cdot\varphi) = C_u([\tilde{b}, v]\cdot[\tilde{b},\varphi]) &= C_u([\tilde{b}, v\cdot\varphi]) \\ &= [\tilde{b}_u, v\cdot\varphi] \\ &= [\tilde{b}_u, v]\cdot[\tilde{b}_u,\varphi] \\ &= e^{-u}X\cdot C_u(\varphi) \end{aligned}$$

For the formulas 2.32, first recall the local expression of the spin connection

$$\nabla^g_X\varphi = \left[\tilde{b}, X\varphi + \frac{1}{2}\sum_{i<j} g(\nabla^g_X e_i, e_j)e_i\cdot e_j\cdot\varphi\right].$$

Under conformal change the Levi–Civita connection transforms as follows:

$$\nabla^{\tilde{g}}_X Y = \nabla^g_X Y + (Xu)Y + (Yu)X - g(X,Y)\operatorname{grad}_g u,$$

see equation 3.10. Hence

$$\nabla^{g^u}_X C_u(\varphi) = \left[\tilde{b}_u, X\varphi + \frac{1}{2}\sum_{i<j} g^u(\nabla^{g^u}_X e_i^u, e_j^u)e_i\cdot e_j\cdot\varphi\right].$$

By the formula above, it follows that

$$\begin{aligned} g^u(\nabla^{g^u}_X e_i^u, e_j^u) &= g^u(\nabla^g_X e_i^u + (Xu)e_i^u + (e_i^u u)X - g(X, e_i^u)\operatorname{grad}_g u, e_j^u) \\ &= g^u(\nabla^g_X e_i^u, e_j^u) + (Xu)\delta_{ij} + (e_i^u u)g^u(X, e_j^u) - (e_j^u u)g^u(X, e_i^u) \\ &= g^u(e^{-u}\nabla^g_X e_i, e^{-u}e_j) + g^u(X(e^{-u})e_i, e_j^u) \\ &\quad + (Xu)\delta_{ij} + (e_i^u u)g^u(X, e_j^u) - (e_j^u u)g^u(X, e_i^u) \\ &= g(\nabla^g_X e_i, e_j) - (Xu)\delta_{ij} + (Xu)\delta_{ij} + (e_i u)g(X, e_j) - (e_j u)g(X, e_i) \\ &= g(\nabla^g_X e_i, e_j) + (e_i u)g(X, e_j) - (e_j u)g(X, e_i) \end{aligned}$$

We then calculate

$$\begin{aligned}
&\sum_{i<j}((e_i u)g(X,e_j) - (e_j u)g(X,e_i))e_i \cdot e_j \cdot \varphi \\
&= \sum_{i\neq j}(e_i u)g(X,e_j)e_i \cdot e_j \cdot \varphi \\
&= \sum_{i,j}(e_i u)g(X,e_j)e_i \cdot e_j \cdot \varphi - \sum_i (e_i u)g(X,e_i)e_i \cdot e_i \cdot \varphi \\
&= \operatorname{grad}_g u \cdot X \cdot \varphi + (Xu)\varphi,
\end{aligned}$$

where we here mean by $\operatorname{grad}_g u$ the vector $\sum_i (e_i u)e_i \in \mathbb{R}^n$ and likewise by X the vector $\sum_i g(X,e_i)e_i \in \mathbb{R}^n$. We then conclude

$$\begin{aligned}
\nabla^{g^u}_X C_u(\varphi) &= \left[\tilde{b}_u, X\varphi + \frac{1}{2}\sum_{i<j} g(\nabla^g_X e_i, e_j)e_i \cdot e_j \cdot \varphi\right] + \left[\tilde{b}_u, \frac{1}{2}\operatorname{grad}_g u \cdot X \cdot \varphi + \frac{1}{2}(Xu)\varphi\right] \\
&= C_u\left(\nabla^g \varphi + \frac{1}{2}\operatorname{grad}_g u \cdot X \cdot \varphi + \frac{1}{2}(Xu) \cdot \varphi\right)
\end{aligned}$$

This is the claimed formula, up to the order of X and $\operatorname{grad}_g u$. For the calculation of the (squared) norm $|\nabla^{g^u} C_u(\varphi)|^2_{g^u}$, first note that

$$|\nabla^{g^u} C_u(\varphi)|^2_{g_u} = e^{-2u}\sum_i |\nabla^{g_u}_{e_i} C_u(\varphi)|^2.$$

We then compute

$$\begin{aligned}
|\nabla^{g_u}_{e_i} C_u(\varphi)|^2 &= \left|\nabla^g_{e_i}\varphi + \frac{1}{2}\operatorname{grad}_g u \cdot e_i \cdot \varphi + \frac{1}{2}(e_i u)\varphi\right|^2 \\
&= |\nabla^g_{e_i}\varphi|^2 + \frac{1}{4}|du|^2_g + \frac{1}{4}|e_i u|^2 \\
&\quad + \langle \nabla^g_{e_i}\varphi, \operatorname{grad}_g u \cdot e_i \cdot \varphi\rangle + \langle \nabla^g_{e_i}\varphi, (e_i u)\varphi\rangle + \frac{1}{2}\langle \operatorname{grad}_g u \cdot e_i \cdot \varphi, (e_i u)\varphi\rangle \\
&= |\nabla^g_{e_i}\varphi|^2 + \frac{1}{4}|du|^2_g + \frac{1}{4}|e_i u|^2 + \langle e_i \cdot \nabla^g_{e_i}\varphi, \operatorname{grad}_g u \cdot \varphi\rangle - \frac{1}{2}|e_i u|^2 \\
&= |\nabla^g_{e_i}\varphi|^2 + \frac{1}{4}|du|^2_g - \frac{1}{4}|e_i u|^2 + \langle e_i \cdot \nabla^g_{e_i}\varphi, \operatorname{grad}_g u \cdot \varphi\rangle
\end{aligned}$$

Summing these terms up then yields

$$|\nabla^{g^u} C_u(\varphi)|^2_{g_u} = e^{-2u}\left(|\nabla^g\varphi|^2_g + \frac{n-1}{4}|du|^2_g + \langle D_g\varphi, \operatorname{grad}_g u \cdot \varphi\rangle\right)$$

as claimed.

We now turn to the Dirac operator. We calculate

$$\begin{aligned}
D_{g^u} C_u(\varphi) &= \sum_i e_i^u \cdot \nabla^{g^u}_{e_i^u} C_u(\varphi) \\
&= \sum_i e_i^u \cdot e^{-u} C_u(\nabla^g_{e_i}\varphi - \frac{1}{2} e_i \cdot \operatorname{grad}_g u \cdot \varphi - \frac{1}{2}(e_i u)\varphi) \\
&= e^{-u} \sum_i C_u(e_i \cdot (\nabla^g_{e_i}\varphi - \frac{1}{2} e_i \cdot \operatorname{grad}_g u \cdot \varphi - \frac{1}{2}(e_i u) e_i \cdot \varphi) \\
&= e^{-u} \sum_i C_u(e_i \cdot \nabla^g_{e_i}\varphi + \frac{1}{2} \operatorname{grad}_g u \cdot \varphi - \frac{1}{2}(e_i u) e_i \cdot \varphi) \\
&= e^{-u} C_u \left(D_g \varphi + \frac{n-1}{2} \operatorname{grad}_g u \cdot \varphi \right)
\end{aligned}$$

as claimed. The formula for the squared norm follows immediately from this. □

Some of these formulas can be found in [8], section 2.3.5 and other sources.

2.6.5 The variation of the spin connection

Let $X \in \Gamma(TM)$. For notational convenience, we define the map

$$K_X : \mathcal{F} \to \mathcal{F}$$

$$(g, \varphi) \mapsto (g, \nabla^g_X \varphi).$$

The calculation of the derivative of K_X is central to the study of the spinor flow. The calculation essentially consists of taking the time derivative of the local representation of the spin connection. For this we will need to understand how an orthonormal frame changes along a time dependent metric. Due to the invariance of an inner product under the orthonormal group, there is a gauge freedom to this problem. In the following lemma, we show how an orthonormal basis of a vector space with time dependent inner product evolves for a canonical choice of orthonormal bases.

Lemma 2.35.
Let V be a vector space, and let $\langle \cdot, \cdot \rangle_t$, $t \in (-\epsilon, \epsilon)$ be a family of inner products depending smoothly on t. Let $e_1, ..., e_n \in V$ be an orthonormal basis of $(V, \langle \cdot, \cdot \rangle_0)$. There exists a unique automorphism $A_t : V \to V$, such that

$$\langle v, w \rangle_t = \langle A_t v, w \rangle \text{ for all } v, w \in V.$$

The vectors $e_i(t) = A_t^{-1/2} e_i$, $1 \leq i \leq n$ form an orthonormal basis of $(V, \langle \cdot, \cdot \rangle_t)$. Define $h : V \times V \to \mathbb{R}$ by

$$h(v, w) = \frac{d}{dt}\Big|_{t=0} \langle v, w \rangle_t.$$

Then
$$\dot e_i = \frac{d}{dt}\Big|_{t=0} e_i(t) = -\frac{1}{2}h(e_i,\cdot)^\sharp = -\frac{1}{2}\sum_j h(e_i,e_j)e_j.$$

Proof. Differentiation yields
$$\frac{d}{dt}\Big|_{t=0}\langle e_i(t), e_j(t)\rangle_t = h(e_i,e_j) + \langle \dot e_i, e_j\rangle_0 + \langle e_i, \dot e_j\rangle_0 = 0$$
On the other hand, the expression
$$\langle \dot e_i, e_j\rangle_0$$
is symmetric in i and j. This can be seen as follows. Denote by $B_t = A_t^{-1/2}$. This is (by construction) a symmetric endomorphism with respect to $\langle\cdot,\cdot\rangle_0$ and hence $\frac{d}{dt}\big|_{t=0}B_t$ is also symmetric. This implies that
$$\langle \dot e_i, e_j\rangle_0 = \left\langle \left(\frac{d}{dt}\Big|_{t=0}B_t\right)e_i, e_j\right\rangle_0 = \left\langle e_i, \left(\frac{d}{dt}\Big|_{t=0}B_t\right)e_j\right\rangle_0 = \langle e_i, \dot e_j\rangle_0.$$
Hence
$$\langle \dot e_i, e_j\rangle_0 = -\frac{1}{2}h(e_i,e_j),$$
which implies the result. □

Unsurprisingly, the derivative of the Levi–Civita connection with respect to a metric variation will also appear in the calculation. This is provided in the next lemma. For a proof see [40], proposition 2.3.1.

Lemma 2.36.
Suppose g_t, $t\in(-\epsilon,\epsilon)$ is a smooth family of metrics. Let $g=g_0$ and $h=\frac{d}{dt}\big|_{t=0}g_t$. Then the time derivative of the Levi–Civita connection ∇^{g_t} is given by
$$g\left(\frac{d}{dt}\Big|_{t=0}\nabla^{g_t}_X Y, Z\right) = \frac{1}{2}\left((\nabla^g_X h)(Y,Z) + (\nabla^g_Y h)(X,Z) - (\nabla^g_Z h)(X,Y)\right),$$
where $X,Y,Z\in\Gamma(TM)$.

Theorem 2.37 (Derivative of the spin connection)**.**
Let $\Phi=(g,\varphi)\in\mathcal{F}$ and let $\dot\Phi\in T_\Phi\mathcal{F}$. Then
$$DK_X(\Phi)\dot\Phi = \left(\dot g, \frac{1}{4}\sum_{i\neq j}(\nabla^g_{e_i}\dot g)(X,e_j)e_i\cdot e_j\cdot\varphi + \nabla^g_X\dot\varphi\right),$$
where e_i is an orthonormal frame.

Proof. We follow the proof given in [3] for lemma 4.11. For a variation of the form $(0, \dot{\varphi})$, the formula is obvious. So we instead suppose $\dot{\Phi} = (\dot{g}, 0) \in T_\Phi \mathcal{F}$. Let $g_t = g + t\dot{g}$. Then there exists a horizontal lift Φ_t of g_t to ΣM. Since g_t is a linear path, the lift is

$$\Phi_t = (g_t, \hat{B}^g_{g_t}\varphi).$$

Suppose $b = (e_1, ..., e_n)$ is a local oriented orthonormal frame with respect to g. Then $b_t = B^g_{g_t} b$ is a local orthonormal frame with respect to g_t and we denote its basis vectors by $e_i(t)$. Let $\tilde{b}$ be a spin frame covering b and extend this to $\tilde{b}_t$, the family covering b_t. Clearly, $\tilde{b}_t = \tilde{B}^g_{g_t}\tilde{b}$. The section Φ is then locally represented by $[\tilde{b}, \varphi]$ for some $\varphi : U \to \Sigma_n$, by some small abuse of notation. Accordingly, Φ_t is represented by $[\tilde{b}_t, \varphi]$. The local expression for $\nabla^{g_t}_X \varphi_t$ (as a section of ΣM) is then

$$\left[\tilde{b}_t, X\varphi + \frac{1}{2}\sum_{i<j} g_t(\nabla^{g_t}_X e_i(t), e_j(t)) e_i \cdot e_j \cdot \varphi\right].$$

The time derivative of this term is then

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} \Phi_t &= \left.\frac{d}{dt}\right|_{t=0} \left[\tilde{b}_t, X\varphi + \frac{1}{2}\sum_{i<j} g_t(\nabla^{g_t}_X e_i(t), e_j(t)) e_i \cdot e_j \cdot \varphi\right] \\
&= \left.\frac{d}{dt}\right|_{t=0} \left[\tilde{b}_t, X\varphi + \frac{1}{2}\sum_{i<j} g(\nabla^{g}_X e_i, e_j) e_i \cdot e_j \cdot \varphi\right] \\
&+ \left.\frac{d}{dt}\right|_{t=0} \left[\tilde{b}, X\varphi + \frac{1}{2}\sum_{i<j} g_t(\nabla^{g_t}_X e_i(t), e_j(t)) e_i \cdot e_j \cdot \varphi\right] \\
&= \left.\frac{d}{dt}\right|_{t=0} \hat{B}^g_{g_t} \nabla^g_X \varphi + \left[\tilde{b}, \left.\frac{d}{dt}\right|_{t=0} \frac{1}{2}\sum_{i<j} g_t(\nabla^{g_t}_X e_i(t), e_j(t)) e_i \cdot e_j \cdot \varphi\right]
\end{aligned}$$

The derivative of $B^g_{g_t}\nabla^g\varphi \in \mathcal{F}$ is just $(\dot{g}, 0)$, as the spinorial part is moving by parallel transport. The remaining work is to find the time derivative

$$\left.\frac{d}{dt}\right|_{t=0} g_t(\nabla^{g_t}_X e_i(t), e_j(t)) = \dot{g}(\nabla^g_X e_i, e_j) + g\left(\left.\frac{d}{dt}\right|_{t=0} \nabla^{g_t}_X e_i, e_j\right) + g(\nabla^g_X \dot{e}_i, e_j) + g(\nabla^g_X e_i, \dot{e}_j).$$

Assuming that $(e_1, ..., e_n)$ is a synchronous orthonormal frame, all terms containing $\nabla^g_X e_i$ vanish, i.e.

$$\left.\frac{d}{dt}\right|_{t=0} g_t(\nabla^{g_t}_X e_i(t), e_j(t)) = g\left(\left.\frac{d}{dt}\right|_{t=0} \nabla^{g_t}_X e_i, e_j\right) + g(\nabla^g_X \dot{e}_i, e_j).$$

For the second term, observe that the first lemma applies, because the construction of $e_i(t)$ in the lemma is exactly the same construction used here. Thus

$$\dot{e}_i = -\frac{1}{2}\sum_k \dot{g}(e_i, e_k) e_k$$

and hence

$$g(\nabla^g_X \dot{e}_i, e_j) = -\frac{1}{2}\sum_k (X\dot{g}(e_i, e_k))g(e_k, e_j) = -\frac{1}{2}X\dot{g}(e_i, e_j),$$

using again the synchronicity of the frame at the point. Moreover,

$$X\dot{g}(e_i, e_j) = (\nabla^g_X \dot{g})(e_i, e_j).$$

The second lemma then implies

$$g\left(\frac{d}{dt}\Big|_{t=0} \nabla^{g_t}_X Y, Z\right) = \frac{1}{2}\left((\nabla^g_X \dot{g})(e_i, e_j) + (\nabla^g_{e_i}\dot{g})(X, e_j) - (\nabla^g_{e_j}\dot{g})(X, e_i)\right).$$

Hence

$$\frac{d}{dt}\Big|_{t=0} g_t(\nabla^{g_t}_X e_i(t), e_j(t)) = \frac{1}{2}\left((\nabla^g_{e_i}\dot{g})(X, e_j) - (\nabla^g_{e_j}\dot{g})(X, e_i)\right).$$

Finally, this yields

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=0} \frac{1}{2}\sum_{i<j} g_t(\nabla^{g_t}_X e_i(t), e_j(t)) e_i \cdot e_j \cdot \varphi \\
&= \frac{1}{4}\sum_{i<j}\left((\nabla^g_{e_i}\dot{g})(X, e_j) - (\nabla^g_{e_j}\dot{g})(X, e_i)\right) e_i \cdot e_j \cdot \varphi \\
&= \frac{1}{4}\sum_{i<j}(\nabla^g_{e_i}\dot{g})(X, e_j) e_i \cdot e_j \cdot \varphi - \frac{1}{4}\sum_{i<j}(\nabla^g_{e_j}\dot{g})(X, e_i) e_i \cdot e_j \cdot \varphi \\
&= \frac{1}{4}\sum_{i<j}(\nabla^g_{e_i}\dot{g})(X, e_j) e_i \cdot e_j \cdot \varphi - \frac{1}{4}\sum_{i>j}(\nabla^g_{e_i}\dot{g})(X, e_j) e_j \cdot e_i \cdot \varphi \\
&= \frac{1}{4}\sum_{i\neq j}(\nabla^g_{e_i}\dot{g})(X, e_j) e_i \cdot e_j \cdot \varphi
\end{aligned}$$

by performing an index switch and using the relation $e_i \cdot e_j = -e_j \cdot e_i$ for $i \neq j$. □

2.7 The spinorial energy functional and the spinor flow

This section introduces the central objects of this thesis: the spinorial energy and the spinor flow. After introducing the spinorial energy, its first and second variation is computed. Given the formula for the variation of the spin connection, these are simple calculations. The first variation formula implies in particular that for manifolds of dimension at least 3, critical points of the spinorial energy functional are pairs of metrics and parallel spinor fields. The space of critical points forms a submanifold of $\mathcal{N}$. Formulas for the gradient are derived with respect to the family of L^2 metrics on $\mathcal{N}$. Since the spinor flow is the negative gradient flow of the spinorial energy, this immediately yields the evolution equation of the spinor flow. Unfortunately, this evolution equation is only weakly parabolic. To obtain short time existence for the spinor flow, we proceed in two steps. First, the spinor flow is transformed to a strongly parabolic system via the DeTurck trick. Since this is a quasilinear, strongly parabolic system, solutions exist for short times by results from chapter 1. The solution of this system can then be pulled back to a solution of the spinor flow by a family of diffeomorphisms. The family of diffeomorphisms also solves a parabolic equation, the mapping flow equation. The material of this section stems from [3].

Definition 2.38. The *spinorial energy functional* is the functional

$$\mathcal{E} : \mathcal{N} \to \mathbb{R}$$
$$\mathcal{E}(\Phi) = \mathcal{E}(g, \varphi) = \frac{1}{2} \int_M |\nabla^g \varphi|_g^2 \operatorname{vol}_g$$

The following theorems document the behavior of the spinorial energy under scaling of the metric and under the action of diffeomorphisms.

Theorem 2.39.
Let $\Phi = (g, \varphi) \in \Sigma M$ and $c \in \mathbb{R}$. Then

$$\mathcal{E}(c^2 g, \hat{B}^g_{c^2 g}(\varphi)) = c^{n-2} \mathcal{E}(g, \varphi)$$

where $n = \dim M$.

In dimension 2, the spinorial energy functional is thus scaling invariant. This leads to a more interesting set of critical points, since in the other cases any $\Phi \in \mathcal{N}$ with $\mathcal{E}(\Phi) \neq 0$ can not be critical.

Theorem 2.40 (Diffeomorphism invariance)**.**
Let $\Phi = (g, \varphi) \in \Sigma M$ and $F \in \widehat{\operatorname{Diff}}_s(M)$. Then

$$\mathcal{E}(\Phi) = \mathcal{E}(F^* \Phi).$$

2.7.1 The first and second variation

Let $\Phi = (g, \varphi) \in \mathcal{N}$. We define the 2-tensor $\langle \nabla^g \varphi \otimes \nabla^g \varphi \rangle$ via

$$\langle \nabla^g \varphi \otimes \nabla^g \varphi \rangle (X, Y) = \langle \nabla^g_X \varphi, \nabla^g_Y \varphi \rangle$$

and the 3-tensor $T_\Phi = T_{g,\varphi}$ by

$$T_{g,\varphi}(X, Y, Z) = \frac{1}{2} \left(\langle X \cdot Y \cdot \varphi, \nabla^g_Z \varphi \rangle + \langle X \cdot Z \cdot \varphi, \nabla^g_Y \varphi \rangle \right).$$

Furthermore, the divergence $\operatorname{div}_g T$ of any tensor T is defined the contraction of the first two entries of $-\nabla^g T$. With respect to an orthonormal frame e_i, we may write

$$\operatorname{div}_g T = -\sum_i (\nabla_{e_i} T)(e_i, \cdot, ..., \cdot).$$

Theorem 2.41 (First variation of the spinorial energy)**.**
Let $\Phi = (g, \varphi) \in \mathcal{N}$ and let $\dot{\Phi} = (\dot{g}, \dot{\varphi}) \in T_\Phi \mathcal{N}$. Suppose $\Phi_t \in \mathcal{N}$ is such that $\Phi_0 = \Phi$ and

$$\frac{d}{dt}\Big|_{t=0} \Phi_t = \dot{\Phi}.$$

Then the first variation of $\mathcal{E}$ is given by

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} \mathcal{E}(\Phi_t) = &\int_M g\left(\frac{1}{4}|\nabla^g \varphi|_g^2 g + \frac{1}{4}\operatorname{div}_g T_{g,\varphi} - \frac{1}{2}\langle \nabla^g \varphi \otimes \nabla^g \varphi\rangle, \dot{g}\right) \operatorname{vol}_g \\ &+ \int_M \langle \nabla^{g*}\nabla^g \varphi, \dot{\varphi}\rangle \operatorname{vol}_g\end{aligned}$$

Proof. We follow the proof of prop 4.13 in [3]. As in the proof of theorem 2.37, we examine the cases $\dot{\Phi} = (\dot{g}, 0)$ and $\dot{\Phi} = (0, \dot{\varphi})$ seperately. Suppose $\dot{\Phi} = (0, \dot{\varphi})$. Using proposition 2.17, we obtain

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} \mathcal{E}(\Phi_t) &= \frac{d}{dt}\Big|_{t=0} \frac{1}{2}\int_M |\nabla^g \varphi|_g^2 \operatorname{vol}_g \\ &= \int_M \langle \nabla^g \varphi, \nabla^g \dot{\varphi}\rangle \operatorname{vol}_g \\ &= \int_M \langle \nabla^{g*}\nabla^g \varphi, \dot{\varphi}\rangle \operatorname{vol}_g,\end{aligned}$$

proving the assertion in this case. Now suppose $\dot{\Phi} = (\dot{g}, 0)$. It is well known that

$$\frac{d}{dt}\Big|_{t=0} \operatorname{vol}_{g_t} = \frac{1}{2}\operatorname{tr}_g \dot{g} \operatorname{vol}_g = \frac{1}{2} g(g, \dot{g}) \operatorname{vol}_g.$$

On the other hand, using theorem 2.37, we find

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} \frac{1}{2}|\nabla^{g_t}\varphi_t|^2 &= \frac{d}{dt}\frac{1}{2}\Big|_{t=0} \sum_k |\nabla^{g_t}_{e_k(t)} \varphi_t|^2 \\ &= \sum_k \left\langle \frac{d}{dt}\Big|_{t=0} (\nabla^{g_t}_{e_k(t)}\varphi_t), \nabla^g_{e_k}\varphi \right\rangle \\ &= \sum_k \left\langle \left(\frac{1}{4}\sum_{i\neq j} (\nabla^g_{e_i}\dot{g})(e_k, e_j) e_i \cdot e_j \cdot \varphi\right) - \frac{1}{2}\sum_l \dot{g}(e_k, e_l)\nabla^g_{e_l}\varphi, \nabla^g_{e_k}\varphi \right\rangle\end{aligned}$$

The second term in the last line can be simplified as follows

$$-\frac{1}{2}\sum_{k,l} \dot{g}(e_k, e_l)\langle \nabla^g_{e_k}\varphi, \nabla^g_{e_l}\varphi\rangle = g\left(\dot{g}, -\frac{1}{2}\langle \nabla^g \varphi \otimes \nabla^g \varphi\rangle\right).$$

For the first term, notice that since φ is a unit spinor, the following identity holds

$$\sum_{i\neq j} \langle e_i \cdot e_j \cdot \varphi, \nabla^g_{e_k}\varphi\rangle = \sum_{i,j} \langle e_i \cdot e_j \cdot \varphi, \nabla^g_{e_k}\varphi\rangle.$$

Then we calculate

$$\sum_k \left\langle \frac{1}{4}\sum_{i\neq j}(\nabla^g_{e_i}\dot g)(e_k,e_j)e_i\cdot e_j\cdot\varphi, \nabla^g_{e_k}\varphi\right\rangle = \frac{1}{4}\sum_{i,j,k}(\nabla^g_{e_i}\dot g)(e_k,e_j)\langle e_i\cdot e_j\cdot\varphi, \nabla^g_{e_k}\varphi\rangle$$
$$= \frac{1}{4}\sum_{i,j,k}(\nabla^g_{e_i}\dot g)(e_k,e_j)T_{g,\varphi}(e_i,e_j,e_k),$$

using the symmetry of $(\nabla^g_{e_i}\dot g)(e_k,e_j)$ in k and j. By definition of the induced metric on tensors, the triple sum is equal to to $g(\nabla^g\dot g, T_{g,\varphi})$. We can then express the time derivative of the energy density as

$$\frac{d}{dt}\Big|_{t=0}\frac{1}{2}|\nabla^{g_t}\varphi_t|^2 = \frac{1}{4}g(\nabla^g\dot g, T_{g,\varphi}) - \frac{1}{2}g(\dot g, \langle\nabla^g\varphi\otimes\nabla^g\varphi\rangle).$$

In conclusion

$$\frac{d}{dt}\Big|_{t=0}\mathcal{E}(\Phi_t) = \int_M \frac{1}{4}|\nabla^g\varphi|^2 g(g,\dot g) + \frac{1}{4}g(T_{g,\varphi},\nabla^g\dot g) - \frac{1}{2}g(\langle\nabla^g\varphi\otimes\nabla^g\varphi\rangle,\dot g)\,\mathrm{vol}_g$$
$$= \int_M g\left(\frac{1}{4}|\nabla^g\varphi|^2 g + \frac{1}{4}\operatorname{div}_g T_{g,\varphi} - \frac{1}{2}\langle\nabla^g\varphi\otimes\nabla^g\varphi\rangle, \dot g\right)\mathrm{vol}_g,$$

where we also used that div_g is the formal adjoint of ∇^g acting on tensors. □

Suppose $\Phi = (g,\varphi)$ is a critical point of $\mathcal{E}$. By the fundamental theorem of the calculus of variations it follows

$$\frac{1}{4}|\nabla^g\varphi|^2_g g + \frac{1}{4}\operatorname{div}_g T_{g,\varphi} - \frac{1}{2}\langle\nabla^g\varphi\otimes\nabla^g\varphi\rangle = 0.$$

Taking the trace of this expression yields

$$\frac{n-2}{4}|\nabla^g\varphi|^2_g + \frac{1}{4}\operatorname{tr}_g\operatorname{div}_g T_{g,\varphi} = 0.$$

The second term is a divergence of a vector field and hence its integral over a closed manifold is zero. We conclude

$$\frac{n-2}{4}\int_M |\nabla^g\varphi|^2_g\,\mathrm{vol}_g = 0.$$

Thus in dimension $n \geq 3$ we have the following characterization of critical points.

Theorem 2.42.
If $\dim M \geq 3$, $\Phi = (g,\varphi) \in \mathcal{N}$ is a critical point of $\mathcal{E}$ if and only if φ is a parallel spinor field with respect to g, i.e.

$$\nabla^g\varphi = 0.$$

By the results of section 2.5.3, the metric g is then Ricci flat and has special holonomy. For the second variation at such a critical point, define the map

$$\kappa_{g,\varphi} : T_{(g,\varphi)}\mathcal{F} \to \Gamma(T^*M \otimes \Sigma_g M)$$

by the condition

$$(\dot{g}, \kappa_{g,\varphi}(\dot{g}, \dot{\varphi})(X)) = DK_X(g,\varphi)(\dot{g}, \dot{\varphi}).$$

Theorem 2.43 (Second variation of the spinorial energy)**.**
Let $\Phi = (g,\varphi) \in \mathcal{N}$ and let $\dot{\Phi} = (\dot{g}, \dot{\varphi}) \in T_\Phi\mathcal{N}$. Suppose $\Phi_t \in \mathcal{N}$ is such that $\Phi_0 = \Phi$ and

$$\frac{d}{dt}\Big|_{t=0} \Phi_t = \dot{\Phi}.$$

Suppose that $\nabla^g \varphi = 0$. Then the second variation of $\mathcal{E}$ is given by

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{E}(\Phi_t) = \int_M |\kappa_{g,\varphi}(\dot{g}, \dot{\varphi})|_g^2 \operatorname{vol}_g .$$

Proof. We follow the proof of prop. 4.14 in [3]. The first derivative can be expressed as

$$\frac{d}{dt}\mathcal{E}(\Phi_t) = \int_M \frac{1}{4}|\nabla^{g_t}\varphi_t|^2 g_t(g_t, \dot{g}_t) + \frac{1}{2}\dot{g}_t(\nabla^{g_t}\varphi_t, \nabla^g \varphi_t) + g_t(\kappa_{g_t,\varphi_t}(\dot{g}_t, \dot{\varphi}_t), \nabla^{g_t}\varphi_t) \operatorname{vol}_{g_t} .$$

To compute the second time derivative, note that the first two terms in the integral are quadratic in $\nabla^{g_t}\varphi_t$ and since $\nabla^g\varphi = 0$, their time derivative at $t = 0$ vanishes. Thus

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{E}(\Phi_t) = \int_M g\left(\kappa_{g,\varphi}(\dot{g}, \dot{\varphi}), \frac{d}{dt}\Big|_{t=0} \nabla^{g_t}\varphi_t\right) \operatorname{vol}_g = \int_M |\kappa_{g,\varphi}(\dot{g}, \dot{\varphi})|_g^2 \operatorname{vol}_g .$$

□

This theorem implies in particular the following characterization of the linearization of the gradient of $\mathcal{E}$ at a critical point, see cor. 4.15 in [3].

Proposition 2.44.
Suppose $\bar{\Phi} = (\bar{g}, \bar{\varphi})$ is a critical point of $\mathcal{E}$, i.e. $\mathcal{E}(\bar{\Phi}) = 0$. Then the linearization $D \operatorname{grad} \mathcal{E}(\bar{\Phi})$ is given by

$$\pi^{T\mathcal{N}} \circ \kappa^*_{\bar{g},\bar{\varphi}} \kappa_{\bar{g},\bar{\varphi}},$$

where $\pi^{T\mathcal{N}} : T\mathcal{F} \to T\mathcal{N}$ is the orthogonal projection. In particular $D \operatorname{grad} \mathcal{E}(\bar{\Phi})$ is formally self-adjoint and non-negative.

Surprisingly from dimension 3 and up, the set of critical points of $\mathcal{E}$ behaves rather nicely: it is a smooth manifold. It has been known that the space of Calabi–Yau metrics is a smooth manifold and also that the space of G_2 and Spin(7) metrics are smooth manifolds. The following theorem has these results as special cases.

Theorem 2.45.
Suppose $\dim M \geq 3$.

$$\mathrm{Crit}(\mathcal{E}) = \{\Phi \in \mathcal{N} : \Phi \textit{ is a critical point of } \mathcal{E}\} = \mathcal{E}^{-1}(0)$$

is a smooth submanifold of $\mathcal{N}$*. In particular* $\mathcal{E}$ *is a Morse–Bott functional.*
Suppose $\Phi \in \mathrm{Crit}(\mathcal{E})$*. Then*

$$T_\Phi \,\mathrm{Crit}(\mathcal{E}) = \ker \kappa_\Phi.$$

The quotient

$$\mathrm{Crit}(\mathcal{E}) / \operatorname{Diff}_0(M)$$

is a smooth, finite-dimensional manifold.

This theorem may be found in [1].

2.7.2 The spinor flow

With the first variation formula at hand, it is now a simple matter to calculate the gradient of the spinorial energy functional. The spinor flow is then simply the negative gradient flow of $\mathcal{E}$.

Proposition 2.46.
Let $\Phi \in \mathcal{N}$*. Then the gradient of* $\mathcal{E}$ *with respect to the* L^2 *metric* $(\cdot,\cdot)_\mu$ *is given by*

$$\operatorname{grad}^\mu \mathcal{E}(\Phi) = \Big(\frac{1}{4}|\nabla^g\varphi|^2 g + \frac{1}{4}\operatorname{div}_g T_{g,\varphi} - \frac{1}{2}\langle \nabla^g\varphi \otimes \nabla^g\varphi\rangle, \tag{2.47}$$

$$\mu(\nabla^{g*}\nabla^g\varphi - |\nabla^g\varphi|^2\varphi)\Big) \tag{2.48}$$

This follows immediately from the formula for the first variation of $\mathcal{E}$, with the exception of the second term in the spinorial part. This is explained by the fact that variations in $\mathcal{N}$ preserve the norm of φ. Hence the gradient is actually the projection of $\mu\nabla^{g*}\nabla^g\varphi$ to $\varphi^\perp$. The calculation

$$\begin{aligned}\langle \nabla^{g*}\nabla^g\varphi, \varphi\rangle &= -\sum_i \langle \nabla^g_{e_i}\nabla^g_{e_i}\varphi, \varphi\rangle \\ &= \sum_i \langle \nabla^g_{e_i}\varphi, \nabla^g_{e_i}\varphi\rangle \\ &= |\nabla^g\varphi|^2\end{aligned}$$

then proves the formula above. In the calculation we used that $|\varphi|^2 = 1$ and its implication $\langle \nabla^g_X\varphi, \varphi\rangle = 0$.

Definition 2.49. The *spinor flow* is the negative gradient flow of the spinorial energy functional, i.e. $\Phi_t \in \mathcal{N}$ solves the spinor flow if

$$\partial_t \Phi_t = -\operatorname{grad}^\mu(\Phi_t).$$

Unless we explicitly specify μ, it is always assumed that $\mu = 1$. For convenience, define

$$Q^\mu(\Phi) = -\operatorname{grad}^\mu(\Phi).$$

Using the decomposition of $T_\Phi\mathcal{N}$, the negative gradient also splits as $Q(\Phi) = (Q_1(\Phi), Q_2(\Phi))$. Then the spinor flow equations for $\Phi_t = (g_t, \varphi_t)$ are

$$\begin{aligned} \partial_t g &= Q_1(g,\varphi) = -\tfrac{1}{4}|\nabla^g\varphi|^2 g - \tfrac{1}{4}\operatorname{div}_g T_{g,\varphi} + \tfrac{1}{2}\langle\nabla^g\varphi \otimes \nabla^g\varphi\rangle \\ \partial_t\varphi &= Q_2(g,\varphi) = \mu\left(-\nabla^{g*}\nabla^g\varphi + |\nabla^g\varphi|^2\varphi\right), \end{aligned} \tag{2.50}$$

where we suppressed the time index for legibility. It can be shown that Q is a quasilinear, second order operator. To get short time existence results for the spinor flow from the results of chapter 1, we would have to show that the linearization of Q satisfies the Legendre–Hadamard condition and has negative definite symbol. This is unfortunately not true. Indeed, the symbol of Q is only negative semi-definite and has an n dimensional kernel. This is a reflection of the diffeomorphism equivariance of Q, which follows from the diffeomorphism invariance of $\mathcal{E}$ and the L^2 metric, as we note in the next proposition.

Proposition 2.51.
If $\Phi \in \mathcal{N}$ and $F \in \widehat{\operatorname{Diff}}_s(M)$, then

$$Q(F^*\Phi) = F^*Q(\Phi).$$

Furthermore, $Q(\Phi)$ is orthogonal to the orbit of $\widehat{\operatorname{Diff}}_s(M)$, equivalently

$$\lambda_\Phi Q(\Phi) = 0. \tag{2.52}$$

The last equation is also called the *Bianchi identity*, because as the classical Bianchi identity for the curvature, it arises from the diffeomorphism invariance. The symbol can be explicitly calculated.

Proposition 2.53.
Let $\Phi = (g,\varphi) \in \mathcal{N}$, $x \in M$, $\xi \in T_x^*M\backslash\{0\}$. Let $e_1, ..., e_n$ be an orthonormal basis of T_xM and suppose $e_1 = \xi^\sharp/|\xi|^2$. The principal symbol of Q at x is given by

$$\sigma(DQ(\Phi))\xi : T_\Phi\mathcal{N}_x \to T_\Phi\mathcal{N}_x$$

$$\sigma(DQ(\Phi))\xi(h,\psi) = \begin{pmatrix} \frac{1}{16}\left(-|\xi|^2 h + \xi \odot h(\xi,\cdot)\right) - \frac{1}{4}\xi \odot \beta_\xi \\ -\frac{1}{4}\xi \wedge h(\xi,\cdot)\cdot\psi - |\xi|^2\psi \end{pmatrix},$$

where $h \in \odot^2 T^*M_x$, $\psi \in \varphi_x^\perp \subset \Sigma_g M_x$, such that $(h,\psi) \in T_\Phi\mathcal{N}_x$. The cotangent vector $\beta_\xi \in T_x^*M$ is defined by

$$\beta_\xi = \sum_i \langle \xi \wedge e_i \cdot \varphi, \psi\rangle e^i.$$

This symbol is negative semidefinite and its kernel is given by

$$\ker\sigma(DQ(\Phi))\xi = \left\{ \left(h, -\frac{1}{4}\xi \wedge h(\xi,\cdot)\cdot\varphi\right) : h(v,w) = 0 \text{ for } v,w \perp \xi^\sharp \right\}.$$

The kernel is n dimensional.

Thus the results from chapter 1 do not apply. The next section will resolve this issue.

2.7.3 The DeTurck trick

In the previous section we saw that the spinor flow is only weakly parabolic and hence the short time existence result from chapter 1 for quasilinear parabolic systems does not apply. It turns out that the spinor flow equation can be transformed in such a way that the resulting equation is strongly parabolic. Then the result from chapter 1 applies and we get a short time solution for this system. This solution can then be transformed back to a solution of the original spinor flow equation.

The basic idea for defining the new system is the following: the degeneracy of Q stems from the diffeomorphism invariance of $\mathcal{E}$ and is reflected by the Bianchi identity

$$\lambda_\Phi Q(\Phi) = 0.$$

Recalling the orthogonal splitting

$$T_\Phi \mathcal{N} = \ker \lambda_\Phi \cap T_\Phi \mathcal{N} \oplus \operatorname{im} \lambda_\Phi^*,$$

the Bianchi identity can be interpreted to say that the spinor flow moves orthogonally to the orbits of the diffeomorphism group. The degeneracy can then be removed by adding a term pointing in the diffeomorphism direction. Solutions of this perturbed system differ from the spinor flow only by a family of diffeomorphism.

We will now implement this approach. Suppose $\bar{g}$ is a metric. Then define

$$X_{\bar{g}} : \Gamma(\odot^2 T^* M) \to \Gamma(TM)$$
$$h \mapsto -2(\delta_{\bar{g}} h)^\sharp.$$

With the help of this map we can define

$$\tilde{Q}_{\bar{g}}(g, \varphi) = Q(g, \varphi) + \lambda^*_{g,\varphi}(X_{\bar{g}}(g)).$$

Proposition 2.54.
For $\bar{\Phi} = (\bar{g}, \bar{\varphi}) \in \mathcal{N}$ the symbol of $D\tilde{Q}_{\bar{g}}(\bar{\Phi})$ is strictly negative definite.

Thus the equation

$$\partial_t \tilde{\Phi}_t = \tilde{Q}_{\bar{g}}(\tilde{\Phi}_t) \text{ and } \tilde{\Phi}_0 = \bar{\Phi} \tag{2.55}$$

is strongly parabolic for initial values $\bar{\Phi} = (\bar{g}, \bar{\varphi})$ and admits a unique maximal short time solution Φ_t defined on an interval $[0, T_{\max})$. This equation is called the *gauged spinor flow* equation. The remaining question is now how this solution relates to the spinor flow. This will be examined in the next section.

2.7.4 The mapping flow

Suppose $\tilde{\Phi}_t = (\tilde{g}_t, \tilde{\varphi}_t)$ is a solution of the gauged spinor flow equation with initial condition $\bar{\Phi} = (\bar{g}, \bar{\varphi})$. Then consider the family of diffeomorphisms $f_t : M \to M$ solving the ordinary differential equation

$$\frac{d}{dt} f_t = -X_{\bar{g}}(\tilde{g}_t) \circ f_t \text{ and } f_0 = \mathrm{id}_M \,.$$

This family can be lifted to a family $F_t \in \widehat{\mathrm{Diff}}_s(M)$. It is then an easy consequence of the definition of the Lie derivative and the operator λ^*_Φ that

$$\partial_t(F_t^* \tilde{\Phi}_t) = Q(F_t^* \tilde{\Phi}_t),$$

i.e. $\Phi_t = F_t^* \tilde{\Phi}_t$ solves the spinor flow equation with initial condition $\bar{\Phi}$.

Theorem 2.56.
For every $\Phi \in \mathcal{N}$ there exists a unique maximal solution of the spinor flow equation

$$\partial_t \Phi_t = Q(\Phi_t)$$

with initial condition Φ, i.e. such that

$$\Phi_0 = \Phi$$

on an interval $[0, T_{\max})$.

The uniqueness requires a seperate argument, which we do not want to give here. The main tool for showing uniqueness is the mapping flow, which we introduce nevertheless, because we will need it for a different purpose. The mapping flow arises, if we try to pass from a solution of the spinor flow equation to a solution of the gauged spinor flow equation. Suppose $\Phi_t = (g_t, \varphi_t) \in \mathcal{N}$ solves the spinor flow equation. Then $F_t^* \Phi_t$ solves the gauged spinor flow equation, if F_t is the lift of the family f_t solving

$$\frac{d}{dt} f_t = -X_{g_0}(f_t^* g_t) \circ f_t \text{ and } f_0 = \mathrm{id}_M \,. \tag{2.57}$$

This equation is called the *mapping flow equation* and it is second order, quasilinear, strongly parabolic equation. It can be rewritten as

$$\frac{d}{dt} f_t = P_{g_t, g_0}(f_t),$$

with

$$\begin{aligned} P_{g,\bar{g}} : \mathcal{C}^\infty(M,M) &\to T\mathcal{C}^\infty(M,M) \\ f &\mapsto -df(X_{f^*\bar{g}}(g)). \end{aligned}$$

For future use, we will need the following expression for the linearization of $P_{g,\bar{g}}$ at id_M:

$$\begin{aligned} DP_{g,\bar{g}}(\mathrm{id}_M) : \Gamma(TM) &\to \Gamma(TM) \\ X &\mapsto -4\left(\delta_{\bar{g}} \delta^*_{\bar{g}} X^\flat\right)^\sharp \end{aligned} \tag{2.58}$$

2.8 The spinorial energy on surfaces

In this section we survey the results from [2], where the behavior of the spinorial energy functional on surfaces is discussed in detail. In the two-dimensional setting the spinorial energy functional differs substantially from the higher dimensional setting. One of the reasons for this is the scaling invariance

$$\mathcal{E}(g, \varphi) = \mathcal{E}(\lambda^2 g, B^g_{\lambda^2 g}(\varphi)).$$

This implies for instance that critical points need not be absolute minimizers. Moreover, the Gauß–Bonnet theorem yields that

$$\begin{aligned}
\mathcal{E}(g, \varphi) &= \frac{1}{2}\|\nabla^g \varphi\|^2_{L^2(M,g)} \\
&= \frac{1}{2}(\nabla^{g*}\nabla^g \varphi, \varphi)_{L^2(M,g)} \\
&= \frac{1}{2}(D_g^2 \varphi, \varphi)_{L^2(M,g)} - \frac{1}{8}\int_M \operatorname{scal}_g |\varphi|^2 \operatorname{vol}_g \\
&= \frac{1}{2}\int_M |D_g \varphi|^2 \operatorname{vol}_g - \frac{\pi}{2}\chi(M).
\end{aligned}$$

This motivates us to introduce the *Dirac energy*

$$\mathcal{D}(g, \varphi) = \frac{1}{2}\int_M |D_g \varphi|^2 \operatorname{vol}_g .$$

We have just shown that $\mathcal{D}$ and $\mathcal{E}$ differ only by a constant, hence their variational properties are the same. One consequence of the formula above is that for negative $\chi(M)$ we get a positive lower bound for $\mathcal{E}(g, \varphi)$. It turns out that for the sphere we also get a positive lower bound of π. Thus

$$\mathcal{E}(g, \varphi) \geq \frac{\pi}{2}|\chi(M)|.$$

The following theorem describes the absolute minimizers of $\mathcal{E}$, depending on the topological type.

Theorem 2.59.
Let M be a closed surfaces with spin structure σ. Then $(g, \varphi) \in \mathcal{N}$ is an absolute minimizer of $\mathcal{E}$ if

1. *$P_g \varphi = 0$, if $\chi(M) = 2$*
2. *$\nabla^g \varphi = 0$, if $\chi(M) = 0$*
3. *$D_g \varphi = 0$, if $\chi(M) < 0$*

The operator P_g is the *twistor operator*, which we will introduce later. In general, *existence* of an absolute minimizer depends on the spin structure σ. Recall that the spin structures of M are classified by

$$H^1(M, \mathbb{Z}_2) = \mathbb{Z}_2^{2\gamma},$$

where g is the genus of M. Thus there are $2^{2\gamma}$ different spin structures on M. Furthermore one can distinguish between *bounding* and *non-bounding* spin structures. A spin structure is bounding, if there exists a spin manifold X with boundary $\partial X = M$ such that the naturally induced spin structure on ∂X coincides with the spin structure on M. There are $2^{\gamma-1}(2^\gamma+1)$ bounding spin structures and $2^{\gamma-1}(2^\gamma - 1)$ non-bounding spin structures.

Theorem 2.60.
*On a spin surface M with spin structure σ and genus γ the infimum of $\mathcal{E}$ **is** attained, if*

- $\gamma = 0$
- $\gamma = 1$ *and σ is non-bounding*
- $\gamma \geq 3$ *and σ is bounding*
- $\gamma \geq 5$, $\gamma \equiv 1 \mod 4$ *and σ is non-bounding*

*The infimum **is not** attained, if*

- $\gamma = 1$ *and σ is a bounding spin structure*
- $\gamma = 2$
- $\gamma = 3, 4$ *and σ is non-bounding*

The case $\gamma \geq 6$, $\gamma \not\equiv 1 \mod 4$ is open.

The spinorial energy functional has a distinct geometric interpretation in dimension two. This is related to the *spinorial Weierstraß representation.* If φ is a unit spinor field on a simply connected spin surface M, such that

$$D_g\varphi = H\varphi,$$

then one can construct an immersion ι of M into $\mathbb{R}^3$, such that H is the mean curvature function of the immersion. Thus if $D_g\varphi = H\varphi$

$$\mathcal{D}(g, \varphi) = \int_M |H|^2 \operatorname{vol}_g = \mathcal{W}(\iota),$$

where $\mathcal{W}(\iota)$ is the *Willmore energy* of the immersion ι. Thus the Dirac energy is in some sense a generalization of the Willmore energy to a larger space, the space of immersions being the subset

$$\{(g, \varphi) \in \mathcal{N} : D_g\varphi = H\varphi \text{ for some } H \in \mathcal{C}^\infty(M)\} \subset \mathcal{N}.$$

If M is not simply connected, we can pass to the universal cover $\tilde{M}$ instead. However, the universal cover $\tilde{M}$ is non-compact and has infinite volume. This means in particular that the functional is no longer well-defined. Nevertheless, we can say that if $\gamma > 0$ and (g, φ) is an absolute minimiser, i.e. satisfies $\nabla^g \varphi = 0$ or $D_g \varphi = 0$, then φ defines a minimal immersion of $\tilde{M}$ into $\mathbb{R}^3$.

On the other hand, suppose that we are given an immersion $\iota : M \hookrightarrow T^3$. Suppose g is a flat metric on T^3, fix a spin structure and let Φ be a parallel unit spinor field on T^3. The spin structure on T^3 induces a spin structure on M and the spinor field Φ can be restricted to a spinor field φ, which satisfies $D\varphi = H\varphi$, where H is the mean curvature function of M. This construction can be used to show that

$$\inf \mathcal{E}(g, \varphi) = \frac{\pi}{2} |\chi(M)|.$$

Indeed, we can construct surfaces of arbitrary genus γ with arbitrarily small Willmore energy in T^3 by joining totally geodesic tori in T^3 by catenoidal necks.

On the other hand, one of the statements of the above theorem is that there exist surfaces, where the infinum is not attained. It is an interesting question how a minimizing sequence then degenerates.

Moreover, it is known that there exist critical points which are not minimizers: one can construct such critical points explicitly on the flat torus with a bounding spin structure.

All this indicates that the landscape of $\mathcal{E}$ is in some sense significantly more complicated than in higher dimension – at least if the genus of M is nonzero. On the sphere, the situation is much simpler.

Theorem 2.61.
If $M = S^2$, then any critical point of $\mathcal{E}$ is an absolute minimiser.

We already mentioned that minimizers on the sphere are twistor spinors. The twistor operator is defined to be

$$P^g : \Gamma(\Sigma_g M) \to \Gamma(T^*M \otimes \Sigma_g M)$$

$$P^g_X \varphi = \nabla^g_X \varphi + \frac{1}{n} X \cdot D_g \varphi.$$

A *twistor spinor* is a spinor field, which satisfies $P^g \varphi = 0$. It turns out that there always exists α, such that if φ is a twistor spinor, then

$$\psi = \cos(\alpha)\varphi + \sin(\alpha)\omega \cdot \varphi$$

is a *real Killing spinor*, i.e.

$$\nabla^g_X \psi = \lambda X \cdot \psi$$

for some $\lambda \in \mathbb{R}$. In particular, ψ is an eigenspinor of the Dirac operator and thus ψ defines a constant mean curvature immersion of the sphere in $\mathbb{R}^3$. In fact, it can be seen by other methods that the Gauß curvature of g must also be constant and that on a sphere it has to be positive, i.e. that (M, g) is isometric to the sphere. In this sense all minimisers are geometrically equivalent.

The spinor flow on the sphere only moves in conformal directions as can be shown fairly easily. Hence it is a good candidate for proving the uniformisation of the sphere. Of course, the usual analytical problems also arise in this setting: the non-compact group of conformal diffeomorphisms.

Chapter 3

Geometric flows on surfaces

This chapter presents general tools to understand geometric flows on surfaces. The first two sections review elementary results concerning complex structures and conformal changes on surfaces. After that we describe how the space of Riemannian metrics on surfaces can be understood as a fibration over the space of constant curvature metrics, where the fibres are conformal classes. The space of constant curvature metrics modulo diffeomorphisms is finite dimensional. This has useful implications for geometric flows: we can decompose a family of metrics g_t on a surface as

$$g_t = e^{2u_t} f_t^* \bar{g}_t,$$

where $\bar{g}_t$ is a family of constant curvature metrics, $u_t \in \mathcal{C}^\infty(M)$ is a family of smooth functions and $f_t : M \to M$ is a family of diffeomorphisms. With the right choice of diffeomorphisms f_t, we have particularly good control on the family $\bar{g}_t$. Once the metric g_t has been split into its constituents, it is necessary to understand the constant curvature metric $\bar{g}_t$ and the conformal factor e^{2u_t}. For constant curvature metrics on surfaces, it is known that under a lower injectivity radius bound the metrics can not degenerate in a certain sense. This is the content of the Mumford compactness theorem. For metrics within a conformal class, there is a theorem of X.X. Chen which describes very precisely how a family of metrics with a bound on the L^2 norm of the curvature can degenerate. We will introduce the Liouville energy of a metric. If either the Liouville energies are bounded from above or the injectivity radius from below, the only possible degenerations are very benign. This gives rise to two different compactness theorems.

In section 3.5, following Buzano and Rupflin we will examine how a geometric flow behaves under this decomposition, i.e. given flow equations for the metric g_t, we will derive equations for $u_t, \bar{g}_t$ and f_t.[11]

In the last section a result of Rupflin and Topping which is very useful to control curves of constant curvature metrics is cited.[35]

3.1 Complex structures and the Hodge star operator

Let (M, g) be an oriented surface. There is a natural construction of a complex structure $J : TM \to TM$ associated to this. Given $x \in M$, take any oriented orthonormal basis $e_1, e_2 \in T_xM$. Then we define

$$J_x : T_xM \to T_xM$$

via

$$J(e_1) = e_2 \text{ and } J(e_2) = -e_1.$$

This definition is independent of the choice of e_1 and e_2 and clearly $J^2 = -\operatorname{id}_{TM}$. This defines apriori an almost complex structure. However, on a surface all complex structures are integrable. Thus J defines a bona fide complex structure.

The orientation also allows us to define the *Hodge star operator*. Let again e_1, e_2 be an oriented orthonormal basis of T_xM and let $e^1, e^2 \in T_x^*M$ be the dual basis. The Hodge star operator $* : \Omega^k(M) \to \Omega^{2-k}(M)$ then acts as follows:

$$*1 = e^1 \wedge e^2 = \mathrm{vol}_g$$

$$*e^1 = e^2$$

$$*e^2 = -e^1$$

$$*e^1 \wedge e^2 = 1$$

This has the following consequences. Let $f \in \mathcal{C}^\infty(M)$, $\alpha, \beta \in \Omega^1(M)$, $\omega \in \Omega^2(M)$. Then

$$* * f = f \tag{3.1}$$

$$* * \alpha = -\alpha \tag{3.2}$$

$$* * \omega = \omega. \tag{3.3}$$

The complex structure and the Hodge star are closely related via the following formula:

$$(*\alpha)(X) = -\alpha(JX). \tag{3.4}$$

Finally, the Hodge star also encodes the metric in the following sense:

$$\alpha \wedge *\beta = g(\alpha, \beta) \, \mathrm{vol}_g \,. \tag{3.5}$$

The *codifferential* $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$ is defined by the relations

$$\begin{array}{ll} \delta = 0, & \text{if } k = 0 \\ \delta = - * d*, & \text{if } k = 1 \\ \delta = *d*, & \text{if } k = 2 \end{array}$$

A straightforward calculation shows $\delta^2 = 0$. On a compact manifold, δ is the formal adjoint of d, i.e.

$$\int_M g(\alpha, d\beta) \operatorname{vol}_g = \int_M g(\delta\alpha, \beta) \operatorname{vol}_g \tag{3.6}$$

for $\alpha \in \Omega^{k+1}(M), \beta \in \Omega^k(M)$. The *Hodge Laplacian* is the operator $\Delta : \Omega^k(M) \to \Omega^k(M)$ defined by

$$\Delta\alpha = (d + \delta)^2\alpha = (\delta d + d\delta)\alpha$$

for $\alpha \in \Omega^k(M)$. On functions it coincides with the ordinary Laplacian. Since δ is the formal adjoint of d, the Hodge Laplacian is symmetric and non-negative, i.e.

$$\int_M g(\Delta\alpha, \beta) \operatorname{vol}_g = \int_M g(d\alpha, d\beta) + g(\delta\alpha, \delta\beta) \operatorname{vol}_g = \int_M g(\alpha, \Delta\beta) \operatorname{vol}_g \tag{3.7}$$

where $\alpha, \beta \in \Omega^k(M)$.

3.2 Conformal changes

Let (M, g) be an oriented surface. Given $u \in \mathcal{C}^\infty(M)$

$$\tilde{g} = e^{2u} g$$

defines a new metric on M. The function e^{2u} is called the *conformal factor* and we say $\tilde{g}$ arises from g by *conformal change*. The *conformal class* of the metric g is the set

$$[g] = \{e^{2u} g : g \in \mathcal{C}^\infty(M)\}.$$

In the following proposition we collect how various geometric objects or quantities derived from the metric behave under conformal change.

Proposition 3.8 ([33], 4.7.14)**.**
Let (M, g) be an oriented Riemannian surface, $u \in \mathcal{C}^\infty(M)$ and $\tilde{g} = e^{2u} g$.

1. (Orthonormal basis) *Suppose $e_1, e_2 \in T_xM$ is an orthonormal basis with respect to g. Then $e^{-u}e_1, e^{-u}e_2$ is an orthonormal with respect to $\tilde{g}$. If e_1^*, e_2^* is the dual basis of e_1, e_2, then $e^u e_2^*, e^u e_2^*$ is the dual basis of $e^{-u}e_1, e^{-u}e_2$.*

2. (The volume element) *The volume element of $\tilde{g}$ is given by*

$$\operatorname{vol}_{\tilde{g}} = e^{2u} \operatorname{vol}_g . \tag{3.9}$$

3. (Levi–Civita connection) *Let $X, Y \in \Gamma(TM)$. Then the Levi–Civita connection of $\tilde{g}$ is given by*

$$\nabla^{\tilde{g}}_X Y = \nabla^g_X Y + (Xu)Y + (Yu)X - g(X, Y) \operatorname{grad}_g u \tag{3.10}$$

4. (Gradient) *The gradient of a function f with respect to $\tilde{g}$ is given by*

$$\operatorname{grad}_{\tilde{g}} f = e^{-2u} \operatorname{grad}_g f \tag{3.11}$$

5. (Hodge star) *Let $\alpha \in \Omega^k(M)$. Then the Hodge star operator of $\tilde{g}$ is given by*

$$*_{\tilde{g}}\alpha = e^{(2-2k)u} *_g \alpha.$$

6. (Laplace operator) *The Laplace operator of $\tilde{g}$ on functions is given by*

$$\Delta_{\tilde{g}} f = e^{-2u} \Delta_g f \tag{3.12}$$

7. (Scalar curvature) *The scalar curvature of $\tilde{g}$ is given by*

$$R_{\tilde{g}} = e^{-2u} \left(2\Delta_g u + R_g\right)$$

3.3 The space of metrics on surfaces

In this section, we discuss the space of metrics on closed surfaces. This material can be found for example in [41]. See also the discussion in [11].

The space of metrics on closed surfaces is remarkably simple in comparison to other dimensions. There are three qualitatively different cases to consider. This is related to the topological classification of closed surfaces. Recall that the genus γ of a closed surface is defined by the relation $2 - 2\gamma = \chi(M)$, where $\chi(M)$ is the Euler characteristic of M.

Theorem (Topological classification of surfaces)**.**
Suppose M is a closed surface of genus γ. Then M is diffeomorphic to the connected sum

$$S^2 \# T^2 \# ... \# T^2,$$

where the number of T^2 factors is given by γ.
In particular, M is diffeomorphic to the sphere S^2 if $\gamma = 0$, and diffeomorphic to T^2 if $\gamma = 1$.

The Gauß–Bonnet theorem relates the integral curvature of a closed Riemannian surface (M, g) to the genus of M:

$$\int_M K_g \operatorname{vol}_g = 2\pi(2 - 2\gamma).$$

The formula implies in particular that any metric on S^2 must have some area of positive curvature, the curvature be zero somewhere on T^2 and on surfaces of genus $\gamma > 1$ must have some area of negative curvature. In particular, the sign of constant curvature metrics on closed surfaces is determined by the genus. The uniformisation theorem establishes existence of such metrics.

Theorem (Uniformisation theorem).
Let M be a closed surface of genus γ and let $[g]$ be a conformal class of metrics on M. Then there exists a metric $g_0 \in [g]$ with constant curvature

$$K_{g_0} \equiv \begin{cases} 1 & \text{if } \gamma = 0, \\ 0 & \text{if } \gamma = 1, \\ -1 & \text{if } \gamma > 1 \end{cases}$$

If $\gamma > 1$, the metric of constant curvature g_0 is unique within the conformal class $[g]$. If $\gamma = 1$, the metric of constant curvature g_0 is unique within the conformal class $[g]$ up to scaling.

The case of the sphere is special. By the classification of simply connected space forms, it is known that any metric of constant curvature on S^2 is isometric to the standard round sphere. Hence up to diffeomorphism, all metrics are conformal to each other. On the other hand, for the standard round metric g on S^2 the group

$$\mathrm{Conf}(S^2, [g]) = \{f : S^2 \to S^2 : f^*[g] = [g]\}$$

is known to be the group of Möbius transformations of the Riemann sphere $\mathbb{C}P^1$, which is isomorphic to $\mathrm{PGL}(2, \mathbb{C})$. Since the condition of constant curvature is diffeomorphism invariant, it follows that for any $f \in \mathrm{Conf}(S^2, [g])$ and any constant curvature metric $g_0 \in [g]$ the metric $f^* g_0$ is also a constant curvature metric conformal to g_0. This explains the failure of uniqueness in the case of the sphere.

The uniformisation theorem gives us a useful parametrization of a conformal class, at least in the case of positive genus: Any metric $\tilde{g}$ in a conformal class $[g]$ can be written *uniquely* as

$$e^{2u} g_0,$$

where $g_0 \in [g]$ is the unique metric of constant curvature 0 or -1. In the case of the torus we assume in addition that g_0 has total volume 1.

To understand the space of metrics on a surface, we thus need to understand the space of conformal structures or equivalently the space of metrics of constant curvature.

Given a closed surface M of genus $\gamma > 1$, the space $\mathcal{M}$ denotes the space of metrics $\Gamma(\odot^2_+ T^* M)$ on M. The space of metrics of constant curvature $\mathcal{M}^{cc} \subset \mathcal{M}$ of volume 1 is defined by

$$\mathcal{M}^{cc} = \{g \in \mathcal{M} : K_g \text{ constant}, \mathrm{Vol}(M, g) = 1\}.$$

By the Gauß-Bonnet theorem, the curvature of a metric $g \in \mathcal{M}^{cc}$ is determined by the genus in the following manner

$$K_g = 2\pi(2 - 2\gamma).$$

As we noted before, the constant curvature condition is diffeomorphism invariant. Thus, the action of $\mathrm{Diff}(M)$ on the space of metrics restricts to $\mathcal{M}^{cc}$.

We already noted that by classical results any metric of constant curvature 1 on the sphere is isometric to the standard sphere, and thus the action of $\operatorname{Diff}(S^2)$ on $\mathcal{M}^{cc}(S^2)$ is transitive. In other words, the moduli space $\mathcal{M}^{cc}(S^2)/\operatorname{Diff}(S^2)$ consists of a single point.

The situation for the torus is more complicated. Any flat torus is isometric to $\mathbb{R}^2/\Lambda$, where $\Lambda = \mathbb{Z}v + \mathbb{Z}w$ with $v, w \in \mathbb{R}^2$. We can rotate and scale the lattice, such that $\Lambda = \mu(\mathbb{Z}(1,0) + \mathbb{Z}\tau)$, where $\tau \in H = \{(x,y) \in \mathbb{R}^2 : y > 0\}$. Denote by g_τ the metric of the torus $\mathbb{R}^2/(\mathbb{Z}(1,0)+\mathbb{Z}\tau)$ rescaled to unit volume. The map

$$H \to \mathcal{M}^{cc}(T^2)/\operatorname{Diff}(T^2)$$

$$\tau \mapsto [g_\tau]$$

is then surjective. By $\operatorname{Diff}_0(M)$ we denote the group of diffeomorphisms on M which are isotopic to the identity. It can be shown that the map

$$H \to \mathcal{M}^{cc}(T^2)/\operatorname{Diff}_0(T^2)$$

is a homeomorphism with respect to the natural topology on $\mathcal{M}^{cc}(T^2)/\operatorname{Diff}_0(T^2)$. It can also be shown that the conformal group $\operatorname{Conf}(T^2,[g])$ is trivial for any conformal class $[g]$ on T^2. Thus there is a decomposition of the space of metrics on the torus given by the following map

$$\mathcal{C}^\infty(M) \times H \times \operatorname{Diff}_0(T^2) \to \mathcal{M}(T^2)$$

$$(u, \tau, f) \mapsto f^*(e^{2u} g_\tau).$$

If M is a surface of genus γ, one can show that $\mathcal{M}^{cc}(M)/\operatorname{Diff}_0(M)$ is homeomorphic to $\mathbb{R}^{6\gamma-6}$. Hence the space of metrics on M can be decomposed into

$$\mathcal{C}^\infty(M) \times \mathbb{R}^{6\gamma-6} \times \operatorname{Diff}_0(M).$$

The following discussion is based on [11]. This decomposition of the space of metrics induces a linear decomposition of its tangent space. Suppose $g \in \mathcal{M}^{cc}$. Then $T_g\mathcal{M}$ can be identified with $\Gamma(\odot^2 T^*M)$. Suppose $f_t : M \to M$ is a family of diffeomorphisms with $f_0 = \mathrm{id}_M$. Then

$$\frac{d}{dt}\bigg|_{t=0} f_t^* g = \mathcal{L}_X g$$

with $X = \frac{d}{dt}\Big|_{t=0} f_t$, i.e. the tangent space of $\operatorname{Diff}(M)\cdot g$ is given by

$$T_g \operatorname{Diff}(M)\cdot g = \{\mathcal{L}_X g : X \in \Gamma(TM)\}.$$

Let $u_t \in \mathcal{C}^\infty(M)$ with $u_0 = 0$. Then

$$\frac{d}{dt}\bigg|_{t=0} e^{2u_t} g = 2(\partial_t u) g.$$

Hence the tangent space of the conformal class $[g]$ at g is given by

$$T_g[g] = \mathcal{C}^\infty(M)g.$$

Now suppose $g_t \in \mathcal{M}^{cc}$, i.e. g_t has constant curvature and volume 1. Then

$$\frac{d}{dt}\Big|_{t=0} \mathrm{vol}(M, g_t) = \frac{1}{2}\int_M \mathrm{tr}_g\, \partial_t g \, \mathrm{vol}_g = 0$$

and

$$\frac{d}{dt}\Big|_{t=0} R_{g_t} = 0.$$

On the other hand

$$\frac{d}{dt}\Big|_{t=0} R_{g_t} = \Delta_g \,\mathrm{tr}_g\, \partial_t g + \delta_g \delta_g \partial_t g - \frac{1}{2} R_g \,\mathrm{tr}_g\, \partial_t g$$

for any $g_t \in \mathcal{M}$. By assumption the scalar curvature is constant and by Gauß–Bonnet theorem it must be $R_g = 4\pi(2 - 2\gamma)$. Hence we obtain

$$T_g\mathcal{M}^{cc} = \left\{ h \in \Gamma(\odot^2 T^*M) : \begin{array}{c} \Delta_g \,\mathrm{tr}_g\, h + \delta_g\delta_g h - \frac{1}{2}R_g \,\mathrm{tr}_g\, h = 0, \\ \int_M \mathrm{tr}_g\, h \,\mathrm{vol}_g = 0 \end{array} \right\}$$

for surfaces of higher genus. The decomposition

$$\mathcal{M}^{cc} \cong (\mathcal{M}^{cc}/\operatorname{Diff}_0(M)) \times \operatorname{Diff}_0(M)$$

is reflected on the infinitesimal level by finding a complement S_g of $T_g \operatorname{Diff}(M) \cdot g$. Then we have a direct sum

$$(S_g \cap T_g\mathcal{M}^{cc}) \oplus T_g \operatorname{Diff}(M) \cdot g.$$

The formal adjoint of the divergence $\delta_g : \Gamma(\odot^2 T^*M) \to \Gamma(T^*M)$ is denoted by $\delta_g^* : \Gamma(T^*M) \to \Gamma(\odot^2 T^*M)$ and one can show that

$$\delta_g^* X^\flat = \frac{1}{2}\mathcal{L}_X g.$$

This means $T_g \operatorname{Diff}(M) \cdot g = \operatorname{im} \delta_g^*$. Furthermore, δ_g^* is overdetermined elliptic and hence there is an L^2 orthogonal decomposition

$$\Gamma(\odot^2 T^*M) = \ker \delta_g \oplus \operatorname{im} \delta_g^*.$$

In particular, we obtain the direct sum decomposition

$$(\ker \delta_g \cap T_g\mathcal{M}^{cc}) \oplus \operatorname{im} \delta_g^*.$$

The space $\ker \delta_g \cap T_g\mathcal{M}^{cc}$ consists of the $h \in \Gamma(\odot^2 T^*M)$, satisfying

$$\Delta_g \,\mathrm{tr}_g\, h + \delta_g\delta_g h - \frac{1}{2}R_g \,\mathrm{tr}_g\, h = 0$$

$$\delta_g h = 0$$

and

$$\int_M \operatorname{tr}_g h \operatorname{vol}_g = 0.$$

On a torus the first equation is equivalent to $\operatorname{tr}_g h$ being harmonic and having average zero. But since the only harmonic functions on the torus are constant, this implies $\operatorname{tr}_g h = 0$. On higher genus surfaces the first equation reduces to the statement that $\operatorname{tr}_g h$ is an eigenfunction for the eigenvalue R_g. But Δ_g is a non-negative operator, thus $\operatorname{tr}_g h$ must be 0 in that case also. Hence we find in both cases

$$\ker \delta_g \cap T_g\mathcal{M}^{cc} = \{h \in \Gamma(\odot^2 T^*M) : \delta_g h = 0, \operatorname{tr}_g h = 0\}.$$

This space is also denoted by

$$\mathcal{H}_g = \{h \in \Gamma(\odot^2 T^*M) : \delta_g h = 0, \operatorname{tr}_g h = 0\}.$$

We can view $\mathcal{H}_g$ as a horizontal connection of the fibration

$$\mathcal{M}^{cc} \to \mathcal{M}^{cc}/\operatorname{Diff}_0(M).$$

In particular, $\mathcal{H}_g$ is isomorphic to $T_{[g]}\mathcal{M}^{cc}/\operatorname{Diff}_0(M)$ and is a two dimensional vector space if M is the torus and a $6\gamma - 6$ dimensional vector space if M is a surface of higher genus.

3.4 Compactness theorems for surfaces with bounded L^2 curvature

Suppose g_i, $i \in I$ is a family of Riemannian metrics on a manifold M. For the study of geometric flows and many other questions, it is important to understand under which geometric conditions on the family g_i, such as bounds on its curvature, volume, diameter etc, the metrics g_i can be controlled well. We will study this question in two cases. In both cases M is a compact surface. In the first case we restrict to a conformal class and assume uniform bounds on the L^2 norm of the curvature and the so called Liouville energy, which we introduce later. In the second case we consider arbitrary metrics on the surface and again assume a uniform bound on the L^2 norm of the curvature and a uniform lower bound on the injectivity radius. In both cases we conclude that the metric can only degenerate in a very benign way.

3.4.1 Within a conformal class

Given a family of conformal metrics $g_n = e^{2u_n} g$, a natural question is under which conditions on the geometry of g_n, we have good estimates of the conformal factors u_n. To answer this question we introduce the Calabi energy and the Liouville energy in a conformal class.

We denote the L^2 norm (squared) of the curvature of any metric g by

$$\mathcal{R}^2(g) = \int_M R_g^2 \operatorname{vol}_g .$$

The functional $\mathcal{R}^2$ is also sometimes called the *Calabi energy*. The *Liouville energy* of $\tilde{g} = e^{2u}g$ relative to g is defined by

$$E_L(\tilde{g}) = E_L(e^{2u}g) = E_L(u) = \frac{1}{2}\int_M |du|_g^2 + R_g u \operatorname{vol}_g .$$

Note that the Liouville energy is only defined within a conformal class.

A family of conformal metrics with uniformly bounded $\mathcal{R}^2$ and fixed volume can not degenerate too badly. This is made precise in the following theorem due to X.X. Chen.[14] We will use a version given by Struwe.

Theorem 3.13 (Concentration compactness principle, [38] Thm. 3.2).
Suppose $g_n = e^{2u_n}g$ is a family of metrics with

$$\operatorname{Vol}(M, g_n) = 1 \text{ and } \mathcal{R}^2(g_n) \leq C$$

for all n. Then either the sequence $(u_n)_n$ is uniformly bounded in $H^2(M,g)$ or there exists $x_1, ..., x_L \in M$ such that for all $r > 0$ and all $l \in \{1, ..., L\}$ there holds

$$\liminf_{n\to\infty} \int_{B_r(x_l)} |R_{g_n}| \operatorname{vol}_{g_n} \geq 4\pi,$$

where $B_r(x)$ denotes the balls with respect to g.

Thus, if we can exclude concentration of curvature, i.e. existence of $x \in M$, such that

$$\liminf_{n\to\infty} \int_{B_r(x_l)} |R_{g_n}| \operatorname{vol}_{g_n} \geq 4\pi$$

for all $r > 0$, then we get a uniform bound of the conformal factors in $H^2(M,g)$. A handy criterium to exclude this concentration of curvature is given by a uniform bound of the Liouville energies of the metrics g_n, at least if the base metric g has non-positive curvature. This approach is due to Struwe, although the following result can not be explicitly found in [38].

Theorem 3.14.
Assume $R_g = 0$ or $R_g = -1$. Given a family of metrics $g_n = e^{2u_n}g$ with

$$\operatorname{Vol}(M, g_n) = 1, \mathcal{R}^2(g_n) \leq C_1 \text{ and } E_L(u_n) \leq C_2$$

for all n. Then the family $(u_n)_n$ is uniformly bounded in $H^2(M,g)$.

Proof. The basic idea of the proof is that the bound on the Liouville energy together with the volume constraint implies an L^p bound on the conformal factor e^{2u} for some $p > 1$. This rules out curvature concentration. Then the concentration compactness principle yields the theorem.

The L^p bound will follow from the Moser–Trudinger inequality 1.3

$$\int_M \exp\left(\alpha\left(\frac{u - \int_M u \operatorname{vol}_g}{\|du\|_{L^2}}\right)^2\right) \operatorname{vol}_g \leq C.$$

The Jensen inequality implies

$$2\int_M u_n \operatorname{vol}_g \leq \log\left(\int_M e^{2u_n} \operatorname{vol}_g\right) = \log\left(\operatorname{Vol}(M, g_n)\right) \leq 0.$$

Using $R_g = 0$ or $R_g = -1$, it follows that

$$E_L(u_n) = \frac{1}{2}\int_M |du_n|_g^2 + R_g u_n \operatorname{vol}_g \geq \frac{1}{2}\int_M |du_n|_g^2 \operatorname{vol}_g = \frac{1}{2}\|du_n\|_{L^2}^2.$$

Hence we have a uniform upper bound

$$\|du_n\|_{L^2}^2 \leq C_2.$$

Denote by

$$\bar{u}_n = \int_M u_n \operatorname{vol}_g .$$

The Moser–Trudinger inequality applied to u_n says

$$\int_M \exp\left(\frac{\alpha}{\|du\|_{L^2}^2}(u_n - \bar{u}_n)^2\right) \operatorname{vol}_g \leq C.$$

The uniform bound on $\|du_n\|_{L^2}$ then implies

$$\int_M \exp\left(\frac{\alpha}{C_2}(u_n - \bar{u}_n)^2\right) \operatorname{vol}_g \leq C.$$

On the other hand, for all $k, \beta \in \mathbb{R}_{>0}$ there exists a constant $D > 0$, such that

$$e^{kt} \leq e^{k|t|} \leq De^{\beta|t|^2}$$

for every $t \in \mathbb{R}$. The previous inequality implies

$$\int_M e^{k(u_n - \bar{u}_n)} \operatorname{vol}_g \leq \int_M e^{k|u_n - \bar{u}_n|} \operatorname{vol}_g \leq C(k).$$

Thus, to find a uniform bound on

$$\int_M e^{k|u_n|} \operatorname{vol}_g$$

it now suffices to bound $|\bar{u}_n|$. We already have established the upper bound

$$\bar{u}_n \le 0.$$

A lower bound on $\bar{u}_n$ is equivalent to an upper bound on $e^{-\bar{u}_n}$ and can be obtained as follows

$$e^{-2\bar{u}_n} = e^{-2\bar{u}_n} \int_M e^{2u_n} \operatorname{vol}_g = \int_M e^{2(u_n - \bar{u}_n)} \operatorname{vol}_g \le C(2).$$

In conclusion we have for every $p \in (1, \infty)$ a uniform estimate

$$\|e^{2u_n}\|_{L^p} = \left(\int_M e^{2p|u_n|} \operatorname{vol}_g \right)^{1/p} \le C(2p)^{1/p}.$$

This bound implies that curvature can not concentrate, as we will show next. Let $B_r(x)$ be any ball in M with respect to g. Then we can estimate

$$\int_{B_r(x)} |R_{g_n}| \operatorname{vol}_{g_n} \le \left(\int_M |R_{g_n}|^2 \operatorname{vol}_g \right)^{1/2} \left(\int_{B_r(x)} \operatorname{vol}_{g_n} \right)^{1/2} \le 2C_1^{1/2} \left(\int_{B_r} \operatorname{vol}_{g_n} \right)^{1/2}$$

using the Cauchy–Schwarz inequality and the bound on $\mathcal{R}^2$. Hence under a $\mathcal{R}^2$ bound curvature concentration implies volume concentration. On the other hand

$$\begin{aligned} \int_{B_r(x)} \operatorname{vol}_{g_n} &= \int_{B_r(x)} e^{2u_n} \operatorname{vol}_g \le \left(\int_M e^{4u_n} \operatorname{vol}_g \right)^{1/2} \left(\int_{B_r(x)} \operatorname{vol}_g \right)^{1/2} \\ &= \|e^{2u_n}\|_{L^2} \operatorname{vol}(B_r(x))^{1/2} \le C(4)^{1/2} \operatorname{vol}(B_r(x))^{1/2}. \end{aligned}$$

Combining these two estimates we get

$$\int_{B_r(x)} |R_{g_n}| \operatorname{vol}_{g_n} \le 2C_1^{1/2} C(4)^{1/2} \operatorname{vol}(B_r(x))^{1/2}.$$

For r small enough, this implies

$$\int_{B_r(x)} |R_{g_n}| \operatorname{vol}_{g_n} < 4\pi.$$

Thus the concentration of curvature in theorem 3.13 does not happen and hence the u_n are uniformly bounded in $H^2(M, g)$. □

3.4.2 Under a lower injectivity radius bound

It turns out that the bound on the Liouville energy in theorem 3.14 can be replaced by a lower bound on the injectivity radius or an upper bound on the Sobolev constant. This criterium is independent of the conformal class and a natural question is whether there is some notion of compactness for the set

$$\left\{ g \in \mathcal{M} : \mathrm{inj}(M, g) \geq \epsilon \text{ and } \int_M |R_g|^2 \operatorname{vol}_g \leq C \right\}.$$

Notice that both conditions are diffeomorphism invariant. When we restricted to a conformal class, this was not an issue, because there are no conformal diffeomorphisms on surfaces of positive genus. Clearly $\mathrm{Diff}(M)$ does not act trivially on $\mathcal{M}$ and any compactness theorem has to reflect this fact.

We will prove the following theorem.

Theorem 3.15.
Suppose M is a closed surface and suppose $\chi(M) \leq 0$. Let g_n be a sequence of Riemannian metrics with

$$\mathrm{Vol}(M, g_n) < V, \mathcal{R}^2(g_n) < K \text{ and } \mathrm{inj}(M, g_n) > \epsilon.$$

Then there exists a subsequence g_{n_k} and a family of $\mathcal{C}^\infty$ diffeomorphisms φ_k with the following significance. Let $\tilde{g}_k = \varphi_k^* g_{n_k}$ and suppose that $\bar{g}_k = e^{-2u_k}\tilde{g}_k$ is the uniformization of $\tilde{g}_k$. Then the sequence $\bar{g}_k$ converges in the $\mathcal{C}^\infty$ topology to a metric $\bar{g}$ and the sequence u_k converges weakly in the H^2 norm.

Its proof will be based on the one hand on the Mumford compactness theorem and on the other hand on the following apriori estimate for solutions of the constant curvature equation.

Theorem 3.16.
Suppose M is closed surface and suppose $\chi(M) \leq 0$. Let g be any Riemannian metric and let $\bar{g} = e^{2u}g$ be the unique metric of constant curvature conformal to g. Then there exists $C > 0$ depending only on the injectivity radius $\mathrm{inj}(M, g)$, the volume $\mathrm{Vol}(M, g)$ and $\mathcal{R}^2(M, g)$ such that

$$\sup_{x \in M} |u| < C.$$

Notice that by "the metric of constant curvature" we mean the following: if $\chi(M) = 0$, then we mean a metric of constant curvature 0 and volume 1. If $\chi(M) < 0$, we mean the metric of curvature -1. The proof is based on the Ricci flow, which solves the constant curvature equation on surfaces. Let g be any metric on a closed surface M with $\chi(M) \leq 0$. Consider the normalized Ricci flow g_t with initial condition g, i.e. the family of metrics g_t satisfying

$$\partial_t g_t = (r - R_{g_t})g_t$$

$$g_0 = g,$$

where

$$r = \mathrm{Vol}(M, g)^{-1} \int_M R_g \operatorname{vol}_g = \mathrm{Vol}(M, g_t)^{-1} \int_M R_{g_t} \operatorname{vol}_{g_t}.$$

It is well known that in this case the normalized Ricci flow exists for all times and that it converges to the unique metric of constant curvature in the conformal class with the same volume as the initial metric. Since all the metrics g_t are conformal, we can write them as $g_t = e^{2u_t} g$. We denote the limit of the metrics g_t as $t \to \infty$ by $g_\infty = e^{2u_\infty} g$. Our goal is to establish an estimate on u_∞. We may as well establish bounds on u_t independent of time. To do this, we proceed in two steps. First we control g_t on some time interval $[0, T]$. At the time T we will have bounds on the maximum of the curvature, rather than only on its L^2 norm. With these bounds we then obtain bounds for all future times $[T, \infty)$. These estimates essentially follow from the following three theorems.

Theorem 3.17 (Yang, [47])**.**
Given a Riemannian metric g on a surface with Sobolev constant $C_S(M, g) = \sigma$, $K = \int_M R_g^2 \operatorname{vol}_g / \operatorname{Vol}(g)$, then considering the Ricci flow g_t with initial condition $g_0 = g$, there exists a $T > 0$ with

$$C_s(M, g_T) \geq \sigma/2$$

$$\int_M R_{g_T}^2 \operatorname{vol}_{g_T} / \operatorname{Vol}(g_T) \leq 2K$$

$$|R_{g_T}| \leq C_1(\sigma, K)$$

$$\|u\|_{C^0} \leq C_2(\sigma, K)$$

Theorem 3.18 (Hamilton, Chow et al)**.**
Suppose M is a closed surface with $\chi(M) \leq 0$. Suppose g_t is the normalized Ricci flow on M with initial condition $g_0 = g$. Then there exists a constant $C > 1$, such that

$$C^{-1} g \leq g_t \leq C g.$$

The constant C depends only on $\max |f|$, where f is the curvature potential of g, i.e. the unique function satisfying

$$\Delta_g f = R_g - r \text{ and } \int_M f \operatorname{vol}_g = 0.$$

This result goes back to Hamilton, but the version we cite can be found in [15], lemma 5.12 and corollary 5.15.

Theorem 3.19 (Calderon, Wang)**.**
Let M be closed surface. For any metric g on M there exists a constant $C > 0$ depending only on $\operatorname{inj}(M, g), \operatorname{Vol}(M, g)$ and $\sup_{x \in M} |R_g(x)|$, such that for $u \in \mathcal{C}^\infty(M)$ with $\int_M u \operatorname{vol}_g = 0$ we have

$$\|u\|_{H^2(M,g)} \leq C \|\Delta_g u\|_{L^2}.$$

The estimate in the previous theorem is known as Calderon inequality. The dependence on only these constants can be found in [44].

Proof of theorem 3.16. Let $g_t = e^{2u_t} g$ be the normalized Ricci flow with initial condition $g_0 = g$. Notice that the Sobolev constant of (M, g) can be estimated solely in terms of $\mathrm{Vol}(M, g)$ and $\mathrm{inj}(M, g)$ by a theorem of Croke.[17] Furthermore, a lower bound on the injectivity radius also implies a lower bound of the volume on a surface, because a lower bound on the Sobolev constant implies a lower bound on the volume of balls. Hence Yang's theorem applies and we obtain a time $T > 0$, such that

$$\sup_{x \in M} |R_{g_T}(x)| < C_1(\mathrm{inj}(M, g), \mathcal{R}^2(M, g)),$$

$$C_S(M, g_T) > C_S(M, g)/2$$

and

$$\|u_T\|_{C^0} \leq C_2(\mathrm{inj}(M, g), \mathcal{R}^2(M, g)).$$

Since the injectivity radius can be bounded from below by the Sobolev constant, the maximal curvature and the volume, we also obtain

$$\mathrm{inj}(M, g) \geq C_3(\mathrm{inj}(M, g), \mathcal{R}^2(M, g), \mathrm{Vol}(M, g)).$$

Now consider the curvature potential of g_T, i.e. the function f satisfying

$$\Delta_{g_T} f = R_{g_T} - r \text{ and } \int_M f \,\mathrm{vol}_g = 0.$$

We have

$$\|R_{g_T}\|_{L^2}^2 = \int_M |R_{g_T}|^2 \,\mathrm{vol}_{g_T} \leq C_1^2 \int_M e^{2u_T} \,\mathrm{vol}_g \leq C_1^2 e^{2C_2} \,\mathrm{Vol}(M, g),$$

in particular there exists $C_4(\mathrm{inj}(M, g), \mathcal{R}^2(M, g), \mathrm{Vol}(M, g)) > 0$, such that

$$\|R_{g_T}\|_{L^2} \leq C_4.$$

By the Calderon inequality we have

$$\|f\|_{H^2(M, g_T)} \leq C(\mathrm{inj}(M, g_T), \mathrm{Vol}(M, g_T), \sup_{x \in M} |R_{g_T}(x)|) \|R_{g_T}\|_{L^2}.$$

Combining all our previous estimates leads to the inequality

$$\|f\|_{H^2(M, g_T)} \leq C_5(\mathrm{inj}(M, g), \mathcal{R}^2(M, g), \mathrm{Vol}(M, g)).$$

Since we have already bounded the Sobolev constant of (M, g_T) in terms of the Sobolev constant of (M, g), it follows by the embedding $H^2(M, g) \hookrightarrow C^0$, that

$$\|f\|_{C^0} = \max |f| \leq C_6(\mathrm{inj}(M, g), \mathcal{R}^2(M, g), \mathrm{Vol}(M, g)).$$

Finally, it follows from theorem 3.18, that there exists a constant $C > 1$ depending only on $\max |f|$ and thus only on $\mathrm{inj}(M, g), \mathcal{R}^2(M, g)$ and $\mathrm{Vol}(M, g)$, such that

$$C^{-1} g_T \leq g_t \leq C g_T$$

for all $t \geq T$. Since $g_t = e^{2u_t} g = e^{2(u_t - u_T)} g_T$, this implies

$$\|u_t - u_T\|_{C^0} \leq \frac{1}{2} |\log C|.$$

We already have a bound on $\|u\|_{C^0}$ and thus we obtain

$$\|u_t\|_{C^0} \leq C(\operatorname{inj}(M,g), \mathcal{R}^2(M,g), \operatorname{Vol}(M,g))$$

independent of t. This implies in particular

$$\|u_\infty\|_{C^0} \leq C(\operatorname{inj}(M,g), \mathcal{R}^2(M,g), \operatorname{Vol}(M,g)),$$

proving the theorem. $\square$

We now turn to the proof of theorem 3.15. The Mumford compactness theorem can be stated as follows.

Theorem 3.20 (Mumford)**.**
Suppose $g_n \in \mathcal{M}^{cc}$ and $\operatorname{inj}(M, g_n) > \epsilon > 0$. Then there exists a subsequence g_{n_k} and a family of smooth diffeomorphisms f_k, such that $f_k^* g_{n_k}$ converges in the $\mathcal{C}^\infty$ topology to some limiting metric $g \in \mathcal{M}^{cc}$.

The Mumford compactness theorem, the a priori estimate from the previous section and the following lemma will yield the theorem 3.15.

Lemma 3.21.
Suppose M is a closed surface. Then for any Riemannian metric g, the diameter can be bounded in terms of the injectivity radius $\operatorname{inj}(M,g)$ and the volume $\operatorname{Vol}(M,g)$.

Proof. We argue by contradiction. First recall that the volume of balls with sufficiently small radius are bounded from below in terms of their radius and the Sobolev constant. By a theorem of Croke, the Sobolev constant on a surface can be bounded in terms of the volume and the injectivity radius of the surface. Now suppose that (M, g_n) is a sequence of complete metrics with $\operatorname{inj}(M, g_n) \geq \epsilon$ and $\operatorname{Vol}(M, g_n) < V$ and $\operatorname{diam}(M, g_n) = D_n \to \infty$. Let x_n, y_n be such that $d(x_n, y_n) = D_n$. Then there exists a minimal geodesic $\gamma_n : [0, D_n] \to M$ such that $\gamma_n(0) = x_n$ and $\gamma_n(D_n) = y_n$. By the former considerations there exists $r > 0$ and $\nu > 0$, such that the volume of any ball $B_r(x) \subset (M, g_n)$ of radius r is greater than ν. We can assume $r < \epsilon$. The balls

$$B_r(\gamma_n(2kr)) \text{ for } 0 \leq k \leq \left\lfloor \frac{D_n}{2r} \right\rfloor$$

are pairwise disjoint. Hence

$$\operatorname{Vol}(M, g_n) \geq \sum_{0 \leq k \leq \left\lfloor \frac{D_n}{2r} \right\rfloor} \int_{B_r(\gamma_n(2kr))} \operatorname{vol}_{g_n} \geq \left\lfloor \frac{D_n}{2r} \right\rfloor \nu.$$

Since $D_n \to \infty$, this implies

$$\mathrm{Vol}(M, g_n) \to \infty.$$

This is a contradiction to our assumptions, and hence proves the lemma. □

Proof of theorem 3.15. Let M be a closed surface with $\chi(M) \le 0$ and let g_n be a family of Riemannian metrics on M with

$$\mathrm{Vol}(M, g_n) < V,\ \mathcal{R}^2(M, g_n) < K \text{ and } \mathrm{inj}(M, g_n) > \epsilon.$$

By the previous lemma we also have

$$\mathrm{diam}(M, g_n) < D(V, \epsilon).$$

From theorem 3.16, it follows that there exists a bound $C > 0$

$$\|\hat{u}_n\|_{C^0} \le C$$

for the uniformized metrics $\hat{g}_n = e^{-2\hat{u}_n} g_n$ independent of n. This implies in particular that

$$\mathrm{diam}(M, \hat{g}_n) \le e^{2C} D(V, \epsilon).$$

On the other hand by Cheeger's lemma a volume bound, a diameter bound and a curvature bound imply a lower injectivity radius bound. Since the $\hat{g}_n$ have constant curvature and fixed volume by assumption, we conclude that there is a lower bound $\bar{\epsilon}$ on the injectivity radii, i.e.

$$\mathrm{inj}(M, \hat{g}_n) > \bar{\epsilon}.$$

Thus we can apply Mumford's compactness theorem to the sequence $\hat{g}_n$. We obtain a subsequence $\hat{g}_{n_k}$ and a family of diffeomorphisms φ_k, such that $\varphi_k^* \hat{g}_{n_k}$ converges in the $\mathcal{C}^\infty$ topology to a metric $\hat{g}$. Denote by $\tilde{g}_k = \varphi_k^* g_{n_k}$ and by $\bar{g}_k = \varphi_k^* \hat{g}_{n_k} = e^{-2u_k} \tilde{g}_k$ the uniformized metrics. Notice that $u_k = \hat{u}_{n_k} \circ \varphi_k$. Hence the estimate

$$\|u_k\|_{C^0} = \|\hat{u}_{n_k} \circ \varphi_k\|_{C^0} \le C$$

still holds. Now consider any fixed metric $\breve{g}$ on M. Since $\bar{g}_k$ converges in the $\mathcal{C}^\infty$ topology, the metrics $\bar{g}_k$ are uniformly equivalent to each other, and hence also uniformly equivalent to $\bar{g}$. In particular, the $L^2(M, \breve{g})$ and $H^2(M, \breve{g})$ norms are uniformly equivalent to the $L^2(M, \bar{g}_k)$ and $H^2(M, \bar{g}_k)$ norms respectively. Now consider the curvature equation for $\tilde{g}_k$ with respect to $\bar{g}_k$:

$$R_{\tilde{g}_k} = 2e^{-2u_k} \left(\Delta_{\bar{g}_k} u_k - 1 \right),$$

or equivalently

$$\Delta_{\bar{g}_k} u_k = \frac{1}{2} R_{\tilde{g}_k} e^{2u_k} + \frac{1}{2}.$$

Since $\|u_k\|_{C^0} \le C$ and $\|R_{\tilde{g}_k}\|_{L^2(M, \tilde{g}_k)} < K$, the right hand side is uniformly bounded in $L^2(M, \bar{g}_k)$. Furthermore, the constant in the Calderon inequality can be chosen independent of k by theorem 3.19. Thus

$$\|u_k\|_{H^2(M, \bar{g}_k)} \le C(\|\Delta_{\bar{g}_k} u_k\|_{L^2(M, \bar{g}_k)} + \|u_k\|_{L^2(M, \bar{g}_k)}) \le \tilde{C},$$

where we also used that u_k is uniformly bounded in the $L^2(M, \check{g})$ norm, because $\|u_k\|_{C^0}$ is. Finally we obtain

$$\|u_k\|_{H^2(M,\check{g})} \leq \check{C}$$

by the uniform equivalence of the norms induced by the $\bar{g}_k$. By the Banach–Alaoglu theorem a bounded set in $H^2(M, \bar{g})$ is weakly precompact, hence the result follows by passing to a weakly convergent subsequence of u_k. $\square$

Remark: If we replace the L^2 norm of R_g by an L^p norm, we get instead boundedness in $W^{2,p}$ by appealing to L^p theory instead.

We will need another version of the compactness theorem which is adapted to smooth curves of metrics.

Theorem 3.22.
Suppose M is a closed surface with $\chi(M) \leq 0$ and let $\hat{g}$ be any Riemannian metric on M. Suppose $g_t \in \mathcal{M}$ is a smooth family defined on an interval $(0, T)$ and suppose

$$\sup_{0<t<T} \operatorname{Vol}(M, g_t) < \infty,$$

$$\sup_{0<t<T} \int_M |R_{g_t}|^2 \operatorname{vol}_{g_t} < \infty$$

and

$$\inf_{0<t<T} \operatorname{inj}(M, g_t) > 0.$$

Then there exist a family of diffeomorphisms f_t, a family of constant curvature metrics $\bar{g}_t$ and a family of conformal factors $u_t \in \mathcal{C}^\infty(M)$, such that

$$g_t = f_t^*(e^{2u_t}\bar{g}_t),$$

$$\partial_t \bar{g}_t \in \mathcal{H}_{\bar{g}_t}$$

and

$$\sup_{0<t<T} \|u_t\|_{H^2(M,\hat{g})} < \infty$$

and

$$\inf_{0<t<T} \operatorname{inj}(M, \bar{g}_t) > 0.$$

Proof of theorem 3.22. Suppose g_t, $t \in (0, T)$ is a family satisfying the conditions of the theorem. By the uniformisation theorem, there exists a family of constant curvature metrics $\hat{g}_t$ of volume 1 and a family of conformal factors u_t, such that

$$g_t = e^{2\hat{u}_t}\hat{g}_t.$$

By the apriori estimate in 3.16, there exists a uniform bound

$$\|\hat{u}_t\|_{C^0} \leq C.$$

By the lemma 3.21 there is also a uniform bound on the diameters of g_t. Together with the uniform bound on the conformal factors u_t, we thus also obtain a uniform bound on the diameters of $\hat{g}_t$. This implies that the injectivity radii of the constant curvature metrics $\bar{g}_t$ are also bounded below. Now using the lemma 3.24, we obtain a family of diffeomorphisms f_t, such that $f_t^* \bar{g}_t = \hat{g}_t$ satisfies $\partial_t \bar{g}_t \in \mathcal{H}_{\bar{g}_t}$. Letting $u_t = \hat{u}_t \circ f_t^{-1}$, we have

$$g_t = f_t^*(e^{2u_t} \bar{g}_t).$$

This implies in particular that

$$\|u_t\|_{C^0} \leq C.$$

The curvature equation is

$$R_{g_t} = e^{-2u_t}(\Delta_{\bar{g}_t} u_t + R_{\bar{g}}),$$

or equivalently

$$\Delta_{\bar{g}_t} u_t = R_{g_t} e^{2u_t} - R_{\bar{g}}.$$

By assumption the right hand side is bounded in $L^2(M, g_t)$. In particular, we can apply the Calderon–Zygmund inequality to $u_t - \int_M u_t \operatorname{vol}_{g_t}$ to obtain

$$\|u_t\|_{H^2(M,g_t)} \leq C.$$

This finishes the proof. $\square$

3.5 Splitting geometric flows on surfaces

Let g_t be a time-dependent family of Riemannian metrics on a closed surface M with $\chi(M) \leq 0$. The uniformization theorem tells us that there is a unique family u_t of smooth functions, such that

$$\bar{g}_t = e^{-2u_t} g_t$$

are metrics of constant curvature 0 with volume 1 if $\chi(M) = 0$, and metrics of constant curvature -1 if $\chi(M) < 0$. In section 3.3 we defined the finite dimensional subspace $\mathcal{H}_{\bar{g}} \subset T_{\bar{g}}\mathcal{M}^{cc}$ for a metric $\bar{g} \in \mathcal{M}$. This space is the orthogonal complement of the tangent space of $\operatorname{Diff}_0(M) \cdot \bar{g} \subset \mathcal{M}^{cc}$. By pulling back the family $\bar{g}_t$ by a family of diffeomorphisms f_t, we can arrange for $\hat{g}_t = f_t^* \bar{g}_t$ that

$$\partial_t(\hat{g}_t) \in \mathcal{H}_{\hat{g}_t} \subset T_{\hat{g}_t}\mathcal{M}^{cc}.$$

It turns out that, provided the injectivity radii of $\bar{g}_t$ are bounded from below, we get very good control on the family $\hat{g}_t$. This can be seen as a quantitative version of the Mumford compactness theorem for time dependent families.

We will use this observation to split a geometric flow into a family of conformal factors and a family of constant curvature metrics, which satisfy the above condition. We will obtain new evolution equations for the flow. This strategy was used by Buzano and Rupflin to study the harmonic Ricci flow. [11] We will very slightly generalize their results. Where they assumed that the geometric flow consists of an evolving metric and a map from the surface into some fixed manifold, we will instead assume that the geometric flow consists of an evolving metric and a section of a fiber bundle, allowing for different transformation behaviors.

3.5.1 Evolution equations for the split flow

We will first define what we mean by a (coupled) geometric flow, we will then introduce the notion of the corresponding split flow. From results of Buzano and Rupflin, existence of the corresponding split flow will follow. Finally, we derive the evolution equations for the split flow.

Let E be a fiber bundle over M. We require that there exists a pullback operation for sections of E for diffeomorphisms, i.e. given a diffeomorphism $f \in \mathrm{Diff}(M)$ and a section $s \in \Gamma(E)$, there should exist $f^*s \in \Gamma(E)$. Furthermore we ask that there is a connection in the sense that we can differentiate families of sections, i.e. given a family of sections s_t, there exists a notion of time derivative

$$\partial_t|_{t=0} s_t \in T_{s_0}\Gamma(E).$$

By $\mathcal{M}$ we denote the space of metrics on M.

Definition 3.23. Suppose $Q : \mathcal{M} \times \Gamma(E) \to T\mathcal{M} \times T\Gamma(E)$ is a diffeomorphism invariant vector field, i.e.

$$f^*Q(g, s) = Q(f^*g, f^*s).$$

We say that Q defines a *coupled geometric flow.* A family $(g_t, s_t) \in \mathcal{M} \times \Gamma(E)$, which satisfies

$$\partial_t(g_t, s_t) = Q(g_t, s_t)$$

is a solution of this coupled geometric flow.

Notice that in this setting we have a Lie derivative for sections of the fiber bundle E:

$$\mathcal{L}_X^E s = \partial_t|_{t=0} f_t^* s,$$

where $s \in \Gamma(E)$ and f_t is the flow of the vector field X.

We will now introduce the corresponding split flow. The following lemma provides that a family of constant curvature metrics can be pulled back to a “canonical form”.

Lemma 3.24 ([11], Lemma 2.2)**.**
Given a family of constant curvature metrics $\bar{g}_t \in \mathcal{M}^{cc}$, there exists a unique family of diffeomorphisms $f_t \in \mathcal{M}^{cc}$, such that

$$\partial_t(f_t^*\bar{g}_t) \in \mathcal{H}_{\bar{g}_t}$$

and $f_0 = \mathrm{id}_M$.

Now suppose $(\tilde{g}_t, \tilde{s}_t)$ is a solution of the coupled geometric flow given by a vector field Q. First we apply the uniformisation theorem to $\tilde{g}_t$ to obtain the unique conformal metric $\hat{g}_t = e^{-2\hat{u}_t}\tilde{g}_t$ of constant curvature. Then, applying the lemma to the family $\hat{g}_t$, we obtain a family of diffeomorphisms f_t, such that $\bar{g}_t = f_t^*\hat{g}_t$ fulfills the above relation. It follows that

$$g_t = f_t^*\tilde{g}_t = e^{2u_t}\bar{g}_t,$$

where $u_t = \hat{u}_t \circ f_t$. We also define $s_t = f_t^*\tilde{s}_t$.

Definition 3.25. The family $(\bar{g}_t, u_t, s_t, f_t)$ is the *split flow* corresponding to the flow $(\tilde{g}_t, \tilde{s}_t)$.

We will now derive the evolution equations of $(\bar{g}_t, u_t, s_t, f_t)$. Let X_t be the time dependent vector field generating the family of diffeomorphisms f_t. Notice that we can split the vector field Q into components $Q = (Q_m, Q_E)$. We start with the evolution of $\bar{g}_t$. We have

$$\begin{aligned}
\partial_t \bar{g}_t &= \partial_t (e^{-2u_t} f_t^* \tilde{g}_t) \\
&= -2e^{-2u_t}(\partial_t u_t) e^{2u_t} \bar{g}_t + e^{-2u_t} \mathcal{L}_{X_t} g_t + e^{-2u_t} f_t^* \partial_t \tilde{g}_t \\
&= -2(\partial_t u_t)\bar{g}_t + \mathcal{L}_{X_t}\bar{g}_t + 2(X_t u_t)\bar{g} + e^{-2u_t} f_t^* Q_m(\tilde{g}_t, \tilde{s}_t) \\
&= (-2\partial_t u_t + 2X_t u_t)\bar{g} + \mathcal{L}_{X_t}\bar{g}_t + e^{-2u_t} Q_m(g_t, s_t) \\
&= \left(-2\partial_t u_t + 2X_t u_t + \frac{1}{2}\operatorname{tr}_{g_t} Q_m(g_t, s_t)\right)\bar{g}_t + \mathcal{L}_{X_t}\bar{g}_t + e^{-2u_t}\mathring{Q}_m(g_t, s_t) \\
&= \rho_t \bar{g}_t + \mathcal{L}_{X_t}\bar{g}_t + e^{-2u_t}\mathring{Q}_m(g_t, s_t)
\end{aligned}$$

where

$$\rho_t = -2\partial_t u_t + 2X_t u_t + \frac{1}{2}\operatorname{tr}_{g_t} Q_m(g_t, s_t).$$

Rewriting this equation also yields

$$\partial_t u_t = \frac{1}{4}\operatorname{tr}_{g_t} Q_m(g_t, s_t) + X_t u_t - \frac{1}{2}\rho_t.$$

It turns out that ρ_t satisfies an elliptic equation on every time slice $M \times \{t\}$, which we will derive now. Observe that $\partial_t \bar{g}_t$ is a variation that preserves the constant curvature condition. Since constant curvature is diffeomorphism invariant, the variation

$$\rho_t \bar{g}_t + e^{-2u_t}\mathring{Q}_m(g_t, s_t)$$

also preserves constant curvature. Let $h = \rho\bar{g}_t + e^{-2u_t}\mathring{Q}_m(g_t, s_t)$. Recall the formula

$$\partial_t|_{t=0} R_{\bar{g}+th} = -\frac{1}{2}R_{\bar{g}}\operatorname{tr}_{\bar{g}} h - \Delta_{\bar{g}}\operatorname{tr}_{\bar{g}} h + \delta_{\bar{g}}\delta_{\bar{g}} h.$$

For our variation h we thus obtain

$$0 = -R_{\bar{g}_t}\rho_t - 2\Delta_{\bar{g}_t}\rho_t + \Delta_{\bar{g}_t}\rho_t + \delta_{\bar{g}_t}\delta_{\bar{g}_t}(e^{-2u_t}\mathring{Q}_m(g_t, s_t)).$$

Thus ρ_t fulfills the equation

$$\Delta_{\bar{g}_t}\rho_t + R_{\bar{g}_t}\rho_t - \delta_{\bar{g}_t}\delta_{\bar{g}_t}(e^{-2u_t}\mathring{Q}_m(g_t, s_t)) = 0$$

at any time t. The relation $\partial_t \bar{g}_t \in \mathcal{H}_{\bar{g}_t}$ implies in particular $\delta_{\bar{g}_t}\partial_t \bar{g}_t = 0$. Using that $\delta_g^* X^\flat = \mathcal{L}_X g$, we find

$$\delta_{\bar{g}}\delta_{\bar{g}}^* X_t^\flat = -\delta_{\bar{g}_t}(e^{-2u_t}(\mathring{Q}_m(g_t, \varphi_t) + \rho_t \bar{g}_t),$$

for every time t. This is also an elliptic equation for X.

Furthermore, denoting by

$$P_{\bar{g}} : \Gamma(\odot^2 T^*M) \to \mathcal{H}_{\bar{g}}$$

the orthogonal projection, the equation for $\partial_t \bar{g}_t$ implies

$$\partial_t \bar{g}_t = P_{\bar{g}_t}(e^{-2u_t} \mathring{Q}_m(g_t, s_t)),$$

because $\operatorname{im} \delta^*_{\bar{g}} + \mathcal{C}^\infty(M)\bar{g}$ is the orthogonal complement of $\mathcal{H}_{\bar{g}}$.

Finally the evolution of s_t is given by

$$\begin{aligned}
\partial_t s_t &= \partial_t (f_t^* \tilde{s}_t) \\
&= f_t^*(\partial_t \tilde{s}_t + \mathcal{L}^F_X \tilde{s}_t) \\
&= f_t^*(Q_E(\tilde{g}_t, \tilde{s}_t) + \mathcal{L}^F_{X_t} \tilde{s}_t) \\
&= Q_E(g_t, s_t) + \mathcal{L}^F_{X_t} s_t.
\end{aligned}$$

It should be noted that the elliptic equations determine X_t and ρ_t uniquely if M has genus greater than 1. If M is the torus this is not the case. To force uniqueness in that case we add the constraints

$$\int_M \rho_t \operatorname{vol}_{\bar{g}_t} = 0$$

and

$$\int_M P_{x,x_0}(X_t(x)) \operatorname{vol}_{\bar{g}_t} = 0,$$

where $P_{x,x_0} : T_x M \to T_{x_0} M$ denotes parallel transport from $T_x M$ to $T_{x_0} M$. This normalization condition was introduced in [11], Remark 2.1.

We sum up these results in the following proposition. Compare [11], prop 2.3.

Proposition 3.26.
Suppose $(\tilde{g}_t, \tilde{s}_t)$ is a coupled geometric flow in the sense of definition 3.23. Let $(\bar{g}_t, u_t, s_t, f_t)$ be the split flow of $(\tilde{g}_t, \tilde{s}_t)$ in the sense of definition 3.25. Let $X_t = \partial_t f_t$ be the generating vector field of f_t. The split flow satisifies the following equations:

$$\partial_t \bar{g}_t = P_{\bar{g}_t}(e^{-2u_t} \mathring{Q}_m(g_t, s_t)) \tag{3.27}$$

$$\partial_t u_t = \frac{1}{4} \operatorname{tr}_{g_t} Q_m(g_t, s_t) + X_t u_t - \frac{1}{2}\rho_t \tag{3.28}$$

$$\partial_t s_t = Q_E(g_t, s_t) + \mathcal{L}^F_{X_t} s_t \tag{3.29}$$

where $P_{\bar{g}_t} : \Gamma(\odot^2 T^*M) \to \mathcal{H}_{\bar{g}_t}$ is the orthogonal projection and ρ_t is the unique solution of

$$\Delta_{\bar{g}_t} \rho_t + R_{\bar{g}_t} \rho_t - \delta_{\bar{g}_t} \delta_{\bar{g}_t}(e^{-2u_t} \mathring{Q}_m(g_t, s_t)) = 0, \tag{3.30}$$

and X_t is the unique solution of

$$\delta_{\bar{g}_t} \delta^*_{\bar{g}_t} X_t^\flat = -\delta_{\bar{g}_t}(e^{-2u_t}(\mathring{Q}_m(g_t, s_t) + \rho_t \bar{g}_t). \tag{3.31}$$

If $\chi(M) = 0$, we additionally impose the normalization conditions introduced above.

3.5.2 Useful estimates

In this section we derive some useful general estimates for the quantities arising in the split flow. We have the following straightforward estimates for X and ρ. The proposition is essentially lemma 2.5 in [11], slightly adapted to our case.

Proposition 3.32.
If ρ solves equation 3.30, then

$$\|\rho\|_{L^p(M,\bar{g})} \leq C(\mathrm{inj}(M,\bar{g}))\|e^{-2u}\mathring{Q}_m(g,s)\|_{L^p(M,\bar{g})} \leq C(\mathrm{inj}(M,\bar{g}),\|u\|_{C^0})\|Q_m(g,s)\|_{L^p(M,g)}.$$

If X solves equation 3.31, then

$$\|X\|_{W^{1,p}(M,\bar{g})} \leq C(\mathrm{inj}(M,\bar{g}))\|e^{-2u}Q_m(g,s)\|_{L^p(M,\bar{g})} \leq C(\mathrm{inj}(M,\bar{g}),\|u\|_{C^0})\|Q_m(g,s)\|_{L^p(M,g)}.$$

If $\chi(M)=0$, we also impose the normalization conditions on X and ρ.

Proof. Observe that

$$\delta_{\bar{g}} : L^p \to W^{-1,p}$$

and

$$\delta_{\bar{g}}\delta_{\bar{g}} : L^p \to W^{-2,p}$$

are continuous. Furthermore, $\Delta_{\bar{g}} + R_{\bar{g}} : L^p \to W^{-2,p}$ and $\delta_g\delta_g^* : W^{1,p} \to W^{-1,p}$ are invertible if $\chi(M) < 0$. If $\chi(M) = 0$, these operators are invertible on the subspaces of functions and vector fields which satisfy the normalization conditions. By the Mumford compactness criterium, the constants of these operators only depend on the injectivity radius of $\bar{g}$. These facts yield the inequality

$$\|\rho\|_{L^p(M,\bar{g})} \leq C(\mathrm{inj}(M,\bar{g}))\|e^{-2u}\mathring{Q}_m(g,\varphi)\|_{L^p(M,\bar{g})}$$

and its vector field analogue. The inequality

$$C(\mathrm{inj}(M,\bar{g}))\|e^{-2u}\mathring{Q}_m(g,\varphi)\|_{L^p(M,\bar{g})} \leq C(\mathrm{inj}(M,\bar{g}),\|u\|_{C^0})\|Q_m(g,\varphi)\|_{L^p(M,g)}$$

and its vector field analogue follows by expressing the volume element of g in terms of the volume element of $\bar{g}$ and estimating e^{2u} by $e^{2\|u\|_{C^0}}$. □

3.6 Uniform control of horizontal curves

Horizontal curves of metrics g_t in $\mathcal{M}^{cc}$ can be controlled very well by the L^2 norm of their velocity and the injectivity radius. This is the content of the following theorem, due to Rupflin and Topping. This can be considered to be a quantitative version of the Mumford compactness theorem.

Theorem 3.33 (Lemma 2.6, [35]).
There exists a constant $C > 0$ depending on the genus of M and $k \in \mathbb{N}$, such that for any family of constant curvature metrics $g_t \in \mathcal{M}^{cc}$, such that $\partial_t g_t \in \mathcal{H}_{g_t}$, we have

$$\|\partial_t g_t\|_{C^k(M,g_t)} \leq C \frac{1}{\operatorname{inj}(M,g_t)^{1/2}} \|\partial_t g_t\|_{L^2(M,g_t)}.$$

From this we immediately obtain uniform estimates along a horizontal curve of metrics on a compact interval.

Lemma 3.34.
Suppose g_t, $t \in [0,T)$, is a family of metrics with $\partial_t g_t \in \mathcal{H}_{g_t}$,

$$\inf_{t\in[0,T)} \operatorname{inj}(M,g_t) > 0$$

and

$$\sup_{t\in[0,T)} \|\partial_t g_t\|_{L^2(M,g_t)} < \infty.$$

Then there exist constants $C_k > 0$ for every k, such that

$$\|g_t - g_0\|_{C^k(M,g_t)} < C_k \text{ for all } t \in [0,T).$$

To apply this theorem to a horizontal curve of metrics which comes from an arbitrary curve of metrics, we need the following result:

Lemma 3.35.
Assume $g_t \in \mathcal{M}$ is a smooth family of metrics on $[0,T)$ and let $\bar{g}_t \in \mathcal{M}^{cc}$ be the corresponding horizontal family of constant curvature metrics, i.e.

$$\bar{g}_t = f_t^*(e^{-2u_t} g_t).$$

If u_t and $\|\partial_t g_t\|_{L^2(M,g_t)}$ are bounded on $[0,T)$, then so is $\|\partial_t \bar{g}_t\|_{L^2(M,\bar{g}_t)}$.

Proof. Let $\hat{g}_t = e^{-2u_t} g_t$. Then

$$\partial_t \hat{g}_t = -2\partial_t u_t \hat{g}_t + e^{-2u_t}\partial_t g_t.$$

The constant curvature condition on $\hat{g}_t$ implies

$$\partial_t u_t = \frac{1}{4} \operatorname{tr}_{g_t} \partial_t g_t.$$

In particular, if u_t and $\|\partial_t g_t\|_{L^2(M,g_t)}$ are uniformly bounded, then so is $\|\partial_t \hat{g}_t\|_{L^2(M,\hat{g}_t)}$. As in the derivation of the split flow equations we find that

$$\delta_{\bar{g}_t}\delta_{\bar{g}_t}^* X_t^\flat = \delta_{\bar{g}_t}(f_t^* \partial_t \hat{g}_t),$$

where X_t is the time dependent vector field generating f_t. Applying elliptic regularity yields

$$\|X\|_{W^{1,2}(M,\bar{g}_t)} \leq C\|f_t^*\partial_t\hat{g}_t\|_{L^2(M,\bar{g}_t)} = C\|\partial_t\hat{g}_t\|_{L^2(M,\hat{g}_t)} \leq C\|\partial_t g_t\|_{L^2(M,g_t)}.$$

By assumption, the last term is uniformly bounded in t. The following calculation then implies

$$\begin{aligned}
\|\partial_t\bar{g}_t\|_{L^2(M,\bar{g}_t)} &= \|\partial_t(f_t^*\hat{g}_t)\|_{L^2(M,\bar{g}_t)} \\
&= \|f_t^*\mathcal{L}_{X_t}\hat{g}_t + f_t^*\partial_t\hat{g}_t\|_{L^2(M,f_t^*\hat{g}_t)} \\
&\leq \|f_t^*\mathcal{L}_{X_t}\hat{g}_t\|_{L^2(M,f_t^*\hat{g}_t)} + \|f_t^*\partial_t\hat{g}_t\|_{L^2(M,f_t^*\hat{g}_t)} \\
&= \|\delta_{\hat{g}_t}^* X_t^\flat\| + \|\partial_t\hat{g}_t\|_{L^2(M,\hat{g}_t)}
\end{aligned}$$

that $\partial_t\bar{g}_t$ is also bounded uniformly in in $L^2(M,\bar{g}_t)$. $\square$

Chapter 4

Stability of the spinor flow

Suppose M is a spin manifold of dimension $n = \dim M \geq 3$, and suppose $(g, \varphi) \in \mathcal{N}$ is a critical point of $\mathcal{E}$, i.e. g is a metric of special holonomy and φ is a parallel spinor field with respect to g, i.e. $\nabla^g \varphi = 0$. In this section we analyze the behavior of the spinor flow nearby such a pair (g, φ). It will turn out that there is a neighborhood of (g, φ), such that the spinor flow with initial condition in that neighborhood exists for all times and converges towards a critical point of $\mathcal{E}$. This property is called the *stability of the spinor flow.* A similar theorem can be proven for volume normalized critical points of $\mathcal{E}$, which will be introduced before proceeding to the proper discussion of the stability theorems. The precise statements of the theorems can be found in section 4.6.

We will briefly describe the strategy for the proof. The key step is to prove the following inequality for the spinorial energy. Suppose $\bar{\Phi}$ is a critical point of $\mathcal{E}$. Then there exists a $C > 0$, such that

$$\mathcal{E}(\Phi) \leq C\|Q(\Phi)\|_{L^2}^2$$

for Φ near $\bar{\Phi}$. This inequality implies that the energy decays exponentially along the spinor flow, as long as Φ_t is in the region where this inequality holds. The next step is to show that this implies long time existence and convergence of the spinor flow. This relies on the one hand on the gradient flow equality

$$\int_{T_1}^{T_2} \|Q(\Phi_t)\|_{L^2}^2 dt = \mathcal{E}(\Phi_{T_1}) - \mathcal{E}(\Phi_{T_2})$$

and on the other hand on the theory of parabolic equations. Observing that $Q(\Phi_t)$ obeys a linear parabolic equation, the bound on

$$\int_{T_1}^{T_2} \|Q(\Phi_t)\|_{L^2}^2 dt$$

should yield a bound on $Q(\Phi_t)$ in higher order Sobolev spaces by parabolic estimates. Such an estimate would then allow us to conclude convergence in the ordinary sense. The spinor flow is however not strongly parabolic, and thus these estimates do not apply. On the other hand, the gauged spinor flow is strongly parabolic. With some technical work, the estimates can be made to work for the gauged spinor flow and we can then use standard parabolic estimates.

An analogous result has been proven for the G_2 heat flow in [46]. Similar results exist for the Ricci flow and the proof of our result is closely related to the proofs in [21], [22], [26].

4.1 Volume normalized spinor flow

Before stating the stability theorems in the next sections, we will first need to introduce the *volume normalized spinor flow*. This is the negative gradient flow of the spinorial energy functional restricted to the set

$$\mathcal{N}^1 = \left\{ \Phi = (g, \varphi) \in \mathcal{N} : \int_M \mathrm{vol}_g = 1 \right\}.$$

The tangent space of the space of metrics of volume 1 is given by

$$V_g = \left\{ h \in \Gamma(\odot^2 T^*M) : \int_M \mathrm{tr}_g\, h \,\mathrm{vol}_g = 0 \right\}$$

and the orthogonal projection onto this space with respect to the L^2 metric is given by

$$\Gamma(\odot^2 T^*M) \to V_g$$
$$h \mapsto \mathring{h} = h - \frac{1}{n} \left(\int_M \mathrm{tr}_g\, h \,\mathrm{vol}_g \right) h.$$

Thus the tangent space of $\mathcal{N}^1$ at $\Phi = (g, \varphi)$ is

$$V_g \oplus \Gamma(\varphi^\perp).$$

Accordingly, the negative gradient of $\mathcal{E}$ on $\mathcal{N}^1$ at $\Phi = (g, \varphi)$ is given by

$$\mathring{Q}(g, \varphi) = (\mathring{Q}_1(g, \varphi), Q_2(g, \varphi)),$$

i.e. the orthogonal projection of the negative gradient Q of $\mathcal{E}$ on $\mathcal{N}$ to the tangent space of $\mathcal{N}^1$. The following calculation will justify the name volume normalized spinor flow. Suppose that $(g, \varphi) \in \mathcal{N}^1$ and let (g_t, φ_t) be a solution of the spinor flow with these initial conditions, i.e. such that $g_0 = g, \varphi_0 = \varphi$. The spinor flow does not leave the volume of the metric invariant, except in dimension 2. Denote by $V(t)$ the volume of the manifold (M, g_t), i.e.

$$V(t) = \int_M \mathrm{vol}_{g_t}.$$

Then for $\mu(t) = V(t)^{-2/n}$ the metric $\mu(t) g_t$ has volume 1. We will now rescale the solution (g_t, φ_t) to $\tilde{\Phi}_t = (\tilde{g}(t), \tilde{\varphi}(t))$, such that it coincides with the negative gradient flow of $\mathcal{E}$ restricted to $\mathcal{N}^1$, i.e. the volume normalized spinor flow. We make the ansatz

$$\tilde{g}(t) = \mu(\tau(t)) g(\tau(t)) \text{ and } \tilde{\varphi}(t) = \hat{B}^{g_t}_{\tilde{g}(t)}(\varphi_{\tau(t)})$$

with $\tau : I \to J$ some homeomorphism between intervals $I, J \subset \mathbb{R}$. Then

$$\partial_t \tilde{g}(t) = \mu'(\tau(t))\tau'(t)g(\tau(t)) + \mu(\tau(t))\tau'(t)(\partial_t g)(\tau(t))$$

and

$$\partial_t \tilde{\varphi}(t) = \hat{B}^{g_{\tau(t)}}_{\tilde{g}(t)} \left(\tau'(t)(\partial_t \varphi)_{\tau(t)}\right).$$

By definition the volume of $(M, \tilde{g}(t))$ is 1 and thus $\partial_t \tilde{g}(t) \in V_{\tilde{g}(t)}$. Hence $\partial_t \tilde{g}(t)$ is precisely the orthogonal projection of $\mu(\tau(t))\tau'(t)(\partial_t g)(\tau(t))$ to $V_{\tilde{g}(t)}$. Moreover, $(\partial_t g)(\tau(t)) = Q_1(g_{\tau(t)}, \varphi_{\tau(t)})$. Thus

$$\partial_t \tilde{g}(t) = \mu(\tau(t))\tau'(t)\mathring{Q}_1(g_{\tau(t)}, \varphi_{\tau(t)}).$$

Recall that

$$Q_1(c^2 g, \hat{B}^g_{c^2 g}(\varphi)) = Q_1(g, \varphi).$$

This implies

$$\partial_t \tilde{g}(t) = \mu(\tau(t))\tau'(t)\mathring{Q}_1(\tilde{g}(t), \tilde{\varphi}(t)).$$

Thus $\tilde{g}(t)$ is the metric evolution of the volume normalized spinor flow, if and only if $\mu(\tau(t))\tau'(t) = 1$. This is a separable ordinary differential equation and μ is manifestly positive. For a given initial value, such as $\tau(0) = 0$, there exists a unique solution $\tau(t)$ satisfying this constraint. Assume that τ solves this differential equation. Then the relation

$$Q_2(c^2 g, \hat{B}^g_{c^2 g}(\varphi)) = c^{-2}\hat{B}^g_{c^2 g}\left(Q_2(g, \varphi)\right)$$

implies

$$Q_2(\tilde{g}(t), \tilde{\varphi}(t)) = \mu(\tau(t))^{-1}\hat{B}^{g_{\tau(t)}}_{\tilde{g}(t)}\left(Q_2(g_{\tau(t)}, \varphi_{\tau(t)})\right).$$

Combining this with the earlier computation of the time derivative of the rescaled spinor field yields

$$\begin{aligned}
\partial_t \tilde{\varphi}(t) &= \hat{B}^{g_{\tau(t)}}_{\tilde{g}(t)}\left(\tau'(t)Q_2(g_{\tau(t)}, \varphi_{\tau(t)})\right) \\
&= \tau'(t)\mu(\tau(t))Q_2(\tilde{g}(t), \tilde{\varphi}(t)) \\
&= Q_2(\tilde{g}(t), \tilde{\varphi}(t)).
\end{aligned}$$

In conclusion, the spatially and temporally rescaled solution $\tilde{\Phi}(t)$ is a solution of the volume normalized spinor flow, i.e.

$$\partial_t \tilde{\Phi}(t) = \mathring{Q}(\tilde{\Phi}(t)).$$

Thus we can view the volume normalized spinor flow to be either a rescaling of the usual spinor flow or as the negative gradient flow of the spinorial energy functional restricted to the class of metrics of constant volume.

4.2 A coordinate chart for the universal spinor bundle

The spinor flow is a dynamical system on the infinite-dimensional space $\mathcal{N} \subset \mathcal{F}$. The space $\mathcal{F}$ of sections of the universal spinor bundle can be considered to be a Fréchet manifold with respect to the $\mathcal{C}^\infty$ topology. This point of view is quite natural, but not very useful for our purposes, because we wish to invoke standard results from analysis, which are usually proven for Sobolev or Hölder spaces of functions or sections. For this reason, we choose a local parametrization of $\mathcal{F}$ near a point $\Phi \in \mathcal{F}$. We will then work within this parametrization, giving the space $\mathcal{F}$ the necessary function space topologies locally.

Recall that for $\Phi = (g, \varphi) \in \mathcal{F}$ the tangent space $T_\Phi\mathcal{F}$ is given by $\Gamma(\odot^2 T^*M \oplus \Sigma_g M)$. With the domain

$$U_g = \{h \in \Gamma(\odot^2 T^*M) : g + h \text{ is a Riemannian metric}\} \times \Gamma(\Sigma_g M)$$

the map

$$\begin{aligned} \Xi = \Xi_{g,\varphi} : U_g &\to \mathcal{F} \\ (h, \psi) &\mapsto \Big(g + h, \hat{B}^g_{g+h}(\varphi + \psi)\Big) \end{aligned}$$

is a parametrization of $\mathcal{F}$. Its inverse is given by

$$\Xi^{-1} : \Gamma(\Sigma M) \to U_g$$

$$(g', \varphi') \mapsto (g' - g, \hat{B}^{g'}_g(\varphi') - \varphi).$$

The metric g induces a natural metric on the vector bundle $\odot^2 T^*M$ and the spinor bundle $\Sigma_g M$ carries its Hermitian metric. These metrics in turn induce Sobolev and Hölder norms on $T_\Phi\mathcal{F} = \Gamma(\odot^2 T^*M \oplus \Sigma_g M)$. We can now interpret the spinorial energy as a function $U_g \to \mathbb{R}$ and the negative gradient as a map $Q : U_g \subset \Gamma(\odot^2 T^*M \oplus \varphi^\perp) \to \Gamma(\odot^2 T^*M \oplus \varphi^\perp)$. In the following, whenever we use a Sobolev norm on the space $\mathcal{N}$ or $\mathcal{F}$ it is to be understood in the following manner. Suppose $\Phi = (g, \varphi) \in \mathcal{N}$ is fixed. Then by

$$\|\tilde{\Phi} - \Phi\|_X \text{ or } d_X(\tilde{\Phi}, \Phi)$$

we mean

$$\|\Xi^{-1}_{g,\varphi}(\tilde{\Phi})\|_X,$$

where X is a Sobolev or Hölder space on the vector bundle $\odot^2 T^*M \oplus \Sigma_g M$.

To analyze the mapping flow, which relates the spinor flow and the gauged spinor flow, we proceed analogously for the space $\mathcal{C}^\infty(M, M)$ of smooth maps. Suppose $f_0 \in \mathcal{C}^\infty(M, M)$ is a diffeomorphism. Then the following map

$$\theta_{f_0} : U \subset \mathcal{C}^\infty(M, M) \to V \subset \Gamma(f_0^* TM)$$

$$f \mapsto (x \mapsto (\exp_{f_0(x)})^{-1}(f(x)))$$

is a local chart of $\mathcal{C}^\infty(M,M)$ near f_0. The exponential map is the Riemannian exponential map with respect to some metric g. Denote by $B_x \subset T_xM$ the largest ball around the origin on which $\exp_x$ is a diffeomorphism. Then we define

$$V = \{s \in \Gamma(f_0^*TM) : s(x) \in B_{f_0(x)} \text{ for every } x \in M\}$$

and

$$U = \{f \in \mathcal{C}^\infty(M,M) : f(x) \in \exp_{f_0(x)}(B_x)\}.$$

Using the induced metric on f_0^*TM we can again define

$$\|f - f_0\|_X \text{ or } d_X(f, f_0)$$

via

$$\|\theta_{f_0}(f)\|_X$$

for X a Sobolev or Hölder space defined on the space of sections of f_0^*TM.

4.3 A slice theorem for the action of the diffeomorphism group

A local *slice* of a group action $G \curvearrowright X$ on a manifold X at a point $x \in X$ is a submanifold $S \subset X$, such that $x \in S$ and such that there is a neighborhood U of x, such that for any $\tilde{x} \in U$ the orbit $G \cdot \tilde{x}$ meets S once and only once, i.e. $S \cap G \cdot \tilde{x} = \{s_{\tilde{x}}\}$. Thus if S is a slice any orbit nearby x can be represented by a unique element $s \in S$ and the tangent space at x of X splits into

$$T_xX = T_x(G \cdot x) \oplus T_xS.$$

One can choose a local parametrization of S by T_xS. We will prove a slice theorem for the action of the spin diffeomorphism group $\widehat{\mathrm{Diff}}_s(M)$ on the set $\mathcal{F} = \Gamma(\Sigma M)$. We will consider these spaces as Banach manifolds with the Hölder space topology. Then the group action is a continuous and differentiable map

$$\widehat{\mathrm{Diff}}_s^{k+1,\alpha}(M) \times \Gamma^{k,\alpha}(\Sigma M) \to \Gamma^{k,\alpha}(\Sigma M).$$

That the diffeomorphism group needs one more differentiability level is explained by the fact that the diffeomorphism is differentiated once in the group action. We will prove a relatively weak, but sufficient version of the slice theorem, which roughly says that for any $\Phi = (g, \varphi) \in \mathcal{F}$ there is a neighborhood U and a slice S, such that for any element $\tilde{\Phi}$ in the neighborhood U there exists an $F \in \widehat{\mathrm{Diff}}_s(M)$, such that $F^*\tilde{\Phi} \in S$. For technical reasons we assume that the isometry group of the metric g is discrete.

Recall that $\mathcal{F}$ is parametrized via the map

$$\Xi_{g,\varphi} : U_g \to \mathcal{F}.$$

The slice we construct is not directly given as a subspace of $\mathcal{F}$, but rather will be parametrized by a subset of U_g. Indeed, the slice will be a neighborhood of the origin in $\ker \lambda_\Phi$. This reflects the fact that $\operatorname{im} \lambda_\Phi^* = T_\Phi \widehat{\mathrm{Diff}}_s(M) \cdot \Phi$ and that $\ker \lambda_\Phi$ is its orthogonal complement.

Proposition 4.1.
Let $\Phi = (g, \varphi) \in \mathcal{F} = \Gamma(\Sigma M)$ and assume g has no Killing fields. Then there exists a $C^{k+1,\alpha}$ neighborhood U of Φ, such that for any $\tilde{\Phi} \in U$, there exists a $C^{k+2,\alpha}$ diffeomorphism $f : M \to M$, such that
$$\lambda_\Phi(\Xi^{-1}(F^*\tilde{\Phi})) = 0,$$
where $\Xi = \Xi_{g,\varphi} : U_g \to \mathcal{F}$.

Proof. The proof is based on [43], where a similar theorem is proven for the diffeomorphism action on the space of metrics. The proof is based on the implicit function theorem for Banach spaces. The diffeomorphism group will be parametrized by the space of vector fields by the map
$$\Gamma(TM) \to \mathrm{Diff}(M)$$
$$X \mapsto f_X,$$
where $f_X : M \to M$ is the time 1 map of the vector field X. Consider the map
$$G : \Gamma^{k+1,\alpha}(\odot^2 T^*M \oplus \Sigma_g M) \times \Gamma^{k+2,\alpha}(TM) \to \Gamma^{k,\alpha}(TM)$$
$$((h, \psi), X) \mapsto \lambda_\Phi\left(f_X^*(g + h, \hat{B}^g_{g+h}(\varphi + \psi))\right).$$
Its derivative at $((0, 0), 0)$ in the second parameter is given by
$$DG((0, 0), 0)(0, V) = \frac{d}{dt}\Big|_{t=0} \lambda_\Phi(f_{tV}^*\Phi) = \lambda_\Phi(\mathcal{L}_V \Phi) = \lambda_\Phi \lambda_\Phi^* V,$$
with $V \in \Gamma(TM)$. The map
$$\lambda_\Phi \lambda_\Phi^* : \Gamma(TM) \to \Gamma(TM)$$
coincides with $\delta_g \delta_g^* X^\flat$. This map is injective if g has no Killing fields, because
$$g(\delta_g \delta_g^* X^\flat, X) = |\delta_g^* X^\flat|^2_{L^2} = |\mathcal{L}_X g|^2_{L^2}.$$
Moreover, $\delta_g \delta_g^*$ is elliptic and by construction self adjoint. Hence it must also be surjective. Thus the implicit function theorem applies to G at $((0, 0), 0)$. This means that there exists a neighborhood $U \subset \Gamma^{k+1,\alpha}(\odot^2 T^*M \oplus \Sigma_g M)$ of $(0, 0)$ and a map $H : U \to \Gamma^{k+2,\alpha}(TM)$, such that for any $(h, \psi) \in U$
$$G((h, \psi), H(h, \psi)) = 0.$$
Suppose $\tilde{\Psi} \in \Xi_\Phi(U)$. Then let $X = H(\Xi_\Phi(\tilde{\Phi}))$ and $f : M \to M$ its time 1 map. Then
$$\lambda_\Phi^*(F^*(\Xi_\Phi^{-1}(\tilde{\Phi}))) = 0$$
by construction. Since the pullback F^* and the parametrization commute, the statement of the proposition follows. $\square$

4.4 Łojasiewic inequality and energy decay

In this section we establish the crucial gradient–energy inequality we mentioned in the introduction. This will then imply exponential decay of the energy near a critical point. It is instructive to consider the finite-dimensional case first. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function and that $0 \in \mathbb{R}^n$ is a critical point of f and suppose $f(0) = 0$. If the critical set

$$\mathrm{Crit}(f) = \{x \in \mathbb{R}^n : df(x) = 0\}$$

is a submanifold of $\mathbb{R}^n$ (near 0), the function f is called Morse–Bott (at 0). In that case a generalization of the Morse lemma applies. This lemma, called the Morse–Bott lemma, says that there exist coordinates $(\tilde{x}_1, ..., \tilde{x}_n)$ centered at the origin, such that

$$f(\tilde{x}_1, ..., \tilde{x}_n) = \tilde{x}_1^2 + ... + \tilde{x}_p^2 - \tilde{x}_{p+1}^2 - ... - \tilde{x}_{p+q}^2,$$

where $n - (p + q) = \dim \mathrm{Crit}(f)$. With respect to the standard Euclidean metric in these coordinates

$$|\operatorname{grad} f(\tilde{x}_1, ..., \tilde{x}_n)|^2 = 4 \sum_{i=1}^{p+q} \tilde{x}_i^2.$$

In particular,

$$|f(\tilde{x}_1, ..., \tilde{x}_n)| \leq \frac{1}{4} \left|\operatorname{grad} f(\tilde{x}_1, ..., \tilde{x}_n)\right|^2.$$

Since an arbitrary Riemannian metric is equivalent to the Euclidean metric on any precompact neighborhood, we find a neighborhood of the origin where

$$|f(x)| \leq C |\operatorname{grad} f(x)|^2$$

holds. The assumption that $\mathrm{Crit}(f)$ is a submanifold near 0 is quite strong. If f is real analytic, a weaker inequality still holds without this assumption. This inequality is due to Łojasiewicz and is a difficult theorem in the theory of semi-analytic sets.[31] In that case there exists a $\theta \in (1, 2)$ and the following inequality holds in a neighborhood of the origin:

$$|f(x)| \leq |\operatorname{grad} f(x)|^\theta.$$

The inequality becomes weaker as θ becomes smaller. For this reason we call the inequality with $\theta = 2$ in the case of Morse–Bott functions the optimal Łojasiewicz inequality. The inequality

$$\mathcal{E}(\Phi) \leq C \|Q(\Phi)\|_{L^2}^2$$

is completely analogous to the finite-dimensional inequality. Indeed, this inequality will be shown by appeal to the finite-dimensional case by a process known as Lyapunov–Schmidt reduction. The Lyapunov–Schmidt reduction exploits that a nonlinear map between Banach spaces whose linearization is Fredholm is in some sense determined by its behavior on a finite dimensional space. Due to the diffeomorphism invariance of $\mathcal{E}$, the linearization of $\mathcal{E}$ or Q are *not* a Fredholm maps. For this reason we will work on the slice provided in the last section.

Although the idea is fairly straightforward, implementing this scheme requires a fair amount of technical work, which is inessential to the broad structure of the argument. The majority of this section is devoted to proving the following to propositions. At the end of this section it is shown how these inequalities imply decay of the energy.

Proposition 4.2 (Optimal Łojasiewicz inequality for absolute minimizers)**.**
Suppose $\bar{\Phi} = (\bar{g}, \bar{\varphi}) \in \mathcal{N}$ *is a critical point of* $\mathcal{E}$*, i.e.* $\bar{\Phi}$ *is an absolute minimizer with* $\mathcal{E}(\bar{\Phi}) = 0$*. Then there exists a* $C^{2,\alpha}$ *neighborhood* $U \subset \mathcal{N}$ *of* $\bar{\Phi}$ *and a* $C > 0$*, such that for any* $\Phi \in U$ *the inequality*

$$\mathcal{E}(\Phi) \leq C\|Q(\Phi)\|_{L^2}^2$$

holds.

Proposition 4.3 (Łojasiewicz inequality for volume constrained critical points)**.**
Suppose $\bar{\Phi} = (\bar{g}, \bar{\varphi}) \in \mathcal{N}$ *is a volume constrained critical point of* $\mathcal{E}$*, i.e. a critical point on the set* $\mathcal{N}^1$*. Then there exists a* $C^{2,\alpha}$ *neighborhood* $U \subset \mathcal{N}$ *of* $\bar{\Phi}$ *and* $\theta \in (1, 2)$*, such that for any* $\Phi \in U$ *the inequality*

$$|\mathcal{E}(\Phi) - \mathcal{E}(\bar{\Phi})| \leq \|Q(\Phi)\|_{L^2}^{\theta}$$

holds. If $\mathcal{E}$ *is Morse–Bott at* $\bar{\Phi}$*, i.e. the critical set is a manifold near* $\bar{\Phi}$*, then this can be improved to*

$$|\mathcal{E}(\Phi) - \mathcal{E}(\bar{\Phi})| \leq C\|Q(\Phi)\|_{L^2}^2$$

for some $C > 0$*.*

Both propositions will be proven by application of a fairly general infinite-dimensional version of the Łojasiewicz inequality due to Colding and Minicozzi II. In the theorem, the function spaces, such as L^2, are to be understood as spaces of sections of some vector bundle.

Theorem 4.4 ([16], Thm. 6.3)**.**

1. *Suppose* $E \subset L^2$ *is a closed subspace,* U *is an open neighborhood of* 0 *in* $C^{2,\alpha} \cap E$*.*
2. *Suppose* $G : U \to \mathbb{R}$ *is an analytic function or that* $G : U \to \mathbb{R}$ *is a smooth function and there is a neighborhood* V *of* 0*, such that* $\{x \in V : \operatorname{grad} G(x) = 0\}$ *is a finite dimensional submanifold.*
3. *Suppose the gradient* $\operatorname{grad} G : U \to C^{\alpha} \cap E$ *is* C^1*,* $\operatorname{grad} G(0) = 0$ *and*

$$\|\operatorname{grad} G(x) - \operatorname{grad} G(y)\|_{L^2} \leq C\|x - y\|_{H^2}$$

4. $L = D \operatorname{grad} G(0)$ *is symmetric, bounded from* $C^{2,\alpha} \cap E$ *to* $C^{\alpha} \cap E$ *and from* $H^2 \cap E$ *to* $L^2 \cap E$ *and Fredholm from* $C^{2,\alpha} \cap E$ *to* $C^{\alpha} \cap E$*.*

Then there exists $\theta \in (1, 2)$ *and a* $C^{2,\alpha} \cap E$ *neighborhood* $\tilde{U} \subset U$*, such that for all* $x \in \tilde{U}$

$$|G(x) - G(0)| \leq \|\operatorname{grad} G(x)\|_{L^2}^{\theta}$$

If there is a neighborhood V of 0, such that $\{x \in V : \operatorname{grad} G(x) = 0\}$ is a finite dimensional submanifold, we get the stronger inequality

$$|G(x) - G(0)| \leq C \| \operatorname{grad} G(x) \|_{L^2}^2$$

for some $C > 0$.

Remark. Colding and Minicozzi II prove this theorem by passing to the finite dimensional case using Lyapunov–Schmidt reduction. The statement of the theorem in [16] only contains the case that G is analytic. The alternative condition we give is that G is Morse–Bott at 0. The proof in that case is the same except that when the finite dimensional Łojasiewicz inequality is invoked, the optimal inequality for Morse–Bott functions is used.

To derive the propositions 4.2, 4.3 from this theorem, we will construct a function, which satisfies the conditions of the theorem, and then show how the Łojasiewicz inequality we obtain for the constructed function implies the Łojasiewicz inequality for the spinorial energy functional. Verifying all the conditions of theorem 4.4 is the longest and most tedious part of this proof.

Theorem 4.4 requires that the function G has Fredholm linearization. We know that this is not the case for the spinorial energy functional $\mathcal{E}$ due to its diffeomorphism invariance. We also know that the gradient of $\mathcal{E}$ or equivalently Q is weakly elliptic and can be made elliptic by a modification contained in the (infinitesimal) diffeomorphism action. These facts can be used to show that the restriction of $\mathcal{E}$ to the slice of the $\widehat{\mathrm{Diff}}_s(M)$ action has Fredholm gradient. The other conditions of theorem 4.4 are straightforward consequences of the definition of $\mathcal{E}$.

The second step, showing that the Łojasiewicz inequality for the restriction of $\mathcal{E}$ to the slice implies a Łojasiewicz inequality for $\mathcal{E}$ on all of $\mathcal{N}$, is then a simple application of the slice theorem and the diffeomorphism invariance of $\mathcal{E}$ and Q.

For the proof of proposition 4.2, we start with the following lemma.

Lemma 4.5.
Suppose $\bar{\Phi} = (\bar{g}, \bar{\varphi}) \in \mathcal{N}$ is a critical point of $\mathcal{E}$ with $\bar{g}$ having no Killing fields. Then let $\iota : \ker \lambda_{\bar{\Phi}} \subset T_{\bar{\Phi}}\mathcal{F} \to T_{\bar{\Phi}}\mathcal{F}$ be the inclusion and $\Xi_{\bar{g},\bar{\varphi}} : U_{\bar{g}} \subset T_{\bar{\Phi}}\mathcal{F} \to \mathcal{F}$ be the parametrization from section 4.2. The functional

$$f = \mathcal{E} \circ \Xi_{\bar{g},\bar{\varphi}} \circ \iota : \ker \lambda_{\bar{\Phi}} \to \mathbb{R}$$

satisfies the conditions of theorem 4.4 and in conclusion there exists a $C^{2,\alpha}$ neighborhood U of 0 and $C > 0$, such that

$$|f(x)| \leq C \| \operatorname{grad} f(x) \|_{L^2}^2.$$

Proof. Observe that $\mathcal{E}(\bar{\Phi}) = 0$ and hence $f(0) = 0$.

The space $T_{\bar{\Phi}}\mathcal{F} = \Gamma(\odot^2 T^*M \oplus \Sigma_g M)$ is equipped with an L^2 norm and $C^{k,\alpha}$ norms by the metric $\bar{g}$. The map

$$\lambda_{\bar{\Phi}} : \Gamma(\odot^2 T^*M \oplus \Sigma_g M) \to \Gamma(TM)$$

is a linear first order differential operator and hence induces a continuous map

$$\lambda_{\bar\Phi} : L^2(\odot^2 T^*M \oplus \Sigma_g M) \to H^{-1}(TM)$$

and hence $\ker \lambda_{\bar\Phi}$ is a closed subspace of $L^2(\odot^2 T^*M \oplus \Sigma_g M)$. The functional $\mathcal{E}$ is well-defined on all of $\ker \lambda_{\bar\Phi} \subset C^{2,\alpha}(\odot^2 T^*M \oplus \Sigma_g M)$, since the definition of $\mathcal{E}$ depends only on the first derivatives of (g, φ). Thus the first condition in theorem 4.4 is met.

The function f is clearly smooth and by theorem 2.45 the set $\mathcal{E}^{-1}(0)$ is a smooth submanifold of $\mathcal{N}$. Hence the set $f^{-1}(0)$ also is a smooth submanifold. In the following we will see that the gradient of f is a Fredholm map. This in particular implies that the dimension of $f^{-1}(0)$ is finite. This will prove the second condition from theorem 4.4.

By assumption $\operatorname{grad} \mathcal{E}(\bar\Phi) = 0$ and this implies that also $\operatorname{grad} f(0) = 0$, since ι is an immersion and $\Xi_{\bar g,\bar\varphi}$ is a diffeomorphism. The gradient can be considered as a nonlinear second order differential operator

$$\operatorname{grad} f : \Gamma^{2,\alpha}(\odot^2 T^*M \oplus \Sigma_g M) \to \Gamma^{\alpha}(\odot^2 T^*M \oplus \Sigma_g M).$$

This operator is smooth, since $\mathcal{E}$ is smooth. Locally, $\operatorname{grad} f$ can be represented as a polynomial expression in the local coefficients of g and φ and their first and second derivatives. Thus we can find a $C^{2,\alpha}$ neighborhood, where

$$\| \operatorname{grad} f(x) - \operatorname{grad} f(y)\|_{L^2} \le C\|x - y\|_{H^2},$$

because we can bound all terms appearing in $\operatorname{grad} f(x)$ by a constant. Hence the third condition in theorem 4.4 is fulfilled.

The last condition is essentially the ellipticity of $\operatorname{grad} \mathcal{E}$ orthogonally to the diffeomorphism action. As in theorem 4.4 we denote $L = D \operatorname{grad} f(0)$. Since

$$\operatorname{grad} f = D(\Xi_{\bar g,\bar\varphi} \circ \iota)^* \operatorname{grad} \mathcal{E},$$

it follows that

$$L = D \operatorname{grad} f(0) = D(\Xi_{\bar g,\bar\varphi} \circ \iota)^* D \operatorname{grad} \mathcal{E}(\bar\Phi).$$

By proposition 2.44 $D \operatorname{grad} \mathcal{E}(\bar\Phi)$ is symmetric and hence so is L. Moreover $\operatorname{grad} \mathcal{E}(\bar\Phi)$ is a linear second order differential operator, which induces continuous maps $H^2 \to L^2$ and $C^{2,\alpha} \to C^\alpha$. Thus L also induces continuous maps $H^2 \to L^2$ and $C^{2,\alpha} \to C^\alpha$. It remains to be seen that these induced maps are Fredholm. By earlier calculations we know that

$$\tilde G : \Phi \mapsto \operatorname{grad} \mathcal{E}(\Phi) + \lambda^*_\Phi(X_{\bar g}(\Phi))$$

is strongly elliptic, i.e. its linearization is elliptic and hence induces Fredholm operators on Sobolev and Hölder spaces. Its linearization is also symmetric. The diffeomorphism invariance of $\mathcal{E}$ implies the Bianchi identity $\lambda_{\bar\Phi} \operatorname{grad} \mathcal{E}(\bar\Phi) = 0$. Linearizing this identity yields

$$\lambda_{\bar\Phi} \circ D \operatorname{grad} \mathcal{E}(\bar\Phi) = 0 \text{ and } D \operatorname{grad} \mathcal{E}(\bar\Phi) \circ \lambda^*_{\bar\Phi} = 0.$$

These identities imply that with respect to the splitting

$$T_{\bar\Phi}\mathcal{N} = \ker \lambda_{\bar\Phi} \oplus \operatorname{im} \lambda^*_{\bar\Phi}$$

the operator $D\tilde{G}(\bar\Phi)$ has the form

$$\begin{array}{c c} & \begin{array}{cc} \ker \lambda_{\bar\Phi} & \operatorname{im} \lambda^*_{\bar\Phi} \end{array} \\ \begin{array}{c} \ker \lambda_{\bar\Phi} \\ \operatorname{im} \lambda^*_{\bar\Phi} \end{array} & \begin{pmatrix} P & 0 \\ 0 & R \end{pmatrix} \end{array}$$

and $D(\operatorname{grad} \mathcal{E})(\bar\Phi)$ has the form

$$\begin{array}{c c} & \begin{array}{cc} \ker \lambda_{\bar\Phi} & \operatorname{im} \lambda^*_{\bar\Phi} \end{array} \\ \begin{array}{c} \ker \lambda_{\bar\Phi} \\ \operatorname{im} \lambda^*_{\bar\Phi} \end{array} & \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} \end{array}.$$

Since $D\tilde{G}(\bar\Phi)$ is Fredholm, so is $P = \pi \circ D \operatorname{grad} \mathcal{E}(\bar\Phi) \circ \iota$, where $\pi : T_{\bar\Phi}\mathcal{N} \to \ker \lambda_{\bar\Phi}$ denotes the orthogonal projection. On the other hand

$$D \operatorname{grad} f(0) = D(\Xi_{\bar g,\bar\varphi} \circ \iota)^* D \operatorname{grad} \mathcal{E}(\bar\Phi) = \pi \circ (D\Xi_{\bar g,\bar\varphi}(0))^* \operatorname{grad} \mathcal{E}(\bar\Phi).$$

Furthermore,

$$D\Xi_{\bar g,\bar\varphi}(0) : T_{\bar\Phi}U_g = T_{\bar\Phi}\mathcal{F} \to T_{\bar\Phi}\mathcal{F}.$$

Hence $D \operatorname{grad} f(0) = P$ is Fredholm and the last condition in theorem 4.4 is met. Since all conditions in theorem 4.4 hold, we conclude that there is a $C^{2,\alpha}$ neighborhood U of 0, such that

$$f(x) \leq C\| \operatorname{grad} f(x)\|^2_{L^2}$$

for all $x \in U$. □

Proof of proposition 4.2. It remains to be shown that the Łojasiewicz inequality for f from the lemma implies the Łojasiewicz inequality for $\mathcal{E}$. By the slice theorem there exists a $C^{2,\alpha}$ neighborhood U of $\bar\Phi$, such that for any $\Phi \in U$ there exists a $C^{3,\alpha}$ diffeomorphism $f : M \to M$, such that

$$\lambda_{\bar\Phi}(\Xi_{\bar g,\bar\varphi}(F_*\Phi)) = 0.$$

The diffeomorphism invariance of $\mathcal{E}$ means

$$\mathcal{E}(\Phi) = \mathcal{E}(F_*\Phi) \text{ and } F_*Q(\Phi) = Q(F_*\Phi).$$

By the diffeomorphism invariance of the L^2 metric

$$\|Q(\Phi)\|_{L^2} = \|F_*Q(\Phi)\|_{L^2}.$$

Hence if

$$\mathcal{E}(\Phi) \leq C\|Q(\Phi)\|_{L^2},$$

then also
$$\mathcal{E}(F_*\Phi) \leq C\|Q(F_*\Phi)\|_{L^2}.$$
In particular, if we show there exists a $C^{2,\alpha}$ neighborhood V of 0 in $\ker \lambda_{\bar{\Phi}}$ such that
$$\mathcal{E}(\Xi_{\bar{g},\bar{\varphi}}(\Psi)) \leq C\|Q(\Xi_{\bar{g},\bar{\varphi}}(\Psi))\|_{L^2}^2$$
for every $\Psi \in V$, then the Łojasiewicz inequality holds on a neighborhood of $\bar{\Phi}$ of the form
$$\{\Phi \in U : \Phi = F_*\Xi_{\bar{g},\bar{\varphi}}(\Psi) \text{ for some } \Psi \in V\}.$$
Since this set is a $C^{2,\alpha}$ neighborhood of $\bar{\Phi}$, knowing the inequality on the slice would prove the theorem. The previous lemma says that there is a $C^{2,\alpha}$ neighborhood V of 0 in $\ker \lambda_{\bar{\Phi}}$, such that
$$|f(\Psi)| \leq C\|\operatorname{grad} f(\Psi)\|_{L^2}^2$$
holds for all $\Psi \in V$. By definition
$$f(\Psi) = \mathcal{E}(\Xi_{\bar{g},\bar{\varphi}}(\Psi)).$$
Furthermore,
$$\operatorname{grad} f(\Psi) = D(\Xi_{\bar{g},\bar{\varphi}} \circ \iota)^* \operatorname{grad} \mathcal{E}(\Xi_{\bar{g},\bar{\varphi}}(\Psi)).$$
Since $D(\Xi_{\bar{g},\bar{\varphi}} \circ \iota)$ is just a (0 order) linear map, it is continuous from L^2 to L^2, i.e. we have
$$\|\operatorname{grad} f(\Psi)\|_{L^2} = \|D(\Xi_{\bar{g},\bar{\varphi}} \circ \iota)^* \operatorname{grad} \mathcal{E}(\Xi_{\bar{g},\bar{\varphi}}(\Psi))\|_{L^2} \leq \hat{C}\|\operatorname{grad} \mathcal{E}(\Xi_{\bar{g},\bar{\varphi}}(\Psi))\|_{L^2}.$$
A point that has been suppressed by our notation is that the L^2 metric on the left hand side is the L^2 metric induced by $\bar{g}$, but the L^2 metric on the right hand side is the metric induced by g, where $(g, \varphi) = \Xi_{\bar{g},\bar{\varphi}}(\Psi)$. But since V is a bounded $C^{2,\alpha}$ neighborhood, all metrics in this neighborhood are uniformly equivalent. Combining the Łojasiewicz inequality from the lemma and this inequality, we obtain
$$\mathcal{E}(\Xi_{\bar{g},\bar{\varphi}}(\Psi)) \leq C\hat{C}^2\|\operatorname{grad} \mathcal{E}(\Xi_{\bar{g},\bar{\varphi}}(\Psi))\|_{L^2}^2 = C\hat{C}^2\|Q(\Xi_{\bar{g},\bar{\varphi}}(\Psi))\|_{L^2}^2,$$
proving the proposition. □

Proof of proposition 4.3. The proof of the Łojasiewicz inequality for volume constrained minimizers proceeds quite similarly to the proof in the case of absolute minimizers. First observe that $\mathcal{E}$ is an analytic functional on $\Gamma^{2,\alpha}(\Sigma M)$, because the maps $g \mapsto \nabla^g$, $g \mapsto \mathrm{vol}_g$, $\Gamma(T^*M \otimes \Sigma M \oplus \odot_+^2 T^*M) \ni (\zeta, g) \mapsto |\varphi|_g^2$ and $\omega \mapsto \int_M \omega$ are analytic.

As in the proof of proposition 4.2, we now construct from $\mathcal{E}$ a function f on the slice, for which theorem 4.4 holds. For this, we first construct a parametrization of $\mathcal{N}^1$. Since 1 is a regular value of the map $\mathrm{Vol} : g \mapsto \int_M \mathrm{vol}_g$ on $\Gamma^{2,\alpha}(\odot_+^2 T^*M)$, by the analytic regular value theorem there exists a neighborhood of 0 in $V_0 \subset \ker D\,\mathrm{Vol}(\bar{g}) = \{h \in \Gamma(\odot^2 T^*M) : \int_M \operatorname{tr}_g h \,\mathrm{vol}_g = 0\}$ and a local parametrization $V_0 \to \mathrm{Vol}^{-1}(1)$. Combining this parametrization with the parametrization $\Xi_{\bar{g},\bar{\varphi}}$ gives an analytic parametrization of a slice of the diffeomorphism group action in $\mathcal{N}^1$ by the set $U = \{(h, \psi) \in \ker \lambda_{\bar{\Phi}} : h \in V_0\}$. This parametrization will be denoted by

$$\chi : U \to S \subset \mathcal{N}^1.$$

Then we define a function $f = \mathcal{E} \circ \chi$. This function fulfills the conditions of theorem 4.4, which can be checked as in the previous lemma. Thus there exists a $C^{2,\alpha}$ neighborhood $W \subset U$ of 0 and a $\theta \in (1, 2)$ if the critical set is not smooth and a constant $C > 0$ if the critical set is smooth, such that

$$|f(x) - f(0)| \leq \| \operatorname{grad} f\|_{L^2}^{\theta} \text{ in the first case}$$

and

$$|f(x) - f(0)| \leq C\| \operatorname{grad} f\|_{L^2}^{2} \text{ in the second case.}$$

Analogously to the previous proof, the Łojasiewicz inequality for $\mathcal{E}$ on $\mathcal{N}^1$ will follow by applying the slice theorem and showing that the inequality for f implies the inequality on the slice. For the last part, we need to show

$$|\mathcal{E}(\chi(x)) - \mathcal{E}(\chi(0))| = |f(x) - f(0)| \leq \| \operatorname{grad} f\|_{L^2}^{\theta} \leq \|\mathring{Q}(\chi(x))\|_{L^2}^{\tilde{\theta}}.$$

The equality follows from the definition of f and the first inequality is the Łojasiewicz inequality for f. Hence we only need to justify the last inequality. We compute

$$\operatorname{grad} f(x) = (D\chi(x))^* \operatorname{grad} \mathcal{E} \circ \iota(\chi(x)) = (D\chi(x))^*(D\iota)^* \operatorname{grad} \mathcal{E}(\chi(x))$$

where $\iota : \mathcal{N}^1 \to \mathcal{N}$ is the inclusion. The adjoint of $D\iota$ is the orthogonal projection

$$\pi : T_\Phi\mathcal{N} \to T_\Phi\mathcal{N}^1$$

and by definition

$$\mathring{Q}(\Phi) = -\pi(\operatorname{grad} \mathcal{E}(\Phi)).$$

Furthermore, by the regular value theorem $D\chi$ is Lipschitz, hence

$$\| \operatorname{grad} f(x)\|_{L^2} = \|(D\chi(x))^*\mathring{Q}(\chi(x))\|_{L^2} \leq C\|\mathring{Q}(\chi(x))\|_{L^2}.$$

Finally, since $\| \operatorname{grad} f(x)\|_{L^2}$ is arbitrary small in a neighborhood of 0, we conclude

$$\| \operatorname{grad} f(x)\|_{L^2}^{\theta} \leq \|\mathring{Q}(\chi(x))\|_{L^2}^{\tilde{\theta}}$$

for some $1 < \tilde{\theta} < \theta$. This proves the proposition. □

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ has a local minimum at the origin with $f(0) = 0$ and suppose that the optimal Łojasiewicz inequality

$$|f(x)| \leq C| \operatorname{grad} f(x)|^2$$

holds on a neighborhood U of the origin. Now suppose that $x(t)$ solves the negative gradient flow equation

$$x'(t) = -\operatorname{grad} f(x(t))$$

and that $x(t) \in U$ for all t. Then

$$\frac{d}{dt} f(x(t)) = -|\operatorname{grad} f(x(t))|^2 \leq -C^{-1} |f(x(t))|.$$

Grönwall's inequaliy implies

$$f(x(t)) \leq f(x(0)) e^{-C^{-1} t},$$

i.e. the function f applied to the solution $x(t)$ decays exponentially. Moreover, since $f(0) = 0$ is a minimum of f on U, this implies that $x(t)$ approaches a minimum of f on U, although not necessarily the origin, since the critical set of f need not be isolated. The following two propositions contain analogous results for the spinor flow and the volume constrained spinor flow. The non-optimal Łojasiewicz inequality also implies decay of the driving function, however the decay is only polynomial and the exponent depends on the constant θ.

Proposition 4.6 (Energy decay for absolute minima)**.**
Suppose $\bar{\Phi} = (\bar{g}, \bar{\varphi}) \in \mathcal{N}$ is a critical point of $\mathcal{E}$, i.e. an absolute minimizer with $\mathcal{E}(\bar{\Phi}) = 0$. Suppose moreover that $\bar{g}$ has no Killing fields. Then there exists a $C^{2,\alpha}$ neighborhood of U in $\mathcal{N}$ and constants $C, \alpha > 0$, such that if Φ_t is a solution of the spinor flow on $[0, T]$ which lies in the neighborhood U, i.e. $\Phi_t \in U$ for all $t \in [0, T]$, then

$$\mathcal{E}(\Phi_t) \leq \mathcal{E}(\Phi_0) e^{-\alpha t},$$

$$\int_t^T \|Q(\Phi_s)\|_{L^2}^2 ds \leq \mathcal{E}(\Phi_0) e^{-\alpha t}$$

and

$$\int_t^T \|Q(\Phi_s)\|_{L^2} ds \leq C \mathcal{E}(\Phi_0)^{1/2} e^{-\alpha t}$$

for all $t \in [0, T]$.

Proof. By proposition 4.2 there exists a $C^{2,\alpha}$ neighborhood U of $\bar{\Phi}$ and $\alpha > 0$, where the Łojasiewicz inequality

$$\mathcal{E}(\Phi) \leq \alpha^{-1} \|Q(\Phi)\|_{L^2}^2$$

holds for all $\Phi \in U$. Now suppose Φ_t is a solution of the spinor flow on an interval $[0, T]$ lying in U. Then

$$\frac{d}{dt} \mathcal{E}(\Phi_t) = d\mathcal{E}(\Phi_t) \partial_t \Phi_t = (\operatorname{grad} \mathcal{E}(\Phi_t), Q(\Phi_t))_{L^2} = -\|Q(\Phi_t)\|_{L^2}^2.$$

Applying the Łojasiewicz inequality then yields

$$\frac{d}{dt} \mathcal{E}(\Phi_t) \leq -\alpha \mathcal{E}(\Phi_t)$$

and the Grönwall inequality implies

$$\mathcal{E}(\Phi_t) \leq \mathcal{E}(\Phi_0)e^{-\alpha t}.$$

Moreover, the time integral of

$$\frac{d}{dt}\mathcal{E}(\Phi_t) = -\|Q(\Phi_t)\|_{L^2}^2$$

from t to T gives the second inequality

$$\int_t^T \|Q(\Phi_s)\|_{L^2}^2 ds = \mathcal{E}(\Phi_t) - \mathcal{E}(\Phi_T) \leq \mathcal{E}(\Phi_t) \leq \mathcal{E}(\Phi_0)e^{-\alpha t}.$$

For the final inequality, we consider the time derivative of $\mathcal{E}(\Phi_t)^{1/2}$:

$$-\frac{d}{dt}\mathcal{E}(\Phi_t)^{1/2} = \frac{1}{2}\frac{1}{\mathcal{E}(\Phi_t)^{1/2}}\|Q(\Phi_t)\|_{L^2}^2.$$

The Łojasiewicz inequality implies

$$\frac{1}{\mathcal{E}(\Phi)^{1/2}} \geq \alpha^{1/2}\frac{1}{\|Q(\Phi)\|_{L^2}}$$

and thus

$$\frac{1}{2}\alpha^{1/2}\|Q(\Phi_t)\|_{L^2} \leq -\frac{d}{dt}\mathcal{E}(\Phi_t)^{1/2}.$$

Integrating this inequality and applying the known bound for $\mathcal{E}(\Phi_t)$

$$\int_t^T \|Q(\Phi_s)\|_{L^2} ds \leq 2\alpha^{-1/2}\mathcal{E}(\Phi_t)^{1/2} \leq 2\alpha^{-1/2}\mathcal{E}(\Phi_0)^{1/2}e^{-\alpha t}.$$

□

The situation for volume constrained minimizers is exactly analogous, if the critical set near the minimizer is smooth. If it is not smooth, we need to slightly adapt the proof to the weaker Łojasiewicz inequality, but the basic argument remains the same.

Proposition 4.7 (Energy decay for volume constrained minimizers)**.**
Suppose $\bar{\Phi} = (\bar{g}, \bar{\varphi}) \in \mathcal{N}^1$ is a volume constrained minimizr of $\mathcal{E}$ and suppose moreover that $\bar{g}$ has no Killing fields. Then there exists a $C^{2,\alpha}$ neighborhood U of $\bar{\Phi}$ in $\mathcal{N}^1$ and constants $C, \alpha, \beta > 0$, such that if Φ_t is a solution of the volume normalized spinor flow which lies in the neighborhood U, then

$$|\mathcal{E}(\Phi_t) - \mathcal{E}(\bar{\Phi})| \leq C\frac{1}{((\mathcal{E}(\Phi_0) - \mathcal{E}(\bar{\Phi}))^{-1/\alpha} + t)^{\alpha}},$$

$$\int_t^T \|\mathring{Q}(\Phi_s)\|_{L^2}^2 ds \leq C\frac{1}{((\mathcal{E}(\Phi_0) - \mathcal{E}(\bar{\Phi}))^{-1/\alpha} + t)^{\alpha}}$$

and

$$\int_t^T \|\mathring{Q}(\Phi_s)\|_{L^2} ds \le C \frac{1}{((\mathcal{E}(\Phi_0) - \mathcal{E}(\bar{\Phi}))^{-1/\alpha} + t)^\beta}$$

for all $t \in [0, T]$.

If the set of critical points of $\mathcal{E}$ *on* $\mathcal{N}^1$ *is a manifold near* $\bar{\Phi}$*, then exponential bounds hold as in proposition 4.6.*

Remark. The constants α and β depend only on the constant θ in the Łojasiewicz inequality. Indeed, $\alpha = \frac{\theta}{2-\theta}$ and $\beta = \frac{\theta-1}{2-\theta}$. As θ tends to 2, α tends to infinity, i.e. the decay rate improves. As θ tends to 1, α tends to 1, i.e. the decay rate gets worse. Likewise, β tends to ∞ if θ tends to 2, but β tends to 0 as θ tends to 1.

Proof. If the critical set near $\bar{\Phi}$ is smooth, the proof proceeds exactly as in the previous proposition, so we assume it is not smooth. By proposition 4.3 there exists a $C^{2,\alpha}$ neighborhood U of $\bar{\Phi}$ and $\theta > 0$, where the Łojasiewicz inequality

$$|\mathcal{E}(\Phi) - \mathcal{E}(\bar{\Phi})| \le \|\mathring{Q}(\Phi)\|_{L^2}^\theta$$

holds for all $\Phi \in U$. Now suppose Φ_t is a solution of the spinor flow on an interval $[0, T]$ lying in U. Then

$$\frac{d}{dt}|\mathcal{E}(\Phi_t) - \mathcal{E}(\bar{\Phi})| = -\|\mathring{Q}(\Phi_t)\|_{L^2}^2 \le -|\mathcal{E}(\Phi_t) - \mathcal{E}(\bar{\Phi})|^{2/\theta},$$

where we also used

$$|\mathcal{E}(\Phi_t) - \mathcal{E}(\bar{\Phi})| = \mathcal{E}(\Phi_t) - \mathcal{E}(\bar{\Phi}),$$

which follows because $\bar{\Phi}$ is a minimizer. Integrating this inequality from 0 to t yields

$$|\mathcal{E}(\Phi_t) - \mathcal{E}(\bar{\Phi})| \le \left(\frac{2}{\theta} - 1\right) \frac{1}{((\mathcal{E}(\Phi_0) - \mathcal{E}(\bar{\Phi}))^{-1/\alpha} + t)^\alpha}$$

with $\alpha = \frac{\theta}{2-\theta}$. The second inequality follows as in the previous proposition. For the last inequality, we calculate

$$\begin{aligned} -\frac{d}{dt}|\mathcal{E}(\Phi_t) - \mathcal{E}(\bar{\Phi})|^{1-1/\theta} &= (1 - 1/\theta)|\mathcal{E}(\Phi_t) - \mathcal{E}(\bar{\Phi})|^{-1/\theta}\|Q(\Phi_t)\|^2 \\ &\ge C\|Q(\Phi_t)\| \end{aligned}$$

Integrating this inequality then yields

$$\int_t^T \|Q(\Phi_s)\| ds \le C|\mathcal{E}(\Phi_t)|^{1-1/\theta} \le C \frac{1}{((\mathcal{E}(\Phi_0) - \mathcal{E}(\bar{\Phi}))^{-1/\alpha} + t)^\beta}$$

with $\beta = (1 - 1/\theta)\alpha = \frac{\theta-1}{2-\theta}$. □

4.5 Mapping flow estimate

In the previous section we derived that the energy near a critical point will exponentially decay provided the flow does not leave a given $C^{2,\alpha}$ neighborhood. It is then imperative to show that the flow does not leave the given $C^{2,\alpha}$ neighborhood to conclude that we in fact have exponential decay of the energy. The idea behind this is that the spinor flow is a gradient flow and hence the energy controls the time integrated L^2 norm of the gradient. Noticing that the gradient fulfills a linear parabolic equation, we would like to conclude by parabolic estimates that indeed higher order norms of the gradient are also controlled. Choosing proper constants this would yield that the flow does not leave the $C^{2,\alpha}$ neighborhood. However, the spinor flow is only weakly parabolic, so this strategy can not be implemented directly. However, the gauged spinor flow *is* strongly parabolic and it is related to the spinor flow by a family of diffeomorphisms. That family of diffeomorphisms is in turn determined by the mapping flow. Solutions of the mapping flow also satisfy a strongly parabolic equation. It turns out that we can control the family of diffeomorphisms. This will allow us to go from the energy estimate for the spinor flow to an estimate of the gauged spinor flow. We will then be able to apply parabolic estimates and from this establish the claimed stability of the spinor flow.

In this section we derive the necessary estimates for the family of diffeomorphisms, or rather their velocities, which fulfill the mapping flow equation. Recall that f_t fulfills the mapping flow equation if the conditions

$$f_0 = \mathrm{id}_M$$
$$\frac{d}{dt} f_t = P_{g_t,g_0}(f_t)$$

are met. The operator

$$P_{\bar{g},g}(f) : \mathcal{C}^\infty(M,M) \to T\mathcal{C}^\infty(M,M)$$
$$f \mapsto -df(X_{f^*\bar{g}}(g))$$

has the following linearization at id_M:

$$DP_{\bar{g},\bar{g}}(\mathrm{id}_M) = X \mapsto -4(\delta_{\bar{g}}\delta^*_{\bar{g}}X^\flat)^\sharp.$$

Lemma 4.8.
Let $\tilde{g} \in \Gamma(\odot^2_+ T^*M)$ and $k > \frac{n}{2} + 2$. Suppose $\tilde{g}$ has no Killing fields. Then there exists a H^k neighborhood $U \times V$ of $(\mathrm{id}_M, \tilde{g}) \in \mathcal{C}^\infty(M,M) \times \Gamma(\odot^2_+ T^*M)$ and constants $C, \lambda > 0$ with the following significance. If g_t is a family of metrics with $g_t \in V$, once differentiable in time and f_t is a solution of the initial value problem

$$f_0 = \mathrm{id}_M$$
$$\frac{d}{dt} f_t = P_{g_t,\tilde{g}}(f_t)$$

we have

$$\int_{t_1}^{t_2} \|P_{g_t,\tilde{g}}\|_{H^{-2}} dt \le C \left(\int_0^{t_1} \|\dot{g}_t\|_{L^2} e^{\lambda(t-t_1)} dt + \int_{t_1}^{t_2} \|\dot{g}_t\|_{L^2} dt + e^{-\lambda t_1} \right)$$

for some $C, \lambda > 0$, provided the flow exists until time t_2 in the neighborhood $U \times V$.

Proof. By the formula for the linearization of $P_{\tilde{g},\tilde{g}}$ at id_M, we have

$$(DP_{\tilde{g},\tilde{g}}(\mathrm{id}_M)X, X)_{L^2} = -4\left(\delta^*_{\tilde{g}}X^\flat, \delta^*_{\tilde{g}}X^\flat\right) = -4\left(\mathcal{L}_X\tilde{g}, \mathcal{L}_X\tilde{g}\right)_{L^2}.$$

Since we assume $\tilde{g}$ has no Killing fields, this implies $DP_{\tilde{g},\tilde{g}}(\mathrm{id}_M)$ is strictly negative definite, i.e. there exists $\mu > 0$, such that

$$(DP_{\tilde{g},\tilde{g}}(\mathrm{id}_M)X, X)_{L^2} \leq -\mu\,(X, X)_{L^2}.$$

Since the coefficients of the operator $P_{g_1,g_2}(f)$ are continuous in f and the first derivatives of g_1 and g_2 and recalling that by the Sobolev embedding theorem H^k continuously embeds in C^2, we conclude that there is a H^k neighborhood U of $\tilde{g}$, a neighborhood V of id_M and a constant $0 < \lambda < \mu$, such that $DP_{g,\tilde{g}}(f)$ is strongly elliptic and strictly negative definite with a constant λ.

Since $L = DP_{\tilde{g},\tilde{g}}(\mathrm{id}_M)$ is strictly negative definite, it induces an invertible operator from $H^{k+2} \to H^k$ for all $k \in \mathbb{Z}$. We have, up to equivalence,

$$\|f\|_{H^{-2}} = \|L^{-1}f\|_{L^2}.$$

This implies, in particular, that $DP_{g,\tilde{g}}(f)$ is also strictly negative definite with respect to the Sobolev inner product $\langle\cdot,\cdot\rangle_{H^{-2}}$.

We will now derive a differential inequality for $\|\dot{f}_t\|^2_{H^{-2}}$, where

$$\dot{f}_t = P_{g_t,\tilde{g}}(f_t).$$

For brevity, we let $P_{g_t,\tilde{g}}(f_t) = P_{g_t}(f_t)$. In what follows, we tacitly assume $g_t \in U$, $f_t \in V$ for all t, as per the statement of the lemma. We calculate

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left\langle P_{g_t}(f_t), P_{g_t}(f_t)\right\rangle_{H^{-2}} &= \left\langle \frac{d}{dt}P_{g_t}(f_t), P_{g_t}(f_t)\right\rangle_{H^{-2}} \\
&= \left\langle P_{\dot{g}_t}(f_t) + DP_{g_t}(f_t)\dot{f}_t, P_{g_t}(f_t)\right\rangle_{H^{-2}} \\
&= \left\langle P_{\dot{g}_t}(f_t), P_{g_t}(f_t)\right\rangle_{H^{-2}} + \left\langle DP_{g_t}(f_t)P_{g_t}(f_t), P_{g_t}(f_t)\right\rangle_{H^{-2}}.
\end{aligned}$$

The map

$$g \mapsto P_g(f) = 2df(\delta_{f^*\tilde{g}}g),$$

is a linear first order differential operator with bounds dependent on $\|f\|_{C^1}$ and $\|\tilde{g}\|_{C^1}$. As such we can estimate, using that bound and the Cauchy-Schwarz inequality

$$|\langle P_{\dot{g}_t}(f_t), P_{g_t}(f_t)\rangle_{H^{-2}}| \leq \|P_{\dot{g}_t}(f_t)\|_{H^{-2}}\|P_{g_t}(f_t)\|_{H^{-2}} \leq C\|\dot{g}_t\|_{L^2}\|P_{g_t}(f_t)\|_{H^{-2}}.$$

Then we obtain for

$$a(t) = \langle P_{g_t}(f_t), P_{g_t}(f_t)\rangle_{H^{-2}}$$

the inequality

$$\frac{1}{2}\dot{a}(t) \leq C\|\dot{g}_t\|_{L^2}\sqrt{a(t)} - \lambda a(t).$$

Let $b(t) = \sqrt{a(t)}$. The function b then satisfies the following differential inequality

$$\dot{b}(t) \leq -\lambda b(t) + \|g_t\|_{L^2}.$$

Define

$$\beta(t) = e^{-\lambda t}\left(b(0) + \int_0^t e^{\lambda s}\|\dot{g}_s\|_{L^2}ds\right).$$

Then we have

$$\dot{\beta}(t) = -\lambda\beta(t) + \|\dot{g}_t\|_{L^2}.$$

We deduce

$$\frac{d}{dt}(b-\beta) \leq -\lambda(b-\beta),$$

and since $b(0) = \beta(0)$, $b(t) \leq \beta(t)$ follows. To obtain the claim of the lemma, we will now estimate the integral of $\beta(t)$. For brevity, we denote $\gamma(t) = \|\dot{g}_t\|_{L^2}$. Define $\chi(s,t) = 1$ if $0 \leq s \leq t$ and $\chi(s,t) = 0$ otherwise. Then we calculate

$$\begin{aligned}
\int_{t_1}^{t_2} e^{-\lambda t}\int_0^t e^{\lambda s}\gamma(s)dsdt &= \int_{t_1}^{t_2}\int_0^t e^{\lambda(s-t)}\gamma(s)dsdt \\
&= \int_{t_1}^{t_2}\int_0^{t_2} \chi(s,t)e^{\lambda(s-t)}\gamma(s)dsdt \\
&= \int_0^{t_2}\gamma(s)\int_{t_1}^{t_2}\chi(s,t)e^{\lambda(s-t)}dtds \\
&= \int_0^{t_2}\gamma(s)\int_{\max\{s,t_1\}}^{t_2} e^{\lambda(s-t)}dtds \\
&= \int_0^{t_1}\gamma(s)\int_{t_1}^{t_2} e^{\lambda(s-t)}dtds + \int_s^{t_2}\gamma(s)\int_{t_1}^{t_2} e^{\lambda(s-t)}dtds \\
&\leq \lambda^{-1}\left(\int_0^{t_1} e^{\lambda(s-t_1)}\gamma(s)ds + \int_{t_1}^{t_2}\gamma(s)ds\right)
\end{aligned}$$

The integral of the term $b(0)e^{-\lambda t}$ is

$$\int_{t_1}^{t_2} b(0)e^{-\lambda t}dt = \lambda^{-1}b(0)\left(e^{-\lambda t_1} - e^{-\lambda t_2}\right).$$

Thus

$$\int_{t_1}^{t_2}\beta(t)dt \leq \lambda^{-1}\left(b(0)e^{-\lambda t_1} + \int_0^{t_1} e^{\lambda(s-t_1)}\gamma(s)ds + \int_{t_1}^{t_2}\gamma(s)ds\right)$$

and the claim of the lemma follows. □

4.6 Long time existence and convergence

In this section we derive from the previous results the long time existence and convergence of the spinor flow for initial values near a critical point.

Theorem 4.9.
Suppose $\bar{\Phi} = (\bar{g}, \bar{\varphi}) \in \mathcal{N}$ is a critical point of $\mathcal{E}$. Assume also that the isometry group of $\bar{g}$ is discrete. Then for any $k > \frac{n}{2} + 5$ and $\epsilon > 0$, there exists a $\delta > 0$, such that for $\Phi \in \mathcal{N}$ with

$$d_{H^k}(\Phi, \bar{\Phi}) < \delta$$

the spinor flow Φ_t with initial condition $\Phi_0 = \Phi$ exists for all times and converges exponentially to a critical point Φ_∞ with

$$d_{H^k}(\bar{\Phi}, \Phi_\infty) < \epsilon.$$

We will formulate and prove an analogous theorem for the volume normalized spinor flow after proving this. The proof of the stability theorem is a simple consequence of the following two lemmas.

Lemma 4.10.
Suppose $\bar{\Phi} = (\bar{g}, \bar{\varphi}) \in \mathcal{N}$ is a critical point of $\mathcal{E}$. Then for any $k > \frac{n}{2} + 2$, $\epsilon > 0$ and $T > 0$, there exists $\delta > 0$, such that if $\Phi \in \mathcal{N}$ with

$$d_{H^k}(\Phi, \bar{\Phi}) < \delta,$$

then the gauged spinor flow Φ_t with initial condition $\Phi_0 = \Phi$ exists on the interval $[0, T]$ and

$$d_{H^k}(\Phi_t, \bar{\Phi}) < \epsilon \text{ for } 0 \leq t \leq T.$$

Lemma 4.11.
Suppose $\bar{\Phi} = (\bar{g}, \bar{\varphi}) \in \mathcal{N}$ is a critical point of $\mathcal{E}$. Then for any $k > \frac{n}{2} + 5$, there exist constants $\delta > 0, C, \alpha > 0$ with the following significance. Suppose that $\Phi_t \in \mathcal{N}$ is a solution of the gauged spinor flow on an interval $[0, T]$, i.e.

$$\partial_t \Phi_t = \tilde{Q}_{\bar{g}}(\Phi_t)$$

with

$$d_{H^k}(\Phi_t, \bar{\Phi}) < \delta \text{ for all } t \in [0, T].$$

Let f_t be the family of diffeomorphisms relating the gauged and the ungauged spinor flow. Suppose that

$$d_{H^k}(f_t, \mathrm{id}_M) < \delta \text{ for all } t \in [0, T].$$

Then

$$\|\tilde{Q}(\Phi_t)\|_{H^k} \leq Ce^{-\alpha t}.$$

Proof of theorem 4.9. Let $r > 0$, such that the conclusions of lemma 4.11 hold with constants C, α for solutions in the H^k ball with radius r around $\bar{\Phi}$. Then using lemma 4.10 choose $\gamma > 0$, such that for $\Phi \in \mathcal{N}$ with $d_{H^k}(\Phi, \bar{\Phi}) < \gamma$ the gauged spinor flow exists until time 1 and remains in the ball of radius r around $\bar{\Phi}$. Then using lemma 4.10 again choose $\delta > 0$, such that for $\Phi \in \mathcal{N}$ with $d_{H^k}(\Phi, \bar{\Phi}) < \delta$ the gauged spinor flow exists until time T and remains in the ball of radius $\gamma/3$, where T is chosen such that

$$\int_T^\infty Ce^{-\alpha t} dt < \gamma/3.$$

Now suppose that $\Phi \in \mathcal{N}$ with $d_{H^k}(\Phi, \bar{\Phi}) < \delta$. Then let Φ_t be the gauged spinor flow with initial condition $\Phi_0 = \Phi$. Let $\hat{T}$ be the first time that

$$d_{H^k}(\Phi_{\hat{T}}, \bar{\Phi}) = \gamma.$$

We will arrive at a contradiction if $\hat{T}$ is any finite time, yielding that the flow exists for all times. Indeed, consider the flow at the point $\Phi_{\hat{T}}$. Then we can compute

$$\begin{aligned}
d_{H^k}(\bar{\Phi}, \Phi_{\hat{T}}) &\leq d_{H^k}(\bar{\Phi}, \Phi_T) + d_{H^k}(\Phi_T, \Phi_{\hat{T}}) \\
&\leq \frac{1}{3}\gamma + \int_T^{\hat{T}} \|\partial_t \Phi_t\|_{H^k} dt \\
&= \frac{1}{3}\gamma + \int_T^{\hat{T}} \|\tilde{Q}(\Phi_t)\|_{H^k} dt \\
&\leq \frac{1}{3}\gamma + \int_T^{\hat{T}} Ce^{-\alpha t} dt \\
&\leq \frac{1}{3}\gamma + \frac{1}{3}\gamma < \gamma.
\end{aligned}$$

This obviously contradicts the assumption and we conclude that the flow exists for all times. Moreover, then

$$\Phi_\infty = \Phi + \int_0^T \tilde{Q}(\Phi_t) dt$$

exists in H^k with

$$d_{H^k}(\bar{\Phi}, \Phi_\infty) < \gamma < \epsilon.$$

Furthermore, the energy $\mathcal{E}(\Phi_t)$ also decays exponentially, and we conclude that the limit Φ_∞ is a global minimum as well. This shows that the *gauged* spinor flow with initial condition $\Phi \in B_\delta$ exists for all time and converges exponentially to a minimum in B_ϵ. To conclude the same for the spinor flow, recall that the mapping flow is a strongly parabolic equation, and thus the velocity along the flow solves a linear strongly parabolic equation to which we can apply the parabolic regularity estimate and the mapping flow estimate to obtain that the mapping flow converges exponentially in any H^k norm. Since the spinor flow is given by $(F_t^{-1})^* \Phi_t$, the spinor flow also converges exponentially. $\square$

To prove lemma 4.10 it has to be shown that the linearization of the spinor flow equation at a critical point defines a Banach space isomorphism between certain anisotropic Sobolev spaces. It is then a matter of applying the inverse function theorem to conclude the theorem as stated. This strategy works more generally for quasilinear equations, see [23], sections 7.2/7.3 and for this specific statement for the G_2 heat flow [46], Cor. 8.6. Since this is well explained in these articles, we do not repeat the proofs here. To prove lemma 4.11, we will need the following interior estimate for parabolic equations.

Lemma 4.12.
If u is a solution of a linear parabolic system on $[0,T]$, i.e.

$$Pu = \partial_t u + L_t u = 0,$$

then for any $\delta > 0$ and any $k, l \in \mathbb{Z}$ there exists a constant (depending on P, but not on u), such that

$$\|u_{t_0}\|_{H^k} \leq C \int_0^T \|u_t\|^2_{H^l} dt.$$

Proof. This follows from parabolic regularity as follows. Let $f : [0,T] \to [0,1]$ be a smooth function, such that $f(0) = 0, f|_{[\delta,T]} = 1$ and $|f'| \leq 2/\delta$. Then

$$P(fu) = f'u + Pu = f'u.$$

Theorem 1.24 implies

$$\begin{aligned}\int_\delta^T \|u\|_{H^{l+2}} dt &\leq \int_0^T \|fu\|^2_{H^{l+2}} dt \\ &\leq C \int_0^T \|P(fu)\|^2_{H^l} + \|fu\|^2_{H^l} dt + \|f(0)u(0)\|^2_{H^{l+1}} \leq \tilde{C} \int_0^T \|u\|^2_{H^l} dt.\end{aligned}$$

Iterating this argument yields

$$\int_\delta^T \|u\|^2_{H^{l+2n}} dt \leq C \int_0^T \|u\|^2_{H^l} dt$$

for any natural number n. Applying theorem 1.24 once again, we obtain

$$\|u_{t_0}\|_{H^k} \leq C \int_0^T \|u\|^2_{H^k} dt$$

for any $t_0 \geq \delta$. □

Proof of lemma 4.11. We will show this estimate by combining the gradient estimate from the Łojasiewicz inequality and the estimate of the mapping flow. This will give us an estimate of the time integral of $\|\tilde{Q}(\tilde{\Phi}_t)\|_{H^s}$ for $s = -3$, which we will then improve via parabolic regularity. We consider the spinor flow

$$\partial_t \Phi_t = Q(\Phi_t), \Phi_0 = \Phi,$$

the gauged spinor flow

$$\partial_t \tilde{\Phi}_t = \tilde{Q}(\tilde{\Phi}_t), \tilde{\Phi}_0 = \Phi$$

and the mapping flow

$$\partial_t f_t = P_{g_t,\bar{g}}(f), f_0 = \mathrm{id}_M \, .$$

Then we have that

$$\tilde{\Phi}_t = F_t^* \Phi_t$$

and hence

$$\begin{aligned} \tilde{Q}(\tilde{\Phi}_t) &= \partial_t(F_t^* \Phi_t) \\ &= F_t^* \tilde{\mathcal{L}}_{X_t} \Phi_t + F_t^* \dot{\Phi}_t \end{aligned}$$

where $X_t = \frac{d}{dt} f_t$ and $\tilde{\mathcal{L}}$ is the spinorial Lie derivative.

Multiplication of Sobolev functions $H^k \times H^s \to H^s$ for negative s and positive k is continous, if $k > -s$ and $k > n/2$, where n is the dimension of the manifold, by theorem 1.6. In particular, our choice of k allows any $s \geq -3$.

We will use this to estimate $\tilde{\mathcal{L}}_{X_t} \Phi_t$ in the H^s norm. Recall that

$$\tilde{\mathcal{L}}_X \Phi = (\mathcal{L}_X g, \tilde{\mathcal{L}}_X \varphi) = (2\delta_g^* X^\flat, \nabla_X^g \varphi - \frac{1}{4} dX^\flat \cdot \varphi).$$

In local coordinates we have

$$\mathcal{L}_X g = p_1(g_{jk}, \partial_l g_{mn}, X^i) + p_2(g_{ij}, \partial_k X^l)$$

for some polynomials p_1, p_2, which are linear in the partial derivative terms and the X^i terms. Likewise we have

$$\tilde{\mathcal{L}}_X \varphi = q_1(X^i, \partial_j \varphi^\alpha) + q_2(g_{ij}, \partial_l g_{mn}, X^k, \varphi^\alpha)$$

for polynomials q_1, q_2, linear in the partial derivative terms and the X^i terms. From this follows, using the multiplication theorem above and the fact that H^{k-1} is a Banach algebra,

$$\begin{aligned} \|\tilde{\mathcal{L}}_X \Phi\|_{H^s} &\leq C \left(\|DX\|_{H^s} \sum_{d=0}^{r} \|\Phi\|_{H^{k-1}}^d + \|X\|_{H^s} \sum_{d=0}^{r} \|D\Phi\|_{H^{k-1}}^d \right) \\ &\leq C \left(\|X\|_{H^{s+1}} \sum_{d=0}^{r} \|\Phi\|_{H^{k-1}}^d + \|X\|_{H^s} \sum_{d=0}^{r} \|\Phi\|_{H^k}^d \right) \\ &\leq C \left(\|X\|_{H^{s+1}} \sum_{d=0}^{r} \|\Phi\|_{H^k}^d \right) \end{aligned}$$

for $k > -s + n/2 + 2$, where r is the maximal degree of the polynomials p_1, p_2, q_1, q_2. Since we will choose $s = -3$ and $k > n/2 + 5$, this will be the case.

Furthermore, given a diffeomorphism $f : M \to M$ and a lift to the topological spin structure $F : \tilde{P} \to \tilde{P}$, we have

$$F^* \Phi = \Phi \circ F,$$

where we view Φ as an equivariant map $\Phi : \tilde{P} \to \left(\widetilde{\mathrm{GL}_n^+} \times \Sigma_n\right) / \operatorname{Spin}(n)$.

Using the transformation rule, we can derive an estimate

$$\|u \circ f\|_{W^{k,p}(M)} \leq \nu(\|f\|_{C^{\max\{k,1\}}})\|u\|_{W^{k,p}(M)}$$

for the integral Sobolev spaces. For negative s, we conclude the following inequality by interpolation and duality

$$\|F^*\Phi\|_{H^s} \leq \tilde{\nu}(\|F\|_{C^{\lceil |s| \rceil}})\|\Phi\|_{H^s},$$

where $\nu, \tilde{\nu} : [0, \infty) \to [0, \infty)$ are continuous functions.

In conclusion we obtain

$$\begin{aligned} \|\tilde{Q}(\tilde{\Phi}_t)\|_{H^s} &= \|F_t^* \tilde{\mathcal{L}}_{X_t} \Phi_t + F_t^* \dot{\Phi}_t\|_{H^s} \\ &\leq C\nu(\|F_t\|_{C^{\lceil |s| \rceil}})(\|X_t\|_{H^{s+1}}\|\Phi_t\|_{H^k} + \|\dot{\Phi}_t\|_{H^s}) \end{aligned}$$

Since we assume both f_t and Φ_t remain in a bounded H^k neighborhood, we can estimate their norms by a constant, hence we obtain

$$\|\tilde{Q}(\tilde{\Phi}_t)\|_{H^s} \leq C(\|\dot{f}_t\|_{H^{s+1}} + \|\dot{\Phi}_t\|_{H^s}).$$

It remains to choose a neighborhood of $\bar{\Phi}$ so that we can also estimate the terms $\|\dot{f}_t\|_{H^{s+1}}$ and $\|\dot{\Phi}_t\|_{H^s}$.

By theorem 4.6 there exists a H^k neighborhood U of $\bar{\Phi}$, such that for any $\Phi \in U$ it holds

$$\int_T^{T_{\max}} \|Q(\Phi_t)\|_{L^2} dt \leq Ce^{-\alpha T}.$$

Choose a neighborhood $U \times V_m$ of $(\mathrm{id}_M, \bar{g})$ such that we have the mapping flow estimate 4.8. Choose a neighborhood V_s of $\bar{\Phi}$, such that we have the L^2 estimate of the gradient along the spinor flow as in theorem 4.6. We may assume that $\pi_\Sigma(V_s) = V_m$. Furthermore, we choose the neighborhoods to be bounded in H^k.

Now choose $\Phi \in V_s$ as initial condition for the spinor and the spinor-DeTurck flow. As above we denote these flows by Φ_t and $\tilde{\Phi}_t$ respectively and by f_t we mean the associated mapping flow. We will now estimate the integral of the H^{-3} norm of $\tilde{Q}(\tilde{\Phi}_t)$. Recall that we have

$$\int_{T_1}^{T_2} \|\dot{\Phi}_t\|_{L^2} dt \leq Ce^{-\alpha T_1}$$

from theorem 4.6. For $\dot{f}_t$ we get the estimate

$$\int_{T_1}^{T_2} \|\dot{f}_t\|_{H^{-2}} dt \leq C\left(\int_0^{T_1} \|\dot{g}_t\|_{L^2} e^{\lambda(t-T_1)} dt + \int_{T_1}^{T_2} \|\dot{g}_t\|_{L^2} dt + e^{-\lambda T_1}\right).$$

The second term can be bounded by $Ce^{-\alpha T_1}$ by the previous estimate, since $\|\dot{g}_t\|_{L^2} \leq \|\dot{\Phi}_t\|_{L^2}$. The first term we decompose into

$$\int_0^{T_1/2} \|\dot{g}_t\|_{L^2} e^{\lambda(t-T_1)} dt < Ce^{-\lambda T_1/2}$$

and

$$\int_{T_1/2}^{T_1} \|\dot{g}_t\|_{L^2} e^{\lambda(t-T_1)} dt < Ce^{-\alpha T_1/2}$$

again using the estimate for $\|\dot{g}_t\|_{L^2}$. Thus

$$\int_{T_1}^{T_2} \|\dot{f}_t\|_{H^{-2}} dt < Ce^{-\mu T}$$

for some $C > 0, \mu > 0$. We will use the same constants in the estimate of $\dot{g}_t$. Putting these estimates together we obtain

$$\begin{aligned}
\int_{T_1}^{T_2} \|\tilde{Q}(\tilde{\Phi}_t)\|_{H^{-3}} dt &\leq C \int_{T_1}^{T_2} \|\dot{f}_t\|_{H^{-2}} + \|\dot{\Phi}_t\|_{H^{-3}} dt \\
&\leq Ce^{-\mu T_1}
\end{aligned}$$

Because H^k embeds into C^3, $\tilde{Q}$ is a locally Lipschitz continuous map from H^k to H^{k-2}. And since $\tilde{\Phi}_t$ remains in a bounded H^k neighborhood, we obtain that $\|\tilde{Q}(\tilde{\Phi}_t)\|_{H^{-3}} \leq \tilde{C}$. Hence we may estimate

$$\int_{T_1}^{T_2} \|\tilde{Q}(\tilde{\Phi}_t)\|^2_{H^{-3}} dt \leq \tilde{C} \int_{T_1}^{T_2} \|\tilde{Q}(\tilde{\Phi}_t)\|_{H^{-3}} dt \leq C\tilde{C}e^{-\mu T_1}.$$

This estimate can be improved using parabolic regularity: the term $\tilde{Q}_t = \tilde{Q}(\tilde{\Phi}_t)$ fulfills the linear parabolic equation

$$\partial_t \tilde{Q}_t = D\tilde{Q}(\tilde{\Phi}_t)\tilde{Q}_t.$$

Since $\tilde{\Phi}_t$ is in a bounded H^k neighborhood, the coefficients of this parabolic equation have uniform bounds. Hence we can apply the interior estimate 4.12 to obtain

$$\|\tilde{Q}_{T+\delta}\|_{H^k} \leq \|\tilde{Q}\|_{L^2([T,\infty),H^{-3})} \leq Ce^{-\mu T} \leq \hat{C}e^{-\mu(T+\delta)},$$

which is the claim of the lemma. □

With the proof of this lemma, the proof of the stability theorem for absolute minimizers is finished. We now turn to the case of minimizers of the energy under a volume constraint. A slight complication arises from the fact that the critical set of the volume constrained energy is not known to be smooth. We introduce the following notion of admissible sets: the critical set of an analytical function is called *admissible*, if the exponent θ in the corresponding Łojasiewicz inequality is strictly larger than $3/2$. The theorem can then be stated as follows.

Theorem 4.13.
Suppose $\bar{\Phi} = (\bar{g}, \bar{\varphi}) \in \mathcal{N}$ is a volume constrained minimizer of $\mathcal{E}$. Suppose that the isometry group of $\bar{g}$ is discrete and suppose that the critical set is admissible. Given $k > \frac{n}{2} + 5$ there exists a H^k neighborhood U of $\bar{\Phi}$, such that the volume constrained spinor flow with initial condition $\Phi \in U$ exists for all times and converges to a volume constrained critical point. If the critical set is smooth the rate of convergence to the critical point is exponential. If the critical set is not smooth, the rate of convergence is $O(T^{-\kappa})$ where $\kappa = \frac{2\theta-3}{2-\theta}$, where θ is the exponent in the Łojasiewicz inequality at $\bar{\Phi}$.

The proof of this theorem is essentially the same as for the case of absolute minimizers. The only serious difference is the fact that if the critical set is not smooth, we have a different Łojasiewicz inequality. Rather than exponential decay as in lemma 4.11, we have instead

$$\|\mathring{\tilde{Q}}(\tilde{\Phi}_t)\|_{H^k} \leq C\frac{1}{1+t^{\beta}}$$

with $\beta = \frac{\theta-1}{2-\theta}$. Following the argument in the proof of 4.9, the spinor flow converges, if

$$\int_T^{\infty} \|\mathring{\tilde{Q}}(\tilde{\Phi}_t)\|_{H^k} dt < \infty.$$

It is a simple matter to check that

$$\int_T^{\infty} \frac{1}{1+t^{\beta}} dt < \infty,$$

if $\theta \geq \frac{3}{2}$. Hence the condition on θ as in the statement of the theorem is sufficient to ensure convergence of the spinor flow to a critical point Φ_∞ and

$$d_{H^k}(\Phi_\infty, \Phi_T) \leq \int_T^{\infty} \|\tilde{Q}(\Phi_t)\|_{H^k} dt \leq C\frac{1}{T^{\kappa}}.$$

Chapter 5

The spinor flow on surfaces

In chapter 2 we have seen that the spinorial energy on surfaces behaves very differently than in higher dimensions. In this chapter we study the analytical behavior of the spinor flow on surfaces. In particular we will give criteria for the blow up of the spinor flow. On a surface, any family of metrics $g(t)$ can be rewritten as $e^{2u(t)}\bar{g}(t)$, where $u(t) \in \mathcal{C}^\infty(M)$ and $\bar{g}(t)$ is a constant curvature metric. Any geometric flow on a surface can be studied by investigating the behaviour of $u(t)$ and $\bar{g}(t)$ along the flow. We will follow this approach for the spinor flow, which has been introduced by Buzano and Rupflin in [11] to study the harmonic Ricci flow and which we have discussed in a general setting in chapter 3.

Before treating the spinor flow itself, we first study the spinor flow restricted to one conformal class. It will turn out that the evolution of the conformal factor $u(t)$ is closely linked to this restricted flow.

Throughout the whole chapter M will be a closed surface of genus $\gamma > 0$ with a fixed topological spin structure. The following theorems are the main results of this chapter.

Theorem 5.1.
Suppose (g_t, φ_t), $t \in [0, T)$ is a solution of the conformal spinor flow on M and suppose

$$\sup_{0 \le t < T} \int_M |R_{g_t}|^2 \operatorname{vol}_{g_t} + \int_M |\nabla^{g_t} \varphi_t|^q \operatorname{vol}_{g_t} < \infty$$

for some $q > 4$. Then the solution (g_t, φ_t) can be extended to a smooth solution on an interval $[0, T + \delta)$ for some $\delta > 0$.

Theorem 5.2.
Suppose (g_t, φ_t), $t \in [0, T)$ is a solution of the spinor flow on M and suppose

$$\sup_{\substack{x \in M, \\ 0 \le t < T}} |\nabla^2 \varphi| < \infty.$$

Then the solution (g_t, φ_t) can be extended to a smooth solution on an interval $[0, T + \delta)$ for some $\delta > 0$.

Theorem 5.3.
Suppose (g_t, φ_t), $t \in [0, T)$ *is a solution of the spinor flow on* M *and suppose*

$$\sup_{0 \le t < T} \int_M |\nabla^2 \varphi_t|^q \operatorname{vol}_{g_t} < \infty$$

for some $q > 8$ *and*

$$\inf_{0 \le t < T} \operatorname{inj}(g_t) > 0.$$

Then the solution (g_t, φ_t) *can be extended to a smooth solution on an interval* $[0, T + \delta)$ *for some* $\delta > 0$.

5.1 Conformal spinor flow

Suppose g is a metric on M. The conformal class of g is denoted by $[g]$ and given by the set

$$\{e^{2u} g : u \in \mathcal{C}^\infty(M)\}.$$

We can restrict the space of sections of the universal spinor bundle $\mathcal{F}$ to this subspace of metrics. We denote that space by $\mathcal{F}^c$,

$$\mathcal{F}^c = \mathcal{F}^c_g = \left\{(e^{2u} g, \varphi) \in \mathcal{F} : u \in \mathcal{C}^\infty(M) \text{ and } \varphi \in \Gamma(\Sigma_{e^{2u}g} M)\right\}.$$

Within $\mathcal{F}^c$, the subset of pairs of metrics and unit spinor fields is given by

$$\mathcal{N}^c = \mathcal{N}^c_g = \{(e^{2u} g, \varphi) \in \mathcal{F}^c : |\varphi| = 1\}.$$

We denote by $\mathcal{E}^c$ the restriction of $\mathcal{E}$ to $\mathcal{N}^c$:

$$\mathcal{E}^c = \mathcal{E}|_{\mathcal{N}^c} : \mathcal{N}^c \to \mathbb{R}$$

and by $\mathcal{D}^c$ the restriction of $\mathcal{D}$ to $\mathcal{N}^c$.

The *conformal spinor flow* is the negative gradient flow of $\mathcal{E}^c$ or equivalently of $\mathcal{D}^c$. The next sections will present the flow equations, evolution equations of certain associated quantities and finally the blow up criterium from the introduction.

5.1.1 Derivation of the gradient

The gradient of $\mathcal{E}^c$ can be computed in two separate ways. The first is to take the known formula for the gradient of $\mathcal{E}$ and project it onto the tangent space of $\mathcal{N}^c$. The second way is to parametrize $\mathcal{F}^c$ by the set $\mathcal{C}^\infty(M) \times \Gamma(\Sigma_g M)$ and directly compute the variation with respect to this parametrization.

It is useful to briefly consider these approaches abstractly. In the first case, let $f : M \to \mathbb{R}$ be some smooth function and let $\iota : N \hookrightarrow M$ be a submanifold. Suppose g is a metric on M. Then equip N with the submanifold metric $\iota^* g$. Then for $v \in T_pN \subset T_pM$ we have

$$g|_N(\operatorname{grad}_N(f \circ \iota), v) = d(f \circ \iota)v = df v = g(\operatorname{grad}_M f, v).$$

This implies

$$\operatorname{grad}_N(f \circ \iota) = P \operatorname{grad}_M f,$$

where

$$P : TM|_N \to TN$$

is the orthogonal projection.

In the second case, assume $f : M \to \mathbb{R}$ is a smooth function and $\varphi : N \to M$ is a diffeomorphism. Let g be a metric on M and denote by $\varphi^* g$ the pullback metric on N. Then we have for $v \in T_pN$

$$\varphi^* g(\operatorname{grad}_N(f \circ \varphi), v) = d(f \circ \varphi)v = (df \circ D\varphi)v = g(\operatorname{grad}_M f, D\varphi v).$$

Hence we have (by definition of the pullback metric)

$$\operatorname{grad}_M f = D\varphi \operatorname{grad}_N(f \circ \varphi).$$

To compute the gradient of $\mathcal{E}^c$ by the first method, first note that the tangent space of $\mathcal{N}^c$ at (g^u, φ), with $g^u = e^{2u} g$, is given by

$$T_{(g^u,\varphi)}\mathcal{N}^c = \{ f g^u : f \in \mathcal{C}^\infty(M)\} \oplus \Gamma(\varphi^\perp).$$

The orthogonal projection

$$\mathcal{P}^c : T_{(g^u,\varphi)}\mathcal{F} \to T_{(g^u,\varphi)}\mathcal{F}^c$$

is given by

$$(h, \psi) \mapsto \left(\frac{1}{2}(\operatorname{tr}_{g^u} h) g^u, \psi \right).$$

Thus to compute the gradient we have to compute the trace of

$$(\operatorname{grad} \mathcal{E})_1 = \frac{1}{4}|\nabla^g \varphi|^2 g + \frac{1}{4} \operatorname{div}_g T_{g,\varphi} - \frac{1}{2}\langle \nabla^g \varphi \otimes \nabla^g \varphi \rangle.$$

The trace of the first term is

$$\frac{1}{2}|\nabla^g \varphi|^2$$

and the trace of the last term is

$$-\frac{1}{2}|\nabla^g \varphi|^2,$$

i.e. both terms cancel. Notice that this is specific to dimension 2! In general

$$\operatorname{tr}_g \left(\frac{1}{4}|\nabla^g \varphi|^2 g \right) = \frac{n}{4}|\nabla^g \varphi|^2$$

and there is no cancellation. The trace of the middle term is computed in the next proposition. This formula can also be found in [3].

Proposition 5.4.
Suppose $(g,\varphi)\in\mathcal{N}$. *Then*

$$\operatorname{tr}_g \operatorname{div}_g T_{g,\varphi} = \langle D_g^2\varphi,\varphi\rangle - |D_g\varphi|^2 \tag{5.5}$$
$$= \frac{1}{4}R_g + |\nabla^g\varphi|^2 - |D_g\varphi|^2 \tag{5.6}$$

Proof. Let e_i be a synchronous orthonormal frame at p. Then we compute at p

$$\begin{aligned}
\operatorname{tr}_g \operatorname{div}_g T_{g,\varphi} &= -\sum_{i,j}(\nabla_{e_j}T_{g,\varphi})(e_j,e_i,e_i)\\
&= -\sum_{i,j} e_j T_{g,\varphi}(e_j,e_i,e_i)\\
&= -\sum_{i,j} e_j\langle e_j\cdot e_i\cdot\varphi,\nabla_{e_i}\varphi\rangle\\
&= -\sum_{i\neq j} e_j\langle e_j\cdot e_i\cdot\varphi,\nabla_{e_i}\varphi\rangle \qquad \text{because } \langle\nabla_X\varphi,\varphi\rangle=0\\
&= -\sum_{i,j}\big(\langle e_j\cdot e_i\cdot\nabla_{e_j}\varphi,\nabla_{e_i}\varphi\rangle + \langle e_j\cdot e_i\cdot\varphi,\nabla_{e_j}\nabla_{e_i}\varphi\rangle\big)\\
&= -\sum_{i,j}\big(\langle e_i\cdot\nabla_{e_i}\varphi, e_j\cdot\nabla_{e_j}\varphi\rangle - \langle e_j\cdot\nabla_{e_j}(e_i\cdot\nabla_{e_i}\varphi),\varphi\rangle\big)\\
&= -|D\varphi|^2 + \langle D^2\varphi,\varphi\rangle.
\end{aligned}$$

The second identity follows immediately from the Lichnerowicz formula

$$D_g^2\varphi = \nabla^{g*}\nabla^g\varphi + \frac{R_g}{4}\varphi$$

and the formula

$$\langle\nabla^{g*}\nabla^g\varphi,\varphi\rangle = |\nabla^g\varphi|^2$$

for unit spinors. $\square$

These calculations imply the following proposition.

Proposition 5.7 (Gradient of $\mathcal{E}^c$)**.**
The gradient of $\mathcal{E}^c$ *at* $(g^u,\varphi)\in\mathcal{N}^c$ *is given by*

$$\operatorname{grad}\mathcal{E}^c(g^u,\varphi) = \left(\frac{1}{8}\left(\langle D_{g^u}^2\varphi,\varphi\rangle - |D_{g_u}\varphi|^2\right)g^u,\quad \nabla^{g_u*}\nabla^{g_u}\varphi - |\nabla^{g_u}\varphi|^2\varphi\right) \tag{5.8}$$

It is worth mentioning that the metric component of the gradient of $\mathcal{E}^c$ depends on φ only up to first order, in contrast to the gradient of $\mathcal{E}$, which contains second order terms. We will later see that the spinorial component of the gradient of $\mathcal{E}^c$ depends on the metric only to first order, which is not obvious here. Thus, restricting to conformal metrics has the surprising effect of partially decoupling the metric and the spinorial component of the gradient. This decoupling will be much more visible in the second approach to computing the gradient, which we will pursue now.

To this end, we introduce the spaces

$$F^c = \mathcal{C}^\infty(M) \times \Gamma(\Sigma_g M)$$

and

$$N^c = \mathcal{C}^\infty(M) \times \Gamma(S(\Sigma_g M)).$$

The tangent space of F^c at any point can be identified with F^c, whereas the tangent space of N^c at $(u, \varphi) \in N^c$ is the space $\mathcal{C}^\infty(M) \times \Gamma(\varphi^\perp)$.

A parametrization of $\mathcal{F}^c$ is given by the mapping

$$\xi : F^c \to \mathcal{F}^c$$

$$(u, \varphi) \mapsto \left(e^{2u} g, \hat{B}^g_{e^{2u}g} \varphi\right)$$

This parametrization restricts to a parametrization of $\mathcal{N}^c$ by N^c. Given $(u, \varphi) \in N^c$ and $(v, \psi) \in T_{(u,\varphi)} F^c = F^c$, the differential of ξ is given by

$$D\xi(u, \varphi)(v, \psi) = \left(2ve^{2u} g, B^g_{e^{2u}g} \psi\right).$$

To compute the pullback metric on F^c, let $(v_i, \psi_i) \in T_{(u,\varphi)} F^c$, $i = 1, 2$. Denoting the pullback metric by $L^2(u)$ we have

$$\begin{aligned}
((v_1, \psi_1), (v_2, \psi_2))_{L^2(u)} &= (D\xi(u, \varphi)(v_1, \psi_1), D\xi(u, \varphi)(v_1, \psi_1))_{L^2(e^{2u}g)} \\
&= \left((2v_1 e^{2u} g, \hat{B}^g_{e^{2u}g}(\psi_1)), (2v_2 e^{2u} g, \hat{B}^g_{e^{2u}g}(\psi_2))\right)_{L^2(e^{2u}g)} \\
&= 4\left(v_1 e^{2u} g, v_2 e^{2u} g\right)_{L^2(e^{2u}g)} + (\psi_1, \psi_2)_{L^2(e^{2u}g)} \\
&= 4 \int_M v_1 v_2 (e^{2u} g)(e^{2u} g, e^{2u} g) \operatorname{vol}_{e^{2u}g} + \int_M \langle \psi_1, \psi_2 \rangle \operatorname{vol}_{e^{2u}g} \\
&= 8 \int_M v_1 v_2 e^{2u} \operatorname{vol}_g + \int_M \langle \psi_1, \psi_2 \rangle e^{2u} \operatorname{vol}_g,
\end{aligned}$$

where to pass to the last line we used $g^u(g^u, g^u) = e^{2u} g(e^{2u} g, e^{2u} g) = 2$.

The functional $\mathcal{D}^c$ can now be computed with respect to the parametrization ξ. Let $(u, \varphi) \in N^c$. Using formula 2.33, we obtain

$$\begin{aligned}
\mathcal{D}^c(\xi(u,\varphi)) &= \frac{1}{2}\int_M |D_{g^u}\hat{B}^g_{g^u}\varphi|^2 \operatorname{vol}_{g^u} \\
&= \frac{1}{2}\int_M \left| e^{-u}\left(D_g\varphi + \frac{1}{2}\operatorname{grad}_g u \cdot \varphi\right)\right|^2 e^{2u} \operatorname{vol}_g \\
&= \frac{1}{2}\int_M |D_g\varphi + \frac{1}{2}\operatorname{grad}_g u \cdot \varphi|^2 \operatorname{vol}_g \\
&= \frac{1}{2}\int_M |D_g\varphi|^2 \operatorname{vol}_g + \frac{1}{2}\int_M \langle D_g\varphi, \operatorname{grad}_g u \cdot \varphi\rangle \operatorname{vol}_g + \frac{1}{8}\int_M |du|^2_g \operatorname{vol}_g
\end{aligned}$$

The following two lemmas help simplify this expression further.

Lemma 5.9.
For any $X \in TM$ and $\varphi \in \mathcal{N}$, the following formula holds

$$\langle D\varphi, X \cdot \varphi\rangle = \beta(JX) = - * \beta(X)$$

where

$$\beta(X) = \langle \nabla^g_X \varphi, \omega \cdot \varphi\rangle.$$

Proof. Suppose $|X| = 1$. The other cases follow by linearity. Then

$$\begin{aligned}
\langle D_g\varphi, X \cdot \varphi\rangle &= \langle X \cdot \nabla^g_X\varphi + JX \cdot \nabla^g_{JX}\varphi, X \cdot \varphi\rangle \\
&= \langle JX \cdot \nabla^g_{JX}\varphi, X \cdot \varphi\rangle \\
&= -\langle \nabla^g_{JX}\varphi, JX \cdot X \cdot \varphi\rangle \\
&= \langle \nabla^g_{JX}\varphi, \omega \cdot \varphi\rangle \\
&= \beta(JX) = -(*\beta)(X)
\end{aligned}$$

□

Lemma 5.10.
For $\varphi \in \mathcal{N}$ and $\beta \in \Omega^1(M)$ defined by

$$\beta(X) = \langle \nabla^g_X \varphi, \omega \cdot \varphi\rangle$$

the following formula holds

$$*d\beta = -\langle D_g^2\varphi, \varphi\rangle + |D_g\varphi|^2.$$

Using this lemma, we can show the following proposition.

Proposition 5.11.
Suppose $(u, \varphi) \in N^c$. Then

$$\mathcal{D}^c(\xi(u,\varphi)) = \frac{1}{2}\int_M |D_g\varphi|^2 \operatorname{vol}_g + \frac{1}{2}\int_M \left(\langle D_g^2\varphi, \varphi\rangle - |D_g\varphi|^2\right) u \operatorname{vol}_g + \frac{1}{8}\int_M |du|^2_g \operatorname{vol}_g .$$

Proof. Only the middle term on the right hand side still needs to be justified. This is now a simple matter of calculation and applying Stokes theorem:

$$\begin{aligned}
\int_M \langle D_g\varphi, \operatorname{grad}_g u \cdot \varphi\rangle \operatorname{vol}_g &= -\int_M *\beta(\operatorname{grad} u) \operatorname{vol}_g \\
&= -\int_M g(du, *\beta) \operatorname{vol}_g \\
&= -\int_M du \wedge * * \beta \\
&= -\int_M du \wedge \beta \\
&= -\int_M d(u\beta) + \int_M u d\beta \\
&= \int_M u \left(\langle D_g^2\varphi, \varphi\rangle - |D_g\varphi|^2\right) \operatorname{vol}_g
\end{aligned}$$

□

Notice the formal similarity between the Liouville energy

$$E_L(g^u) = \frac{1}{2}\int_M |du|_g^2 + R_g u \operatorname{vol}_g$$

and the term

$$\frac{1}{2}\int_M \frac{1}{4}|du|_g^2 + \left(\langle D_g^2\varphi, \varphi\rangle - |D_g\varphi|^2\right) u \operatorname{vol}_g .$$

Indeed, applying the Lichnerowicz formula, we could also write

$$\begin{aligned}
\frac{1}{2}\int_M \frac{1}{4}|du|_g^2 + \left(\langle D_g^2\varphi, \varphi\rangle - |D_g\varphi|^2\right) u \operatorname{vol}_g &= \frac{1}{2}\int_M \frac{1}{4}|du|_g^2 + \left(\frac{1}{4}R_g + |\nabla^g\varphi|^2 - |D_g\varphi|^2\right) u \operatorname{vol}_g \\
&= \frac{1}{4}E_L(g^u) + \frac{1}{2}\int_M \left(|\nabla^g\varphi|^2 - |D_g\varphi|^2\right) u \operatorname{vol}_g
\end{aligned}$$

As mentioned earlier, the decoupling phenomenon becomes clearer in this parametrization. Moreover, we discover a relationship to the Ricci flow on surfaces, since the negative gradient flow of the Liouville energy is precisely the Ricci flow.

Using the proposition, it is now straightforward to calculate the first variation of $\mathcal{D}^c \circ \xi$.

Proposition 5.12 (First variation of $\mathcal{D}^c \circ \xi$).
Suppose $(u, \varphi) \in N^c$ and $(v, \psi) \in T_{(u,\varphi)}N^c$. Then the first variation of $\mathcal{D}^c \circ \xi$ in the conformal direction is is given by

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{D}^c(\xi(u + tv, \varphi)) = \frac{1}{2}\int_M \left(\langle D_g^2\varphi, \varphi\rangle - |D_g\varphi|^2\right) v + \frac{1}{2}\Delta_g uv \operatorname{vol}_g$$

and the first variation in the conformal direction is given by

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{D}^c(\xi(u, \varphi + t\psi)) = \int_M \left\langle D_g^2\varphi - \operatorname{grad}_g u \cdot D_g\varphi - \nabla^g_{\operatorname{grad} u}\varphi, \psi\right\rangle \operatorname{vol}_g .$$

Proof. The variation in the conformal direction is obvious, using the formula

$$\int_M g(du, dv)\,\mathrm{vol}_g = \int_M \Delta_g uv\,\mathrm{vol}_g\,.$$

For the variation in spinorial direction, we first calculate

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\frac{1}{2}\int_M |D_g\varphi|^2\,\mathrm{vol}_g &= \int_M \langle D_g\varphi, D_g\psi\rangle\,\mathrm{vol}_g\\ &= \int_M \langle D_g^2\varphi, \psi\rangle\,\mathrm{vol}_g\,.\end{aligned}$$

For the second term we compute

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\frac{1}{2}\int_M \left(\langle D_g^2\varphi,\varphi\rangle - |D_g\varphi|^2\right)u\,\mathrm{vol}_g &= \frac{1}{2}\int_M \left(\langle D_g^2\psi,\varphi\rangle + \langle D_g^2\varphi,\psi\rangle\right)u\,\mathrm{vol}_g - \int_M \langle D_g\varphi, D_g\psi\rangle u\,\mathrm{vol}_g\\ &= \frac{1}{2}(D_g^2\psi, u\varphi)_{L^2} + \frac{1}{2}(uD_g^2\varphi,\psi)_{L^2} - (uD_g\varphi, D_g\psi)_{L^2}\\ &= \frac{1}{2}(D_g^2(u\varphi),\psi)_{L^2} + \frac{1}{2}(uD_g^2\varphi,\psi)_{L^2} - (D_g(uD_g\varphi),\psi)_{L^2}\\ &= (uD_g^2\varphi,\psi)_{L^2} - (uD_g^2\varphi,\psi)_{L^2} + \frac{1}{2}(\Delta_g u\varphi,\psi)_{L^2}\\ &\quad - (\nabla^g_{\mathrm{grad}_g u}\varphi,\psi)_{L^2} - (\mathrm{grad}_g u\cdot D_g\varphi,\psi)_{L^2}\\ &= (-\nabla^g_{\mathrm{grad}_g u}\varphi - \mathrm{grad}_g u\cdot D_g\varphi,\psi)_{L^2},\end{aligned}$$

where we used the formulas 2.21, 2.23 for $D_g(f\varphi)$ and $D_g^2(f\varphi)$ as well as the fact that $\langle\varphi,\psi\rangle = 0$. □

Finally, we can compute the gradient of $\mathcal{D}^c\circ\xi$ with respect to the pullback metric on N^c. By our initial considerations, the pushforward of this gradient is precisely the gradient of $\mathcal{D}^c$ on $\mathcal{N}^c$.

Proposition 5.13 (Gradient of $\mathcal{D}^c\circ\xi$)**.**
Suppose $(u,\varphi)\in N^c$. *Then*

$$\begin{aligned}(\mathrm{grad}(\mathcal{D}^c\circ\xi))_1(u,\varphi) &= e^{-2u}\left(\frac{1}{32}\Delta_g u + \frac{1}{16}(\langle D_g^2\varphi,\varphi\rangle - |D_g\varphi|^2)\right)\\ (\mathrm{grad}(\mathcal{D}^c\circ\xi))_2(u,\varphi) &= e^{-2u}\big(\nabla^{g*}\nabla^g\varphi - \mathrm{grad}_g u\cdot D_g\varphi - \nabla^g_{\mathrm{grad}_g u}\varphi\\ &\qquad - |\nabla^g\varphi|^2\varphi + \langle\mathrm{grad}_g u\cdot D_g\varphi,\varphi\rangle\varphi\big)\end{aligned}$$

Proof. For the sake of brevity we introduce

$$G_i = (\mathrm{grad}(\mathcal{D}^c\circ\xi))_i(u,\varphi)$$

for $i = 1, 2$. Then we have for $v \in \mathcal{C}^\infty(M)$

$$(G_1, v)_{L^2(u)} = 8 \int_M G_1 v e^{2u} \operatorname{vol}_g = \frac{d}{dt}\Big|_{t=0} \mathcal{D}^c(\xi(u + tv, \varphi)).$$

Thus, by the previous proposition

$$\int_M G_1 v e^{2u} \operatorname{vol}_g = \frac{1}{16} \int_M \left(\langle D_g^2 \varphi, \varphi\rangle - |D_g \varphi|^2\right) v + \frac{1}{2} \Delta_g u v \operatorname{vol}_g .$$

Hence

$$G_1 = e^{-2u} \left(\frac{1}{32} \Delta_g u + \frac{1}{16} \left(\langle D_g^2 \varphi, \varphi\rangle - |D_g \varphi|^2\right) \right)$$

as claimed. Likewise for $\psi \in \Gamma(\varphi^\perp)$,

$$(G_2, \psi)_{L^2(u)} = \int_M \langle G_2, \psi\rangle e^{2u} \operatorname{vol}_g = \frac{d}{dt}\Big|_{t=0} \mathcal{D}^c(\xi(u, \varphi + t\psi)).$$

By the previous proposition

$$\int_M \langle G_2, \psi\rangle e^{2u} \operatorname{vol}_g = \int_M \left\langle D_g^2 \varphi - \operatorname{grad}_g u \cdot D_g \varphi - \nabla^g_{\operatorname{grad} u} \varphi, \psi \right\rangle \operatorname{vol}_g .$$

Thus G_2 is the orthogonal projection of

$$e^{-2u} \left(D_g^2 \varphi - \operatorname{grad}_g u \cdot D_g \varphi - \nabla^g_{\operatorname{grad} u} \varphi\right)$$

onto $\Gamma(\varphi^\perp)$. Because $|\varphi| = 1$, the term $\nabla^g_{\operatorname{grad} u}\varphi$ is already orthogonal to φ. The orthogonal projection of $D_g^2 \varphi$ can be computed to be

$$\nabla^{g*} \nabla^g \varphi - |\nabla^g \varphi|^2 \varphi$$

using $\langle \nabla^{g*} \nabla^g \varphi, \varphi\rangle = |\nabla^g \varphi|^2$ and the Lichnerowicz formula. The projection of $\operatorname{grad}_g u \cdot D_g \varphi$ is given by

$$\operatorname{grad}_g u \cdot D_g \varphi - \langle \operatorname{grad}_g u \cdot D_g \varphi, \varphi\rangle \varphi.$$

This yields the claimed formula for the gradient in spinorial direction:

$$G_2 = e^{-2u} \left(\nabla^{g*} \nabla^g \varphi - \operatorname{grad}_g u \cdot D_g \varphi - \nabla^g_{\operatorname{grad}_g u} \varphi - |\nabla^g \varphi|^2 \varphi + \langle \operatorname{grad}_g u \cdot D_g \varphi, \varphi\rangle \varphi \right) .$$

$\square$

5.1.2 Evolution equation for the Liouville energy

In this very short section we compute the evolution equation for the Liouville energy

$$E_L(g^u) = \frac{1}{2}\int |du|_g^2 + R_g u \operatorname{vol}_g .$$

This is necessary to apply the compactness theorem for metrics within a conformal class. Denote by

$$\sigma = \langle D_g^2\varphi, \varphi\rangle - |D_g\varphi|^2 = \frac{1}{4}R_g + |\nabla^g\varphi|^2 - |D_g\varphi|^2.$$

Proposition 5.14.
Suppose (g_t, φ_t) is a solution of the conformal spinor flow. Then

$$\partial_t E_L(g_t) = -\frac{1}{32}\int_M R_{g_t}\sigma_t \operatorname{vol}_{g_t} = -\frac{1}{128}\int_M R_{g_t}^2 \operatorname{vol}_{g_t} - \frac{1}{32}\int_M R_{g_t}\left(|\nabla^{g_t}\varphi_t|^2 - |D_{g_t}\varphi_t|^2\right)\operatorname{vol}_{g_t} .$$

Proof. Suppose $g_t = e^{2u_t}g$. We then compute

$$\begin{aligned}
\partial_t E_L(g_t) &= \partial_t \frac{1}{2}\int_M |du_t|_g^2 + R_g u_t \operatorname{vol}_g \\
&= \int_M g(du_t, d\partial_t u_t)\operatorname{vol}_g + \frac{1}{2}R_g\partial_t u_t \operatorname{vol}_g \\
&= \int_M (\Delta_g u_t + K_g)\partial_t u_t \operatorname{vol}_g \\
&= \int_M K_{g_t}\partial_t u_t \operatorname{vol}_{g_t} \\
&= -\frac{1}{32}\int_M R_g\sigma_t \operatorname{vol}_g,
\end{aligned}$$

where in the last step we used that

$$\partial_t u_t = -\frac{1}{16}\sigma_t.$$

□

5.1.3 A guide to reading the proofs of the blow up criteria

In the following we will prove different blow up criteria. The proofs follow similar strategies. Certain results from regularity theory are used again and again without referencing the theorem by number every time. To close this gap we give the references here. When we appeal to Schauder theory for parabolic equations, we mean theorems 1.15 and 1.23. For higher derivatives 1.16 comes into play. When we refer to L^p theory for parabolic equations, we think of theorems 1.17 and 1.25. For parabolic equations we will also use the so called Krylov–Safonov estimate which is found in theorem 1.18.

For elliptic equations the Schauder estimates refer to theorem 1.10. This theorem includes higher derivative estimates and the case when the data is in divergence form. The L^p theory for elliptic operators is found in theorem 1.11. This also includes the case when the data is in divergence form. We will also consider time-dependent solutions of elliptic equations, i.e. a function defined defined on a spacetime such that the function satisfies an elliptic equation on every time slice. In that case we need to study the temporal continuity of the solution. These results are found in 1.2.3.

We illustrate how we apply such a theorem in an example. Suppose

$$Pu = f$$

is an elliptic equation. Now suppose we know $f \in C^\alpha$ and $u \in L^2$. Then rather than writing explicitly

$$\|u\|_{C^{2,\alpha}} \leq C \left(\|f\|_{C^\alpha} + \|u\|_{L^2}\right),$$

we will write

$$f \in C^\alpha \text{ and } u \in L^2$$

implies

$$u \in C^{2,\alpha}.$$

Thus $u \in C^{2,\alpha}$ is to be read as u is a member of $C^{2,\alpha}$ *and* there is a bound for $\|u\|_{C^{2,\alpha}}$.

The functions and maps we will consider will all be defined on a spacetime $M \times [0,T]$. For concreteness let $f : M \times [0,T] \to \mathbb{R}$. By

$$f_t \in L^p$$

we will mean that $f_t : M \times \{t\} \to \mathbb{R}$ is in $L^p(M)$ for every $t \in [0,T]$ *and* there is a uniform bound on $\|f_t\|_{L^p(M \times \{t\})}$. On the other hand we will write

$$f \in L^p(M \times [0,T]),$$

when we mean that f is p-integrable over the spacetime.

5.1.4 A blow up criterium for the conformal spinor flow

The evolution of the Liouville energy along the spinor flow can be used to give a blow up criterium for the conformal spinor flow. Before we turn to that, we will prove the following lemma.

Lemma 5.15.
Suppose (g_t, φ_t), $t \in [0,T)$ is a solution of the conformal spinor flow on M and suppose

$$\|u\|_{C_2^{\alpha,\alpha/2}(M\times[0,T))} \leq C,$$

$$\|\varphi\|_{C_2^{\alpha,\alpha/2}(M\times[0,T))} \le C.$$

Then all higher space and time derivatives of (g_t, φ_t) can be bounded in terms of C.

Proof. This is a standard bootstrapping argument. The evolution equation for u is given by

$$\partial_t u_t + \frac{1}{32} e^{-2u_t} \Delta_g u_t = -\frac{1}{16} e^{-2u_t} (\langle D_g^2 \varphi_t, \varphi_t\rangle - |D_g \varphi_t|^2).$$

The left hand side is a strictly parabolic operator with $C_2^{\alpha,\alpha/2}$ coefficients. Since

$$\langle D_g^2 \varphi, \varphi\rangle = |\nabla^g \varphi|^2 + \frac{1}{4} R_g,$$

it follows that the right hand side is in $C_1^{\alpha,\alpha/2}$. (Since g is a fixed smooth metric, the R_g term is in $\mathcal{C}^\infty$.) Hence by parabolic regularity

$$u \in C_1^{2+\alpha,1+\alpha/2} \subset C_3^{\alpha,\alpha/2}.$$

The evolution equation for φ is given by

$$\partial_t \varphi_t + \frac{1}{32} e^{-2u_t} \nabla^{g*}\nabla^g \varphi_t = e^{-2u_t} \left(- \operatorname{grad}_g u_t \cdot D_g \varphi_t - \nabla^g_{\operatorname{grad}_g u_t} \varphi_t - |\nabla^g \varphi_t|^2 \varphi_t + \langle \operatorname{grad}_g u_t \cdot D_g \varphi_t, \varphi_t\rangle \varphi_t \right).$$

Again, the right hand side is in $C_1^{\alpha,\alpha/2}$ and Schauder theory again improves this to

$$\varphi \in C_1^{2+\alpha,1+\alpha/2} \subset C_3^{\alpha,\alpha/2}.$$

It then follows that

$$\partial_t u_t + \frac{1}{32} e^{-2u_t} \Delta_g u_t \in C_2^{\alpha,\alpha/2}$$

and hence

$$u_t \in C_2^{2+\alpha,1+\alpha/2} \subset C_4^{\alpha,\alpha/2}.$$

The same argument applies to φ. Repeating this line of argument inductively yields estimates for all space and time derivatives as claimed. $\square$

Using the lemma, proving the blowup criterium is then a matter of showing a $C_2^{\alpha,\alpha/2}$ bound.

Theorem 5.16.
Suppose (g_t, φ_t), $t \in [0, T)$ is a solution of the conformal spinor flow on M and suppose

$$\sup_{0\le t<T} \int_M |R_{g_t}|^2 \operatorname{vol}_{g_t} + \int_M |\nabla^{g_t} \varphi_t|^q \operatorname{vol}_{g_t} < \infty$$

for $q > 4$. Then the solution (g_t, φ_t) can be extended to a smooth solution on an interval $[0, T + \delta)$ for some $\delta > 0$.

Proof. We can bound the evolution of the Liouville energy

$$\begin{aligned}\partial_t E_L(g_t) &= -\frac{1}{128}\int_M R_g^2 \operatorname{vol}_g - \frac{1}{32}\int_M R_g \left(|\nabla^g\varphi|^2 - |D_g\varphi|^2\right)\operatorname{vol}_g \\ &\leq -\frac{1}{128}\int_M R_g^2 \operatorname{vol}_g + \epsilon\int_M R_g^2\operatorname{vol}_g + C(\epsilon)\int_M \left(|\nabla^g\varphi|^2 - |D_g\varphi|^2\right)^2\operatorname{vol}_g\end{aligned}$$

for any $\epsilon > 0$. In particular, if $\int_M |\nabla^g\varphi|^4 \operatorname{vol}_g$ is uniformly bounded on $[0, T)$, then so is the Liouville energy. Together with the uniform bound on the L^2 norm of the curvature and the fact that the spinor flow on surfaces preserves volume, this implies that theorem 3.14 applies to the family $g_t, 0 \leq t < T$, and hence that there exists a uniform bound

$$\|u_t\|_{H^2(M,g)} \leq C \text{ for all } 0 \leq t < T,$$

where g is the constant curvature metric in the same conformal class as g_t and u_t is the conformal factor of g_t, i.e. $g_t = e^{2u_t}g$. Notice that this bound on u_t also implies a uniform bound

$$\|u_t\|_{C^0} \leq \tilde{C}$$

by Sobolev embedding.

This uniform bound on u_t will allow us to apply parabolic regularity to obtain that all $C^{k,\alpha}$ norms of u_t and φ_t are uniformly bounded on $[0, T)$. This implies that the limits $\lim_{t\to T} u_t$ and $\lim_{t\to T}\varphi_t$ exist in $\mathcal{C}^\infty$. Short time existence for smooth initial data then implies that there exists a solution of the spinor flow on $[T, T+\delta)$ for some $\delta > 0$, proving the theorem.

The evolution equations for u_t and φ_t are given by

$$\partial_t u_t = e^{-2u_t}\left(-\frac{1}{32}\Delta_g u_t - \frac{1}{16}(\langle D_g^2\varphi_t, \varphi_t\rangle - |D_g\varphi_t|^2)\right)$$

$$\partial_t\varphi_t = e^{-2u_t}\left(-\frac{1}{32}\nabla^{g*}\nabla^g\varphi_t - \operatorname{grad}_g u_t\cdot D_g\varphi_t - \nabla^g_{\operatorname{grad}_g u_t}\varphi_t - |\nabla^g\varphi_t|^2\varphi_t + \langle\operatorname{grad}_g u_t\cdot D_g\varphi_t, \varphi_t\rangle\varphi_t\right).$$

We can rewrite the evolution equation for u_t as

$$\partial_t u_t + \frac{1}{32}e^{-2u_t}\Delta_g u_t = -\frac{1}{16}e^{-2u_t}(\langle D_g^2\varphi_t, \varphi_t\rangle - |D_g\varphi_t|^2).$$

Notice that on a surface

$$|\nabla^{g_t}\varphi|^2_{g_t} - |D_{g_t}\varphi|^2 = e^{-2u_t}(|\nabla^g\varphi|^2_g - |D_g\varphi|^2),$$

see equation 2.34. Thus, because the norms g_t are uniformly equivalent to g, we obtain

$$\begin{aligned}\|e^{-2u_t}(|\nabla^g\varphi_t|^2 - |D_g\varphi_t|^2)\|_{L^p(M,g)} &\leq C\|e^{-2u_t}(|\nabla^g\varphi_t|^2 - |D_g\varphi_t|^2)\|_{L^p(M,g_t)} \\ &= C\||\nabla^{g_t}\varphi_t|^2_{g_t} - |D_{g_t}\varphi_t|^2\|_{L^p(M,g_t)} \\ &\leq C\|\nabla^{g_t}\varphi_t\|^2_{L^{2p}(M,g_t)}.\end{aligned}$$

Hence

$$\partial_t u_t + \frac{1}{32} e^{-2u_t} \Delta_g u_t$$

is uniformly bounded in L^p, if $p \leq q/2$. This implies by L^p theory that $u \in W_p^{2,1}$. By the Sobolev embedding

$$W_p^{2,1} \hookrightarrow C^{\alpha,\alpha/2},$$

u is Hölder continuous on $M \times [0,T]$. (Before we only had Hölder continuity on time slices.)

We now consider the evolution of φ_t:

$$\partial_t \varphi_t + \frac{1}{32} e^{-2u_t} \nabla^{g*} \nabla^g \varphi_t = \psi_t,$$

where

$$\psi_t = e^{-2u_t} \Big(-\operatorname{grad}_g u_t \cdot D_g \varphi_t - \nabla^g_{\operatorname{grad}_g u_t} \varphi_t - |\nabla^g \varphi_t|^2 \varphi_t + \langle \operatorname{grad}_g u_t \cdot D_g \varphi_t, \varphi_t \rangle \varphi_t \Big).$$

We want to obtain an L^p bound for the right hand side ψ_t. The e^{-2u} factor is irrelevant, because u is bounded in C^0. The terms $\operatorname{grad}_g u \cdot D_g \varphi$ and $\nabla^g_{\operatorname{grad}_g u} \varphi$ both have the same structure and thus can both be treated in the same way. Since $u \in H^2$, $\operatorname{grad}_g u \in H^1$ and thus $\operatorname{grad}_g u$ is bounded in L^p for every p. On the other hand $D_g \varphi$ and $\nabla^g \varphi$ are bounded in L^q. This implies that $\operatorname{grad}_g u \cdot D_g \varphi$ and $\nabla^g_{\operatorname{grad}_g u} \varphi$ are bounded in $L^{\hat{q}}$ for every $\hat{q} < q$. The same holds for the term $\langle \operatorname{grad}_g u \cdot D_g \varphi, \varphi \rangle \varphi$. Finally, the term $|\nabla^g \varphi|^2 \varphi$ is bounded in $L^{q/2}$. Thus $\partial_t \varphi_t + \frac{1}{32} e^{-2u_t} \nabla^{g*} \nabla^g \varphi_t$ is bounded in $L^{q/2}(M \times [0,T])$ and hence $\varphi \in W^{2,1}_{q/2}(M \times [0,T])$ by L^p theory. In particular, φ is $C^{\alpha,\alpha/2}$ continuous.

To recap, we have shown that u and φ are (spatially and temporally) Hölder continuous. We want to use Schauder estimates to show that u and φ are actually $C_2^{\alpha,\alpha/2}$. To do this we need that the right hand sides in the evolution equations of u_t and φ_t are Hölder continuous. So far we have shown this for no term appearing on the right hand sides, since all of them contain derivatives of either u or φ. Thus we now examine the evolution equations of du_t and $\nabla^g \varphi_t$. We will conclude that these quantities are also Hölder continuous, allowing us to use the Schauder estimates to conclude that indeed u and φ are in $C_2^{\alpha,\alpha/2}$.

We begin with du. Recall that

$$d\Delta_g u = \nabla^{g*} \nabla^g du + R_g du.$$

Differentiating $\partial_t u_t + \frac{1}{32} e^{-2u_t} \Delta_g u_t$ then yields

$$\partial_t du_t + \frac{1}{32} e^{-2u_t} \nabla^{g^*} \nabla^g du_t - \frac{1}{16} e^{-2u_t} (\Delta_g u_t) du_t + \frac{1}{32} e^{-2u_t} R_g du_t.$$

Notice that $\frac{1}{16} e^{-2u} (\Delta_g u) du$ is in $L^p(M \times [0,T])$ for every $p < q/2$ by previous results. Likewise, $\frac{1}{32} e^{-2u_t} R_g du_t$ is in L^p for every p. Differentiating $e^{-2u} \left(|\nabla^g \varphi|^2 - |D_g \varphi|^2 \right)$ yields

$$e^{-2u} (|\nabla^g \varphi|^2 * du + \nabla^g \nabla^g \varphi * \nabla^g \varphi).$$

Since du is in L^p for every p and $|\nabla^g\varphi|^2$ is in $L^{q/2}$, their product is in L^r for every $r < q/2$. We know that $\nabla^g\nabla^g\varphi$ is bounded in $L^{q/2}(M \times [0,T])$. Since $\nabla^g\varphi$ is in $L^p(M \times [0,T])$ for every p, this implies $\nabla^g\nabla^g\varphi * \nabla^g\varphi \in L^r(M \times [0,T])$ for every $r < q/2$. Applying parabolic regularity thus implies

$$du \in W_r^{2,1}$$

for every $r < q/2$. This also implies

$$du \in C^{\alpha,\alpha/2}.$$

Now we differentiate the evolution equation for φ_t in space. For the left hand side $\partial_t\varphi_t + \frac{1}{32}e^{-2u_t}\nabla^{g*}\nabla^g\varphi_t$ we obtain

$$\partial_t\nabla^g\varphi_t - \frac{1}{16}e^{-2u_t}du_t\nabla^{g*}\nabla^g\varphi_t + \frac{1}{32}e^{-2u_t}\nabla^{g*}\nabla^g\nabla^g\varphi_t + R_g * \nabla^g\varphi_t + (dR_g) * \varphi_t.$$

The last two terms come from commuting ∇^g and the connection Laplace $\nabla^{g*}\nabla^g$. Since g is a fixed metric and since $\nabla^g\varphi$ is bounded in $L^p(M \times [0,T])$ for every p, we know that both terms containing the curvature are bounded in $L^p(M \times [0,T])$.

We now differentiate the right hand side ψ spatially. This comes out to

$$\begin{aligned}
e^{2u}\nabla^g\psi = &-2du\psi \\
&- \nabla^g \operatorname{grad}_g u \cdot D_g\varphi - \operatorname{grad}_g u \cdot \nabla^g D_g\varphi \\
&- \nabla^g_{\nabla^g \operatorname{grad} u}\varphi - (\nabla^g\nabla^g\varphi)(\cdot, \operatorname{grad} u) \\
&- d|\nabla^g\varphi|^2\varphi - |\nabla^g\varphi|^2\nabla^g\varphi \\
&+ d(\langle \operatorname{grad}_g u \cdot D_g\varphi, \varphi\rangle)\varphi + \langle \operatorname{grad}_g u \cdot D_g\varphi, \varphi\rangle\nabla^g\varphi
\end{aligned}$$

Notice that ψ is bounded in $L^{q/2}$. Since du is bounded in L^p for every p, we get that $du\psi$ is bounded in every L^r with $r < q/2$. The terms $\operatorname{grad}_g u \cdot D_g\varphi$ and $\nabla^g_{\operatorname{grad}_g u}\varphi$ behave the same way, so we only examine the term $\operatorname{grad}_g u \cdot D_g\varphi$. Its space derivative consists of the terms $\nabla^g \operatorname{grad}_g u \cdot D_g\varphi$ and $\operatorname{grad}_g u \cdot \nabla^g D_g\varphi$. Since $\operatorname{grad}_g u$ is in L^p for every p and since φ is in $W^{2,1}_{q/2}$, we conclude that

$$\operatorname{grad}_g u \cdot \nabla^g D_g\varphi \in L^r \text{ for every } r < q/2.$$

On the other hand, we already know that $du \in W_r^{2,1}$ for every $r < q/2$. This implies $\nabla^g du \in W^{1,r}(M \times [0,T])$, where $W^{1,r}(M \times [0,T])$ is the isotropic Sobolev space on $M \times [0,T]$. By Sobolev embedding $\nabla^g \operatorname{grad}_g u \in L^p(M \times [0,T])$ for every p, and we obtain that

$$\nabla^g \operatorname{grad}_g u \cdot D_g\varphi \in L^p \text{ for every } p.$$

The term $d|\nabla^g\varphi|^2$ is of the form $\nabla^g\nabla^g\varphi * \nabla^g\varphi$. We have already seen that such a term is in L^r for every $r < q/2$. The term $\nabla^g\varphi$ is in $L^p(M \times [0,T])$ for every p, and thus so is $|\nabla^g\varphi|^2\nabla^g\varphi$. The term $d(\langle \operatorname{grad}_g u \cdot D_g\varphi, \varphi\rangle)\varphi$ is in every L^r, $r < q/2$ by the above argument and the term $\langle \operatorname{grad}_g u \cdot D_g\varphi, \varphi\rangle\varphi\nabla^g\varphi$ is in every $L^p(M \times [0,T])$.

In conclusion, we obtain

$$\partial_t \nabla^g \varphi_t + e^{-2u_t} \nabla^{g*} \nabla^g \nabla^g \varphi_t \in L^r(M \times [0,T])$$

for all $r < q/2$. Thus $\nabla^g \varphi \in W^{2,1}_r$ and hence

$$\nabla^g \varphi \in C^{\alpha,\alpha/2}.$$

We have shown that

$$\partial_t u_t + e^{-2u} \Delta_g u_t \in C^{\alpha,\alpha/2}$$

and

$$\partial_t \varphi_t + e^{-2u} \nabla^{g*} \nabla^g \varphi \in C^{\alpha,\alpha/2}.$$

This means that both u_t and φ_t are in $C^{2+\alpha,1+\alpha}$ and applying lemma 5.15 finishes the proof. $\square$

Remark. Had we assumed instead $q > 8$, then we would have had $u, \varphi \in W^{2,1}_p$ with $p > 4$ in the first step. This would have implied

$$u, \varphi \in C^{\alpha,\alpha/2}_1$$

and hence we would have immediately obtained

$$\partial_t u_t + \frac{1}{32} e^{-2u_t} \Delta_g u \in C^{\alpha,\alpha/2}$$

and

$$\partial_t \varphi_t + \frac{1}{32} e^{-2u} \nabla^{g*} \nabla^g \varphi \in C^{\alpha,\alpha/2}.$$

At that point we would have been finished, because then $u, \varphi \in C^{2+\alpha,1+\alpha/2}$.

5.2 Blow up criteria in the general case

In this section we decompose the spinor flow on surfaces using the framework from section 3.5. Assume $(\tilde{g}_t, \tilde{\varphi}_t)$ solves the spinor flow equation on a surface M, i.e.

$$\partial_t \tilde{g}_t = -\frac{1}{4} |\nabla^{\tilde{g}_t} \tilde{\varphi}_t|^2 \tilde{g}_t - \frac{1}{4} \operatorname{div}_{\tilde{g}_t} T_{\tilde{g}_t, \tilde{\varphi}_t} + \frac{1}{2} \langle \nabla^{\tilde{g}_t} \tilde{\varphi}_t \otimes \nabla^{\tilde{g}_t} \tilde{\varphi}_t \rangle$$

$$\partial_t \tilde{\varphi}_t = -\nabla^{\tilde{g}_t *} \nabla^{\tilde{g}_t} \tilde{\varphi}_t + |\nabla^{\tilde{g}_t} \tilde{\varphi}_t|^2 \tilde{\varphi}_t.$$

Now assume that $(\bar{g}_t, u_t, \varphi_t, f_t)$ is the corresponding split flow. We also denote $g_t = e^{2u_t} \bar{g}_t$. Recalling that

$$\operatorname{tr}_g Q_1(g, \varphi) = -\frac{1}{4} \left(\langle D^2_g \varphi, \varphi \rangle - |D_g \varphi|^2 \right)$$

and

$$\tilde{\mathcal{L}}_X\varphi = \nabla^g_X\varphi - \frac{1}{4}dX^\flat \cdot \varphi,$$

we obtain from proposition 3.26 the following evolution equations for the split flow

$$\partial_t \bar{g}_t = P_{\bar{g}_t}\left(e^{-2u_t}\mathring{Q}_1(g_t,\varphi_t)\right)$$

$$\partial_t u_t = -\frac{1}{16}\left(\langle D^2_{g_t}\varphi_t,\varphi_t\rangle - |D_{g_t}\varphi_t|^2\right) - X_t u_t - \frac{1}{2}\rho_t$$

$$\partial_t\varphi_t = -\nabla^{g_t*}\nabla^{g_t}\varphi_t + |\nabla^{g_t}\varphi_t|^2\varphi_t + \nabla^{g_t}_{X_t}\varphi_t - \frac{1}{4}dX_t^\flat\cdot\varphi_t,$$

where the vector field X_t and the function ρ_t are defined by the equations

$$\Delta_{\bar{g}_t}\rho_t + R_{\bar{g}_t}\rho_t = \delta_{\bar{g}_t}\delta_{\bar{g}_t}(e^{-2u_t}\mathring{Q}_1(g_t,\varphi_t))$$

$$\delta_{\bar{g}_t}\delta^*_{\bar{g}_t}X_t^\flat = -\delta_{\bar{g}_t}(e^{-2u_t}\mathring{Q}_1(g_t,\varphi_t) + \rho_t\bar{g}_t)$$

on every time slice $M\times\{t\}$. To apply the parabolic regularity theory as we did in the conformal spinor flow setting, we will need a precise understanding which functions spaces the right hand sides of these equations are members of. For most of the terms this can be read directly from the structure of the term. For the expression $\delta_{\bar{g}_t}(e^{-2u_t}\mathring{Q}_1(g_t,\varphi_t))$ however, we need to use the Bianchi identity 2.52 to obtain better control. This is the content of the following lemma.

Lemma 5.17.
Suppose $\dim M = 2$ *and* $h\in\Gamma(\odot^2T^*M)$ *and* $g,\bar{g}$ *are conformal metrics related by* $g = e^{2u}\bar{g}$. *Then*

$$\delta_{\bar{g}}(e^{-2u}h) = \delta_g h - \operatorname{tr}_g h du + 2h(\operatorname{grad}_g u,\cdot).$$

In particular

$$\delta_{\bar{g}}(e^{-2u}\mathring{Q}_1(g,\varphi)) = \frac{1}{2}d\operatorname{tr}_g Q_1(g,\varphi) + 2\mathring{Q}_1(g,\varphi)(\operatorname{grad}_g u,\cdot).$$

Proof. The Levi–Civita connection behaves under conformal change as follows

$$\nabla^{\bar{g}}_X Y = \nabla^g_X Y - (Xu)Y - (Yu)X + g(X,Y)\operatorname{grad}_g u,$$

whereas given an orthonormal basis e_i for g, the vectors $e^u e_i$ form an orthonormal basis for $\bar{g}$. The divergence is defined by

$$\delta_g h = -\sum_i(\nabla^g_{e_i}h)(e_i,\cdot).$$

This formula implies

$$\delta_g(fh) = f\delta_g h - h(\operatorname{grad}_g f,\cdot).$$

Hence

$$\delta_{\bar{g}}(e^{-2u}h) = e^{-2u}\delta_{\bar{g}}h + 2e^{-2u}h(\operatorname{grad}_{\bar{g}} u,\cdot) = e^{-2u}\delta_{\bar{g}}h + 2h(\operatorname{grad}_g u,\cdot)$$

The induced connection on $\odot^2 T^*M$ satisfies the equation

$$(\nabla^g_X h)(V,W) = Xh(V,W) - h(\nabla^g_X V, W) - h(V, \nabla^g_X W),$$

which implies

$$\begin{aligned}(\nabla^{\bar g}_X h)(V,W) = (\nabla^g_X h)(V,W)\\ &+ h((Xu)V,W) + h((Vu)X,W) - h(g(X,V)\operatorname{grad}_g u, W)\\ &+ h((Xu)W,V) + h((Wu)X,V) - h(g(X,W)\operatorname{grad}_g u, V).\end{aligned}$$

Hence

$$\begin{aligned}(\delta_{\bar g} h)(V) &= -\sum_i (\nabla^{\bar g}_{e^u e_i} h)(e^u e_i, V)\\ &= e^{2u}\sum_i \big(-(\nabla^g_{e_i} h)(e_i,V) - 2h((e_i u)e_i, V) + h(\operatorname{grad}_g u, V)\\ &\qquad - h((e_i u)e_i, V) - (Vu)h(e_i,e_i) + h(\operatorname{grad}_g u, g(e_i,V)e_i)\big)\\ &= e^{2u}\big(\delta_g h(V) + (n-2)h(\operatorname{grad}_g u, V) - du(V)\operatorname{tr}_g h\big)\\ &= e^{2u}\left(\delta_g h(V) - du(V)\operatorname{tr}_g h\right).\end{aligned}$$

From the Bianchi identity 2.52 we get

$$\delta_g Q_1(g,\varphi) = 0.$$

Since

$$\mathring{Q}_1(g,\varphi) = Q_1(g,\varphi) - \frac{1}{2}\operatorname{tr}_g Q_1(g,\varphi) g,$$

it follows that

$$\delta_g \mathring{Q}_1(g,\varphi) = \frac{1}{2} d\operatorname{tr}_g Q_1(g,\varphi).$$

Together with the formula for $\delta_{\bar g}(e^{-2u}h)$ above, the claim follows. □

This lemma implies the following about ρ.

Lemma 5.18.
The solution ρ of

$$\Delta_{\bar g}\rho + R_{\bar g}\rho = \delta_{\bar g}\delta_{\bar g}(e^{-2u}\mathring{Q}_1(g,\varphi))$$

can be decomposed as

$$\rho = \tilde\rho + \frac{1}{2}\operatorname{tr}_g Q_1(g,\varphi),$$

where $\tilde\rho$ solves

$$\Delta_{\bar g}\tilde\rho + R_{\bar g}\tilde\rho = -\frac{1}{2} R_{\bar g}\operatorname{tr}_g Q_1(g,\varphi) + 2\delta_{\bar g}\left(\mathring{Q}_1(g,\varphi)(\operatorname{grad}_g u, \cdot)\right).$$

The lemma follows immediately by substitution. The next lemma computes the term $\delta_{\bar{g}}\mathring{Q}_1(g,\varphi)(\operatorname{grad}_g u,\cdot)$, which is needed to control X.

Lemma 5.19.
Suppose M is 2 dimensional, then

$$\delta_{\bar{g}}\left(\mathring{Q}_1(g,\varphi)(\operatorname{grad}_g u,\cdot)\right) = e^{2u}\left(\frac{1}{2}g(d\operatorname{tr}_g Q_1(g,\varphi),du) - g(\mathring{Q}_1(g,\varphi),\nabla^g du)\right).$$

Proof. First note that for $\alpha \in \Omega^1(M)$

$$\delta_{\bar{g}}\alpha = e^{2u}\left(\delta_g\alpha + (n-2)\alpha(\operatorname{grad}_g u)\right)$$

and hence in 2 dimensions

$$\delta_{\bar{g}}\alpha = e^{2u}\delta_g\alpha.$$

Furthermore for $h \in \Gamma(\odot^2 T^*M)$ and $\alpha \in \Gamma(TM)$

$$\delta_g h(\alpha^\sharp,\cdot) = (\delta_g h)(\alpha^\sharp) - g(h,\nabla^g\alpha).$$

Thus it follows that

$$\delta_{\bar{g}}\left(\mathring{Q}_1(g,\varphi)(\operatorname{grad}_g u,\cdot)\right) = \frac{1}{2}g(d\operatorname{tr}_g Q_1(g,\varphi),du) - g(\mathring{Q}_1(g,\varphi),\nabla^g du)$$

and the claim follows. $\square$

We summarize these results in the following proposition.

Proposition 5.20.
The split flow satisfies

$$\partial_t u_t + \frac{1}{32}(1-R_{\bar{g}})e^{-2u_t}\Delta_{\bar{g}_t}u_t = -\frac{1}{16}(1-R_{\bar{g}})\left(\frac{1}{4}R_{\bar{g}} + |\nabla^{g_t}\varphi_t|^2 - |D_{g_t}\varphi_t|^2\right) - X_t u_t - \frac{1}{2}\tilde{\rho}_t \tag{5.21a}$$

$$\partial_t\varphi_t + \frac{1}{32}e^{-2u_t}\nabla^{\bar{g}_t*}\nabla^{\bar{g}_t}\varphi_t = e^{-2u_t}\psi_t + \nabla^{\bar{g}_t}_{X_t}\varphi_t - \frac{1}{4}dX_t^\flat\cdot\varphi_t \tag{5.21b}$$

$$\psi_t = -\operatorname{grad}_{\bar{g}_t}u_t\cdot D_{\bar{g}_t}\varphi_t - \nabla^{\bar{g}_t}_{\operatorname{grad}_{\bar{g}_t}u_t}\varphi_t - |\nabla^{\bar{g}_t}\varphi_t|^2\varphi_t + \langle\operatorname{grad}_{\bar{g}_t}u_t\cdot D_{\bar{g}_t}\varphi_t,\varphi_t\rangle\varphi_t \tag{5.21c}$$

$$\rho_t = \tilde{\rho}_t + \frac{1}{2}\operatorname{tr}_{g_t}Q_1(g_t,\varphi_t) \tag{5.21d}$$

$$\Delta_{\bar{g}_t}\tilde{\rho}_t + R_{\bar{g}}\tilde{\rho}_t = -\frac{1}{2}R_{\bar{g}}\operatorname{tr}_{g_t}Q_1(g_t,\varphi_t) + 2\delta_{\bar{g}_t}\left(\mathring{Q}_1(g_t,\varphi_t)(\operatorname{grad}_{g_t}u_t,\cdot)\right) \tag{5.21e}$$

$$\delta_{\bar{g}_t}\delta^*_{\bar{g}_t}X_t^\flat = \delta_{\bar{g}_t}(\mathring{Q}_1(g_t,\varphi_t) + \rho_t\bar{g}_t) \tag{5.21f}$$

$$\delta_{\bar{g}_t}\mathring{Q}_1(g_t,\varphi_t)) = \frac{1}{2}e^{2u_t}d\operatorname{tr}_{g_t}Q_1(g_t,\varphi_t) \tag{5.21g}$$

We first prove a lemma stating that $C_2^{\alpha,\alpha/2}$ regularity of u_t and φ_t and some assumptions on $\bar{g}_t$ are enough to obtain a uniformly smooth solution. Compare this to lemma 5.15 where we only needed $C^{\alpha,\alpha/2}$ regularity to conclude the same about the conformal spinor flow.

Lemma 5.22.
Suppose $(\bar{g}_t, u_t, \varphi_t)$, $t \in [0,T)$ is a solution of the split spinor flow equations on M and suppose

$$\operatorname{inj}(\bar{g}_t) > \epsilon > 0,$$

$$\sup_{t\in[0,T)} \|\partial_t \bar{g}_t\|_{L^2} < \infty,$$

$$\|u\|_{C_2^{\alpha,\alpha/2}(M\times[0,T))} \le C$$

and

$$\|\varphi\|_{C_2^{\alpha,\alpha/2}(M\times[0,T))} \le C.$$

Then all higher space and time derivatives of (g_t, φ_t) can be bounded in terms of C.

Proof. By lemma 3.34 the curve $\bar{g}_t$ admits uniform estimates in every C^k norm. In particular, the Laplacians $\Delta_{\bar{g}_t}$ and the spin Laplacians $\nabla^{\bar{g}_t *}\nabla^{\bar{g}_t}$ are all equivalent in the sense that their coefficients admit a uniform bound in every C^k norm. The evolution equation of u_t can be rewritten as

$$\partial_t u_t + \frac{1}{32}e^{-2u_t}\Delta_{\bar{g}_t}u_t = -\frac{1}{16}\left(\frac{1}{4}R_{\bar{g}_t} + |\nabla^{g_t}\varphi_t|^2 - |D_{g_t}\varphi_t|^2\right) - X_t u_t - \frac{1}{2}\rho_t.$$

Since

$$\rho_t = \tilde{\rho}_t + \frac{1}{2}R_{\bar{g}_t}\operatorname{tr}_{g_t} Q_1(g_t,\varphi_t) = \tilde{\rho}_t - R_{\bar{g}_t}\left(\frac{1}{32}R_{g_t} + \frac{1}{8}|\nabla^{g_t}\varphi_t|^2 - \frac{1}{8}|D_{g_t}\varphi_t|^2\right)$$

and $R_g = e^{-2u}(2\Delta_{\bar{g}}u + R_{\bar{g}})$, the evolution equation can be rewritten once more and we obtain

$$\begin{aligned}
&\partial_t u_t + \frac{1}{32}e^{-2u_t}\Delta_{\bar{g}_t}u_t \\
=& -\frac{1}{16}\left(\frac{1}{4}R_{\bar{g}_t} + |\nabla^{g_t}\varphi_t|^2 - |D_{g_t}\varphi_t|^2\right) - X_t u_t - \frac{1}{2}\tilde{\rho}_t + R_{\bar{g}_t}\left(\frac{1}{32}R_{g_t} + \frac{1}{8}|\nabla^{g_t}\varphi_t|^2 - \frac{1}{8}|D_{g_t}\varphi_t|^2\right) \\
=& -\frac{1}{16}(1-R_{\bar{g}_t})\left(\frac{1}{4}R_{\bar{g}_t} + |\nabla^{g_t}\varphi_t|^2 - |D_{g_t}\varphi_t|^2\right) - X_t u_t - \frac{1}{2}\tilde{\rho}_t + R_{\bar{g}_t}\frac{1}{32}e^{-2u_t}\Delta_{\bar{g}_t}u_t
\end{aligned}$$

or equivalently

$$\partial_t u_t + \frac{1}{32}(1-R_{\bar{g}_t})e^{-2u_t}\Delta_{\bar{g}_t}u_t = -\frac{1}{16}(1-R_{\bar{g}_t})\left(\frac{1}{4}R_{\bar{g}_t} + |\nabla^{g_t}\varphi_t|^2 - |D_{g_t}\varphi_t|^2\right) - X_t u_t - \frac{1}{2}\tilde{\rho}_t.$$

Since either $R_{\bar{g}} = 0$ or $R_{\bar{g}} = -1$, it follows that the left hand side is a uniformly parabolic operator with $C_2^{\alpha,\alpha/2}$ coefficients. To gain an improvement in regularity we aim to show that the right hand side is in $C_1^{\alpha,\alpha/2}$. This much is clear for the term in the brackets, because $\varphi \in C_2^{\alpha,\alpha/2}$. Thus it remains to be shown that $\tilde{\rho}$ and Xu are also in $C_1^{\alpha,\alpha/2}$. The function $\tilde{\rho}$ satisfies the elliptic equation

$$\Delta_{\bar{g}_t}\tilde{\rho}_t + R_{\bar{g}_t}\tilde{\rho}_t = -\frac{1}{2}R_{\bar{g}_t}\operatorname{tr}_{g_t} Q_1(g_t,\varphi_t) + \delta_{\bar{g}_t}\mathring{Q}_1(g_t,\varphi_t)(\operatorname{grad}_{\bar{g}_t} u_t,\cdot)$$

on every time slice. Since $Q_1(g,\varphi) \in C^{\alpha,\alpha/2}$ we conclude from Schauder estimates for divergence form data and the results from 1.2.3

$$\tilde{\rho} \in C_1^{\hat{\alpha},\hat{\alpha}/2}$$

for some $0 < \hat{\alpha} < \alpha$. The loss of Hölder regularity here is of no consequence for us. The same holds for X, i.e. since

$$\delta_{\bar{g}_t}\delta^*_{\bar{g}_t}X_t^\flat = \delta_{\bar{g}_t}(\mathring{Q}_1(g_t,\varphi_t) + \rho_t\bar{g}_t)$$

is an elliptic operator and $\mathring{Q}_1(g,\varphi) \in C^{\alpha,\alpha/2}$, it follows that

$$X \in C_1^{\hat{\alpha},\hat{\alpha}/2}.$$

We conclude that indeed

$$\partial_t u_t + \frac{1}{32}(1 - R_{\bar{g}})\Delta_{\bar{g}_t} u_t \in C_1^{\hat{\alpha},\hat{\alpha}/2}$$

and hence by Schauder estimates

$$u \in C_1^{2+\hat{\alpha},1+\hat{\alpha}/2} \subset C_3^{\hat{\alpha},\hat{\alpha}/2}$$

Now we turn to the equation for the spinorial part φ. The equation

$$\partial_t\varphi_t = -\nabla^{g_t *}\nabla^{g_t}\varphi_t + |\nabla^{g_t}\varphi_t|^2\varphi_t + \nabla^{g_t}_{X_t}\varphi_t - \frac{1}{4}dX_t^\flat\cdot\varphi_t$$

can be rewritten as

$$\begin{aligned}
&\partial_t\varphi_t + \frac{1}{32}e^{-2u_t}\nabla^{\bar{g}_t *}\nabla^{\bar{g}_t}\varphi_t \\
=&e^{-2u_t}\Big(-\operatorname{grad}_{\bar{g}_t} u_t\cdot D_{\bar{g}_t}\varphi_t - \nabla^{\bar{g}_t}_{\operatorname{grad}_{\bar{g}_t} u_t}\varphi_t - |\nabla^{\bar{g}_t}\varphi_t|^2\varphi_t + \langle\operatorname{grad}_{\bar{g}_t} u_t\cdot D_{\bar{g}_t}\varphi_t,\varphi_t\rangle\varphi_t\Big) \\
&+ \nabla^{\bar{g}_t}_{X_t}\varphi_t - \frac{1}{4}dX_t^\flat\cdot\varphi_t.
\end{aligned}$$

The left hand side is a parabolic operator with $C_3^{\hat{\alpha},\hat{\alpha}/2}$ coefficients. On the other hand the right hand side consists of terms, which depend on the first derivatives of g and φ, with the exception of the terms involving X. These terms are seen to be $C_1^{\hat{\alpha},\hat{\alpha}/2}$. The term $\nabla^{\bar{g}}_X\varphi$ is $C_1^{\hat{\alpha},\hat{\alpha}/2}$, because $\varphi \in C_2^{\hat{\alpha},\hat{\alpha}/2}$ and $X \in C_1^{\hat{\alpha},\hat{\alpha}/2}$. The term $dX^\flat\cdot\varphi$ is more delicate, because a derivative of X is involved. Thus we need to show that $X \in C_2^{\hat\alpha,\hat\alpha/2}$. Then we can conclude that $dX^\flat \in C_1^{\hat\alpha,\hat\alpha/2}$. However, we already know $u \in C_3^{\hat\alpha,\hat\alpha/2}$. Since X is the solution of

$$\delta_{\bar g}\delta_{\bar g}^* X^\flat = \delta_{\bar g}(\mathring{Q}_1(g,\varphi) + \rho\bar g),$$

and $\rho \in C_1^{\hat\alpha,\hat\alpha/2}$, it remains to be seen that $\delta_{\bar g}\mathring{Q}_1(g,\varphi) \in C^{\hat\alpha,\hat\alpha/2}$ to conclude from Schauder theory that $X \in C_2^{\hat\alpha,\hat\alpha/2}$. That $\delta_{\bar g}\mathring{Q}_1(g,\varphi) \in C^{\hat\alpha,\hat\alpha/2}$ follows from the calculation

$$\delta_{\bar g}\mathring{Q}_1(g,\varphi)) = \frac{1}{2}e^{2u} d\,\mathrm{tr}_g\, Q_1(g,\varphi)$$

and the fact that

$$\mathrm{tr}_g\, Q_1(g,\varphi) = -\frac{1}{4}(R_g/4 + |\nabla^g\varphi|^2 - |D_g\varphi|^2) \in C_1^{\hat\alpha,\hat\alpha/2}.$$

For higher regularity of the solution we can repeat this argument. □

5.2.1 An integral criterium

We will now show that a much weaker assumption suffices to continue the flow.

Theorem 5.23.
Suppose (g_t,φ_t), $t \in [0,T)$ is a smooth solution of the spinor flow on M and suppose

$$\sup_{t\in[0,T)} \int_M |\nabla^2\varphi|^q \,\mathrm{vol}_{g_t} < \infty$$

for some $q > 8$ and

$$\inf_{t\in[0,T)} \mathrm{inj}(g_t) > 0.$$

Then the solution (g_t,φ_t) can be extended to a smooth solution on an interval $[0,T+\delta)$ for some $\delta > 0$.

Proof. The second covariant derivative of φ_t can be orthogonally decomposed into a symmetric and an antisymmetric part:

$$\nabla^g\nabla^g\varphi = (\nabla^g\nabla^g\varphi)^{sym} + (\nabla^g\nabla^g\varphi)^{asym}.$$

The antisymmetric part is the curvature of the spin connection. Since the curvature of the spin connection on a surface is given by

$$R^g(X,Y)\varphi = \frac{R_g}{4} g(X,Y)\omega\cdot\varphi,$$

it follows that for a unit spinor

$$|(\nabla^g\nabla^g\varphi)^{asym}\varphi|^2 = \frac{1}{8}R_g^2.$$

Consequently, a bound on $\int_M |\nabla^2\varphi|^q \operatorname{vol}_g$ implies a bound on $\int_M |R_g|^q \operatorname{vol}_g$.

Now suppose $(\tilde{g}_t, \tilde{\varphi}_t)$ is a smooth solution of the spinor flow on the interval $[0,T)$ satisfying

$$\sup_{0\le t<T} \int_M |\nabla^2\tilde{\varphi}_t|^q \operatorname{vol}_{\tilde{g}_t} < \infty$$

and

$$\inf_{0\le t<T} \operatorname{inj}(\tilde{g}_t) > 0.$$

Consider the corresponding split flow $(\bar{g}_t, u_t, \varphi_t, f_t)$ and denote $g_t = e^{2u_t}\bar{g}_t$. Notice that these bounds are diffeomorphism invariant, so we get the same bounds for (g_t, φ_t). Theorem 3.22 applies to the family g_t and we obtain that $\bar{g}_t$ has injectivity radius bounded from below and that u_t is bounded in $C^0(M, \breve{g})$ for any fixed metric $\breve{g}$. Thus we can apply L^p theory to the curvature equation to conclude that $u \in W^{2,q}$, and in particular in $C^{1,\alpha}$ by Sobolev embedding.

The conditions of lemma 3.35 are met. This is clear from the previous for the injectivity radius and the bound on u. It remains to be seen that $\|\partial_t g_t\|_{L^2(M,g_t)} = \|Q_1(g_t,\varphi_t)\|_{L^2(M,g_t)}$ is bounded. Since Q_1 has roughly the form $\nabla^2\varphi * \varphi + \nabla^g\varphi * \nabla^g\varphi$, this is implied by the bound on $\|\nabla^2\varphi\|_{L^q}$. Thus lemma 3.35 applies and $\|\partial_t\bar{g}_t\|_{L^2(M,\bar{g}_t)}$ is bounded uniformly.

If we also show that $u, \varphi \in C_2^{\alpha,\alpha/2}$, then we can apply lemma 5.22 to get uniform estimates of u_t, φ_t in any C^k norm and thus we may pass to a smooth limit as $t \to T$ and the flow can be restarted at time T, yielding a solution on $[0, T+\delta)$ by the short time existence.

Notice that lemma 3.34 applies to the curve $\bar{g}_t$ and we hence get uniform control of the metrics on the interval $[0,T)$ in any C^k norm.

In the following we indicate the steps we will take to obtain Hölder regularity of u and φ from elliptic and parabolic regularity theory. We denote by r and α regularity exponents, which we will make precise below. Recall the split flow equations from proposition 5.20. In a first step we show that $\tilde{\rho}_t, X_t$ are uniformly bounded in $W^{1,r}$. This will then imply that

$$\partial_t u_t + \frac{1}{32}(1 - R_{\bar{g}_t})e^{-2u_t}\Delta_{\bar{g}_t} u_t \in W^{1,r}.$$

This implies by L^p theory

$$u, du \in W_r^{2,1}.$$

Using the results so far, we then show that

$$\partial_t\varphi_t + \frac{1}{32}e^{-2u_t}\nabla^{\bar{g}_t *}\nabla^{\bar{g}_t}\varphi_t \in L^r,$$

which will imply

$$\varphi, \nabla^{\bar{g}}\varphi \in W_r^{2,1}.$$

If $r > 4$, then the anisotropic Sobolev embedding theorem 1.5 implies

$$u, \varphi \in C_2^{\alpha,\alpha/2}.$$

Step 1. $\tilde{\rho} \in W^{1,q'}$ for every $q' < q$:
Recall that $\tilde{\rho}_t$ satisfies equation 5.21e:

$$\Delta_{\bar{g}_t}\tilde{\rho}_t + R_{\bar{g}_t}\tilde{\rho}_t = -\frac{1}{2}R_{\bar{g}_t}\operatorname{tr}_{g_t} Q_1(g_t,\varphi_t) + 2\delta_{\bar{g}_t}\left(\mathring{Q}_1(g_t,\varphi_t)(\operatorname{grad}_{g_t} u_t,\cdot)\right).$$

Note that $\nabla^g\varphi \in W^{1,2}$. This is because on the one hand $\nabla^g\varphi \in L^2$, because $\|\nabla^g\varphi\|^2_{L^2} = 2\mathcal{E}(g,\varphi)$ is decreasing along the flow. On the other hand $\nabla^g\nabla^g\varphi \in L^q$ by assumption. By Sobolev embedding we then also have $\nabla^g\varphi \in L^q$. This implies $\nabla^g\varphi \in W^{1,q}$. Furthermore $R_g \in L^q$. Since $\operatorname{tr}_g Q_1(g,\varphi) = -\frac{1}{4}(R_g/4 + |\nabla^g\varphi|^2 - |D_g\varphi|^2)$, it follows that $\operatorname{tr}_g Q_1 \in L^q$. Since Q_1 has the structural form $\nabla^g\nabla^g\varphi * \varphi + \nabla^g\varphi * \nabla^g\varphi$, it follows that $\mathring{Q}_1 \in L^q$ and since $du \in W^{1,q}$, it follows that

$$\mathring{Q}_1(g_t,\varphi_t)(\operatorname{grad}_{g_t} u_t,\cdot) \in L^{q'} \text{ for every } q' < q.$$

By elliptic theory it follows that

$$\tilde{\rho}_t \in W^{1,q'} \text{ for every } q' < q.$$

Step 2. $X \in W^{1,q}$:
By the previous step we know $\tilde{\rho}_t \in W^{1,q'}$ and hence by Sobolev inequality $\tilde{\rho}_t \in L^p$ for every p. By equation 5.21d, it follows that $\rho_t \in L^q$. From the equation 5.21f

$$\delta_{\bar{g}_t}\delta^*_{\bar{g}_t}X^\flat_t = \delta_{\bar{g}_t}(\mathring{Q}_1(g_t,\varphi_t) + \rho_t\bar{g}_t),$$

the claim then follows by elliptic theory.
Step 3. $u, du \in W^{2,1}_{q'}$ for every $q' < q$:
First we note that we know u is bounded. Thus by the Krylov–Safonov estimate for parabolic equations 1.18, it suffices to show that

$$\partial_t u_t + \frac{1}{32}(1 - R_{\bar{g}})e^{-2u_t}\Delta_{\bar{g}_t}u_t \in L^3,$$

to conclude that u is Hölder continuous both temporally and spatially. (We already knew that u is Hölder continuous spatially from Sobolev embedding. The new information is the temporal continuity.) Once we have shown this, we can apply the standard L^p theory for equations with Hölder continuous coefficients. Thus the claim will follow from

$$\partial_t u_t + \frac{1}{32}(1 - R_{\bar{g}})e^{-2u_t}\Delta_{\bar{g}_t}u_t \in W^{1,q'},$$

since q' can be chosen to be greater than 3 we can first apply the Krylov–Safonov estimate. Then $u \in W^{2,1}_{q'}$ follows from L^p theory. For $du \in W^{2,1}_{q'}$, notice that if the above term is in $W^{1,q'}$, then

$$\partial_t du_t + \frac{1}{32}(1 - R_{\bar{g}})\left(-2e^{-2u_t}(\Delta_{\bar{g}_t}u_t)du_t + e^{-2u_t}\nabla^{\bar{g}_t *}\nabla^{\bar{g}_t}du_t + \operatorname{Ric}^{\bar{g}_t}(\cdot,\operatorname{grad}_{\bar{g}_t} u)\right) \in L^{q'}.$$

Since the term $e^{-2u_t}(\Delta_{\bar{g}_t} u_t) du_t$ is in $L^{q'}$ and the term $e^{-2u_t} \operatorname{Ric}^{\bar{g}_t}(\cdot, \operatorname{grad}_{\bar{g}_t} u)$ is in L^p for every p, it follows that $du_t \in W^{2,1}_{q'}$ by parabolic regularity.

We now show that $\partial_t u_t + \frac{1}{32}(1 - R_{\bar{g}}) e^{-2u_t} \Delta_{\bar{g}_t} u_t \in W^{1,q'}$. Equation 5.21a says that this term is equal to

$$-\frac{1}{16}(1 - R_{\bar{g}}) \left(\frac{1}{4} R_{\bar{g}} + |\nabla^{g_t} \varphi_t|^2 - |D_{g_t} \varphi_t|^2 \right) - X_t u_t - \frac{1}{2} \tilde{\rho}_t.$$

For the bracketed term, we already know $\nabla^{g_t} \varphi_t \in W^{1,q}(M, g)$ from the last step. Hence $|\nabla^{g_t} \varphi_t|^2, |D_{g_t} \varphi_t|^2 \in W^{1,q/2}$. The term $R_{\bar{g}}$ is constant. We have already seen $\tilde{\rho}_t \in W^{1,q'}$. Furthermore $X_t u_t = du_t(X_t)$ is in $W^{1,q'}$, since $du_t \in L^p$ for every p and $X_t \in W^{1,q}$. This proves the claim.

Step 4. $\varphi \in W^{2,1}_q$:
Recall that φ satisfies equation 5.21b, which says that

$$\partial_t \varphi_t + \frac{1}{32} e^{-2u_t} \nabla^{\bar{g}_t *} \nabla^{\bar{g}_t} \varphi_t$$

equals

$$e^{-2u_t} \psi_t + \nabla^{\bar{g}_t}_{X_t} \varphi_t - \frac{1}{4} dX_t^\flat \cdot \varphi_t.$$

Since $X \in W^{1,q}$, it follows that $dX^\flat \cdot \varphi \in L^q$. Furthermore, since $\nabla^g \varphi \in W^{1,q}$, it follows that $\nabla^{\bar{g}}_X \varphi \in W^{1,q}$, using the Banach algebra property of $W^{1,q}$. The term ψ is given by

$$-\operatorname{grad}_{\bar{g}} u \cdot D_{\bar{g}} \varphi - \nabla^{\bar{g}}_{\operatorname{grad}_{\bar{g}} u} \varphi - |\nabla^{\bar{g}} \varphi|^2 \varphi + \langle \operatorname{grad}_{\bar{g}} u \cdot D_{\bar{g}} \varphi, \varphi \rangle \varphi.$$

Checking term by term and using again the Banach algebra property, we also conclude that $\psi \in W^{1,q}$. Hence

$$\partial_t \varphi_t + \frac{1}{32} e^{-2u_t} \nabla^{\bar{g}_t *} \nabla^{\bar{g}_t} \varphi_t \in L^q$$

and the claim follows by parabolic regularity theory.

Step 5. $\nabla^{\bar{g}} \varphi \in W^{2,1}_{q/2}$:
To prove that $\nabla^{\bar{g}} \varphi \in W^{2,1}_q$ we check that

$$\chi = \partial_t \varphi_t + \frac{1}{32} e^{-2u_t} \nabla^{\bar{g}_t *} \nabla^{\bar{g}_t} \varphi_t$$

satisfies

$$\int_0^T \|\chi\|^{q/2}_{W^{1,q/2}} dt < \infty.$$

The claim will then follow from L^p theory by an argument parallel to the one for $du \in W^{2,1}_q$: we apply $\nabla^{\bar{g}}$ to $\partial_t \varphi_t + \frac{1}{32} e^{-2u_t} \nabla^{\bar{g}*} \nabla^{\bar{g}} \varphi$ and obtain that

$$\partial_t \nabla^{\bar{g}_t} \varphi_t + \nabla^{\bar{g}_t *} \nabla^{\bar{g}_t} (\nabla^{\bar{g}_t} \varphi_t) \in L^{q/2}(M \times [0, T]).$$

We have already shown that the terms $e^{-2u_t}\psi$ and $\nabla^{\bar{g}}_X\varphi$ are uniformly bounded in $W^{1,q}$ in time. In particular the time integral above is bounded. It remains to be checked that $dX^\flat \cdot \varphi$ also satisfies such a bound. First we note that since $\varphi \in W^{2,q}$ we have

$$\|dX^\flat \cdot \varphi\|_{W^{1,q}} \leq \|dX\|_{W^{1,q}} \|\varphi\|_{W^{1,q}} \leq C\|dX\|_{W^{1,q}} \leq C\|X\|_{W^{2,q}}.$$

Thus it suffices to establish a bound on

$$\int_0^T \|X_t\|^{q/2}_{W^{2,q/2}} dt$$

to prove the claim. This can be done by applying elliptic regularity theory to formula 5.21f

$$\delta_{\bar{g}_t}\delta^*_{\bar{g}_t} X^\flat_t = \delta_{\bar{g}_t}(\mathring{Q}_1(g_t, \varphi_t) + \rho_t \bar{g}_t).$$

Thus we need to bound

$$\delta_{\bar{g}_t}(\mathring{Q}_1(g_t, \varphi_t) + \rho_t \bar{g}_t)$$

in $L^{q/2}$. To that end first note that

$$\rho_t = \tilde{\rho}_t + \frac{1}{2}\operatorname{tr}_{g_t} Q_1(g_t, \varphi_t).$$

In step 1 we already saw $\tilde{\rho}_t \in W^{1,q'}$. Moreover

$$\operatorname{tr}_g Q_1(g, \varphi) = -\frac{1}{4}(R_g/4 + |\nabla^g \varphi|^2 - |D_g \varphi|^2)$$

We know that $\nabla^{g_t}\varphi_t \in W^{1,q}$, so that $|\nabla^{g_t}\varphi_t|^2 - |D_{g_t}\varphi_t|^2 \in W^{1,q/2}$. Since $u, du \in W^{2,1}_{q'}$ it follows that

$$R_{g_t} = e^{-2u_t}(2\Delta_{\bar{g}_t} u_t + R_{\bar{g}})$$

satisfies

$$\int_0^T \|R_{g_t}\|^{q'}_{W^{1,q'}} dt < \infty.$$

We conclude

$$\int_0^T \|\operatorname{tr}_{g_t} Q_1(g_t, \varphi_t)\|^{q/2}_{W^{1,q/2}} dt < \infty.$$

Thus in particular

$$\int_0^T \|\delta_{\bar{g}_t}(\operatorname{tr}_{g_t} Q_1(g_t, \varphi_t)\bar{g}_t)\|^{q/2}_{L^{q/2}} dt < \infty.$$

On the other hand we have formula 5.21g

$$\delta_{\bar{g}}\mathring{Q}_1(g, \varphi) = \frac{1}{2} e^{2u} d \operatorname{tr}_g Q_1(g, \varphi).$$

This implies

$$\int_0^T \|\delta_{\bar{g}_t}\mathring{Q}_1(g_t, \varphi_t))\|^{q/2}_{L^{q/2}} dt < \infty.$$

In conclusion we have shown

$$\int_0^T \|\delta_{\bar{g}_t}\delta^*_{\bar{g}_t}X^\flat_t\|^{q/2}_{L^{q/2}}dt < \infty$$

which implies

$$\int_0^T \|X_t\|^{q/2}_{W^{2,q/2}}dt < \infty.$$

In turn we conclude

$$\int_0^T \|\chi_t\|^{q}_{W^{1,q}}dt < \infty,$$

which finishes the proof of this step.

Final step. $u \in C_2^{\alpha,\alpha/2}$ and $\varphi \in C_2^{\alpha,\alpha/2}$:
This claim now follows by applying Sobolev embedding for u, du, φ and $\nabla^g\varphi$ and we thus obtain $u, \varphi \in C_2^{\alpha,\alpha/2}$ for every $\alpha < 1 - \frac{4}{q/2} = 1 - \frac{8}{q}$, see theorem 1.5. We can then apply the bootstrapping lemma 5.22 to obtain the statement of the theorem. $\square$

5.2.2 A pointwise criterium

In this section we prove the following criterium for blow up of the spinor flow in two dimensions.

Theorem 5.24.
Suppose (g_t, φ_t), $t \in [0, T)$ is a solution of the spinor flow on M and suppose

$$\sup_{\substack{x \in M,\\ 0 \le t < T}} |\nabla^2\varphi_t(x)| < \infty.$$

Then the solution (g_t, φ_t) can be extended to a smooth solution on an interval $[0, T)$ for some $\delta > 0$.

To apply theorem 5.23, we need to show on the one hand that

$$\int_M |\nabla^2\varphi_t|^q \operatorname{vol}_{g_t}$$

remains bounded for some $q > 8$. Indeed, this is true for all q, since the spinor flow on surfaces preserves the total volume. On the other hand, we need to check that the injectivity radius stays bounded below. This can be seen as follows. The volume of (M, g_t) remains fixed along the flow and the curvature remains bounded, because of the inequality

$$R_g^2/8 \le |\nabla^2\varphi|^2.$$

Hence by Cheeger's lemma it suffices to check that the diameter remains bounded. If $|\nabla^2\varphi_t(x)|$ is uniformly bounded, then so is $|\partial_t g_t| = |Q_1(g_t, \varphi_t)|$. This implies by integration in time that the metrics g_t are uniformly equivalent along the flow. Thus the diameters are uniformly bounded.

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Erklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift in Inhalt und Form eigenständig erstellt habe — abgesehen von der Beratung durch Hartmut Weiß. Ergebnisse der Dissertation sind bereits in zwei Vorabdrucken von wissenschaftlichen Arbeiten auf dem Dokumentenserver arXiv erschienen:

1. Stability of the Spinor Flow `https://arxiv.org/abs/1706.09292`
2. Blowup criteria for geometric flows on surfaces `https://arxiv.org/abs/1803.05737`

Ich versichere, dass die Arbeit unter Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft entstanden ist und dass sie weder zum Teil noch als Ganzes schon einer anderen Stelle im Rahmen eines Prüfungsverfahrens vorgelegen hat.