MEAN–VARIANCE HEDGING AND OPTIMAL INVESTMENT IN HESTON'S MODEL WITH CORRELATION

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Abstract. This paper solves the mean–variance hedging problem in Heston's model with a stochastic opportunity set moving systematically with the volatility of stock returns. We allow for correlation between stock returns and their volatility (so-called leverage effect).

Our contribution is threefold: using a new concept of opportunity-neutral measure we present a simplified strategy for computing a candidate solution in the correlated case. We then go on to show that this candidate generates the true variance-optimal martingale measure; this step seems to be partially missing in the literature. Finally, we derive formulas for the hedging strategy and the hedging error.

1. Introduction

We examine a classical problem in mathematical finance: how to optimally hedge a given static position in a derivative asset H with pay-off at time T by dynamic trading in the underlying asset S if the hedger wishes to minimize the expected squared hedging error. A crucial step on the way to the optimal hedge is to derive the density process of the so-called variance-optimal martingale measure (VOMM) or, more or less equivalently, the optimal strategy of a pure investment problem with quadratic utility. We focus on a model with stochastic volatility in which the investment opportunity set changes with the volatility level and volatility itself is correlated with the change in stock price.

In the case of zero correlation the problem at hand has been solved by Heath et al. (2001), drawing on the results of Laurent and Pham (1999). In principle, the VOMM in the correlated case has been explicitly determined by Hobson (2004). Our contribution is threefold: using a new concept of opportunity-neutral measure (cf. Černý and Kallsen 2005, hereafter CK05) we present a simplified strategy for computing a candidate solution in the correlated case. We then go on to show that this candidate is the true VOMM. Finally, we derive formulas for the hedging strategy and the hedging error, again based on CK05.

The assumption of zero interest rates is standard in the literature and it entails no loss of generality within the class of models with deterministic interest rates; we shall therefore adopt it here. The task of the hedger is to solve

$$\inf_{\vartheta} E\left((x+\vartheta \bullet S_T - H)^2\right),$$

where x is the initial endowment and ϑ belongs to the set of admissible strategies to be described in Section 2.1. Here $\vartheta \bullet S_T$ stands for $\int_0^T \vartheta_t dS_t$.

Consider the following model for the stock price S and its volatility process Y,

$$\mathcal{L}(S) = \left(\mu Y^2\right) \bullet I + Y \bullet W, \tag{1.1}$$

$$Y^2 = Y_0^2 + \left(\zeta_0 + \zeta_1 Y^2\right) \bullet I + \sigma Y \bullet \left(\rho W + \sqrt{1-\rho^2} U\right), \tag{1.2}$$

where $\mathcal{L}$ denotes stochastic logarithm, W and U are independent Brownian motions, $I_t = t$ is the activity process and $\sigma > 0, \zeta_0 \geq \sigma^2/2, \zeta_1 < 0, \mu, -1 \leq \rho \leq 1$ are real constants. Translated into the dW/dt notation the model reads

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu Y_t^2 dt + Y_t dW_t, \\ dY_t^2 &= \left(\zeta_0 + \zeta_1 Y_t^2\right) dt + \sigma Y_t \left(\rho dW_t + \sqrt{1-\rho^2} dU_t\right). \end{aligned}$$

The model is set up in such a way that the instantaneous Sharpe ratio equals μY and because Y is an autonomous diffusion it follows that the opportunity set is a deterministic function of Y. Conditions on ζ_0 and ζ_1 make sure that the volatility process is strictly positive and has a steady state distribution under P (cf. Cox et al. 1985).

We consider information filtration generated by S, which in the present model coincides with the filtration generated by W and U. In particular, the hedger can back out the current level of volatility from the quadratic variation process of the stock price. In contrast, there is a growing literature in which the volatility is *filtered* from the stock price data, cf. Brigo and Hanzon (1998) and Kim et al. (1998).

1.1. **Computation and verification.** We make use of the structural results reported in CK05. There is an *opportunity process* L and a portfolio process a (called an *adjustment process*) that solve the optimal investment problem in the absence of the contingent claim. The opportunity process has a natural interpretation in that $L_t^{-1} - 1$ equals the square of the maximal Sharpe ratio attainable by *dynamic* trading in asset S from t to maturity. The opportunity set is deterministic when L is a deterministic process.

In general, process L defines the so-called *opportunity-neutral measure* (a non-martingale equivalent measure) $P^\star$ which neutralizes the effect of the stochastic opportunity set and leads to the variance-optimal measure $Q^\star$ and the optimal hedging coefficients. More specifically, $Q^\star$ can be computed as the *minimal* martingale measure relative to $P^\star$. The optimality condition requires that the expected growth rate of L under measure P equals the squared instantaneous Sharpe ratio of the risky asset(s) under $P^\star$.

We guess a candidate opportunity process in the form

$$L = \exp(\varkappa_0 + \varkappa_1 Y^2),$$

where $\varkappa_0$ and $\varkappa_1$ are deterministic functions of time to maturity. For this functional form of L we write down the optimality criterion described above which yields a Riccati equation for $\varkappa_1$ and a first order linear equation for $\varkappa_0$ that are readily solved. With $P^\star$ in hand we evaluate a candidate adjustment process a as the myopic mean-variance stock portfolio weight under $P^\star$,

$$a = b^{S\star}/c^S = (\mu + \varkappa_1 \sigma\rho)/S_t,$$

where $b^{S\star}$ represents the drift of the stock price under measure $P^\star$ and $\sqrt{c^S}$ represents its volatility under P (and hence also under $P^\star$).

The computational procedure described above appears to be significantly shorter, more transparent and economically intuitive than the use of so-called fundamental representation equations proposed in Biagini et al. (2000) and Hobson (2004).

It remains to prove that the candidate adjustment process a corresponds to an admissible trading strategy. To this end we first prove that $L\mathcal{E}(-a \bullet S)$ and $L\left(\mathcal{E}(-a \bullet S)\right)^2$ are martingales which means that the *candidate* variance-optimal martingale measure $Q^\star$,

$$\frac{dQ^\star}{dP} = \frac{\mathcal{E}(-a \bullet S)_T}{L_0}, \tag{1.3}$$

is a martingale measure with square integrable density. This, however, does *not* yet imply that $Q^\star$ is the true VOMM! Merely, we have now constructed an equivalent martingale measure required by Assumption 2.1 in CK05.

In the final step of the verification (this step is left out in the theoretical characterization of Hobson 2004) we show that the process $\mathcal{E}(-a \bullet S)$ is generated by an admissible strategy. This is essentially equivalent to demonstrating that $\mathcal{E}(-a \bullet S)$ is a true martingale under *all* equivalent martingale measures Q such that $E\left((dQ/dP)^2\right) < \infty$. We use Novikov's condition combined with Hölder's inequality to show that $E\left(\mathrm{e}^{((\mu+\varkappa_1\sigma\rho)Y)^2 \bullet I_T}\right) < \infty$ is a sufficient condition for a to be admissible. We then apply the characterization of regular affine processes provided in Duffie et al. (2003), henceforth DFS03, to compute an upper bound of $E\left(\mathrm{e}^{((\mu+\varkappa_1\sigma\rho)Y)^2 \bullet I_T}\right)$ and hence characterize a subset of time horizons T for which a and L described above represent the true solution, and for which $Q^\star$ computed in (1.3) is the true VOMM.

Once we have the true opportunity and adjustment process, the rest of our analysis is a straightforward application of results in CK05. The optimal hedge of the contingent claim H is given by the Föllmer–Schweizer decomposition of H under measure $P^\star$ (cf. Lemma 4.8 in CK05). First we compute the mean value process V as a conditional expectation of H under the variance-optimal measure $Q^\star$, cf. CK05 (4.1). V happens to be a deterministic function of 3 state variables, S, Y^2 and I. The optimal hedge $\varphi = \varphi(x, H)$ is then given by

$$\begin{aligned} \varphi(x,H) &= \xi + a(V - x - \varphi(x,H) \bullet S), \\ \xi &:= c^{VS}/c^S, \end{aligned}$$

where x is the initial capital, c^{VS} represents the instantaneous covariance between V and S and c^S stands for the instantaneous variance of S. The minimal squared hedging error equals

$$\begin{aligned} E\left((x + \varphi(x,H) \bullet S_T - H)^2\right) &= L_0(x - V_0)^2 + \varepsilon_0^2, \\ \varepsilon_0^2 &:= E\left(\left(L\left(c^V - \left(c^{VS}\right)^2/c^S\right)\right) \bullet I_T\right), \end{aligned}$$

where c^V stands for the instantaneous variance of V, cf. CK05 Theorem 4.12.

1.2. **Interpretation.** By CK05, Lemmas 3.1 and 3.7, we have that

$$\varphi_S := \varphi(1,2) = a\mathcal{E}(-a \bullet S) = a(2 - (1 + \varphi_S \bullet S)) \tag{1.4}$$

is a mean-variance efficient strategy for an agent wishing to maximize the unconditional Sharpe ratio of her terminal wealth. The maximal squared Sharpe ratio equals

$$\mathrm{SR}_S^2 = 1/L_0 - 1 = e^{-\varkappa_0(0) - \varkappa_1(0)Y_0^2} - 1,$$

where $\varkappa_0, \varkappa_1$ are non-positive functions of time to maturity computed in Section 3. Thus, in this model, higher volatility means more lucrative dynamic stock investment opportunity.

The optimal stock trading strategy φ_S can be interpreted as a solution to quadratic utility maximization with bliss point at 2 and initial wealth level at 1. At an intermediate point in time the distance of agent's wealth from the bliss point is $2 - (1 + \varphi_S \bullet S)$ which is exactly equal to $\mathcal{E}(-a \bullet S)$. In view of (1.4) we observe that the agent becomes more risk averse as her wealth approaches the bliss point. Vice versa, when the risky investment performs poorly the gap between agent's wealth and the bliss point widens and the agent increases her risky position in direct proportion to the gap size. (The optimal dynamic investment clearly has an element of a doubling strategy and this is why it is important to check admissibility of a candidate solution for a). One can view (1.4) as a *dynamic portfolio insurance* strategy (cf. Black and Jones 1987) in reverse, whereby the investor specifies a fixed ceiling rather than a floor for wealth and uses a state-dependent multiplier a.

We next examine the impact of the stochastic opportunity set on a. When there is no correlation between stock returns and the volatility we have

$$a := \frac{b^{S\star}}{c^S} = \frac{\mu}{S} = \frac{b^S}{c^S},$$

which interestingly means that the investor acts as if the opportunity set were deterministic (or at least predictable, in the sense of L being a predictable process of finite variation) even though this is clearly not the case and $P^\star \neq P$, cf. CK05 Proposition 3.28. Empirical research on equity data finds negative correlation (so called leverage effect) implying that the optimal value of a should be revised upwards by the factor $\varkappa_1 \sigma\rho/S$ relative to the uncorrelated case (cf. equation 3.3).

The mean value process is a sufficiently smooth function of three state variables

$$V_t = f(T-t, Y_t^2, S_t).$$

It represents a price at which an agent holding dynamically efficient portfolio of equities would not wish to buy or sell the option. The optimal hedge $\varphi(x,H)$ consists of two components – the pure hedge ξ and a feedback element $a(V - x - \varphi \bullet S)$. The quantity $(V - x - \varphi \bullet S)$ represents the shortfall of the hedging portfolio relative to the mean value of the derivative asset. Since a is typically positive the optimal strategy tends to overhedge when it is performing poorly and underhedge once it has accumulated a hedging surplus.

The pure hedging coefficient satisfies

$$\xi_t = \frac{c_t^{VS}}{c_t^S} = \frac{\partial f}{\partial x_3}(T-t, Y_t^2, S_t) + \rho\sigma \frac{\partial f}{\partial x_2}(T-t, Y_t^2, S_t)/S_t.$$

The pure hedge therefore has two components: the standard delta hedge using the representative agent price V_t, and a leverage component exploiting the correlation of the representative agent price with the volatility process.

To appreciate the role of the minimal expected squared hedging error ε_0^2 suppose now that in addition to the optimal equity investment the agent is able to sell (issue) an equity option with payoff H at time T at initial price $C_0 > V_0$ (when $C_0 < V_0$ it is optimal to *buy* the option). Suppose that the initial option position is held to maturity and the agent does not trade in any other options, but she is allowed to engage in additional stock trades for hedging purposes. We show in Lemma 5.3 that in order to maximize her Sharpe ratio the agent should sell $\eta = \frac{C_0 - V_0}{\varepsilon_0^2} \frac{1}{1+\mathrm{SR}_{S,H}^2}$ options and hedge them optimally to maturity using the strategy

$$\varphi_H := \varphi(\eta C_0, \eta H) = \eta\xi + a(\eta V - \eta C_0 - \varphi_H \bullet S). \tag{1.5}$$

The unconditional maximal squared Sharpe ratio of the combined strategy $\varphi_S + \varphi_H$ equals

$$\mathrm{SR}_{S,H}^2 := \mathrm{SR}_S^2 + \frac{(C_0 - V_0)^2}{\varepsilon_0^2},$$

which means that $\frac{C_0 - V_0}{\varepsilon_0}$ is an incremental Sharpe ratio generated by trading in the option. Based on this observation we conclude that when ε_0 is very high one may observe a significant deviation of the market price C_0 from the representative agent price V_0 which does not give rise to excessively attractive investment opportunities, beyond the ones that already existed in the market before the option was introduced. One can invert the relationship between the unconditional incremental Sharpe ratio and the selling (buying) price to compute unconditional good-deal price bounds (cf. Černý and Hodges 2002).

1.3. **Organization.** In Section 2 we define the admissible trading strategies. In Section 3 we compute the candidate adjustment and opportunity processes and characterize a time horizon $\tilde{T}$ such that the candidate processes represent the true solution for all $T < \tilde{T}$. In Section 4 we give an explicit formula for the mean value process and the pure hedge. Section 5 concludes by giving an explicit formula for the unconditional expected squared hedging error and the incremental Sharpe ratio of an optimally hedged position.

2. Preliminaries

2.1. **Trading strategies and martingale measures.** We work on a filtered probability space $(\Omega, \mathcal{F}_T, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, P)$ where T is a fixed time horizon. The filtration $\mathbb{F}$ is generated by two uncorrelated Brownian motions U and W which drive the stock price and its volatility as shown in equations (1.1) and (1.2).

Definition 2.1 (Delbaen and Schachermayer 1996)**.** *Semimartingale S is locally in $L^2(P)$ if there is a localizing sequence of stopping times $\{U_n\}_{n\in\mathbb{N}}$ such that*

$$\sup\{E\left(S_\tau^2\right) : \tau \le U_n \text{ stopping time}\} < \infty$$

for any $n \in \mathbb{N}$.

Remark 2.2. *Every continuous semimartingale is locally in $L^2(P)$ since we may take*

$$U_n := \inf\{\tau : S_\tau^2 \geq n\}$$

as the sequence of localizing times.

Definition 2.3. *Consider a price process S locally in $L^2(P)$ with the corresponding localizing sequence $\{U_n\}_{n\in\mathbb{N}}$. A trading strategy ϑ is called* simple *if it is a linear combination of strategies $Y1_{]\!]\tau_1,\tau_2]\!]}$ where $\tau_1 \leq \tau_2$ are stopping times dominated by U_n for some $n \in \mathbb{N}$ and Y is a bounded $\mathcal{F}_{\tau_1}$-measurable random variable. We denote by Θ the set of all simple trading strategies.*

Definition 2.4. *A trading strategy $\vartheta \in L(S)$ is called* admissible *if there is a sequence $\{\vartheta^{(n)}\}_{n\in\mathbb{N}}$ of simple strategies such that*

$$\begin{aligned} \vartheta^{(n)} \bullet S_t &\longrightarrow \vartheta \bullet S_t \text{ in probability for any } t \in [0,T]; \text{ and} \\ \vartheta^{(n)} \bullet S_T &\longrightarrow \vartheta \bullet S_T \text{ in } L^2(P). \end{aligned}$$

We denote the set of all admissible strategies by $\overline{\Theta}$.

Remark 2.5. *The set $\overline{\Theta}$ does not depend on the choice of the localizing sequence $\{U_n\}_{n\in\mathbb{N}}$ in Definition 2.3 (cf. CK05, Remark 2.8).*

The following lemma shows admissible strategies are economically indistinguishable from simple strategies.

Lemma 2.6. *We have*

$$K_2 := \overline{\{\vartheta \bullet S_T : \vartheta \in \Theta\}} = \{\vartheta \bullet S_T : \vartheta \in \overline{\Theta}\},$$

where $\overline{\{.\}}$ denotes closure in $L^2(P)$.

Proof. See CK05, Corollary 2.9. □

We now state a result on the duality between admissible strategies and a suitably chosen class of martingale measures for continuous semimartingales.

Definition 2.7. *Denote by $\mathcal{M}_2^e$ the subset of equivalent martingale measures with square integrable density, i.e.*

$$\mathcal{M}_2^e := \left\{Q \sim P : dQ/dP \in L^2(P), S \text{ is a } Q\text{-local martingale}\right\}.$$

Theorem 2.8. *Let S be a continuous semimartingale and suppose S admits an equivalent martingale measure with square integrable density. Then the following assertions are equivalent:*

(1) $\vartheta \in \overline{\Theta}$
(2) $\vartheta \in L(S)$, $\vartheta \bullet S_T \in L^2(P)$ *and* $\vartheta \bullet S$ *is a Q-martingale for every* $Q \in \mathcal{M}_2^e$.

Proof. (1)⇒(2): This is shown in CK05, Corollary 2.5.

(2)⇒(1): By Delbaen and Schachermayer (1996), Theorems 1.2 and 2.2, we have $\vartheta \bullet S_T \in K_2$. Since martingales are determined by their final value, the claim follows. □

Theorem 2.8 shows that for continuous processes $\overline{\Theta}$ coincides with the class of trading strategies used in Gourieroux et al. (1998). For a general result on the duality between the admissible strategies and (signed) martingale measures we refer the reader to CK05, Lemma 2.4.

2.2. **Semimartingale characteristics.** All processes in this paper are continuous semimartingales. For any $\mathbb{R}^n$-valued process X we write $X = X_0 + B^X + M^X$ for the canonical decomposition of X into a predictable process of finite variation and a local martingale under measure P, and similarly $X = X_0 + B^{X\star} + M^{X\star}$ for the decomposition under $P^\star$. We set

$$C^X_{ij} := \left\langle M^{X_i}, M^{X_j}\right\rangle = \langle X_i, X_j\rangle = [X_i, X_j],$$

where $X_1, \dots, X_n$ denote the components of X. For processes with jumps the three quantities above will generally be different. By Jacod and Shiryaev (2003), II.2.9 there is an increasing predictable process A, a $\mathbb{R}^n$-valued predictable process b^X and $\mathbb{R}^{n\times n}$-valued predictable process c^X whose values are non-negative symmetric matrices such that

$$B^X = b^X \bullet A,\ C^X = c^X \bullet A.$$

We write interchangeably $c^{X_iX_j} := c^X_{ij}, c^{X_i} := c^X_{ii}$.

In this paper the activity process A can be chosen such that $A_t = I_t := t$ and we adopt this convention henceforth. Thus in this paper b^X refers to the drift and $\sqrt{c^X}$ to the volatility when X is a univariate process. For example, for $X = (Y^2, S)$ in (1.1) and (1.2) we have

$$\begin{pmatrix} b^{Y^2} \\ b^S \end{pmatrix} = \begin{pmatrix} \zeta_0 + \zeta_1 Y^2 \\ \mu S Y^2 \end{pmatrix}, \tag{2.1}$$

$$\begin{pmatrix} c^{Y^2} & c^{Y^2S} \\ c^{SY^2} & c^S \end{pmatrix} = \begin{pmatrix} \sigma^2 Y^2 & \rho\sigma S Y^2 \\ \rho\sigma S Y^2 & S^2Y^2 \end{pmatrix}. \tag{2.2}$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be in C^2 and denote by $f_i := \frac{\partial f}{\partial x_i}, f_{ij} := \frac{\partial^2 f}{\partial x_i \partial x_j}$ its derivatives. Consider an $\mathbb{R}^n$-valued semimartingale X. Then $f(X)$ is a semimartingale and the Itô formula in our notation reads

$$\begin{aligned} b^{f(X)} &= \sum_{i=1}^{n} f_i(X) b_i^X + \frac{1}{2} \sum_{i,j=1}^{n} f_{ij}(X) c_{ij}^X, \\ c^{f(X)} &= \sum_{i,j=1}^{n} f_i(X) f_j(X) c_{ij}^X, \\ c^{f(X) X_i} &= \sum_{j=1}^{n} f_j(X) c_{ij}^X. \end{aligned}$$

For a univariate process N its stochastic exponential is given by $\mathcal{E}(N) = e^{N-\frac{1}{2}\langle N \rangle}$. Conversely, for a positive process L its stochastic logarithm equals $\mathcal{L}(L) = \ln L - \ln L_0 + \frac{1}{2} \langle \ln L \rangle$. In terms of characteristics

$$\begin{aligned} b^{\mathcal{E}(N)} &= \mathcal{E}(N) b^N, \qquad c^{\mathcal{E}(N)} = (\mathcal{E}(N))^2 c^N, \\ b^{\mathcal{L}(L)} &= L^{-1} b^L, \qquad c^{\mathcal{L}(L)} = L^{-2} c^L. \end{aligned}$$

Suppose $\mathcal{E}(\eta \bullet M^X)$ is a martingale and define a new measure,

$$dP^\star := \mathcal{E}(\eta \bullet M^X)_T dP. \tag{2.3}$$

Girsanov theorem (cf. Jacod and Shiryaev 2003, III.3.11 and Kallsen 2006, Proposition 2.6) then yields characteristics $b^{X\star}$ and $c^{X\star}$ under $P^\star$ as follows

$$b^{X\star} = b^X + c^X \eta^\intercal, \tag{2.4}$$
$$c^{X\star} = c^X. \tag{2.5}$$

3. The Merton problem

In this section we identify the opportunity process L and the adjustment process a which characterize the dynamically optimal portfolio in the underlying asset and the bank account, as discussed in Section 1.2. This type of dynamically optimal asset allocation is generically referred to as the Merton problem.

Definition 3.1. *We say that L is a* candidate opportunity process *if*

1. *L is a $(0,1]$-valued continuous semimartingale,*
2. $L_T = 1$,
3. *For $K := \mathcal{L}(L)$ we have*

$$b^K = \left(b^S + c^{KS}\right)^2 / c^S. \tag{3.1}$$

In such case we call $a = \left(b^S + c^{KS}\right) / c^S$ the candidate adjustment process *corresponding to L.*

Proposition 3.2. *Set*

$$\begin{aligned} A &= -\mu^2,\ B = \zeta_1 - 2\rho\sigma\mu,\ C = \frac{1}{2}\sigma^2\left(1-2\rho^2\right),\ F = \zeta_0, \\ y_0 &= w_0 = 0, \end{aligned}$$

and for functions w, y and parameter $\tau^\star$ of Lemma 6.1 set

$$\begin{aligned} \varkappa_1(t) &:= w(T-t), \\ \varkappa_0(t) &:= y(T-t), \\ T^\star &:= \tau^\star. \end{aligned}$$

Define

$$L_t = \exp(\varkappa_0(t) + Y_t^2 \varkappa_1(t)), \tag{3.2}$$

$$a_t = \left(\mu + \rho\sigma\varkappa_1(t)\right)/S_t. \tag{3.3}$$

Then L is a candidate opportunity process and a is the corresponding candidate adjustment process for $T < T^\star$.

Proof. The proof proceeds in two steps, i) computation and ii) verification.

i) Consider L in the form (3.2) for as yet unknown functions of calendar time $\varkappa_0$ and $\varkappa_1$, and define $K := \mathcal{L}(L)$. The Itô formula yields

$$\begin{aligned} K &:= \mathcal{L}(L) = \left(\varkappa_0' + Y^2\varkappa_1' + \frac{1}{2}\sigma^2 Y^2 \varkappa_1^2\right) \bullet I + \varkappa_1 \bullet Y^2 \\ &= \left(\varkappa_0' + Y^2\varkappa_1' + \frac{1}{2}\sigma^2 Y^2 \varkappa_1^2 + \left(\zeta_0 + \zeta_1 Y^2\right)\varkappa_1\right) \bullet I \\ &\quad + (\varkappa_1 \sigma Y) \bullet \left(\rho W + \sqrt{1-\rho^2}U\right), \end{aligned}$$

which in terms of characteristics means

$$b^K = L^{-1}b^L = \varkappa_0' + Y^2\varkappa_1' + \frac{1}{2}\sigma^2 Y^2\varkappa_1^2 + \left(\zeta_0 + \zeta_1 Y^2\right)\varkappa_1, \tag{3.4}$$

$$\begin{aligned} c^K &= L^{-2}c^L = \left(\sigma Y \varkappa_1\right)^2, \\ c^{KS} &= \rho\sqrt{c^K c^S} = \rho\sigma S Y^2 \varkappa_1, \\ c^{KY^2} &= \sqrt{c^K c^{Y^2}} = (\sigma Y)^2 \varkappa_1. \end{aligned} \tag{3.5}$$

Substitute from (2.1), (2.2), (3.4) and (3.5) into the local optimality condition (3.1). On collecting powers of Y we obtain

$$\begin{aligned} -\varkappa_0'(t) &= \zeta_0 \varkappa_1(t), \\ -\varkappa_1'(t) &= -\mu^2 + \left(\zeta_1 - 2\rho\sigma\mu\right)\varkappa_1(t) + \frac{1}{2}\sigma^2\left(1-2\rho^2\right)\varkappa_1^2(t), \end{aligned}$$

with terminal conditions $\varkappa_0(T) = \varkappa_1(T) = 0$ implied from $L_T = 1$. The solution for $\varkappa_0, \varkappa_1$ is obtained from Lemma 6.1 in the manner indicated above.

ii) We have $\varkappa_0(t) \leq 0, \varkappa_1(t) \leq 0$, for all $t \in [0,T], T < T^\star$ hence $L \in (0,1]$. Since $\varkappa_0$ and $\varkappa_1$ are continuous and of finite variation L in (3.2) is a continuous semimartingale. By construction of $\varkappa_0, \varkappa_1$ equation (3.1) holds. Therefore L is a candidate opportunity process and

$$a = \frac{b^S + c^{KS}}{c^S} = (\mu + \rho\sigma\varkappa_1)/S_t \tag{3.6}$$

is the corresponding candidate adjustment process. □

The process

$$Z := \frac{L}{L_0}\exp\left(-b^K \bullet I\right) = \mathcal{E}\left(M^K\right) = \mathcal{E}\left((\varkappa_1\sigma Y) \bullet \left(\rho W + \sqrt{1-\rho^2}U\right)\right)$$

is a bounded positive martingale and by virtue of Girsanov's theorem

$$\begin{aligned} W^\star &:= -(\varkappa_1\sigma\rho Y) \bullet I + W, &(3.7)\\ U^\star &:= -\left(\varkappa_1\sigma\sqrt{1-\rho^2}Y\right) \bullet I + U, &(3.8)\end{aligned}$$

are Brownian motions under $P^\star$ with $dP^\star/dP = Z_T$. In view of (1.1), (1.2), (3.7) and (3.8) the $P^\star$-dynamics of S and Y read

$$\begin{aligned} \mathcal{L}(S) &= (\mu + \varkappa_1\sigma\rho)Y^2 \bullet I + Y \bullet W^\star, &(3.9)\\ Y^2 &= Y_0^2 + \left(\zeta_0 + \zeta_1^\star Y^2\right) \bullet I + \sigma Y \bullet \left(\rho W^\star + \sqrt{1-\rho^2}U^\star\right), &(3.10)\\ \zeta_1^\star &:= \zeta_1 + \sigma^2\varkappa_1 &(3.11)\end{aligned}$$

and we have

$$a = b^{S\star}/c^S = (\mu + \varkappa_1\sigma\rho)/S.$$

To be fully in the setup of CK05 we have to verify that the price process S admits an equivalent martingale measure with square integrable density. The following lemma shows that the candidate variance-optimal measure (see equation 1.3) has the desired property.

Lemma 3.3. *For a, L and $T < T^\star$ in Proposition 3.2 define $\hat{Z} := L\mathcal{E}(-a \bullet S)/L_0$, then*

1. *the local martingale $\hat{Z}$ is a martingale,*
2. *the measure $Q^\star$, $dQ^\star = \hat{Z}_T dP$, is an equivalent martingale measure,*
3. *the local martingale $L(\mathcal{E}(-a \bullet S))^2/L_0$ is a martingale and therefore $Q^\star \in \mathcal{M}_2^e$.*

Proof. One can write

$$\begin{aligned} \hat{Z} &= \mathcal{E}(K - a \bullet S - a \bullet \langle K, S\rangle) = \mathcal{E}(M^K - a \bullet M^S + (b^K - a(b^S + c^{KS})) \bullet I)\\ &= \mathcal{E}(M^K - a \bullet M^S),\end{aligned}$$

the last equality a consequence of the local optimality criterion (3.1). Thus $\hat{Z}$ is a local martingale and by Section 5 in Hobson (2004) also a true martingale. Let $b^S_{Q^\star}$ denote the drift of S under measure $Q^\star$, then by Girsanov's theorem (2.3, 2.4),

$$b^S_{Q^\star} = b^S + c^{KS} - ac^S = 0,$$

where the final equality follows from the definition of a, equation (3.6). Consequently, $Q^\star$ is an equivalent martingale measure and $\mathcal{E}(-a \bullet S)$ is a local $Q^\star$-martingale. It follows (cf. Jacod and Shiryaev 2003, III.3.8) that $\hat{Z}\mathcal{E}(-a \bullet S) = L\left(\mathcal{E}\left(-a \bullet S\right)\right)^2/L_0$ is a local martingale and by Section 5 in Hobson (2004) it is a true martingale. □

Remark 3.4. *Under conditions (1)-(3) in Lemma 3.3 Hobson (2004) conjectures that $Q^\star$ is the true VOMM. The validity of such a statement is not obvious in general (cf. Černý and Kallsen 2006). While the conjecture that $Q^\star$ is the VOMM for any $T < T^\star$ may be true in the present model we are not aware of any* proof *to that effect. In general, to conclude that the candidate measure $Q^\star$ is the true VOMM one has to show that $\mathcal{E}\left(-a \bullet S\right)$ is a Q-martingale for all $Q \in \mathcal{M}^e_2$. In the sequel we are able to prove that $Q^\star$ is the true VOMM for sufficiently small T.*

Proposition 3.5. *Take $T < T^\star$ in the notation of Proposition 3.2. If*

$$E\left(e^{\langle a \bullet S\rangle_T}\right) < \infty$$

then L and a in Proposition 3.2 are the true opportunity process and adjustment process, respectively, in the sense of CK05, Definitions 3.3 and 3.8. Consequently, $Q^\star$ defined in Lemma 3.3 (2) is the VOMM.

Proof. Step 1: We show that $\mathcal{E}\left(\left(-a1_{\rrbracket\tau,T\rrbracket}\right) \bullet S\right) L$ is of class (D) for any stopping time τ. Fix a stopping time τ and set $N := K - a \bullet S - \langle K, a \bullet S\rangle$. Lemma 3.3 shows that $\hat{Z} = \mathcal{E}\left(-a \bullet S\right) L/L_0 = \mathcal{E}\left(N\right)$ is a positive martingale. Then

$$\hat{Z}/\hat{Z}^\tau = \mathcal{E}(N)/\mathcal{E}(N^\tau) = \mathcal{E}(N - N^\tau)$$

is a positive local martingale and therefore a supermartingale. Since

$$\begin{aligned} E\left(\left(\hat{Z}/\hat{Z}^\tau\right)_T\right) &= E\left(E\left(\hat{Z}_T/\hat{Z}_\tau|\mathcal{F}_\tau\right)\right) = E\left(E\left(\hat{Z}_T|\mathcal{F}_\tau\right)/\hat{Z}_\tau\right) \\ &= E(\hat{Z}_\tau/\hat{Z}_\tau) = 1 = (\hat{Z}/\hat{Z}^\tau)_0 \end{aligned}$$

$\hat{Z}/\hat{Z}^\tau$ is actually a true martingale and hence of class (D). Since L is bounded

$$\mathcal{E}\left(\left(-a1_{\rrbracket\tau,T\rrbracket}\right) \bullet S\right) L = L^\tau\hat{Z}/\hat{Z}^\tau$$

is of class (D) as well.

Step 2: We show that $\lambda := a1_{]\!]\tau,T]\!]}\mathcal{E}\left(\left(-a1_{]\!]\tau,T]\!]}\right)\bullet S\right)$ is an admissible trading strategy for any stopping time τ. Consider a measure $Q \in \mathcal{M}_2^e$. By Hölder's inequality and hypothesis we have

$$E^Q\left(\mathrm{e}^{\frac{1}{2}\left\langle\left(a1_{]\!]\tau,T]\!]}\right)\bullet S\right\rangle_T}\right) \leq \sqrt{E\left((dQ/dP)^2\right)E\left(\mathrm{e}^{\langle a\bullet S\rangle_T}\right)} < \infty,$$

whereby Novikov condition implies that $\mathcal{E}\left(-\left(a1_{]\!]\tau,T]\!]}\right)\bullet S\right)$ is a Q-martingale for *any* $Q \in \mathcal{M}_2^e$. Noting that $1-\lambda\bullet S = \mathcal{E}\left(\left(-a1_{]\!]\tau,T]\!]}\right)\bullet S\right)$ we conclude that $E^Q(\lambda\bullet S_T) = 0$ for all $Q \in \mathcal{M}_2^e$. By virtue of Lemma 3.3 (1,3) we have

$$\begin{aligned} E((\lambda\bullet S_T)^2) &= E\left(E\left(\left(1-\mathcal{E}\left(\left(-a1_{]\!]\tau,T]\!]}\right)\bullet S\right)_T\right)^2\middle|\mathcal{F}_\tau\right)\right) \\ &= E\left(E\left(\left(1-\frac{\mathcal{E}\left(-a\bullet S\right)_T}{\mathcal{E}\left(-a\bullet S\right)_\tau}\right)^2\middle|\mathcal{F}_\tau\right)\right) = E(1-L_\tau) < 1, \end{aligned}$$

implying $\lambda\bullet S_T \in L^2(P)$. Theorem 2.8 yields $\lambda \in \overline{\Theta}$.

Step 3: We have shown in Proposition 3.2 that conditions 1, 2 and 3 of CK05, Theorem 3.25 are satisfied. Steps 1 and 2 of this proof show that condition 4 (CK05, equations 3.33, 3.34) is satisfied, too. Hence a and L represent the true adjustment and opportunity process, respectively. Proposition 3.13 in CK05 implies that $\hat{Z}$ is the density of the variance-optimal martingale measure. □

Proposition 3.6. *Consider the function $\varkappa_1$ and the parameter $T^\star$ defined in Proposition 3.2. For $T < T^\star$ define*

$$\nu(T) := \max_{t\in[0,T]}\left(\mu+\sigma\rho\varkappa_1(t)\right)^2. \tag{3.12}$$

Set $A = \nu(T)$, $B = \zeta_1$, $C = \frac{1}{2}\sigma^2$ and for the corresponding parameter $\tau^\star \in \mathbb{R}_+ \cup \{\infty\}$ defined in Lemma 6.1 set

$$\tilde{T}(\nu(T)) := \tau^\star.$$

1. *If $\tilde{T}(\nu(T)) > T$ then a, L in Proposition 3.2 represent the true adjustment and opportunity process, respectively.*
2. *The condition*

$$\tilde{T}(\nu(T)) > T$$

is always satisfied for small enough $T > 0$.

Proof. 1) i) We have

$$E(\mathrm{e}^{\langle a\bullet S\rangle_T}) = E\left(\mathrm{e}^{((\mu+\sigma\rho\varkappa_1)Y)^2\bullet I_T}\right) \leq E\left(\mathrm{e}^{\nu(T)\left(Y^2\bullet I_T\right)}\right),$$

with $\nu(T)$ defined in (3.12).

ii) On defining $R := Y^2 \bullet I$ DFS03 (see Theorem 3.2 in Kallsen (2006) for details) yields that $(Y^2, R, \ln S)$ is conservative regular affine and therefore for $\operatorname{Re} z \leq 0$ we have

$$E\left(\mathrm{e}^{zR_t}\right) = \mathrm{e}^{u_0(t)+u_1(t)Y_0^2+u_2(t)R_0+u_3(t)\ln S_0},$$

where the complex functions u_0, u_1, u_2, u_3 satisfy the following system of Riccati equations,

$$\begin{aligned} u_0' &= \zeta_0 u_1,\ u_2' = u_3' = 0, \\ u_1' &= \zeta_1 u_1 + u_2 + (\mu - 1/2)u_3 + \frac{1}{2}\left(\sigma^2 u_1^2 + 2\rho\sigma u_1 u_3 + u_3^2\right), \\ u_0(0) &= u_1(0) = u_3(0) = 0,\ u_2(0) = z. \end{aligned}$$

This implies

$$\begin{aligned} u_3 &= 0, \\ u_2 &= z, \\ u_1' &= z + \zeta_1 u_1 + \frac{1}{2}\sigma^2 u_1^2. \end{aligned}$$

iii) Fix $\nu > 0$ and $t < \tilde{T}(\nu)$. By continuity and monotonicity of $\tilde{T}$ (cf. Lemma 6.2) we have $t < \tilde{T}(\nu + \varepsilon)$ for all $\varepsilon > 0$ sufficiently small. We now show that for all sufficiently small $\varepsilon > 0$ functions $u_0(t)$ and $u_1(t)$ (considered as functions of z) possess analytic extension on the strip $z \in (-1, \nu+\varepsilon) \times \mathrm{i}(-\varepsilon, \varepsilon)$. For $A = z, B = \zeta_1$, and $C = \sigma^2/2$ function g in Lemma 6.1 does not attain the value 0 on $[-1, \nu+\varepsilon]$. Hence for all sufficiently small $\varepsilon > 0$ function g in Lemma 6.1 is bounded away from 0 on $[-1, \nu+\varepsilon]\times \mathrm{i}[-\varepsilon, \varepsilon]$. It follows that both $u_0(t)$ and $u_1(t)$ are analytic on $(-1, \nu+\varepsilon) \times \mathrm{i}\,(-\varepsilon, \varepsilon)$.

iv) By iii) and Lemma A.4 in DFS03 $E\left(\exp\left(\nu(T)Y^2 \bullet I_T\right)\right) < \infty$ for $\tilde{T}(\nu(T)) > T$.

2) Since $\varkappa_1$ is a continuous function and $\varkappa_1(T) = 0$ there is $\varepsilon > 0$ such that for all $T < \varepsilon$ we have $0 < \nu(T) < \mu^2 + 1$. Furthermore $\inf\{\tilde{T}\,(\nu) : 0 \leq \nu \leq \mu^2 + 1\} =: \delta > 0$ because $\tilde{T}$ is a positive function and continuous when not equal to $+\infty$. Consequently for $T < \min(\varepsilon, \delta)$ we have

$$\tilde{T}\,(\nu(T)) > T.$$

□

4. Optimal hedging

From now on fix a time horizon $T > 0$ such that $\tilde{T}\,(\nu(T)) > T$. Existence of such a time horizon is guaranteed by Proposition 3.6. Furthermore, we need to make sure that the contingent claim H has a finite second moment under P. For technical reasons (cf. Proposition 4.1) we restrict our attention to bounded contingent claims such as European put options. This automatically guarantees $H \in L^2(P)$.

The optimal hedge is given by the Föllmer–Schweizer decomposition of H under measure $P^\star$as follows. By Lemma 3.23 in CK05 the variance-optimal measure $Q^\star$ coincides with the minimal measure relative to $P^\star$ (see also equation 3.9)

$$\begin{aligned}\frac{dQ^\star}{dP^\star} &= \mathcal{E}\left(-a \bullet M^{S\star}\right)_T = \mathcal{E}\left(-(aS) \bullet (Y \bullet W^\star)\right)_T \\ &= \mathcal{E}\left(-\left((\mu + \varkappa_1 \sigma\rho)\, Y\right) \bullet W^\star\right)_T .\end{aligned}$$

By virtue of Girsanov's theorem

$$\begin{aligned}\hat{W}^\star &:= \left((\mu + \varkappa_1 \sigma\rho)\, Y\right) \bullet I + W^\star, \\ \hat{U}^\star &:= U^\star\end{aligned}$$

are uncorrelated Brownian motions under $Q^\star$ and therefore the $Q^\star$-dynamics of S and Y read

$$\begin{aligned}\mathcal{L}(S) &= Y \bullet \hat{W}^\star, \\ Y^2 &= Y_0^2 + \left(\zeta_0 + \hat{\zeta}_1^\star Y^2\right) \bullet I + \sigma Y \bullet \left(\rho \hat{W}^\star + \sqrt{1-\rho^2}\,\hat{U}^\star\right), \\ \hat{\zeta}_1^\star &:= \zeta_1^\star - \rho\sigma\left(\mu + \varkappa_1 \sigma\rho\right) = \zeta_1 - \rho\sigma\mu + \varkappa_1 \sigma^2\left(1-\rho^2\right).\end{aligned}$$

Define the mean value process V

$$V_t := E^{Q^\star}\left(H | \mathcal{F}_t\right).$$

Proposition 4.1. *If the contingent claim H is given by $g(Y_T^2, S_T)$ where g is a bounded continuous function then $V_t = f(T-t, Y_t^2, S_t)$ for $f \in \mathcal{C}^{1,2,2}$ and f is the unique classical solution of the PDE*

$$\begin{aligned}0 &= -f_1 + \left(\zeta_0 + \hat{\zeta}_1^\star y\right) f_2 + \frac{1}{2} y \left(\sigma^2 f_{22} + 2\rho\sigma s f_{23} + s^2 f_{33}\right), \\ f(0,y,s) &= g(y,s),\end{aligned}$$

with $f_i := \partial f / \partial x_i, f_{ij} := \partial^2 f / \left(\partial x_i \partial x_j\right)$.

Proof. The proof is given in Heath and Schweizer (2000), Section 2.1 for $\rho = 0$. The reasoning for $\rho \neq 0$ is identical, since in either case $Q^\star$ is equivalent to P and $\hat{\zeta}_1^\star$ is continuously differentiable in time regardless of the value of ρ. □

Proposition 4.1 together with Proposition 4.7 in CK05 and Itô's formula yield an explicit expression for the pure hedge ξ

$$\begin{aligned}\xi_t &:= c_t^{SV}/c_t^S = \frac{f_2(T-t,Y_t^2,S_t)c_t^{Y^2S}+f_3(T-t,Y_t^2,S_t)c_t^S}{c_t^S}\\ &= f_3(T-t,Y_t^2,S_t)+\rho\sigma f_2(T-t,Y_t^2,S_t)/S_t, \end{aligned} \tag{4.1}$$

where $f_i(x_1,x_2,\ldots,x_n):=\partial f/\partial x_i$.

Remark 4.2. *It is possible to provide a more explicit expressions for V and ξ subject to technical conditions whose verification we defer to future research. $(Y^2,\ln S)$ form a time-inhomogeneous conservative regular affine process under $Q^\star$, and one can use the characterization of Filipović (2005) to evaluate their joint characteristic function. For $\operatorname{Re} z=0$ we have*

$$E^{Q^\star}\left(e^{z\ln S_T}|\mathcal{F}_t\right)=e^{v_0(t,z)+v_1(t,z)Y_t^2+z\ln S_t},$$

where both v_i are functions of t and z solving

$$\begin{aligned}-\frac{\partial}{\partial t}v_0(t,z) &= \zeta_0 v_1(t,z)\\ -\frac{\partial}{\partial t}v_1(t,z) &= \frac{1}{2}\left(z^2-z\right)+v_1(t,z)\left(\zeta_1-\sigma\rho\left(\mu-z\right)+\sigma^2\left(1-\rho^2\right)\varkappa_1(t)\right)+\frac{1}{2}\sigma^2v_1^2(t,z),\\ v_0(T,z) &= v_1(T,z)=0.\end{aligned}$$

These Riccati equations are time-dependent and can only be solved numerically. If the $Q^\star$-characteristic function possesses analytic extension for $\operatorname{Re} z>0$ and subject to further technicalities one obtains

$$V_t=E^{Q^\star}\left(\int_{\beta-\mathrm{i}\infty}^{\beta+\mathrm{i}\infty}\pi(z)e^{z\ln S_T}dz\middle|\mathcal{F}_t\right)=\int_{\beta-\mathrm{i}\infty}^{\beta+\mathrm{i}\infty}\pi(z)e^{v_0(t,z)+v_1(t,z)Y_t^2+z\ln S_t}dz, \tag{4.2}$$

where $\beta\in\mathbb{R}$ is a suitably chosen constant and $\pi(z)$ are the Fourier coefficients of the contingent claim (cf. Černý 2006, Hubalek et al. 2006),

$$H=\int_{\beta-\mathrm{i}\infty}^{\beta+\mathrm{i}\infty}\pi(z)e^{z\ln S_T}dz.$$

For example, a European put option with strike e^k yields $\pi(z)=\frac{e^{k(1-z)}}{2\pi\mathrm{i}z(z-1)},\beta<0$.

Subject to additional conditions one can differentiate under the integral sign in (4.2) and from (4.1) obtain

$$\xi_t=S_t^{-1}\int_{\beta-\mathrm{i}\infty}^{\beta+\mathrm{i}\infty}\left(z+\rho\sigma v_1(t,z)\right)\pi(z)e^{v_0(t,z)+v_1(t,z)Y_t^2+z\ln S_t}dz. \tag{4.3}$$

5. Hedging error

Proposition 4.1 and the Itô formula yield

$$\gamma_t := c_t^V - \frac{\left(c_t^{SV}\right)^2}{c_t^S} = \left(f_2(T-t, Y_t^2, S_t)\right)^2 \sigma^2 Y_t^2 \left(1-\rho^2\right).$$

By Theorem 3.22 in CK05 the minimal squared hedging error with initial capital V_0 satisfies

$$\begin{aligned} \varepsilon_0^2 &:= E\left(\left(V_0 + \varphi(V_0, H) \bullet S_T - H\right)^2\right) = E\left((L\gamma) \bullet I_T\right) \\ &= \sigma^2 \left(1-\rho^2\right) E\left(\int_0^T e^{\varkappa_0(t) + \varkappa_1(t) Y_t^2} Y_t^2 \left(f_2(T-t, Y_t^2, S_t)\right)^2 dt\right). \end{aligned}$$

Remark 5.1. *Subject to technical conditions one can use the Fourier expression for the mean value process (4.2) together with the "extended" Fourier transform of Duffie et al. (2000) to write*

$$\begin{aligned} \varepsilon_0^2 &= E\left((L\gamma) \bullet I_T\right) = E\left(\int_0^T \gamma_t L_t dt\right) \\ &= \left(1-\rho^2\right) \sigma^2 \int_0^T dt e^{\varkappa_0(t)} \int_{G^2} \prod_{i=1}^2 \left(dz_i v_1(t, z_i) \pi(z_i) e^{u_0(t, z_i)}\right) \\ &\quad \times \phi\left(t, \varkappa_1(t) + v_1(t, z_1) + v_1(t, z_2), z_1 + z_2\right) \end{aligned} \tag{5.1}$$

where ϕ is computed in Appendix B. We leave the detailed analysis of the technical conditions required to make (5.1) rigorous to future research.

We conclude this section by linking the hedging error ε_0^2 to option prices and performance measures.

Definition 5.2. *We call*

$$\mathrm{SR}_{S,H} := \sup\left\{\frac{E(\vartheta \bullet S_T + \eta\left(C_0 - H\right))}{\sqrt{\mathrm{Var}(\vartheta \bullet S_T + \eta\left(C_0 - H\right))}} : \vartheta \in \overline{\Theta}, \eta \in \mathbb{R}\right\} \tag{5.2}$$

the maximal unconditional Sharpe ratio*, where we set $\frac{0}{0} := 0$.*

Lemma 5.3. *The maximal unconditional Sharpe ratio is given by*

$$\mathrm{SR}_{S,H}^2 = \frac{1}{L_0} - 1 + \frac{(C_0 - V_0)^2}{\epsilon_0^2}, \tag{5.3}$$

with convention $0/0 = 0$.

Proof. Define $X := \vartheta \bullet S_T + \eta\left(C_0 - H\right)$. Easily,

$$\mathrm{SR}^2(X) := \frac{(E(X))^2}{\mathrm{Var}(X)} = \frac{1}{\inf_{\alpha \in \mathbb{R}} \{E\left((1-\alpha X)^2\right)\}} - 1 = \sup_{\alpha \in \mathbb{R}} \left\{\frac{1}{E\left((1-\alpha X)^2\right)} - 1\right\}.$$

Then

$$\begin{aligned}
\mathrm{SR}^2_{S,H} &= \sup_{\vartheta\in\overline{\Theta},\eta\in\mathbb{R}}\{\mathrm{SR}^2(X)\} = \sup_{\alpha\in\mathbb{R},\vartheta\in\overline{\Theta},\eta\in\mathbb{R}}\left\{\frac{1}{E\left((1-\alpha X)^2\right)}-1\right\}\\
&= \frac{1}{\inf_{\vartheta\in\overline{\Theta},\eta\in\mathbb{R}}\{E\left((1-X)^2\right)\}}-1\\
&= \frac{1}{\inf_{\eta\in\mathbb{R}}\left\{\inf_{\vartheta\in\overline{\Theta}}\{E\left((1-X)^2\right)\}\right\}}-1\\
&= \frac{1}{\inf_{\eta\in\mathbb{R}}\left\{L_0(1-\eta(C_0-V_0))^2+\eta^2\epsilon_0^2\right\}}-1,
\end{aligned}$$

where the last equality follows from CK05 Theorem 4.12 with contingent claim $1-\eta(H-C_0)$. By CK05 Theorem 4.10 with contingent claim $1-\eta(H-C_0)$ the optimal investment cum hedging strategy is given by $\varphi_S+\varphi_H$ (see equations 1.4 and 1.5). Straightforward calculations yield the optimal number of shares and the maximal Sharpe ratio,

$$\begin{aligned}
\eta &= \frac{C_0-V_0}{\epsilon_0^2}\frac{1}{1+\mathrm{SR}^2_{S,H}},\\
\mathrm{SR}^2_{S,H} &= 1/L_0-1+(C_0-V_0)^2/\epsilon_0^2.
\end{aligned}$$

□

6. Appendix A

Lemma 6.1. *Consider the following system of ordinary differential equations for* $\tau\geq 0,\ A,B,C,F,w_0,y_0\in\mathbb{C}$

$$w'(\tau) = A+Bw(\tau)+Cw^2(\tau), \tag{6.1}$$
$$w(0) = w_0, \tag{6.2}$$
$$y'(\tau) = Fw(\tau), \tag{6.3}$$
$$y(0) = y_0. \tag{6.4}$$

Define

$$\begin{aligned}
\hat{w}_0 &:= B/2+Cw_0,\\
D &:= \sqrt{B^2-4AC},
\end{aligned}$$

by taking the principal value of the square root with branch cut along the negative real line. Let

$$\tau^\star := \inf\{\tau\geq 0 : w(\tau) \text{ unbounded on } [0,\tau)\}.$$

Then w,y *given below represent a solution of (6.1)-(6.4) on* $[0,\tau^\star)$. *Where* w,y *might be multivalued we take the unique version continuous in* τ *on* $[0,\tau^\star)$ *and satisfying the initial conditions.*

(1) *For* $C = 0, D \neq 0$

$$\begin{aligned} w &= w_0 + \left(\frac{A}{B} + w_0\right)\left(e^{B\tau} - 1\right), \\ y &= y_0 + F\left(w_0\tau + \left(\frac{A}{B} + w_0\right)\left(\frac{e^{B\tau} - 1}{B} - \tau\right)\right), \\ \tau^\star &= +\infty. \end{aligned}$$

(2) *For* $C = 0, D = 0$

$$\begin{aligned} w &= w_0 + A\tau, \\ y &= y_0 + F\left(w_0\tau + \frac{A}{2}\tau^2\right), \\ \tau^\star &= +\infty. \end{aligned}$$

(3) *For* $C \neq 0,\ D = 0$

$$\begin{aligned} w &= C^{-1}\left(\frac{\hat{w}_0}{1 - \hat{w}_0\tau} - \frac{B}{2}\right), \\ y &= y_0 - F\left(\frac{1}{C}\ln\left(1 - \hat{w}_0\tau\right) + \frac{B\tau}{2C}\right), \\ \tau^\star &= +\infty \textit{ for } \operatorname{Im}\left(\hat{w}_0\right) \neq 0, \textit{ or } \hat{w}_0 \leq 0, \\ \tau^\star &= 1/\hat{w}_0 \textit{ for } \hat{w}_0 > 0. \end{aligned}$$

(4) *For* $C \neq 0,\ D \neq 0$

$$\begin{aligned} w &= -\frac{B}{2C} + \frac{D}{2C}\frac{\left(\hat{w}_0 + D/2\right)e^{-D\tau/2} + \left(\hat{w}_0 - D/2\right)e^{D\tau/2}}{\left(\hat{w}_0 + D/2\right)e^{-D\tau/2} - \left(\hat{w}_0 - D/2\right)e^{D\tau/2}}, \\ y &= y_0 + F\left(-\frac{B}{2C}\tau - \frac{1}{C}\ln\left(\frac{\left(\hat{w}_0 + D/2\right)e^{-D\tau/2} - \left(\hat{w}_0 - D/2\right)e^{D\tau/2}}{D}\right)\right), \\ \tau^\star &= \inf\{\tau \geq 0 : \left(\hat{w}_0 + D/2\right)e^{-D\tau/2} - \left(\hat{w}_0 - D/2\right)e^{D\tau/2} = 0\}. \end{aligned}$$

Furthermore, for B, C, w_0 *and* τ *fixed the functions*

$$\begin{aligned} f(A) &:= D\frac{\left(\hat{w}_0 + D/2\right)e^{-D\tau/2} + \left(\hat{w}_0 - D/2\right)e^{D\tau/2}}{\left(\hat{w}_0 + D/2\right)e^{-D\tau/2} - \left(\hat{w}_0 - D/2\right)e^{D\tau/2}}, \\ g(A) &:= \frac{\left(\hat{w}_0 + D/2\right)e^{-D\tau/2} - \left(\hat{w}_0 - D/2\right)e^{D\tau/2}}{D}, \end{aligned}$$

are complex differentiable on the set $\{A \in \mathbb{C} : \tau < \tau^\star\}$.

Proof. Straightforward calculations show that (1)-(4) solve the Riccati equations (6.1)-(6.4). The complex function $f(A)$ is differentiable everywhere apart possibly from the branch cut on the set

$$D^2 = B^2 - 4AC \in \mathbb{R}_-.$$

However, since

$$h(D) := D\frac{(\hat{w}_0 + D/2)\, e^{-D\tau/2} + (\hat{w}_0 - D/2)\, e^{D\tau/2}}{(\hat{w}_0 + D/2)\, e^{-D\tau/2} - (\hat{w}_0 - D/2)\, e^{D\tau/2}} = h(-D)$$

it follows that $f(A)$ is continuous and differentiable also at $D^2 \in \mathbb{R}_-$, and in particular at $D = 0$ where it has a removable singularity. The same argument applies to $g(A)$. □

Lemma 6.2. *Consider the setup of Lemma 6.1 with $A, B, C \in \mathbb{R}, w_0 = 0$. For B, C, w_0 fixed define the function $\tilde{T} : \mathbb{R} \to \mathbb{R}_+ \cup \{\infty\}$ by setting $\tilde{T}(A) := \tau^\star$. Then*

1. *For $C = 0$ we have $\tilde{T}(A) = +\infty$;*
2. *For $C \neq 0$ $\tilde{T}$ is continuous on $\mathbb{R}$, that is there is $A^\star \in \mathbb{R}$ such that*

$$\begin{aligned} \tilde{T}(A) &< \infty \text{ for } AC > A^\star C, \\ \lim_{AC \searrow A^\star C} \tilde{T}(A) &= \infty, \\ \tilde{T}(A) &= \infty \text{ for } AC \leq A^\star C. \end{aligned}$$

Specifically,

(a) *For $B \leq 0$ we have $A^\star C = B^2/4$ and*

$$\tilde{T}(A) = \frac{2\arctan\left(\sqrt{4AC - B^2}/B\right)}{\sqrt{4AC - B^2}} \text{ for } AC > B^2/4;$$

(b) *For $B > 0$ we have $A^\star = 0$ and*

$$\tilde{T}(A) = \begin{cases} \frac{1}{\sqrt{B^2-4AC}} \ln\left(\frac{B+\sqrt{B^2-4AC}}{B-\sqrt{B^2-4AC}}\right) & \text{for } B^2/4 > AC > 0 \\ 2/B & \text{for } B^2/4 = AC \\ \frac{2\arctan\left(\sqrt{4AC-B^2}/B\right)}{\sqrt{4AC-B^2}} & \text{for } AC > B^2/4 \end{cases}.$$

3. *$\tilde{T}$ is differentiable on $\mathbb{R}$ in all points where it is finite valued and*

$$C\tilde{T}'(A) < 0 \text{ for } \tilde{T}(A) < \infty.$$

Proof. Items (1) and (2) follow from Lemma 6.1 by direct calculation. Another calculation shows that the real function $\tilde{T}$ is continuous at $A = \frac{B^2}{4C}$ and differentiable there when finite-valued. We now examine the monotonicity of $\tilde{T}$. For $B > 0$ and $x := D/B \in (0, 1]$ we have

$$\begin{aligned} C\tilde{T}'(A) &= -\frac{2C^2}{B^2 D}\left(x^{-2}\ln\frac{1-x}{1+x} + \frac{2}{x(1-x^2)}\right) \\ &= -\frac{2C^2}{B^2 D}x^{-2}\int_0^x \left(\frac{2z}{1-z^2}\right)^2 dz < 0, \end{aligned}$$

since for $g(z) := \ln\frac{1-z}{1+z} + \frac{2z}{1-z^2}$ we have $g(0) = 0$ and $g'(z) = \left(\frac{2z}{1-z^2}\right)^2$.

For $B^2-4AC<0, B\neq 0$ we obtain for $x:=\frac{1}{B}\sqrt{4AC-B^2}$

$$
\begin{aligned}
C\tilde{T}'(A) &= \frac{2C^2}{B^2\sqrt{4AC-B^2}}\left(-\frac{2\arctan x}{x^2}+\frac{2}{x\,(1+x^2)}\right)\\
&= -\frac{2C^2}{(4AC-B^2)^{3/2}}\int_0^x\left(\frac{2z}{1+z^2}\right)^2 dz<0,
\end{aligned}
$$

since for $g(z):=-2\arctan x+\frac{2x}{1+x^2}$ we have $g(0)=0$ and $g'(z)=\left(\frac{2z}{1+z^2}\right)^2$. □

7. Appendix B

Define $z=(z_1,z_2)$, then DFS03 (see Theorem 3.2 in Kallsen (2006) for details) yields that $(Y^2,\ln S)$ is conservative regular affine and therefore for $\operatorname{Re} z_1\leq 0, \operatorname{Re} z_2=0$ we have

$$\psi(z,t)=E\left(e^{z_1Y_t^2+z_2\ln S_t}\right)=e^{v_0(t,z)+v_1(t,z)Y_0^2+v_2(t,z)\ln S_0},$$

where v_0,v_1,v_2 solve the following system of Riccati equations,

$$
\begin{aligned}
\frac{\partial v_2(t,z)}{\partial t} &= 0, &(7.1)\\
\frac{\partial v_0(t,z)}{\partial t} &= \zeta_0 v_1(t,z), &(7.2)\\
\frac{\partial v_1(t,z)}{\partial t} &= \frac{1}{2}z_2^2+z_2\left(\mu-\frac{1}{2}\right)+(\rho\sigma z_2+\zeta_1)\,v_1(t,z)+\frac{1}{2}\sigma^2 v_1^2(t,z), &(7.3)\\
v_0(0,z) &= 0, v_1(0,z)=z_1, v_2(0,z)=z_2. &(7.4)
\end{aligned}
$$

Set $A=\frac{1}{2}z_2^2+z_2\left(\mu-\frac{1}{2}\right)$, $B=\rho\sigma z_2+\zeta_1, C=\frac{1}{2}\sigma^2, F=\zeta_0$ and take w, y as in Lemma 6.1 with $w_0=z_1, y_0=0$. Then the system (7.1-7.4) is solved by $v_0=y, v_1=w, v_2=z_2$.

Under technical conditions (cf. Duffie et al. 2000) one has

$$
\begin{aligned}
\phi(t,z) &:= E\left(Y_t^2 e^{z_1Y_t^2+z_2\ln S_t}\right)=E\left(\frac{\partial}{\partial z_1}e^{z_1Y_t^2+z_2\ln S_t}\right)\\
&= \frac{\partial}{\partial z_1}E\left(e^{z_1Y_t^2+z_2\ln S_t}\right)=\frac{\partial\psi(t,z)}{\partial z_1}\\
&= e^{v_0(t,z)+v_1(t,z)Y_0^2+z_2\ln S_0}\left(\frac{\partial}{\partial z_1}v_0(t,z)+Y_0^2\frac{\partial}{\partial z_1}v_1(t,z)\right).
\end{aligned}
$$

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